SOME RESULTS ON TRANSFERABILITY IN LATTICES AND SEMILATTICES

by

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ABSTRACT

A finite semilattice $\mathfrak{S}$ is said to be transferable if whenever it can be embedded in a semilattice of ideals, $\mathfrak{J}(\mathfrak{S}^*)$ via ϕ there is an embedding ψ of $\mathfrak{S}$ in $\mathfrak{S}^*$ such that $x\psi \in y\phi$ if and only if $x \leq y$. In Chapter 1 we prove in a constructive manner that every distributive semilattice is transferable. We then give an application of this result showing that four classes of semilattices naturally obtained from an equational class of lattices coalesce to one class for the distributive case.

In Chapter 2, we give a decidable characterization of transferable semilattices. Our method is to construct the freest semilattice $\mathfrak{S}^*$ for which $\mathfrak{S}$ is embeddable in $\mathfrak{J}(\mathfrak{S}^*)$ and to obtain a necessary condition for transferability from this structure. This condition is then proved to be sufficient.

In Chapter 3, we define the concept of transferability for lattices by analogy with the semilattice case. We give a series of four conditions which taken together imply that a finite lattice $\mathfrak{L}$ is transferable. Using methods similar to those in Chapter 2 we show that three of these conditions and a slightly weaker fourth condition are necessary. We then show that for finite distributive lattices these two sets of conditions are equivalent. An alternate characterization of transferable distributive lattices is given in terms of simple conditions on the structure and width of the lattice in question.

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CHAPTER 0

§1. In this thesis we explore a small portion of the following problem first raised by Professor George Grätzer: What properties pass from a lattice $\mathcal{L}$ to its lattice of ideals $\mathfrak{J}(\mathcal{L})$ and conversely? The first observation which can be made is that an equation is valid in $\mathcal{L}$ if and only if it is valid in $\mathfrak{J}(\mathcal{L})$. The next question which naturally arises is: Is it possible to characterize the class Θ of finite lattices $\mathcal{L}'$ such that, for every lattice $\mathcal{L}$, if $\mathcal{L}'$ is embeddable in $\mathfrak{J}(\mathcal{L})$ then $\mathcal{L}'$ is embeddable in $\mathcal{L}$. We have termed such lattices $\mathcal{L}'$ weakly transferable, and it is our interest in the class Θ which motivates this thesis. The best general result on this question, which is due to Grätzer, [1] p. 208, is that if $\mathcal{L} \in \Theta$ then no point of $\mathcal{L}$ is both join and meet reducible. In Theorems 10.1 and 10.2 we give a complete and detailed structural description of those members of Θ which are distributive, based on results in [3].

Since the class of weakly transferable lattices proved very difficult to work with we have studied several simplifications. In Chapter 1 we investigate weak transferability for distributive semilattices, the notion of weakly transferable as applied to semilattices is obtained by analogy to the case for lattices. Our methods for showing that every finite distributive semilattice is transferable are quite different from our attack on the general problem.

In Chapter 2 we characterize the class of transferable semilattices. A semilattice $\mathfrak{S}$ is <u>transferable</u> if for every $\mathfrak{S}^*$ and ϕ embedding $\mathfrak{S}$ in $\mathfrak{I}(\mathfrak{S}^*)$ there is a ψ embedding $\mathfrak{S}$ in $\mathfrak{S}^*$ such that for all x and $y \in S$ $x\psi \in y\phi$ if and only if $x \leq y$. We define transferable lattice by analogy with our definition for semilattices and the first part of Chapter 3, Sections 8 and 9, is devoted to our investigation of this class of lattices. Our best results in this area being Theorems 9.1 and 9.2.

§2. We will need certain preliminary notions before proceeding to our results. Most of the results in this thesis are of an algebraic nature and as a general rule our notation for algebras and algebraic notions will be that of Grätzer [$\underset{\sim}{4}$]. Some of our results, however, belong more to the domain of logic than algebra and for these notions we will depend on Shoenfield [$\underset{\sim}{7}$].

Following Shoenfield, our basic language, which we will denote by λ_0 will have in addition to the usual logical symbols only the binary operation symbol $\underset{\sim}{+}$. We note that for symbols of the formal language we shall place a $\sim$ under the symbol to represent boldface. For convenience if it is clear that we are working in the formal language and no confusion can arise, we will drop the $\sim$. We will use small Greek letters $\underset{\sim}{\tau}, \underset{\sim}{\sigma}, \underset{\sim}{\theta}$ to denote terms in λ_0 or an expanded language, and $\underset{\sim}{\Phi}, \underset{\sim}{\Psi}$ to denote formulas. We will reserve the letters ϕ and ψ for homomorphisms. Our basic theory which we denote by Λ_0 is the theory of semi-lattices having as axioms:

$$\underset{\sim}{\Phi}_1 : (\underset{\sim}{x} \underset{\sim}{+} \underset{\sim}{y}) \underset{\sim}{+} \underset{\sim}{z} \underset{\sim}{=} \underset{\sim}{x} \underset{\sim}{+} (\underset{\sim}{y} \underset{\sim}{+} \underset{\sim}{z})$$

$$\underset{\sim}{\Phi}_2 : \underset{\sim}{x} \underset{\sim}{+} \underset{\sim}{y} \underset{\sim}{=} \underset{\sim}{y} \underset{\sim}{+} \underset{\sim}{x}$$

$$\underset{\sim}{\Phi}_3 : \underset{\sim}{x} \underset{\sim}{+} \underset{\sim}{x} \underset{\sim}{=} \underset{\sim}{x} \quad .$$

By a <u>semilattice</u>, we mean a structure $\mathfrak{S} = \langle S; +\rangle$ which is a model of Λ_0. By a <u>model</u> of a theory Γ we mean a structure in the language of Γ in which each formula of Γ is valid. If $\mathfrak{A}$ is a model of Γ we will write $\mathfrak{A} \models \Gamma$. If $\underset{\sim}{\Phi}$ is valid in $\mathfrak{A}$ we will write $\mathfrak{A} \models \underset{\sim}{\Phi}$. Further, we will write $\Gamma \vdash \underset{\sim}{\Phi}$ in case $\underset{\sim}{\Phi}$ is a logical consequence of Γ.

Associated with Λ_0 is the theory Λ_0' in a language λ_0' having one binary relation symbol, $\underset{\sim}{\leq}$ and satisfying the following nonlogical axioms:

$$\Phi_1' : \underset{\sim}{x} \underset{\sim}{\leq} \underset{\sim}{x}$$

$$\Phi_2' : \underset{\sim}{x} \underset{\sim}{\leq} \underset{\sim}{y} \;\&\; \underset{\sim}{y} \underset{\sim}{\leq} \underset{\sim}{x} \;.\!\rightarrow\!.\; \underset{\sim}{x} = \underset{\sim}{y}$$

$$\Phi_3' : \underset{\sim}{x} \underset{\sim}{\leq} \underset{\sim}{y} \;\&\; \underset{\sim}{y} \underset{\sim}{\leq} \underset{\sim}{z} \;.\!\rightarrow\!.\; \underset{\sim}{x} \underset{\sim}{\leq} \underset{\sim}{z}$$

$$\Phi_4' : \forall \underset{\sim}{x}\, \forall \underset{\sim}{y}\, \exists \underset{\sim}{z}\, \forall \underset{\sim}{w} (\underset{\sim}{x} \underset{\sim}{\leq} \underset{\sim}{z} \;\&\; \underset{\sim}{y} \underset{\sim}{\leq} \underset{\sim}{z}) \;\&\; [(\underset{\sim}{x} \underset{\sim}{\leq} \underset{\sim}{w} \;\&\; \underset{\sim}{y} \underset{\sim}{\leq} \underset{\sim}{w}) \rightarrow \underset{\sim}{z} \underset{\sim}{\leq} \underset{\sim}{w})] \;.$$

A model of Λ_0' is called a <u>semilattice ordering system</u>. It is well known that with any semilattice $\langle S; +\rangle$ we can associate a unique semilattice ordering system, $\langle S; \leq\rangle$ such that for any a and $b \in S$, $a \leq b$ if and only if $a + b = b$. As a general rule, we will assume that in any semilattice we have the relation $\leq$ defined so that $\leq$ and $+$ satisfy the above relationship.

A structure $\langle S; \leq\rangle$ will be termed a <u>quasiordering structure with least upper bounds</u> provided that it satisfies Φ_1', Φ_3' and Φ_4'. If we define $\equiv$ on S by $a \equiv b$ if and only if $a \leq b$ and $b \leq a$, then $\equiv$ is an equivalence relation on S. If we further define $\leq/\equiv$ by $[a]^{\equiv} \leq/\equiv [b]^{\equiv}$ if and only if for some $x \in [a]^{\equiv}$ and $y \in [b]^{\equiv}$ $x \leq y$, then $\langle S/\equiv; \leq/\equiv\rangle$ is a semilattice ordering system.

By a <u>lattice</u>, we mean a structure in the language λ_1 of two binary operation symbols which satisfies in addition to $\Phi_1 - \Phi_3$ the following:

$$\Phi_1^* : (\underset{\sim}{x} \underset{\sim}{\cdot} \underset{\sim}{y}) \underset{\sim}{\cdot} \underset{\sim}{z} \underset{\sim}{=} \underset{\sim}{x} \underset{\sim}{\cdot} (\underset{\sim}{y} \underset{\sim}{\cdot} \underset{\sim}{z})$$

$$\Phi_2^* : \underset{\sim}{x} \underset{\sim}{\cdot} \underset{\sim}{y} \underset{\sim}{=} \underset{\sim}{y} \underset{\sim}{\cdot} \underset{\sim}{x}$$

$$\Phi_3^* : \underset{\sim}{x} \underset{\sim}{\cdot} \underset{\sim}{x} \underset{\sim}{=} \underset{\sim}{x}$$

$$\Phi_4^* : \underset{\sim}{x} \underset{\sim}{\cdot} (\underset{\sim}{y} \underset{\sim}{+} \underset{\sim}{x}) \underset{\sim}{=} \underset{\sim}{x} \underset{\sim}{+} (\underset{\sim}{y} \underset{\sim}{\cdot} \underset{\sim}{x}) \underset{\sim}{=} \underset{\sim}{x} .$$

Henceforth we will write $\underset{\sim}{x}\underset{\sim}{y}$ for $\underset{\sim}{x} \underset{\sim}{\cdot} \underset{\sim}{y}$. We note that if $\langle L; +; \cdot\rangle$ is a lattice and we define $\leq$ on L by $a \leq b$ if and only if $a + b = b$, then $a \leq b$ if and only if $ab = a$. The converse is also valid.

By an <u>ideal</u> of a semilattice $\mathfrak{S}$ we mean a non-empty subset I of S such that (i) a and $b \in I$ imply $a + b \in I$ and (ii) $a \in I$ and $b \leq a$ imply $b \in I$. We will denote the set of ideals of $\mathfrak{S}$ by $I(\mathfrak{S})$. It is well known that the structure $\mathfrak{J}(\mathfrak{S}) = \langle I(\mathfrak{S}); +\rangle$ is a semilattice, where

$$I_1 + I_2 = \{a : a \in S \text{ and } \exists b_1 \in I_1, \exists b_2 \in I_2 \text{ and } a \leq b_1 + b_2\} .$$

Further, if $\mathfrak{S}$ satisfies

$$\underset{\sim}{\Phi}_5 : \forall \underset{\sim}{x}\ \forall \underset{\sim}{y}\ \exists \underset{\sim}{z}\ (\underset{\sim}{z} \leq \underset{\sim}{x}\ \&\ \underset{\sim}{z} \leq \underset{\sim}{y})$$

then the structure $\langle I(\mathfrak{S}); +, \cap\rangle$ is a lattice, where $\cap$ is set theoretic intersection. For our purposes this structure always exists, although if $\underset{\sim}{\Phi}_5$ is not satisfied, then the structure is a partial algebra, hence not a lattice. For lattices we define ideals in the same manner. If $\mathfrak{L}$ is any lattice then $\mathfrak{J}(\mathfrak{L}) = \langle I(\mathfrak{L}); +, \cap\rangle$ is a lattice.

Let K be a class of lattices. If there is some set of equations Λ in the language of lattices and K is the class of all models of Λ, then K is said to be an <u>equational class</u>. Of particular interest is the class Δ of all lattices satisfying the distributive law:

$$\underset{\sim}{\Phi}_6 : (\underset{\sim}{x} + \underset{\sim}{y})\underset{\sim}{z} = \underset{\sim}{x}\underset{\sim}{z} + \underset{\sim}{y}\underset{\sim}{z} .$$

Let K be an equational class. We define Γ_K a collection of formulas in the languages of semilattices by

$$\Gamma_K = \{\underset{\sim}{\Phi} : \underset{\sim}{\Phi} \text{ is valid in each element of } K \text{ and is a sentence of the form } \forall \underset{\sim}{x}_0 \ldots \forall \underset{\sim}{x}_{n-1} \exists \underset{\sim}{y}_0, \ldots, \exists \underset{\sim}{y}_{m-1} \underset{\sim}{\Psi} \text{ where } \underset{\sim}{\Psi} \text{ is open }\}.$$

For any fixed equational class K and positive integer n there is a lattice $\mathcal{L}_n^K$, unique up to isomorphism, satisfying the conditions that (i) $\mathcal{L}_n^K \in K$ and (ii) $\mathcal{L}_n^K$ is generated by elements $\{a_0, \ldots, a_{n-1}\}$ and any map of these generators into an element $\mathcal{L}$ in K has a unique extension to a homomorphism. $\mathcal{L}_n^K$ is the <u>free</u> K <u>lattice</u> on n generators. For a further discussion of free lattices we refer the reader to Chapter 4 of [$\underset{\sim}{4}$].

By a <u>direct family</u> $\mathcal{A}$ we will mean a triple $\langle\{\mathfrak{S}_i : i \in I\}, \langle I; \leq\rangle, \{\phi_i^j : i, j \in I \;\&\; i \leq j\}\rangle$ where $\{\mathfrak{S}_i : i \in I\}$ is a family of similar algebras indexed by I, $\langle I; \leq\rangle$ is a directed partial ordering and $\{\phi_i^j : i, j \in I \;\&\; i \leq j\}$ is a collection of homomorphisms such that ϕ_i^j maps S_i into S_j and, if $i \leq j \leq k$, then $\phi_i^k = \phi_i^j \phi_j^k$. For the direct family $\mathcal{A}$, let $S = \bigcup\{S_i : i \in I\}$ and define $\equiv$ on S by $x \equiv y$ where $x \in S_i$ and $y \in S_j$ if for some $k \in I$ with $i, j \leq k$ we have $x\phi_i^k = y\phi_j^k$. It is easy to see that $\equiv$ is an equivalence relation on S, whence let $S_\infty = S/\equiv$. For $x \in S_i$ let $[x]^\equiv$ denote the equivalence class of x. If $\mathfrak{S}_i = \langle S_i; F\rangle$ and $f \in F$ is n-ary let $[x_0]^\equiv, \ldots, [x_{n-1}]^\equiv \in S_\infty$. We put

$$f([x_0]^\equiv, \ldots, [x_{n-1}]^\equiv) = [f(x_0\phi_{i_0}^k, \ldots, x_{n-1}\phi_{i_{n-1}}^k)]^\equiv$$

where $x_0 \in S_{i_0}, \ldots, x_{n-1} \in S_{i_{n-1}}$ and $i_0, \ldots, i_{n-1} \leq k$. It is easily seen that this definition is independent of our choice of $x_0, \ldots, x_{n-1}$ and k. We set

$$\varinjlim \mathcal{A} = \mathfrak{S}_\infty = \langle S_\infty; F \rangle$$

and call $\mathfrak{S}_\infty$ the direct limit of the direct family $\mathcal{A}$. Note that the map ϕ_i^∞ from $\mathfrak{S}_i$ into $\mathfrak{S}_\infty$ defined by $x \to [x]^{\equiv}$ is clearly a homomorphism. For a complete treatment of direct limits see Grätzer [4], §21.

Natural numbers are regarded as finite ordinals. We will assume that for a given ordinal α, $\alpha = \{\beta : \beta$ is an ordinal and $\beta < \alpha\}$. It follows that the empty set denoted by "0" is the least ordinal. Further, if $P(n)$ is any proposition about a natural number n, then the least natural number for which $P(n)$ is satisfied is given by $\mu n[P(n)]$. Lastly, for any set A, $\mathcal{P}(A)$ will denote the power set of A. Any other notation will be discussed as it arises.

CHAPTER 1

§3. In this chapter we give a constructive solution to the problem of characterizing weakly transferable distributive semilattices. A semilattice is _distributive_ if it satisfies

$$\Phi_6' : \forall \underset{\sim}{x}\, \forall \underset{\sim}{y}\, \forall \underset{\sim}{z}\, \exists \underset{\sim}{u}\, \exists \underset{\sim}{v} [(\underset{\sim}{z} \underset{\sim}{\leq} \underset{\sim}{x} \underset{\sim}{+} \underset{\sim}{y}) \rightarrow (\underset{\sim}{u} \underset{\sim}{\leq} \underset{\sim}{x} \ \& \ \underset{\sim}{v} \underset{\sim}{\leq} \underset{\sim}{y} \ \& \ \underset{\sim}{u} \underset{\sim}{+} \underset{\sim}{v} \underset{\sim}{=} \underset{\sim}{z})].$$

Definition 3.1. Let $\mathfrak{S}$ be a finite semilattice. $\mathfrak{S}$ is _weakly transferable_ if $\mathfrak{S}$ can be embedded in every semilattice $\mathfrak{S}^*$ such that $\mathfrak{S}$ is embeddable in $\mathfrak{I}(\mathfrak{S}^*)$.

We may now state the problem of this section precisely. What finite distributive semilattices are weakly transferable? We will prove that every finite distributive semilattice has this property. In fact we will prove a stronger result, namely that they are all transferable.

Definition 3.2. A finite semilattice $\mathfrak{S}$ is said to be _transferable_ if whenever $\mathfrak{S}$ is embeddable in $\mathfrak{I}(\mathfrak{S}^*)$ via ϕ, there is an embedding ψ of $\mathfrak{S}$ in $\mathfrak{S}^*$ satisfying $x\psi \in y\phi$ if and only if $x \leq y$.

Thus the definition tells us that the embedding ψ is well behaved with respect to the embedding ϕ in the sense that $x\phi \in x\psi$ but $x\psi \notin y\phi$ unless $x \leq y$. Our proof that every finite distributive semilattice is transferable will depend on the following result which was first noticed by Green [5].

Lemma 3.1. Let $\mathfrak{S}$ be any semilattice. Then $\mathfrak{S}$ is distributive if and only if $\mathfrak{J}(\mathfrak{S})$ is a distributive lattice.

This result is easily seen to imply that every finite distributive semilattice $\mathfrak{S}$ is join isomorphic to a distributive lattice $\mathfrak{L}$. In fact $\mathfrak{L}$ may be taken to be $\mathfrak{J}(\mathfrak{S})$. We now have

Lemma 3.2. If $\mathfrak{S}$ is a finite distributive semilattice, then every element has a unique representation as a join of a join irredundant set of join irreducibles.

This is an immediate consequence of our preceding remarks and the analogous result for finite distributive lattices, [2, p.58].

Let $\mathfrak{S}$ be a fixed finite distributive semilattice and $\mathfrak{S}^*$ be any semilattice such that $\mathfrak{S}$ is embeddable in $\mathfrak{J}(\mathfrak{S}^*)$ via ϕ. We let $S_1 = S\phi \subseteq \mathfrak{J}(\mathfrak{S}^*)$. Let $S_2 \subseteq S_1$ consist of exactly those elements of $\langle S_1; +\rangle$ which are join irreducible.

Definition 3.2. A map ψ^* from S_2 into S^* is said to be <u>admissible</u> if (i) for each $x \in S_2$ $x\psi^* \in x$, and (ii) ψ^* is order preserving.

It is an immediate consequence of Lemma 3.2 that each admissible map has a natural extension to a join homomorphism from $\langle S_1; +\rangle$ into S^*. Further we observe that if ϕ^* and ψ^* are admissible maps, and we define $\phi^* \vee \psi^*$ by

$$x(\phi^* \vee \psi^*) = (x\phi^*) + (y\psi^*),$$

then $\phi^* \vee \psi^*$ is an admissible map.

For the remainder, let $S_2 = \{b_0,\ldots,b_{n-1}\}$, where this is a list in non-decreasing order.

Lemma 3.3. If $b \in S_2$ and $a \in b$ then there is an admissible map such that $a \leq b\psi^*$.

Proof. Let $j < n$ and suppose that we have picked $a_i \in b_i$ for each $i < j$ such that (1) if $b = b_i$ then $a \leq a_i$ and (2) if $i \leq s < j$ and $b_i \leq b_s$ then $a_i \leq a_s$. We choose a_j in the following manner. Let a'_j be any element of b_j and set

$$a_j = a'_j + \sum_{b_i \subset b_j} a_i + a,$$

where the last term, a, is included only if $a \in b_j$. It is trivial that $a_j \in b_j$ and satisfies (1) and (2). We set $b_i\psi^* = a_i$ and it is clear that ψ^* is an admissible map. This completes Lemma 3.3.

For each $J \subseteq n$ such that $\{b_i : i \in J\}$ is join irredundant and for each $a \in \sum_{i \in J} b_i$, let A_a^J be any set satisfying the following conditions (i) $A_a^J \subseteq \bigcup\{b_i : i \in J\}$, (ii) $a \leq \sum A_a^J$ and (iii) $|A_a^J| \cap b_i = 1$ for each $i \in J$. For any such J and a, A_a^J must clearly exist.

Lemma 3.4. Let $a \in b \in S_1$. Then there is an admissible map ψ^* such that its extension ψ to a homomorphism satisfies $a \leq b\psi$.

Proof. If $b \in S_2$ we are done. Otherwise b is join reducible. Then there exists unique $J \subseteq n$ such that $b = \Sigma\{b_i : i \in J\}$ and J is minimal. We choose A_a^J and to each $i \in J$ an admissible ψ^*_i

which satisfies $b_i\psi_i^* \geq x \in A_a^J \cap b_i$. Let $\psi^* = \bigvee_{i \in J} \psi_i^*$. Then if ψ is the extension of ψ^* to a join homomorphism, we have

$$a \leq \Sigma A_a^J \leq \sum_{i \in J} (b_i \psi^*) = b\psi,$$

as desired.

Theorem 3.1. Every finite distributive semilattice is transferable.

Proof. To each $x \in S$ pick an $a_x \in S^*$ such that $a_x \in y\phi$ if and only if $x \leq y$. Such choices are clearly possible. Now to each x we can choose a homomorphism $\psi_{a_x} : S_1 \to S^*$ such that $a_x \leq (x\phi)\psi_{a_x}$. We now define ψ on S_1 by

$$\psi = \bigvee_{x \in S} \psi_{a_x}.$$

It is clear that ψ is a homomorphism and the fact that $(x\phi)\psi \in y\phi$ if and only if $x \leq y$ ensures that ψ is 1-1. This completes our proof that $\mathfrak{S}$ is transferable since $\phi\psi$ embeds $\mathfrak{S}$ in $\mathfrak{S}^*$.

§4. We will now diverge from our main theme to give an application of our result for distributive semilattices. One of the most useful notions of modern algebra is that of equational class. While this concept plays a substantial role in lattice theory it unfortunately is completely trivial for semilattices. However the question still remains: Is it possible to usefully distinguish classes of semilattices? Early in our work on this thesis we spent some time considering this question.

Definition 4.1. Let κ be an equational class of lattices. With κ we associate classes of semilattices $\kappa_0, \kappa_1, \kappa_2, \kappa_3$ as follows:

(i) $\mathfrak{S} \in \kappa_0$ if and only if $\mathfrak{I}(\mathfrak{S}) = \langle I(\mathfrak{S}); +; \cap \rangle$ is a lattice in κ.

(ii) $\mathfrak{S} \in \kappa_1$ if and only if every formula in Γ_κ is valid in $\mathfrak{S}$.

(iii) $\mathfrak{S} \in \kappa_2$ if and only if every map of the generators of $\mathfrak{L}_n^\kappa$ into $\mathfrak{S}$ extends to a join homomorphism with the property that the equivalence relation induced on $\mathfrak{L}_n^\kappa$ by this map is a congruence relation.

(iv) $\mathfrak{S} \in \kappa_3$ if and only if $\mathfrak{S}$ is a direct limit of lattices in κ considered as semilattices.

The classes κ_0 and κ_1 are due to earlier authors [5][6] although most take only a finite part of Γ_κ to obtain κ_1. The class κ_3 was suggested by Lachlan and as far as the author knows the class κ_2 was first suggested by Gaskill. It is a consequence of our earlier work that for Δ, the class of distributive lattices, $\Delta_0 = \Delta_1 = \Delta_2 = \Delta_3$. The most general results known to the author are

Theorem 4.1. $\kappa_2 \subseteq \kappa_1 \subseteq \kappa_0$ and $\kappa_3 \subseteq \kappa_0$.

Theorem 4.2. If for each $n \in \omega$, $|\mathfrak{L}_n^\kappa| < \aleph_0$, then $\kappa_1 = \kappa_2 \subseteq \kappa_3$.

Since the proofs of these two results do not fit in with our main theme, we omit them, and continue with our concern in this section, namely, proving that

Theorem 4.3. $\Delta_0 = \Delta_1 = \Delta_2 = \Delta_3$.

Proof. We first show that Δ_1, Δ_2 and $\Delta_3 \subseteq \Delta_0$. For $\Delta_1 \subseteq \Delta_0$, since $\underset{\sim}{\phi}_6'$ is valid in every distributive lattice, we have $\underset{\sim}{\phi}_6' \in \Gamma_\Delta$. Hence, if $\mathfrak{S} \in \Delta_1$ then by Lemma 3.1 $\mathfrak{S} \in \Delta_0$.

To see that $\Delta_2 \subseteq \Delta_0$ we proceed as follows. Let a, b and $c \in S$ where $\langle S; +\rangle \in \Delta_2$ and $a + b \geq c$. Now if $\mathfrak{L}_3^\Delta$ is the free distributive lattice on three generators, x, y and z, let ϕ be any join homomorphism of $\mathfrak{L}_3^\Delta$ into $\mathfrak{S}$ such that $x\phi = a$, $y\phi = b$, $z\phi = c$ and $\equiv_\phi$ is a congruence relation on $\mathfrak{L}_3^\Delta$. Now by the distributive law we have

$$(xz)\phi + (yz)\phi = ((x + y)z)\phi.$$

Observe that we can define a second operation $\wedge$ on $(L_3^\Delta)\phi$ so that $\mathfrak{L}_3^\Delta/\equiv_\phi \approx \langle L_3^\Delta\phi; +, \wedge\rangle$. This is immediate from the fact that ϕ is a join homomorphism and $\equiv_\phi$ is a congruence relation. It follows that $((x + y)z)\phi \geq c$. Since $(xz)\phi + (yz)\phi \leq c$, we have $(xz)\phi + (yz)\phi = c$ and hence the defining formula for distributive semilattices is valid in $\mathfrak{S}$.

