A TOPOLOGY FOR A LATTICE-ORDERED GROUP

by

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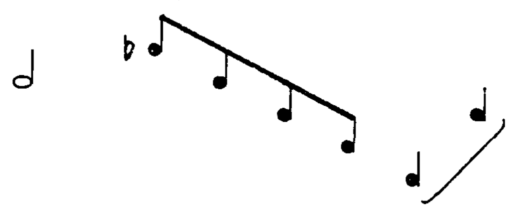
ABSTRACT

In the three editions of Lattice Theory, Birkhoff has suggested topologies for various types of lattices which he and others have investigated for lattice-ordered groups. There have also been several other attempts to topologise lattices in general and lattice-ordered groups in particular. However, the only topologies which have been proved in general to be group topologies (viz. the class of topologies described by Šmarda) fail to reduce to the usual topology on finite cardinal products of the real numbers. The most general of those topologies which have been defined by means of convergence is the topology derived from α-convergence, as developed by Papangelou, Ellis, and Madell. A lattice-ordered group has such a topology if and only if it is completely distributive.

In this dissertation, we define a topology, which we call the $\mathfrak{T}$-topology, on an arbitrary lattice-ordered group G. With respect to the $\mathfrak{T}$-topology, G is both a topological group and a topological lattice. The $\mathfrak{T}$-topology on a totally ordered group is the interval topology, and if G is a cardinal product of lattice-ordered groups, then the $\mathfrak{T}$-topology on G is the (Tychonoff) product of the $\mathfrak{T}$-topologies on the factors. Hence the $\mathfrak{T}$-topology on any cardinal product of the real numbers is the usual topology. The $\mathfrak{T}$-topology is discrete if and only if G is a lexico-sum of lexico-extensions of the integers. We derive necessary and sufficient conditions for the $\mathfrak{T}$-topology to be Hausdorff, and construct a lattice-ordered group which has indiscrete $\mathfrak{T}$-topology. Finally, we investigate convergence with respect to the $\mathfrak{T}$-topology, and when G is completely distributive, we compare the $\mathfrak{T}$-topology with the topology derived from α-convergence.

DEDICATION

To Balthazar, who slept on his cushion



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1. INTRODUCTION

A. *Terminology and Notation*

For the definitions and basic theory of semigroups, groups, (normal) subgroups, and group isomorphisms, see Schenkman [55] (We use the additive notation "+" for most groups even though they may not be commutative.). For the definitions and basic theory of partially ordered sets, directed sets, lattices, sublattices, lattice isomorphisms, topological lattices, partially ordered groups, lattice-ordered groups (ℓ-groups), vector lattices, and Banach lattices, see Birkhoff [8] or Fuchs [30] (except for topological lattice and Banach lattice) (NB: We do not require a topological lattice to be Hausdorff.). For the definitions and basic theory of bases and subbases of topologies and of open sets, closed sets, interiors of sets, neighborhoods, T_1 topologies, T_2 (or Hausdorff) topologies, completely regular topologies, connected sets, one-to-one functions, onto functions, continuous functions, sequences, nets, cofinal subsets, and filter-bases and filter-subbases of filters, see Thron [58] (NB: For a net we use the notation $\{x_\beta \mid \beta \in B\}$ where B is the domain of the net.). For the definition and basic theory of topological groups, see Husain [36] (NB: We do not require a topological group to be Hausdorff.). Zorn's Lemma may be found in [8], [55], or [58]. We consistently use totally ordered set (totally ordered group) to refer to a partially ordered set (partially ordered group) in which every pair of elements is related, i.e. in which $a \leq b$ or $b \leq a$ for every pair of elements a,b. We write all functions, except the projection functions from a product, on the right.

If $\{A_\lambda \mid \lambda \in \Lambda\}$ is a collection of sets, then by the product of the A_λ, $\prod_{\lambda\in\Lambda} A_\lambda$, we mean the set of functions f from Λ to the disjoint union of the A_λ satisfying $\lambda f \in A_\lambda$ for all $\lambda \in \Lambda$. If $\{G_\lambda \mid \lambda \in \Lambda\}$ is a collection of groups, then by the product of the G_λ, $\prod_{\lambda\in\Lambda} G_\lambda$, we mean the set $\prod_{\lambda\in\Lambda} G_\lambda$ with group operation defined by: $f + g$ is the element of $\prod_{\lambda\in\Lambda} G_\lambda$ satisfying $\lambda(f + g) = (\lambda f) + (\lambda g)$. We sometimes write finite products of sets (groups) as $A_1 \times A_2 \times \dots \times A_n$ $(G_1 \times G_2 \times \dots \times G_n)$.

Let G be an ℓ-group and T a totally ordered group. The lexico-graphic product of G and T, $G \overleftarrow{\times} T$, is the group $G \times T$ with order defined by: $(g,t) \leq (a,b)$ if and only if $t < b$ or $t = b$ and $g \leq a$. The lexico-graphic product of an ℓ-group and a totally ordered group is an ℓ-group. If $\{G_\lambda \mid \lambda \in \Lambda\}$ is a collection of ℓ-groups, then the cardinal product of the G_λ, $|\Pi|_{\lambda\in\Lambda} G_\lambda$, is the group $\prod_{\lambda\in\Lambda} G_\lambda$ with order defined by: $f \leq g$ if and only if $\lambda f \leq \lambda g$ for all $\lambda \in \Lambda$. If Λ is finite, we sometimes denote the cardinal product by $G_1 |\times| G_2 |\times| \dots |\times| G_n$. The cardinal sum of the G_λ, $|\Sigma|_{\lambda\in\Lambda} G_\lambda$, is the subgroup of $|\Pi|_{\lambda\in\Lambda} G_\lambda$ consisting of those functions which are 0 for all but a finite number of $\lambda \in \Lambda$. The cardinal product and cardinal sum of ℓ-groups are ℓ-groups.

Let $\{G_\lambda \mid \lambda \in \Lambda\}$ be a collection of sets with topologies $\mathcal{U}_\lambda$. The product topology on $\prod_{\lambda\in\Lambda} G_\lambda$ is the set consisting of the empty set and all subsets of $\prod_{\lambda\in\Lambda} G_\lambda$ which are arbitrary unions of finite intersections of sets of the form $U_\gamma \times \prod_{\lambda\in\Lambda\setminus\{\gamma\}} G_\lambda$ where $U_\gamma \in \mathcal{U}_\gamma$.

We denote the lattice operations of join and meet by $\vee$ and $\wedge$ respectively. In general, if A is a subset of a partially ordered set P and there exists $d \in P$ such that (i) $d \geqq a$ for all $a \in A$ and (ii) $d \leqq b$ whenever $b \geqq a$ for all $a \in A$, then d is written $\vee A$ or $\bigvee_{a\in A} a$. We define $\wedge A$ and $\bigwedge_{a\in A} a$ similarly. A lattice L is said to be conditionally complete if, for all subsets S of L which are bounded above, $\vee S \in L$ and if, for all subsets T of L which are bounded below, $\wedge T \in L$. A lattice L is said to be complete if, for all subsets S of L, $\vee S \in L$ and $\wedge S \in L$. A non-trivial ℓ-group cannot be complete, but there exist ℓ-groups which are conditionally complete [8].

Let L be a lattice. Let L^* be L with greatest and least elements (adjoined if necessary). For $Y \subseteq L^*$, let $u(Y) = \{\ell \in L^* \mid \text{for all } y \in Y,\ \ell \geqq y\}$ and $\ell(Y) = \{\ell \in L^* \mid \text{for all } y \in Y,\ \ell \leqq y\}$. Let

$$\hat{L} = \{Y \mid Y \subseteq L^*,\ Y \text{ non-empty, and } Y = \ell(u(Y))\}.$$

We order $\hat{L}$ by set inclusion and call $\hat{L}$ the completion of L by cuts [8], [42]. We may embed L into the complete lattice $\hat{L}$ by taking $a \in L$ to $\ell(u(\{a\}))$. This map is one-to-one and preserves order and the lattice operations; thus we may assume that L is a subset of $\hat{L}$. The <u>conditional completion</u>, $\bar{L}$, of L is (a) $\hat{L}\setminus\{\vee\hat{L}, \wedge\hat{L}\}$ if $\vee L, \wedge L \notin L$, (b) $\hat{L}\setminus\{\vee\hat{L}\}$ if $\vee L \notin L$ but $\wedge L \in L$, (c) $\hat{L}\setminus\{\wedge\hat{L}\}$ if $\vee L \in L$ but $\wedge L \notin L$, or (d) $\hat{L}$ if both $\vee L, \wedge L \in L$. Then $\bar{L}$ is conditionally complete and L may be considered as a sublattice of $\bar{L}$.

Let L be a lattice. Then L is said to be <u>completely distributive</u> if, whenever $\{\ell_{\alpha\beta} \mid \alpha \in A, \beta \in B\} \subset L$ for arbitrary indexing sets A and B, the equality

$$\bigwedge_{\alpha\in A} \bigvee_{\beta\in B} \ell_{\alpha\beta} = \bigvee_{f\in B^A} \bigwedge_{\alpha\in A} \ell_{\alpha(\alpha f)}$$

holds provided that all the indicated joins and meets exist.

A subset S of a partially ordered set P is said to be <u>convex</u> if $p \in S$ whenever $s \leqq p \leqq r$ for $s,r \in S$ and $p \in P$.

An <u>ℓ-subgroup</u> of an ℓ-group G is a subset S of G which is both a subgroup and a sublattice. Not every subgroup of an ℓ-group is an ℓ-subgroup; a simple example is the following. Let G be the cardinal product of the integers with themselves. Let $S = \{(m,-m) \mid m \text{ is an integer}\}$. Then clearly S is a subgroup of G. However, since $(1,-1) \vee (-1,1) = (1,1) \notin S$, S is not a sublattice of G. A convex normal ℓ-subgroup of an ℓ-group is called an <u>ℓ-ideal</u>. If $\{G_\lambda \mid \lambda \in \Lambda\}$ is a collection of ℓ-groups, then the cardinal sum of the G_λ is an ℓ-ideal of the cardinal product of the G_λ. An ℓ-subgroup is said to be <u>prime</u> if its lattice of left cosets is totally ordered (see [17]). A subgroup N of an ℓ-group G is said to be <u>*L*-closed</u> if whenever $\{g_\alpha \mid \alpha \in A\} \subseteq N$ and $\bigvee_{\alpha\in A} g_\alpha$ exists in G, then $\bigvee_{\alpha\in A} g_\alpha \in N$, and whenever

$\{g_\alpha | \alpha \in A\} \subseteq N$ and $\bigwedge_{\alpha \in A} g_\alpha$ exists in G, then $\bigwedge_{\alpha \in A} g_\alpha \in N$.

Unless otherwise mentioned, we adopt the notation of Birkhoff [8] (see also Fuchs [30]). In particular, (since the notation varies) we emphasize that for any ℓ-group G and any $a \in G$, $a^+ = a \vee 0$ and $a^- = a \wedge 0$. If G is an ℓ-group, $A, B \subseteq G$, and $a, b \in G$ with $a \leq b$, then $[a,b] = \{x \in G | a \leq x \leq b\}$, $(a,b) = \{x \in G | a < x < b\}$; $[a]$ is the ℓ-subgroup generated by a, $G(a)$ is the convex ℓ-subgroup generated by a; $A + B = \{x + y | x \in A, y \in B\}$, $a + B = \{a\} + B$, $-A = \{-a | a \in A\}$, $A \vee B = \{x \vee y | x \in A, y \in B\}$, $A \wedge B = \{x \wedge y | x \in A, y \in B\}$, $a \vee B = \{a\} \vee B$, $a \wedge B = \{a\} \wedge B$; $A^+ = \{x \in A | x \geq 0\}$, and $A^- = \{x \in A | x \leq 0\}$. For $A \subseteq G$, $A^\perp = \{x \in G | |x| \wedge |a| = 0$ for all $a \in A\}$ (read the "polar" of A) is a convex ℓ-subgroup of G (see [18], [56]); for $g \in G$, we let $g^\perp = \{g\}^\perp$. We denote the empty set by $\emptyset$. We denote the additive ℓ-group of the real numbers by R, that of the rational numbers by Q, and that of the integers by Z. We let N denote the natural numbers. The symbol $|\boxtimes|$ at the right hand margin indicates the end of a proof.

We say that two ℓ-groups are <u>ℓ-isomorphic</u> if there is a one-to-one, onto function between them which preserves the group operation and the lattice operations; i.e. if there exists a function between them which is both a group and a lattice isomorphism. We denote ℓ-isomorphism by $\simeq$.

If Ω is a totally ordered set, we let $A(\Omega)$ denote the set of all one-to-one order preserving functions of Ω onto itself.

Functional composition in $A(\Omega)$ is a group operation (which we write multiplicatively), and the partial order defined by

$$f \leqq g \text{ if and only if } \omega f \leqq \omega g \text{ for all } \omega \in \Omega$$

is a lattice order, with $\omega(f \vee g) = (\omega f) \vee (\omega g)$ and $\omega(f \wedge g) = (\omega f) \wedge (\omega g)$. Under this operation and this relation, $A(\Omega)$ is an ℓ-group. We denote the group identity by i, and we say that $A(\Omega)$ is <u>doubly transitive</u> if for all $\alpha,\beta,\gamma,\delta \in \Omega$ with $\alpha < \beta$ and $\gamma < \delta$, there exists $f \in A(\Omega)$ such that $\alpha f = \gamma$ and $\beta f = \delta$.

If Y is a topological space, we let $C(Y)$ denote the set of all continuous real-valued functions of Y. Define addition on $C(Y)$ pointwise, i.e. by

$$y(f + g) = (yf) + (yg) \text{ for all } y \in Y.$$

Order $C(Y)$ by

$$f \leqq g \text{ if and only if } yf \leqq yg \text{ for all } y \in Y.$$

Then $C(Y)$ is an ℓ-group with $y(f \vee g) = (yf) \vee (yg)$ and $y(f \wedge g) = (yf) \wedge (yg)$.

Let $\{a_\gamma | \gamma \in \Gamma\}$ be a net in A, a subset of a set Y with topology $\mathcal{U}$. We say that $\{a_\gamma\}$ <u>converges</u> to $a \in Y$ <u>with respect to $\mathcal{U}$</u> (or <u>$\mathcal{U}$-converges</u> to a) if for all $U \in \mathcal{U}$ with $a \in U$, there is a $\beta \in \Gamma$ such that whenever $\alpha \geqq \beta$, then $a_\alpha \in U$.

Let $\{A_\gamma | \gamma \in \Gamma\}$ be a filter-subbase. We denote the filter generated by $\{A_\gamma | \gamma \in \Gamma\}$ by $F(\{A_\gamma | \gamma \in \Gamma\})$. For notational convenience in the proof of Corollary 5.22, we declare that $F(\emptyset) = \emptyset$.

We assume that the reader is familiar with the basic properties of ℓ-groups found in Birkhoff [8, Chapter XIII], especially those described in § XIII.3 and § XIII.4. In particular, we will use without comment such observations as $|a| = a^+ - a^- = a \vee (-a)$, $a^+ \wedge (-a)^+ = 0$, $a = a^+ + a^-$, and $|a| \wedge |b| = 0$ implies $a + b = b + a$.

The proof that an ℓ-group is a topological group with respect to the $\mathfrak{T}$-topology (to be introduced in Chapter 2) relies on the following theorem from Husain [36, page 46]:

Theorem A: Let G be a group with a filter-base $\mathcal{N}(0)$ satisfying:

(a) Each $H \in \mathcal{N}(0)$ is symmetric.

(b) For each $H \in \mathcal{N}(0)$, there is a $K \in \mathcal{N}(0)$ such that $K + K \subseteq H$.

(c) For each $H \in \mathcal{N}(0)$ and each $a \in G$, there is a $K \in \mathcal{N}(0)$ such that $a + K - a \subseteq H$.

Then there exists a unique topology $\mathfrak{T}$ on G such that, with respect to $\mathfrak{T}$, G is a topological group, and $F(\mathcal{N}(0))$ is the set of $\mathfrak{T}$-neighborhoods of 0. |⌛|

We also use the following result of Husain [36, page 48]:

Theorem B: For a topological group G with group topology $\mathcal{U}$, the following statements are equivalent:

(a) $\mathcal{U}$ is T_1.

(b) $\mathcal{U}$ is Hausdorff.

(c) $\bigcap \mathcal{W} = \{0\}$, where $\mathcal{W}$ is any filter-base for the $\mathcal{U}$-neighborhoods of 0. |⌛|

B. Background

We review here work done on topologising lattices in general and ℓ-groups in particular. For views of the roots of the theory of ℓ-groups, see [5], [9], and [10], and the references there.

Previous to this dissertation, there have been basically four methods of topologising various classes of lattices and lattice-ordered groups: by literally generalising the interval topology on totally ordered sets; by extrapolating the interval topology on totally ordered sets; by attempting to relate the order structure to (topological) convergence; and by considering filters of subgroups.

The <u>interval topology</u> for a lattice L was introduced by Frink in [28]. This topology is defined by taking the (lattice) closed intervals as a subbase for the (topologically) closed sets. Here we understand "closed interval" to mean sets of the form (i) $\{x \mid x \leqq a\}$ for some $a \in L$, (ii) $\{x \mid x \geqq a\}$ for some $a \in L$, (iii) L itself, or (iv) $\{x \mid a \leqq x \leqq b\}$ for some $a,b \in L$. This same definition is a "closed set" definition of the usual interval topology on a totally ordered set. The usual definition uses open intervals as a subbase for the open sets. In the lattice case, however, the closed set version is preferable because the "boundaries" of sets can have more than two points. For example, in $R \,|X|\, R$, $((0,0), (1,1))$ is the unit square without the points $(0,0)$ and $(1,1)$. Intuitively (i.e. in the usual topology), the "open" unit square also fails to include the line segments $[(0,1), (1,1))$, $((0,0), (0,1)]$, $((0,0), (1,0)]$, and $[(1,0), (1,1))$. In a totally ordered set, these line segments collapse into the maximum and minimum points of the interval, which are absent from an "open" interval and present in a "closed" interval. Abstracting the "open set" definition from a totally ordered set has proved fruitful, however, as we shall see when we discuss the open interval topology.

The interval topology defined above has been investigated in [6] (which relied heavily on Frink's paper), [48] (which proved most of the results announced in [47]), [2], [13], [46], [44], [62], [59], [16], [37], [34], and [11].

It is easy to see that a lattice must be T_1 in its interval topology [28]. It was noticed in [6] that the net $\{(r,-r) \mid r \in R^+\}$ in $R \,|X|\, R$ converges to every element of $R \,|X|\, R$ with respect to the interval topology. Therefore the interval topology on $R \,|X|\, R$ is not Hausdorff, and hence by Theorem B, the interval topology is not a group topology on $R \,|X|\, R$.

Many classes of ℓ-groups have been discovered with the property that if G is an ℓ-group in the class and if G is Hausdorff in its interval topology, then G is totally ordered. (See [13], [62], [16], [37], and [11].) Holland [34] has given an example of a non-totally ordered ℓ-group which is a topological lattice and a topological group (and therefore Hausdorff) in its interval topology.

In a 1951 address, Frink [29] suggested that a suitable generalization of the interval topology would be an ideal topology defined as follows: Let P be a partially ordered set. An ideal of P is a subset I of P with the property that if F is a finite subset of I, then the set of all lower bounds of the set of all upper bounds of F is contained in I. A dual ideal, D, defined by interchanging "upper" and "lower" in the definition of ideal, is a filter in the sense of [58, Definition 13.16] if P has a least element and $\phi \neq D \neq P$. A (dual) ideal is completely irreducible if it is not the intersection of a collection of (dual) ideals distinct from it. The ideal topology on a partially ordered set P has as a subbase for the open sets the set of all completely irreducible ideals and dual ideals. The ideal topology reduces to the usual topology on $R |X| R$, the completely irreducible ideals and dual ideals being the open half planes [29]. With the exception of Ward [59], the ideal topology has not been further investigated.

In [32], Guillaume defines topologies similar to the ideal topology as follows: Let P be a partially ordered set. A subset F of P is said to be right-ordered (left-ordered) if for all $x \in F$, $u(\{x\}) \subseteq F$ $(\ell(\{x\}) \subseteq F)$. The open sets of the Td-topology (Tg-topology) are the right-ordered (left-ordered) sets. A subset F of P is called right-closed (left-closed, dg-closed) if whenever S is a non-empty totally ordered subset of F with $\vee S$ $(\wedge S, \wedge S$ and $\vee S)$ existing in P, then $\vee S$ $(\wedge S, \wedge S$ and $\vee S)$ is an element of F. The closed sets of the right-longitudinal topology (left-longitudinal topology, longitudinal topology) are the right-closed (left-closed, dg-closed) subsets of P. Guillaume does not consider algebraic structure in addition to the order structure, and apparently no-one has realized significant theorems in that direction.

Banaschewski [1] has defined a topology on a partially ordered group with a base of closed intervals for the filter of neighborhoods of 0. Let P be a partially ordered group. A set $E \subseteq P$ is called a <u>topological identity</u> if (i) $e \in E$ implies $e > 0$, (ii) for $e, e' \in E$, there is a $d \in E$ such that $d \leqq e, e'$, (iii) for $e \in E$, there is a $d \in E$ such that $d + d \leqq e$, (iv) for $e \in E$ and $x \in P$, there is a $d \in E$ such that $d \leqq x + e - x$, and (v) $\wedge E = 0$. The <u>identity topology</u> on P is defined by taking as a subbase for the neighborhoods of 0 the intervals $[-e,e]$ for $e \in E$, where E is a topological identity of P. Let G be a finite cardinal product of copies of Z and let $E \subseteq G^+ \setminus \{0\}$. Then for all $e \in E$, $(0,e]$ is finite. But if E is a topological identity, then property (iii) implies that for all $e \in E$, $(0,e]$ contains an infinite descending subset. Hence E is not a topological identity, and therefore G cannot have an identity topology defined on it.

In [8], the definition of interval topology was modified as follows: Let L be a lattice. Let $\mathcal{C}$ be the set of all intersections of finite unions of (lattice) closed intervals of L. Here "closed interval" means only sets of the form $[a,b]$ for some $a, b \in L$ with $a \leqq b$. A set S is said to be closed in the <u>new interval topology</u> if $C \in \mathcal{C}$ implies that $S \cap C \in \mathcal{C}$. (We note that

Theorem X.21 of [8] is proven only for the (old) interval topology.) The new interval topology has the advantage of being the usual topology on $R \,|X|\, R$. It is unknown whether an arbitrary ℓ-group is a topological group in its new interval topology (problem 114 of [8]). The relationship between the new interval topology and the ideal topology has not been discovered.

Let G be a partially ordered group. The <u>open interval topology</u> on G is defined by taking the open intervals as a subbase for the open sets. Here "open intervals" are taken as sets of the form (a,b) for $a,b \in G$ with $a < b$. Loy and Miller [41] have defined a class of partially ordered groups called <u>tight Riesz groups</u> by requiring that the order on the group G be directed and that for any elements $a,b,c,d \in G$ with $a,b < c,d$, there exists $x \in G$ such that $a,b < x < c,d$. A non-totally ordered ℓ-group cannot be a tight Riesz group: if $a,b \in G$ are incomparable, then $a \wedge b, a \wedge b < a,b$, but there does not exist an $x \in G$ such that $a \wedge b < x < a$ and $a \wedge b < x < b$. However, Wirth [61] and Reilly [52] have investigated ℓ-groups which permit the existence of a tight Riesz order "compatible" with the lattice order. Loy and Miller [41] proved that the open interval topology on an abelian tight Riesz group G is not discrete and is Hausdorff when G has no pseudozeros ($w \in G$ is <u>pseudopositive</u> if $w \notin G^+$ but $w + (G^+ \setminus \{0\}) \subseteq G^+ \setminus \{0\}$; $w \in G$ is a <u>pseudozero</u> if both w and $-w$ are pseudopositive.)

In [54] (which is an expanded version of [53]), Rennie defines the L-topology of a lattice by extrapolating the interval topology on a totally ordered set as follows: Let L be a lattice. A base of open sets for the L-topology consists of all sets S contained in L such that S is convex and the intersection of S with any maximal totally ordered subset of L is open with respect to the interval topology of the totally ordered subset. A lattice-ordered group is a topological lattice with respect to the L-topology [54]. Rennie did not consider ℓ-groups except to define them; he devoted his attention instead to Banach lattices. We will show in a few paragraphs that not every ℓ-group is a topological group with respect to its L-topology.

One of Rennie's main concerns was the comparison of the L-topology with six other topologies. One of these was the interval topology and the other five were topologies defined by means of various types of lattice convergence. Let P be a partially ordered set. A net $\{x_\beta | \beta \in B\}$ in P is said to order-converge to $x \in P$ in case there are nets $\{a_\beta | \beta \in B\}$ and $\{b_\beta | \beta \in B\}$ in P such that (1) if $\eta \leqq \gamma$, then $a_\eta \leqq a_\gamma$ and $b_\eta \geqq b_\gamma$, (2) $a_\beta \leqq x_\beta \leqq b_\beta$ for all $\beta \in B$, and (3) $\bigvee_{\beta\in B} a_\beta = x = \bigwedge_{\beta\in B} b_\beta$. Order-convergence was introduced by Birkhoff in [3] and [4]; see also Frink [28] and Kantorovich [38]. If L is a lattice, and if a net $\{y_\gamma | \gamma \in \Gamma\}$ in L order-converges to $y \in L$, then [54]

$$y = \vee\{p \mid \text{there is a } \beta \in \Gamma \text{ such that } p \leqq y_\gamma \text{ for all } \gamma \geqq \beta\},$$

and

$$y = \wedge\{p \mid \text{there is a } \beta \in \Gamma \text{ such that } p \geqq y_\gamma \text{ for all } \gamma \geqq \beta\}.$$

We define the order topology on a partially ordered set P by letting the closed sets be exactly those subsets A of P such that the limit of any order-convergent net in A is itself in A. Both Birkhoff [6] and Frink [28] incorrectly assumed that convergence with respect to the order topology was the same as order-convergence. This led Birkhoff to the conclusions [6] that every partially ordered set had Hausdorff order topology and that any conditionally complete ℓ-group was a topological group in its order topology. However, Rennie [53,54] gave an example which showed that order-convergence and convergence with respect to the order topology need not be the same, and Northam announced in [47] (but did not include in [48]) an example of a lattice which was not Hausdorff in its order topology or [redacted] L-topology. Floyd and Klee [27] also pointed out the non-equivalence of the two types of convergence, and in [26], Floyd gave an example of a conditionally complete vector lattice in which addition was not continuous in any topology "compatible" with the order. See also [25].

In particular, the "compatibility" of [26] may be used to show that for lattice-ordered groups (in fact, vector lattices) Rennie's L-topology is not in general a group topology. Let P be a partially ordered set, and let T be a topology for P. Floyd defined T to be <u>σ-compatible</u> with the order on P if and only if whenever $\{x_1, x_2, \ldots\}$ is a sequence in P with

$$x_1 \geqq x_2 \geqq x_3 \geqq \ldots \quad \text{and} \quad \bigwedge_{i=1}^{\infty} x_i = x \in P$$

or

$$x_1 \leqq x_2 \leqq x_3 \leqq \ldots \quad \text{and} \quad \bigvee_{i=1}^{\infty} x_i = x \in P,$$

then the sequence $\{x_i\}$ T-converges to x. He proved that there exists a conditionally complete vector lattice N in which the function $x - y$ is not $\mathcal{T}$-continuous simultaneously in x and y for any T_1 topology $\mathcal{T}$ for N which is σ-compatible with the order on N. (N is the lattice of all continuous real-valued functions on a Stone representation space of the complete Boolean algebra of all regular open subsets of the unit interval, partially ordered by inclusion.)

Let L be a lattice. We will show that Rennie's L-topology is σ-compatible with the order on L: Suppose $\{x_i\}$ is a sequence in L with

$$x_1 \geqq x_2 \geqq x_3 \geqq \ \ldots . \quad \text{and} \quad \bigwedge_{i=1}^{\infty} x_i = x \in L,$$

and let S be a basic open set for the L-topology with $x \in S$. By Zorn's Lemma, there exists a maximal totally ordered set C in L such that $\{x,x_1,x_2,x_3,\ldots .\} \subseteq C$. Since S is a basic open set for the L-topology, $C \cap S$ is an open set of the interval topology of C. Clearly $x \in C \cap S$. Thus there exist $a,b \in C$ such that $x \in (a,b) \subseteq S \cap C$. Suppose that for all i there exists $j \geqq i$ such that $x_j \notin (a,b)$. Then $x_i \geqq b$ for all i, and hence $x = \bigwedge_{i=1}^{\infty} x_i \geqq b$. This contradicts the fact that $x \in (a,b)$. Thus there exists k such that for all $i \geqq k$,

$$x_i \in [x,b) \subseteq (a,b) \subseteq S.$$

Therefore $\{x_i\}$ converges to x with respect to the L-topology. The case for increasing sequences follows similarly and thus the L-topology is σ-compatible with the order on L. Any lattice has T_1 L-topology [54] and hence by Floyd's theorem stated above, for vector lattices (and thus ℓ-groups) the L-topology is not in general a group topology.

Moore [45], Birkhoff [6,7], and Gordon [31] have all studied relative uniform convergence in vector lattices; but this concept has not been generalized to arbitrary ℓ-groups. Ward [59] considered the relations between the interval topology, the ideal topology, and the order topology, and DeMarr [20,21] classified certain lattices in which order-convergence and convergence with respect to the order topology coincide. Birkhoff proved [8, Theorem XIII.26] that in a conditionally complete ℓ-group the operations $+$, $\vee$, and $\wedge$ are "continuous with respect to order-convergence". As he noted later on [8, Theorem XV.14], however, such "continuity" is not necessarily related to topological continuity.

Rennie [54] mentions two methods of defining topologies based on order-convergence but different from the order topology. Instead of using arbitrary nets to define "closed set", one may restrict his attention to nets with totally ordered domains. Alternately, one may embed a lattice in one of its completions (see [8], [54]) and consider the topology inherited from the order topology on the completion. One may also define a new convergence as follows [8]: Let P be a partially ordered set. A net $\{x_\beta \mid \beta \in B\}$ <u>star-converges</u> to $x \in P$ if and only if every subnet of $\{x_\beta\}$ contains a subnet which order-converges to x. However, [54] the order topology on P is equivalent to the topology defined similarly with order-convergence replaced by star-convergence.

The most successful variation on order-convergence as far as ℓ-groups are concerned has been α-convergence. We will discuss the topological ramifications of this idea in Chapter 8. Here we note merely that the definition is due to Papangelou [49,50], who cites a theorem of Löwig [40, Theorem 42], which characterizes Löwig's "interelement" of a sequence as Papangelou's "α-limit" of the sequence. Using α-convergence, one may construct a topology on a completely distributive ℓ-group G with respect to which G is both a topological group and a topological lattice [50], [22], [43].

Further results dealing with the types of convergence mentioned above may be found in [23], [14], and [24].

All the topologies mentioned thus far, except the identity topology, are intrinsic [8] to the ordered set on which they are defined, i.e. they are defined only in terms of the order (and perhaps the lattice operations). The other class of topologies we wish to describe and the topology defined in this dissertation do not have this property. These topologies are defined on ℓ-groups and their definitions require group theoretic concepts.

In [57] Šmarda defines a topology for an ℓ-group G as follows: Let F be a filter in the lattice of all convex ℓ-subgroups of G such that if $H \in F$, then all conjugates of H are in F. Taking F as a base for the neighborhood-filter of 0 defines a topology on G with respect to which G is both a topological group and a topological lattice. Topologies constructed in this manner, however, have the drawback that if an ultrafilter containing the filter F contains a convex ℓ-subgroup which is ℓ-isomorphic to R, then the topology is not Hausdorff [57]. Thus, on finite cardinal products of R, the usual topology cannot be constructed in such a fashion.

In this dissertation, we define a topology, which we call the $\mathfrak{T}$-topology, on an arbitrary ℓ-group G. We prove that, with respect to the $\mathfrak{T}$-topology, G is both a topological group and a topological lattice (Chapter 2). Chapter 3 contains descriptions of the $\mathfrak{T}$-topology for particular examples, and in Chapter 4, we prove that on totally ordered groups the $\mathfrak{T}$-topology is equivalent to the usual (interval) topology. We prove that the $\mathfrak{T}$-topology on a cardinal product of ℓ-groups is the product of the $\mathfrak{T}$-topologies on the factors in Chapter 5, and we investigate the $\mathfrak{T}$-topology on lexico-graphic products. Chapter 6 is devoted to studying ℓ-groups with Hausdorff $\mathfrak{T}$-topologies: we prove that the $\mathfrak{T}$-topology is discrete if and only if G is a lexico-sum of lexico-extensions of the integers, and we derive necessary and sufficient conditions for the $\mathfrak{T}$-topology to be Hausdorff. Chapter 7 contains more examples, including two of ℓ-groups with indiscrete $\mathfrak{T}$-topology. In Chapter 8, we characterize convergence with respect to the $\mathfrak{T}$-topology, and when G is completely distributive, we investigate the relationship of the $\mathfrak{T}$-topology to the topology derived from α-convergence.

2. DEFINITION OF THE TOPOLOGY AND BASIC THEOREMS

We first establish some notation for the definition of the topology. Throughout this chapter, unless otherwise specified, G will denote an arbitrary ℓ-group.

For $g \in G^+$, let

$$\mathcal{T}(g) = \{h \in G^+ \mid \text{there exists } h' \in G \text{ such that } h \wedge h' = 0 \text{ and } h \vee h' = g\}.$$

See figures I, II, and III. For $g \in G^+ \setminus \{0\}$, let

$$N(0,g) = [-g,g] + g^{\perp}.$$

See figures IV and V. Let

$$\mathfrak{U} = \{h \in G^+ \setminus \{0\} \mid \mathcal{T}(h) = \{0,h\}\},$$

$$\mathcal{D}_1 = \{h \in \mathfrak{U} \mid \text{there exist } h_1, h_2, \ldots \in \mathfrak{U} \text{ such that } h_1 = h,\ h_{n+1} + h_{n+1} \leqq h_n, \text{ and } h_n \in h_{n+1}{}^{\perp\perp}\}.$$

We note that if $h \in \mathcal{D}_1$, then there exists $\ell \in \mathcal{D}_1$ such that $0 < \ell < \ell + \ell \leqq h$ and $h \in \ell^{\perp\perp}$: let $\ell = h_2$ in the definition of $\mathcal{D}_1$.

<u>Proposition 2.1</u>: Let $h, \ell \in G^+ \setminus \{0\}$ be such that $h \leqq \ell$. Then the following statements are equivalent:

(i) $\ell \in h^{\perp\perp}$.

