A STUDY OF HOLLAND REPRESENTATION THEOREM

and

FREE LATTICE-ORDERED GROUPS

by

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B.Sc. (Eng.), National Taiwan University, 1966

A THESIS SUBMITTED IN PARTIAL FULFILLMENT

OF THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

in the Department

of

Mathematics

SIMON FRASER UNIVERSITY

December 1972

APPROVAL

Name: Kung Hai Tang

Degree: Master of Science

Title of Thesis: A Study of Holland's Representation Theorem

and

Free Lattice Ordered Groups

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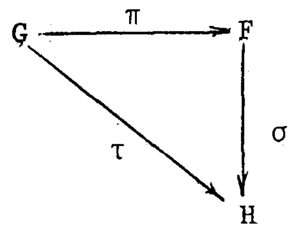
Date Approved: December 5, 1972

ABSTRACT

In this study of lattice-ordered groups, we begin with the fundamental properties as found in the book "Lattice Theory" by G. Birkhoff, and then present Holland's fundamental representation of a lattice-ordered group as a group of order preserving permutations of a totally ordered set. Holland's work is essential to the description of a free lattice-ordered group as given by P. Conrad. P. Conrad's results on free lattice-ordered groups are also reviewed. This work constitutes the major portion of this thesis.

If G is an l-group, then G is l-isomorphic to a subdirect product of l-groups $\{B_g : 0 \neq g \in G\}$ such that each B_g is a transitive l-subgroup of the l-group of automorphisms of a totally ordered set S_g, where S_g is the set of all right cosets of a convex l-subgroup M_g of G which is maximal with respect to not containing g. Furthermore, it then follows that G is also l-isomorphic to an l-subgroup of the l-group of all automorphisms of a totally ordered set.

On the other hand, every free group admits a total order. From this, we have that for a free group G, there is a free l-group F generated by $G\pi$, where π is an o-isomorphism, that is, for every o-homomorphism τ from G into an l-group H, there exists a unique l-homomorphism σ from F into H such that the following diagram



commutes.

Finally, for a po-group G, the following are equivalent:

(1) There exists a free l-group over G.

(2) There exists an o-isomorphism of G into an l-group.

(3) $G^+=\{g:g\geq 0\}$ is the intersection of right orders.

Acknowledgements

The author is grateful to his supervisor Prof. N. R. Reilly, for his suggestion of the topic and very helpful supervision. Writing a mathematical paper under his conscientious assistance is most enjoyable.

The financial aid from Department of Mathematics, Simon Fraser University and President Research Grant, Simon Fraser University is appreciated. I would also like to thank Mr. K. Mak and Mr. R. de la Giroday for doing the typing.

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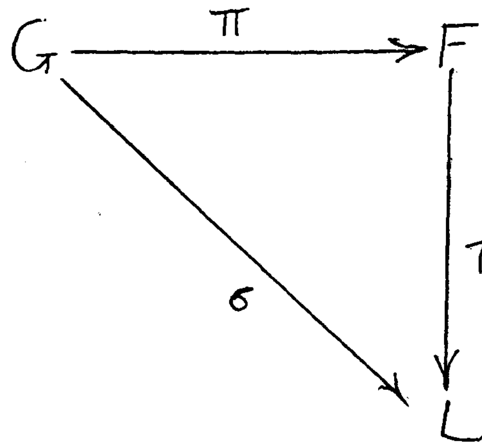
INTRODUCTION

The purpose of this study is to review some of the developments in the theory of lattice ordered groups closely related to the Holland representation for lattice ordered groups and P. Conrad's paper on free lattice ordered groups. In Chapter 1, a detailed discussion of the properties of regular and prime subgroups of an l-group is presented. A subgroup H of a lattice ordered group G is regular if and only if there is an element $g \varepsilon G$ such that H is a maximal convex l-subgroup of G with respect to not containing g; a subgroup M of a lattice ordered group G is prime subgroup if M is the intersection of a chain of regular subgroups of G or M is a convex l-subgroup of G and $a,b \varepsilon G^{+} \setminus M$ implies $a \wedge b \neq 0$.

Prime subgroups are of particular importance in obtaining representation of lattice-ordered groups. If M is prime subgroup of a lattice ordered group G, then the set of cosets of M can be endowed with total order, where $a+M>b+M$, if and only if there exists $m \varepsilon M$ such that $a+m>b$. It follows that if M is both prime and normal (prime subgroup need not be normal, see Example 1.22), then the set of cosets of M is a totally ordered group. The properties of prime subgroups were utilized by C. Holland in his representation theorem which is discussed in Chapter 2. Observe that if M is regular then M is prime (Corollary 1.19), and so we have that every l-group always contains prime subgroups. In Chapter 2, we present the Holland representation of a lattice-ordered group G as a subdirect product of ΠK_β, where each K_β is a transitive l-subgroup of the lattice ordered group of order-preserving permutations on some totally ordered set, where each total order set is the set of cosets of some prime subgroup of G. This answered a problem originally posed by Birkhoff in the second edition of his book on lattice theory, it is an invaluable tool in the study of the nature and occurrence of lattice-ordered groups. In Chapter 3, our main concern is with the groups admitting a linear order, which we shall call O-groups. We prove that all free groups are O-groups, the method is similar to the proof of Simbireva, Neumann Theorem: If a group G has a transfinite central series ending with $C_\tau=\{0\}$ such that all factor groups $C_\alpha/C_{\alpha+1}$ are torsion free, then G is an O-group. (A descending chain $G=G_0\supset G_1\supset G_2\supset\ldots\supset G_\alpha\supset G_{\alpha+1}\supset\ldots..$, with α a variable over ordinals less than a fixed τ is called a transfinite series of G if $G_{\alpha+1}$ is a (normal) subgroup of G_α such that the commutator $[G, G_\alpha]$ is contained in $G_{\alpha+1}$ and for a limit ordinal α, G_α is the intersection of all G_β with $\beta<\alpha$. Clearly, the G_α are normal in G). In the proof, a theorem of Magnus-Witt is used; that is, the lower central series of a free group G terminates at $\{0\}$ after ω steps, where ω denotes the first infinite ordinal, and the factor groups are torsion free. In Chapter 4, let π be an O-isomorphism of a p o-group G into an l-group F. This means that both π and π^{-1} preserve order. Then (F, π) is a free l-group over G if (i) $G\pi$ is a set of generators of the l-group F (that is, no proper l-subgroup of F contains $G\pi$), and (ii) if σ is an O-homomorphism of G into an l-group L, then there exists an l-homomorphism τ of F such that



commutes.

The following two theorems are the main results.

Theorem 4.14. For a p o-group G the following are equivalent:

(1) There exists a free l-group over G.

(2) There exists an O-isomorphism of G into an l-group.

(3) $G^{+}=\{g\varepsilon G;\ g\geq 0\}$ is the intersection of right orders.

In particular, there exists a free l-group over a trivially ordered group G (that is, $G^{+}=\{0\}$) if and only if G admits a right order and this is equivalent to G being a subgroup of an l-group. Thus if G is a free group with S as a free set of generators and a trivial order, then there exists a free l-group (F, π) over G. Moreover F is a free l-group with Sπ as a free set of generators. That is, Sπ generates the l-group F and each mapping of Sπ into an l-group L has a unique extension to an l-homomorphism of F into L.

Suppose that G is a p o-group such that $G^{+}=\bigcap_{\lambda\in\Omega} P_{\lambda}$, where $\{P_{\lambda}\,|\,x\varepsilon\Omega\}$

is the set of all right orders of G such that $P_\lambda \supset G^+$. For each $\lambda\varepsilon\Omega$ let G_λ be the right o-group (G, P_λ) and let $g\to g^\lambda$ be the natural isomorphism of G into the l-group $A(G_\lambda)$ of all order preserving permutations of the totally ordered set G_λ, where

$$xg^\lambda = x+g \qquad \text{for all } x\varepsilon G$$

The direct product $\Pi A(G_\lambda)$ of the l-group $A(G_\lambda)$ with component-wise order is an l-group, the cardinal product of the $A(G_\lambda)$'s. Now let π be the natural map of G onto the subgroup of long constants of the l-group $\Pi A(G_\lambda)$

$$g \xrightarrow{\ \pi\ } (\ldots g^\lambda \ldots)$$

and let F be the l-subgroup of $\Pi A(G_\lambda)$ that is generated by $G\pi$. It then can be shwon that

Theorem 4.13: (F, π) is the free l-group over G.

Note that in our presentation of the Conrad's Generalization of the Method used by Weinberg to construct free abelian l-groups is used to construct free l-groups. The generalization is quite natural and identical with Weinberg's method if we restrict our attention to abelian group. Proposition 4.15 is a generalization of one of the P. Conrad's propositions.

In our construction of the free l-group over a p o-group the key concept is that of a right o-group. In the papers of P. Cohn, Groups of order automorphisms of ordered sets, Mathematika 4 (1957) 41-50 and P. Conrad, Introduction à la thèorie des groups rétinles, Secretariat Mathematiques Paris (1967) there are necessary and sufficient conditions given a group G to admit a right order. In D. Smirnov On right ordered groups (Russian), Akad, Nauk, SSSR Siberian Dept. Algebra and Logic 5 (1966) various right orders of a free group are investigated.

Notation: Throughout the whole thesis, in general we use additive notation as group operation, except in permutation group, we use multiplicative notation.

Chapter One

In this preliminary chapter, we study mainly prime subgroups of an l-group. The reader is referred to Birkhoff [1] and Fuchs [5] for a general theory of lattice-ordered groups. Here we may also assume that the reader is familar with the basic properties of groups.

Definition: By a partially ordered set is meant a system X in which a binnary relation $x \geq y$ is defined, which satisfies

(i) For all x, $x \geq x$. (Reflexive)

(ii) If $x \geq y$ and $y \geq x$, then $x = y$. (Antisymmetric)

(iii) If $x \geq y$ and $y \geq z$, then $x \geq z$. (Transitive)

Definition: A totally ordered set X is a partially ordered set in which either $x \geq y$ or $y \geq x$, for every $x, y \in X$.

Definition: A partially ordered group (G, ≤, +) is such that

(i) (G, ≤) is a partially ordered set.

(ii) (G, +) is a group.

(iii) $x \geq y$ implies $a+x+b \geq a+y+b$, for all $a, b, x, y \in G$.

We define totally ordered group similarly.

Definition: (G, +, ≤) is an right ordered group if and only if

(i) (G, +) is a group.

(ii) (G, ≤) is totally ordered.

(iii) $x \leq y$ implies $x+a \leq y+a$, for every $x, y, a \in G$.

We define left ordered group similarly.

Definition: A lattice is a partially ordered set P any two of whose elements have a greatest lower bound or "meet" $x \wedge y$, and least upper bound or "join" $x \vee y$.

Definition: An l-group $(G, \leq, +)$ is such that

(i) $(G, \leq)$ is a lattice.

(ii) $(G, +)$ is a group.

(iii) $x \geq y$ implies $a+x+b \geq a+y+b$ for all $x, y, a, b \in G$.

Theorem 1.1 Let G be a group with identity o. Let $P \subseteq G$ be such that (i) $o \in P$

(ii) $P+P \subseteq P$

(iii) $P \cup (-P) = G$

(iv) $P \cap (-P) = \{o\}$

(A) Let $\leq$ be defined on G by $g \leq h$ if and only if $h-g \in P$.

(B) Let $\underline{\alpha}$ be defined on G by $g \underline{\alpha} h$ if and only if $-g+h \in P$.

Then the order in (A) is a right total order and the order in (B) is a left total order. Conversely, let $\leq$ and $\underline{\alpha}$ be a right total order and a left total order on G respectively, then $P=\{g;\ o \leq y,\ g \in G\}$ and $P'=\{g;\ o \underline{\alpha} g;\ g \in G\}$ satisfy (i) to (iv).

Proof: For A. (i) $x-x=o \in P$, for every $x \in G$, implies $x \leq x$ for every $x \in G$.

(ii) If $x \leq y$ and $y \leq x$, then $y-x$, $x-y \in P$. But $x-y=-(y-x) \in -P$. Hence $x-y \in P \cap (-P)=\{o\}$. Therefore $x-y=o$. That is $x=y$.

(iii) If $x \leq y$ and $y \leq z$, where $x, y, z \in G$, then $y-x$ and $z-y \in P$. Hence $(z-y)+(y-x) \in P+P \subseteq P$. Therefore $z-x \in P$. Consequently, $x \leq z$.

(iv) Let $x, y \in G$. Then $x-y \in G$. Therefore $x-y \in P$ or $x-y \in (-P)$. So $x \leq y$ or $y \leq x$.

We have thus shown that "$\leq$" is a total order on G. Now suppose $x \leq y$ for some $x, y \in G$, consider $x+a$ and $y+a$ for any $a \in G$, we have $(y+a)-(x+a)=y+a-a-x=y-x \in P$. Hence $x+a \leq y+a$, for any $a \in G$. Consequently

$(G, +, \leq)$ is a right ordered group.

Similarly, the order in (B) is a left order on G.

Conversely, let "$\leq$" be a right order on G and $P=\{g;\ o\leq g,\ g\varepsilon G\}$. Then

(i) $o\leq o$ implies $o\varepsilon P$.

(ii) For any g_1, $g_2\varepsilon P$, we have $o\leq g_1$, therefore $g_1+g_2\geq o+g_2=g_2>o$. That is $g_1+g_2\geq o$. Hence $P+P\subseteq P$.

(iii) Clearly $P\bigcup(-P)\subseteq G$. Let $g\varepsilon G$. Then $g\geq o$ or $g>o$, since "$\leq$" is a total order on G. Therefore $g\varepsilon P$ or $g\varepsilon -P$. Hence $g\varepsilon P\bigcup(-P)$. That is $G\subseteq P\bigcup(-P)$ and hence $P\bigcup(-P)=G$.