To show that $\Delta_3 \subseteq \Delta_0$, let $\mathcal{A} = \langle\{\mathfrak{S}_i : i \in I\}, \langle I; \leq\rangle, \{\phi_i^j : j, i \in I \ \& \ i \leq j\}\rangle$ be a direct family of distributive semilattices. We show that $\varinjlim \mathcal{A}$ is a distributive semilattice. Let $\langle S; +\rangle = \varinjlim \mathcal{A}$, and suppose we have a, b, and $c \in S$ such that $a + b \geq c$. Now there exists $i \in I$ and x, y, and $z \in S_i$ such that $x\phi_i^\infty = a$, $y\phi_i^\infty = b$

and $z\phi_i^\infty = c$. Observe that since $((x + y) + z)\phi_i^\infty = (x + y)\phi_i^\infty$, there exists a $k \in I$ such that $i \leq k$ and $((x + y) + z)\phi_i^k = (x + y)\phi_i^k$. It follows that $z\phi_i^k \leq (x + y)\phi_i^k$, and hence since $\mathfrak{S}_k$ is a distributive semilattice we may choose x' and $y' \in S_k$ such that $x' \leq x\phi_i^k$, $y' \leq y\phi_i^k$ and $x' + y' = z\phi_i^k$. Now if $a' = x'\phi_k^\infty$ and $b' = y'\phi_k^\infty$ we have $a' \leq a$, $b' \leq b$ and $a' + b' = c$. Thus $\mathfrak{S}$ is a distributive semilattice. Now, since every distributive lattice is a distributive semilattice we have immediately that $\Delta_3 \subseteq \Delta_0$.

To complete the proof we must show that $\Delta_0 \subseteq \Delta_1$, Δ_2 and Δ_3. We first claim that in any distributive semilattice $\mathfrak{S}$, if $S_1 \subseteq S$ is finite, then for some finite S_2 with $S_1 \subseteq S_2 \subseteq S$ we have $\langle S_2; +\rangle$ is a distributive semilattice. To see this, we proceed as follows, let ϕ be the injection of $\mathfrak{S}$ in $\mathfrak{J}(\mathfrak{S})$. Now let S'_2 be the lattice closure of $S_1\phi$ in $\mathfrak{J}(\mathfrak{S})$. Since $\mathfrak{J}(\mathfrak{S})$ is a distributive lattice by Lemma 3.1, we have that S'_2 is finite. Now $\langle S'_2; +\rangle$ is a distributive semilattice, whence by Theorem 3.1 we have that it is transferable. Moreover by Lemma 3.4 we may choose ψ embedding $\langle S'_2; +\rangle$ in $\mathfrak{S}$ such that if $x \in S_1$, then $x\phi\psi = x$. Thus $\langle S'_2\psi; +\rangle$ has the desired properties.

To see that $\Delta_0 \subseteq \Delta_2$, recall that every finite distributive semilattice is join isomorphic to a distributive lattice. A glance at the definition of Δ_2 now yields the result.

For $\Delta_0 \subseteq \Delta_1$, we let $\forall x_0 \ldots \forall x_{n-1} \exists y_0 \ldots \exists y_{m-1} \Psi \in \Gamma_D$. Now if $\mathfrak{S} \in \Delta_0$ then we must show that $\forall x_0 \ldots \exists y_{m-1} \Psi$ is valid in $\mathfrak{S}$. Let $a_0, \ldots, a_{n-1}$ be any n-element sequence in S. Then there is a finite $S_2 \supseteq \{a_0, \ldots, a_{n-1}\}$ such that $\langle S_2; +\rangle$ is a distributive semilattice.

Now form $\langle S_2; +, \wedge\rangle$, a distributive lattice. Then $\exists y_0 \ldots \exists y_{m-1}\, \Psi[a_0,\ldots,a_{n-1}]$ is valid in $\langle S_2; +, \wedge\rangle$, hence in $\langle S_2; +\rangle$. It follows that $\exists y_0 \ldots \exists y_{m-1}\, \Psi[a_0,\ldots,a_{n-1}]$ is valid in $\mathfrak{S}$. Since the sequence was arbitrary, the result follows. To obtain $\Delta_0 \subseteq \Delta_3$ we actually prove something stronger, which we state as

Theorem 4.2. $\mathfrak{S}$ is a distributive semilattice if and only if it is a 1-1 direct limit of finite distributive lattices considered as semilattices.

Proof. Sufficiency is a consequence of $\Delta_3 \subseteq \Delta_0$. Now for the converse, let $\mathfrak{S}$ be any distributive semilattice. Recall that every finite subset of S is contained in a finite substructure which is distributive. It is well known that any algebra is a 1-1 direct limit of its finitely generated subalgebras. Observe that since finitely generated implies finite for semilattices, we are done since the finite subsemilattices which are distributive are cofinal in the collection of all finite subsemilattices, and each such subsemilattice is isomorphic to a distributive lattice.

This concludes the proof of Theorem 4.1.

CHAPTER 2

In this chapter we present a characterization of transferable semilattices. Our procedure will be to build from a given finite semilattice $\mathfrak{S}$ a semilattice $\mathfrak{S}^*$ together with a map ϕ embedding $\mathfrak{S}$ in $\mathfrak{J}(\mathfrak{S}^*)$ such that if $\mathfrak{S}$ is not transferable then $\mathfrak{S}^*$ and ϕ constitute a counter example to the transferability of $\mathfrak{S}$. What we actually construct is a relational algebra $\mathfrak{A}^{\mathfrak{S}}$ whose retract $\langle A^{\mathfrak{S}}; +\rangle$ is the desired $\mathfrak{S}^*$. The relations on $\mathfrak{A}^{\mathfrak{S}}$ are all unary and each one constitutes an ideal in $\langle A^{\mathfrak{S}}; +\rangle$. These ideals taken together form an isomorphic copy of $\mathfrak{S}$ and from them we obtain the embedding ϕ.

§5. For the remainder of this Chapter, $\mathfrak{S} = \langle S; +\rangle$ will denote a fixed finite semilattice with $K = \{k_0, \ldots, k_{n-1}\} \subseteq S$ its set of join irreducibles. Further in this Chapter we treat $\mathfrak{J}(\mathfrak{S}^*) = \langle I(\mathfrak{S}^*); +\rangle$ purely as a semilattice.

Definition 5.1. Let $p \in n$ and $J \subseteq n$. The pair $\langle p, J\rangle$ is said to be <u>essential for</u> $\mathfrak{S}$ provided that

(i) $k_p \leq \sum_{i \in J} k_i$,

(ii) for each $i \in J$, $k_p \not\leq k_i$ and,

(iii) $\sum_{i \in J} k_i$ is join irredundant and minimal over k_p in the sense that if $J' \subsetneq J$ then $k_p \not\leq \sum_{J'} k_i$.

The inequality $k_p \leq \sum_{i \in J} k_i$ will be referred to as an <u>essential inequality</u>.

We construct a relational algebra, $\mathfrak{A}^{\mathfrak{S}}$ whose language consists of the following: a binary operation symbol $\underset{\sim}{+}$, a unary operation symbol $\underset{\sim}{f}{}^{J}_{i,p}$ for each triple $\langle p, J, i\rangle$ such that the pair $\langle p, J\rangle$ is essential for $\mathfrak{S}$ and $i \in J$, an individual constant $\underset{\sim}{c}_i$ for each $i \in n$, a unary relation symbol $\underset{\sim}{U}_i$ for each $i \in n$, and a unary relation symbol $\underset{\sim}{V}_s$ for each $s \in S \sim K$.

The construction of $\mathfrak{A}^{\mathfrak{S}}$ will take place in stages according to the following program. At stage m, $\mathfrak{A}_m$ will be a partial system whose retract $\langle A_m; +\rangle$ is a semilattice, for each $m \geq 1$. The $f^J_{i,p}$ will be partial operations with domain A_{m-1}. To obtain $\mathfrak{A}_{m+1}$ from $\mathfrak{A}_m$ we will first extend each operation $f^J_{i,p}$ to A_m, simultaneously extending the universe. Then we extend $+$ and the universe simultaneously so as to obtain a new universe A_{m+1} which is closed under $+$. Finally we will extend the relations U_i and V_s to the universe A_{m+1}. The resultant structure $\mathfrak{A}_{m+1}$ will be an extension of $\mathfrak{A}_m$ and we will set $\mathfrak{A}^{\mathfrak{S}} = \bigcup_{i\in\omega} \mathfrak{A}_i$.

Throughout the remainder of the construction, $B = \{b_0, b_1, b_2, \ldots, b_i, \ldots\}$ will be a fixed infinite set with $b_i = b_j$ only if $i = j$. The structure $\mathfrak{A}_0$ is defined as follows. $A_0 = \{b_0, \ldots, b_{n-1}\}$ and we define the 0-ary operation c_i by $c_i = b_i$. All other operations are empty. We put $b_i \in U_j$ if and only if $k_i \leq k_j$. For $s \in S \sim K$, we put $b_i \in V_s$ if and only if $k_i \leq s$. This completes the definition of $\mathfrak{A}_0$.

Before proceeding to the inductive stage, we define the structure $\mathfrak{A}_1$. We put

$$F^* = \bigcup\{\{\underset{\sim}{f}^J_{i,p}\} \times U_p : \langle p, J\rangle \text{ essential and } i \in J\}$$

and let g be any 1-1 map from F^* into $B \sim A_0$. We set $A_1' = A_0 \cup F^*g$ and define $f^J_{i,p}$ on A_0 by

$$f^J_{i,p}(b_q) = \begin{cases} (\underset{\sim}{f}^J_{i,p}, b_q)g & \text{if } b_q \in U_p \\ b_q & \text{if } b_q \notin U_p . \end{cases}$$

Now let $A^*_1 = \mathcal{P}(A_1') \sim \{0\}$. We define $\leq^*$ on A^*_1 as follows: for H_1 and $H_2 \in A^*_1$ we put $H_1 \leq^* H_2$ provided that if $a \in H_1$ and $a \notin H_2$ then for some essential pair $\langle p, J\rangle$ we have $a \in U_p$ and for each $i \in J$, $f^J_{i,p}a \in H_2$. We prove that $\leq^*$ is a quasi order with least upper bounds. We must show $\leq^*$ is reflexive and transitive. The former is obvious. Let $H_1 \leq^* H_2 \leq^* H_3$, we show $H_1 \leq^* H_3$. Let $a \in H_1$. If $a \in H_2 \cup H_3$ we are done. Suppose not. Then for some essential pair $\langle p, J\rangle$ and each $i \in J$ we have $f^J_{i,p}a \in H_2$. Since $a \in U_p$ we have $f^J_{i,p}a \notin A_0$, whence $f^J_{i,p}a \in H_3$, since $H_2 \leq^* H_3$. But this implies that $H_1 \leq^* H_3$ as desired. Thus $\leq^*$ is a quasi order. Now for any H_1 and H_2 in A^*_1, we claim that $H_1 \cup H_2$ is a $\leq^*$ least upper bound. It is clear that $H_1 \cup H_2$ is an upper bound. Now let H_3 be such that $H_1 \leq^* H_3$ and $H_2 \leq^* H_3$. Since for $a \in H_1 \cup H_2$, we must either have $a \in H_1$ or $a \in H_2$, it is easily seen that $H_1 \cup H_2 \leq^* H_3$. Thus $H_1 \cup H_2$ is a $\leq^*$ least upper bound as desired.

We now define the U_p and V_s on A^*_1. We first extend the U_p and V_s to A_1'. If $b_q \in U_p$, let $f^J_{i,p}b_q$ be in U_r or V_s if and only if $k_i \leq k_r$ or s respectively. For $H \in A^*_1$ put $H \in U_p$ or V_s if and only if $H \subseteq U_p$ or V_s respectively.

We define $\equiv$ on A_1^* by $H_1 \equiv H_2$ if and only if $H_1 \leq^* H_2$ and $H_2 \leq^* H_1$. Let $[H]^{\equiv}$ denote the equivalence class of H with respect to $\equiv$. As pointed out in the introduction the structure $\langle A_1^*/\equiv; \leq^*/\equiv\rangle$ is a semilattice ordering system. In addition, we claim that if $H_1 \equiv H_2$ and $H_1 \in U_r$ or V_s then $H_2 \in U_r$ or V_s respectively. To see this, let $H_2 \equiv H_1 \in U_r$. We fix $a \in H_2$ and show $a \in U_r$ from which the result will follow. Now if $a \notin H_1$ then for some essential pair $\langle p, J\rangle$ and each $i \in J$, $a \in U_p \cap A_0$ and $f_{i,p}^J a \in H_2$. Now $f_{i,p}^J a \in U_r$ if and only if $k_i \leq k_r$. Since this is valid for each $i \in J$, and since $\langle p, J\rangle$ is essential we have $k_p \leq \sum_{i \in J} k_i \leq k_r$. Now $a = b_q$ for some q whence $a \in U_p$ if and only if $k_q \leq k_p$. It follows that $k_q \leq k_r$ and hence that $a \in U_r$ as desired. Similar reasoning yields the result for V_s.

Let U_j, V_s, and $\leq$ denote the relations on $A_1^*/\equiv$ induced by U_j, V_s and $\leq^*$ respectively. It is clear from the definitions that the U_r and V_s are ideals. Further note that if $[H]^{\equiv} \in A_1^*/\equiv \cap U$ and $b_q \in U_r$ or V_s, then $[H]^{\equiv} \in U_r$ or V_s respectively. To see this for U_r, notice from the construction of $\mathfrak{A}_0$ and the definition of U_r on A_1' that $U_q \subseteq U_r$ if and only if $b_q \in U_r$, if and only if $b_q \leq b_r$. It follows that $[H]^{\equiv} \in U_r$. Similar reasoning applied to the case for V_s.

Now choose a map g^* satisfying the following conditions:

(i) $g^* : A_1^*/\equiv \rightarrow B$

(ii) g^* is 1-1

(iii) $[\{a\}]^{\equiv} g^* = a$ for each $a \in A_1'$.

To see that such a map exists, it is sufficient to demonstrate that if $H \equiv \{a\}$ then $H = \{a\}$. If $b \in H$ then either $b \in \{a\}$ or for some pair $\langle p, J\rangle$ with $b \in U_p$ and for each $i \in J$, $f^J_{i,p} \in \{a\}$. The latter condition is clearly impossible since $|J| > 1$ for each pair $\langle p, J\rangle$ and for i and $j \in J$ with $i \neq j$ we have $f^J_{i,p}b \neq f^J_{j,p}b$. It follows that $H = \{a\}$ hence for each $a \in A_1'$, $[\{a\}^{\equiv}] = \{\{a\}\}$.

Let A_1 be the range of g^*. We define $\leq$ on A_1 by $[H]^{\equiv}g^* \leq [H']^{\equiv}g^*$ if and only if $[H]^{\equiv} \leq [H']^{\equiv}$. We put $[H]^{\equiv}g^* \in U_r$ or V_s if and only if $[H]^{\equiv} \in U_r$ or V_s respectively. The operations $f^J_{i,p}$ remain as defined on A_i. The c_i remain the same 0-ary operations as in $\mathfrak{A}_0$. This completes the construction of the structure $\mathfrak{A}_1$ which is easily seen to be an extension of $\mathfrak{A}_0$. Below is a full statement of the inductive hypothesis, (i) - (ix), and the reader will have no trouble ascertaining that all parts are valid with $j = 0$.

Now let us suppose we have constructed $\mathfrak{A}_0, \ldots, \mathfrak{A}_m$ where $n \geq 1$, such that if $j < m$ the relations detailed below are satisfied.

(i) $\mathfrak{A}_{j+1}$ is an extension of $\mathfrak{A}_j$.

(ii) If a and $b \in A_j$ then, $f^J_{i,p}a$ is defined in A_{j+1}, $f^J_{i,p}a \neq a$ if and only if $a \in U_p$, and if $f^J_{i,p}a = f^I_{j,q}b$ with $a \in U_p$ and $b \in U_q$ then $a = b$, $i = j$, $I = J$, and $p = q$.

(iii) $\langle A_{j+1}; +\rangle$ is a semilattice.

(iv) If $a \in A_j$ then $\sum_{i \in J} f^J_{i,p}a \geq a$ in $\mathfrak{A}_{j+1}$.

(v) For each $a \in A_{j+1}$, $a = \Sigma \mathcal{C}a$ where $\mathcal{C}a =$

$\{b : b \in A_{j+1},\ b \leq a$ and b is primitive$\}$.

We say b is <u>primitive</u> if $b \in A_0$ or for some $c \in U_p$ $b = f^J_{i,p}c$.

(vi) If $b \in A_j$ and $f^J_{i,p}b \in A_{j+1} \sim A_j$ then $f^J_{i,p}b \in U_r$ or V_s if and only if $k_i \leq k_r$ or s respectively.

(vii) If $a \in A_{j+1}$ then $a \in U_r$ or V_s if and only if for each $b \in \mathcal{C}a$, $b \in U_r$ or V_s respectively.

(viii) If $b \in U_r \cap A_{j+1}$ and $b_r \in U_p$ or V_s then $b \in U_p$ or V_s respectively.

(ix) Each U_r and V_s is an ideal in $\langle A_{j+1}, +\rangle$.

We construct $\mathfrak{A}_{m+1}$ as follows: Let F' denote the set of unary operation symbols. Let g be any mapping from $F' \times (A_m \sim A_{m-1})$ into $B \sim A_{m-1}$ which satisfies the following three conditions:

1) If $b \notin U_p$ then $(\underset{\sim}{f}{}^J_{i,p}, b)g = b$.

2) If $b \in U_p$ then $(\underset{\sim}{f}{}^J_{i,p} b)g \in B \sim A_m$.

3) If $b_1 \in U_p$ and $b_2 \in U_q$ then

$$(\underset{\sim}{f}{}^L_{j,q}, b_2)g = (\underset{\sim}{f}{}^J_{i,p}, b_1)g$$

if and only if $i = j$, $p = q$, $J = L$ and $b_1 = b_2$.

Let $A'_{m+1} = A_m \cup Rg(g)$. We extend the definition of the operation $f^J_{i,p}$ to A_m by setting $f^J_{i,p}(b) = (f^J_{i,p}, b)g$ for $b \in A_m \sim A_{m-1}$. Clearly this extends each of the operations $f^J_{i,p}$ in such a way that (ii) of the inductive hypothesis is satisfied.

We define U_r and V_s in A'_{m+1} in the following manner. For $a \in A_m$, $a \in U_r$ or V_s if and only if $a \in U_r$ or V_s respectively in $\mathfrak{A}_m$. For $a \in A'_{m+1} \sim A_m$ then $a = f^J_{i,p}b$ for some $b \in A_m \cap U_p$. We put $a \in U_r$ or V_s if and only if $k_i \leq k_r$ or s respectively.

Let $A^*_{m+1} = \mathcal{P}(A'_{m+1}) \sim \{0\}$. Throughout the remainder of this section, we will consistently decompose $H \in A^*_{m+1}$ as follows, $H = I \cup I'$ where $I = H \cap A_m$ and $I' = H \cup (A'_{m+1} \sim A_m)$. Further if $I' \subseteq A'_{m+1} \sim A_m$ then

$$\mathcal{B}(I') = \{b : b \in A_m \text{ and for some essential pair } \langle p, J\rangle \text{ and each } i \in J,\ f^J_{i,p}b \in I'\}\ .$$

We wish to define a semilattice order on A^*_{m+1}. To this end with $\mathcal{C}$ defined as in (v) of the inductive hypothesis with $j = m-1$, we set

$$\mathcal{C}^*H = I' \cup \mathcal{C}\Sigma\ (I \cup \mathcal{B}(I')),$$

and then put $H_1 \leq^* H_2$ if and only if $\mathcal{C}^*H_1 \subseteq \mathcal{C}^*H_2$. It is straight forward from the definition that $\leq^*$ is reflexive and transitive and hence $\leq^*$ is a quasi order. Before proceeding it will be useful to demonstrate that $\mathcal{C}^*\mathcal{C}^*H = \mathcal{C}^*H$. We have that

$$\mathcal{C}^*\mathcal{C}^*H = I' \cup \mathcal{C}\ \Sigma\ [\mathcal{C}\ \Sigma\ (I \cup \mathcal{B}(I')) \cup \mathcal{B}(I')].$$

Now the desired result will follow provided we can show that

$$\mathcal{C}\,\Sigma\,[\mathcal{C}\;\Sigma\;(I \cup \mathcal{B}(I'))\cup \mathcal{B}(I')] = \mathcal{C}\;\Sigma\;(I \cup \mathcal{B}(I'))\;.$$

Recall that (v) of the inductive hypothesis asserts that $a = \Sigma\mathcal{C}\,a$. We therefore have that

$$\Sigma\mathcal{C}\;\Sigma\;(I \cup \mathcal{B}(I')) = \Sigma\;(I \cup \mathcal{B}(I')).$$

From the above result and the semilattice laws, we have

$$\Sigma\;[\mathcal{C}\,\Sigma\;(I \cup \mathcal{B}(I'))\cup \mathcal{B}(I')] = \Sigma\,\mathcal{C}\;\Sigma\;(I \cup \mathcal{B}(I')).$$

These two equations yield

$$\mathcal{C}\;\Sigma\;[\mathcal{C}\;\Sigma\;(I \cup \mathcal{B}(I'))\cup \mathcal{B}(I')] = \mathcal{C}\;\Sigma\;(I \cup \mathcal{B}(I'))$$

which immediately gives $\mathcal{C}^*\,\mathcal{C}^*H = \mathcal{C}^*H$ as desired.

We now show that $H_1 \cup H_2$ is a $\leq^*$ least upper bound for H_1 and H_2. We will demonstrate that if H_1 and $H_2 \leq^* H_3$ then $H_1 \cup H_2 \leq^* H_3$. We recall that $H_1 \cup H_2 \leq^* H_3$ if and only if $\mathcal{C}^*(H_1 \cup H_2) \subseteq \mathcal{C}^*H_3$. Now we have $H_1 = I_1' \cup I_1$, $H_2 = I_2' \cup I_2$ and $H_3 = I_3' \cup I_3$. Our task is to prove the following inclusion:

$$I_1' \cup I_2' \cup \mathcal{C}\;\Sigma\;(I_1 \cup I_2 \cup \mathcal{B}(I_1' \cup I_2')) \subseteq I_3' \cup \mathcal{C}\,\Sigma\;(I_3 \cup \mathcal{B}(I_3')).$$

By hypothesis I_1' and $I_2' \subseteq I_3'$, hence we have only to show that

$$\mathcal{C}\,\Sigma\;(I_1 \cup I_2 \cup \mathcal{B}(I_1' \cup I_2')) \subseteq \mathcal{C}\;\Sigma\;(I_3 \cup \mathcal{B}(I_3')).$$

Now $\mathcal{B}(I_1' \cup I_2') \subseteq \mathcal{B}(I_3')$ since as noted above $I_1' \cup I_2' \subseteq I_3'$. Further, since $\mathcal{C}^*H_1 \subseteq \mathcal{C}^*H_3$ it is easy to see that $\Sigma\;I_1 \leq \Sigma\;(I_3 \cup \mathcal{B}(I_3'))$. A similar statement holds for $\Sigma\;I_2$, whence

$$\Sigma\ (I_1 \cup I_2 \cup \mathcal{D}(I_1' \cup I_2')) \leq \Sigma\ (I_3 \cup \mathcal{D}(I_3')).$$

Now it follows directly from the definition of $\mathcal{C}$ that $a \leq b$ implies $\mathcal{C}a \subseteq \mathcal{C}b$. We therefore conclude that the necessary inclusion is valid and hence $H_1 \cup H_2 \leq^* H_3$ as desired.

We have shown that $\leq^*$ is a quasi order with least upper bounds. We therefore have that $\langle A^*_{m+1}/\equiv;\ \leq^*/\equiv\rangle$ is a semilattice ordering system, where $H_1 \equiv H_2$ if and only if $H_1 \leq^* H_2$ and $H_2 \leq^* H_1$. We will denote the equivalence class of H under $\equiv$ by $[H]^{\equiv}$. It is a consequence of the idempotence of $\mathcal{C}^*$ that $H \equiv \mathcal{C}^*H$. We define $U_{p/\equiv}$ on $A^*_{m+1}/\equiv$ by $[H]^{\equiv} \in U_{p/\equiv}$ if and only if $\mathcal{C}^*H \subseteq U_p$, recall that U_p has already been defined on A'_{m+1}. Since $H_1 \equiv H_2$ implies that $\mathcal{C}^*H_1 = \mathcal{C}^*H_2$, this definition is independent of the choice of H. In a similar manner, we define V_s.

Our task now is to obtain a structure which is an extension of $\mathfrak{A}_m$. We have two structures with respective universes A'_{m+1} and $A^*_{m+1}/\equiv$. The former has the desired properties with respect to the constants, unary operations and unary relations, but it is not a semilattice. The second structure has the desired semilattice properties. We will use the second structure to expand the first to obtain $\mathfrak{A}_{m+1}$.

It will facilitate our considerations if we first show that if a and $b \in A'_{m+1}$ and $\{a\} \equiv \{b\}$ then $a = b$. We first observe that if $a \in A'_{m+1} \sim A_m$, then $\mathcal{C}^*\{a\} = \{a\}$ since for no essential pair $\langle p, J\rangle$ is J a singleton. Further, if $a \in A_m$ then $\mathcal{C}^*\{a\} \subseteq A_m$ and we may therefore assume that a and $b \in A_m$. Since $\{a\} \equiv \{b\}$ we have

$$\mathcal{C}^*\{a\} = \{a\} = \{b\} = \mathcal{C}^*\{b\}.$$

Thus

$$a = \Sigma\, \mathcal{C} a = \Sigma \mathcal{C} b = b$$

by (v) of the inductive hypothesis. It is now clear that a map $g^* : A^*_{m+1}/\equiv \to B$ exists which satisfies (i) g^* is 1-1 and (ii) $[\{a\}]g^* = a$ for each $a \in A'_{m+1}$. We set $A_{m+1} = (A^*_{m+1}/\equiv)g^*$. We define $\leq$ on A_{m+1} by putting $[H]^{\equiv}g^* \leq [H']^{\equiv}g^*$ if and only if $[H]^{\equiv} \leq^*/\equiv [H']^{\equiv}$. This is obviously a semilattice order and we define $+$ in the usual way. We let $f^J_{i,p}$ be that partial operation on A_{m+1} with domain A_m and range in A'_{m+1} which has already been defined. We put $[H]^{\equiv}g^*$ in U_r or V_s in case $[H]^{\equiv}$ is in U_r or V_s respectively. We let c_i be that constant operation whose value is b_i. This completes the construction of the partial system $\langle A_{m+1}; F, R\rangle = \mathfrak{A}_{m+1}$. It remains our task to verify the inductive hypothesis for $j = n$. We treat the various parts in order. First we show that $\mathfrak{A}_{m+1}$ is an extension of $\mathfrak{A}_m$. That the constant and unary operations are indeed extensions of their counterparts on $\mathfrak{A}_m$ is evident from their definition on $\mathfrak{A}_{m+1}$ and we leave this to the reader. For the remainder it suffices to show that if a and b are in A_m then

(1) if $c \in A_{m+1} \sim A_m$ then $c \not\leq a$,

(2) $a \leq b$ in $\mathfrak{A}_{m+1}$ if and only if $a \leq b$ in $\mathfrak{A}_m$ and,

(3) $a \in U_r$ or V_s in $\mathfrak{A}_{m+1}$ if and only if $a \in U_r$ or V_s respectively in $\mathfrak{A}_m$.