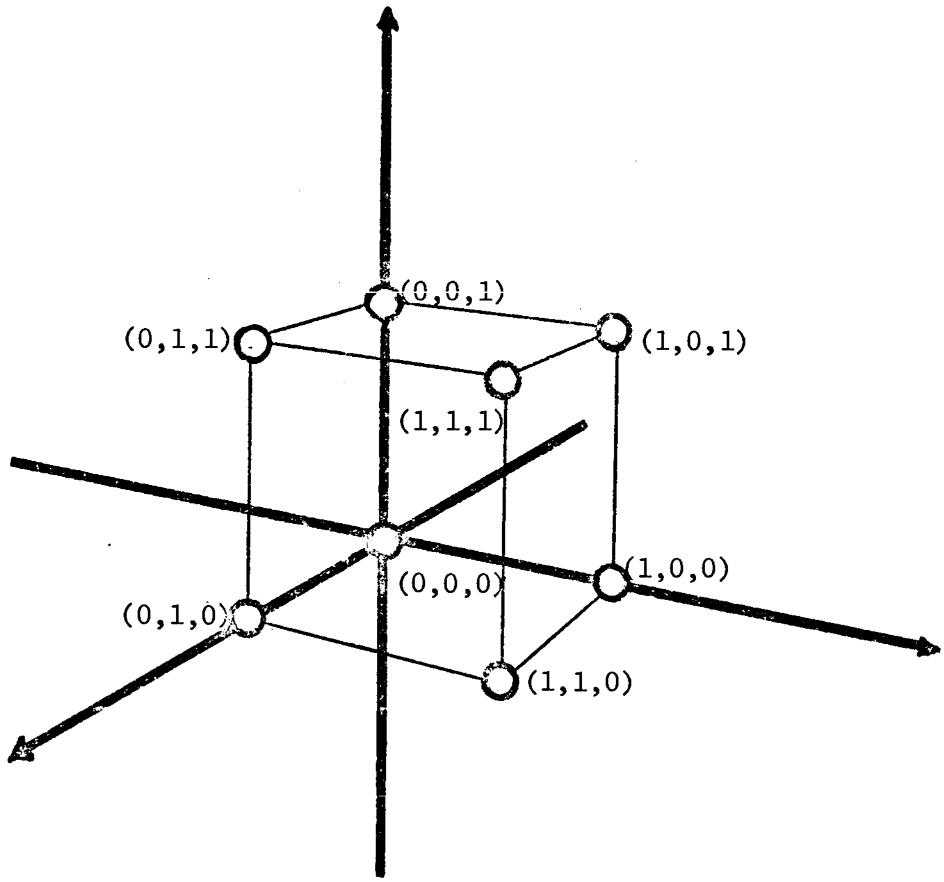
(ii) $\ell^{\perp} = h^{\perp}$.

(iii) $\ell^{\perp} \supseteq h^{\perp}$.

<u>Proof</u>: Suppose (i) holds and let $k \in h^{\perp}$. Then since $\ell \in h^{\perp\perp}$, $\ell \wedge |k| = 0$. Hence $k \in \ell^{\perp}$. This proves (iii).

Suppose (iii) holds. Since $0 < h \leqq \ell$, $0 \leqq |k| \wedge h \leqq |k| \wedge \ell$ for all $k \in G$. If $k \in \ell^{\perp}$, then $|k| \wedge \ell = 0$, and thus $k \in h^{\perp}$. By figure I

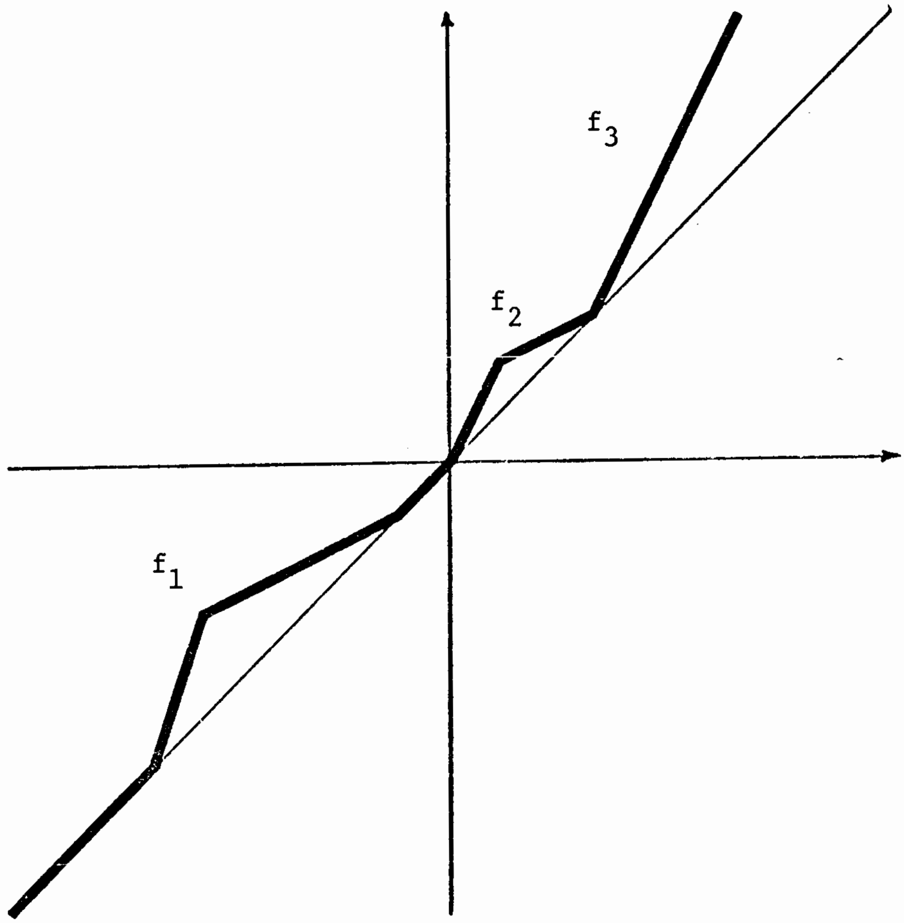
$$G = \left|\prod_{1}^{3}\right| R$$



$T((1,1,1)) = \{(1,1,1), (1,1,0), (1,0,1), (0,1,1), (1,0,0), (0,1,0), (0,0,1), (0,0,0)\}$

figure II

$G = A(R)$



$$xf_1 = \begin{cases} 3x + 12 & \text{if } x \in [-6,-5) \\ \frac{1}{2}(x - 1) & \text{if } x \in [-5,-1) \\ x & \text{otherwise} \end{cases} \qquad xf_2 = \begin{cases} 2x & \text{if } x \in [0,1) \\ \frac{1}{2}(x + 3) & \text{if } x \in [1,3) \\ x & \text{otherwise} \end{cases}$$

$$xf_3 = \begin{cases} 2x - 3 & \text{if } x \in [3,\infty) \\ x & \text{otherwise} \end{cases} \qquad f = f_1 \vee f_2 \vee f_3$$

$$T(f) = \{f,\ f_1 \vee f_2,\ f_1 \vee f_3,\ f_2 \vee f_3,\ f_1,\ f_2,\ f_3,\ i\}$$

<u>figure III</u>

$$G = \prod_{1}^{4} R$$

$g = (a,b,c,d)$

$a,b,c,d > 0$

$T(g)$:

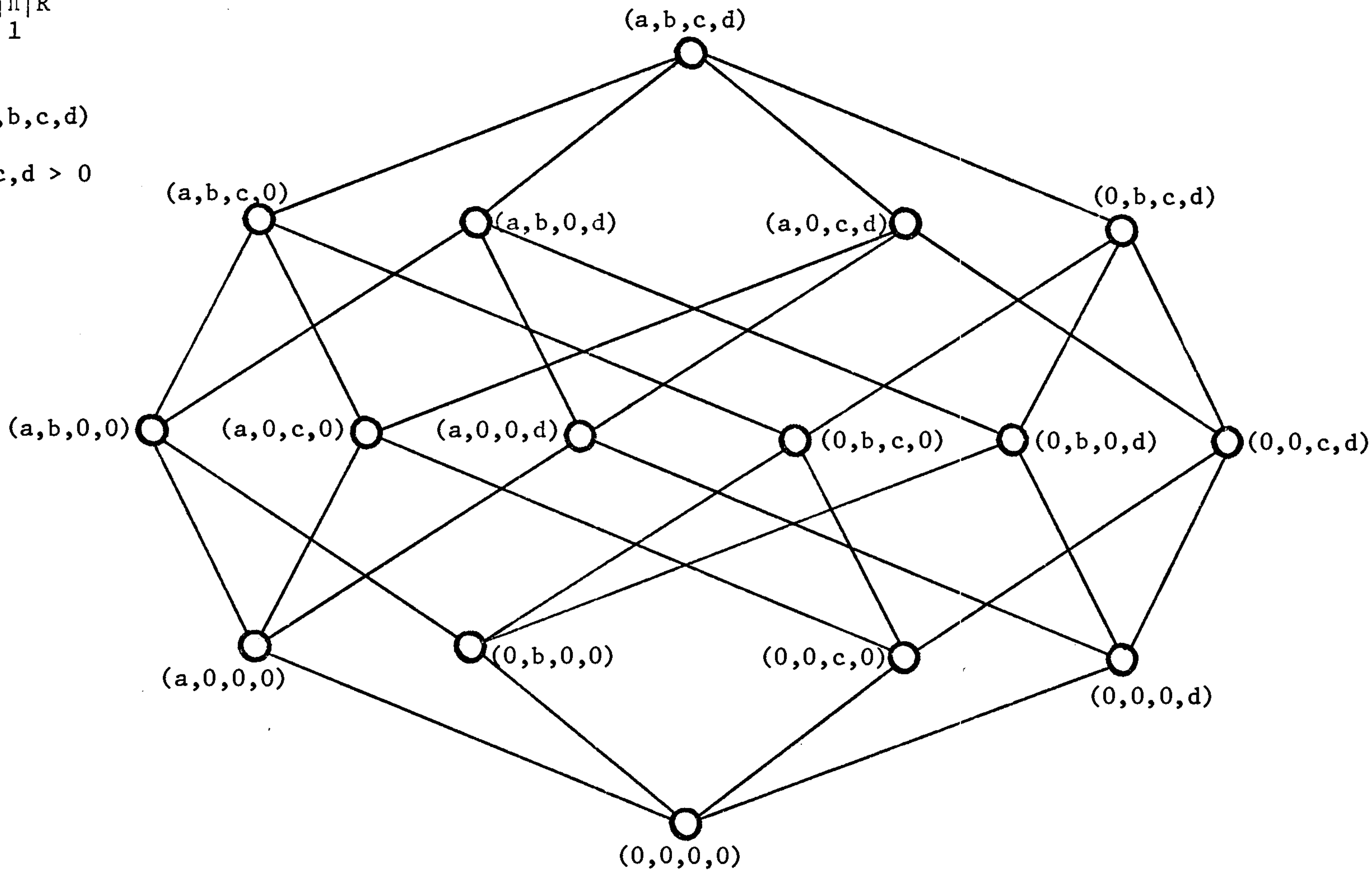


figure IV

$G = R\ |X|\ R$

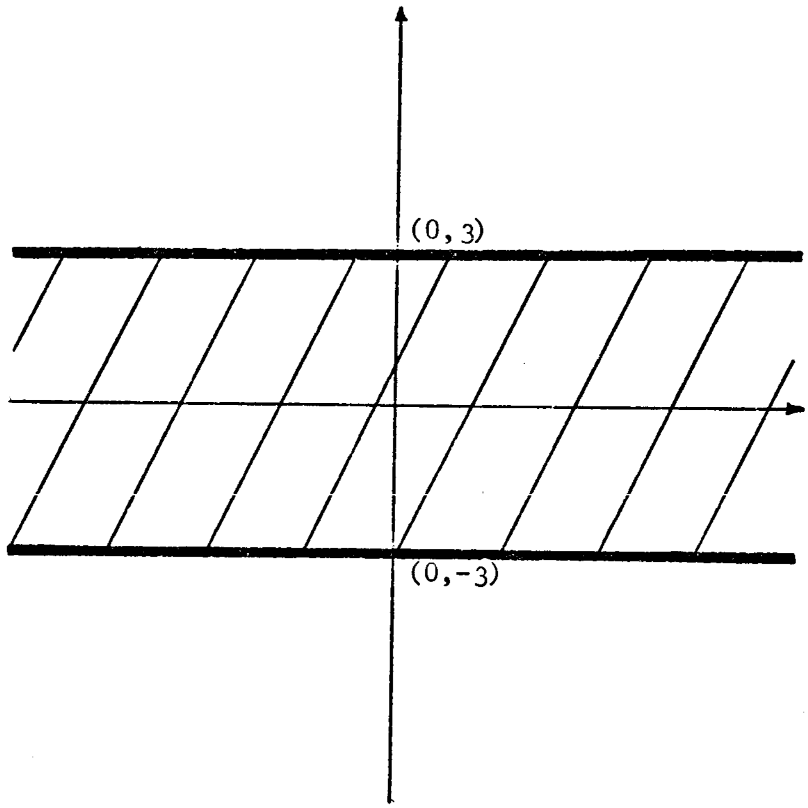
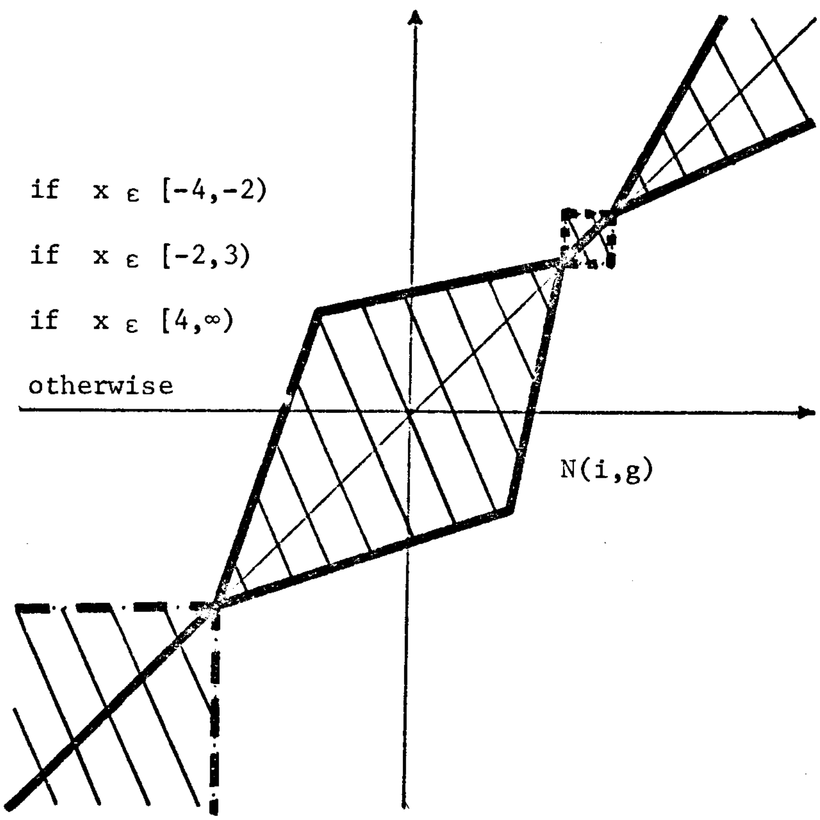


figure V

$G = A(R)$

$$xg = \begin{cases} 3x + 8 & \text{if } x \in [-4,-2) \\ \frac{1}{5}(x + 12) & \text{if } x \in [-2,3) \\ 2x - 4 & \text{if } x \in [4,\infty) \\ x & \text{otherwise} \end{cases}$$



hypothesis, $h^{\perp} \subseteq \ell^{\perp}$. This proves (ii).

Suppose (ii) holds. Since $\ell^{\perp} = h^{\perp}$, then $\ell^{\perp\perp} = h^{\perp\perp}$ and hence $\ell \in \ell^{\perp\perp} = h^{\perp\perp}$. This proves (i). |⌛|

Therefore,

$$\mathcal{D}_1 = \{h \in \mathfrak{U} \mid \text{there exist } h_1, h_2, \ldots \in \mathfrak{U} \text{ such that } h = h_1,\ h_{n+1} + h_{n+1} \leqq h_n, \text{ and } h_{n+1}{}^{\perp} = h_n{}^{\perp}\}.$$

Let

$$\mathcal{D}_2 = \{h \in G^+ \setminus \{0\} \mid \text{there exists a convex } \ell\text{-subgroup } C \subseteq G \text{ such that (i) for all } c \in C,\ c < h, \text{ and (ii) if } a<t<b \text{ for } a, -h+b \in C, \text{ then } t \in C \cup (h+C)\}.$$

Lemma 2.2: Let $h \in G^+ \setminus \{0\}$. Let C be a convex ℓ-subgroup of G such that if $c \in C$, then $c < h$. Then

(a) for all $c,d \in C$, $h+d > c$,

(b) $C \subseteq [-h,h]$.

Proof: (a) Since $c-d \in C$, then $(c-d) \wedge h = c-d$. Thus

$$c \wedge (h+d) = [(c-d) \wedge h] + d = (c-d) + d = c,$$

and hence $h+d \geqq c$. If $h+d = c$, $h \in C$, which contradicts the condition on C. Hence $h+d > c$.

(b) Let $c \in C$. Then $c < h$. Since C is an ℓ-subgroup, $-c \in C$. Hence $-c < h$, i.e. $-h < c$. Thus $C \subseteq [-h,h]$. |⌛|

Proposition 2.3: $\mathcal{D}_2 \subseteq \mathfrak{U}$.

Proof: Let $h \in \mathcal{D}_2$, and let C be a convex ℓ-subgroup satisfying (i) and (ii) of the definition of $\mathcal{D}_2$. Suppose $\ell, k \in G$ are such that $\ell \wedge k = 0$ and $\ell \vee k = h$. Then $\ell + k = h = k + \ell$. Since $h \notin C$ and since $\ell + k = h$, we cannot have both $k \in C$ and $\ell \in C$. Suppose $\ell \notin C$. Then since $\ell \in [0,h]$, we have by (ii) that $\ell \in h+C$. Thus there is a

$b \in C$ such that $\ell = h + b$. Since $h = \ell + k$, this implies that $0 = k + b$, i.e. $k \in C$. Thus by Lemma 2.2(a), $\ell > k$. Therefore, $k = \ell \wedge k = 0$, and $\ell = \ell + k = h$. If $k \notin C$, we have similarly that $k = h$, and thus $h \in \mathfrak{U}$. |▯|

Proposition 2.4: For $h \in \mathcal{D}_2$, there is a unique C satisfying conditions (i) and (ii) in the definition of $\mathcal{D}_2$.

Proof: Suppose C and C' are two convex ℓ-subgroups of G satisfying conditions (i) and (ii) in the definition of $\mathcal{D}_2$. Let $d \in C'$. Then either $d \in C$ or $d \notin C$. Suppose $d \notin C$. Then either $d^+ \notin C$ or $d^- \notin C$; for otherwise, since $d^- \leqq d \leqq d^+$ and C is convex, $d \in C$. If $d^+ \notin C$, let $b = d^+$. If $d^+ \in C$, let $b = -d^-$. Then $b \in C'$, $b \notin C$, and $0 < b < h$. Since $-h + h = 0 \in C$, then by (ii) $b \in C \cup (h+C)$. Since $b \notin C$, $b \in h+C$, i.e. $-h + b \in C$. Since C is an ℓ-subgroup, thus $-b + h \in C$. If $-b + h \in C'$, then $-b \notin C'$ since $h \notin C'$. But this contradicts $b \in C'$. Hence $-b + h \notin C'$. Since $0 < b < h$, $0 < -b+h < h$. Thus $-b + h \in C$, $-b+h \notin C'$, and $0 < -b+h < h$. Then by (ii) applied to C', $-b+h \in h+C'$. Thus by Lemma 2.2(a), $-b+h > k$ for all $k \in C'$. By Lemma 2.2(b), $C' \subseteq [-h+b, -b+h]$. Since C is convex and $-b+h \in C$, $C' \subseteq C$. But we assumed $d \in C' \setminus C$. This is a contradiction. Hence $C' \subseteq C$. Similarly we may show that $C \subseteq C'$. |▯|

For $h \in \mathcal{D}_2$, let the unique convex ℓ-subgroup satisfying (i) and (ii) be denoted by D(h).

We digress for a few paragraphs to connect the set $\mathcal{D}_2$ with some standard ideas in the theory of lattice-ordered groups. We note that since we do not use the uniqueness of D(h) in the proof of Proposition 2.7 (let D(h) in the proof be any ℓ-subgroup satisfying (i) and (ii)), then Proposition 2.4 is a corollary of

Proposition 2.7.

Conrad [15] makes the following definitions: Let N be a convex ℓ-subgroup of an ℓ-group G. Then G is a lexico-extension of N if and only if N is a normal subgroup of G, G/N is a totally ordered group, and each positive element in $G \setminus N$ exceeds every element in N. If $g \in G$, then a convex ℓ-subgroup M of G which is maximal with respect to not containing g is called a value of g. If $g \in G$ has only one value, then g is called special. Clearly, every $g \neq 0$ has at least one value.

The following proposition is proven in [17].

Proposition 2.5: For $g \in G \setminus \{0\}$, the following are equivalent:

(a) G(g) is a lexico-extension of a proper ℓ-ideal,

(b) g is special in G(g).

If this is the case and if N is the unique value of g in G(g), then G(g) is a lexico-extension of N. |X|

An element $a \in G$ is a non-unit if $a > 0$ and $a \wedge b = 0$ for some $b \in G^+ \setminus \{0\}$. The subgroup generated by the non-units of G, denoted Lex(G), is an ℓ-ideal and is called the lex-kernel of G.

The following proposition is proven in [18].

Proposition 2.6: For $g \in G^+ \setminus \{0\}$, the following are equivalent:

(a) $g \notin \mathrm{Lex}(G)$,

(b) g is special and $g^\perp = \{0\}$. |X|

If G is an ℓ-group and H is a convex ℓ-subgroup of G, we let L(G,H) denote the partially ordered set of left cosets of H in G. (L(G,H) is ordered by $x+H \leq y+H$ if and only if there exists $h \in H$ such that $x \leq y + h$.)

<u>Proposition 2.7</u>: An element $h \in \mathcal{D}_2$ if and only if h satisfies the following three properties:

(a) $h > 0$,

(b) h is special in G(h),

(c) if N is the unique value of h in G(h), then in L(G(h),N) $[N,h+N] = \{N,h+N\}$.

We first prove four lemmas.

<u>Lemma 2.8</u>: Let $h \in \mathcal{D}_2$. Then for all $d \in D(h)$, $(h+d)^\perp = h^\perp$.

<u>Proof</u>: Let $k \in h^\perp$. Then $h \wedge |k| = 0$. Since $d \in D(h)$, then by Lemma 2.2(a) $h > |d| \geqq 0$, and $h + |d| \geqq h + d > 0$. Since $h > |d| \geqq 0$, $|d| \wedge |k| = 0$. Thus $(h + |d|) \wedge |k| = 0$. Since $h + |d| \geqq h + d > 0$, this implies that $(h + d) \wedge |k| = 0$, i.e. $k \in (h+d)^\perp$.

Let $k \in (h+d)^\perp$. Then $(h + d) \wedge |k| = 0$. By Lemma 2.2(a) $h + d > |-d| \geqq 0$ and hence $|-d| \wedge |k| = 0$. Hence $(h + d + |-d|) \wedge |k| = 0$. Since $|-d| \geqq -d$, $h + d + |-d| \geqq h + d + (-d) = h > 0$. Thus $h \wedge |k| = 0$, i.e. $k \in h^\perp$. |X|

<u>Lemma 2.9</u>: Let $h \in \mathcal{D}_2$. Let $a \in G(h)$. If $h \wedge a \in D(h)$, then $a \in D(h)$.

<u>Proof</u>: Let $h \wedge a = c \in D(h)$. Then $(h - c) \wedge (a - c) = 0$, and thus $a-c \in (h-c)^\perp$. By Lemma 2.8, $a-c \in h^\perp$, i.e. $h \wedge (a-c) = 0$. Since $a \in G(h)$, $a-c \in G(h)$. Hence $a-c = 0$, i.e. $a = c \in D(h)$. |X|

<u>Lemma 2.10</u>: Let $h \in \mathcal{D}_2$. Then D(h) is a value of h in G(h).

<u>Proof</u>: Let $k \in G(h)^+ \setminus D(h)$. Let H be the convex ℓ-subgroup of G(h) generated by $\{k, D(h)\}$. If $k \wedge h \in D(h)$, then by Lemma 2.9 $k \in D(h)$. Thus $k \wedge h \notin D(h)$, and by (ii) $k \wedge h = h + c$ for some $c \in D(h)$. Thus $h = k \wedge h - c$. Since $0 \leqq k \wedge h \leqq k$ and H is convex, then $k \wedge h \in H$.

Since $c \in D(h) \subseteq H$, $h = k \wedge h - c \in H$. Thus $D(h)$ is a value of h in $G(h)$. |∎|

Lemma 2.11: If $h \in \mathcal{D}_2$, then $h \notin \mathrm{Lex}(G(h))$.

Proof: Let a be a non-unit of $G(h)$. Suppose $b \in G(h)^+ \backslash \{0\}$ is such that $a \wedge b = 0$. Then $(h \wedge a) \wedge (h \wedge b) = 0$. If $h \wedge a \notin D(h)$, then $h \wedge a \in h+D(h)$. If $h \wedge b \in D(h)$, then $h \wedge a > h \wedge b$ by Lemma 2.2(a), and hence $h \wedge b = 0$. Since $b \in G(h)$, this implies that $b = 0$, which contradicts our choice of b. Suppose $h \wedge b \notin D(h)$. Then $h \wedge b \in h+D(h)$. If $D(h) = \{0\}$, then $h \wedge a = h = h \wedge b$ and hence $a \wedge b \geq h > 0$, which contradicts our choice of b. If $D(h) \neq \{0\}$, there is a $c \in D(h)^+ \backslash \{0\}$. By Lemma 2.2(a), $h \wedge a > c$ and $h \wedge b > c$. Then

$$a \wedge b \geq (h \wedge a) \wedge (h \wedge b) \geq c > 0.$$

This also contradicts our choice of b. Hence $h \wedge a \in D(h)$. Therefore, $D(h)$ contains all the non-units of $G(h)$. Thus $\mathrm{Lex}(G(h)) \subseteq D(h)$, and hence $h \notin \mathrm{Lex}(G(h))$. |∎|

Proof of Proposition 2.7: Suppose $h \in \mathcal{D}_2$. Clearly $h > 0$. By Lemma 2.11, $h \notin \mathrm{Lex}(G(h))$. Hence by Lemma 2.6, h is special in $G(h)$. By Lemma 2.10, $D(h)$ is a value of h. Since h is special, $D(h)$ is the unique value of h in $G(h)$. In $L(G(h),D(h))$, suppose that $d+D(h) \in [D(h),h+D(h)]$. Then for some $a \in D(h)$ and some $b \in h+D(h)$, $a < d < b$. Since $b \in h+D(h)$, $-h+b \in D(h)$. Hence by (ii) $d \in D(h) \cup (h+D(h))$. Thus $d+D(h) \in \{D(h),h+D(h)\}$, and hence $[D(h),h+D(h)] = \{D(h),h+D(h)\}$.

Suppose h satisfies (a), (b), and (c) of the proposition. By Proposition 2.5, $G(h)$ is a lexico-extension of the ℓ-ideal N which is the unique value of h in G(h). Since $h \in G(h)^{+} \setminus N$, by definition of a lexico-extension $h > n$ for all $n \in N$. Suppose that $a < t < b$ for $a, -h+b \in N$. Then $a+N = N$ and $b+N = h+N$. We have that $a+N \leq t+N \leq b+N$ and hence $N \leq t+N \leq h+N$. By (c), $t+N \in \{N, h+N\}$, i.e. $t \in N \cup (h+N)$. Thus, $h \in \mathcal{D}_2$. |X|

This ends our digression; we continue with the development of the topology. The following example shows that $\mathcal{D}_1 \cap \mathcal{D}_2$ need not be empty. Let $G = R \overleftarrow{X} Z$. Clearly $(0,1) \in \mathcal{D}_2$ with $D((0,1)) = R \times \{0\}$. Let $h_1 = (0,1)$. For $n > 1$, let $h_n = (1/2^n, 0)$. Then clearly $h_1, h_2, \ldots \in \mathfrak{A}$, $h_1 = (0,1)$, $h_{n+1} + h_{n+1} \leqq h_n$, and $h_n^{\perp} = 0 = h_{n+1}^{\perp}$. Hence $(0,1) \in \mathcal{D}_1$.

Let $\mathcal{D}^* = \mathcal{D}_1 \cup \mathcal{D}_2$. By Proposition 2.3, $\mathcal{D}^* \subseteq \mathfrak{A}$. If $\mathcal{D}^* \neq \emptyset$, let

$$\mathcal{N}_1(0) = \{N(0,g) \mid g \in \mathcal{D}_1\}$$

$$\mathcal{N}_2(0) = \{D(h) + h^{\perp} \mid h \in \mathcal{D}_2\}$$

$$\mathcal{N}_3(0) = \mathcal{N}_1(0) \cup \mathcal{N}_2(0)$$

$$\mathcal{N}(0) = \{\bigcap_{i=1}^{n} H_i \mid H_i \in \mathcal{N}_3(0) \text{ for all } i = 1,2, \ldots, n\}.$$

If $\mathcal{D}^* = \emptyset$, let

$$\mathcal{N}(0) = \{G\}.$$

If $G = R \;|X|\; R \;|X|\; R$, then $\mathcal{D}_2 = \emptyset$ and

$$\mathfrak{A} = \mathcal{D}_1 = \mathcal{D}^* = \{(a,0,0) \mid a>0\} \cup \{(0,b,0) \mid b>0\} \cup \{(0,0,c) \mid c>0\},$$

and if $g = (a,0,0)$ for $a>0$, then

$$N(0,g) = [-a,a] \times R \times R.$$

If $g \in \{(0,b,0) \mid b>0\}$ and $h \in \{(0,0,c) \mid c>0\}$, then

$$N(0,g) = R \times [-b,b] \times R$$

$$N(0,h) = R \times R \times [-c,c].$$

The elements of $\mathcal{N}(0)$, therefore, have the form

$[-a,a] \times R \times R$, $R \times [-b,b] \times R$, $R \times R \times [-c,c]$,

$[-a,a] \times [-b,b] \times R = [(-a,-b),(a,b)] \times R$

$R \times [-b,b] \times [-c,c] = R \times [(-b,-c),(b,c)]$

$[-a,a] \times R \times [-c,c]$, or

$[-a,a] \times [-b,b] \times [-c,c] = [(-a,-b,-c),(a,b,c)]$

where $a,b,c > 0$. See figures VI and VII.