(iv) Let $g\varepsilon P\bigcap(-P)$. Then $g\geq o$ and $g\leq o$. Hence $g=o$ (by antisymmetry), so $P\bigcap(-P)=\{o\}$. The rest is clear. Similarly,

Theorem 1.2. Let G be a p o-group and let $G^+=\{g;\ g\varepsilon G,\ g\geq o\}$ be the set of all positive elements in G. The following three conditions are equivalent.

(1) $a\leq b$

(2) $b-a\varepsilon G^+$

(3) $-a+b\varepsilon G^+$

Moreover,

(i) $o\varepsilon G^+$

(ii) $G^+ + G^+\subseteq G^+$

(iii) $a+G^+=G^++a$, for all $a\varepsilon G$

(iv) $G^+\bigcap(-G^+)=\{o\}$

On the other hand, suppose that G^+ is a subset of G which possesses The properties (i) —— (iv). Then it is easy to see that G becomes a p o-group if one defines a relation $\leq$ on G by $f\leq g$ if and only if $g-f$ and $-f+g\varepsilon G^+$.

Proposition 1.3. A p o-group is an l-group if and only if for all $a\epsilon G$, $a\vee o$ exists.

Proof: If G is an l-group, then obviously $a\vee o$ exists for all $a\epsilon G$. Conversely, let G be any p o-group in which $a\vee o$ exists, for all a. Then $(a-b)\vee o+b$ exists, it is not hard to see that $(a-b)\vee o+b$ is the least upper bound of a and b. Similarly $a\wedge b$ always exists in G.

Theorem 1.2 and proposition 1.3 gives

Theorem 1.4. For any l-group G, let G^+ be the set of its positive elements, that is $G^+=\{g;\ g\geq o,\ g\epsilon G\}$. The following three conditions are equivalent.

(1) $a\leq b$.

(2) $b-a\epsilon G^+$.

(3) $-a+b\epsilon G^+$.

Moreover,

(i) $o\epsilon G^+$.

(ii) $G^+ + G^+ \subseteq G^+$.

(iii) $a+G^+=G^++a$, for all $a\epsilon G$.

(iv) $G^+\cap(-G^+)=\{o\}$.

(v) $a\vee o\epsilon G^+$, that is $a\vee o$ exists, for all $a\epsilon G$.

On the other hand, it is easy to see that if G^+ is a subset of G and has the properties (i) —— (v), then G becomes an l-group if one defines a relation $\leq$ on G by $f\leq g$ if and only if $g-f$ and $-f+g\epsilon G^+$.

Notation: Thereafter, a partial order P or an right order P on G is meant that $P=\{g\geq o,\ g\epsilon G\}$ under that particular order.

Definition: A subgroup C of $(G, \leq, +)$ is convex, provided C contains along with $x\geq o$ also all y's such that $x\geq y\geq o$. Hence C is convex if and only if for any c_1, $c_2 \varepsilon C$, $c_1 \leq c \leq c_2$, we have $c \varepsilon C$.

Lemma 1.5. Let C be a convex subgroup of a partially ordered group G. Let $R(C)=\{C+g \mid g \varepsilon G\}$ be the set of all right cosets of C in G. If we define $C+g \leq C+h$ to mean that there exists $c \varepsilon C$ with $c+g \leq h$, then this defines a partial order on the set R(C).

Proof: (i) $C+g \leq C+g$, for every $g \varepsilon G$. (Reflexive)

(ii) If $C+g_1 \leq C+g_2$ and $C+g_2 \leq C+g_1$. Then there exist c_1, $c_2 \varepsilon C$, such that $c_1+g_1 \leq g_2$ and $c_2+g_2 \leq g_1$. Therefore $c_2+c_1+g_1 \leq c_2+g_2 \leq g_1$. Hence $o \leq c_2+g_2-g_1-c_1-c_2 \leq g_1-g_1-c_1-c_2=-c_1-c_2$. Consequently $c_2+g_2-g_1-c_1-c_2 \varepsilon C$, by convexity. So $g_2-g_1 \varepsilon C$. Hence $C+g_2-g_1=C$. It follows that $C+g_2=C+g_1$ (antisymmetric).

(iii) If $C+g_1 \leq C+g_2$ and $C+g_2 \leq C+g_3$. Then there exist c_1, $c_2 \varepsilon C$ such that $c_1+g_1 \leq g_2$ and $c_2+g_2 \leq g_3$. Therefore $c_2+c_1+g_1 \leq c_2+g_2 \leq g_3$. So $c_2+c_1+g_1 \leq g_3$. Hence $C+g_1 \leq C+g_3$ (Transitive). Consequently "$\leq$" is a partial order on R(C).

In an l-group, the following properties are trivially verified:

(i) $a+(x \vee y)=(a+x) \vee (a+y)$ and $(x \vee y)+b=(x+b) \vee (y+b)$.

(ii) $a+(x \wedge y)=(a+x) \wedge (a+y)$ and $(x \wedge y)+b=(x+b) \wedge (y+b)$.

(iii) $-(a \vee b)=(-a) \wedge (-b)$ and $-(a \wedge b)=(-a) \vee (-b)$.

Definition: A subgroup of an l-group G which is also a sublattice, is an l-subgroup. A convex l-subgroup is a convex subgroup and also a sublattice.

Lemma 1.6: In lemma 1.5, if G is an l-group and C is a sublattice, then R(C) is a lattice with $(C+x) \vee (C+y)=c+x \vee y$ and $(C+x) \wedge (C+y)=C+x \wedge y$.

Proof: Note that $C+x \vee y \geq C+x$, $C+y$. Now suppose $C+g \geq C+x$, $C+y$. Then there exist c_1, $c_2 \varepsilon C$, such that $c_1+x \leq g$, $c_2+y \leq g$. Hence $x \leq -c_1+g$ and $y \leq -c_2+g$. Therefore $x \vee y \leq (-c_1+g) \vee (-c_2+g) = (-c_1) \vee (-c_2)+g = c+g$, where $c=(-c_1) \vee (-c_2) \varepsilon C$, since C is a sublattice. Hence $x \vee y \leq c+g$. Therefore $C+x \vee y \leq C+g$. Consequently $C+x \vee y=(C+x) \vee (C+y)$. Dually $C+x \wedge y=(C+x) \wedge (C+y)$. Hence R(C) is a lattice.

Example 1.7: Let C be the set of all complex numbers, then $(C, \leq, +)$ is a partially ordered group in which $a+bi \leq c+di$ if and only if $b=d$ and $a \leq c$. But $(C, \leq, +)$ is not an l-group.

Example 1.8: Let R be the set of all real numbers, then $(R, \leq, +)$ is an l-group with usual order.

Example 1.9: Let Z be the set of all integers, then $(Z \boxplus Z, \leq, \boxplus)$ is an l-group with partial order defined by $(a, b) \leq (c, d)$, if and only if $a \leq c$ and $b \leq d$.

The following proposition 1.10 is important in Chapter 2, and also is an example of non-abelian l-group.

Definition: Let S be a totally ordered set, and let G be the group of all functions $f: S \longrightarrow S$ such that f is one to one, onto and the inequality $x<y$ $(x, y \varepsilon S)$ implies that $xf<yf$. We call such a function an automorphism of S.

Proposition 1.10: If f and g are automorphisms of S, then define $f \leq g$ if $xf \leq xg$ for all $x \varepsilon S$. Then this defines a partial order on G under which G is a lattice-ordered group.

Proof: Let G be the group of all automorphisms of S. Then

(i) $f \leq f$, for every $f \varepsilon G$.

(ii) If $f \leq g$ and $g \leq f$, then $xf \leq xg$ and $xg \leq xf$, for every $x \varepsilon S$. Hence $xf=xg$, for every $x \varepsilon S$; since S is totally ordered. Therefore $f=g$.

(iii) Suppose $f \leq g$, and $g \leq h$. Then $xf \leq xg$ and $xg \leq xh$, for every $x \in S$. Therefore $xf \leq xh$, for every $x \in S$. Consequently $f \leq h$.

Therefore "$\leq$" is a partial order on G.

Now given any f, $g \in G$, we define F by $xF = sf V xg$, for every $x \in S$. Suppose $x > y$, x, $y \in S$, then $xF = xf V xg > yf V yg = yF$, since S is totally ordered. Therefore F is order-preserving, and hence F is one-one. Furthermore, if $y \in S$, then there exist x_1, $x_2 \in S$, such that $x_1 f = y$, $x_2 g = y$. Without loss of generarity, say $x_1 \geq x_2$, we get $x_2 f \leq x_1 f = y = x_2 g$. That is $x_2 g \geq x_2 f$. Hence $x_2 F = x_2 f V x_2 g = x_2 g = y$. Therefore F is onto. Consequently $F \in G$. Note that $F \geq f$, g. Now let $h \geq f$, g. Then $h(x) \geq f(x)$ and $h(x) \geq g(x)$, for every $x \in S$. Therefore $xh \geq xf V xg$, for every $x \in S$. Hence $h \geq F$. Consequently $F = f V g$. Similarly $f \Lambda g \in S$. Hence G is a lattice. Finally, let $f \geq g \in G$. Consider $h_1 f h_2$ and $h_1 g h_2$, where h_1, $h_2 \in G$. Then $xh_1 = xh_1$ for every $x \in S$. Therefore $xh_1 f \geq xh_1 g$, for every $x \in S$. So $xh_1 f h_2 \geq xh_1 g h_2$, for every $x \in S$. That is $h_1 f h_2 \geq h_1 g h_2$. Hence G is an l-group.

Definition: A subgroup H of G is transitive, if and only if for every x, y $\in S$, there exists $h \in H$, such that $xh = y$.

Notation; F or $x \in G$, and G is an l-group. $|x| = xV(-x)$.

Lemma 1.11: Let G be an l-group and a $\in G$, x, $y \in G^+$.

(i) $a = aVo + a\Lambda o$.

(ii) $(aVo)\Lambda((-a)Vo) = o$.

(iii) $|a| = aVo + (-a)Vo$.

(iv) $x \Lambda y = o$ implies $x + y = y + x$.

(v) Let b, $c \in G^+$ be such that $b \Lambda c = o$ and $a = b - c$. Then $b = aVo$, $c = (-a)Vo$. In other words, $b = aVo$, $c = (-a)Vo$ are the unique elements of G such that $b\wedge c=o$, $a=b-c$.

Proof: (i) $a-a\wedge o=a+(-a)\vee o=(a-a)\vee(a+o)=o\vee a$. Hence $a=a\vee o+a\wedge o$.

(ii) $(a\vee o)\wedge((-a)\vee o)=(a+o\vee(-a))\wedge((-a)\vee o)=((a-(a\wedge o))\wedge(-(a\wedge o))=a\wedge o-a\wedge o=o$.

(iii) $a\vee(-a)=(a\vee(-a))\vee o+(a\vee(-a))\wedge o$ by (i)

$=a\vee(-a)\vee o-((-a)\wedge a)\vee o\geq o$. Hence $a\vee o+(-a)\vee o=(a+(-a)\vee o)\vee(o+(-a)\vee o)$

$=((a-a)\vee a)\vee((-a)\vee o)=o\vee a\vee(-a)\vee o=a\vee(-a)=|a|$.

(iv) $x\wedge y=o$ implies $x+y=x+x\wedge y+y=x+(-x)\vee(-y)+y=(x-x+y)\vee(x-y+y)$

$=y\vee x=x\vee y=y+x$, since $x\wedge y=y\wedge x=o$. Hence $x+y=y+x$, if $x\wedge y=o$.

(v) If $a=b-c$, then $b=a+c$. Hence $o=b\wedge c=(a+c)\wedge c=a\wedge o+c$. Therefore

$c=-(a\wedge o)=(-a)\vee o$, and $b=a+c=a+(-a)\vee o=o\vee a$.

Corollary 1.12: If G is an l-group. Let H be a subgroup of G, such that $H\supset G^+$. Then H=G.

Proof: For every $a\in G$, we have $a\vee o$, $-(a\wedge o)\in H$. Hence $a=a\vee o+a\wedge o\in H$. That is, H=G.

Lemma 1.13: Let G be an l-group, $x, y, z\in G^+$. Then $x\wedge(y+z)\leq x\wedge y+x\wedge z$.

Proof: $x\wedge y+x\wedge z=(x+x\wedge z)\wedge(y+x\wedge z)=\{(x+x)\wedge(x+z)\}\wedge\{(y+x)\wedge(y+z)\}$

$=\{(x+x)\wedge(x+z)\wedge(y+x)\}\wedge(y+z)\geq x\wedge(y+z)$.

To prove Lemma 1.14, we need the result from Birkhoff [1] P.134 that "A lattice G is distributive if and only if $a\wedge x=a\wedge y$ and $a\vee x=a\vee y$ implies $x=y$, for every $a, x, y\in G$.

Lemma 1.14: Any l-group is a distributive lattice.

Proof: Let G be an l-group, $a, x, y\in G$ and $a\wedge x=a\wedge y$, $a\vee x=a\vee y$.

We know that $a-a\wedge x+x=a+(-a)\vee(-x)+x=(a-a+x)\vee(a-x+x)=x\vee a$. Therefore

$x=a\wedge x-a+x\vee a=a\wedge y-a+y\vee a$ by assumption

$=y$.

The following lemma is useful.