From (1) and (2) $\mathfrak{A}_{m+1}$ is an extension of $\mathfrak{A}_m$ with respect to $+$. To see (1), let $c = [H]^{\equiv}g^*$ where $H \subseteq A'_m$. If $c \leq a$ in $\mathfrak{A}_{m+1}$ and $a \in A_m$ then $H \leq^* \{a\}$ whence $H \subseteq A_m$. Since $[H]^{\equiv} = [\{\Sigma\, H\}]$, we have $c = [\{\Sigma\, H\}]g^* = \Sigma\, H \in A_m$. For (2), we have $a \leq b$ in $\mathfrak{A}_{m+1}$ if and only if $\mathcal{C}^*\{a\} \subseteq \mathcal{C}^*\{b\}$, that is, if and only if $\mathcal{C}a \subseteq \mathcal{C}b$. From the definition of $\mathcal{C}$ and (v), the last statement holds if and only if $a \leq b$ in $\mathfrak{A}_m$. For (3), $a \in U_r$ in $\mathfrak{A}_{m+1}$ if and only if $\mathcal{C}^*\{a\} \subseteq U_r$, that is, if and only if $\mathcal{C}a \subseteq U_r$, which is equivalent to $a \in U_r$ by (vii) of the inductive hypothesis. A similar line of reasoning yields the result for V_s.

Part (ii) of the inductive hypothesis for $j = m$ is a matter of recalling the definition of the extension to A'_{m+1} and the resultant definition of the partial operations on A_{m+1}. It is a direct consequence of the fact that $\leq^*/\equiv$ is a semilattice order, that $\langle A_{m+1}; +\rangle$ is a semilattice. This brings us to part (iv) of the inductive hypothesis which states that if $a \in A_m$ then $a \leq \sum_{i \in J} f^J_{i,p}a$. Now if $a \notin U_p$ then $f^J_{i,p}a = a$ and we are done. Suppose $a \in U_p$. We must show that $\mathcal{C}^*\{a\} \subseteq \mathcal{C}^*\{f^J_{i,p}a : i \in J\}$. Now

$$\mathcal{C}^*\{f^J_{i,p}a : i \in J\} = \{f^J_{i,p}a : i \in J\}$$

$$\cup\, \mathcal{C}\, \Sigma\, \mathcal{D}\{f^J_{i,p}a : i \in J\}.$$

By definition of $\mathcal{D}$ we have $a \in \mathcal{D}\{f^J_{i,p}a : i \in J\}$ and hence

$$\mathcal{C}^*\{a\} = \mathcal{C}a \subseteq \mathcal{C}\, \Sigma\, \mathcal{D}\{f^J_{i,p}a : i \in J\}.$$

This immediately yields (iv) as desired.

For part (v) we must show that $\Sigma \mathcal{C}a = a$. We first observe that for $a \in A_{m+1}$, a is primitive if and only if $a \in A_m$ and a is primitive in $\mathfrak{A}_m$, or $a \in A'_{m+1} \sim A_m$. For arbitrary $b \in A_{m+1}$ we have $b = [H]^{\equiv} g^*$ for some $H \in A^*_{m+1}$. Now $H \equiv \mathcal{C}^*H = \bigcup\{\{a\} : a \in \mathcal{C}^*H\}$. We claim that $\mathcal{C}b = \mathcal{C}^*H$. To see this observe that if $a \in \mathcal{C}^*H$ then a is primitive in $\mathfrak{A}_{m+1}$. Since $\{a\} \subseteq \mathcal{C}^*H$, we have

$$\mathcal{C}^*\{a\} \subseteq \mathcal{C}^*(\mathcal{C}^*H) = \mathcal{C}^*H.$$

It follows that

$$a = \{\{a\}\}g^* \leq [H]^{\equiv} g^* = b.$$

Conversely, if $a \in \mathcal{C}b$ then $\mathcal{C}^*\{a\} \subseteq \mathcal{C}^*H$. Since a is either primitive in $\mathfrak{A}_m$ or an element of $A'_{m+1} \sim A_m$, we see that $a \in \mathcal{C}^*\{a\}$ from the definition of $\mathcal{C}^*$. Thus $\mathcal{C}^*H = \mathcal{C}b$ as desired. It follows that

$$\Sigma \mathcal{C}b = \Sigma \{a : a \in \mathcal{C}^*H\} = [\mathcal{C}^*H]^{\equiv} g^* = b$$

as desired.

Part (vi) of the inductive hypothesis asserts that for each $a \in A_m$ and $f^J_{i,p}a \in A_{m+1} \sim A_m$ we have $f^J_{i,p}a \in U_r$ or V_s if and only if $k_i \leq k_r$ or s respectively. It is simply a matter of recalling the relevant definitions to check that this is the case.

For part (vii) we must prove that if $a \in A_{m+1}$ then $a \in U_r$ or V_s if and only if $a \subseteq U_r$ or V_s respectively. We have already remarked that if $[H]^{\equiv} g^* = a$ then $\mathcal{C}a = \mathcal{C}^*H \equiv H$. Since $a \in U_r$ if and only if $[H]^{\equiv} \in U_r/\equiv$, which is equivalent to $\mathcal{C}^*H \subseteq U_r$, we are done. Similar reasoning applies to V_s.

We must now show that for each $b \in A_{m+1} \cap U_r$, if $b_r \in U_p$ or V_s then $b \in U_p$ or V_s respectively. This will demonstrate (viii). This property follows immediately from the corresponding property of A'_{m+1}. For A'_{m+1}, the property comes from (viii) for $j = m - 1$ and the definitions of U_r and V_s on A'_{m+1}.

Lastly we must show that each U_r and V_s is an ideal in $\langle A_{m+1}; + \rangle$. Part (vii) and the definition of $\mathcal{C}$ easily yield the fact that $a \leq b \in U_r$ or V_s implies $a \in U_r$ or V_s respectively. Thus we have only to show that if a and $b \in U_r$ then $a + b \in U_r$. This will follow readily provided that we can show, if $[H_1]^{\equiv}$ and $[H_2]^{\equiv} \in U_r/\equiv$ then $[H_1 \cup H_2]^{\equiv} \in U_r$. But this is equivalent to showing that $\mathcal{C}^*H_1$ and $\mathcal{C}^*H_2 \subseteq U_r$ imply that $\mathcal{C}^*(H_1 \cup H_2) \subseteq U_r$. Recall that

$$\mathcal{C}^*(H_1 \cup H_2) = (I'_1 \cup I'_2) \cup \mathcal{C}\,\Sigma\,(I_1 \cup I_2 \cup \mathcal{D}(I'_1 \cup I'_2))$$

where $H_1 = I'_1 \cup I_1$ and $H_2 = I'_2 \cup I_2$ are the standard decompositions of H_1 and H_2. By hypothesis

$$\mathcal{C}^*H_1 = I'_1 \cup \mathcal{C}\,\Sigma\,(I_1 \cup \mathcal{D}(I'_1)) \subseteq U_r$$

and similarly for $\mathcal{C}^*H_2$. Thus it is enough to show

$$\mathcal{C}\,\Sigma\,(I_1 \cup I_2 \cup \mathcal{D}(I'_1 \cup I'_2)) \subseteq U_r\ .$$

Since U_r is an ideal in $\mathfrak{A}_m$, it is sufficient to show that each of I_1, I_2, $\mathcal{D}(I'_1 \cup I'_2)$ is included in U_r. From above,

$$\mathcal{C}\,\Sigma\,I_1 \subseteq \mathcal{C}\,\Sigma\,(I_1 \cup \mathcal{D}(I'_1)) \subseteq U_r\ .$$

Since $\Sigma\mathcal{C}$ is the identity by (v), we get $\Sigma I_1 \subseteq U_r$ and similarly $\Sigma I_2 \subseteq U_r$. Thus we have only to show that $\mathcal{B}(I_1' \cup I_2') \subseteq U_r$. Hence let $b \in \mathcal{B}(I_1' \cup I_2')$. Then for some essential pair $\langle p, J\rangle$ and each $i \in J$, $f^J_{i,p} b \in I_1' \cup I_2'$. It follows that $k_p \leq \sum_{I \in J} k_i \leq k_r$, since $f^J_{i,p} b \in U_r$ if and only if $k_i \subseteq k_r$. Now $b \in U_p$ since $f^J_{i,p} b \neq b$. Since $k_p \leq k_r$ implies $b_p \in U_r$, we have $b \in U_r$ by (viii). This completes the proof of (ix) since a similar argument yields the result for V_s.

We complete the construction by setting $\mathfrak{A}^{\mathfrak{S}} = \bigcup_{i \in \omega} \mathfrak{A}_i$. It is trivial that all parts of the inductive hypothesis carry over directly to $\mathfrak{A}^{\mathfrak{S}}$.

Before continuing we make some observations about $\langle A^{\mathfrak{S}}; +\rangle$. The central observation is that if $b \in A_q$ and $a \in A \sim A_q$ then $a \nleq b$. This was shown for $\langle A_{m+1}; +\rangle$ extending $\langle A_m; +\rangle$ and is easily seen to extend to the general case. The notion of primitive extends to $\mathfrak{A}^{\mathfrak{S}}$ and we note that a primitive $b \in A_m$ is minimal in $\langle A_m; +\rangle$ whence it is minimal in $\langle A^{\mathfrak{S}}; +\rangle$. Thus each primitive is join-irreducible. If $b \in A_m$ and b is not primitive then b is join reducible, whence join irreducible is equivalent to primitive. Now from the above we see that we can generalize the definition of $\mathcal{C}$ to $A^{\mathfrak{S}}$ so that

$$\mathcal{C}a = \{x : x \text{ is primitive, } x \in A \text{ and } x \leq a\}.$$

Further we define $\mathcal{B}$ on the finite subsets of primitives in $A^{\mathfrak{S}}$ as follows:

$$\mathcal{B}(H) = \{b : \text{for some essential pair } \langle p, J\rangle$$

$$b \in U_p \text{ and for each } i \in J \ f^J_{i,p} b \in H\}.$$

Lastly we note that if H_1 and H_2 are sets of primitives $H_1, H_2 \subseteq A_{m+1}$ and $\Sigma H_1 = \Sigma H_2$ then $I_1' = I_2'$ and

$$\Sigma [I_1 \cup \mathcal{D}(I_1')] = \Sigma [I_2 \cup \mathcal{D}(I_2')]$$

where $I_1' = H_1 \sim A_m$, $I_1 = H_1 \cap A_m$, $I_2' = H_2 \sim A_m$ and $I_2' = H_2 \cap A_m$. This is because in extending $\langle A_m; +\rangle$ to $\langle A_{m+1}; +\rangle$ we ensured that for H_1 and $H_2 \subseteq A'_{m+1}$, if $H_1 \equiv H_2$ then $\mathcal{C}^*H_1 = \mathcal{C}^*H_2$. Together with (v), we see that this implies the result above. We conclude this section with a theorem showing $\langle A^{\mathfrak{S}}; +\rangle$ has a free mapping property. To this end we prove the following lemma.

Lemma 5.1. If $a \in A^{\mathfrak{S}}$ is join reducible, then there is a unique set of primitives Q_a which is join irredundant such that (i) $\Sigma Q_a = a$ and (ii) if $\Sigma H = a$ then to each $y \in Q_a$ there is an $x \in H$ with $y \leq x$.

Proof. Suppose for each j such that $0 \leq j \leq m + 1$, if $a \in A_j$, then such a Q_a exists. Observe that $Q_a \subseteq A_j$, and by the above remarks we may confine ourselves in the induction step to looking at $\langle A_{m+1}; +\rangle$. Let $a \in A_{m+1} \sim A_m$ be join reducible, with $I' = \mathcal{C}a \sim A_m$ and $I = \mathcal{C}a \cap A_m$. We set

$$Q_a = I' \cup I^*$$

where

$$I^* = \{b : b \in Q_{\Sigma I} \text{ and } b \not\leq \Sigma \mathcal{D}(I')\} .$$

Clearly,

$$\Sigma\ I^* + \Sigma\ \mathcal{B}(I') = \Sigma\ I\ ,$$

whence $\Sigma\ Q_a = \Sigma\ \mathcal{C}a = a$. Now let $\Sigma\ H = a$ for any $H \subseteq A_{m+1}$. Again we note for any $J \subseteq A$ if $\Sigma\ J = a$ then $J \subseteq A_{m+1}$, so we are dealing with the most general case. Now set

$$H_1 = \{x : x \text{ is primitive and } x \leq b \in H \text{ for some } b\}.$$

Thus $\Sigma\ H_1 = a$. Let $I_1' = H_1 \sim A_m$ and $I_1 = H_1 \cap A_m$. Since $I_1' \subseteq A_{m+1} \sim A_m$ and $I' \subseteq A_{m+1} \sim A_m$ and $a \in A_{m+1}$, we conclude that $I_1' = I'$. Now $\Sigma\ \mathcal{B}(I') \in A_m$ and $\Sigma\ \mathcal{B}(I') \leq a$, whence

$$\Sigma\ \mathcal{C}\ \Sigma\ \mathcal{B}(I') = \Sigma\ \mathcal{B}(I')$$

by (v) of the inductive hypothesis. Thus $\Sigma\ \mathcal{B}(I') \leq \Sigma\ I$, whence

$$\Sigma\ I = \Sigma\ [\mathcal{B}(I') \cup I] = \Sigma\ [\mathcal{B}(I_1') \cup I_1]$$

since $\Sigma\ H_1 = a = \Sigma\,\mathcal{C}a$. Thus to each $x \in Q_{\Sigma I}$ there is a $y \in \mathcal{B}(I_1') \cup I_1$ with $x \leq y$. For such an $x \in I^* \subseteq I$, $y \notin \mathcal{B}(I')$ whence $y \in I_1$. Since I_1 is a set of primitives and every element is minimal, $I^* \subseteq I_1$. Thus $Q_a \subseteq H_1$ and the result follows.

Theorem 5.1. Let $\mathfrak{S}^*$ be any semilattice such that $\mathfrak{S}$ can be embedded in $\mathfrak{J}(\mathfrak{S}^*)$ via ψ. Let ϕ^* be any map such that $\phi^* : K \to S^*$ and $k_i\phi^* \in x\psi$ if and only if $k_i \leq x$. Then there is an extension ϕ of ϕ^* mapping $\langle A^{\mathfrak{S}} ; + \rangle$ into $\mathfrak{S}^*$ such that ϕ is a homomorphism and for each $x \in A^{\mathfrak{S}}$, $x\phi \in k_i\psi$ or $s\psi$ if and only if $x \in U_i$ or V_s respectively.

Proof. We define $\phi^0 \subseteq \phi^1 \subseteq \ldots \phi^i \subseteq \ldots$ for each $i \in \omega$ such that if $1 \leq i$ ϕ^i is a homomorphism of $\langle A_i; +\rangle$ into $\mathfrak{S}^*$ satisfying the conditions of the theorem. We let $\phi^0 = \phi^*$. Suppose that for each j such that $0 \leq j < m+1$ ϕ^j is defined and satisfies the above conditions. Let $\langle p, J\rangle$ be an arbitrary essential pair, and $a \in U_p \cap (A_m \sim A_{m-1})$. For each $i \in J$ we pick $a_{\langle p,J,i\rangle} \in k_i\psi$ such that (i) $a_{\langle p,J,i\rangle} \in x\psi$ if and only if $k_i \leq x$ and (ii) $a\phi^m \leq \sum_{i \in J} a_{\langle p,J,i\rangle}$. Such choices are possible since ψ embeds $\mathfrak{S}$ in $\mathfrak{J}(\mathfrak{S}^*)$, and $a\phi^m \in k_p\psi$. We define ϕ^{m+1} by

$$b\phi^{m+1} = \begin{cases} b\phi^m & \text{if } b \in A_m \\ a_{\langle p,J,i\rangle} & \text{if } b = f^J_{i,p}a \in A_{m+1} \sim A_m \text{ and } a \in A_m \\ \Sigma[(Q_b)\phi^{m+1}] & \text{if } b \text{ is join reducible } b \in A_{m+1} \sim A_m. \end{cases}$$

To see that ϕ^{m+1} is a homomorphism, we have only to show that for each join reducible $a \in A_{m+1} \sim A_m$ if H is a set of join irreducibles and $\Sigma H = a$ then $\Sigma (H\phi^{m+1}) = a\phi^{m+1}$. Now join irreducible is the same as primitive. Further since every primitive is minimal if H is a set of primitives and $\Sigma H = a$ then $Q_a \subseteq H$ by Lemma 5.1 whence

$$a\phi^{m+1} \leq \Sigma (H\phi^{m+1}).$$

Thus it is sufficient to show that if $b \in \mathcal{C}a$ then

$$b\phi^{m+1} \leq \Sigma (Q_a\phi^{m+1}) = a\phi^{m+1}.$$

Let $b \in \mathcal{C}a \sim Q_a$. Then $b \in \mathcal{C}a \cap A_m$ since $b \in \mathcal{C}a \sim A_m$ implies $b \in Q_a$ as shown in the proof of Lemma 5.1. Now

$$I^* = Q_a \cap A_m = \{d : d \in Q_{\Sigma I} \text{ and } d \not\leq \Sigma \mathcal{B}(I'))\}$$

where $I = \mathcal{C}a \cap A_m$ and $I' = \mathcal{C}a \sim A_m$, whence $b \leq \Sigma\, \mathcal{B}(I')$. Now $\mathcal{B}(I') \subseteq A_m$ whence

$$b\phi^{m+1} = b\phi^m \leq \Sigma\, [\mathcal{B}(I')\phi^m] = \Sigma\, [\mathcal{B}(I')\phi^{m+1}].$$

Thus we will be done if we show that for each $c \in \mathcal{B}(I')$

$$c\phi^{m+1} \leq \Sigma\, (Q_a\phi^{m+1}).$$

Now if $c \in \mathcal{B}(I')$, then for some essential pair $\langle p, J\rangle$ $c \in U_p \cap (A_m \sim A_{m-1})$ and for each $i \in J$ $f^J_{i,p}c \in I'$. Now

$$c\phi^{m+1} = c\phi^m \leq \sum_{i\in J} c_{\langle p,J,i\rangle} = \sum_{i\in J} [(f^J_{i,p}c)\phi^{m+1}]$$

by choice of the $c_{\langle p,J,i\rangle}$ for $i \in J$. Thus

$$c\phi^{m+1} \leq \Sigma\, (I'\phi^{m+1})$$

for each $c \in \mathcal{B}(I')$ whence we obtain that

$$\Sigma\, [\mathcal{B}(I')\phi^{m+1}] \leq \Sigma\, (I'\phi^{m+1}) \leq \Sigma\, (Q_a\phi^{m+1})$$

and hence that $b \leq \Sigma\, (Q_a\phi^{m+1})$ as desired.

Thus ϕ^{m+1} is a homomorphism. Now let $a \in A_{m+1} \sim A_m$. We show that $a \in U_r$ or V_s if and only if $a\phi^{m+1} \in k_r\psi$ or $s\psi$ respectively. If $a \in A_{m+1} \sim A_m$ is primitive the result follows by choice of $a\phi^{m+1}$. For join reducible a, if $a \in U_r$ or V_s then $Q_a \subseteq U_r$ or V_s respectively and the result follows. Now if $a \notin U_r$ or V_s then for some $b \in Q_a$,

$b \notin U_r$ or V_s and again the result follows. If we let $\phi = \bigcup_{i \in \omega} \phi^i$ we have the desired homomorphism.

§6. We will now present our characterization of transferable semilattices.

Definition 6.1. Let $s \in S$ and $H \subseteq S$, we will say that H is minimal over s in case $s \leq \Sigma H$, but if $H' \subsetneq H$ then $s \not\leq \Sigma H'$. Further if $p \in n$ and $J \subseteq n$, with $|J| \geq 2$, we will say that $\langle p, J \rangle$ is a minimal pair in case

(1) $\{k_i : i \in J\}$ is minimal over k_p and,

(2) if $\{k_i : i \in J^*\}$ is minimal over k_p and to each $j \in J^*$ there exists an $i \in J$ such that $k_j \leq k_i$, then $J^* = J$.

Definition 6.2. $\mathfrak{S}$ is strictly transferable in case for some linear order $\prec$ of the join irreducibles $k_0, \ldots, k_{n-1}$, every minimal pair $\langle p, J \rangle$ satisfies $k_p \prec k_i$ for each $i \in J$.

Theorem 6.1. Let $\mathfrak{S}$ be a finite semilattice. Then $\mathfrak{S}$ is transferable if and only if $\mathfrak{S}$ is strictly transferable.

Proof. We prove first that strict transferability implies transferability. Hence let ϕ^* be any embedding of $\mathfrak{S}$ in $\mathfrak{J}(\mathfrak{S}^*)$. Without loss of generality we assume that $k_i \prec k_j$ if and only if $i < j$, where $\prec$ is the linear order witnessing the strict transferability of $\mathfrak{S}$, For each $i \in n$ choose $a_i \in k_i\phi^*$ such that for any $J \subseteq n$ and $s \in S$,

$\sum_{i\in J} a_i \in s\phi^*$ if and only if $\sum_{i\in J} k_i \leq s$. It is easy to see that such choices can be made. For $i \in n$ define $k_i^0 = a_i$. By induction on j, we define $k_0^j, \ldots, k_{n-1}^j$ for $j \in n$ such that $k_i^j \in k_i\phi^*$. In the induction step we let $k_i^{j+1} = k_i^j$ for each $i \leq j$ and we choose k_{j+1}^{j+1}, $k_{j+2}^{j+1}, \ldots, k_{n-1}^{j+1}$ in $k_{j+1}\phi^*, k_{j+2}\phi^*, \ldots, k_{n-1}\phi^*$ respectively such that $k_i^{j+1} \geq k_i^j$ for $i \in n$ and such that for any $J \subseteq n$, if $J \cap (j+1) = 0$ and $\langle j, J\rangle$ is a minimal pair, then $k_j^j \leq \sum_{i\in J} k_i^{j+1}$. We can choose $k_{j+1}^{j+1}, \ldots, k_{n-1}^{j+1}$ suitably because $k_j \leq \sum_{i\in J} k_i$ implies $k_j\phi^* \leq \sum_{i\in J}(k_i\phi^*)$, since ϕ^* is an embedding. Define $\phi : K \to S^*$ by $k_i\phi = \Sigma\{k_j^n : k_j \leq k_i\}$. Our first claim is that if $\langle p, J\rangle$ is a minimal pair then $k_p\phi \leq \sum_{i\in J}(k_i\phi)$. We must show that if $k_q \leq k_p$ then $k_q^n \leq \sum_{i\in J}(k_i\phi)$. This is obvious if $k_q \leq k_i$ for some $i \in J$. Otherwise, we have that there is a minimal pair $\langle q, J^*\rangle$ such that to each $j \in J^*$ there is an $i \in J$ such that $k_j \leq k_i$. That such a pair $\langle q, J^*\rangle$ exists is a consequence of the finiteness of $\mathfrak{S}$. Now

$$k_q^n \leq \sum_{j\in J^*} k_j^n \leq \sum_{i\in J}(k_i\phi)$$

by the definition of ϕ which establishes the claim. Clearly ϕ preserves the partial order on K and every inequality $k_p \leq \sum_{i\in J} k_i$ where $\langle p; J\rangle$ is a minimal pair. Let $k_p \leq \sum_{i\in J} k_i$ be an arbitrary inequality valid in $\mathfrak{S}$. If $k_p \not\leq k_i$ for each $i \in J$ then we find a minimal pair $\langle p, J^*\rangle$ satisfying the condition that for each $j \in J^*$ there is an $i \in J$ such that $k_j \leq k_i$. It follows immediately that $k_p\phi \leq \sum_{i\in J}(k_i\phi)$. Since $a_i \leq k_i\phi \in k_i\phi^*$ for each $i \in n$, ϕ clearly satisfies $x\phi \in y\phi^*$ if and only if $x \leq y$, whence ϕ is 1-1. This shows that ϕ is an embedding of $\mathfrak{S}$ in $\mathfrak{S}^*$, and thus $\mathfrak{S}$ is transferable.

Conversely, let us suppose that $\mathfrak{S}$ is transferable. Now $\mathfrak{S}$ is embeddable via ϕ^* in $\mathfrak{J}(\mathfrak{A}^{\mathfrak{S}})$, with $s\phi^* = V_s$ and $k_i\phi^* = U_i$, for this embedding. By the definition there is an embedding ϕ of $\mathfrak{S}$ in $\mathfrak{A}^{\mathfrak{S}}$ which satisfies $x\phi \in U_i$ or V_s if and only if $x \leq k_i$ or s respectively. Now let

$$R_i = \{x : x \in \mathcal{C}(k_i\phi) \ \& \ x \in y\phi^* \text{ if and only if } U_i \subseteq y\phi^*\}.$$

We will first demonstrate that $R_i \neq 0$ for each $i \in n$. First observe that with each primitive element a we can associate a unique $i_a \in n$ such that $a \in U_j$ or V_s if and only if $k_{i_a} \leq k_j$ or s respectively. Hence it is sufficient to find $a \in \mathcal{C}(k_i\phi)$ such that $i_a = i$. Now if $R_i = 0$ then let $s = \Sigma\{k_{i_a} : a \in \mathcal{C}(k_i\phi)\}$. Since k_i is join irreducible and $k_{i_a} < k_i$ for each $a \in \mathcal{C}(k_i\phi)$ we have $s < k_i$. It is clear from the construction that $k_i\phi \in V_s$ (or U_j respectively if $s = k_j$), which is contrary to our hypothesis about ϕ. Thus $R_i \neq 0$ as desired. We now set

$$m_i = \mu j[R_i \subseteq A_j],$$

and define $\prec$ by:

$$k_i \prec k_j \text{ if and only if } \quad \text{i) } m_i < m_j \text{ or}$$
$$\text{ii) } m_i = m_j \text{ and } i < j.$$

We must show that every minimal pair $\langle p, J\rangle$ satisfies $k_p \prec k_i$ for each $i \in J$.