If $G = Z \,|X|\, Z$,

$$\mathfrak{U} = \{(a,0)\,|\,a>0\} \cup \{(0,b)\,|\,b>0\};$$

$$\mathfrak{U} \supseteq \mathcal{D}_2 = \mathcal{D}^* = \{(1,0),(0,1)\};\ \mathcal{D}_1 = \phi.$$

Thus, $\mathcal{N}_1(0) = \phi$, and

$$\mathcal{N}_2(0) = \{Z \times \{0\},\ \{0\} \times Z\}.$$

Hence

$$\mathcal{N}(0) = \{Z \times \{0\},\ \{0\} \times Z,\ \{0\} \times \{0\} = \{(0,0)\}\}.$$

If $G = (Z \,|X|\, R) \overleftarrow{X} Z$,

$$\mathfrak{U} = \{(a,0,0)\,|\,a>0\} \cup \{(0,b,0)\,|\,b>0\} \cup \{(a,b,c)\,|\,c>0\},$$

$$\mathcal{D}_1 = \{(0,b,0)\,|\,b>0\},$$

$$\mathcal{D}_2 = \{(1,0,0)\} \cup \{(a,b,1)\,|\,a\varepsilon Z,\ b\varepsilon R\};$$

$$\mathfrak{U} \supseteq \mathcal{D}^*,\ \text{but}\ \mathfrak{U} \neq \mathcal{D}^*.$$

The elements of $\mathcal{N}_1(0)$ have the form

$$N(0,(0,b,0)) = Z \times [-b,b] \times \{0\}$$

for $b > 0$, and

$$\mathcal{N}_2(0) = \{\{0\} \times R \times \{0\},\ Z \times R \times \{0\}\}.$$

Hence the elements of $\mathcal{N}(0)$ have the form

$Z \times [-b,b] \times \{0\}$, $\{0\} \times R \times \{0\}$, $Z \times R \times \{0\}$,

$\{0\} \times [-b,b] \times \{0\} = [(0,-b,0),(0,b,0)]$

for $b > 0$. See figure VIII.

figure VI

$$G = \prod_{1}^{3} R$$

[-a,a] X R X [-c,c]

a,c > 0

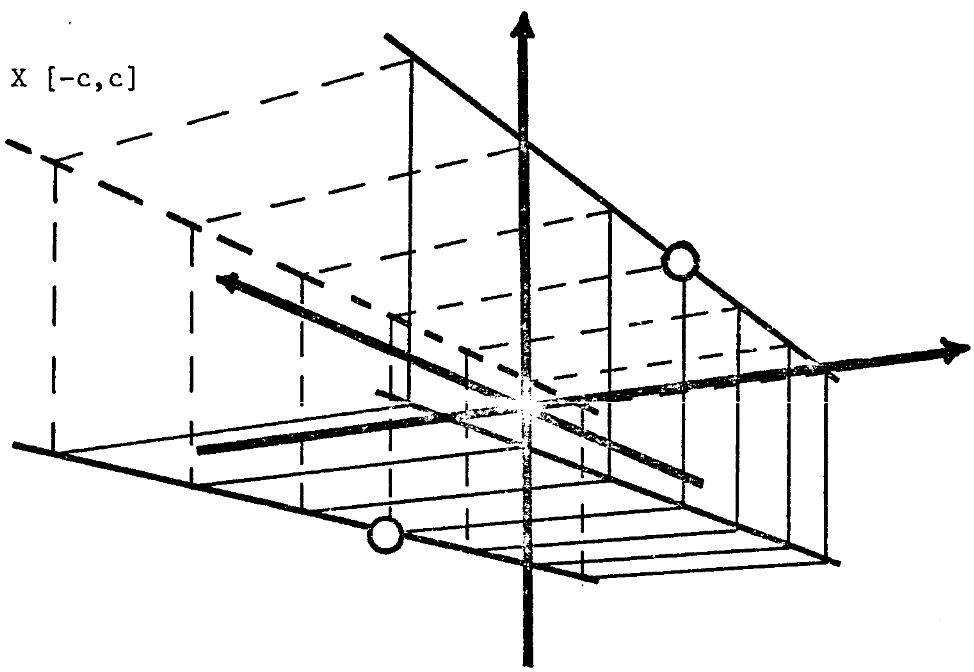


figure VII

$$G = \prod_{1}^{3} R$$

[-a,-b,-c] X [a,b,c]

a,b,c > 0

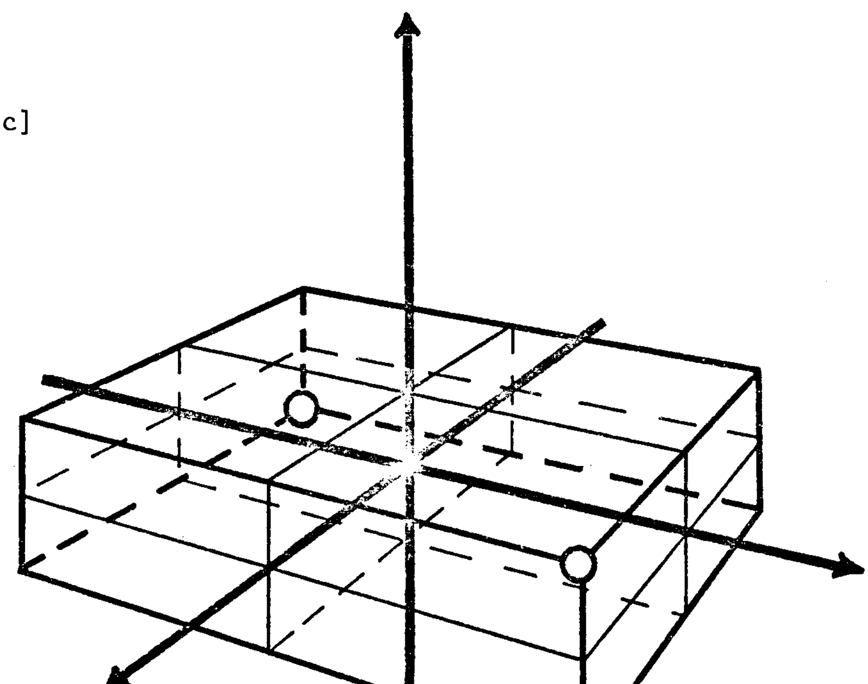
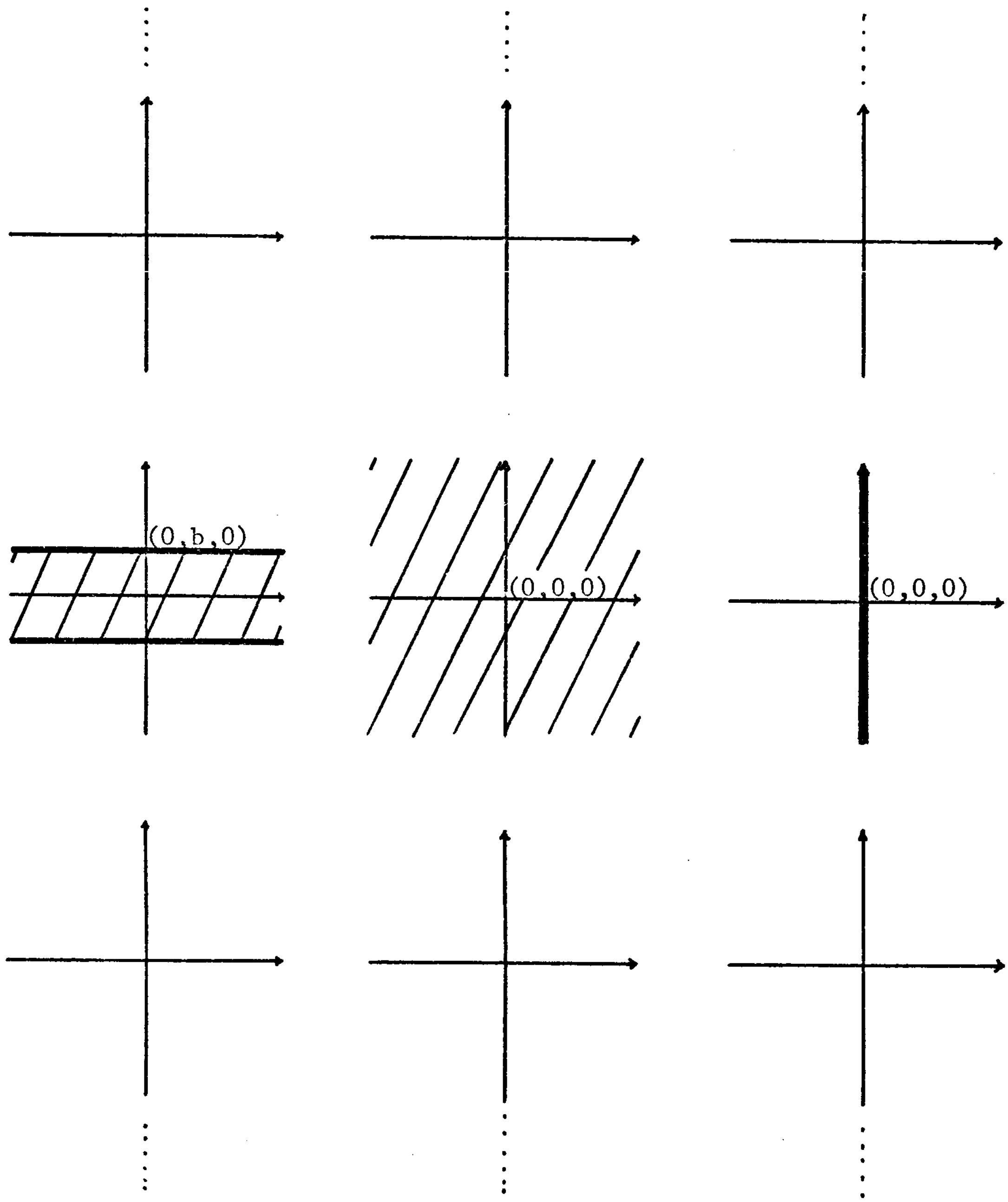


figure VIII

$G = (Z \,|X|\, R) \overleftarrow{X} Z$



Z X [-b,b] X {0}   Z X R X {0}   {0} X R X {0}

b > 0

The main lemma in the construction of the topology for G is the following.

Lemma 2.12: $N(0)$ is a filter-base satisfying:

(a) Each $H \in N(0)$ is symmetric.

(b) For each $H \in N(0)$, there is a $K \in N(0)$ such that $K + K \subseteq H$.

(c) For each $H \in N(0)$ and each $a \in G$, there is a $K \in N(0)$ such that $a + K - a \subseteq H$.

For the proof of Lemma 2.12, we need the following four lemmas.

Lemma 2.13: Let $g \in G^+\setminus\{0\}$. Let $a \in [-g,g]$ and $b \in g^\perp$.

(a) If $a + b \in G^+$, then $a \wedge b = 0$.

(b) $a + b = b + a$.

Proof: (a) Let $h \in [-g,g] \cap g^\perp$. Since $h \in [-g,g]$, then $-h \in [-g,g]$ and hence $|h| = (h) \vee (-h) = (g \wedge h) \vee (g \wedge (-h)) = g \wedge ((h) \vee (-h)) = g \wedge |h|$. Since $h \in g^\perp$, $|h| = |h| \wedge g = 0$. Thus $h = 0$, and hence $[-g,g] \cap g^\perp = \{0\}$.

Since $a + b \geqq 0$, $a \geqq -b$ and hence $a^+ \geqq (-b)^+ \geqq 0$. By assumption $b \in g^\perp$ and hence $(-b)^+ \in g^\perp$. Since $a \in [-g,g]$, $a^+ = a \vee 0 \in [-g,g]$. Thus, $(-b)^+ \in [-g,g] \cap g^\perp$. By the above, $(-b)^+ = 0$, i.e. $b \geqq 0$. Similarly, $a \geqq 0$. Thus, $a \wedge b \geqq 0$, and hence $a \wedge b \in [-g,g] \cap g^\perp$. Therefore, $a \wedge b=0$.

(b) Since $a \in [-g,g]$, $|a| = a \vee (-a) \leqq g \vee g = g$. Thus, since $|b| \wedge g = 0$, $|a| \wedge |b| = 0$. Hence $a + b = b + a$. |X|

Lemma 2.14: For all $a,k \in G$,

(a) $-a + |k| + a = |-a + k + a|$,

(b) $a + (-a + k + a)^\perp - a = k^\perp$.

Proof:

(a) $-a + |k| + a = -a + [k \vee (-k)] + a$

$= (-a + k + a) \vee (-a - k + a)$

$= (-a + k + a) \vee (-(-a + k + a))$

$= |-a + k + a|.$

(b) $k^{\perp} = \{h \mid |h| \wedge |k| = 0\}$

$= \{h \mid (-a + |h| + a) \wedge (-a + |k| + a) = 0\}$

$= \{h \mid |-a + h + a| \wedge |-a + k + a| = 0\}$ (by (a))

$= \{h \mid -a + h + a \in (-a + k + a)^{\perp}\}$

$= a + (-a + k + a)^{\perp} - a$ |X|

Lemma 2.15: For $\ell \in G^{+} \setminus \{0\}$ and $a \in G$, $k \in \mathcal{T}(\ell)$ if and only if $-a + k + a \in \mathcal{T}(-a + \ell + a)$. Therefore, $g \in \mathfrak{U}$ if and only if $-a + g + a \in \mathfrak{U}$.

Proof: Suppose $k \in \mathcal{T}(\ell)$ and let $k' \in G$ be such that $k \wedge k' = 0$ and $k \vee k' = \ell$. Then

$(-a + k + a) \wedge (-a + k' + a) = -a + (k \wedge k') + a = 0,$

$(-a + k + a) \vee (-a + k' + a) = -a + (k \vee k') + a = -a + \ell + a.$

Thus $-a + k + a \in \mathcal{T}(-a + \ell + a)$. Similarly, if $-a + k + a \in \mathcal{T}(-a + \ell + a)$, then $k \in \mathcal{T}(\ell)$.

Let $g \in \mathfrak{U}$ and let $h = -a + g + a$. If $\ell \in \mathcal{T}(h) \setminus \{0\}$, then $a + \ell - a \in \mathcal{T}(g) \setminus \{0\}$ by the above. Since $g \in \mathfrak{U}$ and $\ell > 0$, $a + \ell - a = g$ and thus $\ell = h$. Therefore, $h \in \mathfrak{U}$. Similarly, if $-a + g + a \in \mathfrak{U}$, then $g \in \mathfrak{U}$. |X|

Lemma 2.16: Let $h \in \mathcal{D}_2$ and $a \in G$. Then $-a + h + a \in \mathcal{D}_2$ and $-a + D(h) + a = D(-a + h + a)$.

Proof: Let $\ell = -a + h + a$ and $L = -a + D(h) + a$. If $b \in L$, then $b = -a + b' + a$ for some $b' \in D(h)$. Thus $b = -a + b' + a <$ $-a + h + a = \ell$. Suppose $x < t < y$ for $x, -\ell + y \in L$. Then $x = -a + x' + a$ and $-\ell + y = -a + y' + a$ for $x', y' \in D(h)$. Thus

$$-a + x' + a < t < (-a + h + a) + (-a + y' + a).$$

Hence

$$x' < a + t - a < h + y'.$$

Thus, since $h \in \mathcal{D}_2$ and $x', y' \in D(h)$, $a + t - a \in D(h) \cup (h+D(h))$. Hence

$$t \in (-a + D(h) + a) \cup ((-a + h + a) + (-a + D(h) + a))$$
$$= L \cup (\ell + L).$$

Since $D(h)$ is a convex ℓ-subgroup and since L is a conjugate of $D(h)$, L is a convex ℓ-subgroup. Therefore, $\ell \in \mathcal{D}_2$ and $D(\ell) = L$. |⌧|

Proof of Lemma 2.12: If $\mathcal{D}^* = \phi$, then $N(0) = \{G\}$, and clearly the lemma holds for $N(0)$.

For the remainder of the proof we assume that $\mathcal{D}^* \neq \phi$. For all $g \in G^+ \setminus \{0\}$, $0 \in g^{\perp} \subseteq [-g,g] + g^{\perp}$. Hence for all $H \in N(0)$, $H \neq \phi$. By definition, $N(0)$ is closed under finite intersections. Thus $N(0)$ is a filter-base.

(a) If $H \in N_1(0)$, then $H = N(0,g)$ for some $g \in \mathcal{D}_1$. If $p \in N(0,g)$, then $p = p_1 + p_2$ where $p_1 \in [-g,g]$ and $p_2 \in g^{\perp}$. By Lemma 2.13(b) $p = p_1 + p_2 = p_2 + p_1$. Thus $-p = -p_1 - p_2$. Clearly $-p_2 \in g^{\perp}$. Since $-g \leqq p_1 \leqq g$, then $g \geqq -p_1 \geqq -g$. Thus $-p_1 \in [-g,g]$ and hence $-p \in N(0,g)$. Therefore H is symmetric.

If $H \in N_2(0)$, then $H = D(h) + h^{\perp}$ for some $h \in \mathcal{D}_2$. Since $D(h) \subseteq [-h,h]$, we have by Lemma 2.13(b) that for all $p \in D(h) + h^{\perp}$ there exist $p_1 \in D(h)$ and $p_2 \in h^{\perp}$ such that $p = p_1 + p_2 = p_2 + p_1$. Thus $-p = -p_1 - p_2$. Clearly $-p_1 \in D(h)$ and $-p_2 \in h^{\perp}$. Therefore

$-p \in D(h) + h^{\perp}$ and thus H is symmetric.

Therefore, for all $H \in \mathcal{N}_3(0)$, H is symmetric, and hence for all $H \in \mathcal{N}(0)$, H is symmetric.

(b) If $L \in \mathcal{N}_1(0)$, then $L = N(0,g)$ for some $g \in \mathcal{D}_1$. Since $g \in \mathcal{D}_1$, there is an $h \in \mathcal{D}_1$ such that $h + h \leq g$ and $h^{\perp} \subseteq g^{\perp}$. Let $L^* = N(0,h)$. Since $h \in \mathcal{D}_1$, $L^* \in \mathcal{N}_1(0)$. If $p \in L^* + L^*$, then $p = a + b + c + d$ where $a,c \in [-h,h]$ and $b,d \in h^{\perp}$. By Lemma 2.13(b), $p = (a + c) + (b + d)$. Since $h^{\perp} \subseteq g^{\perp}$, $b + d \in g^{\perp}$, and since $a,c \in [-h,h]$, $a + c \in [-h-h,h+h] \subseteq [-g,g]$. Thus $p \in L$ and therefore $L^* + L^* \subseteq L$.

If $L \in \mathcal{N}_2(0)$, then $L = D(h) + h^{\perp}$ for some $h \in \mathcal{D}_2$. Let $L^* = L$. If $p \in L^* + L^*$, then $p = a + b + c + d$ for $a,c \in D(h)$ and $b,d \in h^{\perp}$. Since $D(h) \subseteq [-h,h]$, by Lemma 2.13(b) $p = (a + c) + (b + d)$. Clearly $a + c \in D(h)$ and $b + d \in h^{\perp}$. Thus $p \in L$, and therefore $L^* + L^* \subseteq L$. (Along with (a), this shows that L is in fact a subgroup.)

If $H \in \mathcal{N}(0)$, then $H = \bigcap_{i=1}^{n} L_i$ for $L_i \in \mathcal{N}_3(0)$. Let $K = \bigcap_{i=1}^{n} L_i^*$. Then $K \in \mathcal{N}(0)$. If $\ell \in K + K$, $\ell = \ell_1 + \ell_2$ for $\ell_1, \ell_2 \in \bigcap_{i=1}^{n} L_i^*$. Since $\ell_1, \ell_2 \in L_i^*$ for all $i = 1,2, \ldots,n$, we have by the above that $\ell_1 + \ell_2 \in L_i^* + L_i^* \subseteq L_i$ for all $i = 1,2, \ldots,n$. Thus $\ell \in H$. Therefore $K + K \subseteq H$.

(c) Let $a \in G$.

If $H \in \mathcal{N}_1(0)$, then $H = N(0,g)$ for $g \in \mathcal{D}_1$. Let $h = -a + g + a$. Since $g \in \mathcal{D}_1$, there are $g_1, g_2, \ldots \in \mathfrak{U}$ such that $g = g_1$, $g_{n+1} + g_{n+1} \leq g_n$, and $g_{n+1}^{\perp} \subseteq g_n^{\perp}$. Consider $-a + g_1 + a$, $-a + g_2 + a, \ldots$. By definition, $-a + g_1 + a = h$. For any n we have the following: By Lemma 2.15, $-a + g_n + a \in \mathfrak{U}$. Since

$g_{n+1} + g_{n+1} \leqq g_n$, $(-a + g_{n+1} + a) + (-a + g_{n+1} + a) \leqq -a + g_n + a$. Since $g_{n+1}^{\perp} \subseteq g_n^{\perp}$, $-a + g_{n+1}^{\perp} + a \subseteq -a + g_n^{\perp} + a$. Hence by Lemma 2.14(b) $(-a + g_{n+1} + a)^{\perp} \subseteq (-a + g_n + a)^{\perp}$. Thus $h \in \mathcal{D}_1$. Let $H^a = N(0,h)$. Then $H^a \in \mathcal{N}_1(0)$. Let $p \in a + H^a - a$. Then $p = a + h_1 + h_2 - a$ for $h_1 \in [-h,h]$ and $h_2 \in h^{\perp}$. Since $h_1 \in [-h,h]$ and $h = -a + g + a$, then $a + h_1 - a \in [-g,g]$. By Lemma 2.14(b) $g^{\perp} = a + h^{\perp} - a$. Since $h_2 \in h^{\perp}$, $a + h_2 - a \in g^{\perp}$. Therefore $p = (a + h_1 - a) + (a + h_2 - a) \in H$. Hence $a + H^a - a \subseteq H$.

If $H \in \mathcal{N}_2(0)$, then $H = D(h) + h^{\perp}$ for $h \in \mathcal{D}_2$. Let $H^a = -a + H + a$. Then $H^a = (-a + D(h) + a) + (-a + h^{\perp} + a)$. By Lemma 2.14(b) $-a + h^{\perp} + a = (-a + h + a)^{\perp}$. By Lemma 2.16 $-a + h + a \in \mathcal{D}_2$ and $-a + D(h) + a = D(-a + h + a)$. Hence $H^a = D(-a + h + a) + (-a + h + a)^{\perp} \in \mathcal{N}_2(0)$. Clearly $a + H^a - a = H$.

Let $H \in \mathcal{N}(0)$. Then $H = \bigcap_{i=1}^{n} H_i$ for $H_i \in \mathcal{N}_3(0)$. Let $K = \bigcap_{i=1}^{n} H_i^a$. Then $K \in \mathcal{N}(0)$. Since $\bigcap_{i=1}^{n} H_i^a \subseteq H_i^a$ for all $i = 1,2,\ldots,n$,

$$a + K - a = a + \bigcap_{i=1}^{n} H_i^a - a \subseteq \bigcap_{i=1}^{n} (a + H_i^a - a) \subseteq \bigcap_{i=1}^{n} H_i = H. \quad |\!\boxtimes\!|$$

For $g \in G\setminus\{0\}$, let

$$\mathcal{N}(g) = \{g+H \mid H\in\mathcal{N}(0)\}.$$

Let

$$\mathfrak{T} = \{W \subseteq G \mid \text{ for all } x\in W,\ W\in F(\mathcal{N}(x))\}.$$

Theorem 2.17: The ℓ-group G is a topological group with respect to the topology $\mathfrak{T}$.

Proof: The theorem follows immediately from Lemma 2.12 and Theorem A. $|\!\boxtimes\!|$

We call $\mathfrak{T}$ the $\mathfrak{T}$-topology on G.

We prove that G is a topological lattice with respect to its $\mathfrak{T}$-topology by establishing a criterion for an ℓ-group with a topology to be a topological lattice.

Lemma 2.18: Let G be an ℓ-group, and let $U \subseteq G$. If U is a convex sublattice of G, then for $r,s \in G$

(a) $(r+U) \wedge (s+U) = (r \wedge s)+U$,

(b) $(r+U) \vee (s+U) = (r \vee s)+U$.

Proof: (a) Let $u \in U$. Then

$$(r \wedge s) + u = (r + u) \wedge (s + u) \in (r+U) \wedge (s+U).$$

Thus $((r \wedge s)+U) \subseteq (r+U) \wedge (s+U)$.

Let $u,w \in U$. Then

$$(r + (u \wedge w)) \wedge (s + (u \wedge w)) \leqq (r + u) \wedge (s + w)$$
$$\leqq (r + (u \vee w)) \wedge (s + (u \vee w)).$$

Hence

$$u \wedge w \leqq -(r \wedge s) + (r + u) \wedge (s + w) \leqq u \vee w.$$

Since U is a convex sublattice of G, this implies that

$-(r \wedge s) + (r + u) \wedge (s + w) \in U$, i.e.

$(r + u) \wedge (s + w) \in (r \wedge s)+U$.

Thus $(r+U) \wedge (s+U) \subseteq (r \wedge s)+U$. Therefore (a) holds.

Similarly (b) holds. |X|

(NB: Lemma 2.18 says that the partially ordered set of left cosets of a convex ℓ-subgroup of G is a lattice.)

Let L be an ℓ-group and let $\mathcal{U}$ be a collection of subsets of L. Let

$$\mathcal{V}(\mathcal{U}) = \{W \subseteq L \mid \text{for all } x \in W, \text{ there exists } U \in \mathcal{U} \text{ such that } x + U \subseteq W\}.$$

Lemma 2.19: If $\mathcal{U}$ is a filter-base, then $V(\mathcal{U})$ is a topology on L.

Proof: Let $\{W_\alpha \mid \alpha \in A\} \subseteq V(\mathcal{U})$. Let $x \in \bigcup_{\alpha\in A} W_\alpha$. Then $x \in W_\alpha$ for some $\alpha \in A$. Since $W_\alpha \in V(\mathcal{U})$, there exists $U \in \mathcal{U}$ such that $x + U \subseteq W_\alpha \subseteq \bigcup_{\alpha\in A} W_\alpha$. Thus $\bigcup_{\alpha\in A} W_\alpha \in V(\mathcal{U})$. Let $\{W_1,\ldots,W_n\} \subseteq V(\mathcal{U})$, and suppose $x \in \bigcap_{i=1}^n W_i$. Then $x \in W_i$ for all $i = 1,\ldots,n$. Since $W_i \in V(\mathcal{U})$ for all i, there exists $U_i \in \mathcal{U}$ for all i such that $x + U_i \subseteq W_i$. Since $x + \bigcap_{i=1}^n U_i \subseteq x + U_i$ for all i,

$$x + \bigcap_{i=1}^n U_i \subseteq \bigcap_{i=1}^n (x + U_i) \subseteq \bigcap_{i=1}^n W_i.$$

Since $\mathcal{U}$ is a filter-base, there exists $U \in \mathcal{U}$ such that $U \subseteq \bigcap_{i=1}^n U_i$.

Hence

$$x + U \subseteq x + \bigcap_{i=1}^n U_i \subseteq \bigcap_{i=1}^n W_i.$$

Therefore $\bigcap_{i=1}^n W_i \in V(\mathcal{U})$. Clearly $\phi \in V(\mathcal{U})$. Since $\mathcal{U}$ is a filter-base, $\mathcal{U} \neq \phi$ and hence $G \in V(\mathcal{U})$. Thus $V(\mathcal{U})$ is a topology on L. |X|

Proposition 2.20: Let G be an ℓ-group and let $\mathcal{U}$ be a collection of subsets satisfying:

(a) $\mathcal{U}$ is a filter-base.

(b) Each $U \in \mathcal{U}$ is a convex sublattice of G.

(c) If $U \in \mathcal{U}$, then there is a $V \in V(\mathcal{U})$ such that $0 \in V \subseteq U$.

Then G is a topological lattice with respect to the topology $V(\mathcal{U})$.

Proof: By (a) and Lemma 2.19 $V(\mathcal{U})$ is a topology for G. Let $W \in V(\mathcal{U})$ and suppose $r,s \in G$ are such that $r \wedge s \in W$. Since $W \in V(\mathcal{U})$, there exists $U \in \mathcal{U}$ such that $(r\wedge s) + U \subseteq W$. By (c) there exists $V \in \mathcal{V}(\mathcal{U})$ such that $0 \in V \subseteq U$. Let $W(r) = r + V$ and $W(s) = s + V$. Since $0 \in V$, $r \in W(r)$ and $s \in W(s)$. By (b) and Lemma 2.18(a)

$$W(r) \wedge W(s) \subseteq (r + U) \wedge (s + U) = (r \wedge s) + U \subseteq W$$

If $x \in W(r)$, then $-r + x \in V$ and since $V \in \mathcal{V}(\mathcal{U})$, there is a $U \in \mathcal{U}$ such that $-r + x + U \subseteq V$. Thus $x + U \subseteq r + V$. Hence $W(r) \in \mathcal{V}(\mathcal{U})$. Similarly $W(s) \in \mathcal{V}(\mathcal{U})$.

Similarly, if $W \in \mathcal{V}(\mathcal{U})$ and $r,s \in G$ are such that $r \vee s \in W$, then we may construct sets $W(r), W(s) \in \mathcal{V}(\mathcal{U})$ such that $r \in W(r)$, $s \in W(s)$, and $W(r) \vee W(s) \subseteq W$.

Therefore G is a topological lattice with respect to the topology $\mathcal{V}(\mathcal{U})$. |⊠|

Using the criterion established in Proposition 2.20, we now prove that an ℓ-group is a topological lattice with respect to its $\mathfrak{T}$-topology.

Let G be an ℓ-group and suppose that there is a topology on G. For all subsets A of G, the <u>interior</u> of A, denoted $\mathrm{Int}(A)$, is the union of all the open sets contained in A.

Let G be an ℓ-group and let $\mathfrak{T}$ be its $\mathfrak{T}$-topology. In proving the following proposition, we use only the definition of $N(0)$, and of $\mathfrak{T}$ in terms of $N(0)$, and property (b) of Lemma 2.12.

<u>Proposition 2.21</u>: For all $A \subseteq G$,

$$\mathrm{Int}(A) = \{r \in G \mid \text{there exists } H \in N(0) \text{ such that } r + H \subseteq A\}.$$

<u>Proof</u>: Let $S = \{r \in G \mid$ there exists $H \in N(0)$ such that $r + H \subseteq A\}$. Since $0 \in H$ for all $H \in N(0)$, $S \subseteq A$. Let $r \in S$ and let $H \in N(0)$ be such that $r + H \subseteq A$. Let $H^* \in N(0)$ be

such that $H^* + H^* \subseteq H$. Then for all $h \in H^*$,

$$r + h + H^* \subseteq r + H^* + H^* \subseteq r + H \subseteq A.$$

Hence $r + h \in S$, and thus $r + H \subseteq S$. Thus $S \in \mathfrak{T}$, and hence $S \subseteq \text{Int}(A)$. If $s \in \text{Int}(A)$, then there is an $H \in N(0)$ such that $s + H \subseteq \text{Int}(A) \subseteq A$. Thus $s \in S$. Therefore $S = \text{Int}(A)$. |X|

<u>Corollary 2.22</u>: For all $H \in N(0)$, $0 \in \text{Int}(H)$, and consequently $\text{Int}(H) \neq \emptyset$.

<u>Proof</u>: For all $H \in N(0)$, $0 + H = H$. |X|

<u>Lemma 2.23</u>: Let $0 \in A \subseteq B \subseteq G$. Suppose A is a convex sublattice of G. Then $A + B^{\perp}$ is a convex sublattice of G. In particular, each $H \in N(0)$ is a convex sublattice of G.

<u>Proof</u>: Suppose $0 \leqq y \leqq \ell_1 + \ell_2$ for $\ell_1 \in A^+$, $\ell_2 \in (B^{\perp})^+$. Since $A \subseteq B$, $A^{\perp} \supseteq B^{\perp}$, and hence $\ell_1 \wedge \ell_2 = 0$. Since A is convex, $y \wedge \ell_1 \in A^+$. Similarly $y \wedge \ell_2 \in (B^{\perp})^+$. Further

$$(y \wedge \ell_1) \wedge (y \wedge \ell_2) = y \wedge (\ell_1 \wedge \ell_2) = 0.$$

Hence

$$(y \wedge \ell_1) + (y \wedge \ell_2) = (y \wedge \ell_1) \vee (y \wedge \ell_2)$$
$$= y \wedge (\ell_1 \vee \ell_2) = y.$$

Thus $y = y_1 + y_2$ where $y_1 \in A^+$, $y_2 \in (B^{\perp})^+$. Similarly, if $\ell_1 + \ell_2 \leqq y \leqq 0$ for $\ell_1 \in A^-$, $\ell_2 \in (B^{\perp})^-$, then $y = y_1 + y_2$ for $y_1 \in A^-$, $y_2 \in (B^{\perp})^-$.

Let $y \in G$ be such that $p \leqq y \leqq q$ for $p,q \in A + B^{\perp}$. Then $p = a + b$, $q = c + d$ for $a,c \in A$, $b,d \in B^{\perp}$, and

$$a^- + b^- \leqq p^- \leqq y^- \leqq 0 \leqq y^+ \leqq q^+ \leqq c^+ + d^+.$$

Since $0,a,c \in A$ and A is a sublattice, $a^- \in A^-$ and $c^+ \in A^+$.

Similarly $b^- \in (B^{\perp})^-$ and $d^+ \in (B^{\perp})^+$. Therefore

$$y^+ = u + v \quad \text{for} \quad u \in A^+ \quad \text{and} \quad v \in (B^{\perp})^+,$$

$$y^- = x + w \quad \text{for} \quad x \in A^- \quad \text{and} \quad w \in (B^{\perp})^-.$$

By Lemma 2.13(b) $y = u + v + x + w = (u + x) + (v + w)$. Clearly $v + w \in B^{\perp}$. Since $x \leqq 0 \leqq u$, $x \leqq u + x \leqq u$ and hence, since A is convex and $x,u \in A$, $u + x \in A$. Therefore $y \in A + B^{\perp}$ and hence $A + B^{\perp}$ is convex.

We note that for $a,b,c,d \in G$,

$$(a \wedge c) + (b \wedge d) \leqq (a + b) \wedge (c + d)$$
$$\leqq (a + b) \vee (c + d) \leqq (a \vee c) + (b \vee d).$$

Thus, since A and $B^{\perp}$ are convex sublattices of G, then $A + B^{\perp}$ is a sublattice of G.

If $H \in N_1(0)$, then $H = N(0,g)$ for $g \in \mathcal{D}_1$. Let $A = [-g,g] = B$. Then $0 \in A \subseteq B$. Clearly A is a convex sublattice of G. Clearly $B^{\perp} = g^{\perp}$. Thus $H = [-g,g] + g^{\perp}$ is a convex sublattice of G.

If $H \in N_2(0)$, then $H = D(h) + h^{\perp}$ for $h \in \mathcal{D}_2$. Let $A = D(h)$ and $B = [-h,h]$. Then $0 \in A \subseteq B$, and A is a convex sublattice. Since $B^{\perp} = g^{\perp}$, H is a convex sublattice of G.

If $H \in N(0)$, then $H = \bigcap_{i=1}^{n} H_i$ for $H_i \in N_3(0)$, or $H = G$. Since each H_i is a convex sublattice of G, H is a convex sublattice of G. Clearly G is a convex sublattice of itself. |X|

Theorem 2.24: An ℓ-group G is a topological lattice with respect to its $\mathfrak{T}$-topology.

Proof: By Lemma 2.12 $N(0)$ is a filter-base. By Lemma 2.23 each $H \in N(0)$ is a convex sublattice of G. By Corollary 2.22, for each $H \varepsilon N(0)$, $0 \varepsilon \mathrm{Int}(H) \subseteq H$. If $\mathfrak{T}$ is the $\mathfrak{T}$-topology on G, then $\mathrm{Int}(H) \varepsilon \mathfrak{T}$. Clearly $\mathfrak{T} = V(N(0))$. Therefore by Proposition 2.20, G is a topological lattice with respect to $\mathfrak{T}$. |X|

In Chapter 6 we will investigate ℓ-groups with Hausdorff $\mathfrak{T}$-topology in some detail. Here we wish to use Theorems 2.17 and 2.24 to discover when (lattice) closed intervals and the elements of $N(0)$ are ($\mathfrak{T}$-topologically) closed. It turns out that the key property is that the $\mathfrak{T}$-topology be Hausdorff.

We use the following theorems from topology ([58, page 75], [58, page 74], and [58, page 96] respectively):

<u>Proposition 2.25</u>: A function f from a topological space W with topology U to a topological space Y with topology V is continuous if and only if for every net $\{x_\beta | \beta \varepsilon B\}$ in W converging to $x \varepsilon W$, the net $\{x_\beta f | \beta \varepsilon B\}$ in Y converges to $xf \varepsilon Y$. |X|

<u>Proposition 2.26</u>: Let Y be a topological space with topology V and let $A \subseteq Y$. Then A is closed with respect to V if and only if whenever $\{x_\beta | \beta \varepsilon B\}$ is a net in A which converges to $x \varepsilon Y$, then $x \varepsilon A$. |X|

<u>Proposition 2.27</u>: A topological space has Hausdorff topology if and only if all nets in the space which have limits have unique limits. |X|

Our method is to prove the desired result in general and then derive the result on the $\mathfrak{T}$-topology as a corollary.

Let L be a lattice with topology $\mathcal{V}$ with respect to which L is a topological lattice. Recall (Chapter 1) that the interval topology on L takes the (lattice) closed intervals as a subbase for the closed sets, where the (lattice) closed intervals are the sets of the form $[a,b]$, $\{x \in L \mid x \leqq a\}$, $\{x \in L \mid x \geqq a\}$, and L itself for $a,b \in L$ with $a \leqq b$. Let $\mathcal{J}$ be the interval topology on L.

Proposition 2.28: If $\mathcal{V}$ is Hausdorff, then $\mathcal{J} \subseteq \mathcal{V}$.