Lemma 1.15 (A. H. Clifford [4]): Let G be an l-group. Let M be a convex l-subgroup of G. Let a, $b\varepsilon G^{+}$. For any x, let G(M, x) denote the smallest convex l-subgroup of G containing M and x. If x>o, then $G(M, x)=\{g; |g|\leq m_1+x+m_2+x+.....+m_n+x+m_{n+1}$, for some $m_i\varepsilon M^{+}\}$. Moreover $G(M, a)\cap G(M, b)=G(M, a\wedge b)$.

Proof: Let $A=\{g; |g|\leq m_1+x+m_2+x+.....+m_n+x+m_{n+1}, m_i\varepsilon M^{+}\}$. Since G(M, x) is a subgroup and contains M and x, it is clear that G(M,x) contains all expressions of the form $m_1+x+m_2+x+.....+m_n+x+m_{n+1}$. Moreover $o\leq|g|$, for every $g\varepsilon G$, hence $G(m, x)\supset A$, by convexity.

Now if $|g_1|\leq m_1+x+m_2+x+.....+m_p+x+m_{p+1}$, and $|g_2|\leq n_1+x+n_2+x+.....n_q+x+n_{q+1}$, m_i, $n_j\varepsilon M^{+}$. Then $(g_1Vg_2)V(-(g_1Vg_2))=g_1Vg_2V((-g_1)\wedge(-g_2))\leq(g_1Vg_2)V(-g_1V-g_2)$ $=(g_1V(-g_1)Vg_2V(-g_2)=|g_1|V|g_2|\leq m_1Vn_1+x+n_2Vm_2+x+.....+m_pVn_p+x+.....+n_q+x+n_{q+1}$, since $m_iVn_i\geq m_i$, n_i, and assuming that $q\geq p$. This implies $g_1Vg_2\varepsilon A$. Furthermore, suppose g_1, $g_2\varepsilon A$, and $g_1\leq g\leq g_2$, then $-g_2\leq -g\leq -g_1$. Hence $|g|=gV(-g)\leq g_2V(-g_1)\varepsilon A$, since $-g_1\varepsilon A$. Therefore $g\varepsilon A$. Moreover, $|g_1-g_2|=(g_1-g_2)V(g_2-g_1)\leq g_1V(-g_1)+g_2V(-g_2)=|g_1|+|g_2|$. Therefore $g_1-g_2\varepsilon A$. Consequently, A is a convex l-subgroup of G containing M and x. That is $G(M, x)=A$. Now suppose that a, $b\varepsilon G^{+}$; we get $o\leq a\wedge b\leq a$ and $o\leq a\wedge b\leq b$. Hence $G(M, a)\supset G(M, a\wedge b)$ and $G(M, b)\supset G(M, a\wedge b)$, by convexity, that is, $G(M, a)\cap G(M, b)\supset G(M, a\wedge b)$. Let g G(M, a)$\cap$G(M, b). Then $|g|\leq m_1+a+m_2+a+.....+m_p+a+m_{p+1}$ and $|g|\leq n_1+b+n_2+b+.....+n_q+b+n_{q+1}$, where m_i, $n_j\varepsilon M^{+}$, therefore

$$|g|\leq(m_1+a+m_2+a+.....+m_p+a+m_{p+1})\wedge(n_1+b+n_2+b+.....+n_q+b+n_{q+1})$$

$$\leq m_1\wedge(n_1+b+n_2+b+.....+n_q+b+n_{q+1})+a\wedge(n_1+b+n_2+b+.....+n_q+b+n_{q+1})+.....$$

$$+m_p\wedge(n_1+b+n_2+b+.....+n_q+b+n_{q+1})+a\wedge(n_1+b+n_2+b+.....+n_q+b+n_{q+1})$$

$+m_{p+1}\wedge(n_1+b+n_2+b+\ldots\ldots+n_q+b+n_{q+1})\leq m_1+(n_1+a\wedge b+n_2+a\wedge b+\ldots\ldots+n_q+a\wedge b+n_{q+1})$
$+\ldots\ldots+m_p+(n_1+a\wedge b+n_2+a\wedge b+\ldots\ldots+n_q+a\wedge b+n_{q+1})+m_{p+1}\varepsilon G(M,\ a\wedge b)$.
Hence $G(m,\ a)\cap G(M,\ b)=G(M,\ a\wedge b)$.

We now consider some properties of prime subgroups of an l-group, which are important in the respresentation of an l-group as a transitive l-subgroup of the l-group of automorphisms of an ordered set to be observed in Chapter 2.

Definition: Let H be a convex l-subgroup of an l-group G. If there is an element $g\varepsilon G$ such that H is a maximal convex l-subgroup of G with respect to not containing g, then H is called a regular subgroup.

Theorem 1.16: For a convex l-subgroup M of an l-group G, the following are equivalent.

(i) If A and B are convex l-subgroups of G such that $A\cap B\subseteq M$, then either $A\subseteq M$ or $B\subseteq M$.

(ii) If A, B are convex l-subgroups of G, $A\supsetneq M$ and $B\supsetneq M$, then $A\cap B\supsetneq M$.

(iii) If $a,\ b\varepsilon G^+\setminus M$, then $a\wedge b\varepsilon G^+\setminus M$.

(iv) If $a,\ b\varepsilon G^+\setminus M$, then $a\wedge b\neq o$.

(v) The lattice R(M) of right cosets of M is totally ordered.

(vi) The set of convex l-subgroups of G which contain M form a totally ordered set (under inclusion).

(vii) M is the intersection of a chain of regular subgroups.

Proof: We prove that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) $\Rightarrow$ (vii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii). If $A \cap B = M$, then, by (i), $A \subseteq M$ or $B \subseteq M$, contradiction. Hence $A \cap B \supsetneq M$.

(ii) $\Rightarrow$ (iii). Since $G(M, a) \supsetneq M$, $G(M, b) \supsetneq M$, we have by (ii) $G(M, a) \cap G(M, b) = G(M, a \wedge b) \supsetneq M$. Therefore $a \wedge b \notin M$, and so $a \wedge b \in G^+ \setminus M$.

(iii) $\Rightarrow$ (iv) Let $a, b \in G^+ \setminus M$. Then $a \wedge b \in G^+ \setminus M$, by (iii). Hence $a \wedge b \neq o$.

(iv) $\Rightarrow$ (v). By way of contradiction, if neither $M+g \leq M+f$ nor $M+f \leq M+g$, then $o < g - g \wedge f \notin M$, since otherwise $M+g = M+g \wedge f \leq M+f$. Likewise $o < f - g \wedge f \notin M$. But $(g - g \wedge f) \wedge (f - g \wedge f) = g \wedge f - g \wedge f = o$, a contradiction. Hence $R(M)$ is totally ordered.

(v) $\Rightarrow$ (vi). Let M_1, M_2 be convex l-subgroups of G which contain M. Suppose M_1 and M_2 are incomparable, then there exist m_1, $m_2 \in G^+$, such that $m_1 \in M_1^+ \setminus M_2$ and $m_2 \in M_2^+ \setminus M_1$ by Corollary 1.12. Without lost of generality, we may assume that $M+m_1 \geq M+m_2 > M$, then there exist m, $m' \in M$, such that $m+m_1 \geq m_2 \geq m'$. This implies $m_2 \in M_1$, by convexity, a contradiction. Hence the set of convex l-subgroup of G which contain M form a totally ordered set.

(vi) $\Rightarrow$ (vii). If M is a maximal convex l-subgroup of G with respect to not containing some $g \in G$, we are done. For every $o \neq g \in G$ and $g \notin M$, there exists a maximal convex l-subgroup of G with respect to not containing g in the set of (vi). For every $g \neq o$, $g \notin M$, we denote one of such subgroups by C_g. Then $\bigcap_{\substack{o \neq g \in G \\ g \notin M}} C_g \supset M$. Clearly, the set of all C_g's is totally ordered, since C_g's are in the set of (vi). Let $g \notin M$, $g \in G$. Then $g \notin C_g$. Hence $g \notin \bigcap_{\substack{o \neq g \in G \\ g \notin M}} C_g$. That is,

$\bigcap_{\substack{g\in G \\ g\notin M}} C_g = M.$

(vii) $\Rightarrow$ (i). Suppose M is regular. If there exist a, b such that $a\in A^+\setminus M$ and $b\in B^+\setminus M$, then $a\neq b$, by hypothesis. Now $G(M, a)\cap G(M, b)=G(M, a\wedge b)=M$, since $a\wedge b\in A\cap B$ and $A\cap B\subseteq M$. Moreover $G(M, a)\supsetneq M$ and $G(M, b)\supsetneq M$, but then M is not regular. Because $g\notin M$ implies that $g\notin G(M, a)$ or $g\notin G(M, b)$. Consequently, $A\subseteq M$ or $B\subseteq M$.

Now suppose M is an intersection of a chain of a regular subgroups C_g where C_g is a regular subgroup with respect to not containing $g\in G$. That is, suppose that $A\cap B\subseteq M=\bigcap_{\substack{g\notin M \\ g\in G}} C_g$. Then $A\cap B\subseteq C_g$, for every such C_g. Therefore $A\subseteq C_g$ or $B\subseteq C_g$ by previous argument. If there exist C_{g_1} and C_{g_2} of such type such that $A\subseteq C_{g_1}$, $B\nsubseteq C_{g_1}$, and $B\subseteq C_{g_2}$, $A\nsubseteq C_{g_2}$, then from $A\subseteq C_{g_1}$ and $A\nsubseteq C_{g_2}$, we have $C_{g_2}\subsetneq C_{g_1}$, since the set of such Cg's are totally ordered. Similarly, $C_{g_1}\subsetneq C_{g_2}$, a contradiction. Hence $A\subseteq C_g$ for every such C_g or $B\subseteq C_g$ for every such C_g. That is $A\subseteq \bigcap C_g$ or $B\subseteq \bigcap C_g$. Hence $A\subseteq M$ or $B\subseteq M$.

Definition: (a) Let G and H be p o-groups. Then an o-homomorphism θ from G into H is an isotone homomorphism: that is to say θ is a group homomorphism such that for any x, $y\in G$, if $x\leq y$, then $x\theta\leq y\theta$.

(b) If G and H are l-groups, then an l-homomorphism θ from G into H is an o-homomorphism such that for any x, $y\in G$,

$(x\vee y)\theta=x\theta\vee y\theta$.....(i)

$(x\wedge y)\theta=x\theta\wedge y\theta$.....(ii)

(d) An l-isomorphism θ from l-group G into l-group H is an o-isomorphism from G into H such that (i) and (ii) of (c) hold.

Remark: An o-homomorphism need not be an l-homomorphism. Consider the l-group G of all continuous functions from [o, 1] into reals. Let H be the l-group of all linear functions from [o, 1] into reals. Then H is a subgroups of G. The inclusion mapping from H into G is an o-homomorphism but not an l-homomorphism.

Definition: The Cardinal product of l-groups A_i, $i\varepsilon I$, denoted by $\left|\prod_{i\varepsilon I}\right| A_i$ is the direct product $\prod_{i\varepsilon I} A_i$ with the partial order defined by $a \geq o$ if and only if $a_i \geq o$, for every $i\varepsilon I$, where $a \varepsilon \prod_{i\varepsilon I} A_i$ and $a_i \varepsilon A_i$. To verify that $\left|\prod_{i\varepsilon I}\right| A_i$ is an l-group is routine.

Definition: G is l-isomorphic to a subdirect product of l-groups $\{A_i;\ i\varepsilon I\}$ if and only if there is an l-group monomorphism $k:G \longrightarrow A=\left|\prod_{i\varepsilon I}\right| A_i$ such that $k\pi_i$ is an epimorphism for all $i\varepsilon I$, where $\pi_i : A \longrightarrow A_i$ is the i^{th} projection.

Corollary 1.17: Every abelian l-group is a subdirect product of abelian o-groups.

Proof: Let C_g be a maximal convex l-subgroup with respect to not containing $g\varepsilon G$, then $C_g \Delta G$ and $G/C_g = R(C_g)$. Hence G/C_g is an abelian o-group. Define $f:G \longrightarrow R=\left|\prod_{o\neq g\varepsilon G}\right| R(C_g)$ by $xf=(\ldots,\ C_g+x,\ \ldots)$, for every $x\varepsilon G$.

Clearly f is an l-homomorphism. Furthermore, $o\neq g\varepsilon G$, then $C_g+g\neq C_g$. Hence $g\notin \ker f$. It follows that f is one to one. Moreover fp_g is an epimorphism for all $g\neq o\varepsilon G$, where p_g is the projection from R onto $R(C_g)$. Consequently, G is a subdirect product of G/C_g, for every $o\neq g\varepsilon G$.

Definition: Any convex l-subgroup of an l-group G which satisfies one of the conditions in Theorem 1.10 is called prime.

Corollary 1.18: If A and B are primes, then $A\cap B$ is prime, if and only if A and B are comparable.

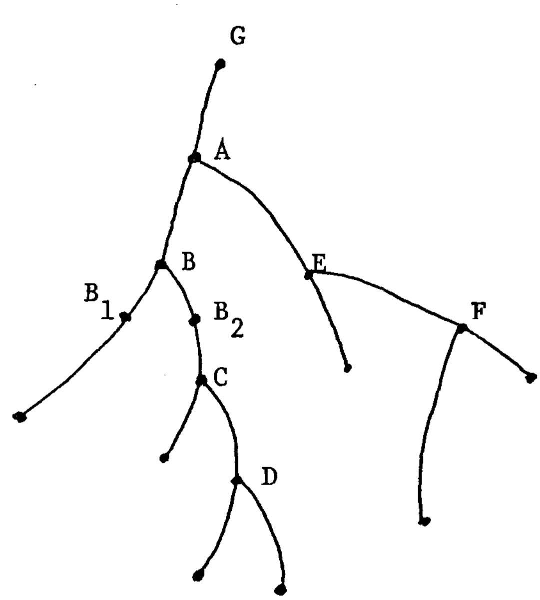
Proof: $A\cap B$ is prime, then if $A \supsetneq A\cap B$ and $B \supsetneq A\cap B$, then $A\cap B \supsetneq A\cap B$, by (ii) of Theorem 1.16, a contradiction. Therefore, $A=A\cap B$ or $B=A\cap B$. Hence $A\supset B$ or $B\subset A$. Conversely, if $A\supset B$ or $B\subset A$, then $A\cap B=B$or A which is prime.