For the remainder, we fix $a \in R_p$ such that $m_p = \mu j[a \in A_j]$. Now $a \le \Sigma(\bigcup_{i \in J} \mathcal{C}(k_i \phi))$. Let $H \subseteq \bigcup_{i \in J} \mathcal{C}(k_i \phi)$ be such that H is minimal over a. Observe that for each $b \in H$ there is an $i \in J$ such that $k_{i_b} \le k_i$. Further we assert that $k_p \le \Sigma\{k_{i_b} : b \in H\}$. To see this, let $s = \Sigma\{k_{i_b} : b \in H\}$. Then $a \in V_s$. But $a \in V_s$ if and only if $U_p \subseteq V_s$, which is equivalent to $k_p \le s$. Now since $\langle p, J \rangle$ is a minimal pair, for each $i \in J$ there is a $b \in H$ such that $i_b = i$. This is an immediate consequence of the fact that there is an $H^* \subseteq \{k_{i_b} : b \in H\}$ such that H^* is minimal over k_p, and that $\langle p, J \rangle$ is a minimal pair. It is immediate from the lemma proved below that $m_p < \mu_j[b \in A_j]$ for each $b \in H$. But this implies that $m_p < m_i$ for each $i \in J$, whence $k_p < k_i$ for each $i \in J$. This completes the proof.

Lemma 6.1. Let a, b be distinct primitive elements, and let $a \in A_m$ where $m = \mu i[b \in A_i]$. For any set H of primitive elements, if $b \le \Sigma H$, then $b \le \Sigma (H \sim \{a\})$.

Proof. We carry out an induction on q where $H \subseteq A_{q+m}$. For $q = 0$, we see from the definition of $\le^*$ in A^*_m that $b \le \Sigma H$ if and only if $b \in H$. Hence let us assume the result in A_{m+q}. Let $a \in H \subseteq A_{m+q+1}$. Now from the definitions of $\le^*$ and $\mathcal{C}^*$ and the inductive hypothesis part (v), if $b \le \Sigma H$, we have

$$b \in \mathcal{C}^*(H) = H' \cup \mathcal{C}(\Sigma[(H^*) \cup \mathcal{D}(H')]),$$

$$= H' \cup \mathcal{C}(\Sigma[(H^*) \cup \mathcal{C}(\Sigma \mathcal{D}(H'))]),$$

where $H^* = H \cap A_{q+m}$ and $H' = H \sim H^*$. Since $b \notin H'$,

$$b \in \mathcal{C}(\Sigma[H^* \cup \mathcal{C}(\Sigma \mathcal{B}(H'))] \subseteq A_{m+q}.$$

Thus,

$$b \leq \Sigma[H^* \cup \mathcal{C}(\Sigma\ (\mathcal{B}(H'))].$$

By the inductive hypothesis,

$$b \leq \Sigma\ ([H^* \sim \{a\}] \cup \mathcal{C}[\Sigma^* \mathcal{B}(H'))].$$

It follows that

$$b \in \mathcal{C}\,\Sigma\ ([H^* \sim \{a\}] \cup \mathcal{C}[\Sigma \mathcal{B}(H')]) \subseteq \mathcal{C}\ \Sigma\ (H \sim \{a\}).$$

We therefore have $b \leq \Sigma\ (H \sim \{a\})$ as desired.

§7. As pointed out in the introduction, it remains an open question whether transferable and weakly transferable are equivalent for semilattices. We have examined all semilattices having at most three join irreducibles, and in all cases have found the two notions to be the same. We now present a proof that a particular semilattice is not weakly transferable. The construction of $*$ for this case is interesting since similar techniques work for some larger finite semilattices.

Example 7.1. We consider the semilattice $\mathfrak{S}_1$ depicted in Fig. 1. Now it is easily seen that $\mathfrak{S}_1$ is not transferable. Note however that a somewhat similar semilattice $\mathfrak{S}_2$ is transferable. We show that $\mathfrak{S}_1$ is not weakly transferable. We do this by construction of $\mathfrak{S}^*$, see Fig. 1.

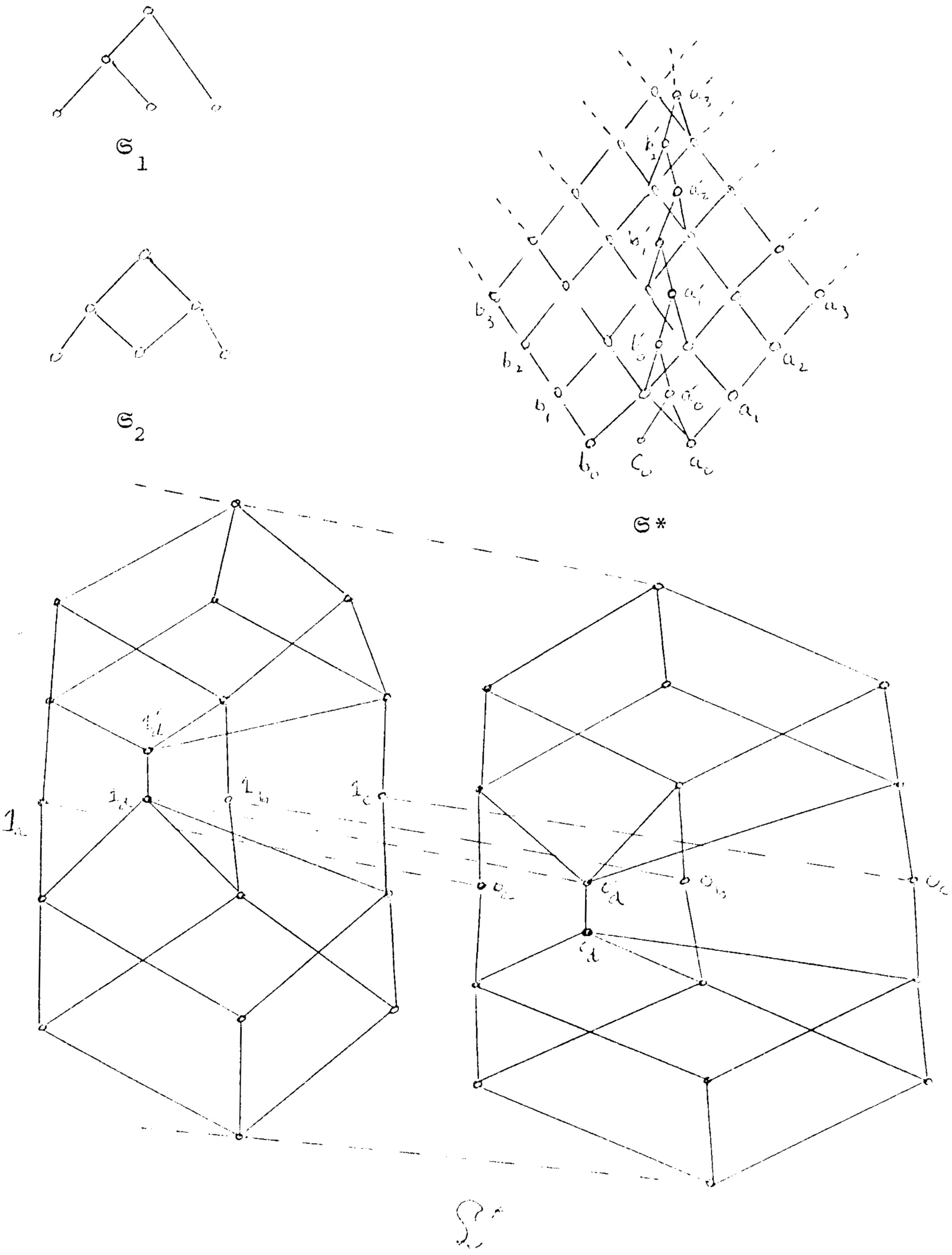


Figure 7.1

For the construction of $\mathfrak{S}^*$, let $A = \{a_0, a_1, a_2, \ldots\}$, $B = \{b_0, b_1, b_2, \ldots\}$ and $C = \{c_0, a_0', b_0', a_1', b_1', a_2', b_2', \ldots\}$. We define $\leq$ on A so that $\langle A; \leq\rangle$ is isomorphic to $\langle \omega, \leq\rangle$ in the obvious manner. Similarly we define $\leq$ on B and C to form $\langle B; \leq\rangle$ and $\langle C; \leq\rangle$. Now consider $\langle A; +\rangle$ and $\langle B; +\rangle$ as semilattices. Let $\overline{A \cup B} = A \times B \cup A \cup B$. We define $+$ on $\overline{A \cup B}$ by the following rule and the commutative law:

$$x + y = \begin{cases} x + y & \text{if } x \text{ and } y \in A \text{ or } x \text{ and } y \in B, \\ (x + a, b) & \text{if } x \in A \text{ and } y = (a, b) \in A \times B, \\ (a, b + y) & \text{if } x = (a, b) \in A \times B \text{ and } y \in B, \\ (a + c, b + d) & \text{if } x = (a, b) \in A \times B \text{ and} \\ & \quad y = (c, d) \in A \times B. \end{cases}$$

We leave it to the reader to verify that $+$ is a semilattice operation. We now set $S^* = \overline{A \cup B} \cup C$ and define $+$ by

$$x + y = \begin{cases} x + y & \text{if } x \text{ and } y \in \overline{A \cup B}, \text{ or } x \text{ and } y \in C, \\ a_m' + b_n' + c & \text{where } x = (a_m, b_n) \in \overline{A \cup B} \text{ and} \\ & \quad y = c \in C. \end{cases}$$

We note that $\langle S^*; +\rangle$ is a semilattice. It can be shown that $\mathfrak{S}_1$ is not embeddable in $\mathfrak{S}^*$, but the ideals A, B and the principal ideal $\{c_0\}$ generate a copy of $\mathfrak{S}_1$ in $\mathfrak{J}(\mathfrak{S}^*)$.

In the next chapter we consider the problem of transferability for lattices. We will show that if $\mathfrak{L} = \langle L; +, \cdot\rangle$ is a transferable lattice then $\langle L; +\rangle$ is a transferable semilattice. Our next example shows that the converse of this theorem fails.

Example 7.2. We construct a lattice $\mathcal{L}^*$ to show that $\mathcal{L}$, the free distributive lattice on three generators, is not transferable as a lattice. That $\mathcal{L}$ is transferable as a semilattice may be verified by the reader. Let a, b and c be the three free generators of $\mathcal{L}$. Let $d = (a + b)(a + c)(b + c)$. Now for each $x \in L$, for which $x \neq d$, let $\{0_x, 1_x, 2_x, \ldots\} = \omega_x$ be a copy of ω. For d we have $\omega_d = \{0_d, 0'_d, 1_d, 1'_d, 2_d, 2'_d, \ldots\}$, also a copy of ω. Let $L^* = \bigcup_{x \in L} \omega_x$. Order each ω_x according to the enumeration displayed. We define $\leq$ on L^* as follows: $a \leq b$ if and only if (i) $a = n_x$, $b = m_y$, $x \leq y$ and $m \leq n$ or (ii) $a = n_x$, $b = m'_d$, $x \leq d$ and $n \leq m$ or (iii) $a = n'_d$, $b = m_y$, $d \leq y$ and $n \leq m$ or (iv) $a = n'_d$, $b = m'_d$ and $n \leq m$. Now we claim that $\leq$ is a lattice order. It is trivial that it is a partial order. Further it is easy to see that least upper bounds and greatest lower bound exist. For example the least upper bound of the pair $\{n_x, m_y\}$ is k_{x+y} where $k = \max[n, m]$. If the map $\phi : \mathcal{L}$ into $\mathfrak{I}(\mathcal{L}^*)$ is defined by $x\phi = [\omega_x]$ where

$$[\omega_x] = \{z : z \in L^* \text{ and for some } n_x,\ z \leq n_x\},$$

then it is clear that ϕ is an isomorphism. Observe that $\mathcal{L}$ cannot be transferable since

$$(n_a m_b) + (n_a k_c) + (m_b k_c) < (n_a + m_b)(n_a + k_c)(m_b + k_c).$$

CHAPTER 3

In this chapter we present some results concerning transferability of finite lattices. In Section 9 we define the concept of weak stability. We then prove that a transferable lattice is transferable as a join semilattice, weakly stable, dual weakly stable, and that the set of meet irreducibles satisfies a linear order condition which is like the dual of that given in the definition of strict transferability in Chapter 2. These results are summarized in Theorem 9.1, p. 79. In Theorem 9.2, p. 80, we show that if a lattice satisfies all of the above conditions and an additional condition which we have termed the "join-meet condition", then it is transferable. Lastly in Section 10 we give a complete characterization of transferable distributive lattices.

§8. In this section we essentially repeat the construction of §6. However, as we are dealing with lattices instead of semilattices the construction is much more complicated.

The following lemma will considerably ease the construction. We note that it is a consequence of Grätzer's observation about weak transferability.

Lemma 8.1. If $\mathfrak{L}$ is a finite lattice and some element of L is both join and meet reducible then $\mathfrak{L}$ is not transferable.

Proof. Let $a \in L$ with a both join and meet reducible. Let $\omega' = \{0, 0', 1, 1', 2, 2', \ldots\}$ be naturally ordered by the given enumeration. Let

$$L^* = (\{a\} \times \omega') \cup \bigsqcup \{\{x\} \times \omega : x \neq a \text{ and } x \in L\}.$$

Now let b and $c \in L^*$. We define $\leq$ on L^* by: $b \leq c$ if and only if (1) $b = \langle x, n \rangle$, $c = \langle y, m \rangle$ and $x \leq y$ and $n \leq m$ or (2) $b = \langle a, n' \rangle$, $c = \langle y, m \rangle$ and either $a \leq y$ and $n < m$, or $a < y$ and $n = m$, or (3) $b = \langle x, n \rangle$, $c = \langle a, m' \rangle$ and $x \leq a$ and $n \leq m$ or (4) $b = \langle a, n' \rangle$ and $c = \langle a, m' \rangle$ and $n \leq m$. One readily observes that $\leq$ is a partial ordering. We show in detail that $\{b, c\}$ has a least upper bound. Let $b = \langle x, y \rangle$ and $c = \langle z, w \rangle$.

Case 1. y and $w \in \omega$. Let $d = \langle x + z, \max (y, w) \rangle$, then d is clearly an upper bound. Let $\langle u, v \rangle$ be any other upper bound. Then $x + z \leq u$. If $v \in \omega$ we are done. Otherwise $v = m' \in \omega'$ and y and $w \leq m$ whence $d \leq \langle u, v \rangle$ in either case.

Case 2. $y = m' \in \omega'$, $w \in \omega$. Then $b = \langle a, m' \rangle$. If $c \leq b$ we are done. Otherwise either $a < a + z$ and $m \leq w$, or $a = z$ and $m < w$. Let $d = (a + z, \max (m, w))$. Clearly d is an upper bound. Let b and $c \leq \langle u, v \rangle$. Then $a + z \leq u$. If $v \in \omega$ then $m \leq v$ and $w \leq v$. If $v = n' \in \omega'$ then $a = u$, $m \leq n$, and $w \leq n$. Thus $d \leq \langle u, v \rangle$ in either case.

The only other case is that in which y and w are both in ω' and $x = z = a$. In this case either $b \leq c$ or $c \leq b$. This completes the proof that least upper bounds exist. In a similar manner we show the existence of greatest lower bounds. Let $*$ be the resulting lattice.

We define $\phi : L \to \mathfrak{I}(\mathcal{L}^*)$ by

$$x\phi = \{\langle z, y \rangle \in L^* : z \leq x\}.$$

It is easily seen that ϕ is an isomorphism. Now let $H \subseteq L$ with $\Pi H = a$ but $a \notin H$. Suppose $H = \{x_0, \ldots, x_{n-1}\}$. Consider

$H^* = \{\langle x_0, y_0\rangle, \ldots, \langle x_{n-1}, y_{n-1}\rangle\}$, where $y_i \in \omega$. Then $\Pi H^* = \langle a, [\min(y_0, \ldots, y_{n-1})]'\rangle$. Let $a \notin I = \{u_0, \ldots, u_{n-1}\} \subseteq L$ with $a = \Sigma I$. Let $I^* = \{\langle u_0, v_0\rangle, \ldots, \langle u_{m-1}, v_{m-1}\rangle\}$ where $v_i \in \omega$. Then $\Sigma I^* = \langle a, \max(v_0, \ldots, v_{m-1})\rangle$. It is easy to see that if $\Sigma I^* \leq \Pi H^*$ then $\Sigma I^* < \Pi H^*$. Now if $\mathfrak{L}$ is transferable there exists ψ embedding $\mathfrak{L}$ in $\mathfrak{L}^*$ such that for each $x \in H \cup I$, there is an $m \in \omega$ with $x\psi = \langle x, m\rangle$. It follows from the above that $\Sigma(I\psi) < \Pi(H\psi)$ whence ψ is not an isomorphism.

For the remainder of this chapter $\mathfrak{L} = \langle L; +, \cdot\rangle$ is a fixed finite lattice with $K = \{k_0, \ldots, k_{n-1}\} \subseteq L$ its set of join irreducibles. Further we suppose that no element of L is both join and meet reducible.

We construct the relational algebra $\mathfrak{A}^{\mathfrak{L}}$ whose language is that of the structure $\mathfrak{A}^{\mathfrak{S}}$ defined in the last chapter augmented by one additional binary operation symbol $\underset{\sim}{\cdot}$. As in the construction of $\mathfrak{A}^{\mathfrak{S}}$, we let $B = \{b_0, \ldots, b_i, \ldots\}$ be a fixed infinite set, with $b_i \neq b_j$ if $i \neq j$. For the universe of $\mathfrak{A}_0$ we take $\{b_0, \ldots, b_{n-1}\}$. We put $b_i \in U_j$ or V_s provided $k_i \leq k_j$ or s respectively. The individual constant c_i has value b_i. All other operations are empty.

It will be useful to have $\mathfrak{A}_1$ and $\mathfrak{A}_2$ before attempting the inductive step. Let $A_1^* = \mathcal{P}(A_0) \sim \{0\}$, and let g be any map from A_1^* into B which is 1-1 and such that for any $b \in A_0$ $\{b\}g = b$. We define $\leq$ on $A_1 = A_1^* g$ by:

$$a \leq b \text{ if and only if } bg^{-1} \subseteq ag^{-1}.$$

This obviously a semilattice order. We define $\cdot$ on A_1 from $\leq$ so that $\langle A_1; \cdot\rangle$ is a meet semilattice. Now define the operator $\mathcal{S} : A_1 \to L$

as follows:

$$\mathcal{S}x = \begin{cases} k_i & \text{if } x = b_i \\ \Pi\{k_i : b_i \in xg^{-1} \ \& \ i < n\} & \text{if } x \in A_1 \sim A_0. \end{cases}$$

We put $x \in U_i$ or V_s, if $\mathcal{S}x \leq k_i$ or s respectively. The operations $+$ and $f^J_{i,p}$ remain empty.

Let F' be the set of unary operation symbols and

$$F^* = \bigcup\{\{\underset{\sim}{f}^J_{i,p}\} \times U_p : \underset{\sim}{f}^J_{i,p} \in F'\}$$

where U_p denotes the unary relation on A_1. Let g be any 1-1 map from F^* into $B \sim A_1$. We set $A_1' = A_1 \cup F^*g$, and define $f^J_{i,p}$ on A_1 by

$$f^J_{i,p}(x) \begin{cases} (\underset{\sim}{f}^J_{i,p}, b_q)g & \text{if } b_q \in U_p \\ b_q & \text{if } b_q \notin U_p . \end{cases}$$

We extend the operator $\mathcal{S}$ to A_1' by setting

$$\mathcal{S}((\underset{\sim}{f}^J_{i,p}, b_q)g) = k_i ,$$

when $b_q \in U_p$.

Now let $A_2^* = \mathcal{P}(A_1') \sim \{0\}$. For $H \in A_2^*$, let $H' = A_1 \cap H$ and $H^* = H \sim H'$. With respect to this decomposition of H, we define the operators $\mathcal{D}$, $\mathcal{C}^1$ and $\mathcal{C}^*$ as follows:

$$\mathcal{D}(H) = \{a : a \in A_1 \text{ and for some essential pair } \langle p, J\rangle \text{ and each } i \in J,\ f^J_{i,p}a \in H^*\},$$

$$\mathcal{C}^1(H) = \{x : x \in A_1 \text{ and for some } y \in H' \cup \mathcal{D}(H),\quad y \leq x\},$$

$$\mathcal{C}^*(H) = H^* \cup \mathcal{C}^1(H).$$

Now for H_1 and H_2 in A_2^*, we put $H_1 \leq^* H_2$ in case $\mathcal{C}^*(H_1) \subseteq \mathcal{C}^*(H_2)$. We claim that $\leq^*$ is a quasi order with least upper bounds. First we observe that it is obviously a quasi order. We also note that

$$\mathcal{C}^*(\mathcal{C}^*(H)) = \mathcal{C}^*(H).$$

To see this, observe first that $[\mathcal{C}^*(H)]^* = H^*$ since $\mathcal{C}^1(H) \subseteq A_1$. Thus $\mathcal{D}(\mathcal{C}^*(H)) = \mathcal{D}(H)$ whence

$$\mathcal{C}^1(\mathcal{C}^*(H)) = \{x : x \in A_1 \text{ and for some } y \in \mathcal{C}^1(H) \cup \mathcal{D}(H)$$
$$xy = x\} = \mathcal{C}^1(H).$$

From this we observe that $\mathcal{C}^*(H) \leq^* H$ and vice versa. Now we claim that $H_1 \cup H_2$ is a least upper bound for H_1 and H_2. It is clearly an upper bound. Let H_1 and $H_2 \leq^* H_3$. Then

$$\mathcal{C}^*(H_1) \cup \mathcal{C}^*(H_2) \subseteq \mathcal{C}^*(H_3).$$

It is immediate that

$$(H_1 \cup H_2)^* = H_1^* \cup H_2^* \subseteq H_3^*$$

whence we have only to show

$$\mathcal{C}^1(H_1 \cup H_2) \subseteq \mathcal{C}^1(H_3) .$$

Now if $x \in \mathcal{C}^1(H_1 \cup H_2) \sim [\mathcal{C}^1(H_1) \cup \mathcal{C}^1(H_2)]$, then for some $y \in \mathcal{D}(H_1 \cup H_2)$, $xy = x$. But $\mathcal{D}(H_1 \cup H_2) \subseteq \mathcal{D}(H_3)$ since $H_1^* \cup H_2^* \subseteq H_3^*$, whence $x \in \mathcal{C}^1(H_3)$. It is now clear that $\leq^*$ is a quasi order with least upper bounds. We define $\equiv$ by $H_1 \equiv H_2$ if and only if $H_1 \leq^* H_2$ and $H_2 \leq^* H_1$, then we have that $H_1 \equiv \mathcal{C}^*(H_1)$ and $H_1 \equiv H_2$ implies $\mathcal{C}^*(H_1) = \mathcal{C}^*(H_2)$. Now

$\langle A_2^*/\Xi;\ \leq^*/\Xi\rangle$ is a semilattice ordering system.

We claim that there is a 1-1 map g^* from A_2^*/Ξ into B such that for each $b \in A_1'$, $[\{b\}]^{\Xi} g^* = b$. To see this we have only to show that for b_1 and $b_2 \in A_1'$ if $\{b_1\} \Xi \{b_2\}$ then $b_1 = b_2$. For such a b_1 and b_2, $\mathcal{C}^*\{b_1\} = \mathcal{C}^*\{b_2\}$. Now since $|J| \geq 2$ for each $f^J_{i,p}$, we conclude that $\mathcal{D}(\{b_1\}) = \mathcal{D}(\{b_2\}) = 0$. Now $\{b_1\}^* = \{b_2\}^*$ and if $\{b_1\}' \neq 0$ then $\{b_1\} = \{b_1\}^* = \{b_2\}^* = \{b_2\}$ whence $b_1 = b_2$. If $\{b_1\}^* = \{b_2\}^* = 0$, then since b_1 and $b_2 \in A_1$, $b \in \mathcal{C}^1(\{b_2\})$, whence $b_1 \leq b_2$ by definition of $\mathcal{C}^1$. Similarly $b_2 \leq b_1$ whence $b_1 = b_2$.

It follows that g^* exists as desired. For such a g^*, let $A_2 = A_2^* g^*$. We define $\leq$ on A_2 by setting $a \leq b$ if and only if $ag^{*-1} \leq^*/\Xi\ bg^{*-1}$. We define $+$ on A_2 from $\leq$ in the usual way and extend $\mathcal{S}$ to A_2 as follows. For each $x \in A_2$, choose $H \in xg^{*-1}$ and let

$$\mathcal{S}x = \Sigma\{\ \mathcal{S}y : y \in \mathcal{C}^*(H)\}.$$

This is independent of the choice of H because $H_1 \Xi H_2$ implies $\mathcal{C}^*H_1 = \mathcal{C}^*H_2$. We put $x \in U_r$ or V_s if $\mathcal{S}x \leq k_i$ or s respectively. This completes the definition of $\mathfrak{A}_2$.

Below we list an extensive inductive hypothesis. Most of the clauses have not been verified for $\mathfrak{A}_1$ and $\mathfrak{A}_2$, however their validity for these structures will become clear as we proceed through the inductive step of the construction. Hence suppose that we have the structures $\mathfrak{A}_0, \ldots, \mathfrak{A}_{2m}$, $m \geq 1$, that $\mathcal{S} : A_j \to L$ is defined for $j \leq 2m$ and that the following conditions are satisfied.

(i) $\mathfrak{A}_{j+1}$ is an extension of $\mathfrak{A}_j$. Further if $a + b$ or ab is defined in $\mathfrak{A}_j$ then $a + b$ is the $\leq$ least upper bound of $\{a, b\}$ in $\mathfrak{A}_{j+1}$ and ab is the $\leq$ greatest lower bound of $\{a, b\}$. We will term this property the <u>strong extension</u> property.

(ii) In $\mathfrak{A}_{2j}$, each $f^J_{i,p}$ is defined as a partial operation with domain A_{2j-1}, and for each $a \in A_{2j-1}$ we have $f^J_{i,p}a \neq a$ if and only if $a \in U_p$. Further, if $a \in U_p$ and $b \in U_q$ and $f^J_{i,p}a = f^I_{j,q}b$ then $a = b$, $i = j$, $I = J$ and $p = q$.

(iii) $\langle A_{2j+1}; \cdot \rangle$ is a meet semilattice, $\langle A_{2j}; + \rangle$ is a join semilattice. The domain of $+$ in $\mathfrak{A}_{2j+1}$ is A_{2j} and the domain of $\cdot$ in $\mathfrak{A}_{2j}$ is A_{2j-1}.

(iv) If $a \in A_{2j-1}$ then $a \leq \sum_{i \in J} f^J_{i,p}a$ in $\mathfrak{A}_{2j}$.

An element $b \in A_j$ is <u>primitive</u> if $b \in A_0$ or for some $c \in U_p \cap A_j$ and unary operation $f^J_{i,p}$, $f^J_{i,p}c = b$.