Proof: Since $\mathcal{V}$ is a topology, L is closed with respect to $\mathcal{V}$. Let $a \in L$ and consider $[a,\infty)$. Let $\{x_\beta \mid \beta \in B\}$ be a net in $[a,\infty)$ $\mathcal{V}$-converging to $x \in L$. Since L is a topological lattice with respect to $\mathcal{V}$, the function taking an element $\ell \in L$ to $x \vee \ell$ is continuous. Thus by Proposition 2.25, $\{x \vee x_\beta\}$ converges to $x \vee x = x$. Clearly $x \leqq x \vee x_\beta$ for all $\beta \in B$. Suppose that there exists $z \in L$ such that $x \leqq z \leqq x \vee x_\beta$ for all $\beta \in B$. Let $\{z_\beta\}$ be a net in L defined by $z_\beta = z$ for all $\beta \in B$. Clearly $\{z_\beta\}$ converges to z with respect to $\mathcal{V}$. Since L is a topological lattice with respect to $\mathcal{V}$, we must have that $\{(z_\beta) \vee (x \vee x_\beta)\}$ converges to $z \vee x = z$. But $z_\beta \vee x \vee x_\beta = z \vee x \vee x_\beta = x \vee x_\beta$ for all $\beta \in B$. Thus $\{x \vee x_\beta\}$ converges to z. Since $\mathcal{V}$ is Hausdorff, by Proposition 2.27 $z = x$. Therefore $\bigwedge_{\beta \in B}(x \vee x_\beta)$ exists in L and equals x. But $x_\beta \geqq a$ for all $\beta \in B$ and thus $x \vee x_\beta \geqq a$ for all $\beta \in B$. Hence $x = \bigwedge_{\beta \in B}(x \vee x_\beta) \geqq a$, i.e. $x \in [a,\infty)$. Therefore by Proposition 2.26, $[a,\infty)$ is closed with respect to $\mathcal{V}$. Similarly, $(-\infty,a]$ is closed with respect to $\mathcal{V}$. Since $\mathcal{V}$ is a topology,

$[a,b] = (-\infty,b] \cap [a,\infty)$ is closed with respect to $\mathcal{V}$. Therefore $\mathcal{J} \subseteq \mathcal{V}$. |⌧|

Let G be an ℓ-group with a topology $\mathcal{U}$ with respect to which G is both a topological group and a topological lattice. The group structure is enough to give us the converse of Proposition 2.28.

Proposition 2.29: $\mathcal{J} \subseteq \mathcal{U}$ if and only if $\mathcal{U}$ is Hausdorff.

Proof: If $\mathcal{U}$ is Hausdorff, then by Proposition 2.28 $\mathcal{J} \subseteq \mathcal{U}$, since G is a topological lattice with respect to $\mathcal{U}$.

Conversely, suppose that $\mathcal{J} \subseteq \mathcal{U}$. As we noted in Chapter 1, $\mathcal{J}$ is T_1. Thus since $\mathcal{J} \subseteq \mathcal{U}$, $\mathcal{U}$ is T_1. Then by Theorem B, since $\mathcal{U}$ is a group topology for G, $\mathcal{U}$ is Hausdorff. |⌧|

Applying Proposition 2.29 to the $\mathfrak{T}$-topology, we have the following result.

Corollary 2.30: Let G be an ℓ-group with $\mathfrak{T}$-topology $\mathfrak{T}$. Then the sets $[a,b]$, $\{x \in G \mid x \geqq a\}$, and $\{x \in G \mid x \leqq a\}$ are closed with respect to $\mathfrak{T}$ for all $a,b \in G$ with $a \leqq b$ if and only if $\mathfrak{T}$ is Hausdorff.

Proof: By Theorems 2.17 and 2.24, $\mathfrak{T}$ is a group and lattice topology for G. |⌧|

We investigate the elements of $\mathcal{N}(0)$ with respect to $\mathfrak{T}$-closure by first considering $\mathcal{N}_1(0)$ and $\mathcal{N}_2(0)$. The case for the elements of $\mathcal{N}_2(0)$ is straightforward, and we can use it to get a theorem analagous to Corollary 2.30 (in one direction only) for the sets D(h) where $h \in \mathcal{D}_2$.

Proposition 2.31: Let G be an ℓ-group with $\mathfrak{T}$-topology $\mathfrak{T}$. Let $h \in \mathcal{D}_2$. Then $D(h) + h^{\perp}$ is both open and closed with respect to $\mathfrak{T}$.

Proof: In the proof of Lemma 2.12(b) we proved that $D(h) + h^{\perp}$ is a subgroup of G. Thus if $d \in D(h) + h^{\perp}$, then

$$d + D(h) + h^{\perp} \subseteq D(h) + h^{\perp}.$$

Hence by Proposition 2.21, $D(h) + h^{\perp}$ is open with respect to $\mathfrak{T}$. Therefore by [36, page 54], since $D(h) + h^{\perp}$ is a subgroup, $D(h) + h^{\perp}$ is closed with respect to $\mathfrak{T}$. |■|

Proposition 2.32: If G is an ℓ-group with Hausdorff $\mathfrak{T}$-topology $\mathfrak{T}$, then for all $h \in \mathcal{D}_2$, $D(h)$ is closed with respect to $\mathfrak{T}$.

Proof: Clearly $D(h) \subseteq D(h) + h^{\perp}$. By Lemma 2.2(b), $D(h) \subseteq [-h,h]$, and hence $D(h) \subseteq (D(h) + h^{\perp}) \cap [-h,h]$.

Conversely, let $d \in (D(h) + h^{\perp}) \cap [-h,h]$. By Lemmas 2.12(a) and 2.23, $D(h) + h^{\perp}$ is a symmetric sublattice of G. Since clearly $[-h,h]$ is a symmetric sublattice of G, this implies that $(D(h) + h^{\perp}) \cap [-h,h]$ is a symmetric sublattice of G. Therefore

$$|d| = d \vee (-d) \in (D(h) + h^{\perp}) \cap [-h,h].$$

Since $|d| \in D(h) + h^{\perp}$, by Lemma 2.13 $|d| = a + b = a \vee b$ for $a \in D(h)^+$ and $b \in (h^{\perp})^+$. Since $|d| \in [-h,h]$, we have that

$$|d| = |d| \wedge h = (a \vee b) \wedge h = (a \wedge h) \vee (b \wedge h)$$
$$= (a \wedge h) \vee 0 = a \wedge h = a.$$

Hence $|d| \in D(h)$. Since $D(h)$ is a convex ℓ-subgroup of G and since $-|d| \leqq d \leqq |d|$, this implies that $d \in D(h)$. Therefore $D(h) \supseteq (D(h) + h^{\perp}) \cap [-h,h]$ and hence $D(h) = (D(h) + h^{\perp}) \cap [-h,h]$.

Since $\mathfrak{T}$ is Hausdorff, by Proposition 2.30 $[-h,h]$ is closed with respect to $\mathfrak{T}$, and by Proposition 2.31 $D(h) + h^{\perp}$ is closed with respect to $\mathfrak{T}$. Therefore, since $D(h) = (D(h) + h^{\perp}) \cap [-h,h]$, $D(h)$ is closed with respect to $\mathfrak{T}$. |X|

Example 7.11 shows that in general the converse of Proposition 2.32 fails to hold.

We now turn our attention to the elements of $N_1(0)$. As in the case of intervals, we first prove a general theorem and get the result for the $\mathfrak{T}$-topology as a corollary.

Proposition 2.33: Let G be an ℓ-group with group and lattice topology $\mathcal{U}$. Let $g \in G^+$ and let $0 \in A \subseteq [-g,g]$. Suppose that A is a convex symmetric sublattice of G. If $\mathcal{U}$ is Hausdorff and if A is closed with respect to $\mathcal{U}$, then $A + g^{\perp}$ is closed with respect to $\mathcal{U}$.

Proof: Let $\{x_\beta | \beta \in B\}$ be a net in $A + g^{\perp}$ converging (with respect to $\mathcal{U}$) to $f \in G$. Since A is a symmetric sublattice of G, $|x_\beta| = (x_\beta) \vee (-x_\beta) \in A + g^{\perp}$ for all $\beta \in B$ (2.13(b) and 2.23). Thus $|x_\beta| = a_\beta + b_\beta$ for $a_\beta \in A^+$ and $b_\beta \in (g^{\perp})^+$. Since $\mathcal{U}$ is a group and lattice topology for G, by Proposition 2.25 $\{|x_\beta|\}$ converges to $|f|$. Hence $\{a_\beta\} = \{|x_\beta| \wedge g\}$ converges to $|f| \wedge g$. Since $\{a_\beta\} \subseteq A$ and A is closed, by Proposition 2.26 $|f| \wedge g \in A$. Let $d = -(|f| \wedge g) + |f|$. Then $\{b_\beta\} = \{-a_\beta + |x_\beta|\}$ converges to d, and thus $\{b_\beta \wedge g\}$ converges to $d \wedge g$. Let $0_\beta = 0$ for all $\beta \in B$; then $\{0_\beta\}$ converges to 0. Since $\mathcal{U}$ is Hausdorff and since $b_\beta \wedge g = 0_\beta$ for all $\beta \in B$, by

Proposition 2.27 $d \wedge g = 0$, i.e. $d \in g^{\perp}$. Therefore

$$|f| = |f| \wedge g + d \in A + g^{\perp}.$$

Since A is a convex sublattice of G containing 0, by Lemma 2.23 $A + g^{\perp}$ is a convex sublattice of G. Since A is symmetric, as in the proof of Lemma 2.12(a) $A + g^{\perp}$ is symmetric. Hence $-|f| \in A + g^{\perp}$, and since $-|f| \leqq f \leqq |f|$, $f \in A + g^{\perp}$. Therefore by Proposition 2.26, $A + g^{\perp}$ is closed with respect to $\mathcal{U}$. |X|

Corollary 2.34: Let G be an ℓ-group with $\mathfrak{T}$-topology $\mathfrak{T}$. If $\mathfrak{T}$ is Hausdorff, then every $H \in \mathcal{N}_1(0)$ is closed with respect to $\mathfrak{T}$.

Proof: Let $\mathfrak{T} = \mathcal{U}$ and $[-g,g] = A$ in Proposition 2.33 above, and apply Corollary 2.30. |X|

Proposition 2.35: Let G be an ℓ-group with $\mathfrak{T}$-topology $\mathfrak{T}$. If $\mathfrak{T}$ is Hausdorff, then every $H \in \mathcal{N}(0)$ is closed with respect to $\mathfrak{T}$.

Proof: Clearly G is closed. If $H \in \mathcal{N}(0) \setminus \{G\}$, then H is a finite intersection of elements of $\mathcal{N}_1(0) \cup \mathcal{N}_2(0)$. By Proposition 2.31 and Corollary 2.34, the elements of $\mathcal{N}_1(0) \cup \mathcal{N}_2(0)$ are closed. Thus H is closed with respect to $\mathfrak{T}$. |X|

Example 7.11 shows that in general the converse of Proposition 2.35 fails to hold.

3. SOME EXAMPLES

This chapter is devoted to investigating in some detail two examples of ℓ-groups and their $\mathfrak{T}$-topologies. We do not include the most obvious examples, cardinal products of the real numbers, because in Chapters 4 and 5 we will prove two theorems which together will imply that an arbitrary cardinal product of the real numbers has for its $\mathfrak{T}$-topology the usual topology (i.e. the product of the interval topologies on the factors).

Example 3.1: The first example we consider is A(R). Intuitively, the elements of $\mathfrak{U}$ in A(R) have the form of the functions in figures IX and X; that is, they have only one "bump". We can make this idea more precise in the general case as follows.

Let Ω be a totally ordered set. Let $\hat{\Omega}$, $\bar{\Omega}$ be the completion of Ω and the conditional completion of Ω, respectively, as defined in Chapter 1. In [35] Holland proves that the map from $A(\Omega)$ to $A(\bar{\Omega})$ which takes $g \in A(\Omega)$ to the element $\bar{g} \in A(\bar{\Omega})$ defined by

$$\bar{\alpha}\bar{g} = \vee \{\beta g \mid \beta \in \Omega,\ \beta \leq \bar{\alpha}\}$$

for $\bar{\alpha} \in \bar{\Omega}$ is an ℓ-isomorphism. Clearly $\bar{g}|_{\Omega} = g$. For $f \in A(\Omega)$, let

$$S(\bar{f}) = \{\bar{\eta} \in \bar{\Omega} \mid \bar{\eta}\bar{f} \neq \bar{\eta}\}$$

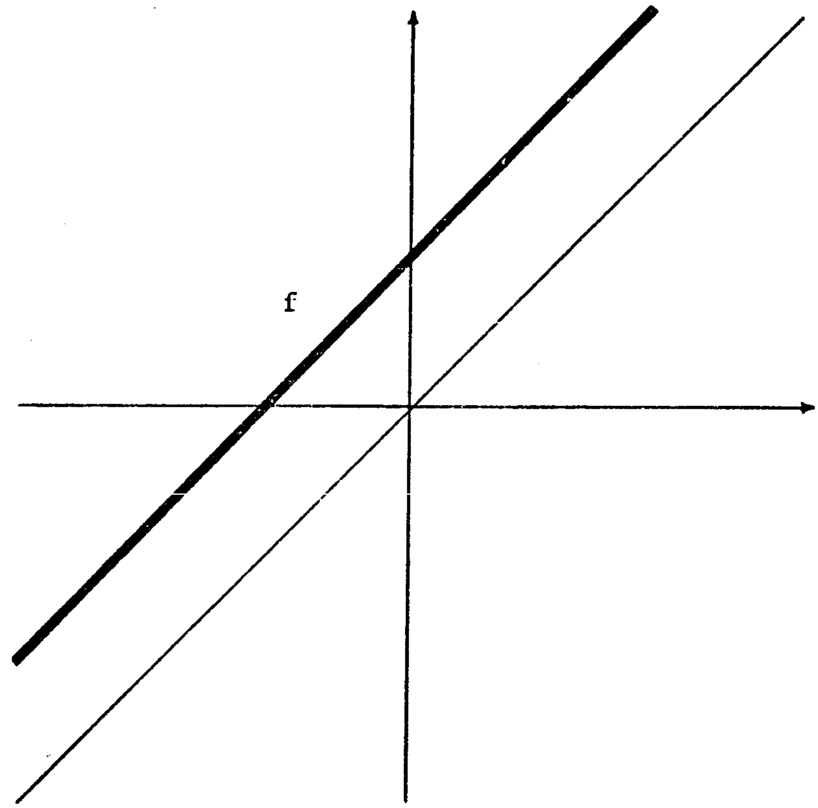
denote the support of $\bar{f}$.

The general case of the comment above is then the following proposition.

figure IX

G = A(R)

$xf = x + 3$



$$xg = \begin{cases} 2x + 3 & \text{if} \quad x \in [-3,0) \\ \frac{1}{4}x + 3 & \text{if} \quad x \in [0,4) \\ x & \text{otherwise} \end{cases}$$

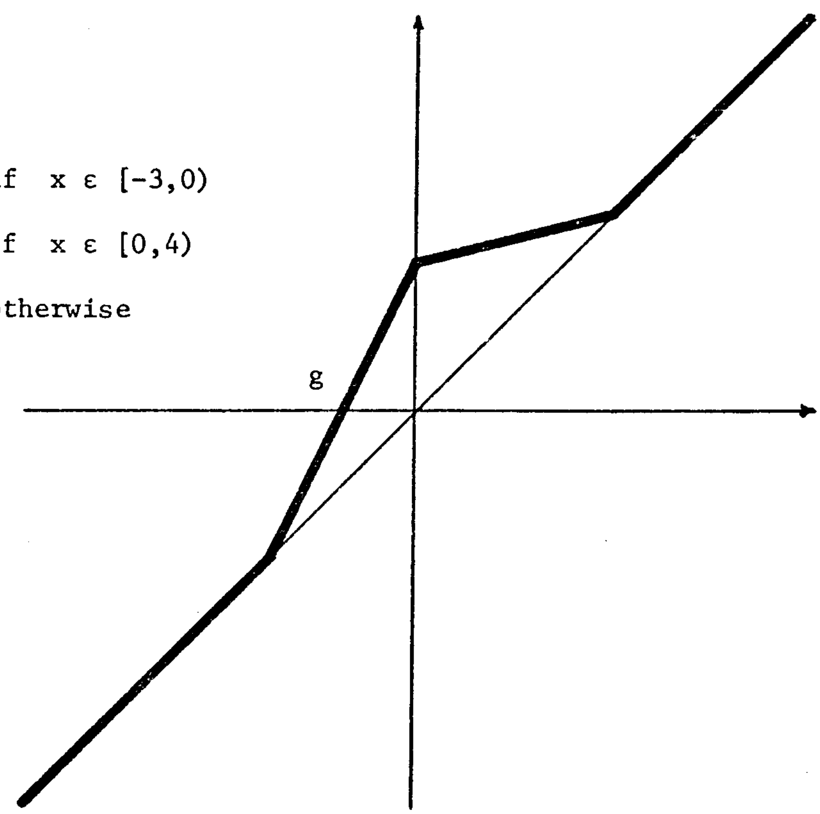
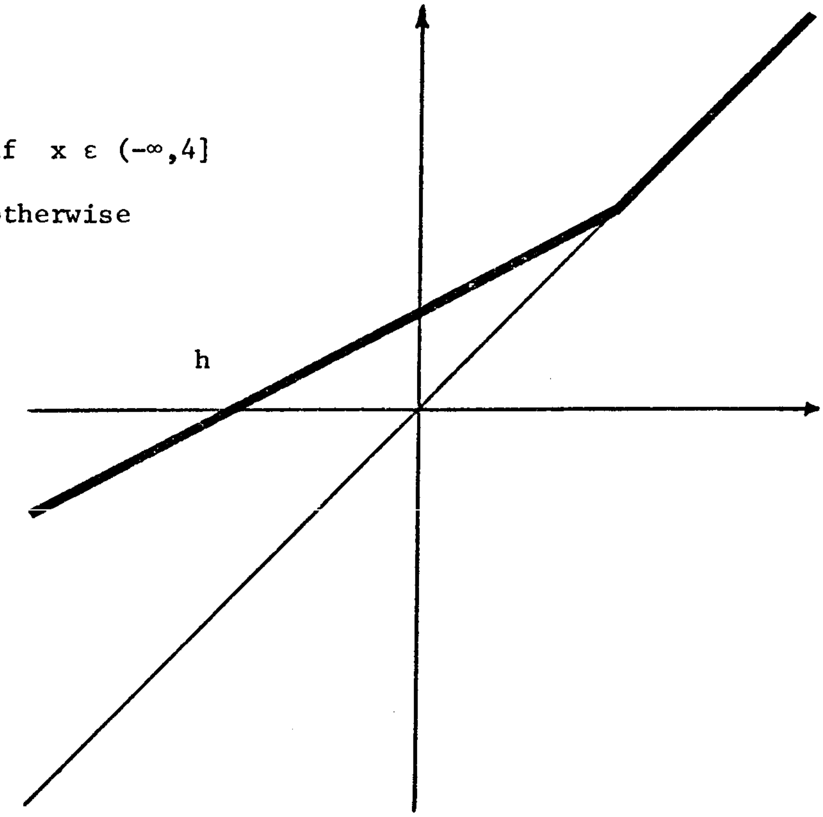


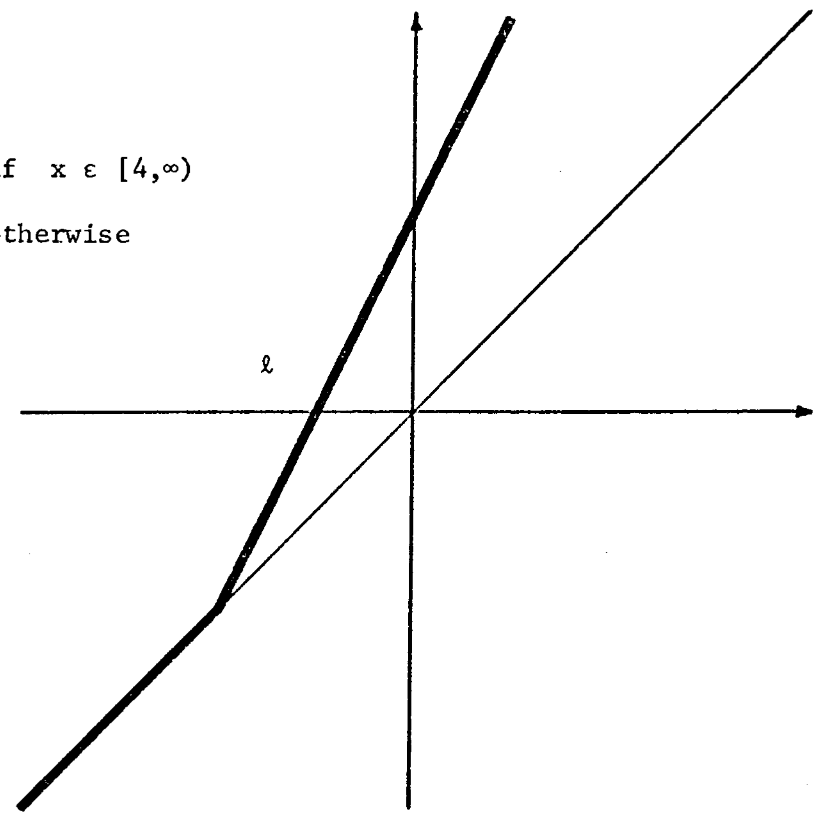
figure X

G = A(R)

$$xh = \begin{cases} \frac{1}{2}x + 2 & \text{if } x \in (-\infty, 4] \\ x & \text{otherwise} \end{cases}$$



$$x\ell = \begin{cases} 2x + 4 & \text{if } x \in [4, \infty) \\ x & \text{otherwise} \end{cases}$$



Proposition 3.2: Let Ω be a totally ordered set. Then for the ℓ-group $A(\Omega)$, $f \in \mathfrak{A}$ if and only if $f \in A(\Omega)^+ \setminus \{i\}$ and there exist $\bar{\alpha}, \bar{\beta} \in \hat{\Omega}$ such that $S(\bar{f}) = (\bar{\alpha}, \bar{\beta})$.

We first note the following lemma, whose proof is straightforward.

Lemma 3.3: Let $f \in A(\Omega)^+$. Let $\bar{\alpha}, \bar{\beta} \in \hat{\Omega} \setminus S(\bar{f})$. Let f_1, f_2 be functions from Ω to Ω defined by

$$\omega f_1 = \begin{cases} \omega & \text{if } \omega \in (\bar{\alpha}, \bar{\beta}) \\ \omega f & \text{otherwise} \end{cases}$$

$$\omega f_2 = \begin{cases} \omega f & \text{if } \omega \in (\bar{\alpha}, \bar{\beta}) \\ \omega & \text{otherwise.} \end{cases}$$

Then $f_1, f_2 \in A(\Omega)^+$. |X|

Proof of Proposition 3.2: Suppose $f \in \mathfrak{A}$. Clearly $f \in A(\Omega)^+ \setminus \{i\}$. Let $\eta(f) = \vee S(\bar{f})$ and $\mu(f) = \wedge S(\bar{f})$. Clearly $\eta(f), \mu(f) \in \hat{\Omega}$, and clearly $\eta(f)\bar{f} = \eta(f)$ and $\mu(f)\bar{f} = \mu(f)$. Therefore, for all $\bar{\alpha} \in \bar{\Omega} \setminus (\mu(f), \eta(f))$, $\bar{\alpha}\bar{f} = \bar{\alpha}$. Suppose there exists $\bar{\alpha} \in (\mu(f), \eta(f))$ such that $\bar{\alpha}\bar{f} = \bar{\alpha}$. Let f_1, f_2 be functions from Ω to Ω defined by

$$\omega f_1 = \begin{cases} \omega & \text{if } \omega > \bar{\alpha} \\ \omega f & \text{otherwise} \end{cases}$$

$$\omega f_2 = \begin{cases} \omega f & \text{if } \omega > \bar{\alpha} \\ \omega & \text{otherwise.} \end{cases}$$

By Lemma 3.3, $f_1, f_2 \in A(\Omega)^+$. We will show that $f_1 > i$. If $[\mu(f), \bar{\alpha}] = \{\mu(f), \bar{\alpha}\}$, then $\mu(f) \geqq \bar{\alpha}$ by definition of $\mu(f)$. This contradicts our choice of $\bar{\alpha}$ and thus $(\mu(f), \bar{\alpha}) \neq \emptyset$. Hence $(\mu(f), \bar{\alpha}) \cap \Omega \neq \emptyset$. Suppose that for all $\gamma \in (\mu(f), \bar{\alpha}) \cap \Omega$, $\gamma f = \gamma$.

Then for all $\bar{\gamma} \in (\mu(f),\bar{\alpha})$, $\bar{\gamma}\bar{f} = \bar{\gamma}$, and hence $\bar{\alpha} \leq \mu(f)$. This contradicts our choice of $\bar{\alpha}$ and thus there exists $\gamma \in (\mu(f),\bar{\alpha}) \cap \Omega$ such that $\gamma f \neq \gamma$. Hence $\gamma f_1 \neq \gamma$ and therefore $f_1 > i$. Similarly we may show that $f_2 > i$. Clearly $f_1 \wedge f_2 = i$ and $f_1 \vee f_2 = f$. This contradicts the fact that $f \in \mathfrak{U}$. Therefore, for all $\bar{\alpha} \in (\mu(f),\eta(f))$, $\bar{\alpha}\bar{f} \neq \bar{\alpha}$, i.e. $S(\bar{f}) = (\mu(f),\eta(f))$.

Conversely, suppose that $f > i$ and that there exist $\bar{\alpha},\bar{\beta} \in \hat{\Omega}$ such that $S(\bar{f}) = (\bar{\alpha},\bar{\beta})$. Let $\ell,\ell' \in A(\Omega)$ be such that $\ell \wedge \ell' = i$ and $\ell \vee \ell' = f$. Then clearly $S(\bar{\ell}) \cup S(\overline{\ell'}) \subseteq S(\bar{f})$. Suppose that $\bar{\upsilon} \in S(\bar{\ell}) \cap S(\overline{\ell'})$. If $\bar{\xi} = \bar{\upsilon}(\bar{\ell})^{-1} \vee \bar{\upsilon}(\overline{\ell'})^{-1}$, then $\bar{\xi} < \bar{\upsilon}$. Clearly there exists $\kappa \in [\bar{\xi},\bar{\upsilon}) \cap \Omega$. But then

$$\kappa(\ell \wedge \ell') = (\kappa\ell) \wedge (\kappa\ell') \geq (\bar{\xi}\bar{\ell}) \wedge (\bar{\xi}\overline{\ell'}) = \bar{\upsilon} > \kappa.$$

This contradicts our choice of the pair ℓ,ℓ'; thus $S(\bar{\ell}) \cap S(\overline{\ell'}) = \emptyset$. Suppose $\ell \neq i$ and $\ell' \neq i$. Then there exist $\bar{\gamma} \in S(\bar{\ell})$ and $\bar{\delta} \in S(\overline{\ell'})$. Without loss of generality, we may assume that $\bar{\gamma} < \bar{\delta}$. We have $\bar{\gamma},\bar{\delta} \in S(\bar{\ell}) \cup S(\overline{\ell'}) \subseteq S(\bar{f})$, and thus, since $S(\bar{f}) = (\bar{\alpha},\bar{\beta})$, $(\bar{\gamma},\bar{\delta}) \subseteq S(\bar{f})$. Following [33], we let

$$I(\bar{\gamma},\bar{\ell}) = \{\bar{\tau} \in \bar{\Omega} \mid \text{there exist integers } m,n \text{ such that } \bar{\gamma}\bar{\ell}^m \leq \bar{\tau} \leq \bar{\gamma}\bar{\ell}^n\}.$$

Clearly $I(\bar{\gamma},\bar{\ell})$ is convex and since $\bar{\gamma}\bar{\ell} > \bar{\gamma}$, $\bar{\tau}\bar{\ell} > \bar{\tau}$ for all $\bar{\tau} \in I(\bar{\gamma},\bar{\ell})$. Since $\bar{\delta} \in S(\overline{\ell'})$ and $S(\bar{\ell}) \cap S(\overline{\ell'}) = \emptyset$, $\bar{\delta} \notin S(\bar{\ell})$. Thus $\bar{\delta}$ is an upper bound for $I(\bar{\gamma},\bar{\ell})$ and hence $\vee I(\bar{\gamma},\bar{\ell}) \in \bar{\Omega}$. Let $\bar{\rho} = \vee I(\bar{\gamma},\bar{\ell}) \in (\bar{\gamma},\bar{\delta})$. Clearly $\bar{\rho}\bar{\ell} = \bar{\rho}$. Suppose that $\bar{\rho}\overline{\ell'} \neq \bar{\rho}$. Then $\bar{\rho}(\overline{\ell'})^{-1} < \bar{\rho} < \bar{\rho}\overline{\ell'}$, since $\overline{\ell'} > i$. Let $\bar{\sigma} = \bar{\gamma} \vee (\bar{\rho}(\overline{\ell'})^{-1})$. Then $\bar{\sigma} \in I(\bar{\gamma},\bar{\ell}) \subseteq S(\bar{\ell})$. However, since $\bar{\rho}(\overline{\ell'})^{-1} \leq \bar{\sigma} < \bar{\rho}\overline{\ell'}$, $\bar{\sigma} < \bar{\rho}\overline{\ell'} \leq \bar{\sigma}\overline{\ell'}\overline{\ell'}$. Hence $\bar{\sigma} \in S(\overline{\ell'})$. This contradicts the fact that

$S(\bar{\ell}) \cap S(\overline{\ell'}) = \emptyset$. Therefore $\bar{\rho}\overline{\ell'} = \bar{\rho}$, and hence $\bar{\rho}\bar{f} = \bar{\rho}(\bar{\ell} \vee \overline{\ell'}) = \bar{\rho}$. But we chose $\bar{\rho} \in (\bar{\gamma},\bar{\delta}) \subseteq S(\bar{f})$. This is a contradiction, and hence $f \in \mathfrak{A}$. |⌧|

With another assumption about $A(\Omega)$ we can describe $\mathcal{D}_1$ exactly (Proposition 3.5).

Proposition 3.4: If G is a divisible ℓ-group, then $\mathcal{D}_1 = \mathfrak{A}$.

Proof: By definition $\mathfrak{A} \supseteq \mathcal{D}_1$. Conversely, let $a \in \mathfrak{A}$. Then there exists $b \in G^+$ such that $b + b = a$. Clearly $b > 0$. Suppose $b \notin \mathfrak{A}$. Then there exist $h,\ell > 0$ such that $h \wedge \ell = 0$ and $h \vee \ell = b$. Then $h \vee \ell = h + \ell = \ell + h$. Hence

$$a = b + b = (h \vee \ell) + (h \vee \ell) = h + \ell + h + \ell = (h + h) + (\ell + \ell).$$

But since $h \wedge \ell = 0$, $(h + h) \wedge \ell = 0$ and hence $(h + h) \wedge (\ell + \ell) = 0$. Thus $a \notin \mathfrak{A}$. This contradicts our choice of a. Hence $b \in \mathfrak{A}$. Since $0 < b < a$, $b^{\perp} \supseteq a^{\perp}$. Let $k \in b^{\perp}$. Then $|k| \wedge b = 0$. Hence $|k| \wedge a = |k| \wedge (b + b) = 0$. Thus $k \in a^{\perp}$. Hence $b^{\perp} = a^{\perp}$. Therefore, for all $a \in \mathfrak{A}$, there exists $b \in \mathfrak{A}$ such that $b + b = a$ and $b^{\perp} = a^{\perp}$. If $a \in \mathfrak{A}$, define a sequence $\{b_1,b_2, \dots\} \subseteq \mathfrak{A}$ by $a = b_1$ and $b_n \in \mathfrak{A}$ is such that $b_n + b_n = b_{n-1}$ and $b_n^{\perp} = b_{n-1}^{\perp}$ for $n > 1$. Thus $a \in \mathcal{D}_1$. Therefore $\mathfrak{A} = \mathcal{D}_1$. |⌧|

Proposition 3.5: Let Ω be a totally ordered set. If $A(\Omega)$ is doubly transitive, then $\mathfrak{A} = \mathcal{D}_1$.