Corollary 1.19: If the convex l-subgroup M is regular, then M is prime.

Proof: By (vii) of Theorem 1.16.

Corollary 1.20: If A, B, C are regular subgroups with respect to not containing some $g\neq o\varepsilon G$, and A, B, C are distinct, then $A\cap B \not\subseteq C$.

Remark: From Corollary 1.18, we know that the p. o. set of prime subgroups looks like the roots of a tree in picture, we call it a root system. The following diagram is a root system.



where B_1 and B_2 are regular, and B covers B_1, B_2. That is there is no prime subgroup between B, B_1 and B, B_2 in the diagram.

Proposition 1.21 For any prime subgroup A of an l-group G, there exists a minimal prime subgroup M with $M \subseteq A$.

Proof: Let $B = \bigcap_{i \in I} A_i$, where $\{A_i : i \in I\}$ is a chain of prime subgroups. Then if $a, b \in G^+ \setminus B$ we have $a, b \in G^+ \setminus A_i$ for some i. Hence $a \wedge b \in G^+ \setminus A_i$. That is $a \wedge b \notin A_i$. Hence $a \wedge b \notin B$. Consequently $a \wedge b \in G^+ \setminus B$. That is, every chain of prime subgroups is bounded below. Hence by Zorn's Lemma, for any prime subgroup A, there exists a minimal prime M with $M \subseteq A$.

Example 1.22: Let G be the stablizer of "x" in A(X), the l-group of all antomorphisms of totally ordered set X. Then if $a, b \in A(X)^+ \setminus G$, we have $xa \neq x \neq xb$. Hence $x(a \wedge b) = xa \wedge xb = xa$ or $xb \neq x$, since X is totally ordered. That is $a \wedge b \neq i$. Therefore G is prime.

We claim that if X is actually the set of real numbers then G is also regular. Now suppose $g \notin G$, then $xg \neq x$. For any $x \neq a \in X$, we have

(i) If $x > xg > a$, then there exists $g_0 \in G$ such that $xgg_0 = a$.

(ii) If $x > a > xg$, then there exists $g_0 \in G$, such that $xgg_0 = a$.

(iii) If $a > x > xg$, then either $a > xg^{-1} > x$ or $xg^{-1} > a > x$.

In either case, there exists $g_0 \in G$, such that $xg^{-1}g_0 = a$.

(iv) Note that the remaining case is $xg > x > a$. In this case, we have either $x > xg^{-1} > a$ or $x > a > xg^{-1}$. Hence, there exists $g_0 \in G$, such that $xg^{-1}g_0 = a$. Therefore, $A(X) = G \bigcup_{g_0 \in G} gg_0G \bigcup_{g_0 \in G} g^{-1}g_0G$

$$= G \cup gG \cup g^{-1}G \subset \langle G, g \rangle$$

That is, $A(X) = \langle G, g \rangle$. Consequently, G is regular.

Remark: Let $g \varepsilon G$, $f \varepsilon A(X) \setminus G$. Then $xf \neq x$. Hence $xfg \neq x$. Therefore $xfgf^{-1} \neq x$. That is $fgf^{-1} \notin G$. Hence G is not normal.

Proposition 1.23. If every minimal prime subgroup of G is normal, then G is a subdirect product of o-groups.

Pf: The proof is similar to that of Corollary 1.17. Note that for every $g \varepsilon G$, there exists regular subgroup C_g of G withrespect to not containing g. Hence there exists a minimal prime subgroup $M_g \subseteq C_g$ and $M+g \neq M$.

Lemma 1.24: If G is an l-group, the following are equivalent:

(1) G is totally ordered.

(2) Every convex l-subgroup of G is prime.

(3) The set of all prime subgroups of G, $\Gamma(G)$ is totally ordered.

Pf: Suppose G is totally ordered and M is a convex l-subgroup of G. Let $a,b \varepsilon G^{+} \setminus M$. Then $a \wedge b = a$ or b. Hence $a \wedge b \varepsilon G^{+} \setminus M$. Therefore M is prime. Conversely, if every convex l-subgroup of G is prime, then $\{0\}$ is prime, implies G is totally ordered. This converse also implies $\Gamma(G)$ is totally ordered. Now suppose $M' \neq \{0\}$ is a minimal prime subgroup of G, let C'_g be regular in M' with respect to not containing g, $g \varepsilon M'$. Then if C'_g is not regular in G, we have C_g regular in G with respect to not containing g, hence $C_g \supset M'$, contradiction. That is C'_g is regular in G, consequently, C'_g is prime, contradiction. Therefore $M' = \{0\}$. Hence every convex l-subgroup of G is prime.

Example 1.25: Let $H = \prod_{r \varepsilon R} Z_r$, $Z_r = Z$. Let G ={ $h \varepsilon H$, such that the support of h satisfies asscending chain condition $\} \cup \{0\}$.

Let $g,h \varepsilon G$, $h \neq g$, $D = \{r;\ rg \neq rh\} \subseteq$ Support of $g \cup$ Support of h

Clearly, Support of $g \cup$ Support of h satisfies asscending chain condition (A.C.C.). Hence D satisfies A.C.C.

Therefore D has a maximum member r. Then define $g>h$ if and only if $rg>rh$. Now $0\in G$. Let $f,g\in G$, then $D = \{r, rg \neq rf\}$ satisfies A.C.C. we have $f-g\in G$. G is a group. Furthermore

(i) $g\geq g$ for every $g\in G$. Reflexive.

(ii) Suppose $g\geq h$, $h\geq g$, then it is immediate that $g = h$ Antisymmetric.

(iii) Suppose $g>h$ and $h>f$, then $r_1 g>r_1 h$ and $r_2 h>r_2 f$ where $r_1 = \max\{r; rg \neq rh\}$ $r_2 = \max\{r; rh \neq rf\}$

Let $r_3 = \max\{r; rf \neq rg\}$. To show that $r_3 = \max\{r_1,r_2\}$ is routine. Hence $g>f$.

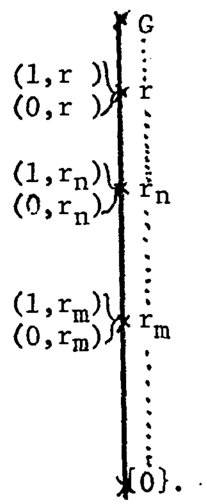
Consider any two elements of G which are comparable, that is $g\geq h$ and $g,h\in G$, let $r_0 = \max\{r; rg \neq rh\}$. Then $r_0 g\geq r_0 h$, and note that $r_0 = \max\{r; r(f+g+k) \neq r(f+g+k)\}$ for every $f,k\in G$. Hence

$$\begin{aligned} r_0(f+g+k) &= r_0 f+r_0 g+r_0 k \\ &\geq r_0 f+r_0 h+r_0 k \\ &= r_0(f+g+k) \end{aligned}$$

Consequently $f+g+k \geq f+h+k$. Therefore, G is totally ordered abelian group. This implies $\Gamma(G)$ is totally ordered by Lemma 1.24.

Now we are going to discuss $\Gamma(G)$. For every $r\in R$, let $G_r = \{h;$ max. element of support of $h<r, h\in G\} \cup \{0\}$. Then $0\in G_r$, and if $f,g\in G_r$, clearly $f-g\in G_r$ and $f\vee g$, $f\wedge g\in G_r$. If $f\leq g_1\leq g$, then $g_1\in G_r$. Consequently, Gr is a regular subgroup of G with respect to not containing k where k is any element whose max. element of support is r. On the other hand $r_1\geq r_2$ if and only if $G_{r_1}\geq G_{r_2}$ under inclusion. Conversely, if C_g is regular with respect to not containing g, let r be the max. element of support of g. Then for every element f in C_g, the max. element of support of f is less than r. Hence we have thus shown that there

exists a one-one order preserving mapping from R onto the set of all regular subgroup of G. Let (0,r) denote Gr and (1,r) denote the intersection of all regular subgroups $Gr' \supsetneqq Gr$. Then (1,r')is prime and $(1,r)>(o,r)$ for every r. Hence $\Gamma(G) = \{(i,r),\ i = 0,1\} \cup \{G,\{0\}\}$. where $(i_1,r_1)>(i_2,r_2)$ if and only if $r_1>r_2$ or $r_1=r_2$ and $i_1>i_2$ and $G>(i,r)$, $(i,r)>\{0\}$ for every i, r. The picture of $\Gamma(G)$ is as follow:



We have thus shown that G contains prime subgroups that are not regular such as (1,r') for every $r' \varepsilon R$.

Chapter Two

Lemma 2.1: Let C be a convex l-subgroups of an l-group G, and let $0\leq a\epsilon G$. Define $C^*(a)=\{x\epsilon G: a\wedge|x|\epsilon C\}$. Then $C^*(a)$ is a convex l-subgroup of G and $C\subseteq C^*(a)$.

Proof: $a\wedge|0|=0\epsilon C$. Hence $0\epsilon C^*(a)$. Consider $d_1,d_2\epsilon C^*(a)$, then $0\leq a\wedge|d_1-d_2|\leq a\wedge(|d_1|+|-d_2|)\leq(a\wedge|d_1|)+(a\wedge|-d_2|)\epsilon C$. Hence $a\wedge|d_1-d_2|\epsilon C$. That is, $d_1-d_2\epsilon C^*(a)$. Consequently $C^*(a)$ is a subgroup. Furthermore, $(a\wedge|d_1|)\vee(a\wedge|d_2|)=a\wedge(|d_1|\vee|d_2|)\geq a\wedge|d_1\vee d_2|\geq 0$, implies $a\wedge|d_1\vee d_2|\epsilon C$. Hence $d_1\vee d_2\epsilon C^*(a)$. Therefore $d_1\wedge d_2\epsilon C^*(a)$ dually. Consequently, $C^*(a)$ is a sublattice. Now if $d\geq 0\epsilon C^*(a)$ and $x\epsilon G$ such that $d\geq x\geq 0$, then $a\wedge|d|\geq a\wedge|x|\geq 0$ implies $a\wedge|x|\epsilon C$, that is $x\epsilon C^*(a)$. Hence $C^*(a)$ is a convex l-subgroup. Finally, if $g\epsilon C$, then $|g|\epsilon C$, but $0\leq a\wedge|g|\leq|g|$. Hence $a\wedge|g|\epsilon C$, and therefore $g\epsilon C^*(a)$. Consequently $C\subseteq C^*(a)$.

Lemma 2.2: Let C be a convex subgroup of l-group G, and suppose R(C) is totally ordered. Then each $g\epsilon G$ induces an automorphism $\beta(g,C)$ of R(C) defined by $(C+x)\beta(g,C)=C+x+g$.

Proof: Straight-forward.

If C is a convex l-subgroup of G and if R(C) is totally ordered, we let AR(C) denote the l-group of all automorphisms of R(C).

Lemma 2.3: If C is a convex l-subgroup of G and if R(C) is totally ordered, then the mapping $\alpha(C):G\longrightarrow AR(C)$ defined by $g\alpha(C)=\beta(g,C)$ is an l-group homomorphism of G onto a transitive l-subgroup of AR(C).

Proof: The only non-trivial part of the proof is to show that the lattice operation are preserved. We must show that $(g\vee 0)\alpha(C)=\beta(g,C)\vee i$, where i denotes the identity function in AR(C). In other words, if we can show that for any right coset C+x, $(C+x)\star(g\vee 0)=(C+x+g)\vee(C+x)$, we are done. But this follows immediately from Lemma 1.6.

Example 2.4: Let $G=R\boxplus R$, then if H is a convex l-subgroup of G,then $H\varepsilon\{\{0\}\boxplus\{0\}, R\boxplus\{0\}, \{0\}\boxplus R, R\boxplus R\}$. Note that $R\boxplus R\supset R\boxplus\{0\}\supset\{0\}\boxplus\{0\}$ and $R\boxplus R\supset\{0\}\boxplus R\supset\{0\}\boxplus\{0\}$. Moreover $R\boxplus\{0\}$ and $\{0\}\boxplus R$ are maximal convex l-subgroups with respect to not containing (0,a),(b,0), respectively, where $a\neq 0\neq b$.

Theorem 2.5(Holland [6]): If G is an l-group, then G is l-isomorphic to a subdirect product of l-groups $\{B_g: 0\neq g\varepsilon G\}$ such that each B_g is a transitive l-subgroup of the l-group of automorphisms of a totally ordered set S_g.

Proof: For each $0\neq g\varepsilon G$, by Zorn's Lemma, there exists a convex l-subgroup C_g of G which is maximal with respect to not containing g. By Theorem 1.16 and Corollary 1.19, the set of convex l-subgroups of G which contains C_g form a tower. Let $S_g=R(C_g)$. Then by Theorem 1.16, S_g is totally ordered. By Lemma 2.3, the mapping $\alpha_g: G\longrightarrow A(S_g)$, where $(C_g+x)\alpha_g(k)=C_g+x+k, k\varepsilon G$,is an l-homomorphism of G onto a transitive l-subgroup B_g of $A(S_g)$. Let $\sigma: G\longrightarrow B=\prod_{\substack{g\neq 0\\ g\varepsilon G}} B_g$ defined by $\sigma(k)_g=\alpha_g(k);k\varepsilon G$. Clearly,since each α_g is an l-homomorphism,σ is an l-homomorphism. Furthermore, if $0\neq g\varepsilon G$, then $C_g+g\neq C_g$. Hence $g\notin Ker\alpha_g$, that is $g\notin\bigcap_{0\neq x\varepsilon G} Ker\alpha_x=Ker\sigma$. Consequently σ is one to one. Note that $\sigma\pi_g$ is an epimorphism for all $g\neq 0\varepsilon G$, where $\pi_g: B\longrightarrow B_g$ is a projection. Hence the proof is complete.