(v) If $a \in A_{2k} \sim A_{2k-1}$ and $2k \leq j$ then a is either primitive or join reducible, but not meet reducible in $\mathfrak{A}_j$. If $a \in A_{2k+1} \sim A_{2k}$ and $2k + 1 \leq j$ then a is meet reducible but not primitive or join reducible in $\mathfrak{A}_j$. If $a \in A_{2k} \sim A_{2k-1}$, $2k \leq j$ and a is primitive in $\mathfrak{A}_{2k}$ then a is primitive in $\mathfrak{A}_j$ and a is both join and meet irreducible in $\mathfrak{A}_j$.

(vi) If x and $y \in A_j$ and $x = f^J_{i,p}y \neq y$, then $x \in U_r$ or V_s if and only if $k_i \leq k_r$ or s respectively.

(vii) For each x and $y \in A_j$, the operator $\mathcal{S}$ satisfies the following three conditions (1) $x \leq k_r$ or s if and only if $x \in U_r$ or V_s, (2) if $x + y$ is defined, then $\mathcal{S}(x + y) = \mathcal{S}x + \mathcal{S}y$ and (3) if xy is defined then $\mathcal{S}(xy) = \mathcal{S}x\,\mathcal{S}y$. Note that (2) and (3) imply that if $x \le y$ then $\mathcal{S}x \le \mathcal{S}y$.

(viii) Each U_r or V_s is an ideal in $\langle A_{2j}; +\rangle$.

In each of the above clauses, the range of values of j is the maximum consistent with the fact that $\mathfrak{A}_i$ is not yet meaningful for $i > 2m$.

Our first task is the construction of $\mathfrak{A}_{2m+1}$. Let $A^*_{2m+1} = \mathcal{P}(A_{2m}) \sim \{0\}$. Now for $H \in A^*_{2m+1}$, let $H' = H \cap A_{2m-1}$ and $H^* = H \sim H'$. We set

$$\mathcal{M}(H) = \{x : x \in A_{2m-1},\ x \text{ is meet irreducible in } \langle A_{2m-1}; \cdot\rangle \text{ and for some } y \in H^* \cup \{\Pi H'\},\ y \le x\}$$

and

$$\mathcal{N}(H) = \{x : x \in A_{2m} \sim A_{2m-1} \text{ and for some } y \in H^* \cup \{\Pi\,\mathcal{M}(H)\} \text{ we have } y \le x\}.$$

By convention we leave empty products and sums undefined. Thus if $H = 0$ then $\{\Pi H\} = \{\Sigma H\} = 0$. For H_1 and $H_2 \in A^*_{2m+1}$, we put

$$H_1 \le^* H_2 \text{ if and only if } (1)\ \mathcal{N}(H_2) \subseteq \mathcal{N}(H_1) \text{ and}$$
$$(2) \text{ if } \mathcal{M}(H_2) \ne 0 \text{ then } \mathcal{M}(H_1) \ne 0 \text{ and } \Pi\,\mathcal{M}(H_1) \le \Pi\,\mathcal{M}(H_2).$$

We show that $\le^*$ is a quasi order with greatest lower bounds. That $\le^*$ is a quasi order is clear. Let H_1 and H_2 be given. We show that $H_1 \cup H_2$ is a greatest lower bound for H_1 and H_2. It is easily seen to be a lower bound. Let H_3 be any other lower bound. We show $H_3 \le^* H_1 \cup H_2$. Our first task is to show $\mathcal{N}(H_1 \cup H_2) \subseteq \mathcal{N}(H_3)$. Now if $x \in \mathcal{N}(H_1 \cup H_2)$, then for some $y \in H_1^* \cup H_2^* \cup \{\Pi(\mathcal{M}(H_1 \cup H_2))\}$ we have $y \le x$. Now $H_1^* \cup H_2^* \subseteq \mathcal{N}(H_3)$ since $H_1^* \subseteq \mathcal{N}(H_1) \subseteq \mathcal{N}(H_3)$ and similarly for H_2^*. Hence the result will follow easily if we can only show that

$$\Pi \mathcal{M}(H_3) \leq \Pi \mathcal{M}(H_1 \cup H_2)$$

when $\mathcal{M}(H_1)$, $\mathcal{M}(H_2)$ and $\mathcal{M}(H_3)$ are all $\neq 0$; the other cases are easily handled. This will be obvious if we show that

$$\Pi \mathcal{M}(H_1) \; \Pi \mathcal{M}(H_2) \leq \Pi \mathcal{M}(H_1 \cup H_2) \; .$$

Let $x \in \mathcal{M}(H_1 \cup H_2)$. Then x is meet irreducible, $x \in A_{2m-1}$ and there is a $y \leq x$ such that $y \in H_1^* \cup H_2^* \cup \{\Pi(H_1' \cup H_2')\}$. Consider this y. If $y \in H_1^*$, then

$$\Pi \mathcal{M}(H_1) \; \Pi \mathcal{M}(H_2) \leq \Pi \mathcal{M}(H_1) \leq x$$

since $x \in \mathcal{M}(H_1)$. If $y \in H_2^*$ a similar result is obtained. Otherwise $y = \Pi(H_1' \cup H_2')$, whence

$$\Pi \mathcal{M}(H_1) \; \Pi \mathcal{M}(H_2) \leq \Pi H_1' \; \Pi H_2' = \Pi(H_1' \cup H_2') \leq x.$$

Thus for each $x \in \mathcal{M}(H_1 \cup H_2)$, $\Pi \mathcal{M}(H_1) \; \Pi \mathcal{M}(H_2) \leq x$. This completes the proof that $\Pi \mathcal{M}(H_1 \cup H_2) \geq \Pi \mathcal{M}(H_1) \; \Pi \mathcal{M}(H_2)$ whence $H_3 \leq^* H_1 \cup H_2$ as desired. It follows that $H_1 \cup H_2$ is a $\leq^*$ greatest lower bound for H_1 and H_2.

We define $\equiv$ on A_{2m+1}^* by $H_1 \equiv H_2$ if and only if $H_1 \leq^* H_2$ and $H_2 \leq^* H_1$. Let $\leq^*/\equiv$ be defined on $A_{2m+1}^*/\equiv$ in the usual way.

We claim that there is a 1-1 function $g : A_{2m+1}^*/\equiv \to B$ which satisfies $[\{x\}]^{\equiv} = x$ for each $x \in A_{2m}$. To see that such a function exists, we have only to show that if x and $y \in A_{2m}$ and $[\{x\}]^{\equiv} = [\{y\}]^{\equiv}$ then $x = y$. This will be trivial from

Lemma 8.2. *If x and $y \in A_{2m}$ and $\{x\} \leq^* \{y\}$ then $x \leq y$.*

Proof. Let x and $y \in A_{2m}$ with $\{x\} \leq^* \{y\}$.

Case 1. $y \in \mathcal{N}(\{y\})$. Note that this case includes the case when $\mathcal{M}(\{y\}) = 0$. Now, $y \in \mathcal{N}(\{y\}) \subseteq \mathcal{N}(\{x\})$ whence for some $z \in \{x\}^* \cup \{\Pi\, \mathcal{M}(\{x\})\}$, $z \leq y$. Now if $\mathcal{M}(\{x\}) \neq 0$ then it is easy to see that $x \leq \Pi\, \mathcal{M}(\{x\})$. Further if $\{x\}^* \neq 0$ then $\{x\}^* = \{x\}$ whence we conclude that $x \leq z \leq y$.

Case 2. $y \notin \mathcal{N}(\{y\})$. For this case, we have $y \in A_{2m-1}$ whence $\mathcal{M}(\{y\}) \neq 0$ and $y = \Pi\, \mathcal{M}(\{y\})$. Since

$$x \leq \Pi\, \mathcal{M}(\{x\}) \leq \Pi\, \mathcal{M}(\{y\}) = y$$

we are done. This completes the lemma.

Now fix $g : A^*_{2m+1}/\equiv \to B$ such that if $x \in A_{2m}$ then $[\{x\}]g = x$ and g is 1-1. Let $A_{2m+1} = (A_{2m+1}/\equiv)g$. We define a binary relation $\leq^\bullet$ on A_{2m+1} by: $a \leq^\bullet b$ if and only if $ag^{-1} \leq^*/\equiv bg^{-1}$. We make the following claims about $\leq^\bullet$:

(1) if a and $b \in A_{2m}$, then $a \leq b$ if and only if $a \leq^\bullet b$,

(2) if $a \in A_{2m+1}$ and $a \leq^\bullet x$ for each $x \in I_1 \subseteq A_{2m-1}$, then $a \leq^\bullet \Pi I_1$ where ΠI_1 is computed in $\mathfrak{A}_{2m-1}$, and

(3) if $a \in A_{2m+1}$ and $x \leq^\bullet a$ for each $x \in I_2 \subseteq A_{2m}$, then $\Sigma I_2 \leq^\bullet a$ where ΣI_2 is computed in $\mathfrak{A}_{2m}$.

For (1), if $a \leq^\bullet b$ then $a \leq b$ by Lemma 8.2. If $a \leq b$ where a and $b \in A_{2m}$ it is easy to see from the definitions of $\mathcal{M}$ and $\mathcal{N}$ that $\mathcal{N}(\{b\}) \subseteq \mathcal{N}(\{a\})$ and $\mathcal{M}(\{b\}) \subseteq \mathcal{M}(\{a\})$. It is now immediate that $\{a\} \leq^* \{b\}$, whence $a \leq^\bullet b$.

Let $I_1 \subseteq A_{2m-1}$ and $a \in A_{2m+1}$ with $a \leq^\bullet x$ for each $x \in I_1$. Fix

$H \in ag^{-1}$. Then $H \leq^* \{x\}$ for each $x \in I_1$, whence $H \leq^* I_1$ since I_1 is a $\leq^*$ greatest lower bound for $\{\{x\} : x \in I_1\}$. Now $I_1^* = 0$, from which we can easily deduce that

$$\mathfrak{m} I_1 = \mathfrak{m}\{\Pi I_1\} .$$

This equality yields $\eta(\{\Pi I_1\}) = \eta(I_1)$, whence $\{\Pi I_1\} \equiv I_1$. Thus $ag^{-1} \leq^*/\equiv [\{\Pi I_1\}]$ whence $a \leq^{\bullet} \Pi I_1$ as claimed.

For (3), let $I_2 \subseteq A_{2m}$, $a \in A_{2m+1}$ and suppose $x \leq^{\bullet} a$ for each $x \in I_2$. Choose $H \in ag^{-1}$ and fix $x \in I_2$ and $y \in H$. Now

$$\{x\} \leq^* H \leq^* \{y\} .$$

By Lemma 8.2 we have $x \leq y$ whence $\Sigma I_2 \leq y$. Thus $\{\Sigma I_2\} \leq^* \{y\}$ for each $y \in H$ whence $\{\Sigma I_2\} \leq^* H$ since H is a $\leq^*$ greatest lower bound for $\{\{y\} : y \in H\}$. It follows that $\Sigma I_2 \leq^{\bullet} a$ as claimed.

From (1) - (3) we conclude that $\langle A_{2m+1}; \leq^{\bullet} \rangle$ is a strong extension of $\langle A_{2m}; \leq \rangle$. Further we no longer need maintain a distinction between $\leq^{\bullet}$ and $\leq$. We define $\cdot$ from $\leq$ in the usual way to obtain a meet semilattice $\langle A_{2m+1}; \cdot \rangle$. From (3) we have that this extends $\langle A_{2m-1}; \cdot \rangle$. All other operations remain as in $\mathfrak{A}_{2m}$. We extend $\mathcal{S}$ to A_{2m+1} by choosing H in xg^{-1} and letting

$$\mathcal{S}x = \begin{cases} x & \text{if } x \in A_{2m} \\ \Pi\{\mathcal{S}y : y \in \eta(H) \cup \{\Pi\, \mathfrak{m}(H)\}\} & \text{if } x \in A_{2m+1} \sim A_{2m} . \end{cases}$$

This definition is independent of H, since $H_1 \equiv H_2$ implies $\eta(H_1) = \eta(H_2)$ and $\Pi\, \mathfrak{m}(H_1) = \Pi\, \mathfrak{m}(H_2)$. We extend U_r and V_s to $\mathfrak{A}_{2m+1}$ by adding $a \in A_{2m+1} \sim A_{2m}$ to U_r or V_s if and only if $\mathcal{S}a \leq k_r$ or s respectively. This completes the definition of $\mathfrak{A}_{2m+1}$.

We must verify the following clauses of the inductive hypothesis, (iii), (v), and (vii). We note that we have already treated (i) and (iii), whence we come to (v). We easily note that if $a \in A_{2m+1} \sim A_{2m}$ then a is meet reducible. Further if $a \in A_{2m+1} \sim A_{2m}$ then a is neither join reducible in $\mathfrak{A}_{2m+1}$ nor primitive. Let $b \in A_{2j} \sim A_{2j-1}$, $j \leq m$, then b is meet irreducible in $\mathfrak{A}_{2m}$ from the inductive hypothesis. We claim that b is meet irreducible in $\mathfrak{A}_{2m+1}$. To demonstrate this fact, we have only to show that whenever $H \subseteq A_{2m}$ and $\Pi H = b$, that $b \in H$. This is sufficient since if $a \in A_{2m+1}$ then $a = \Pi I$ for some $I \subseteq A_{2m}$. There are two cases.

Case 1. $j < m$. Thus $b \in A_{2m-1}$, whence we have $b = \Pi\mathcal{M}(\{b\})$. Since b is meet irreducible in $\langle A_{2m-1}; \cdot \rangle$ by (v), $b \in \mathcal{M}(\{b\})$. Further $z \in \mathcal{M}(\{b\})$ implies $b \leq z$. Now if $H \subseteq A_{2m}$ and $H \equiv \{b\}$ then

$$b = \Pi\, \mathcal{M}(\{b\}) = \Pi\, \mathcal{M}(H) .$$

Now since b is meet irreducible in $\mathfrak{A}_{2m-1}$ we have $b \in \mathcal{M}(H)$. Thus for some $z \in H^* \cup \{\Pi H'\}$, $z \leq b$. For such a z, $z = b$ since $b \leq x$ for each $x \in H$, whence $z \notin H^* \subseteq A_{2m} \sim A_{2m-1}$, whence $b = \Pi H'$. Since $H' \subseteq A_{2m-1}$ we conclude by (v) that $b \in H' \subseteq H$. Thus b is meet irreducible in $\mathfrak{A}_{2m+1}$.

Case 2. $j = m$. Thus $b \in A_{2m} \sim A_{2m-1}$, whence if $H \equiv \{b\}$ with $H \subseteq A_{2m}$, then $b \in \mathcal{N}(\{b\}) = \mathcal{N}(H)$, and if $\Pi\, \mathcal{M}(\{b\})$ exists we have

$$b < \Pi\, \mathcal{M}(\{b\}) = \Pi\, \mathcal{M}(H) .$$

Thus from the definition of $\mathcal{N}$ we have that for some $z \in H^*$, $z \leq b$, whence $b \in H^* \subseteq H$ since for each $x \in H$, $b \leq x$. Thus again $b \in H$

which completes the proof that b is meet irreducible in $\mathfrak{A}_{2m+1}$.

This brings us to (vii). We must first show that $\mathcal{S}x \in k_r$ or s if and only if $x \in U_r$ or V_s. This is seen to be satisfied by definition. Next we must show that $\mathcal{S}x\,\mathcal{S}y = \mathcal{S}(xy)$. For this it is sufficient to show that if $H \subseteq A_{2m}$ then

$$\mathcal{S}(\Pi H) = \mathcal{S}([H]^{\equiv} g) = \Pi(\mathcal{S} H),$$

because every element of A_{2m+1} is the meet of elements in A_{2m-1}. We divide the proof into three cases.

Case 1. $\Pi H \in A_{2m-1}$. For this case, let $a = \Pi H$. Now we have already shown that $a = \Pi\, \mathcal{M}(\{a\})$, whence since $\{a\} \equiv H$ we have $\Pi\, \mathcal{M}(H) = a$. Now for each $x \in \mathcal{M}(H)$ there exists $y \in H^* \cup \{\Pi H'\}$ with $y \leq x$. Further $a \leq y$ since $y = \Pi H'$. By the induction hypothesis we have

$$\mathcal{S} a \leq \mathcal{S} y \leq \mathcal{S} x$$

whence

$$\mathcal{S} a \leq \Pi\, \mathcal{S}(H^* \cup \{\Pi H'\}) \leq \Pi \mathcal{S} \mathcal{M}(H).$$

Applying the induction hypothesis to $H' \subseteq A_{2m-1}$ we have $\mathcal{S}\Pi H' = \Pi \mathcal{S} H'$ whence $\Pi\, \mathcal{S}(H^* \cup \{\Pi H'\}) = \Pi \mathcal{S} H$. Also applying the induction hypothesis to $\mathcal{M}(H) \subseteq A_{2m-1}$ we get

$$\Pi\, \mathcal{S}(\mathcal{M}(H)) = \mathcal{S}\Pi(\mathcal{M}(H)) = \mathcal{S} a,$$

whence $\mathcal{S}\Pi(H) = \Pi(\mathcal{S} H)$ for this case as desired.

Case 2. $\Pi H \in A_{2m} \sim A_{2m-1}$. Let $a = \Pi H$. Since $a \in A_{2m} \sim A_{2m-1}$, a is meet irreducible whence $a = \Pi H$ if and only if $a \in H$ and $a \leq y$ for each $y \in H$. This clearly yields

$$\mathfrak{S}a = \mathfrak{S}(\Pi H) = \Pi(\mathfrak{S} H),$$

as desired.

Case 3. $\Pi H \in A_{2m+1} \sim A_{2m}$. In this case by definition

$$\mathfrak{S}(\Pi H) = \Pi\{\mathfrak{S}y : y \in \mathcal{N}(H) \cup \{\Pi \mathcal{M}(H)\}\} .$$

Exactly as in Case 1 we have

$$\Pi \mathfrak{S}(H^* \cup \{\Pi H'\}) \leq \Pi \mathfrak{S} M(H)$$

and $\Pi \mathfrak{S}(H^* \cup \{\Pi H'\}) = \Pi(\mathfrak{S} H)$. If $x \in \mathcal{N}(H)$ then for some $y \in H^* \cup \{\Pi \mathcal{M}(H)\}$, $y \leq x$, whence $\mathfrak{S}y \leq \mathfrak{S}x$, whence

$$\Pi \mathfrak{S}(H) \leq \Pi(\mathfrak{S} \mathcal{N}(H)) .$$

Thus

$$\Pi \mathfrak{S}(H) \leq \Pi \mathfrak{S} (\mathcal{N}(H) \cup \{\Pi \mathcal{M}(H)\}) = \mathfrak{S}(\Pi H) .$$

Conversely $H^* \subseteq \mathcal{N}(H)$ and $\Pi \mathcal{M}(H) \leq \Pi H'$ from which we see

$$\mathfrak{S}(\Pi H) = \Pi \mathfrak{S}(\mathcal{N}(H) \cup \{\Pi \mathcal{M}(H)\}) \leq \Pi \mathfrak{S}(H^* \cup \{\Pi H'\}) = \Pi \mathfrak{S} H .$$

We conclude that $\mathfrak{S}\Pi H = \Pi(\mathfrak{S}H)$ in any case. This concludes the construction of $\mathfrak{A}_{2m+1}$.

The task remaining is the construction of $\mathfrak{A}_{2m+2}$. As before, let F' denote the set of unary operation symbols. Let g^* be any mapping from $F' \times (A_{2m+1} \sim A_{2m-1})$ into $B \sim A_{2m-1}$ which satisfies the following three conditions:

1) If $b \notin U_p$ then $(\underset{\sim}{f}^J_{i,p}, b)g^* = b$.

2) If $b_1 \in U_p$ and $b_2 \in U_q$ then

$$(\underset{\sim}{f}^J_{i,p}, b_1)g^* = (\underset{\sim}{f}^J_{j,q}, b_2)g^*$$

if and only if $i = j$, $p = q$, $J = J'$ and $b_1 = b_2$.

3) If $b \in U_p$ then $(\underset{\sim}{f}{}^J_{i,p}, b)g^* \in B \sim A_{2m+1}$.

Let $A'_{2m+2} = A_{2m+1} \cup Rg(g^*)$. We extend the domain of definition of the operation $f^J_{i,p}$ to A_{2m+1} by setting

$$f^J_{i,p}x = \begin{cases} f^J_{i,p}x & \text{if } x \in A_{2m-1} \\ (\underset{\sim}{f}{}^J_{i,p}, x)g^* & \text{if } x \in A_{2m+1} \sim A_{2m-1} . \end{cases}$$

Clearly this definition satisfies condition (ii) of the inductive hypothesis.

We extend the U_p and V_s to A'_{2m+2} by the following:

$x \in U_r$ or V_s if and only if (1) $x \in A_{2m+1}$ and $x \in U_r$ or V_s respectively, or (2) $x = f^J_{i,p}b \in A'_{2m+2} \sim A_{2m+1}$ and $k_i \leq k_r$ or s respectively.

This definition satisfies (vi) of the inductive hypothesis and each $f^J_{i,p}$ extends the definition of $f^J_{i,p}$ on A_{2m}. Now we extend $\mathcal{S}$ to A'_{2m+2} by setting

$$\mathcal{S}x = \begin{cases} \mathcal{S}x & \text{if } x \in A_{2m+1} \text{ where } \mathcal{S} \text{ is defined on } A_{2m+1} \\ k_i & \text{if } x = f^J_{i,p}y \in A'_{2m+2} \sim A_{2m+1} . \end{cases}$$

It is trivial that for each $x \in A'_{2m+1}$, $x \in U_r$ or V_s if and only if $\mathcal{S}x \leq k_r$ or s respectively.

We now extend the universe of our structure so as to make it an upper semilattice. To this end, let $A^*_{2m+2} = \mathcal{P}(A'_{2m+1}) \sim \{0\}$, and for $H \in A^*_{2m+2}$, let $H' = A_{2m+1} \cap H$ and $H^* = H \sim H'$. Now we define the operators $\mathcal{B}$,

$\mathcal{J}$ and $\mathcal{C}^{2m+1}$ as follows:

$\mathcal{B}(H) = \{x : x \in A_{2m+1}$ and for some essential pair $\langle p, J\rangle$ and each $i \in J$ $f^J_{i,p}x \in H^*\}$,

$\mathcal{J}(H) = \{x : x \in A_{2m}$ and for some $y \in H' \cup \mathcal{B}(H)$, $x \leq y\}$,

$\mathcal{C}^{2m+1}(H) = \{x : x \in A_{2m+1}$, x is primitive or meet reducible , and for some $y \in H' \cup \mathcal{B}(H) \cup \{\Sigma\, \mathcal{J}(H)\}$, $x \leq y\}$.

Set

$$\mathcal{C}^*(H) = H^* \cup \mathcal{C}^{2m+1}(H).$$

Note that $\mathcal{B}(H) = \mathcal{B}(H^*)$.

We claim that $\mathcal{C}^*$ is idempotent as an operator. Since $[\mathcal{C}^*(H)]^* = H^*$, it is sufficient to show that

$$\mathcal{C}^{2m+1}(\mathcal{C}^*(H)) = \mathcal{C}^{2m+1}(H).$$

We first observe that $\mathcal{B}(\mathcal{C}^*(H)) = \mathcal{C}(H)$, whence

$\mathcal{J}(\mathcal{C}^*(H)) = \{x : x \in A_{2m}$ and for some $y \in H^* \cup \mathcal{B}(H) \cup \mathcal{C}^{2m+1}(H)$, $x \leq y\}$.

Now let $x \in \mathcal{J}(\mathcal{C}^*(H))$, with witness y. If $y \in H^* \cup \mathcal{B}(H)$, then $x \in \mathcal{J}(H)$. Otherwise $y \in \mathcal{C}^{2m+1}(H)$. For this case there exists $z \in H' \cup \mathcal{B}(H) \cup \{\Sigma\, \mathcal{J}(H)\}$ such that $y \leq z$. Now if $z \in H' \cup \mathcal{B}(H)$ then $x \leq z$ whence $x \in \mathcal{J}(H)$. Otherwise $x \leq \Sigma\mathcal{J}(H)$. Thus in any case this last inequality is valid. Thus

$$\Sigma\, \mathcal{J}(\mathcal{C}^*(H)) \leq \Sigma\, \mathcal{J}(H).$$

Since $\mathcal{J}(H) \subseteq \mathcal{J}(\mathcal{C}^*(H))$ is obvious, we can conclude that $\Sigma\, \mathcal{J}(\mathcal{C}^*(H)) = \Sigma\, \mathcal{J}(H)$ and hence that

$$\mathcal{B}(H) \cup \{\Sigma\, \mathcal{J}(H)\} = \mathcal{B}(\mathcal{C}^*(H)) \cup \{\Sigma\, \mathcal{J}(\mathcal{C}^*(H))\}.$$

Now if $x \in \mathcal{C}^{2m+1}(H)$ with witness $y \in H'$ then it is clear that $x \in \mathcal{C}^{2m+1}(\mathcal{C}^*(H))$, since

$$\mathcal{C}^{2m+1}(\mathcal{C}^*(H)) = \{x : x \text{ is primitive or meet reducible and for some } y \in \mathcal{C}^{2m+1}(H) \cup \mathcal{D}(\mathcal{C}^*(H)) \cup \{\Sigma \mathcal{J}(\mathcal{C}^*(H))\},\ x \leq y\}.$$

Conversely, if $x \in \mathcal{C}^{2m+1}(\mathcal{C}^*(H))$ with witness $y \in \mathcal{C}^{2m+1}(H)$, it is trivial that $x \in \mathcal{C}^{2m+1}(H)$. Thus

$$\mathcal{C}^{2m+1}(\mathcal{C}^*(H)) = \mathcal{C}^{2m+1}(H).$$

We now define $\leq^*$ on A^*_{2m+2} by:

$$H_1 \leq^* H_2 \text{ if and only if } \mathcal{C}^*(H_1) \subseteq \mathcal{C}^*(H_2).$$

We claim that $\leq^*$ is a quasi order with least upper bounds. That it is a quasi order is trivial. We show that $H_1 \cup H_2$ is a least upper bound for H_1 and H_2. Let H_1 and $H_2 \leq^* H_3$. Then $\mathcal{C}^*(H_1)$ and $\mathcal{C}^*(H_2) \subseteq \mathcal{C}^*(H_3)$. Now $\mathcal{C}^*(H) = H^* \cup \mathcal{C}^{2m+1}(H)$. Since $H^* \subseteq A'_{2m+2} \sim A_{2m+1}$ and $\mathcal{C}^{2m+1}(H) \subseteq A_{2m+1}$ we conclude that $H_1^* \cup H_2^* \subseteq H_3^*$. Since $\mathcal{D}(H) = \mathcal{D}(H^*)$ we have that $\mathcal{D}(H_1 \cup H_2) \subseteq \mathcal{D}(H_3)$. Now to complete the argument, we must show that

$$\mathcal{C}^{2m+1}(H_1 \cup H_2) \subseteq \mathcal{C}^{2m+1}(H_3).$$

Now

$$\mathcal{C}^{2m+1}(H_1 \cup H_2) = \{x : x \in A_{2m+1},\ x \text{ is primitive or meet reducible and for some } y \in H_1' \cup H_2' \cup \mathcal{D}(H_1 \cup H_2) \cup \{\Sigma \mathcal{J}(H_1 \cup H_2)\},\ x \leq y\}.$$

Let $a \in \mathcal{C}^{2m+1}(H_1 \cup H_2)$ with witness b. If $b \in H_1'$, then $a \in \mathcal{C}^{2m+1}(H_1) \subseteq \mathcal{C}^{2m+1}(H_3)$ and we are done. Similarly if $b \in H_2'$. If $b \in \mathcal{D}(H_1 \cup H_2) \subseteq \mathcal{D}(H_3)$ we are done by the definition of $\mathcal{C}^{2m+1}(H_3)$.