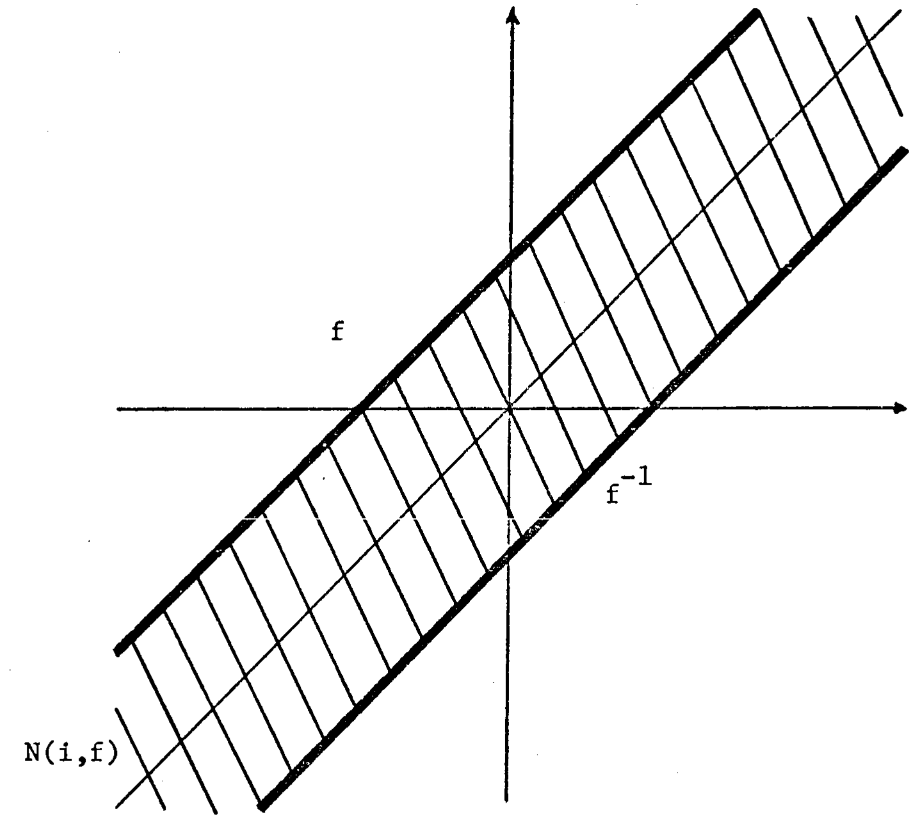
Proof: By [33], since $A(\Omega)$ is doubly transitive, $A(\Omega)$ is divisible. By Proposition 3.4, $\mathfrak{A} = \mathcal{D}_1$. |⌧|

In $A(R)$, clearly $\mathcal{D}_2 = \emptyset$. Hence the sets of $N_3(0) = N_1(0)$ are of the form of those in figures XI and XII.

figure XI

$G = A(R)$

$xf = x + 3$



$$xg = \begin{cases} 2x + 3 & \text{if} \quad x \quad [-3,0) \\ \frac{1}{4}x + 3 & \text{if} \quad x \quad [0,4) \\ x & \text{otherwise} \end{cases}$$

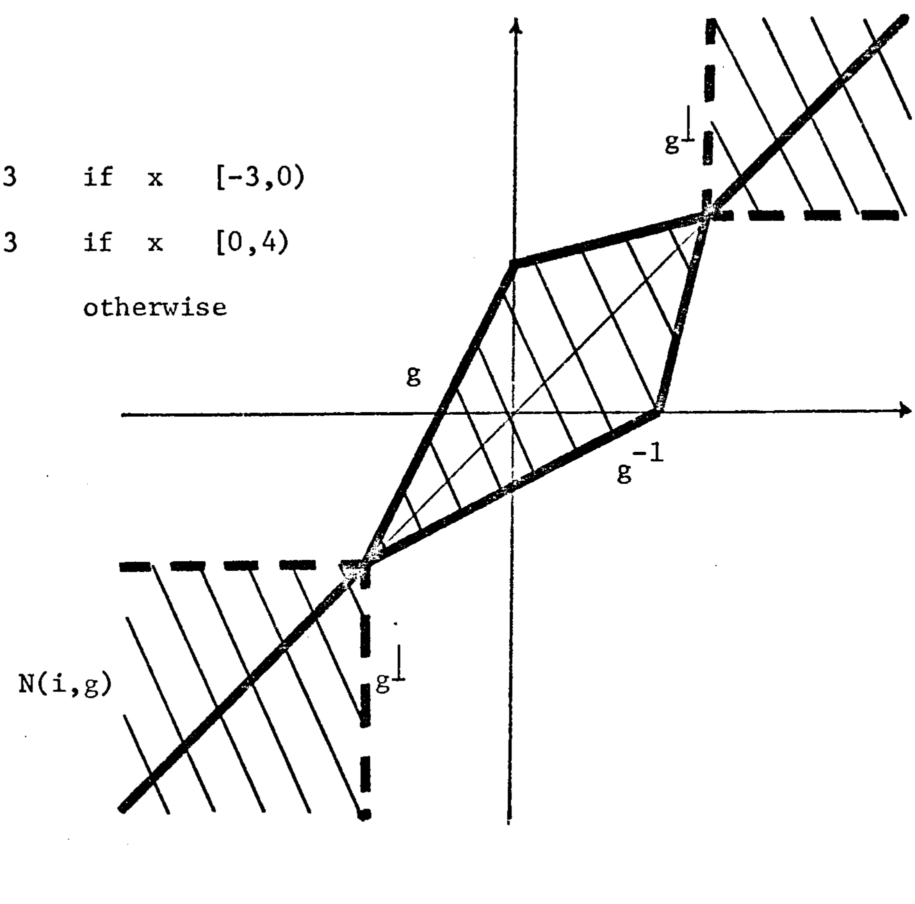
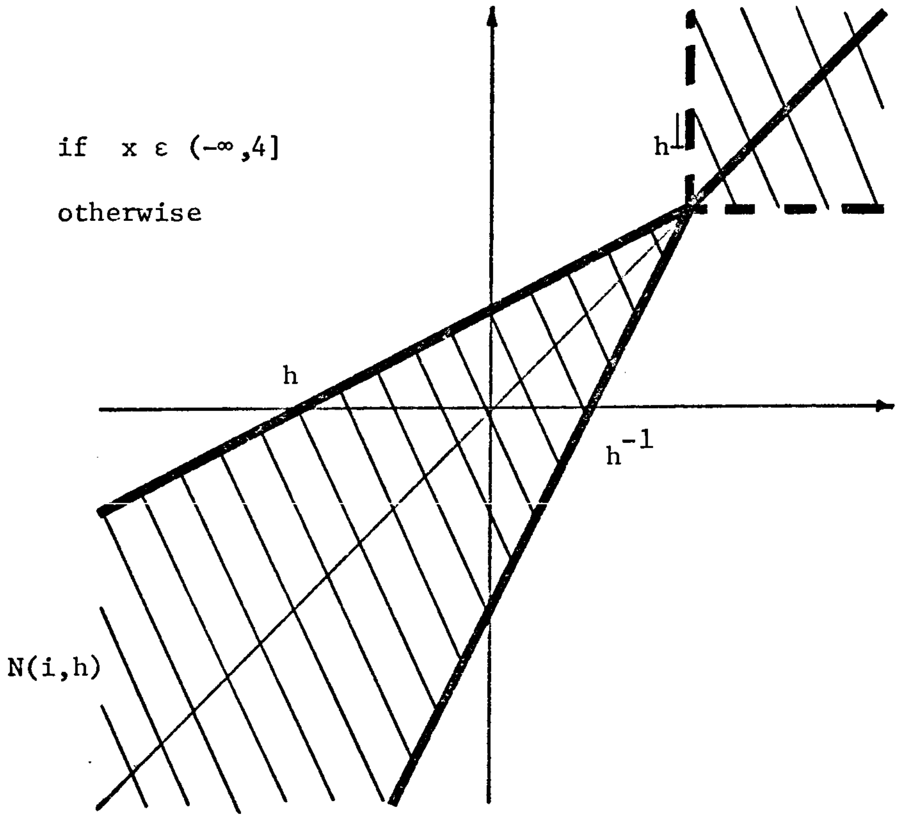


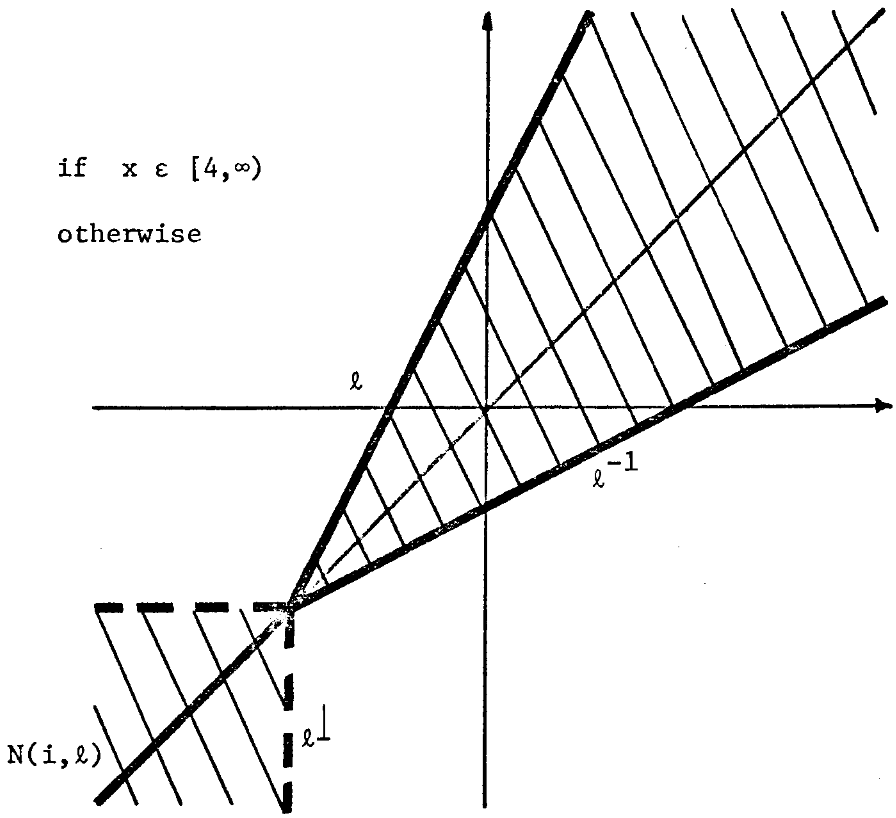
figure XII

$G = A(R)$

$$xh = \begin{cases} \frac{1}{2}x + 2 & \text{if } x \in (-\infty, 4] \\ x & \text{otherwise} \end{cases}$$



$$x\ell = \begin{cases} 2x + 4 & \text{if } x \in [4, \infty) \\ x & \text{otherwise} \end{cases}$$



In $R \,|X|\, R$, if $h, \ell \in \mathfrak{A} = \mathcal{D}_1$, there always exists $k \in G^+$ such that

$$N(0,h) \cap N(0,\ell) = N(0,k).$$

See figure XIII.

However, in $A(R)$, it may happen that for $f,g \in \mathfrak{A}$, there does not exist an $h \in A(R)^+$ such that $N(0,f) \cap N(0,g) = N(0,h)$. Let f,g be as in figure XIV. Suppose that such an h exists (cf. figure XV). Since $12k = 12$ for all $k \in N(0,f)$, $12h = 12$. Since $15k = 15$ for all $k \in N(0,g)$, $15h = 15$. Suppose $rh > r$ for some $r \in [12,15]$. Then $12 \neq r \neq 15$. If $z \in (12,15)$, let $\ell_z \in A(R)$ be defined by

$$x\ell_z = \begin{cases} \frac{1}{3}\left(\frac{z - 6}{z - 12}\right)x + 8\left(\frac{z - 15}{z - 12}\right) & \text{if } x \in [12,z) \\ \frac{1}{3}x + 10 & \text{if } x \in [z,15) \\ x & \text{otherwise.} \end{cases}$$

Then $x\ell_z = xg$ if $x \in [z,15)$, and clearly for $z \in (12,15)$, $\ell_z \in N(0,f) \cap N(0,g)$. If $z \in (12,15)$ is such that $z < zh < zg$, then $\ell_z \notin N(0,h)$. Hence, since we are assuming there exists $r \in (12,15)$ such that $r < rh$, we have that $zh = zg$ for all $z \in (12,15)$. Therefore

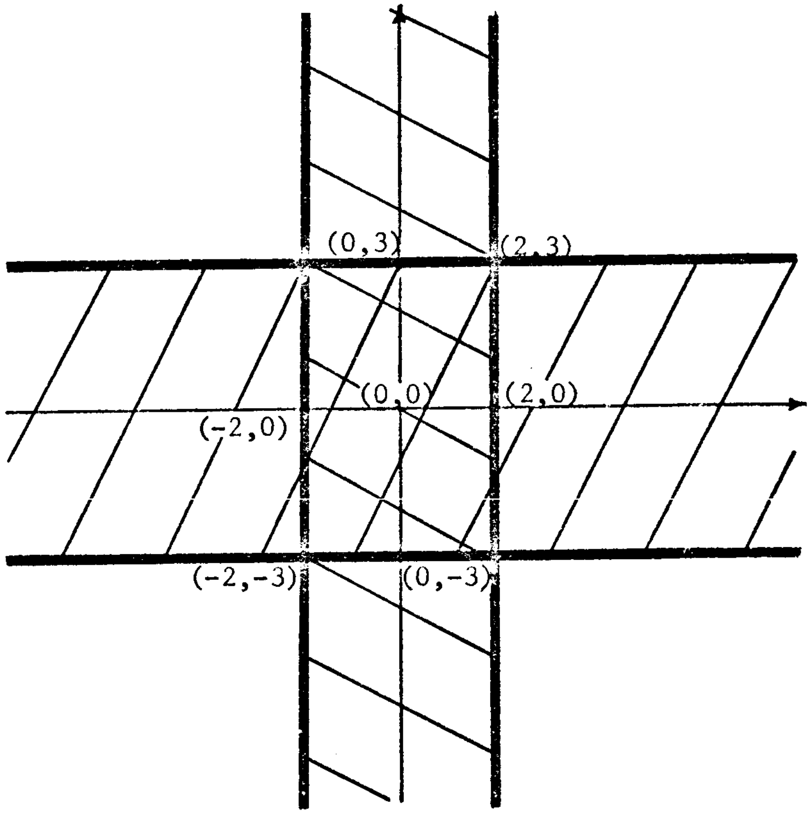
$$14 = 12g = (\wedge\{z \mid z \in (12,15)\})g = \wedge\{zg \mid z \in (12,15)\}$$
$$= \wedge\{zh \mid z \in (12,15)\} = (\wedge\{z \mid z \in (12,15)\})h = 12h = 12.$$

This is a contradiction, and hence $zh = z$ for all $z \in [12,15]$. Let $\ell \in A(R)$ be defined by

$$x\ell = \begin{cases} 5x - 48 & \text{if } x \in [12,12\tfrac{1}{2}) \\ \frac{1}{5}x + \frac{56}{5} & \text{if } x \in [12\tfrac{1}{2},15) \\ x & \text{otherwise.} \end{cases}$$

<u>figure XIII</u>

G = R |X| R



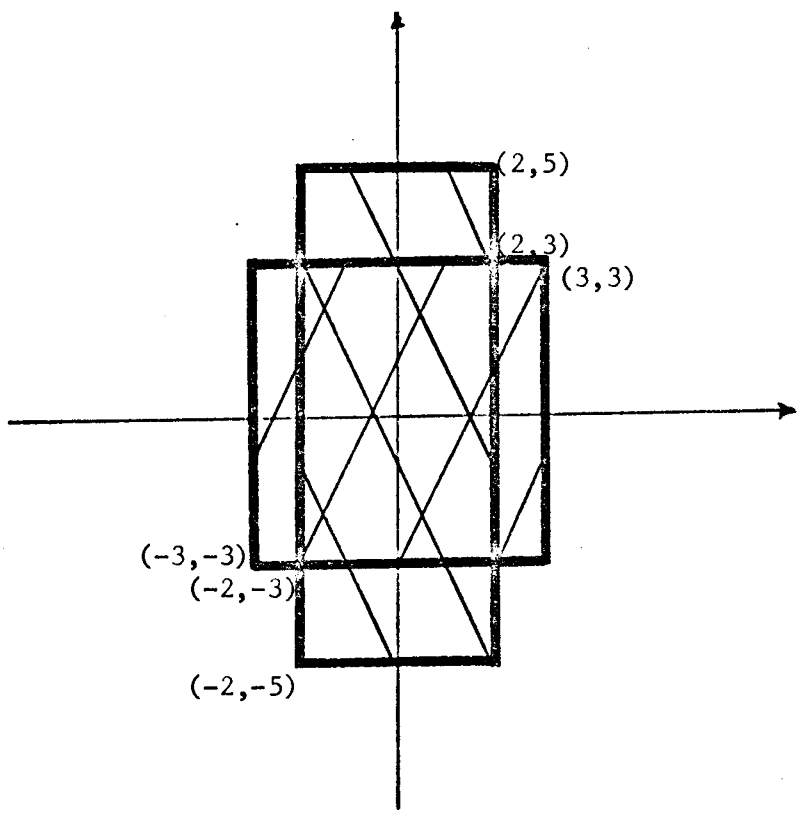
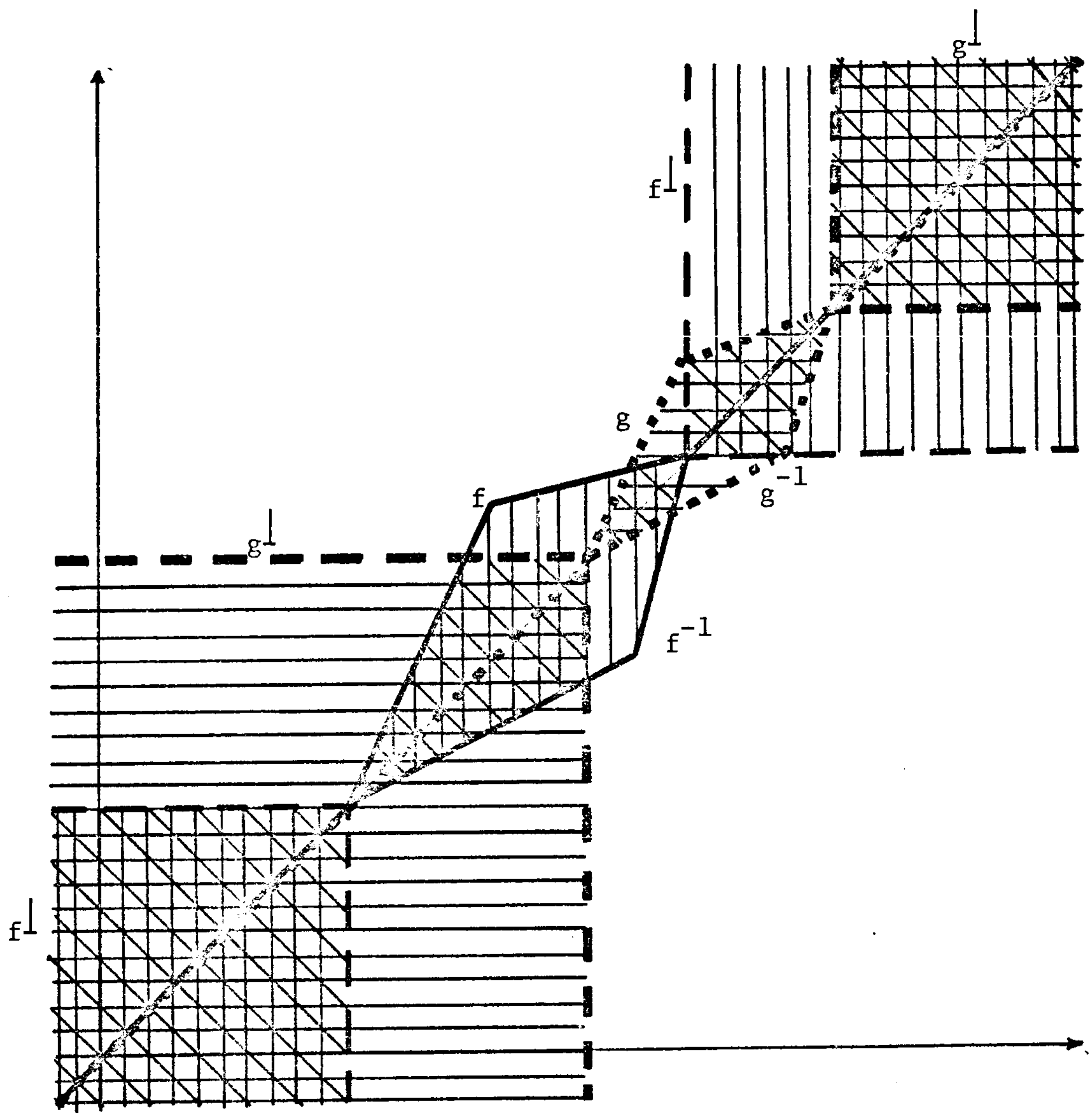


figure XIV

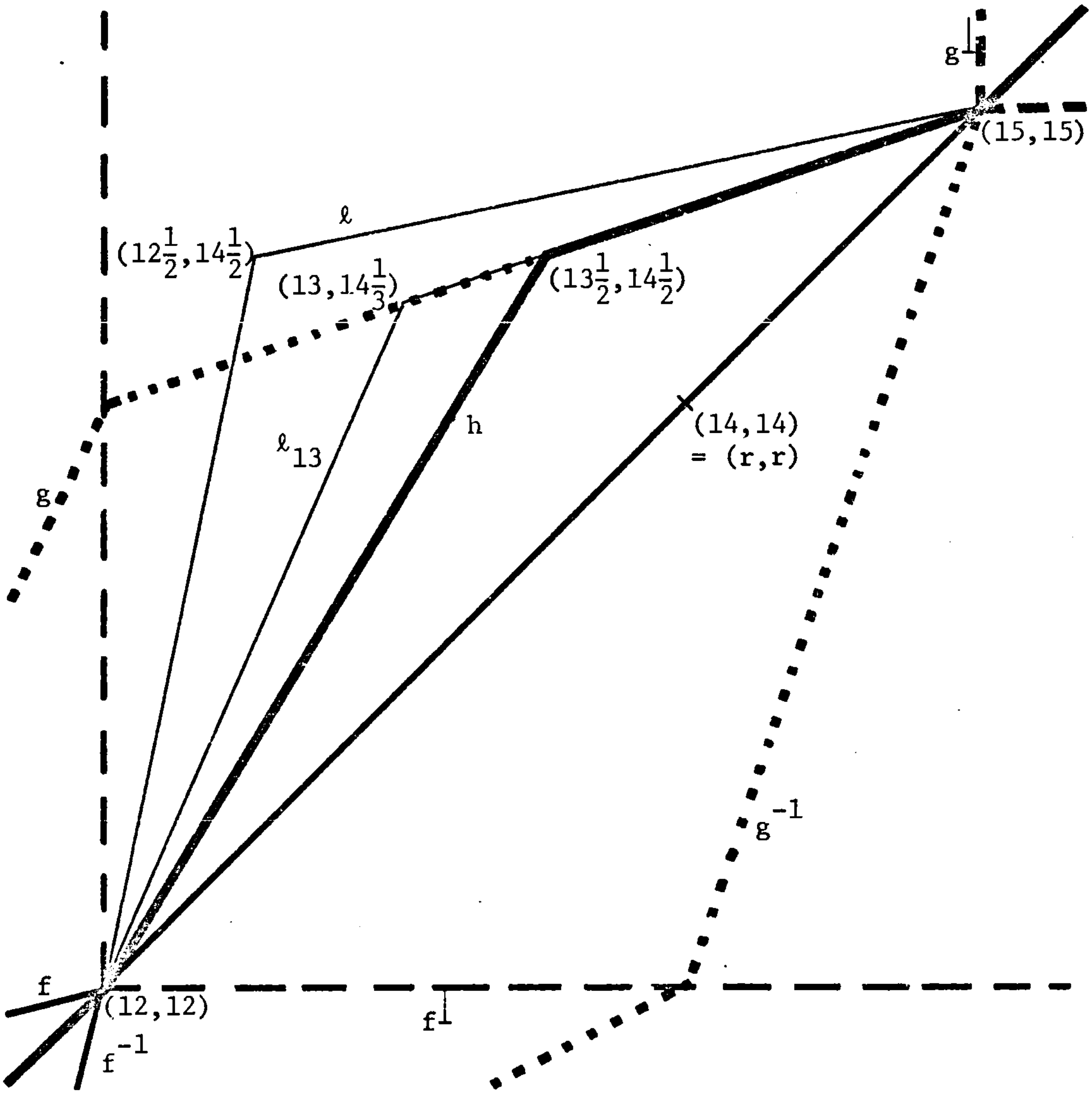
G = A(R)



$$xf = \begin{cases} 2x - 5 & \text{if } x \in [5,8) \\ \frac{1}{4}x + 9 & \text{if } x \in [8,12) \\ x & \text{otherwise} \end{cases} \qquad xg = \begin{cases} 2x - 10 & \text{if } x \in [10,12) \\ \frac{1}{3}x + 10 & \text{if } x \in [12,15) \\ x & \text{otherwise} \end{cases}$$

figure XV

G = A(R)



$$xh = \begin{cases} \frac{5}{3}x - 8 & \text{if } x \in [12,13\frac{1}{2}) \\ \frac{1}{3}x + 10 & \text{if } x \in [13\frac{1}{2},15) \\ x & \text{otherwise} \end{cases}$$

Then $\ell \notin N(0,g)$, but since $x\ell = x$ if $x \notin [12,15]$, $\ell \wedge h = i$. Thus $\ell \in h \subseteq N(0,h)$. Therefore, for all $h \in A(\mathbb{R})^+$

$$N(0,g) \cap N(0,f) \neq N(0,h).$$

However, even though such an h does not exist, we do have the following: Let $h \in A(R)$ be defined by

$$xh = \begin{cases} 2x - 5 & \text{if } x \in [5,7) \\ \frac{1}{3}x + \frac{20}{3} & \text{if } x \in [7,10) \\ 2x - 10 & \text{if } x \in [10,\frac{76}{7}) \\ \frac{1}{4}x + 9 & \text{if } x \in [\frac{76}{7},12) \\ 2x - 12 & \text{if } x \in [12,13) \\ \frac{1}{2}x + \frac{15}{2} & \text{if } x \in [13,15) \\ x & \text{otherwise.} \end{cases}$$

Then for f,g as in figure XIV,

$$N(0,f) \cap N(0,g) \supseteq N(0,h).$$

Here $N(0,h)$ is a finite intersection of elements from $\mathcal{N}_1(0)$ and hence $N(0,h) \in \mathcal{N}(0)$. See figure XVI.

Example 3.6: Our second example is of an ℓ-group G which contains positive elements f,g such that for all $h \in G^+$,

$$N(0,f) \cap N(0,g) \not\supseteq N(0,h);$$

f,g will be elements of $\mathfrak{A}$ and of $\mathcal{D}_1$; $\mathcal{D}_2$ will be empty.

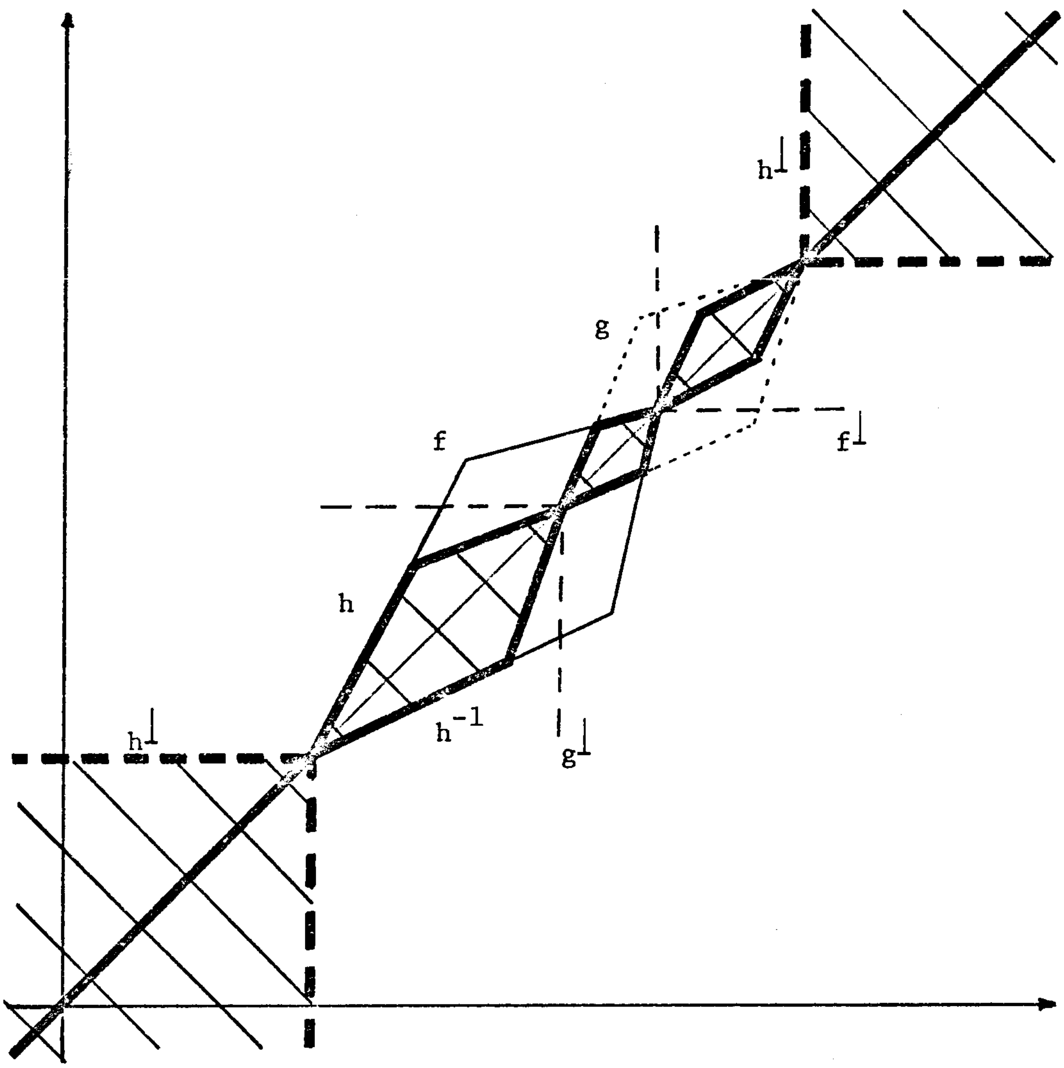
First we note the following standard result.

Proposition 3.7: Let A be a subgroup of an ℓ-group G. Let $[A]$ be the ℓ-subgroup of G generated by A. Then

$$[A] = \{\bigvee_{k\in K} \bigwedge_{\ell\in L} a_{k\ell} \mid a_{k\ell} \in A \text{ for all } k \in K \text{ and } \ell \in L; \text{ and } K \text{ and } L \text{ are finite}\}.$$

figure XVI

G = A(R)



$$xh = \begin{cases} 2x - 5 & \text{if } x \in [5,7) \\ \frac{1}{3}x + \frac{20}{3} & \text{if } x \in [7,10) \\ 2x - 10 & \text{if } x \in [10,\frac{76}{7}) \\ \frac{1}{4}x + 9 & \text{if } x \in [\frac{76}{7},12) \\ 2x - 12 & \text{if } x \in [12,13) \\ \frac{1}{2}x + \frac{15}{2} & \text{if } x \in [13,15) \\ x & \text{otherwise} \end{cases}$$

Proof: An ℓ-group is a (finitely) distributive lattice and addition distributes over joins and meets. Conrad [19] notes that repeated applications of these two properties prove the proposition. $|\boxtimes|$

The ℓ-group we will construct is an ℓ-subgroup of $C(R)$, the ℓ-group of all continuous functions from the real numbers to themselves. Compare what follows with figure XVII.

For $n \in Z^+$, define $h_n \in C(R)^+$ by

$$xh_n = \begin{cases} x - 2(1 + 4n) & \text{if } x \in [2 + 8n, 6 + 8n) \\ -x + 2(5 + 4n) & \text{if } x \in [6 + 8n, 10 + 8n) \\ 0 & \text{otherwise.} \end{cases}$$

Also define $\ell_n \in C(R)^+$ by

$$x\ell_n = \begin{cases} x + 2(5 + 4n) & \text{if } x \in [-10 - 8n, -6 - 8n) \\ -x - 2(1 + 4n) & \text{if } x \in [-6 -8n, -2 - 8n) \\ 0 & \text{otherwise,} \end{cases}$$

and let $H_n = G(h_n)$ and $L_n = G(\ell_n)$, the convex ℓ-subgroups generated by h_n and ℓ_n respectively. For $p \in Q$, define $f_p, g_p \in C(R)$ by

$$xf_p = \begin{cases} \frac{p}{4} x + \frac{p}{2} & \text{if } x \in [-2,2] \\ p & \text{if } x > 2 \\ 0 & \text{if } x < -2, \end{cases}$$

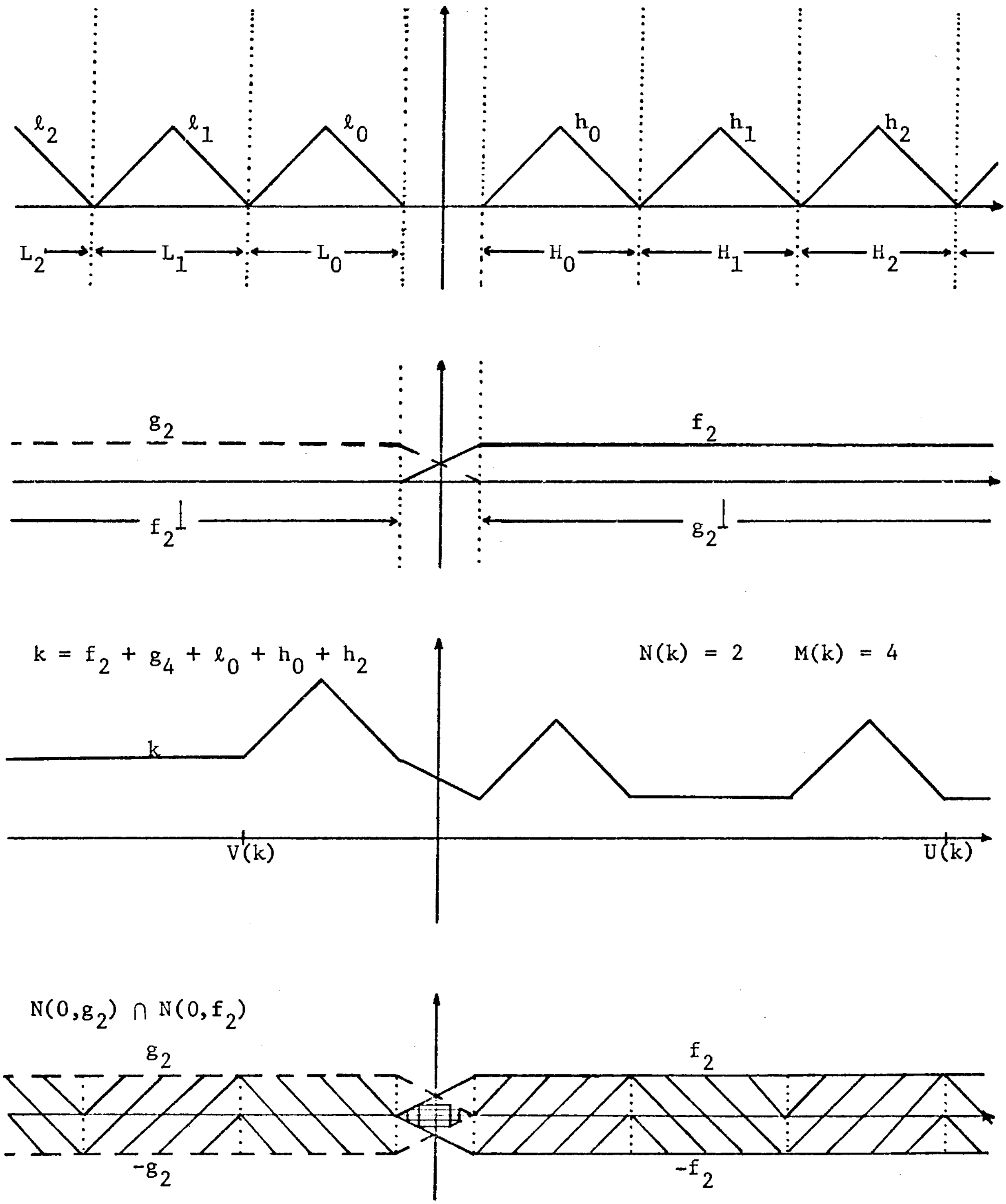
$$xg_p = \begin{cases} -\frac{p}{4} x + \frac{p}{2} & \text{if } x \in [-2,2] \\ p & \text{if } x < -2 \\ 0 & \text{if } x > 2. \end{cases}$$

Let $F = \{f_p \mid p \in Q\}$ and $P = \{g_p \mid p \in Q\}$. Then clearly F and P are ℓ-subgroups of $C(R)$ which are ℓ-isomorphic to Q. Let

$$H = \left|\sum_{n=1}^{\infty}\right| H_n, \qquad L = \left|\sum_{n=1}^{\infty}\right| L_n.$$

figure XVII

G = C(R)



Finally, let $G = [H,L,F,P]$, the ℓ-subgroup of $C(R)$ generated by H, L, F, and P. We note that $(2)f_p = p$ and $(-2)g_p = p$.