Example 2.6: The maximal convex l-subgroups of $Z\boxplus Z$ without (a,b),where $a\neq 0$or $b\neq 0$ are $\{0\}\boxplus Z$, $Z\boxplus\{0\}$. A transitive l-subgroup B_1,of $AR(\{0\}\boxplus Z)$ is $\{\alpha_z: z\varepsilon Z, (\{0\}\boxplus Z)\alpha_z=\{z\}\boxplus Z\}$ and a transitive l-group B_2 of $AR(Z\boxplus\{0\})$ is $\{\beta_z: z\varepsilon Z, (Z\boxplus\{0\})\beta_z=Z\boxplus\{z\}\}$. Define $f: Z\boxplus Z\longrightarrow B_1\times B_2$ by

$(z_1,z_2)f=(\alpha_{z_1},\beta_{z_2})$. Then f is an l-monomorphism and $f\pi_i$ is an epimorphism, where π_i is the i^{th} projection of $B_1 \times B_2$.

Example 2.7: The convex l-subgroup of $(Z \boxplus Z)\overleftarrow{\times} Z$ are;

(i) $\{0\} \boxplus \{0\} \times \{0\}$ (ii) $(Z \boxplus \{0\}) \times \{0\}$

(iii) $(\{0\} \boxplus Z) \times \{0\}$ (iv) $(Z \boxplus Z) \times \{0\}$

(v) $(Z \boxplus Z)\overleftarrow{\times} Z$

The regular subgroups are;

(i) $Z \boxplus Z \times \{0\}$ (ii) $\{0\} \boxplus Z \times \{0\}$

(iii) $Z \boxplus \{0\} \times \{0\}$

A transitive l-subgroup B_1 of $AR(\{0\} \boxplus Z \times \{0\})$ is $\{\alpha_{z_1 z_3} : z_1, z_2 \varepsilon Z, (\{0\} \boxplus Z \times \{0\})\alpha_{z_1 z_2} = (\{z_1\} \boxplus Z)\overleftarrow{\times}\{z_3\}\}$, which is o-isomorphic to $Z\overleftarrow{\times}Z$, and B_2 of $AR(Z \boxplus \{0\} \times \{0\})$ is $\{\beta_{z_2 z_3} : z_2, z_3 \varepsilon Z, (Z \boxplus \{0\} \times \{0\})\beta_{z_2 z_3} = (Z \boxplus \{z_2\})\overleftarrow{\times}\{z_3\}\}$, which is also o-isomorphic to $Z\overleftarrow{\times}Z$, and B_3 of $AR(Z \boxplus Z \times \{0\})$ is $\{\gamma_{z_3} : z_3 \varepsilon Z, (Z \boxplus Z \times \{0\})\gamma_{z_3} = (Z \boxplus Z)\overleftarrow{\times}\{z_3\}\}$ which is o-isomorphic to Z.

Define $f : (Z \boxplus Z)\overleftarrow{\times} Z \longrightarrow (B_1 \boxplus B_2)\overleftarrow{\times} B_3$ by $f(z_1,z_2,z_3)=(\alpha_{z_1 z_3},\beta_{z_2 z_3},\gamma_{z_3})$.

Clearly $Z \boxplus Z\overleftarrow{\times}Z$ is l-isomorphic to a subdirect product of B_1,B_2,B_3.

If H is the direct product of l-groups B_α and if B_α is the l-group of automorphisms of an ordered set S_α where $S_\alpha \cap S_\beta = \emptyset$ for $\alpha \neq \beta$, then we may totally order the set $\bigcup S_\alpha$ as follows: first order the collection of sets S_α in any way; for example, it may be well-ordered. Then for $x,y \varepsilon \bigcup S_\alpha$, let $x<y$ if $x,y \varepsilon S_\alpha$ and $x<y$ as elements of S_α, or if $x \varepsilon S_\alpha$ and $y \varepsilon S_\beta$ where

$S_\alpha < S_\beta$. If $\Phi \varepsilon H$, then Φ induces an automorphism of the ordered set $\bigcup S_\alpha$ in the following way: $x\Phi' = x\Phi_\alpha$, where $x \varepsilon S_\alpha$ and Φ_α is the α^{th} component of Φ. From this and Theorem 2.5 we have the following theorem.

Theorem 2.8: If G is an l-group, G is l-isomorphic to an l-subgroup of the l-group of all automorphisms of an ordered set.

Definition: By an l-ideal of an l-group G is meant a normal subgroup of G provided it contains any a, then also all x with $|x| \leq |a|$.

Theorem 2.9: An l-group G is l-isomorphic to a transitive l-subgroup of the l-group of all automorphisms of an ordered set if and only if there exists a convex l-subgroup C of G such that;

(1) the set of convex l-subgroups of G containing C is totally ordered under inclusion, and

(2) the only l-ideal of G contained in C is $\{0\}$.

Proof: If G is a transitive l-subgroup of the l-group of automorphisms of an ordered set L, and if $x \varepsilon L$, then $C = \{g \varepsilon G: xg = x\}$ is clearly a convex l-subgroup of G. C contains no l-ideals of G. For if $0 \neq g \varepsilon C$, then $yg \neq y$, for some $y \varepsilon L$. Hence, as G is transitive, there exists $f \varepsilon G$ such that $xf = y$. Therefore $xfgf^{-1} = ygf^{-1} \neq yf^{-1} = x$ and so $fgf^{-1} \notin C$. The convex l-subgroups of G containing C form a tower, for otherwise, by Theorem 1.16, there exists $a, b \notin C$, such that $a \wedge b = 1$. That is, $xa \neq x \neq xb$, and yet $x = x1 = x(a \wedge b) = xa \wedge xb$, which is impossible since L is totally ordered.

Conversely, if C is such a subgroup of G, then by Theorem 1.16, R(C) is totally ordered, and by Lemma 2.3, the mapping $\alpha(C)$ is an l-homomorphism of G onto a transitive subgroup of AR(C). If g is in the kernel of $\alpha(C)$, then $C+g=C$; thus the kernel is contained in C. As the kernel is an l-ideal of G, the kernel is $\{0\}$ and $\alpha(C)$ is one-to-one.

Corollary 2.10: If there exists an l-ideal $K \neq \{0\}$ of G such that every l-ideal ($\neq \{0\}$) of G contains K, then G is a transitive l-group of automorphisms of an ordered set.

Proof: Let $0 \neq g \in K$, and let C_g be a convex l-subgroup of G maximal without g. Then C_g satisfies conclusions (1) and (2) of the above theorem.

Corollary 2.11: A simple l-group (without proper l-ideal) is a transitive l-group of automorphisms of an ordered set.

Corollary 2.12: If G is abelian and is a transitive l-group of automorphisms of an ordered set, then G is totally ordered.

Proof: Any such C in Theorem 2.8 is an l-ideal. Hence $C=\{0\}$, and G is isomorphic as an ordered set to $R(C)$, which is totally ordered.

Chapter Three

Let $S(a_1,a_2,\ldots\ldots,a_n)$ denote the normal subsemigroup of a group G that is generated by the elements $a_1,a_2,\ldots\ldots,a_n(\varepsilon G)$, and define $S'(a_1,a_2,\ldots\ldots,a_n)$ as $S(a_1,a_2,\ldots\ldots,a_n)$ with 0 adjoined. These normal subsemigroups will play an important role in dealing with extensions partial orders P, that is, for some partial order "$\leq$" on G, $P=\{g\geq 0,\ g\varepsilon G\}$. This is due to the fact that they obey the following rules:

(a) $a\varepsilon P$ implies $S'(a)\subseteq P$;

(b) $a\varepsilon P$, $a\neq 0$, implies $P\cap S(-a)=\phi$;

(c) $S'(a_1,a_2,\ldots\ldots,a_n)=S'(a_1)+S'(a_2)+\ldots\ldots+S'(a_n)$;

(d) $-S(a_1,a_2,\ldots\ldots,a_n)=S(-a_1,-a_2,\ldots\ldots,-a_n)$.

The next result has numerous consequences.

Theorem 3.1 [Fuchs (5)]. A partial order P of a group G can be extended to a full order of G, if and only if, it has the property:

(*) for every finite set of elements $a_1,a_2,\ldots\ldots,a_n$ in G, $(a_i\neq 0)$, the signs $\varepsilon_1,\varepsilon_2,\varepsilon_3,\ldots\ldots,\varepsilon_n(\varepsilon_i=0,\text{or}1)$ can be chosen such that $P\cap S((-1)^{\varepsilon_1}a_1,(-1)^{\varepsilon_2}a_2,\ldots\ldots,(-1)^{\varepsilon_n}a_n)=\phi$.

Proof: If P can be extended to a full order Q, then let ε_i be chosen such that $-(-1)^{\varepsilon_i}a_i\varepsilon Q$. Now $-S((-1)^{\varepsilon_1}a_1,(-1)^{\varepsilon_2}a_2,\ldots\ldots,(-1)^{\varepsilon_n}a_n)$ $=S(-(-1)^{\varepsilon_1}a_1,-(-1)^{\varepsilon_2}a_2,\ldots\ldots,-(-1)^{\varepsilon_n}a_n)\subseteq Q$, and so $P\cap S((-1)^{\varepsilon_1}a_1,(-1)^{\varepsilon_2}a_2,\ldots\ldots,(-1)^{\varepsilon_n}a_n)\subseteq Q\cap S((-1)^{\varepsilon_1}a_1,(-1)^{\varepsilon_2}a_2,\ldots,(-1)^{\varepsilon_n}a_n)$ $=\phi$

For the proof of the sufficiency we need the following lemma.

Lemma 3.2. If P satisfies (*) and $a\varepsilon G$, then either $P+S'(a)$ or $P+S'(-a)$

defines a partial order P' in G which again satisfies (*).

Proof: Suppose that G contains elements $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots b_m$ ($\neq 0$) such that for every choice of the signs ε_i, η_j one has $P \cap S(a, (-1)^{\varepsilon_1}a_1, (-1)^{\varepsilon_2}a_2, \ldots, (-1)^{\varepsilon_n}a_n) \neq \phi$, and $P \cap S(-a, (-1)^{\eta_1}b_1, (-1)^{\eta_2}b_2, \ldots, (-1)^{\eta_m}b_m) \neq \phi$, then the intersection of P with $S((-1)^{\varepsilon}a, (-1)^{\varepsilon_1}a_1, \ldots, (-1)^{\varepsilon_n}a_n, (-1)^{\eta_1}b_1, (-1)^{\eta_2}b_2, \ldots, (-1)^{\eta_m}b_m)$ is never void, contrary to (*). Thus either (i) to every finite set $a_1, a_2, \ldots, a_n (\neq 0)$ in G there are signs $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ such that $P \cap S(a, (-1)^{\varepsilon_1}a_1, (-1)^{\varepsilon_2}a_2, \ldots, (-1)^{\varepsilon_n}a_n) = \phi$; we then put $P' = P + S'(-a)$; or (ii) to every finite set $a_1, a_2, \ldots, a_n$ ($\neq 0$) in G there are signs $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ such that $P \cap S(-a, (-1)^{\varepsilon_1}a_1, (-1)^{\varepsilon_2}a_2, \ldots, (-1)^{\varepsilon_n}a_n) = \phi$; in this case we put $P' = P + S'(a)$. (If both (i) and (ii) are true, we can choose either.) Now in case (i) for example, P' is evidently a normal subsemigroup with "0", which moreover satisfies (*); for $(P+S'(-a)) \cap S((-1)^{\varepsilon_1}a_1, (-1)^{\varepsilon_2}a_2, \ldots, (-1)^{\varepsilon_n}a_n) \neq \phi$ implies $P \cap S(a, (-1)^{\varepsilon_1}a_1, (-1)^{\varepsilon_2}a_2, \ldots, (-1)^{\varepsilon_n}a_n) \neq \phi$.

Property (*) of P' shows that, for all $b(\neq 0)$ in G, $P' \cap S((-1)^{\varepsilon}b) = \phi$ for $\varepsilon = 0$ or 1, that is, either $b \notin P'$ or $-b \notin P'$. Thus P' is a partial order of G.

To complete the proof of Theorem 3.1, let Q be a maximal element in the set B of all partial orders of G which are extensions of P and satisfies (*). Such Q exists, by Zorn Lemma, for (*) is satisfied by union of an ascending chain of partial orders provided it is satisfied by the member of the chain. By the Lemma 3.2, for every $a \varepsilon G$, either $Q + S'(a)$ or $Q + S'(-a)$ also belong to B. Therefore $Q + S'(a)$ or $Q + S'(-a)$ coincides with Q, that is $a \varepsilon Q$ or $-a \varepsilon Q$, proving that Q defines a full order on G.

Our main concern now is with the group admitting a linear order. Following Neumann [5] we shall call these groups O-groups (orderable groups). A necessary and sufficient condition for having this property can read directly from Theorem 3.1.

Theorem 3.3 [Los', Ohnishi (5)]. A group G is an O-group if, and only if, given $a_1, a_2, \ldots, a_n$ in G with $a_i \neq 0$, for at least one choice of the sign $\varepsilon_i = 0$ or 1, one has $0 \notin S((-1)^{\varepsilon_1}a_1, (-1)^{\varepsilon_2}a_2, \ldots, (-1)^{\varepsilon_n}a_n)$.