Thus let $b = \Sigma \mathcal{J}(H_1 \cup H_2)$. We will show that if $c \in \mathcal{J}(H_1 \cup H_2)$ then $c \leq \Sigma \mathcal{J}(H_3)$. Now for such a c there is a $d \in H_1 \cup H_2 \cup \mathcal{B}(H_1 \cup H_2)$ with $c \leq d$. Now if $d \in \mathcal{B}(H_1 \cup H_2)$ then $d \in \mathcal{B}(H_3)$ whence $c \leq \Sigma \mathcal{J}(H_3)$. Hence let $d \in H_1$ such that for no $d_1 \in \mathcal{B}(H_1) \subseteq \mathcal{B}(H_1 \cup H_2)$ is it the case that $d \leq d_1$. Now if $d \in H_1^* \subseteq H_3^*$ then $c \leq \Sigma \mathcal{J}(H_3)$. Thus we may assume that $d \in A_{2m+1}$. If $d \in A_{2m+1} \sim A_{2m}$, then by (v), $d \in \mathcal{C}^{2m+1}(H_1) \subseteq \mathcal{C}^{2m+1}(H_3)$ whence for some $d_1' \in H_3' \cup \mathcal{B}(H_3) \cup \{\Sigma \mathcal{J}(H_3)\}$ we have $d \leq d_1'$. Now $c \in A_{2m}$ whence we again conclude that $c \leq \Sigma \mathcal{J}(H_3)$. Lastly suppose $d \in A_{2m}$. For such a d, we obtain

$$d = \Sigma\{x : x \in A_{2m} \quad x \text{ is primitive or meet reducible and } x \leq d\}.$$

This is a consequence of the inductive hypothesis part (v) and the fact that $\langle A_{2m}; +\rangle$ is a semilattice. For $d_1 \leq d$, with d_1 primitive or meet reducible, $d_1 \in \mathcal{C}^{2m+1}(H_3)$ whence for some $d_2 \in H_3 \cup \mathcal{B}(H_3) \cup \{\Sigma \mathcal{J}(H_3)\}$, $d_1 \leq d_2$. Thus from the definition of $\mathcal{J}$ we conclude $d_1 \leq \Sigma \mathcal{J}(H_3)$. We therefore have

$$\begin{aligned} c \leq d &= \Sigma\{x : x \in A_{2m} \;\&\; x \text{ is primitive or meet reducible } \&\; x \leq d\} \\ &\leq \Sigma \mathcal{J}(H_3) . \end{aligned}$$

This completes the proof that if $c \in \mathcal{J}(H_1 \cup H_2)$ then $c \leq \Sigma \mathcal{J}(H_3)$. We conclude that

$$b = \Sigma \mathcal{J}(H_1 \cup H_2) \leq \Sigma \mathcal{J}(H_3) .$$

It follows that

$$\mathcal{C}^{2m+1}(H_1 \cup H_2) \subseteq \mathcal{C}^{2m+1}(H_3)$$

as desired, whence $H_1 \cup H_2$ is a least upper bound for H_1 and H_2 with respect to $\leq^*$.

We define $\equiv$ on A^*_{2m+2} by $H_1 \equiv H_2$ if and only if $H_1 \leq^* H_2$ and $H_2 \leq^* H_1$. The following are immediate: (1) $H_1 \equiv H_2$ if and only if $\mathcal{C}^*(H_1) = \mathcal{C}^*(H_2)$, (2) $H \equiv \mathcal{C}^*(H)$ and (3) $\langle A^*_{2m+2}/\equiv; \leq/\equiv \rangle$ is a join semilattice ordering system.

We will now show that if a and $b \in A'_{2m+2}$ and $\{a\} \equiv \{b\}$ then $a = b$. First observe that $\mathcal{B}(\{a\}) = \mathcal{B}(\{b\}) = 0$, since $|\{a\}^*|$ and $|\{b\}^*| \leq 1$. Now by hypothesis $\mathcal{C}^*(\{a\}) = \mathcal{C}^*(\{b\})$ whence $\{a\}^* = \{b\}^*$ and $\mathcal{C}^{2m+1}(\{a\}) = \mathcal{C}^{2m+1}(\{b\})$. Now if $\{a\}^* \neq 0$ then

$$\{a\} = \{a\}^* = \{b\}^* = \{b\}$$

whence $a = b$. Similarly if $\{b\}^* \neq 0$. Hence, we suppose a and $b \in A_{2m+1}$. Now we consider two cases, first $a \in A_{2m+1} \sim A_{2m}$ and second $a \in A_{2m}$. In the first case a is meet reducible, whence $a \in \mathcal{C}^{2m+1}(\{a\})$. In the second case, since $\langle A_{2m}; + \rangle$ is a join semilattice, and since by (v) join irreducible is equivalent to "primitive or meet reducible" in $\mathfrak{A}_{2m}$, we have $a = \Sigma[\mathcal{C}^{2m+1}(\{a\}) \cap A_{2m}]$. Since

$$\mathcal{C}^{2m+1}\{a\} \subseteq \mathcal{C}^{2m+1}\{b\} \subseteq \{x : x \in A_{2m+1} \text{ and } x \leq b\},$$

we have in either case that $a \leq b$ follows from $\{a\} \leq^* \{b\}$. We isolate this fact as

Lemma 8.3. If a and $b \in A_{2m+1}$ and $\{a\} \leq^* \{b\}$ then $a \leq b$.

Thus from Lemma 8.3 and our preceding remarks, if $\{a\} \equiv \{b\}$ with a and $b \in A'_{2m+2}$, we have $a = b$ as claimed. It follows that there is a 1-1 function g from $A^*_{2m+2}/\equiv$ into B such that $[\{a\}]^{\equiv} = a$ for each $a \in A'_{2m+2}$. Fix such a g and let $A_{2m+2} = (A^*_{2m+2}/\equiv)g$.

We define a binary relation $\leq^+$ on A_{2m+2} so that g is order preserving. We make the following assertions about $\leq^+$:

(1) if a and $b \in A_{2m+1}$ then $a \leq b$ if and only if $a \leq^+ b$,

(2) if $a \in A_{2m+2}$ and $x \leq^+ a$ for each $x \in I_1 \subseteq A_{2m}$, then $\Sigma I_1 \leq^+ a$, where ΣI_1 is computed in $\mathfrak{A}_{2m}$, and,

(3) if $a \in A_{2m+2}$ and $a \leq^+ x$ for each $x \in I_2 \subseteq A_{2m+1}$, then $a \leq^+ \Pi I_2$ where ΠI_2 is computed in $\mathfrak{A}_{2m+1}$.

For (1) recall the observation made above that if a and $b \in A_{2m+1}$ and $\{a\} \leq^* \{b\}$ then $a \leq b$. Conversely let a and $b \in A_{2m+1}$ with $a \leq b$. Then $\mathcal{C}^*(\{a\}) = \mathcal{C}^{2m+1}(\{a\})$. Further by (i) we have that $\Sigma\, \mathcal{J}(\{a\}) \leq a$, whence for each $x \in \mathcal{C}^{2m+1}(\{a\})$ $x \leq a$, whence $x \leq b$, whence $x \in \mathcal{C}^*(\{b\})$. Thus $\mathcal{C}^*(\{a\}) \subseteq \mathcal{C}^*(\{b\})$ whence $\{a\} \leq^* \{b\}$.

For (2), let $a \in A_{2m+2}$ and $I_1 \subseteq A_{2m}$ with $x \leq a$ for each $x \in I_1$. Let $H \in ag^{-1}$. Then $\{x\} \leq^* H$ for each $x \in I_1$. Since $I_1 = \bigcup\{\{x\} : x \in I_1\}$ is a $\leq^*$ least upper bound for $\{\{x\} : x \in I_1\}$ we have $I_1 \leq^* H$. Hence it is sufficient to show that $\{\Sigma I_1\} \equiv I_1$. Now $I_1^* = 0 = \{\Sigma I_1\}^*$ whence by definition of $\mathcal{J}$ we have $\Sigma\, \mathcal{J}(I_1) = \Sigma\, \mathcal{J}\{\Sigma I_1\} = \Sigma I_1$. It follows that

$$\mathcal{C}^{2m+1}(I_1) = \mathcal{C}^{2m+1}(\{\Sigma I_1\})$$

whence $I_1 \equiv \{\Sigma I_1\}$. This completes the proof of (2).

For (3), let $a \in A_{2m+2}$ with $a \leq^+ x$ for each $x \in I_2 \subseteq A_{2m+1}$. Fix $H \in ag^{-1}$. Then for each $x \in I_2$ and $y \in H$ we have

$$\{y\} \leq^* H \leq^* \{x\}.$$

Now $y \notin A'_{2m+2} \sim A_{2m+1}$ since if this were the case we would have $y \in \{x\}^* = 0$ since $x \in A_{2m+1}$. Thus $y \in A_{2m+1}$ and by Lemma 8.3 we have $y \leq x$. We conclude that for each $y \in H$, $y \leq \Pi I_2$ whence by (1) we have $\{y\} \leq^* \{\Pi I_2\}$. Since H is a $\leq^*$ least upper bound for $\{\{y\} : y \in H\}$ we have $H \leq^* \{\Pi I_2\}$ whence $a \leq^+ \Pi I_2$ as claimed.

From (1) - (3) above we see that $\langle A_{2m+2}; \leq^+ \rangle$ is a strong extension of $\langle A_{2m+1}; \leq \rangle$. Since $\leq^+$ agrees with $\leq$ on A_{2m+1} we cease the distinction. We define $+$ on A_{2m+2} from $\leq$ in the usual way and it is clear that $\langle A_{2m+2}; + \rangle$ is an extension of $\langle A_{2m}; + \rangle$. The partial operation $\cdot$ remains as on $\mathfrak{A}_{2m+1}$, as do the constant operations. The unary operations remain as in the construction of A'_{2m+2}, with domain A_{2m+1}. For $a \in A_{2m+2} \sim A'_{2m+2}$ we choose $H \in ag^{-1}$ and set

$$\mathcal{S}a = \Sigma\{ \mathcal{S}b : b \in \mathcal{C}^*(H)\}.$$

This definition is independent of our choice of H and extends the definition of $\mathcal{S}$ to A_{2m+2}. For $a \in A_{2m+2} \sim A'_{2m+2}$ we add a to U_r or V_s if and only if $\mathcal{S}a \leq k_r$ or s respectively. This completes the construction of $\mathfrak{A}_{2m+2}$ and it is clear that $\mathfrak{A}_{2m+2}$ is a strong extension of $\mathfrak{A}_{2m+1}$.

We now verify the various clauses of the inductive hypothesis. We have already dealt with (i) - (iii). For (iv) if $a \in (A_{2m+1} \cap U_p) \sim A_{2m-1}$, then

$$a \in \mathcal{D}(\{f^J_{i,p}a : i \in J\})$$

for the essential pair $\langle p, J \rangle$. Thus

$$\{a\} \leq^* \{f^J_{i,p}a : i \in J\}$$

whence $a \leq \sum_{i \in J} f^J_{i,p}a$. If $a \notin U_p$ then $f^J_{i,p}a = a$ and the result follows. If $a \in A_{2m-1}$, we have the result from the inductive hypothesis.

For (v), we note first that if $a \in A_{2m+2} \sim A_{2m+1}$ then a is either primitive or join reducible but not meet reducible since the domain of $\cdot$ is A_{2m+1} and A_{2m+1} is closed under this operation. Further if a is primitive, then $\{a\}^* = \{a\} \subseteq H^*$ whenever $\{a\} \leq^* H$ whence it is immediate that a is not join reducible. We will be done if we show that if a is primitive or meet reducible in $\mathfrak{A}_{2m+1}$ then a is not join reducible in $\mathfrak{A}_{2m+2}$. Fix such an $a \in A_{2m+1}$. Recall that if a is join reducible in $\mathfrak{A}_{2m+2}$ then there is an $H \subseteq A'_{2m+2}$ such that $H \equiv \{a\}$ and $a \notin H$. Fix such an H. Now $\{a\}^* = 0$, whence $\mathcal{C}^*(\{a\}) \subseteq \mathcal{C}^{2m+1}(\{a\}) \subseteq A_{2m+1}$. It follows that $H^* = 0$, since $H^* \subseteq \mathcal{C}^*(H) = \mathcal{C}^*(\{a\})$. Thus $H \subseteq A_{2m+1}$ whence $\mathcal{C}^*(H) = \mathcal{C}^{2m+1}(H)$. Further, we note that if $x \in H$ then

$$\{x\} \leq^* H \leq^* \{a\} ,$$

whence since x and $b \in A_{2m+1}$ we conclude that $x \leq b$ by Lemma 8.3. Next observe that since $H^* = 0$, $\mathcal{B}(H) = 0$, whence if $y \in \mathcal{J}(H)$ then for some $x \in H \cup \mathcal{B}(H) = H$, we have $y \leq x \leq a$. Thus if $\mathcal{J}(H) \neq 0$ then

$$\Sigma \mathcal{J}(H) \leq a$$

by (i) of the inductive hypothesis. There are two cases.

Case 1. $a \in A_{2m}$. Now a is either primitive or meet reducible in $\mathfrak{A}_{2m+1}$, whence $a \in \mathcal{C}^{2m+1}(\{a\}) = \mathcal{C}^{2m+1}(H)$. Thus for some

$$z \in H' \cup \mathcal{B}(H) \cup \{\Sigma \mathcal{J}(H)\} = H \cup \{\Sigma \mathcal{J}(H)\} ,$$

$a \leq z$. If $z = \Sigma \mathcal{J}(H)$, then $a = \Sigma \mathcal{J}(H)$, since as noted above

$\Sigma\, \mathcal{J}(H) \leq a$, whence since $\mathcal{J}(H) \subseteq A_{2m}$ and a is join irreducible in $\langle A_{2m+1}; + \rangle$, we have that $a \in \mathcal{J}(H)$. Now by our above remarks, $a \in \mathcal{J}(H)$ implies that for some $x \in H$ $a \leq x$, whence $a = x$ since $x \leq a$. Thus if $a \in A_{2m}$ and a is join irreducible in $\mathfrak{A}_{2m+1}$ and $H \subseteq A'_{2m+1}$ with $H \equiv \{a\}$, then $a \in H$, whence a is join irreducible in $\mathfrak{A}_{2m+2}$.

Case 2. $a \in A_{2m+1} \sim A_{2m}$. Since $a \in A_{2m+1} \sim A_{2m}$, a is meet reducible by (v) whence again $a \in \mathcal{C}^{2m+1}(\{a\})$. Thus as above we conclude that for some $z \in H \cup \{\Sigma\, \mathcal{J}(H)\}$, $a \leq z$. Since $a \in A_{2m+1} \sim A_{2m}$, by (i) and our remarks above, we conclude that $\Sigma\, \mathcal{J}(H) < a$. Thus $a \leq z \in H$ for some z. Since $z \leq a$ for such a z, we conclude that $a \in H$. Thus if $a \in A_{2m+1} \sim A_{2m}$ and $H \subseteq A'_{2m+2}$ and $H \equiv \{a\}$ then $a \in H$ whence a is join irreducible in $\mathfrak{A}_{2m+2}$. This completes the proof of (v).

We have already shown (vi) hence we consider (vii). Since no new meets are defined we have only to show that $\mathcal{S}x + \mathcal{S}y = \mathcal{S}(x + y)$. Since for each $a \in A_{2m+2}$ there exists $H \subseteq A'_{2m+2}$ with $\Sigma H = a$ it is sufficient to show that if $H \subseteq A'_{2m+2}$ then $\mathcal{S}(\Sigma H) = \Sigma(\mathcal{S} H)$. If ΣH is join irreducible the result is trivial. Hence suppose $\Sigma H = a$ is join reducible.

Case 1. $a \in A_{2m}$. Let $\Sigma H = a$. Now $\{a\} \equiv H$, whence $\mathcal{C}^*(\{a\}) = \mathcal{C}^*(H)$. Since $\{a\}^* = 0$, we have $H^* = 0$. Thus,

$$\mathcal{C}^*(\{a\}) = \mathcal{C}^{2m+1}(\{a\}) = \mathcal{C}^{2m+1}(H).$$

Now if $x \in H$, then $x \leq a$ by Lemma 8.3 whence $x \leq \mathcal{S}a$. Thus

$$\Sigma\, \mathcal{S} H \leq \mathcal{S} a = \mathcal{S}(\Sigma H).$$

Now if $x \in \mathcal{J}(H)$, then since $\mathcal{D}(H) = 0$, there is a $y \in H = H'$ such that $x \leq y$. Hence $\Sigma \mathcal{S}\mathcal{J}(H) \leq \Sigma \mathcal{S}(H)$. Since $\mathcal{J}(H) \subseteq A_{2m}$, by the induction hypothesis $\mathcal{S}(\Sigma \mathcal{J}(H)) = \Sigma(\mathcal{S}\mathcal{J}(H))$. It is now immediate from the definition of $\mathcal{C}^{2m+1}$ that

$$\Sigma \mathcal{S}(\mathcal{C}^{2m+1}(H)) \leq \Sigma \mathcal{S}(H).$$

Since $a = \Sigma(\mathcal{C}^{2m+1}(H) \cap A_{2m})$ we have

$$\mathcal{S}a = \Sigma \mathcal{S}(\mathcal{C}^{2m+1}(H) \cap A_{2m}) \leq \Sigma(\mathcal{S}H),$$

which concludes this case.

Case 2. $a \in A_{2m+2} \sim A_{2m+1}$, a join reducible. Choose $H \in ag^{-1}$. We must show that $\mathcal{S}a = \Sigma(\mathcal{S}H)$. Now

$$\mathcal{S}a = \Sigma[\mathcal{S}\mathcal{C}^*(H)] \geq \Sigma \mathcal{S}(H^*),$$

since $H^* \subseteq \mathcal{C}^*(H)$. Further, if $b \in H' \sim A_{2m}$, then $b \in \mathcal{C}^{2m+1}(H)$ since b is meet reducible, whence $\mathcal{S}b \leq \mathcal{S}a$. If $b \in H' \cap A_{2m}$, then $\{b\} \leq^* H$ whence $\mathcal{C}^{2m+1}(\{b\}) \subseteq \mathcal{C}^{2m+1}(H)$. In this case $\mathcal{S}b = \Sigma\mathcal{S}(\mathcal{C}^{2m+1}(\{b\}))$ by Case 1. Thus

$$\Sigma(\mathcal{S}H') \leq \Sigma \mathcal{S}(\mathcal{C}^{2m+1}(H)) \leq \mathcal{S}a.$$

We therefore obtain that $\Sigma \mathcal{S}H \leq \mathcal{S}(\Sigma H) = \mathcal{S}a$.

To see that $\mathcal{S}a \leq \Sigma(\mathcal{S}H)$ we first note that if $b \in \mathcal{D}(H)$, then for some essential pair $\langle p, J \rangle$ and each $i \in J$, $f^J_{i,p} b \in H^*$, with $b \in U_p$. From this we have

$$\mathcal{S}b \leq k_p \leq \sum_{i \in J} k_i \leq \Sigma \mathcal{S}(H^*) \leq \mathcal{S}a .$$

It is immediate that if $y \in \mathcal{J}(H)$ then $\mathcal{S}y \leq \Sigma(\mathcal{S}H)$ whence

$\Sigma(\mathfrak{S}\mathcal{J}(H)) \leq \Sigma\,\mathfrak{S}(H)$. Since $\mathcal{J}(H) \subseteq A_{2m}$, by the induction hypothesis

$$\mathfrak{S}(\Sigma\mathcal{J}(H)) = \Sigma(\mathfrak{S}\mathcal{J}(H)) \leq \Sigma(\mathfrak{S}H).$$

Now if $x \in \mathcal{C}^{2m+1}(H)$, then for some $y \in H' \cup \{\Sigma\mathcal{J}(H)\}$ $x \leq y$, whence $\mathfrak{S}x \leq \mathfrak{S}y \leq \Sigma\,\mathfrak{S}(H)$. Thus

$$\Sigma\,\mathfrak{S}\mathcal{C}^{2m+1}(H) \leq \Sigma\,\mathfrak{S}(H).$$

Since $H^* \subseteq H$ we have

$$\mathfrak{S}a = \Sigma\,\mathfrak{S}(H^* \cup \mathcal{C}^{2m+1}(H)) \leq \Sigma(\mathfrak{S}H) \leq \mathfrak{S}a$$

which concludes Case 2. Thus for all $H \subseteq A'_{2m+2}$, $\Sigma\,\mathfrak{S}H = \mathfrak{S}\,\Sigma H$. This completes the verification of (vii).

Lastly we come to part (viii). Let $c \leq a + b$, with a and $b \in U_r$. Then $\mathfrak{S}c \leq \mathfrak{S}a + \mathfrak{S}b$. Since $\mathfrak{S}a$ and $\mathfrak{S}b \leq k_r$ we have $\mathfrak{S}c \leq k_r$ whence $c \in U_r$ by (vii). Thus U_r is an ideal. A similar argument shows V_s is an ideal.

We conclude the construction by setting $\mathfrak{A}^{\mathcal{L}} = \bigcup_{i\in\omega} \mathfrak{A}_i$. It is trivial that $\langle A^{\mathcal{L}}; +, \cdot\rangle$ is a lattice. In our work with semilattices we were able to generalize the operator $\mathcal{C}$ to act on $A^{\mathfrak{S}}$. This is not possible with $\mathcal{M}$, $\mathcal{N}$, $\mathcal{J}$, $\mathcal{D}$ and $\mathcal{C}^*$, since their natural extensions yield infinite sets. However, we have these operators defined in each structure $\mathfrak{A}_i$ and since at any given time we are only interested in a finite part of $\mathfrak{A}^{\mathcal{L}}$ we can relativize our arguments to an $\mathfrak{A}_i$ and make full use of the appropriate operators. We conclude this section with

Theorem 8.1. Let ϕ be the map of $\mathcal{L}$ into $\langle \mathfrak{I}(A^{\mathcal{L}}); +, \cdot\rangle$ defined by

$$x\phi = \begin{cases} U_r & \text{if } x = k_r, \\ V_s & \text{if } x = s \in L \sim K, \end{cases}$$

then ϕ is an isomorphism.

Proof. First recall that if $x \in A^{\mathfrak{L}}$ then $x \in U_r$ or V_s if and only if $\mathcal{S}x \leq k_r$ or s respectively. We restate this as: for any $a \in L$ and $x \in A^{\mathfrak{L}}$, $x \in a\phi$ if and only if $\mathcal{S}x \leq a$. Let a and $b \in L$. We show that $(ab)\phi = a\phi \cap b\phi$. Now $x \in a\phi \cap b\phi$ if and only if $\mathcal{S}x \leq a$ and $\mathcal{S}x \leq b$, that if, if and only if $\mathcal{S}x \leq ab$ which is equivalent to $x \in (ab)\phi$. We must also show that $(a + b)\phi = a\phi + b\phi$. Now it is sufficient to show that if $H \subseteq L$ is a set of join irreducibles with $\Sigma H = s$ join reducible, then $s\phi = \Sigma(H\phi)$. Note that $\Sigma(H\phi)$ refers to the sum in $\mathfrak{J}(\langle A^{\mathfrak{L}}; +\rangle)$. Fix H. Without loss of generality we may assume that H is join irredundant. Since $s\phi \supseteq \Sigma(H\phi)$ is clear, it is sufficient to prove by induction on j that $V_s \cap A_j \subseteq \Sigma(H\phi)$. Hence we suppose that for each $j < m$ if $a \in A_{2j+2} \cap V_s$ then $a \in \Sigma(H\phi)$. Let $a \in A_{2m+2}$. We first assume that $m = 0$ and $a \in A_0$. Then $a = b_p$ where $k_p \leq \sum_{i \in J} k_i$, $k_p \leq s$, and $J = \{i : k_i \in H\}$. Now suppose $k_p \not\leq k_i$ for each $i \in J$ because the other case is trivial. Then there is an essential pair $\langle p, J^* \rangle$ such that to each $j \in J^*$ there is an $i \in J$ with $k_j \leq k_i$. Now for such a $j \in J^*$ and $i \in J$, $f^{J^*}_{j,p} b_p \in U_i$, whence we easily conclude from the construction of $\mathfrak{A}_2$ that $b_p \in \sum_{i \in J}(k_i\phi)$. For the remaining cases we drop the assumption that $m = 0$, and suppose that $a \in A_{2m+2} \sim A_{2m}$. Now without loss of generality we may assume that a is join irreducible, since V_s is an ideal and for each $x \in V_s$

$$x = \Sigma\{y : y \leq x \text{ and } y \text{ is join irreducible}\}.$$

Case 1. $a \in A_{2m+1} - A_{2m}$. Since a is meet reducible, there is an $I \subseteq A_{2m}$ such that $\Pi I = a$. If $x \in V_s$ for some $x \in I$ then the inductive hypothesis yields the result. Hence suppose that for each $x \in I$, $x \notin V_s$. Now $\Pi(\mathcal{S} I) = \mathcal{S}(\Pi I) \leq s$ whence

$$\Pi(\mathcal{S}\, I) = k_p < \Sigma H = s,$$

since by assumption $\mathcal{L}$ has no member which is both join and meet reducible. Let $J = \{i : k_i \in H\}$ as above. If $k_p \leq k_i$ for some $i \in J$ it is obvious that $\Pi I \in \sum_{i \in J} k_i$. Otherwise we obtain J^* as before, and since $\Pi I \in U_p$ we obtain that $\Pi I \in \sum_{i \in J} (k_i \phi)$ in the same manner as for k_p above.

Case 2. $a \in A_{2m+2} \sim A_{2m+1}$. Since a is join irreducible, we have by (v) that a is primitive, whence $a = f^{J^*}_{q,p} b$ for some essential pair $\langle p, J^* \rangle$, some $q \in J^*$, and some $b \in U_p$. Since $f^{J^*}_{q,p} b \in V_s$ we have $k_q \leq \sum_{i \in J} k_i$. If $k_q \not\leq k_i$ for each $i \in J$ we proceed as above to obtain $a \in \sum_{i \in J} (k_i \phi)$ as desired.