For $k \in C(R)$, let

$$S(k) = \{r \in R \mid rk \neq 0\}$$

denote the support of k. Then if $k \in H \cup L$, $S(k)$ is bounded above and below. Thus, since $[H,L] = H \,|X|\, L$, $S(k)$ is bounded above and below for all $k \in [H,L]$. Let $k \in G$. By Proposition 3.7,

$$k = \bigvee_{i\in I} \bigwedge_{j\in J} \left(\sum_{n=1}^{m(i,j)} a_n^{ij} \right)$$

where $a_n^{ij} \in H \cup L \cup F \cup P$ for all i, j, n and where I and J are finite. Since $C(R)$ is commutative,

$$\left(\sum_{n=1}^{m(i,j)} a_n^{ij}\right) = \left(\sum_{n=1}^{m(i,j)} f_{p_n}\right) + \left(\sum_{n=1}^{m(i,j)} g_{r_n}\right) + a_{ij}$$

where $a_{ij} \in [H,L]$, and $p_n, r_n \in Q$ for all $n = 1,\ldots,m(i,j)$.

By the comment above, $S(a_{ij})$ is bounded above and below. Let

$$U(k) = \bigvee(\{t \mid t \in S(a_{ij}) \text{ for some } i,j\} \cup \{2\}),$$

$$V(k) = \bigwedge(\{t \mid t \in S(a_{ij}) \text{ for some } i,j\} \cup \{-2\}).$$

Let $n_{ij} = \sum_{n=1}^{m(i,j)} p_n$ and $m_{ij} = \sum_{n=1}^{m(i,j)} r_n$. Then by the construction of F and P,

$$\sum_{n=1}^{m(i,j)} f_{p_n} = f_{n_{ij}} \quad \text{and} \quad \sum_{n=1}^{m(i,j)} g_{r_n} = g_{m_{ij}}.$$

Let

$$N(k) = \bigvee_{i\in I} \bigwedge_{j\in J} (n_{ij}),$$

$$M(k) = \bigvee_{i\in I} \bigwedge_{j\in J} (m_{ij}).$$

Then for all $i \in I$ and $j \in J$, if $u \geqq U(k)$, then $(u)a_{ij} = 0$ and $(u)g_{m_{ij}} = 0$, and if $v \leqq V(k)$, then $(v)a_{ij} = 0$ and $(v)f_{n_{ij}} = 0$.

Suppose $u \geqq U(k)$. Then

$$(u)k = (u)\bigvee_{i\varepsilon I}\bigwedge_{j\varepsilon J}(f_{n_{ij}} + g_{m_{ij}} + a_{ij})$$

$$= \bigvee_{i\varepsilon I}\bigwedge_{j\varepsilon J}((u)f_{n_{ij}} + (u)g_{m_{ij}} + (u)a_{ij})$$

$$= \bigvee_{i\varepsilon I}\bigwedge_{j\varepsilon J}(u)f_{n_{ij}}$$

$$= (u)\bigvee_{i\varepsilon I}\bigwedge_{j\varepsilon J}f_{n_{ij}}$$

$$= (u)f_{N(k)}.$$

Similarly, if $v \leqq V(k)$, then

$$(v)k = (v)g_{M(k)}.$$

Since $a_{ij} \varepsilon [H,L]$, $(2)a_{ij} = 0 = (-2)a_{ij}$ and by construction, $(2)g_{m_{ij}} = 0 = (-2)f_{n_{ij}}$. Thus, as above,

$$(2)k = (2)f_{N(k)},$$

$$(-2)k = (-2)g_{M(k)}.$$

Suppose $k \varepsilon g_2^{\perp}$. Since k is continuous, $(2)k = 0$. Hence $(2)f_{N(k)} = 0$, i.e. $N(k) = 0$. Therefore, if $u \geqq U(k)$, $(u)k = (u)f_{N(k)} = 0$. Thus $S(k)$ is bounded above by $U(k)$. If $k \varepsilon [-g_2,g_2]$, then clearly $S(k)$ is bounded above by 2. Hence if $k \varepsilon N(0,g_2)$, $S(k)$ is bounded above by $U(k)$. Similarly, if $k \varepsilon N(0,f_2)$, then $S(k)$ is bounded below by $V(k)$.

Let $k \varepsilon N(0,g_2) \cap N(0,f_2)$. Then $S(k)$ is bounded above by $U(k)$ and below by $V(k)$. Hence $H_{U(k)} \subseteq k^{\perp}$ and $L_{V(k)} \subseteq k^{\perp}$. But clearly $H_{U(k)} \nsubseteq N(0,f_2)$ and $L_{V(k)} \nsubseteq N(0,g_2)$. Therefore, if $k \varepsilon (N(0,f_2) \cap N(0,g_2))^{+}\setminus\{0\}$, then

$$N(0,k) \nsubseteq N(0,f_2) \cap N(0,g_2).$$

Suppose $k \varepsilon G^{+}\setminus\{0\}$, and $N(0,k) \subseteq N(0,f_2) \cap N(0,g_2)$. Since $0 \varepsilon k^{\perp}$, $k \varepsilon N(0,k)$. Hence $k \varepsilon N(0,f_2) \cap N(0,g_2)$. But this contradicts the above argument. Hence for all $k \in G^+\backslash\{0\}$,

$$N(0,k) \not\supseteq N(0,f_2) \cap N(0,g_2).$$

Let $p \in Q^+\backslash\{0\}$. Suppose that $f_p \notin \mathfrak{A}$. Then there exist $k,\ell \in G^+\backslash\{0\}$ such that $k \wedge \ell = 0$ and $k \vee \ell = f_p$. Let $r \in S(k) \subseteq S(f_p)$ and $s \in S(\ell) \subseteq S(f_p)$. Then without loss of generality we may assume that $r < s$. Let $t = \vee(S(k) \cap [r,s])$. Since R is conditionally complete, $t \in R$. Since k is continuous, $tk = 0$. If $t\ell \neq 0$, then there exist $a,b \in R$ such that $t \in (a,b) \subseteq S(\ell)$. But by definition of t, $S(k) \cap (a,b) \neq \emptyset$. This contradicts the fact that $k \wedge \ell = 0$, and thus $t\ell = 0$. Therefore $tf_p = t(k \vee \ell) = 0$, and since $r,s \in S(f_p)$, this contradicts the definition of f_p. Therefore $f_p \in \mathfrak{A}$. Similarly $g_p \in \mathfrak{A}$ for all $p \in Q^+\backslash\{0\}$. Therefore $(F \cup P)^+\backslash\{0\} \subseteq \mathfrak{A}$.

Clearly H, L, F, and P are divisible. Therefore G is divisible, and hence by Proposition 3.4 and the above

$$(F \cup P)^+\backslash\{0\} \subseteq \mathfrak{A} = \mathcal{D}_1.$$

We conclude that $f_2, g_2 \in \mathcal{D}_1 \subseteq \mathfrak{A}$ and that $\mathcal{D}_2 = \emptyset$.

Let

$$N_1^*(0) = \{N(0,g) \mid g = \bigvee_{i=1}^{n} g_i \text{ for } g_i \in \mathcal{D}_1\},$$

$$N_2^*(0) = \{D(h) + h^{\perp} \mid h = \bigvee_{i=1}^{n} h_i \text{ for } h_i \in \mathcal{D}_2\},$$

$$N_3^*(0) = N_1^*(0) \cup N_2^*(0).$$

If $G = R \,|X|\, R$, $N_3^*(0)$ is a filter-base which generates $F(N(0))$ (see figure XIII). If $G = A(R)$, this is also the case. For example, in figures XIV and XVI, we exhibited an element $h \in A(R)$ such that $N(0,f) \cap N(0,g) \supseteq N(0,h)$ and $N(0,h) \in N_3^*(0)$. However, Example 3.6 shows that in general $N_3^*(0)$ will not be a filter-base.

In Chapter 7 (Example 7.8) we show that the ℓ-group of Example 3.6 has Hausdorff $\mathfrak{T}$-topology.

4. TOTALLY ORDERED GROUPS

Let T be a totally ordered group. Let L denote the interval topology on T. Then $U \in L$ if and only if for all $x \in U$, there exist $a,b \in T$ such that $x \in (a,b) \subseteq U$. The main result of this chapter is that the $\mathfrak{T}$-topology on T is equivalent to the interval topology, L.

The proof of this equivalence requires the fact that if $g \in T^+ \setminus \{0\}$ is such that $(0,g) \neq \phi$, then there exists a $g' \in T$ such that $0 < g' < g' + g' \leqq g$. This "semi-divisibility" is straightforward for totally ordered groups. As a prelude to the theorem that $\mathfrak{T} = L$ on T, we will investigate this "semi-divisibility" in an arbitrary ℓ-group.

<u>Lemma 4.1</u>: Let G be an ℓ-group. Let $g \in G^+ \setminus \{0\}$. Then there exists $h \in G$ such that $0 < h < h + h \leqq g$ if and only if there exists $h' \in G$ such that $0 < h' < g \leqq h' + h'$.

<u>Proof</u>: Suppose $h \in G$ is such that $0 < h < h + h \leqq g$. Let $h' = g - h$. Then $0 < h' < g$ and since $h + h \leqq g$, $0 \leqq -h + g - h$. Thus $g \leqq g - h + g - h = h' + h'$. Conversely, suppose that $h' \in G$ is such that $0 < h' < g \leqq h' + h'$. If $h = g - h'$, then $0 < h < g$ and since $g \leqq h' + h'$, $-h' + g - h' \leqq 0$. Thus

$$g - h' + g - h' = h + h \leqq g. \qquad |\mathbb{X}|$$

Proposition 4.2: Let G be an ℓ-group. Let $g \in G^+\backslash\{0\}$. Then there exists $h \in G$ such that $0 < h < h + h \leqq g$ if and only if $[0,g] \neq T(g)$.

Proof: Suppose $h \in G$ is such that $0 < h < h + h \leqq g$. Since $h + h \leqq g$, $h \leqq g - h$. If $h \in T(g)$, then $h \wedge (g - h) = 0$, and hence $h = 0$. This contradicts our choice of h and hence $h \notin T(g)$. Thus $[0,g] \neq T(g)$..

Conversely, if $[0,g] \neq T(g)$, let $\ell \in [0,g]\backslash T(g)$. (Such an ℓ exists since $[0,g] \supsetneq T(g)$.) Then let $h = \ell \wedge (g - \ell)$. If $h = 0$, $\ell \wedge (g - \ell) = 0$ and hence $(g - \ell) \vee \ell = g - \ell + \ell = g$. Thus $\ell \in T(g)$, which contradicts our choice of ℓ. Thus $h > 0$. Furthermore,

$$h + h = \ell \wedge (g - \ell) + \ell \wedge (g - \ell) \leqq g - \ell + \ell = g. \qquad |\boxtimes|$$

In many ℓ-groups $[0,g] \neq T(g)$ for all $g \in G^+\backslash\{0\}$. For instance, $R \,|X|\, R$, $A(R)$, and $R \overleftarrow{X} Z$, but not Z or $Z \overleftarrow{X} R$. If $g \in \mathfrak{A}$ and $[0,g] \neq T(g)$, then Proposition 4.2 indicates that the important criterion that g must fulfill to be a member of $\mathcal{D}_1$ is that some $0 < h < g$ must be "big enough" to have $h^\perp \subseteq g^\perp$ as well as $h + h \leqq g$. Let G be the ℓ-group of Example 3.6. Let L be the ℓ-subgroup of G generated by H and $G(f_2)$. Then if $0 < k < f_2$, by an argument similar to that in the discussion of Example 3.6, $k^\perp \nsubseteq f_2^\perp$. However, $T(f_2) = \{0,f_2\}$ and $[0,f_2]$ is uncountable.

We return to the investigation of totally ordered groups.

Lemma 4.3: If T is a totally ordered group, then $T^+\backslash\{0\} = \mathfrak{A}$.

Proof: Clearly $\mathfrak{A} \subseteq T^+\backslash\{0\}$. Conversely, let $t \in T^+\backslash\{0\}$. If $s \in T(t)$, then there exists $s' \in T$ such that $s \vee s' = t$ and $s \wedge s' = 0$. Since T is totally ordered, $s \wedge s' = s$ or $s \wedge s' = s'$. If $s \wedge s' = s$, $s = 0$; if $s \wedge s' = s'$, $s = t$. Hence $t \in \mathfrak{A}$. $|\boxtimes|$

Proposition 4.4: Let T be a totally ordered group. Let $t \in T^+ \setminus \{0\}$. Then there exists $h \in T$ such that $0 < h < h + h \leqq t$ if and only if $(0,t) \neq \phi$.

Proof: By Proposition 4.2, there exists such an h if and only if $[0,t] \neq \mathcal{T}(t)$. By Lemma 4.3, $[0,t] \neq \mathcal{T}(t)$ if and only if $[0,t] \neq \{0,t\}$. Clearly $[0,t] \neq \{0,t\}$ if and only if $(0,t) \neq \phi$. |X|

Theorem 4.5: If T is a totally ordered group, then the $\mathfrak{T}$-topology on T is equivalent to the interval topology.

Proof: (a) Suppose there exists $h \in T^+ \setminus \{0\}$ such that $[0,h] = \{0,h\}$. Then clearly $h \in \mathcal{D}_2$ with $D(h) = \{0\}$. Hence $D(h) + h^{\perp} = h^{\perp} \in \mathcal{N}_2(0)$. But if $g \in (h^{\perp})^+ \setminus \{0\}$, then g is not comparable to h. Since T is totally ordered, no such g exists and hence $h^{\perp} = \{0\}$. Thus $\{0\} \in \mathcal{N}(0)$, i.e. the $\mathfrak{T}$-topology is discrete. Also, $(-h,h) \in \mathcal{L}$, the interval topology. Thus, since $(-h,h) = \{0\}$, $\{0\} \in \mathcal{L}$ and $\mathcal{L}$ is discrete. Therefore $\mathfrak{T} = \mathcal{L}$.

(b) Suppose that for all $t \in T^+ \setminus \{0\}$, $(0,t) \neq \phi$. If $t \in T^+ \setminus \{0\}$, then by Proposition 4.4 there is a $t_2 \in T$ such that $0 < t_2 < t_2 + t_2 \leqq t$. By Lemma 4.3, $t, t_2 \in \mathfrak{A}$. As in (a), $t^{\perp} = \{0\} = t_2^{\perp}$. Therefore, since $(0,t) \neq \phi$ for all $t \in T^+ \setminus \{0\}$, then for all $t \in T^+ \setminus \{0\}$ there exist $t_1, t_2, \ldots \in \mathfrak{A}$ such that $t = t_1$, $t_{n+1} + t_{n+1} \leqq t_n$, and $t_{n+1}^{\perp} = t_n^{\perp}$. Thus $T^+ \setminus \{0\} \subseteq \mathcal{D}_1$. Since $T^+ \setminus \{0\} \supseteq \mathcal{D}_1$, $T^+ \setminus \{0\} = \mathcal{D}_1$. Hence, if $T \neq \{0\}$,

$$\mathcal{N}(0) = \{[-t,t] \mid t \in T^+ \setminus \{0\}\}.$$

Let $U \in \mathcal{L}$. Let $u \in U$. Then there exist $a, b \in T$ such that $u \in (a,b) \subseteq U$. Let $c = (-u + b) \wedge (-a + u)$. Since $a \neq u \neq b$,

$c \neq 0$. Since $a < u < b$, $c > 0$. Thus there is a $c' \in T$ such that $0 < c' < c$. Then

$$u \in [u - c', u + c'] \subseteq (u - c, u + c).$$

But $u + c \leqq u + (-u + b) = b$ and $u - c \geqq u + (-u + a) = a$. Thus

$$u \in u + [-c', c'] \subseteq (a,b) \subseteq U.$$

Since $c' > 0$, $c' \in \mathcal{D}_1$. Hence $u + [-c', c'] \in N(u)$. Since u was an arbitrary element of U, $U \in \mathfrak{T}$.

Let $U \in \mathfrak{T}$. Let $u \in U$. Then there is a $g \in T^+ \setminus \{0\}$ such that $u \in u + [-g, g] \subseteq U$. Hence $u \in (u - g, u + g) \subseteq U$. Since u was an arbitrary element of U, $U \in L$.

If $T = \{0\}$, $\mathfrak{T} = \{\{0\}\} = L$. |X|

5. PRODUCTS OF ℓ-GROUPS

In this chapter we characterize the $\mathfrak{T}$-topology on cardinal and lexico-graphic products of ℓ-groups in terms of the $\mathfrak{T}$-topologies on the factors. We first prove the result for cardinal products:

Theorem 5.1: Let $\{G_\lambda \mid \lambda \in \Lambda\}$ be a collection of ℓ-groups. For $\lambda \in \Lambda$ let $\mathfrak{T}_\lambda$ be the $\mathfrak{T}$-topology on G_λ. Let G be an ℓ-subgroup of $\underset{\lambda\in\Lambda}{|\Pi|} G_\lambda$ which contains $\underset{\lambda\in\Lambda}{|\Sigma|} G_\lambda$. Let $\mathfrak{T}$ be the $\mathfrak{T}$-topology on G and let P be the topology on G inherited from the product topology on $\underset{\lambda\in\Lambda}{|\Pi|} G_\lambda$. Then $\mathfrak{T} = P$.

We will use the notation introduced in the statement of Theorem 5.1 throughout the proofs of the theorem and the preceding lemmas. Additionally, we let $p_\gamma: \underset{\lambda\in\Lambda}{|\Pi|} G_\lambda \to G_\gamma$ denote the γ^{th} projection. Then $U \in P$ if and only if for all $g \in U$ there exist $U_{\gamma_i} \in \mathfrak{T}_{\gamma_i}$, $i = 1,\ldots n$, such that $g \in \bigcap_{i=1}^{n} (p_{\gamma_i}^{-1}(U_{\gamma_i}) \cap G) \subseteq U$.

To denote the sets $\mathfrak{U}$, $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}^*$, $N_1(0)$, $N_2(0)$, $N_3(0)$, and $N(0)$ in G_λ, we use $\mathfrak{U}_\lambda$, $\mathcal{D}_{1\lambda}$, $\mathcal{D}_{2\lambda}$, etc. To denote them in G, we use $\mathfrak{U}$, $\mathcal{D}_1$, $\mathcal{D}_2$, etc. Then $U \in \mathfrak{T}$ if and only if for all $g \in U$ there exists $H \in N(0)$ such that $g \in g + H \subseteq U$. The notation "$\perp$" will refer to $\underset{\lambda\in\Lambda}{|\Pi|} G_\lambda$. Thus if $A \subseteq \underset{\lambda\in\Lambda}{|\Pi|} G_\lambda$,

$$A^\perp = \{g \mid g \in \underset{\lambda\in\Lambda}{|\Pi|} G_\lambda \text{ and } |g| \wedge |a| = 0 \text{ for all } a \in A\}.$$

Hence if $A \subseteq G$, the polar of A in G will be

$$A^\perp \cap G = \{g \mid g \in G \text{ and } |g| \wedge |a| = 0 \text{ for all } a \in A\}.$$

For $h_\gamma \in G_\gamma$, let $\overline{h_\gamma} \in \underset{\lambda\in\Lambda}{|\Sigma|} G_\lambda \subseteq G$ be defined by

$$\lambda\overline{h_\gamma} = \begin{cases} 0 & \text{if } \lambda \neq \gamma \\ h_\gamma & \text{if } \lambda = \gamma. \end{cases}$$

If $L_\gamma \subseteq G_\gamma$, let

$$\overline{L_\gamma} = \{\overline{h} \mid h \in L_\gamma\}.$$

If L_γ is a convex ℓ-subgroup of G_γ, then clearly $\overline{L_\gamma}$ is a convex ℓ-subgroup of G.

Lemma 5.2: Let A_γ and B_γ be subsets of G_γ. Then

$$p^{-1}(A_\gamma + B_\gamma^\perp) \cap G = \overline{A_\gamma} + (\overline{B_\gamma}^\perp \cap G).$$

Proof: Let $g \in p_\gamma^{-1}(A_\gamma + B_\gamma^\perp) \cap G$. Then $\gamma g = h_\gamma + k_\gamma$ for $h_\gamma \in A_\gamma$ and $k_\gamma \in B_\gamma^\perp$. Let $g' \in \underset{\lambda\in\Lambda}{|\Pi|} G$ be defined by

$$\lambda g' = \begin{cases} k_\gamma & \text{if } \lambda = \gamma \\ \lambda g & \text{if } \lambda \neq \gamma. \end{cases}$$

Then if $\gamma \neq \lambda \in \Lambda$, $\lambda g = 0 + \lambda g = \lambda\overline{h_\gamma} + \lambda g'$, and $\gamma g = h_\gamma + k_\gamma = \gamma\overline{h_\gamma} + \gamma g'$. Thus $g = \overline{h_\gamma} + g'$, i.e. $g' = g - \overline{h_\gamma}$. Since $g, \overline{h_\gamma} \in G$, then $g' \in G$. Let $b \in \overline{B_\gamma}$. Then $b = \overline{\gamma b}$ and $\gamma b \in B_\gamma$. Since $k_\gamma \in B_\gamma^\perp$, $|k_\gamma| \wedge |\gamma b| = 0$. Since $\lambda b = 0$ when $\lambda \neq \gamma$, this implies that $|\lambda g'| \wedge |\lambda b| = 0$ for all $\lambda \in \Lambda$. Hence $|g'| \wedge |b| = 0$ and thus $g' \in \overline{B_\gamma}^\perp \cap G$. Clearly $\overline{h_\gamma} \in \overline{A_\gamma}$ and therefore since $g = \overline{h_\gamma} + g'$, $g \in \overline{A_\gamma} + (\overline{B_\gamma}^\perp \cap G)$.

Conversely, suppose that $g \in \overline{A_\gamma} + (\overline{B_\gamma}^\perp \cap G)$. Since $\overline{A_\gamma} \subseteq \underset{\lambda\in\Lambda}{|\Sigma|} G_\lambda$, $\overline{A_\gamma} \subseteq G$. Thus $g \in G$. Also $g = h + \ell$ for $h \in \overline{A_\gamma}$ and $\ell \in \overline{B_\gamma}^\perp$. Then $\gamma h \in A_\gamma$ and $|\ell| \wedge |\overline{b_\gamma}| = 0$ for all $b_\gamma \in B_\gamma$. Thus $|\gamma\ell| \wedge |b_\gamma| = 0$ for all $b_\gamma \in B_\gamma$, i.e. $\gamma\ell \in B_\gamma^\perp$. Hence $\gamma g = \gamma h + \gamma\ell \in A_\gamma + B_\gamma^\perp$. Therefore $g \in p^{-1}(A_\gamma + B_\gamma^\perp) \cap G$. |$\boxtimes$|

Lemma 5.3: If $h_\gamma \in \mathfrak{A}_\gamma$, then $\overline{h_\gamma} \in \mathfrak{A}$.

Proof: Let $g \in T(\overline{h_\gamma}) \setminus \{0\}$. Since $0 < g \leqq \overline{h_\gamma}$, $\eta g = 0$ for all $\eta \neq \gamma$, i.e. $\overline{\gamma g} = g$. Since $g \in T(\overline{h_\gamma})$, there exists $g' \in G$ such that $g \wedge g' = 0$ and $g \vee g' = \overline{h_\gamma}$. Thus $(\gamma g) \vee (\gamma g') = h_\gamma$ and $(\gamma g) \wedge (\gamma g') = 0$. Then $\gamma g \in T(h_\gamma)$. Since $h_\gamma \in \mathfrak{A}_\gamma$, $\gamma g = 0$ or $\gamma g = h_\gamma$. If $\gamma g = 0$, then $g = 0$ which contradicts our choice of g. Thus $\gamma g = h_\gamma$ and hence $g = \overline{h_\gamma}$. $|\boxtimes|$

Lemma 5.4: If $h_\gamma \in \mathcal{D}_{1\gamma}$, then $\overline{h_\gamma} \in \mathcal{D}_1$.

Proof: Since $h_\gamma \in \mathcal{D}_{1\gamma}$, there exist $h_1, h_2, \ldots \in \mathfrak{A}_\gamma$ such that $h_\gamma = h_1$, $h_{n+1} + h_{n+1} \leqq h_n$, and $h_n^\perp \supseteq h_{n+1}^\perp$. Since $h_\gamma = h_1$, $\overline{h_\gamma} = \overline{h_1}$. Since $h_n \in \mathfrak{A}_\gamma$, $\overline{h_n} \in \mathfrak{A}$ by Lemma 5.3. Since $h_{n+1} + h_{n+1} \leqq h_n$, $\overline{h_{n+1}} + \overline{h_{n+1}} \leqq \overline{h_n}$. If $g \in \overline{h_{n+1}}^\perp$, then $|g| \wedge \overline{h_{n+1}} = 0$. Hence $\gamma g \in h_{n+1}^\perp$ and since $h_{n+1}^\perp \subseteq h_n^\perp$, $\gamma g \in h_n^\perp$. Thus, since $\lambda \overline{h_n} = 0$ for all $\lambda \neq \gamma$, $|g| \wedge \overline{h_n} = 0$. Hence $g \in \overline{h_n}^\perp$. Thus $\overline{h_{n+1}}^\perp \subseteq \overline{h_n}^\perp$. Therefore $\overline{h_\gamma} \in \mathcal{D}_1$. $|\boxtimes|$

Lemma 5.5: If $h_\gamma \in \mathcal{D}_{2\gamma}$, then $\overline{h_\gamma} \in \mathcal{D}_2$ and $D(\overline{h_\gamma}) = \overline{D(h_\gamma)}$.

Proof: As noted above, $\overline{D(h_\gamma)}$ is a convex ℓ-subgroup of G. Let $g \in \overline{D(h_\gamma)}$. Then $\lambda g = 0$ if $\lambda \neq \gamma$, i.e. $g = \overline{\gamma g}$. Further, $\gamma g \in D(h_\gamma)$ and thus $\gamma g < h_\gamma$. Hence $g = \overline{\gamma g} < \overline{h_\gamma}$. Suppose that $h < t < b$ for $h, -\overline{h_\gamma} + b \in \overline{D(h_\gamma)}$. Then $\lambda h = 0 = 0 + \lambda b$ if $\lambda \neq \gamma$, and hence $0 = \lambda t$ if $\lambda \neq \gamma$. Thus $\overline{\gamma h} = h$, $\overline{\gamma t} = t$, and $\overline{\gamma b} = b$. Further, γh, $-h_\gamma + \gamma b \in D(h_\gamma)$, and $\gamma h < \gamma t < \gamma b$. Hence $\gamma t \in D(h_\gamma) \cup (h_\gamma + D(h_\gamma))$. Therefore $\overline{\gamma t} \in \overline{D(h_\gamma)} \cup \overline{(h_\gamma + D(h_\gamma))}$ and hence $t \in \overline{D(h_\gamma)} \cup (\overline{h_\gamma} + \overline{D(h_\gamma)})$. Thus $\overline{h_\gamma} \in \mathcal{D}_2$ and $D(\overline{h_\gamma}) = \overline{D(h_\gamma)}$. $|\boxtimes|$

Lemma 5.6: If $H \in \mathcal{N}_{3\gamma}(0)$, then $p^{-1}(H) \cap G \in \mathcal{N}_3(0)$.

Proof: Suppose $H \in \mathcal{N}_{1\gamma}(0)$. Then $H = N(0,h_\gamma)$ for $h_\gamma \in \mathcal{D}_{1\gamma}$. By Lemma 5.4 $\overline{h_\gamma} \in \mathcal{D}_1$. By Lemma 5.2 $p_\gamma^{-1}([-h_\gamma,h_\gamma] + h_\gamma^{\perp}) \cap G = \overline{[-h_\gamma,h_\gamma]} + (\overline{h_\gamma}^{\perp} \cap G)$. Since $\overline{[-h_\gamma,h_\gamma]} = [-\overline{h_\gamma},\overline{h_\gamma}]$ and since the polar of $\overline{h_\gamma}$ in G is $\overline{h_\gamma}^{\perp} \cap G$, then $p_\gamma^{-1}(N(0,h_\gamma)) \cap G = N(0,\overline{h_\gamma})$. Hence $p_\gamma^{-1}(N(0,h_\gamma)) \cap G \in \mathcal{N}_1(0)$.

Suppose $H \in \mathcal{N}_{2\gamma}(0)$. Then $H = D(h_\gamma) + h_\gamma^{\perp}$ for some $h_\gamma \in \mathcal{D}_{2\gamma}$. By Lemma 5.5 $\overline{h_\gamma} \in \mathcal{D}_2$. By Lemma 5.2 $p_\gamma^{-1}(D(h_\gamma) + h_\gamma^{\perp}) \cap G = \overline{D(h_\gamma)} + (\overline{h_\gamma}^{\perp} \cap G)$. Thus, by Lemma 5.5, $p_\gamma^{-1}(D(h_\gamma) + h_\gamma^{\perp}) \cap G = D(\overline{h_\gamma}) + (\overline{h_\gamma}^{\perp} \cap G)$, and since the polar of $\overline{h_\gamma}$ in G is $\overline{h_\gamma}^{\perp} \cap G$, then $p_\gamma^{-1}(D(h_\gamma) + h_\gamma^{\perp}) \cap G \in \mathcal{N}_2(0)$. |☒|

Lemma 5.7: Let $\{H_1,\ldots,H_n\}$ be a finite collection of subsets of G_γ. Let $h \in G$. Then

$$p_\gamma^{-1}(\gamma h + \bigcap_{i=1}^{n} H_i) \cap G = h + \bigcap_{i=1}^{n} [p_\gamma^{-1}(H_i) \cap G].$$

Proof:

$$\begin{aligned} p_\gamma^{-1}(\gamma h + \bigcap_{i=1}^{n} H_i) \cap G &= \{f \mid f \in G \text{ and } \gamma f \in \gamma h + \bigcap_{i=1}^{n} H_i\} \\ &= \{f \mid f \in G \text{ and } -\gamma h + \gamma f \in H_i \text{ for all } i\} \\ &= \{f \mid f \in G \text{ and } -h + f \in p_\gamma^{-1}(H_i) \text{ for all } i\} \end{aligned}$$

Since $h \in G$, $-h \in G$ and thus $-h + f \in G$ if and only if $f \in G$. Thus

$$\begin{aligned} p_\gamma^{-1}(\gamma h + \bigcap_{i=1}^{n} H_i) \cap G &= \{f \mid -h + f \in p_\gamma^{-1}(H_i) \cap G \text{ for all } i\} \\ &= \{f \mid f \in h + \bigcap_{i=1}^{n} [p_\gamma^{-1}(H_i) \cap G]\} \\ &= h + \bigcap_{i=1}^{n} [p_\gamma^{-1}(H_i) \cap G] \end{aligned}$$

|☒|

Lemma 5.8: If $h \in \mathfrak{A}$, then there is a $\gamma \in \Lambda$ such that $\overline{\gamma h} = h$ and $\gamma h \in \mathfrak{A}_\gamma$.

Proof: Suppose that there exist $\alpha, \beta \in \Lambda$ such that $\alpha \neq \beta$ and $\alpha h \neq 0 \neq \beta h$. Let $h' \in |\Pi|_{\lambda\in\Lambda} G_\lambda$ be defined by

$$\lambda h' = \begin{cases} \lambda h & \text{if } \lambda \neq \alpha \\ 0 & \text{if } \lambda = \alpha. \end{cases}$$

Then as in the proof of Lemma 5.2, we have that if $\lambda \neq \alpha$, $\lambda h = \lambda h + 0 = \lambda h' + \lambda\overline{\alpha h}$ and $\alpha h = 0 + \alpha h = \alpha h' + \alpha\overline{\alpha h}$. Thus $h = h' + \overline{\alpha h}$ and since $\overline{\alpha h}, h \in G$, $h' \in G$. Clearly $h' \wedge \overline{\alpha h} = 0$ and $h' \vee \overline{\alpha h} = h$. Since $\alpha h \neq 0$, $\overline{\alpha h} \neq 0$, and since $\beta h \neq 0$, $h' \neq 0$. This contradicts the assumption that $h \in \mathfrak{A}$. Hence there is a $\gamma \in \Lambda$ such that $\lambda h = 0$ for all $\lambda \neq \gamma$, i.e. $\overline{\gamma h} = h$. Let $\mathrm{h} \in T(\gamma h)\setminus\{0\}$. Then there is an $\mathrm{h}' \in G_\gamma$ such that $\mathrm{h} \wedge \mathrm{h}' = 0$ and $\mathrm{h} \vee \mathrm{h}' = \gamma h$. Then $\bar{\mathrm{h}} \wedge \overline{\mathrm{h}'} = 0$ and $\bar{\mathrm{h}} \vee \overline{\mathrm{h}'} = \overline{\gamma h} = h$. Since $h \in \mathfrak{A}$ and $\mathrm{h} \neq 0$, $\bar{\mathrm{h}} = h$. Thus $\mathrm{h} = \gamma h$ and $\gamma h \in \mathfrak{A}_\gamma$. |⌛|

Lemma 5.9: Let $h \in \mathcal{D}_1$. Then there exists a $\gamma \in \Lambda$ and an $\mathrm{h}_\gamma \in \mathcal{D}_{1\gamma}$ such that $p_\gamma^{-1}(N(0,\mathrm{h}_\gamma)) \cap G = N(0,h)$.