In a group G, the intersection of the 2^n subsemigroups $S((-1)^{\varepsilon_1}a_1, (-1)^{\varepsilon_2}a_2, \ldots, (-1)^{\varepsilon_n}a_n)$ with fixed $a_1, a_2, \ldots, a_n$ and varing the signs $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ is either a subgroup or void, therefore another formulation of the Theorem 3.3 is

Theorem 3.4 [Lorenzen(5)]. A necessary and sufficient condition for a group G to be an O-group is that, for every finite set $a_1, a_2, \ldots, a_n$ in G $(a_i \neq 0)$, the intersection of the 2^n subsemigroup $S((-1)^{\varepsilon_1}a_1, (-1)^{\varepsilon_2}a_2, \ldots, (-1)^{\varepsilon_n}a_n)$ taken from all choices of signs $\varepsilon_i = 0$ or 1 is void.

Corollary 3.5 [Neumann (5)]. In order that G be an O-group it is necessary and sufficient that every finitely generated subgroup of G be an O-group.

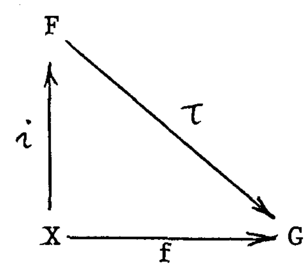
Assume that H is a finitely generated abelian group. If H is an O-group, then it must be torsion-free. If it is torsion-free, then it is a direct sum of n copies of its generators, hence it can be given a lexicographic order, that is, it is an O-group.

Corollary 3.5 implies

Corollary 3.6 [Levi (5)] An abelian group is an 0-group if and only if it is torsion free.

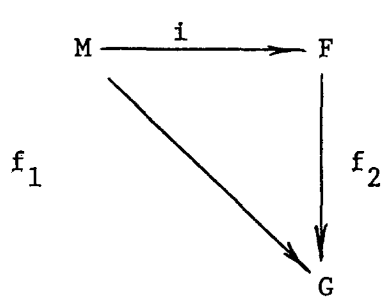
Definition: F is free abelian group on$\{x_k\}$ in case F is a direct sum of infinite cyclic group Z_k, where $Z_k=[x_k]$.

Definition: Let X be a set and F a group containing X; F is free on X if, for every group G, every function $f:X\longrightarrow G$ has a unique extension to a homomorphism of F into G.

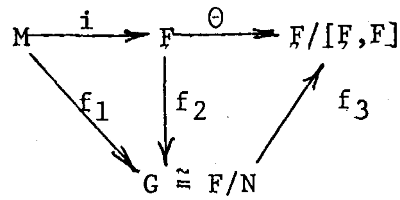


Proposition 3.7 For every free group F, F/[F,F] is a free abelian group.

Pf: Let M be the free set of generators of free group F and free abelian group G, that is



commutes, where f_2 is the unique homomorphism from F to G such that $xif_2=xf_1$, for every $x\varepsilon M$. Since F and G have the same free set of generators ,it is immediate that $f_2$is onto. Therefore there exists $N\triangleleft F$, such that $F/N\simeq G$. We know that $x+y+N=y+x+N$, for every $x,y\varepsilon F$. Hence $-x-y+x+y+N=N$, so $-x-y+x+y\varepsilon N$, for every $x,y\varepsilon F$. Therefore $[F,F]\subseteq N$. Now consider the diagram



where Θ is the natural mapping, and f_3 is a homomorphism from G to F/[F,F], such that $gf_2f_3 = (g+N)f_3 = g+[F,F]$ for every $g\varepsilon F$, and $g\notin N$.
Note that $mf_1 = mif_2$, $mf_1f_3 = mi\Theta$. Hence $mif_2f_3 = mf_1f_3 = mi\Theta$, that is $gf_2f_3 = g\Theta$ for every $g\varepsilon F$. Consequently $f_2f_3 = \Theta$. Therefore $[F,F] =$ Ker $\Theta =$ Ker $f_2f_3 \supseteq N$. Hence $[F,F] \geq N$, that is $[F,F] = N$.

Recall that the members of the lower central series of group G are defined by $G_0 = G$, $G_{n+1} = [G,G_n]$.
Theorem 3.8: Let G be a free group, for each total order of the free abelian group G/[G,G], there is a total order of G so that the natural map of G onto G/[G,G] is an o-homomorphism.
Proof: Note that all the factor G_n/G_{n+1} , $n = 0,1,2,\cdots$ of the lower central chain of a free group G are free abelian groups [9], and hence G_n/G_{n+1} is an O-group. Thus by corollary 3.6, there exists a total order P_n on G_n/G_{n+1}. For $0 \neq g \varepsilon G$, if n is the integer defined by $g\varepsilon G_n \setminus G_{n+1}$ (such an integer exists, because by a theorem of Magnns-Witt[8], the lower central series $\{G_n\}$ of a free group G is such that $\bigcap_{n=1}^{w} G_n = \{ 0 \}$, where w denotes the first infinite ordinal), let P_0 be the given order on G/[G,G], we define P in G, $P_0 \subset P$ and P consists of 0 and all such $g\varepsilon G$, where $g+G_{n+1} \ \varepsilon \ P_n$. Then for each $g\varepsilon G$, either $g\varepsilon P$ or $-g\varepsilon P$, but not

both unless $g = 0$. If $g,h \in P$ and m,n are the integers with $g \in G_m \smallsetminus G_{m+1}$ and $h \in G_n \smallsetminus G_{n+1}$, then $g+G_{m+1} \in P_m$, $h+G_{n+1} \in P_n$. Without loss of generality, we assume that $m \geq n$. Then we have $g+h \in G_n \smallsetminus G_{n+1}$ and $g+h+G_{n+1} \in P_n$. Hence $g+h \in P$. In the same way, we also get $h+g \in P$. Finally, if g is as before and $x \in G$ is arbitrary, then $-x+g+x = g+[g,x] \in g+G_{n+1}$, that is $-x+g+x$ again belongs to P. Consequently, P defines a full order on G.

Now consider the natural map f from G onto $G/[G,G]$ under order P. Then if $x,y \in G$ and $x>y$, then $-y+x>0$. Therefore there exists an n, such that $-y+x+G_{n+1} \in P_n$. Hence $-y+x+[G,G] \geq [G,G]$, for otherwise if $-y+x+[G,G]<[G,G]$, then $-x+y+[G,G]>[G,G]$, that is $-x+y>0$, in other words, $y>x$, a contradiction. Hence we have $x+[G,G] \geq y+[G,G]$. We have thus shown that f is an o-homomorphism.

Let G be a free group, and $K \Delta G$, let G/K be totally ordered. We define a total order on K as in theorem 3.8, that is, let $G_0, G_1, G_2, \cdots$ be lower central series of free group G, say $k \in P_k$ if $k \in G_n \smallsetminus G_{n+1}$ and $k+G_{n+1} \in P_n$, where P_n is some fixed order on G_n/G_{n+1}. Now define $g>h$ if and only if $g+K>h+K$ in G/K or $gK = hK$ and $-h+g \in P_k$. Then

(i) Let P consists of 0 and all g with the property: $g \in P_k$ or $g+K>K$ in G/K. Then $g \in P$ or $-g \in P$, for every $g \in G$, but not both, unless $g=e$.

(ii) Let $h,g \in P$.

(A) If $h,g \in K$, then $h+g$, $g+h \in P_k$. Hence $h+g$, $g+h \in P$.

(B) Without loss of generality, let $h \in K$ and $g \notin K$, then $g+h+K = g+K>K$. Hence $g+h \in P$ and $h+g+K = g+K>K$ (by normality) Hence $h+g \in P$.

(C) If $h,g \notin K$, without loss of generality, we may assume that $h+K \geq g+K$. Hence $g+h+K \geq g+g+K>g+K>K$, that is $g+h \in P$ and $g+K>K$ implies

$h+g+K>h+K>K$, that is $h+g+K>K$. Hence $h+g\varepsilon P$.

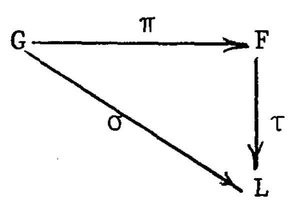
(iii) Let $g\varepsilon P$, for any $x\varepsilon G$, if $g\varepsilon K$ implies $g\varepsilon P_k$ and $x+g-x\varepsilon K$, but in theorem 3.8, $x+g-x$ is a positive element, hence $x+g-x\varepsilon P_k$. That is $x+g-x\varepsilon P$. If $g\notin K$, then $g+K>K$. Hence $x+g-x+K>K$. Consequently $x+g-x\varepsilon P$. We have thus shown that G is totally ordered. Furthermore, $g>h$ implies $g+K\geq h+K$, for otherwise if $g+K<h+K$, then $-h+g+K<K$ implies $-h+g<0$ in P, that is $g<h$ Therefore the natural map of G onto G/K is an o-homomorphism.

Chapter Four

Definition: Let π be an o-isomorphism of a p o-group G into an l-group F. This means that both π and π^{-1} preserve order. Then (F,π) is a free l-group over G if

(i) $G\pi$ is a set of generators of the l-group F (that is, no proper l-subgroup of F contains $G\pi$), and

(ii) if σ is an o-homomorphism of G into an l-group L, then there exists an l-homomorphism τ of F into L such that the diagram



commutes.

Proposition 4.1: In any distributive lattice and hence in any l-group $\vee_I \wedge_J a_{ij} = \wedge_{J^I} \vee_I a_{if(i)}$, for I and J finite sets and dually.

Proof: Let J be arbitrary finite set, we prove the statement by induction on $|I|$. If $|I|=1$, the proposition holds clearly. Suppose it is true if $|I|=n$. Now if $|I|=n+1$, then $\vee_I \wedge_J a_{ij} = (\vee_{I'} \wedge_J a_{ij}) \vee (\wedge_J a_{i_{n+1}j})$, where $I'=I-\{i_{n+1}\}$.

Thus $\vee_I \wedge_J a_{ij} = (\wedge_{J^{I'}} \vee_{I'} a_{if(i)}) \vee (\wedge_{J^{\{i_{n+1}\}}} a_{i_{n+1}f(i_{n+1})})$, by induction

$$= \wedge_{J^{I'}} ((\vee_{I'} a_{if(i)}) \vee (\wedge_{J^{\{i_{n+1}\}}} a_{i_{n+1}f(i_{n+1})}))$$

$$= \wedge_{J^{I'}} (\wedge_{J^{\{i_{n+1}\}}} ((a_{i_{n+1}f(i_{n+1})}) \vee (\vee_{I'} a_{if(i)})))$$

$$= \wedge_{J^{I'} \cup J^{\{i_{n+1}\}}} \vee_I a_{if(i)}$$

$$=\Lambda_J V_I a_{if(i)}$$

Hence, $V_I \Lambda_J a_{ij} = \Lambda_J V_I a_{if(i)}$ and dually.

Proposition 4.2: If S is a subgroup of an l-group L, then $T=\{V_A \Lambda_B S_{\alpha\beta} : S_{\alpha\beta} \varepsilon S, \alpha\varepsilon A, \beta\varepsilon B$, and A and B are finite sets$\}$ is the l-subgroup of L that is generated by S. If S is abelian, then so is T. Now either $S=\{0\}$ or $|S|$ is infinite and so S and T have the same cardinality.

Proof: Clearly, T is a sublattice. On the other hand, $0\varepsilon T$. Let $t_1, t_2 \varepsilon T$, that is, $t_1 = V_A \Lambda_B S_{\alpha\beta}$, $t_2 = V_{A'} \Lambda_{B'} S_{\alpha'\beta'}$. Then $t_1 - t_2 = V_A \Lambda_B S_{\alpha\beta} - V_{A'} \Lambda_{B'} S_{\alpha'\beta'}$. Thus

$$\begin{aligned} t_1 - t_2 &= V_A(\Lambda_B S_{\alpha\beta} - V_{A'} \Lambda_{B'} S_{\alpha'\beta'}) \\ &= V_A(\Lambda_B(S_{\alpha\beta} - V_{A'} \Lambda_{B'} S_{\alpha'\beta'})) \\ &= V_A \Lambda_B(S_{\alpha\beta} + \Lambda_{A'} V_{B'}(-S_{\alpha'\beta'})) \\ &= V_A \Lambda_B \Lambda_{A'}(S_{\alpha\beta} + V_{B'}(-S_{\alpha'\beta'})) \\ &= V_A \Lambda_B \Lambda_{A'} V_{B'}(S_{\alpha\beta} - S_{\alpha'\beta'}) \ \varepsilon\ T, \text{by Proposition 4.1.} \end{aligned}$$

Hence T is a subgroup. Finally for any $x,y\varepsilon T$ and $x \geq y$, we have $t_1+x+t_2 \geq t_1+y+t_2$ in L, for any $t_1, t_2 \varepsilon T$. Hence $t_1+x+t_2 \geq t_1+y+t_2$ in T, for any $t_1, t_2 \varepsilon T$. We have thus shown that T is an l-subgroup of L. Now if S is abelian, then

$$\begin{aligned} V_{A'} \Lambda_{B'} S_{\alpha'\beta'} + V_A \Lambda_B S_{\alpha\beta} &= V_{A'}(\Lambda_{B'} S_{\alpha'\beta'} + V_A \Lambda_B S_{\alpha\beta}) \\ &= V_{A'}(V_A(\Lambda_{B'} S_{\alpha'\beta'} + \Lambda_B S_{\alpha\beta})) \\ &= V_{A'}(V_A(\Lambda_{B'}(S_{\alpha'\beta'} + \Lambda_B S_{\alpha\beta}))) \\ &= V_{A'}(V_A(\Lambda_{B'}(\Lambda_B(S_{\alpha'\beta'} + S_{\alpha\beta})))) \\ &= V_{A'}(V_A(\Lambda_{B'}(\Lambda_B(S_{\alpha\beta} + S_{\alpha'\beta'})))) \\ &= V_{A'}(V_A(\Lambda_B(\Lambda_{B'}(S_{\alpha\beta} + S_{\alpha'\beta'})))) \\ &= V_A(V_{A'}(\Lambda_B(S_{\alpha\beta} + \Lambda_{B'} S_{\alpha'\beta'}))) \\ &= V_A(V_{A'}(\Lambda_B S_{\alpha\beta} + \Lambda_{B'} S_{\alpha'\beta'})) \end{aligned}$$

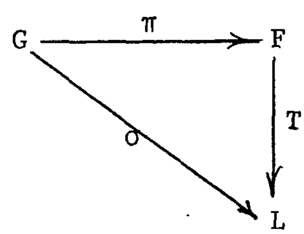
$$= V_A(\Lambda_B s_{\alpha\beta} + V_{A'}\Lambda_{B'} s_{\alpha'\beta'})$$
$$= V_A\Lambda_B s_{\alpha\beta} + V_{A'}\Lambda_{B'} s_{\alpha'\beta'}.$$

Let (F,π) be a free l-group over the p o-group G.