We conclude that $V_s \subseteq \Sigma(H\phi)$. The reverse inequality is an immediate consequence of (vii) which asserts that if x and $y \in A^{\mathcal{L}}$ then $\mathcal{S} x + \mathcal{S} y = \mathcal{S}(x + y)$, and that $x \in U_r$ or V_s if and only if $\mathcal{S} x \leq k_r$ or s respectively. From the definition of $\mathfrak{A}_0$, we easily conclude that the restriction of ϕ to the join irreducibles is an order isomorphism. Thus ϕ is a 1-1 homomorphism. This completes the proof of Theorem 8.1.

§9. In this section we examine the properties of $\mathfrak{A}^{\mathcal{L}}$. Our object is to obtain as much information about transferability as possible.

Lemma 9.1. Let $\mathcal{L}$ be a transferable lattice. Then $\langle L; + \rangle$ is a transferable semilattice.

Proof. Let $\mathfrak{S} = \langle L; + \rangle$. We form the structure $\mathfrak{A}^{\mathfrak{S}}$ as in the preceding chapter. To $A^{\mathfrak{S}}$ we adjoin the distinguished element $\mathcal{O}$ to form $A^{\mathfrak{S}} \cup \{\mathcal{O}\}$, and put $\mathcal{O} \leq x$ for all $x \in A^{\mathfrak{S}}$. Now we claim that $\langle A^{\mathfrak{S}} \cup \{\mathcal{O}\}; \leq \rangle$ is a lattice ordering structure. It is clearly a join semilattice ordering structure. In the construction of $\langle A_{m+1}; + \rangle$ from $\langle A_m; + \rangle$ we showed that if $c \in A_{m+1} \sim A_m$ then for each $a \in A_m$, $c \not\leq a$, see p. 25. Thus if $b \in A^{\mathfrak{S}} \sim A_m$ and $a \in A_m$ then $b \not\leq a$, whence A_m is an ideal in $\langle A^{\mathfrak{S}}; + \rangle$. We obtain then that $\langle A_m \cup \{\mathcal{O}\}; + \rangle$ is a join semilattice with $A_m \cup \{\mathcal{O}\}$ a finite ideal in $\langle A^{\mathfrak{S}} \cup \{\mathcal{O}\}; + \rangle$. Now for a and $b \in A_m$ let

$$c = \Sigma\{x : x \leq a \text{ and } x \leq b\}.$$

Since $\mathcal{O} \leq a$ and $\mathcal{O} \leq b$, c exists and is easily seen to be the greatest lower bound for $\{a, b\}$. It follows that $\langle A^{\mathfrak{S}} \cup \{\mathcal{O}\}; \leq \rangle$ is a lattice ordering structure as claimed.

Now define $\psi : L \to A$ by

$$x\psi = \begin{cases} U_r \cup \{\mathcal{O}\} & \text{if } x = k_r \in K \\ V_s \cup \{\mathcal{O}\} & \text{if } x = s \in L \sim K . \end{cases}$$

We claim that ψ is a lattice isomorphism. It is trivial that $(x + y)\psi = x\psi + y\psi$ for each x and $y \in L$. Let x, y and $z \in L$ with $z = xy$. Now if $a \in z\psi$ then $a \in x\psi$ and $y\psi$ whence $z\psi \subseteq x\psi \cap y\psi$.

The reverse inclusion, that $x\psi \cap y\psi \subseteq z\psi$, will follow if we show that if $a \in x\psi$ and $b \in y\psi$ then $ab \in z\psi$. Fix such an a and $b \in A_{m+1} \cup \{\mathcal{O}\}$. For such an a and b if either is $\mathcal{O}$ then $ab = \mathcal{O}$ and we are done. Thus we assume a and $b \in A_{m+1}$. Now from the construction of $\mathfrak{A}_{m+1}$, we have that $a \in U_r$ or V_s if and only if $\mathcal{C}a \subseteq U_r$ or V_s respectively by (vii) p.21 . For $d \in A_{m+1}$, $d \le a$ and $d \le b$ if and only if $\mathcal{C}d \subseteq \mathcal{C}a \cap \mathcal{C}b$ by (v) p.21 . Since $d = \Sigma\, \mathcal{C}d$, again by (v), it is easy to see that

$$ab = \Sigma[(\mathcal{C}a \cap \mathcal{C}b) \cup \{\mathcal{O}\}] .$$

Further, from the above, we have that if $ab \in A_{m+1}$, then $\mathcal{C}a \cap \mathcal{C}b = \mathcal{C}(ab)$. Thus we have only to show if $a \in x\psi$ and $b \in y\psi$ and $d \in \mathcal{C}a \cap \mathcal{C}b$ then $d \in z\psi$. Now d is primitive. From the definition of the U_r and V_s on A_0, p.17 and (vii) p.21 , we see that for some unique k_p, $d \in U_r$ or V_s if and only if $k_p \le k_r$ or s respectively. Thus $d \in x\psi \cap y\psi$ if and only if for the unique k_p given above, $k_p \le x$ and $k_p \le y$. Since $\mathcal{L}$ is a lattice $k_p \le xy$ whence $d \in z\psi$ as desired. Thus ψ is a lattice isomorphism.

Since $\mathcal{L}$ is transferable, we obtain ϕ embedding $\mathcal{L}$ in $\langle A^{\mathfrak{S}} \cup \{\mathcal{O}\}; +, \cdot\rangle$ satisfying $x\phi \in y\psi$ if and only if $x \le y$. Now if $L\phi \subseteq A^{\mathfrak{S}}$, then $\langle L; +\rangle$ is transferable. If $L\phi \not\subseteq A^{\mathfrak{S}}$, then we conclude that k_0, the least element of L, satisfies $k_0\phi = \mathcal{O}$. No other element can be mapped to $\mathcal{O}$ by the condition that ϕ is 1-1 and order preserving. Now k_0 is join irreducible, whence $b_0 \in A_0 \cap U_0$. Also $b_0 \in U_r$ and V_s for each k_r and $s \in L$. Define ϕ' by

$$x\phi' = x\phi + b_0 .$$

Then ϕ' is a join homomorphism and satisfies $x\phi' \in y\psi$ if and only if $x \leq y$. Since $L\phi' \subseteq A^{\mathfrak{S}}$, $\langle L; +\rangle$ is transferable. This concludes the lemma.

Definition 9.1. A semilattice $\langle L; \cdot\rangle$ is said to be <u>weakly stable</u> if for each meet reducible $x \in L$ there exists a $T_x \subseteq L$ such that (i) T_x is a meet irredundant set of meet irreducibles, (ii) $\Pi T_x = x$ and (iii) if H is any set of meet irreducibles such that $\Pi H = x$ then for each $y \in T_x$ there is a $z \in H$ with $z \leq y$.

Lemma 9.2. $\langle A^{\mathcal{S}}; +, \cdot\rangle$ is weakly stable.

Proof. We proceed by induction. For our induction hypothesis, we assume that for each $j < m$, if b is meet reducible and $b \in A_{2j+1} \sim A_{2j}$ then $T_b \subseteq A_{2j}$ exists and satisfies (i)-(iii) of Definition 9.1. Thus let $b \in A_{2m+1} \sim A_{2m-1}$ with b meet reducible. By (v) p.48, we have $b \in A_{2m+1} \sim A_{2m}$. We calculate $\mathfrak{M}$ and $\mathfrak{N}$ as in the construction of $\mathfrak{A}_{2m+1}$. Now from the construction of $\langle A_{2m+1}; \cdot\rangle$ there is an $H \subseteq A_{2m}$ such that $\Pi H = b$. Fix such an H and let $H^* = H \sim A_{2m-1}$ and $H' = H \cap A_{2m-1}$. We treat the case when $\mathfrak{M}(H) \neq 0$, the proof for the other case being similar. We also assume that $m > 0$, the case $m = 0$ being easily checked. Let $b_0 = \Pi\mathfrak{M}(H)$, then $b_0 \in A_{2m-1}$. If b_0 is meet irreducible, let $T_{b_0} = \{b_0\}$, and otherwise let T_{b_0} be given as in the induction hypothesis. In either case (i), (ii) and (iii) are satisfied when $x = b_0$. Also, $T_{b_0} \subseteq A_{2m-2}$, by (v) if b_0 is meet irreducible, and by the induction hypothesis otherwise. Let

$$T_b^* = \{x : x \in H^*, b_0 \not\leq x \text{ and if } z \in H^* \text{ then } z \not< x\}$$

and

$$T_b' = \{x : x \in T_{b_0} \text{ and for each } y \in T_b^*, y \not\leq x\}.$$

We set $T_b = T_b' \cup T_b^*$. Before proceeding, we note that for the case when $\mathcal{M}(H) = 0$, we simply set

$$T_b = \{x : x \in H^* \text{ and for each } z \in H^*, z \not< x\}.$$

To continue, we first claim that $T_b \equiv H$. First note that if $x \in T_{b_0}$ then either $x \in T_b'$ or for some $z \in T_b^*$ $z \leq x$. Thus $\mathcal{M}(T_b) \supseteq T_{b_0}$ by definition of $\mathcal{M}$ and (v), whence

$$\Pi\, \mathcal{M}(T_b) \leq \Pi T_{b_0} = \Pi\, \mathcal{M}(H).$$

Now if $x \in \mathcal{N}(H)$, then for some $y \in H^* \cup \{\Pi\, \mathcal{M}(H)\}$, $y \leq x$. If $b_0 = \Pi\mathcal{M}(H) \leq x$ then $\Pi\mathcal{M}(T_b) \leq x$, whence $x \in \mathcal{N}(T_b)$ by definition of $\mathcal{N}$. Otherwise $y \in H^*$ and $z \leq y \leq x$ for some $z \in T_b^*$, whence again $x \in \mathcal{N}(T_b)$. Thus $\mathcal{N}(H) \subseteq \mathcal{N}(T_b)$ whence $T_b \leq^* H$. Conversely, if $x \in T_b$ then either $x \in H^*$ or $x \in T_{b_0}$. For such an x it is easily seen that $H \leq^* \{x\}$ whence we obtain $H \leq^* \cup \{\{x\} : x \in T_b\} = T_b$. Thus $T_b \equiv H$. It is immediate that $\Pi T_b = b$, which establishes (ii).

For (i) we must show that T_b is a meet irredundant set of meet irreducibles. If $x \in T_b$ then $x \in T_{b_0}$ or $x \in A_{2m} \sim A_{2m-1}$. If the former then x is meet irreducible by the induction hypothesis. If the latter then x is meet irreducible by (v) p. 48. To obtain that T_b is meet irredundant, we have only to show that T_b is a set of pairwise incomparables whence the meet irredundancy of T_b will follow from the fact that T_b satisfies (iii). From their definitions, it is easy to see that T_b' and T_b^* are sets of incomparables, the former requiring an application of (i) of the inductive hypothesis. Now let $a_1 \in T_b^*$ and $a_2 \in T_b'$. If $a_1 \leq a_2$ then $a_2 \notin T_b'$. If $a_2 \leq a_1$, then $b_0 \leq a_1$, whence $a_1 \notin T_b^*$. Thus T_b is a set of pairwise incomparables.

For (iii) let H_1 be any set of meet irreducibles such that $H_1 \subseteq A_{2m}$ and $H_1 \equiv T_b$. First note that $\mathcal{N}(H_1) = \mathcal{N}(T_b)$ and $\Pi\mathcal{M}(H_1) = \Pi\mathcal{M}(T_b) = b_0$. From the definition of T_b^* and the definition of $\mathcal{N}$ it is immediate that $T_b^* \subseteq H_1^*$. Now let $c \in T_b'$. Then $c \in T_{b_0}$. Now $\Pi\mathcal{M}(H_1) = b_0$ whence to each $x \in T_{b_0}$ there is a $z \in \mathcal{M}(H_1)$ with $z \leq x$ by (iii) of the inductive hypothesis. Since x is meet irreducible it follows from (v) p. 48 and the definition of $\mathcal{M}$ that $x \in \mathcal{M}(H_1)$. Thus $c \in T_{b_0} \subseteq \mathcal{M}(H_1)$ whence for some $z \in H_1^* \cup \{\Pi H_1'\}$, $z \leq c$. If $z \in H_1^*$ we are done, whence suppose $\Pi H_1' \leq c$. Now if $x \in H_1'$ $b_0 \leq x$, whence if

$$H_2 = \{y : y \in A_{2m-1},\ y \text{ is meet irreducible and } z \leq y \text{ for some } z \in H_1'\}$$

then

$$b_0 = \Pi[(T_{b_0} \sim \{c\}) \cup H_2].$$

Thus for some $y \in H_2$ $y \leq c$, since $c \in T_{b_0}$. It follows that for some $z \in H_1'$ $z \leq c$. Thus T_b satisfies (iii) in $\langle A_{2m+1}; \cdot \rangle$. Now let $H \subseteq A_{2q}$, $q > m$, be a finite set of meet irreducibles such that $H \equiv \{b\}$. Further assume that if $m \leq k \leq q$ then T_b satisfies (iii) in $\langle A_{2k+1}; \cdot \rangle$. We show that T_b satisfies (iii) in $\langle A_{2q+1}; \cdot \rangle$. Note that if H is any set of meet irreducibles in A_{2q+1} then by (v) $H \subseteq A_{2q}$. We calculate $\mathcal{M}$ and $\mathcal{N}$ as in the construction of $\mathfrak{A}_{2q+1}$. Since

Since $H \equiv \{b\}$, we have that $\Pi\, \mathcal{M}(H) = \Pi\, \mathcal{M}(\{b\}) = b$ since $b \in A_{2q-1}$. Now $\mathcal{M}(H) \subseteq A_{2q-1}$ is a set of meet irreducibles whence to each $z \in T_b$ there is a $y \in \mathcal{M}(H)$ with $y \leq z$. Thus to each $z \in T_b$ there is an $x \in H^* \cup \{\Pi H'\}$ such that $x \leq z$. If $x \in H^*$ we are done. If $x = \Pi H'$ let

$$H_2 = \{w : w \text{ is meet irreducible, } w \in A_{2q-1} \text{ and } y \leq w \text{ for some } y \in H'\}$$

Then $b = \Pi[(T_b \sim \{z\}) \cup H_2] = \Pi T_b$. As above we conclude for some $y \in H'$, $y \leq z$. Thus $\langle A_{2q+1}; \cdot\rangle$ satisfies (iii). This completes the proof of Lemma 9.2.

Lemma 9.3. If $\mathcal{L}$ is transferable, then $\langle L; \cdot\rangle$ is weakly stable.

Proof. Since $\mathcal{L}$ is transferable, there is an embedding ψ of $\mathcal{L}$ in $\langle A^{\mathcal{L}}; +, \cdot\rangle$ such that $x\psi \in U_r$ or V_s if and only if $x \leq k_r$ or s respectively. This is a consequence of Theorem 8.1 and the definition of transferable. Note that for such a ψ, $\mathcal{S}(y\psi) = y$ for each $y \in L$ by (vii) p.48. Let $x \in L$ meet reducible. We must show that there is a $T_x \subseteq L$ such that (i) T_x is a meet irredundant set of meet irreducibles, (ii) $\Pi T_x = x$ and (iii) if H is any set of meet irreducibles and $\Pi H = x$ then for each $y \in T_x$ there is a $z \in H$ with $z \leq y$. Let $a = x\psi$. Then by Lemma 9.2 T_a exists satisfying (i) - (iii) above with x replaced by a. Set

$$T_x = \{y : y \in \mathcal{S}T_a \text{ and for each } z \in \mathcal{S}T_a,\ z \not< y\}.$$

Since $\mathcal{S}a = x$, we conclude by (vii) p.48 that $\Pi T_x = x$. Thus (ii) is satisfied. For (iii), let $H \subseteq L$ be a set of meet irreducibles such that $\Pi H = x$. Fix $y \in T_x$, then $y = \mathcal{S}u$ for some $u \in T_a$. Since $\Pi(H\psi) = a$, for some $v \in H\psi$, $v \leq u$ since T_a satisfies (iii). Now $\mathcal{S}v \in \mathcal{S}(H\psi) = H$ and $\mathcal{S}v \leq \mathcal{S}u = y$ by (vii). Thus T_x satisfies (iii). Now T_x is a set of pairwise incomparables by definition whence it is immediate that since T_x satisfies (iii) it is meet irredundant. Further if some $y \in T_x$ were meet reducible then T_x could not satisfy (iii). Thus T_x satisfies (i) which concludes the proof of Lemma 9.3.

For the remainder let $\mathcal{L}$ be weakly stable and let

$$T^* = \{y : y \in L \text{ and for some meet reducible } x,\ y \in T_x\}.$$

We list T^* as $t_0, \ldots, t_{m-1}$. Let $x \in L$ and $H \subseteq L$. We will say that H is <u>minimal under</u> x if $\Pi H \leq x$ but if $H' \subsetneq H$ then $\Pi H' \not\leq x$. Let $p \in m$ and $J \subseteq M$, we say that $\langle p, J\rangle$ is a <u>dual minimal pair</u> in case

(1) $\{t_i : i \in J\}$ is minimal under t_p and,

(2) if $\{t_j : i \in J^*\}$ is minimal under t_p and to each $j \in J^*$ there is an $i \in J$ with $k_i \leq k_j$, then $J = J^*$.

Lemma 9.4. If $\mathcal{L}$ is transferable then there is a linear order $\prec$ of T^* such that for each dual minimal pair $\langle p, J\rangle$, if $i \in J$ then $t_i \prec t_p$.

Proof. Since $\mathcal{L}$ is transferable, there is an embedding ψ of $\mathcal{L}$ in $\langle A^{\mathcal{L}}; +, \cdot\rangle$ such that $x\phi \in U_r$ or V_s if and only if $x \leq k_r$ or s respectively. We define the rank of $x \in T^*$ as follows:

$$Rk(x) = p_i[(A_i \cap \overline{x\phi} \cap \underline{x\phi}) \neq 0]$$

where $\overline{x\phi} = \{y : \mathcal{S}y = \mathcal{S}(x\phi)\}$ and $\underline{x\phi} = \{y : x\phi \leq y\}$. Let $\langle p, J\rangle$ be a dual minimal pair; we claim that for each $i \in J$ $Rk(t_i) < Rk(t_p)$. To see this, let $Rk(t_p) = n_0$ and let $a \in A_{n_0} \cap \overline{t_p\phi} \cap \underline{t_p}\phi$. Further set

$$m = \mu j[\Pi(A_j \cap (\cup\{\underline{t_i}\phi : i \in J\})) \leq a].$$

Since $|A_j| < \aleph_0$ for each $j \in \omega$ this product exists, whence m exists. Since $\langle p, J\rangle$ is a dual minimal pair, we have for each $i \in J$ that $\underline{t_i}\phi \cap \overline{t_i\phi} \cap A_m \neq 0$. Suppose otherwise. Let

$$b = \Pi\{A_m \cap (\cup\{\underline{t_i}\phi : i \in J\})\} .$$

Then $T_b \subseteq A \cap (\cup\{\underline{t_i}\phi : i \in J\})$. Now $\mathcal{S}b \leq \mathcal{S}a = t_p$. Further $\Pi(\mathcal{S}T_b) = \mathcal{S}b$ by (vii) p.48, and to each $x \in \mathcal{S}T_b$ there is a $j \in J$ with $t_j \leq x$. Let $J' = \{i : t_i \in T_{\mathcal{S}b}\}$, then to each $i \in J'$ there is a $j \in J$ with $t_j \leq t_i$. Now we easily obtain $J^* \subseteq J'$ such that $\{t_i : i \in J^*\}$ is minimal under t_p. We conclude that $J^* = J$ since $\langle p, J\rangle$ is a dual minimal pair. Thus for each $i \in J$, there is an $x \in T_b$ with $\mathcal{S}x = t_i$. For such an x, $x \in A_m \cap \overline{t_i\phi} \cap \underline{t_i}\phi$. Thus for each $i \in J$ $A_m \cap \underline{k_i}\phi \cap \overline{k_i\phi} \neq 0$ as claimed. It is immediate that $Rk(t_i) \leq m$ for each $i \in J$.

We now claim that $m < n_0$. Suppose not, then $n_0 \leq m$. Let H be any meet irredundant set of meet irreducibles such that $\Pi H \leq a$ and $H \subseteq A_m \cap \bigcup_{i \in J} \underline{t_i}\phi$. Such an H exists by choice of m above. Observe that $\Pi H \in A_{m+1} \sim A_m$, otherwise $H \subseteq A_{m-1}$ contradicting the choice of m. Thus there are meet irreducibles in $A_m \sim A_{m-1}$ whence $m = 2m_0$ for some m_0 by (v) p.48. We calculate $\mathfrak{M}$ and $\mathfrak{N}$ as in the construction $\mathfrak{A}_{2m_0+1}$. Recall that we have assumed for proof by contradiction that $n_0 \leq m$. Since a was chosen in A_{n_0}, $a \in A_{2m_0} = A_m$.

Thus from the definition of $\leq^*$ and since $\Pi H \leq a$ we have $H \leq^* \{a\}$, that is $\mathcal{N}(\{a\}) \subseteq \mathcal{N}(H)$ and if $\mathcal{M}(\{a\}) \neq 0$ then $\Pi\mathcal{M}(H) \leq \Pi\mathcal{M}(\{a\})$. Now we claim that $\Pi\mathcal{M}(H) \not\leq a$. Suppose not. Then let

$$H_1 = \{x : x \in A_{2m_0-1} \text{ and for some } y \in H, \ y \leq x\}.$$

Now

$$\mathcal{M}(H) \subseteq \{x : x \in A_{2m_0-1} \text{ and for some } y \in H^* \cup \{\Pi H'\}, \ y \leq x\}.$$

It is easily seem from the lattice postulates that

$$\Pi H_1 \leq \Pi\mathcal{M}(H) .$$

Thus if $\Pi\mathcal{M}(H) \leq a$ then $\Pi H_1 \leq a$. Since $H_1 \subseteq A_{m-1} \cap \bigcup_{i \in J} \underline{t_i \phi}$ the choice of m is contradicted. It follows that $a \in A_{2m_0} \sim A_{2m_0-1}$ since otherwise

$$\Pi\mathcal{M}(H) \leq \Pi\mathcal{M}(\{a\}) = a.$$

Thus $a \in \mathcal{N}(\{a\}) \subseteq \mathcal{N}(H)$, whence for some $x \in H^* \cup \{\Pi\mathcal{M}(H)\}$, $x \leq a$. Such an x must satisfy $x \in H^*$. Now if $x \leq a$, then $\mathcal{S}x \leq \mathcal{S}a = t_p$ by (vii) p.48. Since for some $i \in J$, $x \in \underline{t_i \phi}$ whence $t_i \leq x$, we conclude that $t_i = \mathcal{S}(t_i \phi) \leq t_p$. This contradicts the choice of $\langle p, J \rangle$ as a dual minimal pair. We must therefore have that $m < n_0$ as claimed.

To complete the lemma, recall that this yields $Rk(t_i) < Rk(t_p)$ for each $i \in J$. If we now define $\prec$ on T^* by:

$t_i \prec t_j$ if and only if (i) $Rk(t_i) < Rk(t_j)$ or

(ii) $Rk(t_i) = Rk(t_j)$ and $i < j$,

then $\prec$ satisfies the conclusion of the lemma.

Lemma 9.5. If $\mathcal{L}$ is transferable then $\mathcal{L}$ is dual weakly stable.

Proof. By Lemma 9.1 and Theorem 6.1 $\langle L; +\rangle$ is strictly transferable. Let $<$ be the linear order of K witnessing the strict transferability of $\langle L; +\rangle$. We assume $k_0 < k_1 < \ldots < k_{n-1}$. Let $s \in L$ be join reducible. Set $Q_s^{-1} = 0$. For $i \in n$, let $Q_s^i = Q_s^{i-1} \cup \{k_i\}$ if (1) below is satisfied, $Q_s^i = Q_s^{i-1}$ otherwise.

(1) Every join irredundant $H \subseteq K$ which is contained in $Q_s^{i-1} \cup \{k_j : i \leq j\}$ and sums to s contains $\{k_i\}$.

We set $Q_s = Q_s^{n-1}$. Now $Q_s \subseteq K$ and by definition is join irredundant. Further it is clear that $\Sigma Q_s = s$. To see this we first note that if $k_j \in Q_s$ then $k_j \leq s$. Thus $\Sigma Q_s \leq s$. The reverse inequality will follow if we can show that for each $i \in n$

$$s \leq \Sigma(Q_s^i \cup \{k_j : i < j\}).$$

Consider Q_s^0. If $Q_s^0 \neq 0$ then we have $s \leq \Sigma K$. If $Q_s^0 = 0$ then there is an $H \subseteq K \sim \{k_0\}$ such that $\Sigma H = s$, whence $s \leq \Sigma(K \sim \{k_0\})$ as claimed. Now suppose that

$$s \leq \Sigma(Q_s^m \cup \{k_j : m < j\}).$$

If $k_{m+1} \in Q_s^{m+1}$ the result follows by the inductive hypothesis. If $Q_s^{m+1} = Q_s^m$ then the result is a consequence of the failure of (1) for the case $i = m$. Thus $\Sigma Q_s = s$ as desired.

Lastly let $H \subseteq K$ be join irredundant such that $\Sigma H = s$. Further suppose that for some $k_p \in Q_s$ there is no $x \in H$ such that $k_p \leq x$. Then we can obtain $J \subseteq n$ such that for each $j \in J$ there is an $x \in H$ with $k_j \leq x$ and $\langle p, J\rangle$ is a minimal pair. Now $k_p < x$ for each

$j \in J$ whence by choice of Q_s, $k_p \notin Q_s$. This is a contradiction. Thus to each $k_p \in Q_s$ there is an $x \in H$ with $k_p \leq x$. This completes the proof that $\mathcal{L}$ is dual weakly stable.

We summarize these results as

Theorem 9.1. If $\mathcal{L}$ is transferable then the following conditions are satisfied:

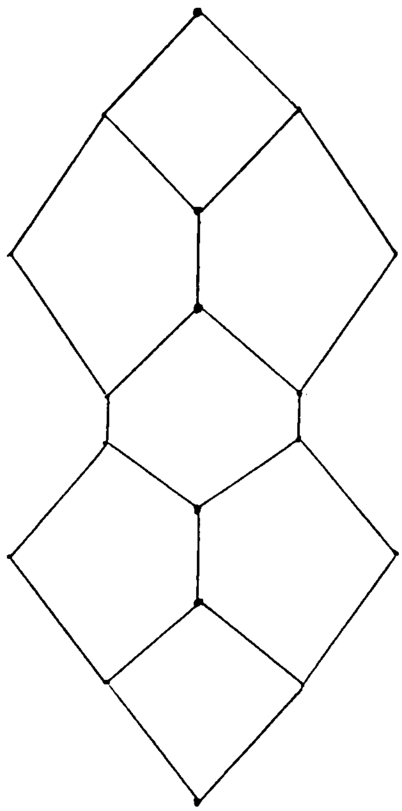
(i) $\langle L; +\rangle$ is a transferable semilattice.