Proof: Since $\mathcal{D}_1 \subseteq \mathfrak{A}$, by Lemma 5.8 $h = \overline{\gamma h}$ for some $\gamma \in \Lambda$. Let $\mathrm{h}_\gamma = \gamma h$. Since $h \in \mathcal{D}_1$, there exist $h_1, h_2, \ldots \in \mathfrak{A}$ such that $h = h_1$, $h_{n+1} + h_{n+1} \leqq h_n$ and $h_{n+1}{}^{\perp} \cap G \subseteq h_n{}^{\perp} \cap G$. Since $h_1 = h$, $\gamma h_1 = \mathrm{h}_\gamma$. Since $0 < h_n \leqq h$ and since $h = \overline{\gamma h}$, $h_n = \overline{\gamma h_n}$. Since $h_1, h_2, \ldots \in \mathfrak{A}$, this implies by Lemma 5.8 that $\gamma h_1, \gamma h_2, \ldots \in \mathfrak{A}_\gamma$. Since $h_{n+1} + h_{n+1} \leqq h_n$, $\gamma h_{n+1} + \gamma h_{n+1} \leqq \gamma h_n$. If $\mathrm{f} \in (\gamma h_{n+1})^{\perp}$, then $|\bar{\mathrm{f}}| \wedge \overline{\gamma h_{n+1}} = 0$, and hence $\bar{\mathrm{f}} \in \overline{h_{n+1}}^{\perp} \cap G$. Since $h_{n+1}{}^{\perp} \cap G \subseteq h_n{}^{\perp} \cap G$, $\bar{\mathrm{f}} \in h_n{}^{\perp} \cap G$. Thus $|\bar{\mathrm{f}}| \wedge \overline{\gamma h_n} = 0$ and hence $f \in (\gamma h_n)^{\perp}$. Thus $(\gamma h_{n+1})^{\perp} \subseteq (\gamma h_n)^{\perp}$ and hence $h_\gamma \in \mathcal{D}_{1\gamma}$. Further, by Lemma 5.2, $p_\gamma^{-1}([-h_\gamma, h_\gamma] + h_\gamma^{\perp}) \cap G = \overline{[-h_\gamma, h_\gamma]} + (\overline{h_\gamma}^{\perp} \cap G)$. Thus $p_\gamma^{-1}(N(0,h_\gamma)) \cap G = [-h,h] + (h^{\perp} \cap G) = N(0,h)$. |☒|

<u>Lemma 5.10</u>: Let $h \in \mathcal{D}_2$. Then there exists a $\gamma \in \Lambda$ and an $h_\gamma \in \mathcal{D}_2$ such that $p_\gamma^{-1}(D(h_\gamma) + h_\gamma^{\perp}) \cap G = D(h) + (h^{\perp} \cap G)$.

<u>Proof</u>: By Proposition 2.3, $h \in \mathfrak{U}$, and thus by Lemma 5.8, $h = \overline{\gamma h}$ for some $\gamma \in \Lambda$. Let $h_\gamma = \gamma h$. Let $D' = \{\gamma\delta \mid \delta \in D(h)\}$. Then clearly D' is a convex ℓ-subgroup of G and since $h = \overline{h_\gamma}$ and $D(h) \subseteq [-h,h]$, then $\overline{D'} = D(h)$. If $d \in D'$, then $d = \gamma\delta < \gamma h = h_\gamma$ since $\delta \in D(h)$. Suppose $a < t < b$ for $a, -h_\gamma + b \in D'$. Then $\bar{a} < \bar{t} < \bar{b}$ and $\bar{a}, \overline{-h_\gamma} + \bar{b} \in \overline{D'}$. Hence $\bar{a}, -h + \bar{b} \in D(h)$ and thus $\bar{t} \in D(h) \cup (h + D(h))$. Then $t = \gamma\bar{t} \in D' \cup (h_\gamma + D')$. Therefore $h_\gamma \in \mathcal{D}_{2\gamma}$ and $D' = D(h_\gamma)$. By Lemma 5.2 $p_\gamma^{-1}(D(h_\gamma) + h_\gamma^{\perp}) \cap G = \overline{D'} + (\overline{h_\gamma}^{\perp} \cap G)$. Thus $p_\gamma^{-1}(D(h_\gamma) + h_\gamma^{\perp}) \cap G = D(h) + (h^{\perp} \cap G) \in N_2(0)$. |☒|

<u>Lemma 5.11</u>: Let $H \in N_3(0)$. Let $\delta \in G$. The there exists a $\gamma \in \Lambda$ and an $H_\gamma \in N_{3\gamma}(0)$ such that $p^{-1}(\gamma\delta + H_\gamma) \cap G = \delta + H$.

<u>Proof</u>: If $H \in N_1(0)$, then $H = N(0,h)$ for some $h \in \mathcal{D}_1$. By Lemma 5.9, there exists a $\gamma \in \Lambda$ and an $h_\gamma \in \mathcal{D}_{1\gamma}$ such that $p_\gamma^{-1}(N(0,h_\gamma)) \cap G = H$. Let $H_\gamma = N(0,h_\gamma)$. Since $h_\gamma \in \mathcal{D}_{1\gamma}$, $H_\gamma \in N_{1\gamma}(0)$. If $H \in N_2(0)$, then $H = D(h) + (h^{\perp} \cap G)$ for some $h \in \mathcal{D}_2$. By Lemma 5.10, ther exists a $\gamma \in \Lambda$ and an $h_\gamma \in \mathcal{D}_{2\gamma}$ such that $p^{-1}(D(h_\gamma) + h_\gamma^{\perp}) \cap G = H$. Let $H_\gamma = D(h_\gamma) + h_\gamma^{\perp}$. Since $h_\gamma \in \mathcal{D}_{2\gamma}$, $H_\gamma \in N_{2\gamma}(0)$. Then by Lemma 5.7, $p_\gamma^{-1}(\gamma\delta + H_\gamma) \cap G = \delta + (p_\gamma^{-1}(H_\gamma) \cap G)$. Hence $p_\gamma^{-1}(\gamma\delta + H_\gamma) \cap G = \delta + H$. |☒|

<u>Proof of Theorem 5.1</u>: Let $U \in \mathcal{P}$, and $\delta \in U$. Then there exist $\{\gamma_1,\ldots,\gamma_n\} \subseteq \Lambda$ such that $\delta \in \bigcap_{i=1}^{n}[p_{\gamma_i}^{-1}(U_{\gamma_i}) \cap G] \subseteq U$ for $U_{\gamma_i} \in \mathfrak{T}_{\gamma_i}$. If $U_{\gamma_i} = G_{\gamma_i}$ for all $i = 1,\ldots,n$, then $\bigcap_{i=1}^{n} p_{\gamma_i}^{-1}(U_{\gamma_i}) = \left|\Pi\right|_{\lambda\in\Lambda} G_\lambda$ and thus $\bigcap_{i=1}^{n}[p_{\gamma_i}^{-1}(U_{\gamma_i}) \cap G] = G \subseteq U$. Hence $U = G$, and thus $U \in \mathfrak{T}$. Therefore we may assume that $U_{\gamma_i} \neq G_{\gamma_i}$ for at least one i. But then if $U_{\gamma_\ell} = G_{\gamma_\ell}$ for some ℓ, $1 \leqq \ell \leqq n$, we have that

$$\bigcap_{i=1}^{n}[p_{\gamma_i}^{-1}(U_{\gamma_i}) \cap G] = \bigcap_{\substack{i=1\\ i\neq\ell}}^{n}[p_{\gamma_i}^{-1}(U_{\gamma_i}) \cap G].$$

Therefore, we may in fact assume that $U_{\gamma_i} \neq G_{\gamma_i}$ for all $i = 1,\ldots n$.

Since $\delta \in \bigcap_{i=1}^{n}[p_{\gamma_i}^{-1}(U_{\gamma_i}) \cap G]$, $\gamma_i\delta \in U_{\gamma_i}$ for all i. Since $U_{\gamma_i} \in \mathfrak{T}_{\gamma_i}$ for all i, then for each i there exists an integer $m(i) \geqq 1$ such that $H_{ij} \in N_{3\gamma}(0)$ for $j = 1,\ldots m(i)$ and

$$\gamma_i\delta \in \gamma_i\delta + \bigcap_{j=1}^{m(i)} H_{ij} \subseteq U_{\gamma_i}.$$

Thus

$$\delta \in p_{\gamma_i}^{-1}(\gamma_i\delta + \bigcap_{j=1}^{m(i)} H_{ij}) \cap G \subseteq p_{\gamma_i}^{-1}(U_{\gamma_i}) \cap G.$$

By Lemma 5.7

$$\delta \in \delta + \bigcap_{j=1}^{m(i)} [p_{\gamma_i}^{-1}(H_{ij}) \cap G] \subseteq p_{\gamma_i}^{-1}(U_{\gamma_i}) \cap G.$$

By Lemma 5.6 $p_{\gamma_i}^{-1}(H_{ij}) \cap G \in N_3(0)$ for all $j = 1,\ldots,m(i)$ and all $i = 1,\ldots,n$. Thus $\bigcap_{i=1}^{n}\bigcap_{j=1}^{m(i)} [p_{\gamma_i}^{-1}(H_{ij}) \cap G] \in N(0)$. Therefore, since

$$\delta \in \delta + \bigcap_{i=1}^{n}\bigcap_{j=1}^{m(i)} [p_{\gamma_i}^{-1}(H_{ij}) \cap G] \subseteq \bigcap_{i=1}^{n}[p_{\gamma_i}^{-1}(U_{\gamma_i}) \cap G] \subseteq U,$$

and since δ was an arbitrary element of U, $U \in \mathfrak{T}$.

Let $U \in \mathfrak{T}$, and $\mathfrak{f} \in U$. If $U = G$, then $U \in \mathcal{P}$. Otherwise, there exist an integer $n \geqq 1$ and an $H_i \in N_3(0)$ for $i = 1,\ldots,n$ such that $\mathfrak{f} \in \mathfrak{f} + \bigcap_{i=1}^{n} H_i \subseteq U$. By Lemma 5.11, for each i there exist a $\gamma_i \in \Lambda$ and an $H_{\gamma_i} \in N_{3\gamma_i}(0)$ such that

$$p_{\gamma_i}^{-1}(\gamma_i \mathfrak{f} + H_{\gamma_i}) \cap G = \mathfrak{f} + H_i.$$

Thus by Lemma 5.7

$$\mathfrak{f} \in \bigcap_{i=1}^{n} [p_{\gamma_i}^{-1}(\gamma_i \mathfrak{f} + H_{\gamma_i}) \cap G] = \mathfrak{f} + \bigcap_{i=1}^{n} H_i \subseteq U.$$

Let $U_i = \gamma_i \mathfrak{f} + \mathrm{Int}(H_{\gamma_i})$. By Corollary 2.22, $0 \in \mathrm{Int}(H_{\gamma_i}) \in \mathfrak{T}_{\gamma_i}$. Thus $\gamma_i \mathfrak{f} \in U_i \in \mathfrak{T}_{\gamma_i}$, and hence

$$\mathfrak{f} \in \bigcap_{i=1}^{n} [p_{\gamma_i}^{-1}(U_i) \cap G] \subseteq \bigcap_{i=1}^{n} [p_{\gamma_i}^{-1}(\gamma_i \mathfrak{f} + H_{\gamma_i}) \cap G] \subseteq U.$$

Since $\mathfrak{f}$ was an arbitrary element of U, $U \in \mathcal{P}$.

Therefore $\mathfrak{T} = \mathcal{P}$, which proves Theorem 5.1. |▣|

Another natural topology on a product, $\prod_{\lambda \in \Lambda} K_\lambda$, of topological spaces K_λ may be defined as follows: For each $\lambda \in \Lambda$ let K_λ have topology $\mathcal{U}_\lambda$. Define a topology $\mathcal{P}'$ on $\prod_{\lambda \in \Lambda} K_\lambda$ by: $U \in \mathcal{P}'$ if and only if $p_\lambda(U) \in \mathcal{U}_\lambda$ for all $\lambda \in \Lambda$. For an arbitrary ℓ-group G, let

$$N'(0) = \{\bigcap_{\delta \in \Delta} H_\delta \mid H_\delta \in N_3(0) \text{ for all } \delta \in \Delta\}$$

where Δ may be infinite. For $g \in G$ {0}, let

$$N'(g) = \{g + H \mid H \in N'(0)\}.$$

Let

$$\mathfrak{T}' = \{W \subseteq G \mid \text{for all } x \in W,\ W \in F(N'(x))\}.$$

Similarly to the proofs in Chapter 2 we can show that $\mathfrak{T}'$ is a group and a lattice topology for G. Call $\mathfrak{T}'$ the $\mathfrak{T}'$-topology.

If, in Theorem 5.1, we let $\mathfrak{T}_\lambda$ be the $\mathfrak{T}'$-topology and take $\mathcal{P}'$ defined above in place of $\mathcal{P}$, then $\mathfrak{T}' = \mathcal{P}'$ where $\mathfrak{T}'$ is the $\mathfrak{T}'$-topology on $\mathcal{G}$.

Combining Theorem 5.1 with Theorem 4.5, we have the following corollaries.

Corollary 5.12: Let G be an ℓ-group and suppose there exists an ℓ-isomorphism of G into a cardinal product of totally ordered groups. If the sum of the totally ordered groups is contained in the image of G, then the $\mathfrak{T}$-topology on G is equivalent to the topology that G inherits from the product of the interval topologies on the factors. |⊠|

In particular, we have:

Corollary 5.13: The $\mathfrak{T}$-topology on any cardinal product of the real numbers is the usual topology (i.e. the product of the interval topologies on the factors). |⊠|

We now turn to consideration of lexico-graphic products. Let G be an arbitrary ℓ-group and let T be an arbitrary totally ordered group. To denote the sets $\mathfrak{U}$, $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}^*$, $N_1(0)$, $N_2(0)$, $N_3(0)$, and $N(0)$ in G, we use $\mathfrak{U}_G$, $\mathcal{D}_{1G}$, $\mathcal{D}_{2G}$, etc. To denote them in T we use $\mathfrak{U}_T$, $\mathcal{D}_{1T}$, $\mathcal{D}_{2T}$, etc., and in $\mathcal{G} = G \overleftarrow{X} T$ we use $\mathfrak{U}$, $\mathcal{D}_1$, $\mathcal{D}_2$, etc. We denote the $\mathfrak{T}$-topology on G by $\mathfrak{T}_G$; on T by $\mathfrak{T}_T$; and on $\mathcal{G}$ by $\mathfrak{T}$. Then $U \in \mathfrak{T}$ if and only if for all $\mathfrak{g} \in U$ there exists $H \in N(0)$ such that $\mathfrak{g} \in \mathfrak{g} + H \subseteq U$. We let $p_G: G \overleftarrow{X} T \to G$ denote the projection of $\mathcal{G}$ onto G, and $p_T: G \overleftarrow{X} T \to T$ denote the projection of $\mathcal{G}$ onto T. We sometimes write the elements of G as ordered pairs; thus if $g \in G$, $g = (p_G(g), p_T(g))$. We note that p_T preserves order, but that p_G does not. For instance, in $R \overleftarrow{X} Z$, $(0,1) < (-100,2)$, but $0 > -100$. Both p_T and p_G, of course, preserve the group operation.

We prove a theorem (5.21) that characterizes $F(N(0))$ in terms of $N_G(0)$, $\mathcal{D}_{1T}$, and $\mathcal{D}_{2T}$. The proof of the theorem is straightforward but somewhat long to write out. We prove seven preliminary lemmas characterizing $\mathcal{D}_1$ and $\mathcal{D}_2$ in terms of their projections to G and T. A case-by-case argument proves the theorem. We then prove two corollaries (5.22 and 5.23) which give extremely simple characterizations of $F(N(0))$. One of the two corollaries applies to any $G = G \overleftarrow{X} T$.

<u>Lemma 5.14</u>: If $g \in G \times (T^+ \setminus \{0\})$, then $g^{\perp} = \{0\}$.

<u>Proof</u>: Suppose $g \in G \times \{t\}$ for $t \in T^+ \setminus \{0\}$. Suppose $h \in G^+$ is such that $h \wedge g = 0$. If $p_T(h) = 0$, then since $p_T(g) = t > 0 = p_T(h)$, $g > h$ and hence $h = 0$. Suppose $p_T(h) > 0$. Then $h > \ell$ for $\ell \in G \times \{0\}$. Since $p_T(g) > 0$, $g > \ell$ for $\ell \in G \times \{0\}$. If $G \neq \{0\}$, then there exists an element $p \in G^+ \setminus \{0\}$. Then $(p,0) \in G \times \{0\}$ and $(p,0) > 0$, and hence $h \wedge g > (p,0) > 0$. This contradicts our choice of h. Thus $G = \{0\}$. Then G is totally ordered and thus $h \wedge g = h$ or $h \wedge g = g$. These contradict our choices of g and h. Thus $p_T(h) = 0$, i.e. $h = 0$. Hence $g^{\perp} = \{0\}$. |⊠|

<u>Lemma 5.15</u>: Let $g \in G$. Then $g \in \mathfrak{A}$ if and only if $p_T(g) > 0$ or $p_T(g) = 0$ and $p_G(g) \in \mathfrak{A}_G$.

Proof: Suppose $g \in \mathfrak{A}$. Since $g > 0$, then $p_T(g) > 0$ or $p_T(g) = 0$ and $p_G(g) > 0$. Thus suppose $p_T(g) = 0$, and let $\ell \in T(p_G(g)) \setminus \{0\}$. Then there exists $\ell' \in G$ such that $\ell \wedge \ell' = 0$ and $\ell \vee \ell' = p_G(g)$. Thus $(\ell,0) \vee (\ell',0) = (p_G(g),0) = g$ and $(\ell,0) \wedge (\ell',0) = (0,0)$. Since $\ell \neq 0$, $(\ell,0) \neq 0$ and hence $(\ell,0) = g$, i.e. $p_G(g) = \ell$. Therefore $p_G(g) \in \mathfrak{A}_G$.

Conversely, suppose that $p_T(g) = 0$ and $p_G(g) \in \mathfrak{A}_G$. Let $h \in T(g) \setminus \{0\}$. Let $h' \in G$ be such that $h \vee h' = g$ and $h \wedge h' = 0$. Then since $p_T(h) = 0 = p_T(h')$, $p_G(h) \wedge p_G(h') = 0$ and $p_G(h) \vee p_G(h') = p_G(g)$. Since $h \neq 0$, $p_G(h) \neq 0$ and thus since $p_G(g) \in \mathfrak{A}_G$, $p_G(h) = p_G(g)$. Therefore $g = (p_G(g),0) = (p_G(h),0) = h$ and hence $g \in \mathfrak{A}$. Suppose that $p_T(g) > 0$. Let $h,h' \in G$ be such that $h \wedge h' = 0$ and $h \vee h' = g$. Then $h + h' = g$. If both $h,h' \notin G \times \{0\}$, then as in the proof of Lemma 5.14, $h \wedge h' > 0$. Hence we may assume that $h \in G \times \{0\}$. Then $p_T(h) = 0$. But since $h + h' = g$, $p_T(h) + p_T(h') = p_T(g)$. Thus $p_T(h') = p_T(g) > 0 = p_T(h)$. Hence $h' > h$. Therefore $h = 0$ and $h' = g$. Hence $g \in \mathfrak{A}$. |X|

Lemma 5.16: Let $g \in G$ and suppose $p_T(g) > 0$. Then $g \in \mathcal{D}_1$ if and only if $p_T(g) \in \mathcal{D}_{1T}$ or there exists $f \in \mathcal{D}_{1G}$ such that $f^{\perp} = \{0\}$.

Proof: Suppose that there exists $f \in \mathcal{D}_{1G}$ such that $f^{\perp} = \{0\}$. Then there exist $f_1, f_2, \ldots \in \mathfrak{A}_G$ such that $f_1 = f$, $f_{n+1} + f_{n+1} \leqq f_n$, and $f_n^{\perp} = f_{n+1}^{\perp}$. Hence $f_n^{\perp} = \{0\}$ for all n. Since $f_1 = f$, $(f_1,0) = (f,0)$. Since $f_{n+1} + f_{n+1} \leqq f_n$, $(f_{n+1},0) + (f_{n+1},0) \leqq (f_n,0)$. Suppose $|(a,b)| \wedge (f_n,0) = 0$. If $|b| > 0$, then $|(a,b)| > (f_n,0)$.

Since $(f_n,0) > 0$, thus $|b| = 0$, i.e. $b = 0$. Then $(|a|,0) \wedge (f_n,0) = 0$ implies $|a| \wedge f_n = 0$ and hence, since $f_n^{\perp} = \{0\}$, $a = 0$. Thus $(f_n,0)^{\perp} = \{0\}$. Since $p_T(g) > 0$, $g > (f + f, 0) = (f,0) + (f,0)$. Let $g_1 = g$, and $g_n = (f_{n-1},0)$ for $n \geqq 2$. Then $g_{n+1} + g_{n+1} \leqq g_n$, and by Lemma 5.14 $g_1^{\perp} = g^{\perp} = \{0\} = g_n^{\perp}$ for all $n \geqq 2$. Thus $g \in \mathcal{D}_1$. If $p_T(g) \in \mathcal{D}_{1T}$, then there exist $h_1, h_2, \ldots \in \mathfrak{U}_T$ such that $h_1 = p_T(g)$, $h_{n+1} + h_{n+1} \leqq h_n$, and $h_n^{\perp} = h_{n+1}^{\perp}$. Let $g_1 = g$ and $g_n = (0,h_n)$ for $n \geqq 2$. Then $g_{n+1} + g_{n+1} \leqq g_n$ and by Lemma 5.14, since $p_T(g_n) = h_n > 0$, $g_n^{\perp} = \{0\}$ for all n. Therefore, $g \in \mathcal{D}_1$.

Conversely, suppose that $g \in \mathcal{D}_1$ and that for all $f \in \mathcal{D}_{1G}$, $f^{\perp} \neq \{0\}$. Let $g_1, g_2, \ldots \in \mathfrak{U}$ be such that $g_1 = g$, $g_{n+1} + g_{n+1} \leqq g_n$, and $g_n^{\perp} = g_{n+1}^{\perp}$. Since $p_T(g) > 0$, by Lemma 5.14 $g^{\perp} = \{0\}$. Thus $g_n^{\perp} = \{0\}$ for all n. Suppose there exists m such that $p_T(g_m) = 0$. Then since $g_{k+1} \leqq g_k$ for all k, $p_T(g_k) = 0$ for all $k \geqq m$. Let $f_n = p_G(g_{m+n-1})$ for all $n \geqq 1$. Clearly $f_{n+1} + f_{n+1} \leqq f_n$. If $|k| \wedge f_n = 0$, then $(|k|,0) \wedge (f_n,0) = 0$ and hence $|(k,0)| \wedge g_{m+n-1} = 0$. Thus $(k,0) \in g_{m+n-1}^{\perp}$ and hence $k = 0$. Thus $f_n^{\perp} = \{0\}$ for all n. We conclude that $f_1 \in \mathcal{D}_{1G}$ and $f_1^{\perp} = \{0\}$ which contradicts our assumption on G. Thus $p_T(g_n) > 0$ for all n. Clearly $p_T(g_{n+1}) + p_T(g_{n+1}) \leqq p_T(g_n)$ and since T is totally ordered, $p_T(g_{n+1})^{\perp} = \{0\} = p_T(g_n)^{\perp}$. Thus since $p_T(g) = p_T(g_1)$, $p_T(g) \in \mathcal{D}_{1T}$. |⊠|

Lemma 5.17: Let $g \in G$ and suppose $p_T(g) > 0$. Then $g \in \mathcal{D}_2$ if and only if $p_T(g) \in \mathcal{D}_{2T}$. In this case, $D(g) = G \times D(p_T(g))$.

<u>Proof</u>: Suppose $g \in \mathcal{D}_2$. Clearly $p_T(D(g))$ is a convex ℓ-subgroup of T. Let $d \in p_T(D(g))$. Since T is totally ordered, $p_T(g) > d$ or $p_T(g) \leqq d$. Suppose $p_T(g) \leqq d$, and let $k \in G$ be such that $(k,d) \in D(g)$. Then $(k,d) < g$. Thus $d = p_T(g)$. But $(k,d) + (k,d) = (k + k, d + d) \in D(g)$. Thus $d + d \leqq p_T(g)$, i.e. $p_T(g) \leqq 0$. This contradicts our choice of g. Thus $p_T(g) > d$. Suppose $a < t < b$ in T with $a, -p_T(g) + b \in p_T(D(g))$. Let $a', b' \in G$ be such that $(a',a), -g + (b',b) \in D(g)$. Then $(a',a) < (0,t) < (b',b)$ and hence $(0,t) \in D(g) \cup (g + D(g))$. Thus $t \in p_T(D(g)) \cup (p_T(g) + p_T(D(g)))$. Hence $p_T(g) \in \mathcal{D}_2$.

Conversely, suppose $p_T(g) \in \mathcal{D}_{2T}$ and let $D' = G \times D(p_T(g))$. Clearly D' is a convex ℓ-subgroup of G. Let $d \in D'$. Then $p_T(d) \in D(p_T(g))$. Hence $p_T(d) < p_T(g)$ and thus $d = (p_G(d), p_T(d)) < (p_G(g), p_T(g)) = g$. Let $a < t < b$ in G with $a, -g + b \in D'$. Then $p_T(a), -p_T(g) + p_T(b) \in D(p_T(g))$, and $p_T(a) \leqq p_T(t) \leqq p_T(b)$. Since $p_T(g) \in \mathcal{D}_2$, then $p_T(t) \in D(p_T(g)) \cup [p_T(g) + D(p_T(g))]$. Therefore $t = (p_G(t), p_T(t)) \in D' \cup (g + D')$. Hence $g \in \mathcal{D}_2$ and $D(g) = D' = G \times D(p_T(g))$. |X|

<u>Lemma 5.18</u>: Let $h \in G^+ \setminus \{0\}$. Then $N(0,(h,0)) = N(0,h) \times \{0\}$.

<u>Proof</u>: If $g \in [-(h,0),(h,0)]$, clearly $p_T(g) = 0$ and $p_G(g) \in [-h,h]$. If $\ell \in (h,0)^\perp$, then $|\ell| \wedge (h,0) = 0$. If $|p_T(\ell)| > 0$, $\ell > (h,0)$. Hence $|p_T(\ell)| = 0$, i.e. $p_T(\ell) = 0$. Hence $|p_G(\ell)| \wedge h = 0$, i.e. $p_G(\ell) \in h^\perp$. If $k \in N(0,(h,0))$, then $k = g + \ell$ for $g \in [-(h,0),(h,0)]$ and $\ell \in (h,0)^\perp$. Then $p_T(k) = p_T(g) + p_T(\ell) = 0$ and $p_G(k) = p_G(g) + p_G(\ell) \in N(0,h)$. Thus $k \in N(0,h) \times \{0\}$. Conversely, suppose that $p \in N(0,h)$.

Then $p = r + s$ for $r \in [-h,h]$ and $s \in h^{\perp}$. Hence $(p,0) = (r,0) + (s,0)$ and clearly $(r,0) \in [-(h,0),(h,0)]$. Also $|(s,0)| \wedge (h,0) = (|s|,0) \wedge (h,0) = (|s| \wedge h,0) = 0$. Thus $(s,0) \in (h,0)^{\perp}$. Hence $(p,0) \in N(0,(h,0))$. |X|

Lemma 5.19: Let $g \in G$ be such that $p_T(g) = 0$. Then $g \in \mathcal{D}_1$ if and only if $p_G(g) \in \mathcal{D}_{1G}$.

Proof: Let $g \in \mathcal{D}_1$. Let $g_1, g_2, \ldots \in \mathfrak{U}$ be such that $g_1 = g$, $g_{n+1} + g_{n+1} \leqq g_n$, and $g_{n+1}^{\perp} \subseteq g_n^{\perp}$. Then since $p_T(g) = 0$, we have by Lemma 5.15 that $p_G(g_n) \in \mathfrak{U}_G$ for all n. Clearly $p_G(g_1) = p_G(g)$ and $p_G(g_{n+1}) + p_G(g_{n+1}) \leqq p_G(g_n)$. If $f \in p_G(g_{n+1})^{\perp}$, then $(f,0) \in g_{n+1}^{\perp} \subseteq g_n^{\perp}$ and hence $f \in p_G(g_n)^{\perp}$. Thus $p_G(g) \in \mathcal{D}_{1G}$.

If $p_G(g) \in \mathcal{D}_{1G}$, let $h_1, h_2, \ldots \in \mathfrak{U}_G$ be such that $h_1 = p_G(g)$, $h_{n+1} + h_{n+1} \leqq h_n$, and $h_{n+1}^{\perp} \subseteq h_n^{\perp}$. By Lemma 5.15, $(h_n,0) \in \mathfrak{U}$ for all n. Clearly $(h_1,0) = g$ and $(h_{n+1},0) + (h_{n+1},0) \leqq (h_n,0)$. If $k \in (h_{n+1},0)^{\perp}$, then as in the proof of Lemma 5.18, $p_G(k) \in h_{n+1}^{\perp}$ and $p_T(k) = 0$. Thus $p_G(k) \in h_n^{\perp}$ and since $p_T(k) = 0$, $k \in h_n^{\perp}$. Hence $g \in \mathcal{D}_1$. |X|

Lemma 5.20: Let $g \in G$ be such that $p_T(g) = 0$. Then $g \in \mathcal{D}_2$ if and only if $p_G(g) \in \mathcal{D}_{2G}$. In this case, $D(g) + g^{\perp} = [D(p_G(g)) + p_G(g)^{\perp}] \times \{0\}$.

Proof: Suppose $g \in \mathcal{D}_2$. If $h \in p_G(D(g))$, then $(h,\ell) \in D(g)$ for some $\ell \in T$. Then $(h,\ell) < g$ and since $p_T(g) = 0$ and $D(g) \subseteq [-g,g]$, $\ell = 0$. Thus $\ell = p_T(g)$ and $p_G(g) > h$. Suppose $a < t < b$ for $a, -p_G(g) + b \in p_G(D(g))$. Then $(a,0) < (t,0) < (b,0)$ and $(a,0)$, $-g + (b,0) \in D(g)$. Hence $(t,0) \in D(g) \cup (g + D(g))$, and thus $t = p_G((t,0)) \in p_G(D(g)) \cup [p_G(g) + p_G(D(g))]$. Hence $p_G(g) \in \mathcal{D}_{2G}$.

Suppose $p_G(g) \in \mathcal{D}_{2G}$. Let $D' = D(p_G(g)) \times \{0\}$. If $\ell \in D'$, then $p_G(\ell) \in D(p_G(g))$ and hence $p_G(\ell) < p_G(g)$. Thus $\ell = (p_G(\ell),0) < (p_G(g),0) = g$. Let $a < t < b$ for $a, -g + b \in D'$. Then $p_G(a) < p_G(t) < p_G(b)$ since $p_T(a) = 0 = p_T(t) = p_T(b)$, and $p_G(a)$, $-p_G(g) + p_G(b) \in D(p_G(g))$. Thus $p_G(t) \in D(p_G(g)) \cup [p_G(g) + D(p_G(g))]$ and hence $t = (p_G(t),0) \in D' \cup (g + D')$. Therefore $g \in \mathcal{D}_2$ and $D(g) = D(p_G(g)) \times \{0\}$.

As in the proof of Lemma 5.18, $g^{\perp} \subseteq G \times \{0\}$. Thus $g^{\perp} = p_G(g)^{\perp} \times \{0\}$ and hence

$$D(g) + g^{\perp} = [D(p_G(g)) + p_G(g)^{\perp}] \times \{0\}. \qquad |\boxtimes|$$

<u>Theorem 5.21</u>: Let G be an ℓ-group, T a totally ordered group, and let $\mathcal{G} = G \overleftarrow{\times} T$. Then

$$\begin{aligned} F(N(0)) = F(&\{V \times \{0\} \mid V \in N_G(0)\setminus\{G\}\} \\ &\cup \{[-g,g] \mid p_T(g) \in \mathcal{D}_{1T}\} \\ &\cup \{G \times D(h) \mid h \in \mathcal{D}_{2T}\} \\ &\cup \{\mathcal{G}\}). \end{aligned}$$

<u>Proof</u>: If $\mathcal{D}^* = \phi$, $N(0) = \{\mathcal{G}\}$. By Lemmas 5.19 and 5.20, $N_G(0) = \{G\}$ and thus $\{V \times \{0\} \mid V \in N_G(0)\setminus\{G\}\} = \phi$. By Lemma 5.16, $\{[-g,g] \mid p_T(g) \in \mathcal{D}_{1T}\} = \phi$. By Lemma 5.17, $\{G \times D(h) \mid h \in \mathcal{D}_{2T}\} = \phi$. Thus the right side of the equation in the statement of the theorem is $F(\{\mathcal{G}\})$.

Therefore we may assume that $\mathcal{D}^* \neq \phi$. Clearly $\mathcal{G} \in F(N(0))$. If $p_T(g) \in \mathcal{D}_{2T}$, then by Lemma 5.17 $D(g) = G \times D(p_T(g))$. Since

$p_T(g) \in \mathcal{D}_{2T}$, $p_T(g) > 0$ and by Lemma 5.14, $g^{\perp} = \{0\}$. Hence $D(g) + g^{\perp} = G \times D(p_T(g))$. Therefore, if $h \in \mathcal{D}_{2T}$, $G \times D(h) \in N(0)$. If $p_T(g) \in \mathcal{D}_{1T}$, then by Lemma 5.16 $g \in \mathcal{D}_1$ and thus $N(0,g) \in N(0)$. But by Lemma 5.14, $g^{\perp} = \{0\}$. Thus $[-g,g] \in N(0)$. Suppose that $V \in N_G(0) \setminus \{G\}$. If $V \in N_{1G}(0)$, then $V = N(0,h)$ for $h \in \mathcal{D}_{1G}$. By Lemma 5.19, $(h,0) \in \mathcal{D}_1$ and hence by Lemma 5.18, $V \times \{0\} = N(0,(h,0)) \in N(0)$. If $V \in N_{2G}(0)$, then $V = D(h) + h^{\perp}$ for $h \in \mathcal{D}_{2G}$. By Lemma 5.20, $(h,0) \in \mathcal{D}_2$ and $V \times \{0\} = D((h,0)) + (h,0)^{\perp} \in N(0)$. If $V = \bigcap_{i=1}^{n} H_i$ for $H_i \in N_{3G}(0)$, then $V \times \{0\} = \bigcap_{i=1}^{n}(H_i \times \{0\}) \in N(0)$ since by the above $H_i \in N_{3G}(0)$ implies $H_i \times \{0\} \in N(0)$. Therefore $F(N(0))$ contains the right side of the equation.