Proposition 4.3: If G is abelian, then so if F. If S is a set of generators for the group G, then $S\pi$ is a set of generators for the l-group F.

Proof: G is abelian, implies F is abelian by Proposition 4.2. Let S be a set of generators of G. Then $S\pi$ is a set of group generators for $G\pi$ and $G\pi$ generates F (as an l-group). Hence $S\pi$ generates F.

Proposition 4.4: If σ is an o-homomorphism of the group G into an l-group L, then there exists a unique l-homomorphism T of F into L such that the diagram

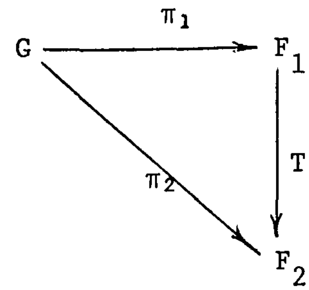


commutes.

Proof: Suppose that T_1 and T_2 are two such l-homomorphisms and consider $f=V_A\Lambda_B(g_{\alpha\beta}\pi)\varepsilon F$, where $\alpha\varepsilon A, \beta\varepsilon B$ and A and B are finite sets. Then

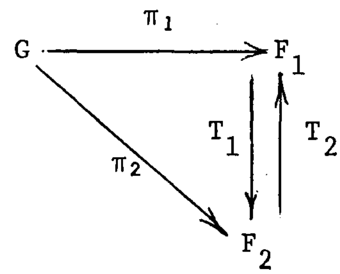
$$fT_1=(V_A\Lambda_B(g_{\alpha\beta}\pi))T_1 = V_A\Lambda_B(g_{\alpha\beta}\pi T_1)= V_A\Lambda_B(g_{\alpha\beta}\sigma)= V_A\Lambda_B g_{\alpha\beta}\pi T_2$$
$$= (V_A\Lambda_B g_{\alpha\beta}\pi)T_2= fT_2$$

Proposition 4.5: If (F_1,π_1) and (F_2,π_2) are free l-groups over the p o-group G, then there exists a unique l-isomorphism of F_1 onto F_2 such that the diagram



commutes.

Proof: By Proposition 4.4, there exist l-homomorphisms T_1 and T_2 such that



commutes.

Consider $f=V_A \Lambda_B g_{\alpha\beta}\pi_1 \varepsilon F_1$ where A and B are finite sets, $\alpha\varepsilon A$, $\beta\varepsilon B$.

$fT_1T_2=(V_A\Lambda_B g_{\alpha\beta}\pi_1 T_1)T_2 = (V_A\Lambda_B g_{\alpha\beta}\pi_2)T_2 = V_A\Lambda_B(g_{\alpha\beta}\pi_2 T_2) = V_A\Lambda_B g_{\alpha\beta}\pi_1 = f.$

Thus T_1T_2 is the identity on F_1 and similarly T_2T_1 is the identity on F_2. Therefore T_1 is an l-isomorphism.

Proposition 4.6: If (F,π) is the free l-group over the trivially ordered free group G and S is a free set of generators for G, then F is a free l-group with $S\pi$ as a free set of generators.

Proof: By Proposition 4.3, $S\pi$ is a set of generators of the l-group F. Let η be a mapping of $S\pi$ into an l-group L. Then $\pi\eta$ is a map of S into L and since S is a free set of generators of G there exists a unique homomorphism σ of G into L such that $s\pi\eta=s\sigma$ for $s\varepsilon S$. Since G is trivially ordered σ is an o-homomorphism and hence there exists a unique l-homomorphism τ of F into L such that $s\pi\eta=s\sigma=s\pi\tau$ for all $s\pi\varepsilon S\pi$.

Proposition 4.7: If L is an l-group, then $L^+=\{g\varepsilon L:g\geq 0\}$ is the intersection of right orders on L, that is, $L^+=\bigcap_{\lambda\in\Omega} P_\lambda$, for some Ω, where P_λ is the set of positive elements of some right order on L.

Proof: By Theorem 2.8, we may assume that L is an l-subgroup of the l-group A(T) of all o-permutations of a totally ordered set T. Let e denote the identity in A(T). Well order T and for each $e\neq\alpha\in A(T)$, let $f(\alpha)$ be the first element in this well ordering such that $f(\alpha)\alpha\neq f(\alpha)$. Define α to be positive if $f(\alpha)\alpha\geq f(\alpha)$. Let $B(T)'$ be the set of all such positive elements then $B(T)^+=B(T)'\cup\{e\}$ is a subset of A(T) satisfying;

(i) $e\in B(T)^+$

(ii) If $x,y\in B(T)^+$, then clearly $f(xy)=\text{Min.}\{f(x),f(y)\}$, since $\text{Min.}\{f(x),f(y)\}$ is moved by xy, and also if $z<f(x)$ and $f(y)$ in this well order, then $zxy=zy=z$.

Therefore

$$f(xy)xy \begin{cases} >f(xy)y=f(xy) & \text{if } f(x)<f(y) \\ =f(xy)y>f(xy) & \text{if } f(x)>f(y) \\ >f(xy)y>f(xy) & \text{if } f(x)=f(y) \end{cases}$$

Hence $xy>e$. That is $xy\in B(T)^+$ and consequently $B(T)^+B(T)^+\subseteq B(T)^+$.

(iii) Clearly, $B(T)^+\cup(B(T)^+)^{-1}\subseteq A(T)$. Now consider $e\neq g\in A(T)$, if $g\in B(T)^+$, fine; if not, then $f(g)g<f(g)$ by hypothesis. But $f(g)=f(g^{-1})$, thus $f(g^{-1})<f(g^{-1})g^{-1}$. Hence $g^{-1}\in B(T)^+$ and therefore $g\in(B(T)^+)^{-1}$, with the consequence that $B(T)^+\cup(B(T)^+)^{-1}=A(T)$.

(iv) Consider $e\neq g\in B(T)^+\cap(B(T)^+)^{-1}$. That is $f(g)g<f(g)$ and $f(g)g>f(g)$. This is impossible, hence $B(T)^+\cap(B(T)^+)^{-1}=\{e\}$. Furthermore, let "$\leq$" be defined on A(T) by $g\leq h$ if and only if $hg^{-1}\in B(T)^+$. Then "$\leq$" is a right total order on A(T) by Theorem 1.1. Finally, let $\alpha\in A(T)^+$, then $t\alpha\geq t$, for all t. Then $\alpha\in B(T)^+$, for every such $B(T)^+$. Hence $\alpha\in\bigcap$ right order of this type on

A(T). That is $A(T)^+ \subseteq \bigcap$ right order of this type on A(T). Now let $\alpha \in \bigcap$ right order of this type on A(T). Suppose there exist $t \in T$ such that $t\alpha < t$, then there exist a right order $B(T)^+$ of this type such that $\alpha \notin B(T)^+$, a contradiction. (For instance, we can let such t to be the first element of T in this particular well order). Hence $t\alpha \geq t$, for every $t \in T$. That is $\alpha \in A(T)^+$. Therefore, $A(T)^+ = \bigcap$ right order of this type on A(T). Thus

$L^+ = L \cap A(T)^+ = L \cap (\bigcap$ right orders of this type on A(T))

$= \bigcap (L \cap$ right orders of this type on A(T))

$= \bigcap$ right orders on L.

Example 4.8: Consider the l-group $G = R \boxplus R$, the cardinal sum of reals. Let $G_{\lambda_1} = R \overleftarrow{\oplus} R$ and $G_{\lambda_2} = R \overrightarrow{\oplus} R$ with lexiographic orders. Then $G^+ = G_{\lambda_1}{}^+ \cap G_{\lambda_2}{}^+$. Furthermore, let $P_{\lambda_3} = G^+ \cup \{(x,y): x \leq 0, y \geq 0, |x| \leq y\} \cup \{(x,y): x \geq 0, y \leq 0, |y| < x\}$. Define a new relation in G, let $a \leq b$ in G if and only if b-a P_{λ_3}. Then G is also a totally ordered group under P_{λ_3}. Note that $G^+ = G_{\lambda_1}{}^+ \cap G_{\lambda_2}{}^+ \cap G_{\lambda_3}{}^+$ holds also.

Proposition 4.9: A p o-group G is o-isomorphic to a subgroup of an l-group if and only if G^+ is the intersection of right orders.

Proof: We may assume that G is a subgroup of an l-group L with the order of G induced by the order of L. We know that $L^+ = \bigcap P_\lambda$, where P_λ are right orders of L. Thus $G^+ = G \cap L^+ = G \cap (\bigcap P_\lambda) = \bigcap (G \cap P_\lambda)$ and each $G \cap P_\lambda$ is a right order for G.

Conversely, if $G^+ = \bigcap P_\lambda$, where P_λ's are right orders for G, let G_λ be the right ordered group under the right order P_λ on G, and let $g \longrightarrow g^\lambda$ be the right regular respresentation of G in $A(G_\lambda)$, which is a one-one homomorphism, where $xg^\lambda = x+g$, for every $x \in G$. If $g \in G^+$, then $x+g \geq x$, for every $x \in G$. Hence $x+g \geq x$ for every x in G_λ and for each λ. Therefore $xg^\lambda \geq x$ for every x in G_λ and for each λ. That is g^λ is positive, for each λ. If each g^λ is positive, then $e \le eg^\lambda = g$ in G ,and so $g \varepsilon \bigcap P_\lambda = G^+$. Thus the map $g \xrightarrow{\pi} (\ldots, g^\lambda, \ldots)$ is an o-isomorphism of G into the l-group $\Pi A(G_\lambda)$.

Proposition 4.10: If $V_A \Lambda_B a_{\alpha\beta} \neq 0$ in the l-group L,where A and B are finite sets,$\alpha\varepsilon A$,$\beta\varepsilon B$,then there is a right order of L which extends the given lattice-order and such that $V_A \Lambda_B a_{\alpha\beta} \neq 0$ in this right o-group L.

Proof: We may,by the Holland embedding theorem,assume that L is an l-subgroup of an l-group A(T) of all o-permutations of a totally ordered set T and $V_A \Lambda_B a_{\alpha\beta} \neq e$ in A(T).

Case I; There exists an α such that $\Lambda_B a_{\alpha\beta} \nleq e$. Then $t < t(\Lambda_B a_{\alpha\beta}) = \text{Min.}\{ta_{\alpha\beta} : \beta\varepsilon B\}$ for some $t\varepsilon T$. Now well order T so that this t is the first element in the well ordering. This determines a right order of A(T),(see the proof of Proposition 4.7),that extends the given lattice order and so that $\Lambda_B a_{\alpha\beta} > e$ and hence $V_A \Lambda_B a_{\alpha\beta} > e$ in the right o-group A(T).

Case II; For each α,$\Lambda_B a_{\alpha\beta} \le e$. Then $V_A \Lambda_B a_{\alpha\beta} < e$,since $V_A \Lambda_B a_{\alpha\beta} \neq e$, and so $t > t(V_A \Lambda_B a_{\alpha\beta})$ for some $t\varepsilon T$. Thus for each α,$t > t(\Lambda_B a_{\alpha\beta}) = \text{Min.}\{ta_{\alpha\beta} : \beta\varepsilon B\}$. Let t be the first element in a well ordering of T. Then in the corresponding right order of A(T) we have $V_A \Lambda_B a_{\alpha\beta} < e$ in the right o-group A(T).

Proposition 4.11: If G is a right o-group,$V_A \Lambda_B a_{\alpha\beta} \neq 0$ in G,where A and B are finite sets,$\alpha\varepsilon A$,$\beta\varepsilon B$, and $g \longrightarrow \tilde{g}$ is the right regular representation of G in A(G),then $V_A \Lambda_B \tilde{a}_{\alpha\beta} \neq 0$ in the l-group A(G).

Proof: Case I; $V_A \Lambda_B a_{\alpha\beta} > 0$ in G. Then for some α,$0 < \Lambda_B a_{\alpha\beta} = \text{Min.}\{a_{\alpha\beta} : \beta\varepsilon B\}$. Thus $0(\Lambda_B \tilde{a}_{\alpha\beta}) = \text{Min.}\{a_{\alpha\beta} : \beta\varepsilon B\} > 0$ and so $0(V_A \Lambda_B \tilde{a}_{\alpha\beta})$ is the largest element in a finite subset of G, where at least one element in this subset is strictly positive. Thus $V_A \Lambda_B \tilde{a}_{\alpha\beta} \neq e$.