(ii) $\mathcal{L}$ is weakly stable.

(iii) If $T^* = \{x_0, \ldots, x_{n-1}\}$ is the set of meet irreducibles witnessing the weak stability of $\mathcal{L}$, then there is a linear order $<$ of T^* such that for each dual minimal pair $\langle p, J\rangle$ and each $i \in J$, $x_i < x_p$.

Definition 9.2. A lattice $\mathcal{L}$ satisfies the <u>join-meet condition</u> if for any $a, b, c, d \in L$ if $cd \leq a+b$ then one of the following four conditions is satisfied: $cd \leq a$, $cd \leq b$, $c \leq a+b$ or $d \leq a+b$.

In Theorem 9.2 we show that the join-meet condition together with (i) - (iii) above are enough to ensure transferability. The example $\mathcal{L}_0$ of Figure 9.1 shows that conditions (i) - (iii) together with the added condition that no point is both join and meet reducible are not sufficient for transferability. We note however that the proof that $\mathcal{L}_0$ is not transferable is somewhat difficult.



$\mathcal{L}_0$

Figure 9.1

Theorem 9.2. Let $\mathcal{L}$ be a finite lattice. $\mathcal{L}$ is transferable provided that the following conditions are satisfied:

(i) $\langle L; +\rangle$ is transferable.

(ii) $\mathcal{L}$ is weakly stable.

(iii) If $T^* = \{x_0, \ldots, x_{m-1}\}$ is the set of meet irreducibles witnessing the weak stability of $\mathcal{L}$, then there is a linear order $<$ of T^* such that for each dual minimal pair $\langle p, J\rangle$ and each $i \in J$, $x_i < x_p$.

(iv) $\mathcal{L}$ satisfies the join-meet condition.

Proof. Let $\mathcal{L}^*$ and ϕ be such that ϕ embeds $\mathcal{L}$ in $\mathfrak{J}(\mathcal{L}^*)$. Since $\langle L; +\rangle$ is transferable, let ψ_0 be an embedding of $\langle L; +\rangle$ in $\langle L^*; +\rangle$ such that $x\psi_0 \in y\phi$ if and only if $x \le y$. We define maps $\psi_1, \ldots, \psi_m$ where $m = |T^*|$ as follows. For $1 \le j \le m$, if x is meet reducible we set

$$x\psi_j = \Pi(T_x\psi_{j-1}).$$

If x is meet irreducible we put

$$x\psi_j = x\psi_{j-1} + \Sigma\{y\psi_j : y \text{ is meet reducible and } y \le x\}.$$

Without loss of generality we assume that $T^* = \{x_0, \ldots, x_{m-1}\}$ and that $x_0 < \ldots < x_{m-1}$. We claim that $\psi_m = \psi$ is a meet isomorphism. To see this, we first note that if $x_i \le x_j$ then $x_i\psi_k \le x_j\psi_k$ for $0 \le k \le m$. This follows from the fact that $x_i\psi_0 \le x_j\psi_0$ by choice of ψ_0. Further if y is meet reducible and $y \le x_i$ then $y \le x_j$ whence by the definition of ψ_k we easily conclude $x_i\psi_k \le x_j\psi_k$ if $0 < k \le m$.

Our next assertion is that if $i \le k < m$, then $x_i\psi_k = x_i\psi_{k+1}$. For $i = 0$, we must show that $x_0\psi_k = x_0\psi_{k+1}$, or equivalently that

$$x_0\psi_k = x_0\psi_k + \Sigma\{y\psi_{k+1} : y \text{ is meet reducible and } y \leq x_0\}.$$

If y is meet reducible and $\leq x_0$, then for some $x_p \in T_y$, $x_p \leq x_0$. Otherwise there is a dual minimal pair of the form $\langle J, 0\rangle$. But from (iii) of the hypothesis each $i \in J$ satisfies $0 \leq i \leq 0$ which is impossible. It follows that if y is meet reducible and $\leq x_0$ then

$$y\psi_{k+1} = \Pi(T_y\psi_k) \leq x_p\psi_k \leq x_0\psi_k .$$

This makes it clear that the assertion is true for $k = 0$.

We now suppose that for some q such that $0 < q \leq m$, if $i < q$ and $i \leq k < m$ then $x_i\psi_k = x_i\psi_{k+1}$. Consider a particular k, $q \leq k < m$. Now

$$x_q\psi_{k+1} = x_q\psi_k + \Sigma\{y\psi_{k+1} : y \text{ is meet reducible and } y \leq x_q\}.$$

Let $y \leq x_q$ with y meet reducible. If $x_i \leq x_q$ for some $x_i \in T_y$ then $y\psi_{k+1} \leq x_q\psi_k$ just as $y\psi_{k+1} \leq x_0\psi_k$ above. Now suppose $x_i \not\leq x_q$ for each $x_i \in T_y$. Then there is a dual minimal pair $\langle q, J\rangle$ such that

$$y \leq \prod_{i\in J} x_i \leq x_q ,$$

and such that to each $i \in J$ there is an $x \in T_y$ with $x \leq x_i$. That such a pair exists is a consequence of the finiteness of $\mathcal{L}$. Now from (iii) $i < q$ for each $i \in J$, whence $x_i\psi_{k-1} = x_i\psi_k$. Now let $b = \prod_{i\in J} x_i$. Since $\langle q, J\rangle$ is a dual minimal pair, we conclude that $T_b = \{x_i : i \in J\}$. By the induction hypothesis

$$b\psi_{k+1} = \prod_{i\in J} x_i\psi_k = \prod_{i\in J} x_i\psi_{k-1} = b\psi_k.$$

Since we have already shown that $x_i \leq x_j$ implies $x_i\psi_k \leq x_j\psi_k$ for $0 \leq k \leq m$, we obtain

$$y\psi_{k+1} \leq b\psi_{k+1} = b\psi_k \leq x_q\psi_k.$$

Further since we have shown that $y\psi_{k+1} \leq x_q\psi_k$ for every meet reducible, $y \leq x_q$, we have $x_q\psi_{k+1} = x_q\psi_k$. Thus, $x_i\psi_k = x_i\psi_{k+1}$ whenever $i \leq k < m$.

Let $\psi = \psi_m$. We saw above that if x and y are meet irreducible and $x \leq y$ then $x\psi_m \leq y\psi_m$. Let H be any set of meet irreducibles such that $\Pi H = z$ is meet reducible. Then from the definition of ψ_m, $z\psi_m \leq \Pi(H\psi_m)$ and $z\psi_m = \Pi(T_z\psi_{m-1})$. From the weak stability of $\mathcal{L}$ for each $x \in T_z$ there exists $y \in H$ such that $y \leq x$. Thus $\Pi(H\psi_m) \leq \Pi(T_z\psi_m)$. But since we have shown that $x_i\psi_k = x_i\psi_{k+1}$ for each k, $i \leq k < m$, we have $\Pi(T_z\psi_m) = \Pi(T_z\psi_{m-1})$. It is now clear that $z\psi_m = \Pi(H\psi_m)$ whence ψ_m is a meet isomorphism.

We wish to show that ψ is a join isomorphism. Suppose for proof by contradiction that ψ is not a join isomorphism then for some z_0 and z_1 in L we have $z\psi \neq z_0\psi + z_1\psi$ where $z = z_0 + z_1$. Since ψ is order preserving we may assume that $z \notin \{z_0, z_1\}$ whence z is join reducible. Further, $z_0\psi + z_1\psi \leq z\psi$ whence $z\psi \not\leq z_0\psi + z_1\psi$. Consider the least j such that $z\psi_j \not\leq z_0\psi + z_1\psi$. Note that $j > 1$ since ψ_0 is a join embedding. By the join-meet condition z, being join reducible, is meet irreducible. Recall from the definition of ψ_j that

$$z\psi_j = z\psi_{j-1} + \Sigma\{y\psi_j : y \text{ is meet reducible and } y \leq z\}.$$

By choice of j, $z\psi_{j-1} \leq z_0\psi + z_1\psi$. Thus $y\psi_j \not\leq z_0\psi + z_1\psi$ for some meet reducible $y \leq z$. Fix such y. By the join-meet condition there are two cases.

Case 1. $x \leq z$ for some $x \in T_y$ then $y\psi_j \leq x\psi_{j-1} \leq z\psi_{j-1}$. This contradicts the choice of j.

Case 2. $y \leq z_0$ or $y \leq z_1$. In this case since ψ is order preserving $y\psi_j \leq y\psi \leq z_0\psi + z_1\psi$. Thus the choice of y is contradicted.

This completes the proof that ψ is a join isomorphism. The theorem is now immediate.

§10. In this section we give a complete description of those distributive lattices which are transferable. For the remainder of this section let $\mathcal{L}$ be a finite distributive lattice. From Lemma 3.2, we see that if $\{a\} \cup H$ is a set of join irreducibles and $a \leq \Sigma H$ then for some $b \in H$, $a \leq b$. Suppose for the moment that $\mathcal{L}$ is transferable. By Lemma 8.1 no point of L is both join and meet reducible whence if d is any meet reducible point then d is join irreducible. Thus, if $d \leq a + b$, let

$$H = \{x : x \in L, \ x \text{ is join irreducible and } x \leq a \text{ or } x \leq b\}.$$

Thus, $d \leq \Sigma H$ and the above result yields $d \leq a$ or $d \leq b$. Thus the join-meet condition is satisfied. Since the dual of the above condition on joins is valid it is trivial that (ii) and (iii) of Theorem 9.2, p.80, hold for any finite distributive lattice $\mathcal{L}$. In Chapter 1 we showed that every finite distributive semilattice $\langle L; + \rangle$ was transferable. We therefore have

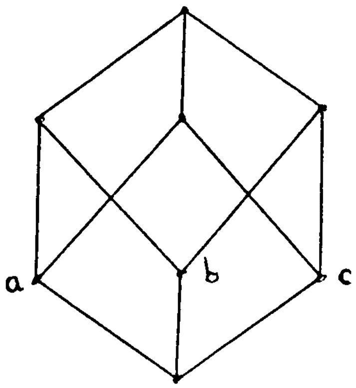
Theorem 10.1. $\mathcal{L}$ is transferable if and only if no point of $\mathcal{L}$ is both join and meet reducible.

We will now give some very precise structure conditions which will give a much better picture of those finite distributive lattices which are transferable. These results, Lemmas 10.1 - 10.4 were first shown by Galvin and Jónsson in [3].

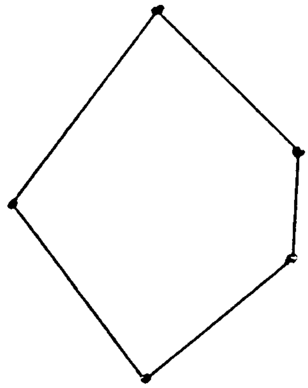
Lemma 10.1. Let $\mathcal{L}$ be transferable.

If a, b and $c \in L$ are mutually incomparable then the sublattice generated by a, b and c is the eight element Boolean lattice $\mathfrak{B}_8$ of Figure 10.1.

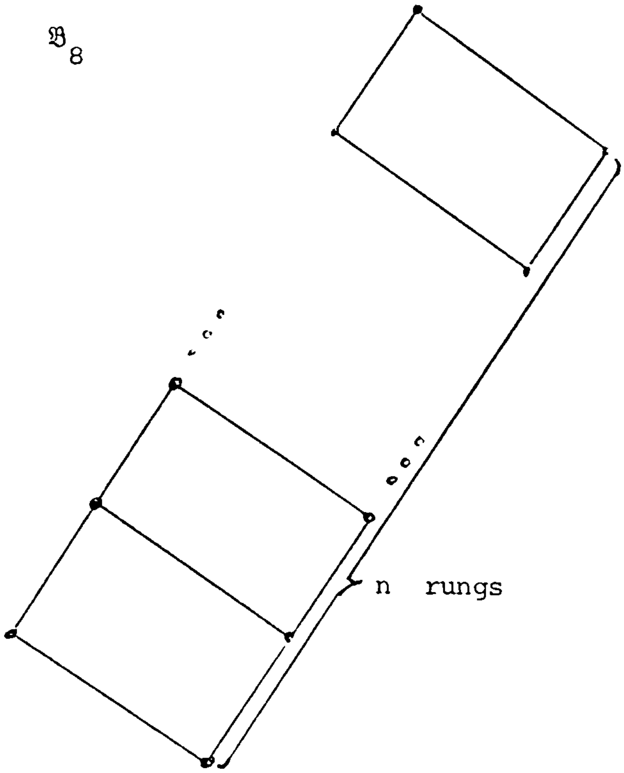
Proof. Let a, b and $c \in L$ be mutually incomparable, generating a sublattice $\mathcal{L}_1$. Now since $\mathcal{L}$ is distributive,



$\mathfrak{B}_8$



$\mathfrak{N}_5$



n-runged diagonal ladder

Figure 10.1

$$(a + b)(a + c)(b + c) = ab + ac + bc = d \ .$$

By Lemma 8.1 we have that d is not both join and meet reducible. By duality, we may assume d is not join reducible. Hence assume that $ab + ac + bc = bc$, whence we have

$$ab = ac \leq bc \ .$$

Now $a + bc = (a + b)(a + c)$. Since no element is both join and meet reducible, one of these inequalities

$$a \leq bc, \quad bc \leq a, \quad a + b \leq a + c, \quad a + c \leq a + b$$

holds. If $a + b \leq a + c$, then $b \leq a + c$ whence,

$$b = (a + c)b = ab + cb = bc \ ,$$

yielding $b \leq c$. Similarly, if $a + c \leq a + b$ then $c \leq b$. Thus we must have $bc \leq a$ or the pairwise incomparability of a, b and c is upset. It follows that $ab = ac = bc = abc$. This suffices for the proof of the lemma.

We define the <u>width</u> of a lattice $\mathcal{L}$ as follows:

$$w(\mathcal{L}) = \sup\{|H| : H \subseteq L, \ H \text{ is a set of pairwise incomparables}\}.$$

Lemma 10.2. If $\mathcal{L}$ is a transferable distributive lattice then $w(\mathcal{L}) \leq 3$.

Proof. Let a, b, c be a set of three pairwise incomparables. We claim that for all $d \in L$, d is comparable with at least one of a, b and c. For proof by contradiction, suppose that a, b, c and d are pairwise incomparable. As in the previous lemma, by duality we may assume that $ab = ac = bc$. We look at the sublattice generated by d, a and b. Since a, b and d are pairwise incomparables and generate $\mathfrak{B}_8$, there are two cases.

Case 1. a, b and d are join reducible. For this case we already know that a and b are meet reducible from above. Thus this case is ruled out.

Case 2. $ad = ab = bd$. For this case, applying the distributive law, we obtain $a + cd = (a + c)(a + d)$. Now if $a + c \leq a + d$ then

$$c = c(a + d) = ac + bc = bc$$

since $ac = bc$ whence $c \leq b$, contrary to assumption. Similarly $a + d \leq a + c$. Since $a \not\geq cd$ we obtain $cd \leq a$ whence by distributivity,

$$((a + b) + c)((a + b) + d) = (a + b) + cd = a + b \ .$$

Since $a + b$ is join reducible, one of $a + b + c$ and $a + b + d$ is $a + b$. Thus either $c \leq a + b$ or $d \leq a + b$, both of which are contrary to assumption. This completes the proof of the lemma.

Lemma 10.3. Let $\mathfrak{L}$ be transferable. If a, b and $c \in L$ are three pairwise incomparables and $d \in L$ such that d is not in the sublattice generated by a, b and c, then either $d \leq abc$ or $a + b + c \leq d$.

Proof. By duality we again assume that $ab = ac = bc$. For proof by contradiction we assume that there is a $d \in L$ which is not in the sublattice generated by a, b and c satisfying $a + b + c \not\leq d$ and $d \not\leq abc$. By Lemma 10.2 d is comparable with at least one of a, b and c, and d is comparable with at least one of $a + b$, $a + c$ and

$b + c$. By symmetry and duality only the following three cases need be examined.

Case 1. $abc < d < a$. Now $bd = ad$ whence $b + d < a + b$, for otherwise d, a, b, ab and $a + b$ would form $\mathfrak{N}_5$, see Figure 10.1, and no distributive lattice contains $\mathfrak{N}_5$ as a sublattice. Further, $b + c < b + c + d < a + b + c$, since otherwise $a + b, b + d$ and $b + c$ generate a copy of $\mathfrak{N}_5$. We therefore obtain

$$\begin{aligned}(a + b)(d + c + b) &= (a + b)d + (a + b)c + (a + b)b \\ &= d + abc + b \\ &= b + d\ .\end{aligned}$$

Since $b + d$ is join reducible we conclude that one of $a + b$ and $d + c + b$ is $d + b$ contrary to assumption. This completes Case 1.

Case 2. $a < d < a + b$. For this case we obtain $abc < bd < b$ since otherwise a, b and d generate a copy of $\mathfrak{N}_5$. Thus we are now in a situation which is symmetric to Case 1. This concludes Case 2.

Case 3. $abc \nleq d < a$. For this case we have $abc < d + abc \leq a$. Thus either a is both join and meet reducible or we are in Case 1. Either possibility yields that $\mathfrak{L}$ is not transferable.

Lemma 10.4. If $\mathfrak{L}$ is transferable of width 2 and for each $x \in L \sim \{0,1\}$, where 0 and 1 are the least and greatest elements of L respectively, there is a $y \in L$ such that x and y are incomparable, then $\mathfrak{L}$ is an n-runged diagonal ladder for some $n \geq 2$.

Proof. We proceed by induction on the order of $\mathfrak{L}$. Since $\mathfrak{L}$ has width 2 the order of $\mathcal{L} \geq 4$. If $\mathcal{L}$ has order four, then $\mathcal{L}$ is the diamond which is the 2-runged diagonal ladder. We suppose that if has order i where $4 \leq i < m$ and $\mathcal{L}$ satisfies the hypothesis of the lemma, then it satisfies the conclusion.

For the remainder of the proof, let $\mathcal{L}$ be fixed such that satisfies the hypothesis of the lemma and has a minimal order $\geq m$. Since an n+1-runged diagonal ladder has only 2 more elements than an n-runged diagonal ladder, and every diagonal ladder satisfies the hypothesis of the lemma, we have that the order of $\mathcal{L}$ is at most $m+1$. Now there are exactly 2 elements of L which cover 0. To see this, note that the width condition implies there are at most 2, while the order condition together with the fact that for each $x \in L \sim \{0,1\}$ there is a $y \in L \sim \{0,1\}$ with x and y incomparable imply there are at least 2. Fix a and b covering 0. Suppose a and b are both meet reducible. We show that this leads to a contradiction. Clearly we can choose distinct x and y covering a. Similarly, choose z and w covering b. Now x, y and z are not pairwise incomparable. Without loss of generality assume $x \leq z$. Similarly x, y and w are not incomparable. Since w comparable with x yields a contradiction either $w \leq y$ or $y \leq w$. If $y \leq w$ then

$$a = xy \leq wz = b$$

contrary to hypothesis, whence we have $w \leq y$. Now if $x < z$ then since z covers b and b covers 0 we have that a, b and x generate $\mathfrak{N}_5$. We must therefore have that $x = z$. Similarly we obtain $y = w$, whence $a = xy = wz = b$ contrary to hypothesis. Thus one of a and b is meet irreducible.

Let a be meet irreducible. We now assert that $a + b$ covers a. Suppose for contradiction that there is a d such that either $a < d < a + b$ or $b < d < a + b$. Since a and b are the only elements covering 0, we conclude that a, b and d would generate $\mathfrak{N}_5$ whence $\mathfrak{L}$ would not be distributive. Thus $a + b$ covers a and so must be the unique cover of a since a is meet irreducible. Let $[b, 1] = L \sim \{0,a\}$. Then $\langle [b, 1]; +, \cdot \rangle$ is a distributive lattice of width 2, since $|[b, 1]| \geq 4$. Further if $x \in [b, 1] \sim \{b,1\}$ then for some $y \in L$ x and y are incomparable. Also if $y = a$ then x and $a + b$ are incomparable, because $x < a + b$ implies $x \in \{0,a,b\}$ and $x \geq a + b$ implies $x \geq y$ both of which are contrary to hypothesis. Since we can not have $x \leq a + b$, we are done. Lastly, no point of $[b, 1]$ is both join and meet reducible. Thus $\langle [b, 1]; +, \cdot \rangle$ satisfies the hypothesis of the lemma. Now $|[b, 1]| < m$ whence $\langle [b, 1]; +, \cdot \rangle$ is an n-runged diagonal ladder for some n. We now have two cases.

Case 1. $n = 2$. For this case it is clear that $\mathfrak{L}$ is the 3-runged diagonal ladder.

Case 2. $n \geq 3$. For this case we obtain that $\mathfrak{L}$ is one of the lattices pictured in Figure 10.2. Since only the n+1-runged diagonal ladder is transferable we are done. This completes the lemma.

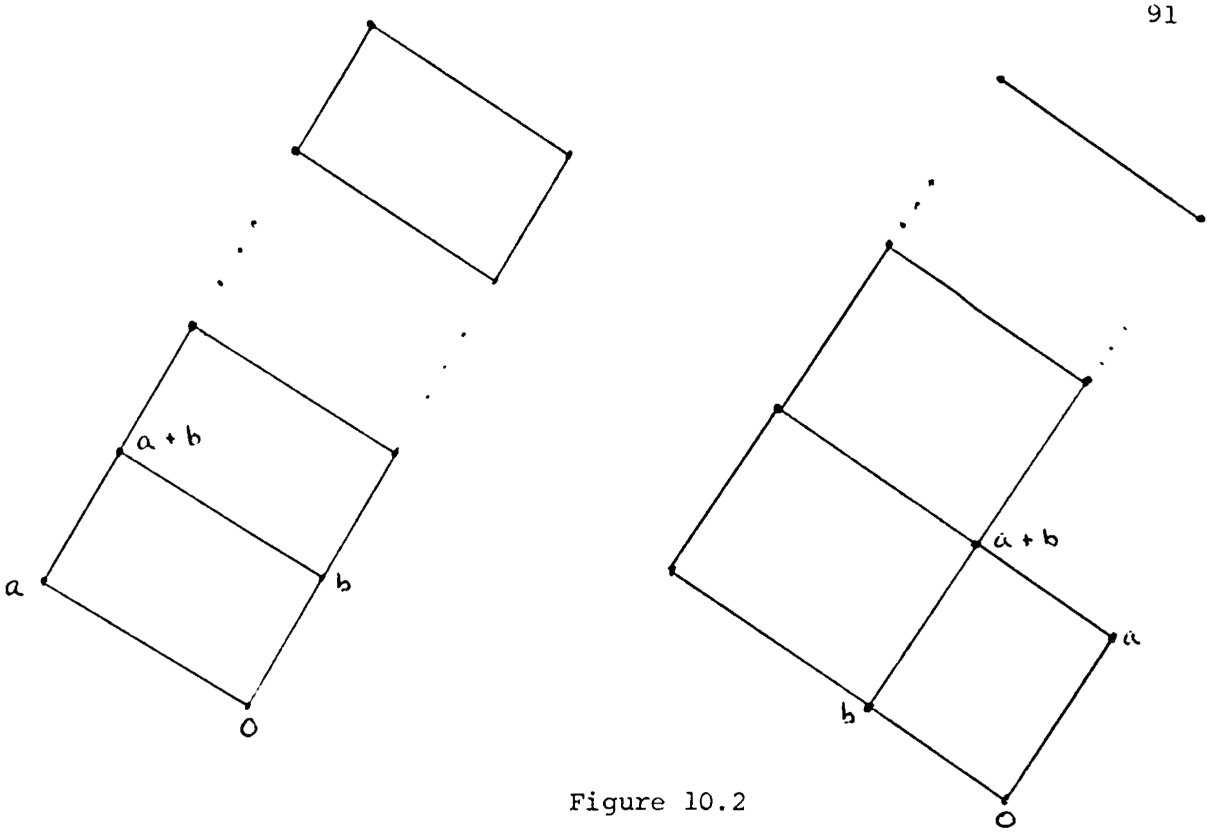


Figure 10.2

Theorem 10.2. $\mathcal{L}$ is transferable if and only if for some positive integer n there is a function, f, such that

(1) for each $i \in n$ $f(i)$ is a sublattice $\mathcal{L}_i$ of $\mathcal{L}$ such that $\mathcal{L}_i$ is either $\mathfrak{B}_8$ or an m-runged diagonal ladder or the 1-element lattice,

(2) $L = \bigcup_{i \in n} L_i$ and

(3) if $i < j \in n$ and $x \in L_i$ and $y \in L_j$ then $x < y$.

Proof. It is easy to see that if such a function exists for $\mathcal{L}$ then no point of $\mathcal{L}$ is both join and meet reducible whence $\mathcal{L}$ is transferable.

For the converse we proceed by induction on the order of $\mathcal{L}$. Clearly the result is valid if $\mathcal{L}$ has order 1. We suppose that if $1 \leq i < m$ and $\mathcal{L}$ is transferable of order i then the result is valid for $\mathfrak{L}$. Let $\mathfrak{L}$ be fixed of order m with $\mathfrak{L}$ transferable. Such an $\mathfrak{L}$ exists since the m-element chain is transferable. Let a be the least element of L other than 0 such that for all $x \in L$ either $x \leq a$ or $a \leq x$. Since 1 satisfies this condition such an a must exist. If $a = 1$ we are done since it is easy to see from Lemmas 10.1 - 10.4 that $\mathfrak{L}$ is either the 2-element chain or $\mathfrak{B}_8$ or a k-runged diagonal ladder. Thus suppose $a < 1$. Since $\mathfrak{L}$ is transferable, a is either join or meet irreducible. If a is join reducible then let b the the unique element covering a. Were b not unique a would be meet reducible. Then $\langle [b, 1]; +, \cdot \rangle$ has order $< m$ and is transferable. Further $\langle [0, a]; +, \cdot \rangle$ is transferable and as shown above $\langle [0, a]; +, \cdot \rangle$ is either $\mathfrak{B}_8$ or a k-runged diagonal ladder. Let f have domain n and satisfy (1) - (3) for $\langle [b, 1]; +, \cdot \rangle$, we define g on $n + 1$ by

$$g(i) = \begin{cases} \langle [0, a]; +, \cdot \rangle & \text{if } i = 0 \\ f(i - 1) & \text{if } i > 0 . \end{cases}$$

It is clear that g satisfies (1) - (3) for $\mathfrak{L}$.

We now consider the case when a is join irreducible. Then a covers 0 by choice of a. We then apply the induction hypothesis to $\langle [a, 1]; +, \cdot \rangle$ and proceed as above. This completes the proof of the theorem.

The results given above also serve to characterize weakly transferable distributive lattices. To see this we have only to recall Grätzer's observation that if $\mathfrak{L}'$ is weakly transferable then no point of $\mathfrak{L}'$ is both join and meet reducible. We therefore obtain, Theorem 10.3. $\mathcal{L}$ is transferable if and only if $\mathcal{L}$ is weakly transferable.

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