Case II; $V_A \Lambda_B a_{\alpha\beta} < 0$ in G. Then for each α,$\Lambda_B a_{\alpha\beta} < 0$ in G, $\text{Min.}\{a_{\alpha\beta} : \beta\varepsilon B\} < 0$.

Thus $0(\vee_A \wedge_B \tilde{a}_{\alpha\beta})$ is the largest element in a finite set of strictly negative element in G and so it is strictly negative, since $\vee_A \wedge_B a_{\alpha\beta} \neq 0$ in G. Therefore $\vee_A \wedge_B \tilde{a}_{\alpha\beta} \neq e$.

Throughout the following let G be a p o-group with $G^+ = \bigcap_{\lambda\in\Omega} P_\lambda$, Ω the set of all λ, such that P_λ is a right order on G, and $P_\lambda \supseteq G^+$. For each $\lambda \in \Omega$ let G_λ be the right o-group (G, P_λ) and let $g \longrightarrow g^\lambda$ be the right regular representation of G as a subgroup of the l-group $A(G_\lambda)$, where $xg^\lambda = x+g$ for all $x \in G$. Let L_λ be the l-subgroup of $A(G_\lambda)$ generated by the image of G under this isomorphism.

Lemma 4.12: If σ is an o-homomorphism of G into an l-group L and $\vee_A \wedge_B a_{\alpha\beta}\sigma \neq 0$ for $\{a_{\alpha\beta} \in G : \alpha \in A, \beta \in B$ and A and B are finite set$\}$, then $\vee_A \wedge_B a_{\alpha\beta}^{\lambda} \neq e$ in some L_λ.

Proof: By Proposition 4.10 we can extend the lattice-order of L to a right order so that $\vee_A \wedge_B a_{\alpha\beta}\sigma \neq 0$ in right o-group L. Now $G/K(\sigma) \simeq$ subgroup of L, where $K(\sigma)$ is the kernel of σ, $K(\sigma)+g \longrightarrow g\sigma$ and this isomorphism defines a right order on $G/K(\sigma)$. Let P_α be one of the right orders of G such that $P_\alpha \supseteq G^+$. Then $K(\sigma) \cap P_\alpha$ is a right order for $K(\sigma)$ that extends the given partial order of $K(\sigma)$. Define $g \in G$ to be positive if $g \notin K(\sigma)$ and $K(\sigma)+g$ is positive in the right o-group $G/K(\sigma)$, or $g \in K(\sigma)$ and g is positive in the right o-group $K(\sigma)$. By using Theorem 1.1, it is not hard to prove that this is a right order for G that extends the given partial order and hence it is one of the P_λ. Also the natural map of G into $G/K(\sigma)$ is an o-homomorphism with respect to this right order P_λ. Thus $K(\sigma) \neq \vee_A \wedge_B (K(\sigma)+a_{\alpha\beta})$ $= K(\sigma) + \vee_A \wedge_B a_{\alpha\beta}$ and hence $\vee_A \wedge_B a_{\alpha\beta} \neq 0$ in G_λ. Now embed G in $A(G_\lambda)$. Then by Proposition 4.11, $\vee_A \wedge_B a_{\alpha\beta}^{\lambda} \neq e$ in L_λ.

Now let π be the natural map of the p o-group G onto the subgroup of long constants of $\Pi L_\lambda \subset \Pi A(G_\lambda), g \longrightarrow (\ldots, g^\lambda, \ldots)$. Then π is an o-isomorphism (see the proof of Proposition 4.8)

Theorem 4.13: The l-subgroup F of ΠL_λ generated by $G\pi$ is the free l-group over the partially ordered group G.

Proof: Suppose that σ is an o-homomorphism of G into an l-group L. Consider $V_A \Lambda_B a_{\alpha\beta} \pi \varepsilon F$ and define $(V_A \Lambda_B (a_{\alpha\beta}\pi))\tau = V_A \Lambda_B (a_{\alpha\beta}\sigma)$. If $V_A \Lambda_B k_{\alpha\beta}\sigma \neq V_C \Lambda_D g_{\gamma\delta}\sigma$, then $0 \neq V_A \Lambda_B k_{\alpha\beta}\sigma - V_C \Lambda_D g_{\gamma\delta}\sigma = V_A \Lambda_B k_{\alpha\beta}\sigma + \Lambda_C V_D(-g_{\gamma\delta}\sigma) = V_A \Lambda_B V_{{}_D C} \Lambda_C (k_{\alpha\beta} - g_{\gamma f(\gamma)})\sigma$

$= V_{A \cup (D^C)^B} \Lambda_{B \cup C} (k_{\alpha\beta} - g_{\gamma q(\beta)})\sigma$. By Lemma 4.12 there exists an L_λ such that in L_λ, $V_{A \cup (D^C)^B} \Lambda_{B \cup C} (k_{\alpha\beta} - g_{\gamma q(\beta)})^\lambda \neq e$. Therefore $V_A \Lambda_B k_{\alpha\beta}^\lambda \neq V_C \Lambda_D g_{\gamma\delta}^\lambda$ in L_λ and hence, $V_A \Lambda_B (k_{\alpha\beta}\pi) \neq V_C \Lambda_D (g_{\gamma\delta}\pi)$. Therefore τ is single-valued. Next consider $k = V_A \Lambda_B k_{\alpha\beta}\pi$ and $g = V_C \Lambda_D g_{\gamma\delta}\pi$ in F. Then $(k-g)\tau = (V_A \Lambda_B k_{\alpha\beta}\pi - V_C \Lambda_D g_{\gamma\delta}\pi)\tau$
$= (V_{A \cup (D^C)^B} \Lambda_{B \cup C} (k_{\alpha\beta} - g_{\gamma q(\beta)})\pi)\tau = V_{A \cup (D^C)^B} \Lambda_{B \cup C} (k_{\alpha\beta} - g_{\gamma q(\beta)})\sigma$

$= V_A \Lambda_B k_{\alpha\beta}\sigma - V_C \Lambda_D g_{\gamma\delta}\sigma = k\tau - g\tau$. Thus τ is a group homomorphism. Furthermore, $(kVg)\tau = (V_A \Lambda_B V_C \Lambda_D (k_{\alpha\beta}\pi V g_{\gamma\delta}\pi))\tau = (V_P \Lambda_Q c_{pq}\pi)\tau = V_P \Lambda_Q c_{pq}\sigma = (V_A \Lambda_B k_{\alpha\beta}\sigma) V (V_C \Lambda_D g_{\gamma\delta}\sigma)$
$= k\tau V g\tau$, where $c_{pq} = k_{\alpha\beta}$ or $g_{\gamma\delta}$. Therefore τ is an l-homomorphism of F into L and $\pi\tau = \sigma$.

Theorem 4.14: For a p o-group G, the following are equivalent;

(1) There exists a free l-group over G.

(2) There exists an o-isomorphism of G into an l-group.

(3) $G^+ = \{g \varepsilon G : g \geq 0\}$ is a intersection of right orders.

Proof: By Proposition 4.9, (2) and (3) are equivalent and clearly (1) implies (2). It follows from Theorem 4.13 that (3) implies (1).

Proposition 4.15: If K is a free l-group with S a free set of generators, then S is a free set of generators for the subgroup [S] of K generated by S.

Proof: Let G be a free group with S as a free set of generators. Let $H=\Pi A(G_\lambda)$ and let $g \longrightarrow (\ldots,g^\lambda,\ldots)$ be the o-isomorphism of G into the long constants of H. Then the l-subgroup F of H generated by $G\pi$ is the free l-group with $S\pi$ as a free set of generators. Clearly, there exists an l-isomorphism τ of F onto K such that $s\pi\tau=s$ for all $s\varepsilon S$ and hence $(G\pi)\tau=[S]$. Since $S\pi$ freely generates $G\pi$ is follows that S freely generates [S].

Proposition 4.16: Let G be a free group with S as a free set of generators and let $H=\Pi A(G_\lambda)$. Let K_n be the l-subgroups of H generated by the long constants from G_n (where $G_0=G, G_{n+1}=[G,G_n]$). Let C_n be a free set of generators for G_n. Then K_n is the free l-group (really $(K_n,\pi|G_n)$) over G_n with $C_n\pi$ as a free set of generators.

Proof: Let $-a-b+a+b\varepsilon[G,G_{n-1}], b\varepsilon G_{n-1}, a\varepsilon G$ and let $x\varepsilon G$. From $x-a-b+a+b-x=(x-a-x)+(x-b-x)+(x+a-x)+(x+b-x)$, we have $G_n \lhd G$. Furthermore, $x+y+[G_{n-1},G_{n-1}]=y+x+[G_{n-1},G_{n-1}]$, for every $x,y\varepsilon G_{n-1}$. Hence $x+y+[G,G_{n-1}]=y+x+[G,G_{n-1}]$, since $[G_{n-1},G_{n-1}]\subset[G,G_{n-1}]$. It can then be shown that $G_{n-1}/[G,G_{n-1}]$ is free abelian, by a theorem of Magnus-Witt.[8]. Now suppose that σ is a homomorphism of $[G,G_{n-1}]$ into an l-group L and $V_A \Lambda_B(a_{\alpha\beta}\sigma)\neq 0$ in L. By Lemma 4.12, there exists a right order for $[G,G_{n-1}]$ so that in the l-group $A(G_n), V_A\Lambda_B(\tilde{a}_{\alpha\beta})\neq e$, where $x \longrightarrow \tilde{x}$ is the right regular representation of G_n in $A(G_n)$. Pick a total order for G_{n-1}/G_n, then the lexicographic extension of the right o-group G_n by the o-group G_{n-1}/G_n is a right order for G_{n-1}. Therefore by induction we get a right order for G(which induces the right order on G_{n-1}), say P_λ. For each $a_{\alpha\beta}\varepsilon G_n, a_{\alpha\beta}^\lambda$ maps G_n onto itself. Thus there is a $t\varepsilon G_n$ such that $tV_A\Lambda_B(a_{\alpha\beta}^\lambda)\neq t$ and so

$\vee_A \wedge_B a_{\alpha\beta}^{\lambda} \neq e$ in the l-subgroup of $P(G_\lambda)$ generated by G_n^λ. Thus by the proof of Theorem 4.13 it follows that K_n is the free l-group on the free set of generators $C_n\pi$.

We are now going to give some conditions for an l-group G to be free. Let (F,π) be the free l-group over the l-group G constructed as in Theorem 4.12. If G is not an o-group, then there exists right orders $P_{\lambda_1} \neq P_{\lambda_2}$, of G such that $P_{\lambda_1} \cap P_{\lambda_2} \supset G^+$. If $g \in P_{\lambda_1} \setminus P_{\lambda_2}$, then $0g^{\lambda_1} = g > 0$ and $0g^{\lambda_2} = g < 0$. Thus $0(g^{\lambda_1} \vee e) = g$ and $0(g^{\lambda_2} \vee e) = 0$ and so $g\pi \vee e\pi = (\ldots, g^\lambda \vee e, \ldots) \neq (g \vee 0)\pi$ because $0(g \vee 0)^{\lambda_1} = g \vee 0 > 0$ and $0(g \vee 0)^{\lambda_2} = g \vee 0 > 0$. Thus π is an o-isomorphism of G into F, but not an l-isomorphism and so $G\pi \subset F$.

Theorem 4.17: Let (F,π) be the free l-group over l-group G. Then the following are equivalent;

(1) $G\pi = F$.

(2) G is an o-group.

(3) Each o-homomorphism of G into an l-group is an l-homomorphism.

Proof: We have shown that (1) implies (2) and clearly (2) implies (3). (3) $\Longrightarrow$ (1). π is an o-isomorphism of G into l-group F and so π is an l-isomorphism. Thus $G\pi$ is an l-subgroup of F and hence $G\pi = F$.

Proposition 4.18: Let G be an l-group and (F,π) be the free l-group over G. For every element $a,b \in G$, $(a \vee b)\pi = a\pi \vee b\pi$ if and only if a,b are comparable.

Proof: Clearly if a,b are comparable, $(a \vee b)\pi = a\pi \vee b\pi$.

Conversely, it is trivial if G is an o-group. Now suppose G is not an o-group. Then $G^+ = \bigcap_{i \in I} P_{\lambda_i}$ for some index I, and P_{λ_i} is a right order on G, such that $P_{\lambda_i} \supset G^+$. Let $h = a-b$. (i) If $h \| 0$, then $h \notin \bigcap_{i \in I} P_{\lambda_i}$, that is,

$h \notin P_{\lambda_i}$, for some $i \in I$. If $h \in P_{\lambda_j}$ for some $j \in I$, then $h \in P_{\lambda_j} \setminus P_{\lambda_i}$. Therefore, $Oh^{\lambda_j} = h > 0$ in G_{λ_j} and $Oh^{\lambda_i} = h < 0$ in G_{λ_i}. Thus $O(h^{\lambda_j} Ve) = h$, and $O(h^{\lambda_i} Ve) = 0$ and so $h\pi Ve\pi = (\ldots, h^{\lambda} Ve, \ldots) \neq (hVe)\pi$, because $O(hVe)^{\lambda} = hVe > 0$ for all λ. Hence $(aVb)\pi \neq a\pi Vb\pi$ in this case. On the other hand, if $h \notin P_{\lambda_i}$ for all $i \in I$, then $-h \in P_{\lambda_i}$ for all $i \in I$. Hence $-h \in G^{+}$, that is $-h \geq 0$. Therefore $h \leq 0$, contradiction Consequently, if $h || 0$, then $(aVb)\pi \neq a\pi Vb\pi$. The proof is complete.

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