Let $H \in N(0)$. If $H \in N_1(0)$, then $H = N(0,g)$ for $g \in \mathcal{D}_1$. If $p_T(g) > 0$, then by Lemma 5.16 $p_T(g) \in \mathcal{D}_{1T}$ or there exists $f \in \mathcal{D}_{1G}$ such that $f^{\perp} = \{0\}$. If $p_T(g) \in \mathcal{D}_{1T}$, then by Lemma 5.14 $g^{\perp} = \{0\}$ and hence $H = [-g,g] \in \{[-g,g] \mid p_T(g) \in \mathcal{D}_{1T}\}$. If $f \in \mathcal{D}_{1G}$ is such that $f^{\perp} = \{0\}$, then $(f,0) < g$ since $p_T(g) > 0$, and hence $[-f,f] \times \{0\} = [-(f,0),(f,0)] \subseteq [-g,g]$. Since $g^{\perp} = \{0\}$ and $f^{\perp} = \{0\}$, this implies that $N(0,f) \times \{0\} \subseteq N(0,g)$. Thus $H \in F(\{V \times \{0\} \mid V \in N_G(0) \setminus \{G\}\})$. If $p_T(g) = 0$, then by Lemma 5.19 $p_G(g) \in \mathcal{D}_{1G}$ and by Lemma 5.18 $H = N(0,p_G(g)) \times \{0\} \in \{V \times \{0\} \mid V \in N_G(0) \setminus \{G\}\}$. Suppose that $H \in N_2(0)$. Then $H = D(h) + h^{\perp}$ for some $h \in \mathcal{D}_2$. If $p_T(h) > 0$, then by Lemma 5.17 $p_T(h) \in \mathcal{D}_{2T}$ and by Lemma 5.14 $h^{\perp} = \{0\}$ so that $D(h) + h^{\perp} = D(h) = G \times D(p_T(h)) \in \{G \times D(h) \mid h \in \mathcal{D}_{2T}\}$. If $p_T(h) = 0$, then by Lemma 5.20

$p_G(h) \in \mathcal{D}_2$ and $D(h) + h^{\perp} = [D(p_G(h)) + p_G(h)^{\perp}] \times \{0\} \in \{V \times \{0\} |$ $V \in N_G(0) \setminus \{G\}\}$. Suppose $H = \bigcap_{i=1}^{n} H_i$ for $H_i \in N_3(0)$. Since by the above each H_i is a member of the right side of the equation in the statement of the theorem and since the right side of the equation is a filter, H must be a member of the right side of the equation. If $H = G$, then $H \in \{G\}$.

Therefore

$$\begin{aligned} F(N(0)) = F(\{V \times \{0\} | \; V \in N_G(0) \setminus \{G\}\} \\ \cup \{[-g,g] | \; p_T(g) \in \mathcal{D}_{1T}\} \\ \cup \{G \times D(h) | \; h \in \mathcal{D}_{2T}\} \\ \cup \{G\}). \end{aligned}$$

|⊠|

Theorem 5.21 characterizes the $\mathfrak{T}$-topology on a lexico-graphic product. This characterization can be simplified depending on the structures of G and T.

Corollary 5.22: Let G be an ℓ-group, T a totally ordered group, and let $G = G \overleftarrow{X} T$. If (a) $\mathcal{D}_G^* \neq \emptyset$, or if (b) there exists $t \in T^+ \setminus \{0\}$ such that the closed interval $[0,t] = \{0,t\}$, or if (c) $T = \{0\}$, then

$$F(N(0)) = F(\{V \times \{0\} | \; V \in N_G(0)\}).$$

Proof: Clearly if (c) $T = \{0\}$, then $F(N(0)) = F(\{V \times \{0\} |$ $V \in N_G(0)\})$.

If $V \in N_G(0)$, then clearly $V \times \{0\} \subseteq [-g,g] \subseteq G$ when $p_T(g) > 0$. Suppose $h \in \mathcal{D}_{2T}$. Since $V \subseteq G$ and $\{0\} \subseteq D(h)$ then $V \times \{0\} \subseteq G \times D(h)$. Hence by Theorem 5.21 (with the convention that $F(\emptyset) = \emptyset$)

$$F(\{V \times \{0\} | \; V \in N_G(0)\}) \supseteq F(N(0)) \supseteq F(\{V \times \{0\} | \; V \in N_G(0) \setminus \{G\}\}).$$

If (a) $\mathcal{D}^*_G \neq \phi$, $\{V \times \{0\} | V \in N_G(0)\} = \{V \times \{0\} | V \in N_G(0) \setminus \{G\}\}$, and hence $F(N(0)) = F(\{V \times \{0\} | V \in N_G(0)\})$.

Suppose that (b) there exists $t \in T^+ \setminus \{0\}$ such that the closed interval $[0,t] = \{0,t\}$. If $\mathcal{D}^*_G \neq \phi$, then by (a) the corollary holds. Thus, suppose that $\mathcal{D}^*_G = \phi$. Then $N_G(0) = \{G\}$. Hence $\{V \times \{0\} | V \in N_G(0)\} = \{G \times \{0\}\}$. Clearly $t \in \mathcal{D}_{2T}$ and $D(t) = \{0\}$. Thus $G \times \{0\} \in \{G \times D(h) | h \in \mathcal{D}_{2T}\}$. Thus by Theorem 5.21

$$F(N(0)) \supseteq F(\{G \times D(h) | h \in \mathcal{D}_{2T}\}) \supseteq F(\{G \times \{0\}\}).$$

Then by the above

$$F(\{V \times \{0\} | V \in N_G(0)\}) \supseteq F(N(0)) \supseteq F(\{G \times \{0\}\})$$
$$= F(\{V \times \{0\} | V \in N_G(0)\}).$$

Therefore $F(N(0)) = F(\{V \times \{0\} | V \in N_G(0)\})$. |X|

In case the hypotheses of Corollary 5.22 are not satisfied we may apply the following corollary.

Corollary 5.23: Let G be an ℓ-group, $\{0\} \neq T$ a totally ordered group, and let $G = G \overleftarrow{\times} T$. If $\mathcal{D}^*_G = \phi$ and for all $t \in T^+ \setminus \{0\}$ the closed interval $[0,t] \neq \{0,t\}$, then

$$F(N(0)) = F(\{[-g,g] | \ p_T(g) \in T^+ \setminus \{0\}\}).$$

Proof: By the proof of Theorem 4.5, under the conditions on T in the corollary, $T^+ \setminus \{0\} = \mathcal{D}_{1T}$. Thus by Theorem 5.21

$$F(N(0)) \supseteq F(\{[-g,g] | \ p_T(g) \in \mathcal{D}_{1T}\}) = F(\{[-g,g] | \ p_T(g) \in T^+ \setminus \{0\}\}).$$

Since $\mathcal{D}^*_G = \phi$, $\{V \times \{0\} | V \in N_G(0) \setminus \{G\}\} = \phi$. If $h \in \mathcal{D}_{2T}$, there exists $\ell \in T$ such that $0 < \ell < h$. Then ℓ, $h - \ell \in T^+ \setminus \{0\}$. If $\ell \in D(h)$, then $[-\ell,\ell] \subseteq D(h)$. If $\ell \notin D(h)$, then $\ell \in h + D(h)$, i.e. $-h + \ell \in D(h)$. Hence $h - \ell = -(-h + \ell) \in D(h)$ and thus

$[-(h - \ell), h - \ell] \subseteq D(h)$. Since $T \neq \{0\}$, $G \in F(\{[-g,g] \mid g \in T^+ \setminus \{0\}\})$. Thus by Theorem 5.21

$$F(\{[-g,g] \mid g \in T^+ \setminus \{0\}\}) \supseteq F(N(0)),$$

and hence $F(\{[-g,g] \mid g \in T^+ \setminus \{0\}\}) = F(N(0))$. |X|

The generality of the definition of $\mathcal{D}_2$ comes into play in the case of lexico-graphic products. Instead of $\mathcal{D}_2$ we could have defined

$$\mathcal{D}_2^\P = \{h \mid h \in G^+ \setminus \{0\}, \text{ and } [0,h] = \{0,h\}\}$$
$$= \{h \mid h \in \mathcal{D}_2, \text{ and } D(h) = \{0\}\},$$

and then for $x \in G^+ \setminus \{0\}$

$$N_2^\P(0) = \{h^\perp \mid h \in \mathcal{D}_2^\P\},$$
$$N_3^\P(0) = N_1(0) \cup N_2^\P(0),$$
$$N^\P(0) = \{\bigcap_{i=1}^n H_i \mid H_i \in N_3^\P(0) \text{ for all } i = 1,\ldots,n\},$$
$$N^\P(x) = \{x + H \mid H \in N^\P(0)\},$$
$$\mathfrak{T}^\P = \{W \subseteq G \mid \text{for all } x \in W, \ W \in F(N^\P(x))\}.$$

The theorems in Chapters 2, 3, and 4 as well as the theorem on cardinal products would have remained true [51]. However, if G is an ℓ-group such that $\mathcal{D}^* = \phi$ (we will construct two in Chapter 7), then the topology $\mathfrak{T}^\P$ is indiscrete on $G \overleftarrow{X} Z$. More naturally, the filter of $\mathfrak{T}$-neighborhoods of 0 is, by Corollary 5.22,

$$F(\{G \times \{0\}\}).$$

In [51] the $\mathfrak{T}^\P$-topology is discussed in more detail. As an example of the close relationship between the $\mathfrak{T}$-topology and the $\mathfrak{T}^\P$-topology, we note the following proposition, which is proved in [51]:

<u>Proposition 5.24</u>: If $D(h) \cap (\mathcal{D}_1 \cup \mathcal{D}_2^\P) \neq \phi$ for all $h \in \mathcal{D}_2 \setminus \mathcal{D}_2^\P$, then $\mathfrak{T} = \mathfrak{T}^\P$. |X|

6. GROUPS WITH HAUSDORFF $\mathfrak{T}$-TOPOLOGY

In this chapter we investigate the Hausdorff separation axiom on ℓ-groups with $\mathfrak{T}$-topology. We first characterize those ℓ-groups with discrete $\mathfrak{T}$-topology; and then derive necessary and sufficient conditions for an ℓ-group to have a Hausdorff $\mathfrak{T}$-topology. In Chapter 7 we construct an ℓ-group with indiscrete $\mathfrak{T}$-topology.

Recall (Conrad [15] and Chapter 2) that an ℓ-group L is a lexico-extension of an ℓ-group S if and only if S is an ℓ-ideal of L, L/S is a totally ordered group, and every positive element in $L\setminus S$ exceeds every element in S. If $A_1,\ldots,A_n$ are totally ordered groups, then by a finite alternating sequence of cardinal summations and lexico-extensions, we can construct ℓ-groups from the A_i, in which each A_i is used exactly once to make a cardinal extension and in which the lexico-extensions are arbitrary. Such ℓ-groups are <u>lexico-sums</u> of the A_i. In forming a lexico-sum the first operation must be cardinal summation. Thus, if $n = 3$, there are two ways of constructing lexico-sums of A_1, A_2, and A_3 in this order, namely $\langle A_1 \,|X|\, \langle A_2 \,|X|\, A_3\rangle\rangle$ and $\langle\langle A_1 \,|X|\, A_2\rangle \,|X|\, A_3\rangle$, where $\langle L\rangle$ denotes a lexico-extension of the ℓ-group L.

The first theorem of this chapter characterizes those ℓ-groups with discrete $\mathfrak{T}$-topology as lexico-sums of lexico-extensions of the integers.

Let G be an arbitrary ℓ-group. As in the preceding chapter, let

$$\mathcal{D}_2^{\P} = \{h \mid h \in G^+\setminus\{0\}, \text{ and } [0,h] = \{0,h\}\}$$
$$= \{h \mid h \in \mathcal{D}_2, \text{ and } D(h) = \{0\}\}.$$

Lemma 6.1: Let $h \in \mathcal{D}_2\setminus\mathcal{D}_2^{\P}$. Let $k \in G$. If $k \wedge d = 0$ for all $d \in D(h)^+$, then $k \wedge h = 0$.

Proof: Since $D(h) \neq \{0\}$ and $k \wedge d = 0$ for all $d \in D(h)^+$, then $k \geqq 0$. Hence $0 \leqq k \wedge h \leqq h$. Thus $k \wedge h \in D(h) \cup (h + D(h))$. If $k \wedge h \in h + D(h)$, then by Lemma 2.2(a) $k \wedge h > d$ for all $d \in D(h)$. Thus $d = k \wedge h \wedge d = k \wedge d = 0$ for all $d \in D(h)^+$. Since $D(h)$ is a subgroup, this implies that $D(h) = \{0\}$, i.e $h \in \mathcal{D}_2^{\P}$. This contradicts our choice of h. Thus $k \wedge h \in D(h)$. Then $k \wedge h = k \wedge k \wedge h = 0$. |⊠|

Therefore, if $h \in \mathcal{D}_2\setminus\mathcal{D}_2^{\P}$ and $k \wedge h > 0$, then there exists $d \in D(h)^+$ such that $k \wedge d > 0$. We denote the meet of k with one such $d \in D(h)^+$ by $d(h,k)$. Thus $d(h,k) \in D(h)$ and $k \wedge d(h,k) = d(h,k) > 0$.

Lemma 6.2: $\{0\} \in N(0)$ if and only if there exist $\{h_1,\ldots,h_n\} \subseteq \mathcal{D}_2^{\P}$ such that $(\bigvee_{i=1}^{n} h_i)^{\perp\perp} = G$.

Proof: Suppose there exist $\{h_1,\ldots,h_n\} \subseteq \mathcal{D}_2^{\P}$ such that $(\bigvee_{i=1}^{n} h_i)^{\perp\perp} = G$. Then for all $i = 1,\ldots,n$, $h_i \in \mathcal{D}_2$ with $D(h_i) = \{0\}$ and hence $h_i^{\perp} \in N_2(0)$. Thus $\bigcap_{i=1}^{n}(h_i^{\perp}) \in N(0)$. But $\bigcap_{i=1}^{n}(h_i^{\perp}) = (\bigvee_{i=1}^{n} h_i)^{\perp}$ and since $(\bigvee_{i=1}^{n} h_i)^{\perp\perp} = G$, $(\bigvee_{i=1}^{n} h_i)^{\perp} = \{0\}$. Thus $\{0\} \in N(0)$.

Suppose that for all finite subsets $\{h_1,\ldots h_n\} \subseteq \mathcal{D}_2^{\P}$, $(\bigvee_{i=1}^{n} h_i)^{\perp\perp} \neq G$. Then for all $\{h_1,\ldots h_n\} \subseteq \mathcal{D}_2^{\P}$, $((\bigvee_{i=1}^{n} h_i)^{\perp})^{+}\setminus\{0\} \neq \emptyset$.

Let $\{H_1,\ldots,H_n\} \subseteq N_3(0)$ and consider $\bigcap_{i=1}^{n} H_i$. In the most general case, we may assume that $H_1 = h^{\perp}$ for $h = \bigvee_{i=1}^{p} h_i$ where $\{h_1,\ldots h_p\} \subseteq \mathcal{D}_2^{\P}$; for $i = 2,\ldots,m$, $m < n$, $H_i = D(g_i) + g_i^{\perp}$ for $g_i \in \mathcal{D}_2 \setminus \mathcal{D}_2^{\P}$; and for $i = m+1,\ldots,n$, $H_i = N(0,f_i)$ for $f_i \in \mathcal{D}_1$. If $\mathcal{D}_2^{\P} \cap \{h_1,\ldots,h_n\} = \phi$, $(\mathcal{D}_2 \setminus \mathcal{D}_2^{\P}) \cap \{h_1,\ldots,h_n\} = \phi$, or $\mathcal{D}_1 \cap \{h_1,\ldots,h_n\} = \phi$, then the following argument may be appropriately simplified. Let $\ell_1 \in (h^{\perp})^{+} \setminus \{0\}$, and define $\ell_2,\ldots,\ell_n$ as follows:

$$\ell_i = \begin{cases} d(g_i,\ell_{i-1}) & \text{if } \ell_{i-1} \wedge g_i > 0 \\ \ell_{i-1} & \text{if } \ell_{i-1} \wedge g_i = 0 \end{cases}$$

for $2 \leqq i \leqq m$ and $d(g_i,\ell_{i-1})$ as defined preceding the statement of the lemma, and

$$\ell_i = \begin{cases} \ell_{i-1} \wedge f_i & \text{if } \ell_{i-1} \wedge f_i > 0 \\ \ell_{i-1} & \text{if } \ell_{i-1} \wedge f_i = 0 \end{cases}$$

for $m+1 \leqq i \leqq n$. Clearly $0 < \ell_n \leqq \ell_{n-1} \leqq \cdots \leqq \ell_2 \leqq \ell_1$. Thus $\ell_n \in H_1$. Suppose $2 \leqq i \leqq m$. If $\ell_{i-1} \wedge g_i > 0$, then $\ell_i = d(g_i,\ell_{i-1}) \in D(g_i)$. If $\ell_{i-1} \wedge g_i = 0$, then $\ell_i = \ell_{i-1} \in g_i^{\perp}$. Thus $\ell_n \in H_i$. Suppose $m+1 \leqq i \leqq n$. If $\ell_{i-1} \wedge f_i > 0$, then $\ell_i = \ell_{i-1} \wedge f_i \in [-f_i,f_i]$. If $\ell_{i-1} \wedge f_i = 0$, then $\ell_i = \ell_{i-1} \in f_i^{\perp}$. Hence $\ell_n \in H_i$. Thus $\ell_n \in H_i$ for all $i = 1,\ldots,n$, i.e. $\ell_n \in \bigcap_{i=1}^{n} H_i$. Since $\ell_n > 0$, $\bigcap_{i=1}^{n} H_i \neq \{0\}$. Therefore

$\{0\} \notin N(0)$. |X|

<u>Corollary 6.3</u>: G has discrete $\mathfrak{T}$-topology if and only if there exist $\{h_1,\ldots h_n\} \subseteq \mathcal{D}_2^{\P}$ such that $(\bigvee_{i=1}^{n} h_i)^{\perp\perp} = G$. |X|

<u>Lemma 6.4</u>: If $h \in \mathcal{D}_2^{\P}$, then $h^{\perp\perp}$ is a lexico-extension of $[h] \simeq Z$.

Proof: Let $\ell \in h^{\perp\perp}$. Then $\ell + h - \ell \in h^{\perp\perp}$, and thus $(\ell + h - \ell) \wedge h > 0$. Since $h \in \mathcal{D}_2^{\P}$, this implies that $(\ell + h - \ell) \wedge h = h$. Hence $\ell + h - \ell \geqq h$, i.e. $h \geqq -\ell + h + \ell$. But $-\ell + h + \ell > 0$ and thus $h = -\ell + h + \ell$, i.e. $-\ell + h + \ell \in [h]$. Therefore $[h]$ is a normal subgroup of $h^{\perp\perp}$.

Since $h \in \mathcal{D}_2^{\P}$, clearly $[h] \simeq Z$. Let $k, \ell \in h^{\perp\perp}$. If $(\ell - k)^+ > 0$, then $(\ell - k)^+ \wedge h = h$ and if $(-(\ell - k))^+ > 0$, then $(-(\ell - k))^+ \wedge h = h$. Thus, since $(\ell - k)^+ \wedge (-(\ell - k))^+ = 0$, either $(\ell - k)^+ = 0$ or $(-(\ell - k))^+ = 0$, i.e. either $\ell \leqq k$ or $k \leqq \ell$. Hence $h^{\perp\perp}$ is totally ordered. Thus $G(h)$ is totally ordered, and hence if $k \in G(h)$, then $nh \leqq k < (n+1)h$ for some $n \in Z$. Thus $0 \leqq k - nh < h$, i.e. $k = nh$. Hence $G(h) = [h]$, and thus $[h]$ is convex.

Therefore $[h]$ is an ℓ-ideal of $h^{\perp\perp}$, and since $h^{\perp\perp}$ is totally ordered, every element $0 < a \in h^{\perp\perp} \setminus [h]$ exceeds every element of $[h]$. Hence by [15, Lemma 1.1], $h^{\perp\perp}$ is a lexico-extension of $[h]$. |X|

Lemma 6.5: If $\{h_1, \ldots, h_n\} \subseteq \mathcal{D}_2^{\P}$, then $(\bigvee_{i=1}^{n} h_i)^{\perp\perp}$ is a lexico-sum of $h_1^{\perp\perp}, h_2^{\perp\perp}, \ldots, h_n^{\perp\perp}$.

Proof: If $x \in ((\bigvee_{i=1}^{n} h_i)^{\perp\perp})^+ \setminus \{0\}$, then clearly $x \wedge h_i > 0$ for at least one h_i. Hence $x \geqq h_i$ for at least one h_i. Thus $(\bigvee_{i=1}^{n} h_i)^{\perp\perp}$ cannot contain more than n disjoint elements; for otherwise there would exist x, y, i such that $x \wedge y = 0$ but $x \geqq h_i$ and $y \geqq h_i$. Also $\{h_1, \ldots, h_n\} \subseteq (\bigvee_{i=1}^{n} h_i)^{\perp\perp}$ is disjoint since $\{h_1, \ldots, h_n\} \subseteq \mathcal{D}_2^{\P}$. For each $i = 1, \ldots, n$, let A_i be the subgroup of $(\bigvee_{i=1}^{n} h_i)^{\perp\perp}$ generated by $\{x \in (\bigvee_{i=1}^{n} h_i)^{\perp\perp} \mid x \wedge h_j = 0$ for all $j \neq i\}$. Then by [15, Theorem 1], $(\bigvee_{i=1}^{n} h_i)^{\perp\perp}$ is a lexico-sum of the totally ordered groups $A_1, \ldots, A_n$.

If $x \in (h_i^{\perp\perp})^+$, then $x \wedge h_j = 0$ for all $j \neq i$. Hence $x \in A_i$ and thus $h_i^{\perp\perp} \subseteq A_i$.

If $x \wedge h_j = 0 = y \wedge h_j$ for all $j \neq i$, then $(x + y) \wedge h_j = 0$ for all $j \neq i$. Thus $\{x \in (\bigvee_{i=1}^{n} h_i)^{\perp\perp} \mid x \wedge h_j = 0$ for all $j \neq i\}$ is a convex subsemigroup of positive elements of $(\bigvee_{i=1}^{n} h_i)^{\perp\perp}$ that contains 0 and hence by [15, Lemma 2.3], $A_i^+ = \{x \in (\bigvee_{i=1}^{n} h_i)^{\perp\perp} \mid x \wedge h_j = 0$ for all $j \neq i\}$. If $y \in (h_k^{\perp} \cap (\bigvee_{i=1}^{n} h_i)^{\perp\perp}) \setminus \{0\}$, then clearly $|y| \wedge h_j > 0$ for some $j \neq k$. Hence $A_i \cap h_i^{\perp} = \{0\}$. Let $x \in A_i^+$. If $x \notin (h_i^{\perp\perp})^+$, then there exists $k \in h_i^{\perp}$ such that $k \wedge x > 0$. Since A_i^+ is convex, $k \wedge x \in A_i$. But $k \wedge x \wedge h_i = 0$ and hence $k \wedge x \in h_i^{\perp}$. Thus $k \wedge x = 0$, which contradicts our choice of k. Thus $h_i^{\perp\perp} \supseteq A_i$.

Therefore $A_i = h_i^{\perp\perp}$ for all i. $|\boxtimes|$

Theorem 6.6: An ℓ-group G has discrete $\mathfrak{X}$-topology if and only if G is ℓ-isomorphic to a lexico-sum of lexico-extensions of the integers.

Proof: Suppose that G is ℓ-isomorphic to a lexico-sum of lexico-extensions of Z. Then for $i = 1, \ldots, n$, there exists $A_i \subseteq G$ such that A_i is ℓ-isomorphic to a lexico-extension of Z and G is the lexico-sum of $A_1, \ldots, A_n$. Let $h_i \in A_i$ correspond to $1 \in Z$ under the ℓ-isomorphism. Then $h_i \in \mathcal{D}_2^{\P}$ for all $i = 1, \ldots, n$ and clearly $(\bigvee_{i=1}^{n} h_i)^{\perp} = \{0\}$. Hence $(\bigvee_{i=1}^{n} h_i)^{\perp\perp} = G$ so that by Corollary 6.3 G has discrete $\mathfrak{X}$-topology.

Suppose that G has discrete $\mathfrak{T}$-topology. Then by Corollary 6.3, $G = (\bigvee_{i=1}^{n} h_i)^{\perp\perp}$ for $\{h_1,\ldots,h_n\} \subseteq \mathcal{D}_2^{\P}$. By Lemma 6.4, each $h_i^{\perp\perp}$ is ℓ-isomorphic to a lexico-extension of Z. By Lemma 6.5, G is a lexico-sum of $h_1^{\perp\perp},\ldots,h_n^{\perp\perp}$. |X|

We now turn our attention to criteria for the $\mathfrak{T}$-topology to be Hausdorff.

<u>Theorem 6.7</u>: If G is an ℓ-group with $\mathfrak{T}$-topology $\mathfrak{T}$, then the following statements are equivalent:

(a) $\mathfrak{T}$ is Hausdorff.

(b) $\cap N(0) = \{0\}$.

(c) For all $g \in G^+\setminus\{0\}$, there exists $H \in N_3(0)$ such that $g \notin H$.

<u>Proof</u>: By Theorem B, (a) is equivalent to (b). Suppose (b) holds, and let $g \in G^+\setminus\{0\}$. Then $g \notin \cap N(0)$, and hence there exists $H \in N(0)$ such that $g \notin H$. Since $H \in N(0)$, $H = \bigcap_{i=1}^{n} H_i$ for $H_i \in N_3(0)$. Since $g \notin H$, $g \notin H_i$ for at least one i. Thus (c) holds. Conversely, suppose that (c) holds and let $g \in \cap N(0)$. If $g \neq 0$, $|g| > 0$. Hence by (c), $|g| \notin \cap N_3(0) \supseteq \cap N(0)$. But by Lemmas 2.12 and 2.23, every $H \in N(0)$ is a symmetric sublattice of G. Hence since $g \in \cap N(0)$, $|g| = g \vee (-g) \in \cap N(0)$. This is a contradiction. Therefore $g = 0$ and (b) holds. |X|

<u>Corollary 6.8</u>: If $\mathcal{D}^* \cap [0,g] \neq \emptyset$ for all $g \in G^+\setminus\{0\}$, then G has Hausdorff $\mathfrak{T}$-topology.

Proof: Let $g \in G^+ \setminus \{0\}$. Let $h' \in \mathcal{D}^* \cap [0,g]$. If $h' \in \mathcal{D}_1$, let $h \in \mathcal{D}_1$ be such that $h + h \leqq h'$ and $h' \in h^{\perp\perp}$, and let $H = N(0,h) \in \mathcal{N}_3(0)$. If $h' \in \mathcal{D}_2 \setminus \mathcal{D}_1$, let $h = h'$ and $H = D(h) + h^{\perp}$ $\in \mathcal{N}_3(0)$. Then by our choice of h, $g \notin H$. Thus by Theorem 6.7, G has Hausdorff $\mathfrak{T}$-topology. |☒|

Example 7.10 in Chapter 7 shows that the converse of Corollary 6.8 fails to hold in general. In the remainder of this chapter, we derive a condition which is similar in character to that in Corollary 6.8 and which is necessary and sufficient for the $\mathfrak{T}$-topology to be Hausdorff. We have to find a larger set to replace $[0,g]$ in the hypothesis of Corollary 6.8.

Let $g \in G^+ \setminus \{0\}$. For $h \in \mathcal{T}(g)$, we adopt the notation h' for the element $h' \in G$ such that $h \wedge h' = 0$ and $h \vee h' = g$, and we let

$$M(g,h) = G^+ \setminus [(h,\infty) \cap (h'^{\perp})].$$

Clearly $\{0,g\} \subseteq \mathcal{T}(g)$ and

$$M(g,0) = (G^+ \setminus (g^{\perp})) \cup \{0\} = G^+ \setminus [(g^{\perp}) \setminus \{0\}],$$

$$M(g,g) = G^+ \setminus (g,\infty).$$

We note that $G^+ \setminus M(g,h) = (h,\infty) \cap (h'^{\perp})$ and hence

$$G^+ \setminus \Big(\bigcap_{h \in \mathcal{T}(g)} M(g,h) \Big) = \bigcup_{h \in \mathcal{T}(g)} [(h,\infty) \cap (h'^{\perp})].$$

Thus $\bigcap_{h \in \mathcal{T}(g)} M(g,h)$ is intuitively those elements of G^+ which are not "directly above" some "factor" of g. See figures XVIII through XXII as elaborated below.

In all the figures XVIII through XXII, the shaded areas include all points on the boundaries except the "base" points, i.e. the point "h" is not included in the set $G^+ \setminus M(g,h)$. Compare figures XVIII and XIX with figures VI and VII respectively. In figures XX, XXI, and XXII, the shaded areas are to be interpreted as follows: to be an element of the set depicted by a shaded area, a function must have its graph lying within the shaded area; the boundary of the shaded area is to be interpreted as above. Thus a function is in the complement of a set depicted by a shaded area if some point of its graph lies outside the shaded area or if it is a "base" point of the set. In figure XXII, the dashed lines are meant as reference lines for the other figures; they are not involved in interpreting the shading of the figure. For instance, the function $k \varepsilon A(R)$ defined by $xk = xh_1 + 1$ is in the shaded area of figure XXII; the function $k \varepsilon A(R)$ defined by $xk = xh_1 + (1/2)$ is not in the shaded area. The "base" points i, g, h_1, h_2 are not in the shaded area; however, functions such as ℓ defined by

$$x\ell = \begin{cases} x & \text{if } x \varepsilon (-\infty,3) \\ 2x - 3 & \text{if } x \varepsilon [3,4) \\ x + 1 & \text{if } x \varepsilon [4,\infty) \end{cases}$$

are.

The set which successfully replaces $[0,g]$ in the hypothesis of Corollary 6.8 is $\bigcap_{h \varepsilon T(g)} M(g,h)$. The theorem corresponding to Corollary 6.8 is Theorem 6.15; we prove six preliminary lemmas.

figure XVIII

$$G = \left|\prod_{1}^{3}\right| R$$

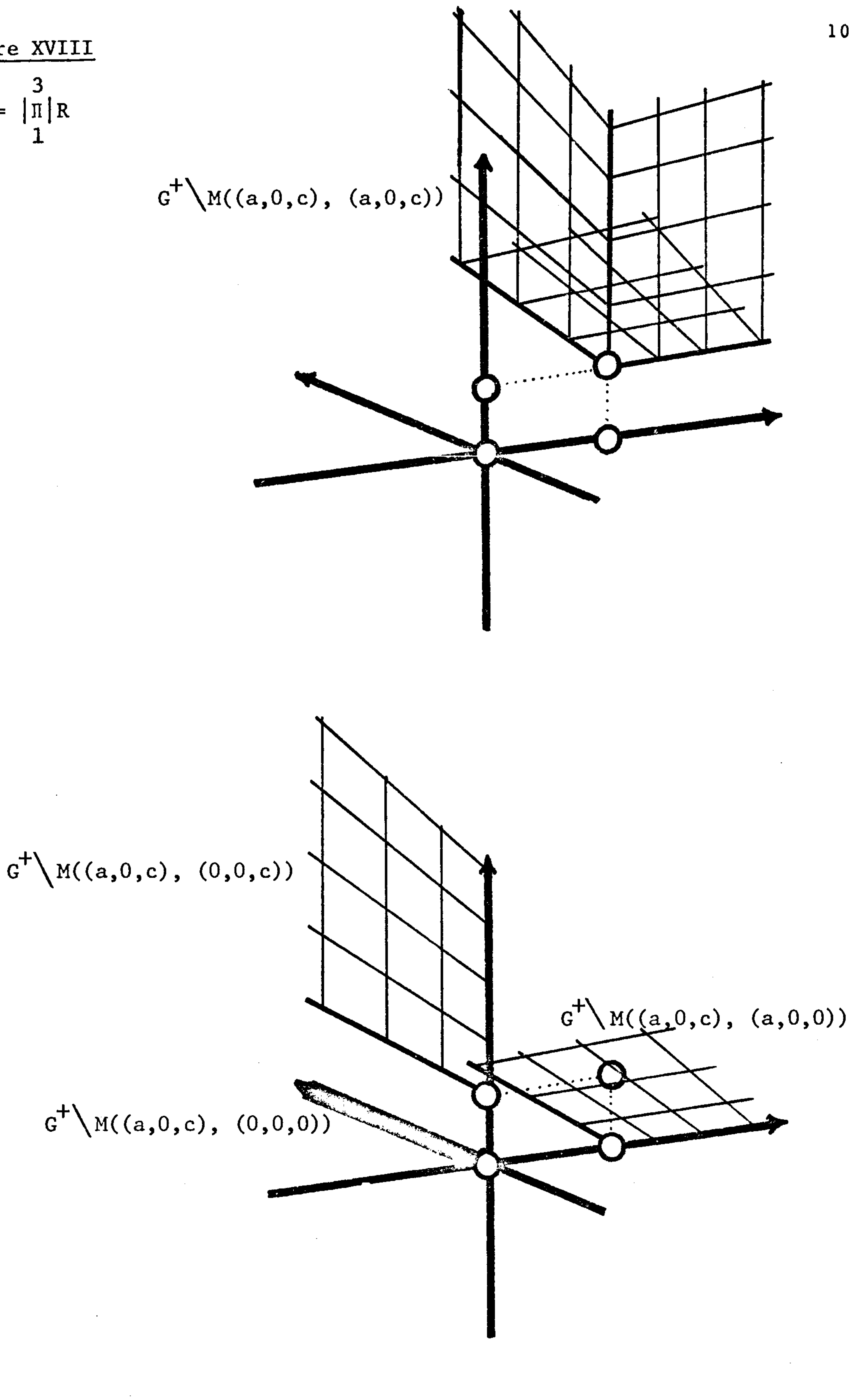
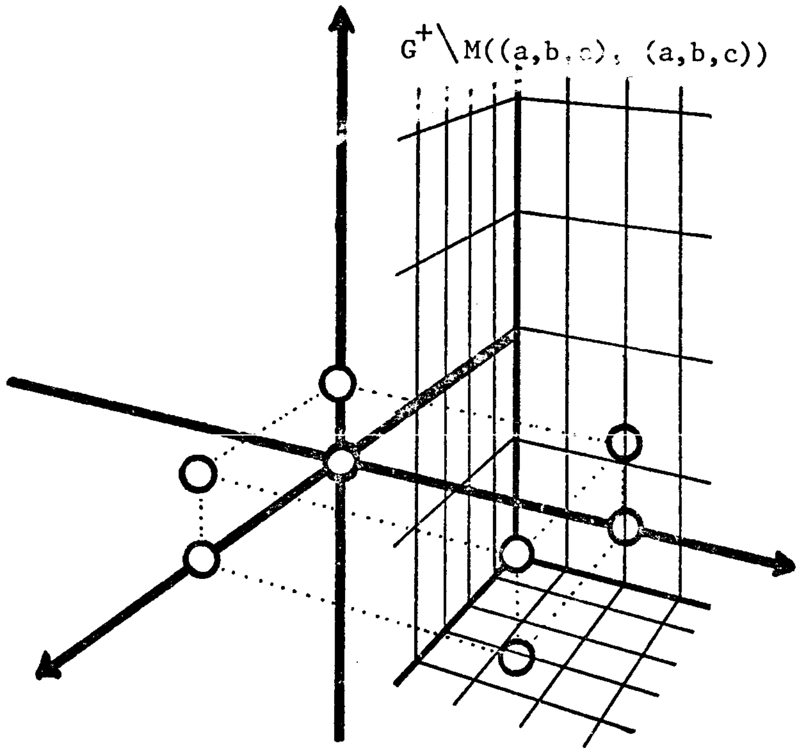


figure XIX

$$G = \left|\prod_{1}^{3}\right| R$$



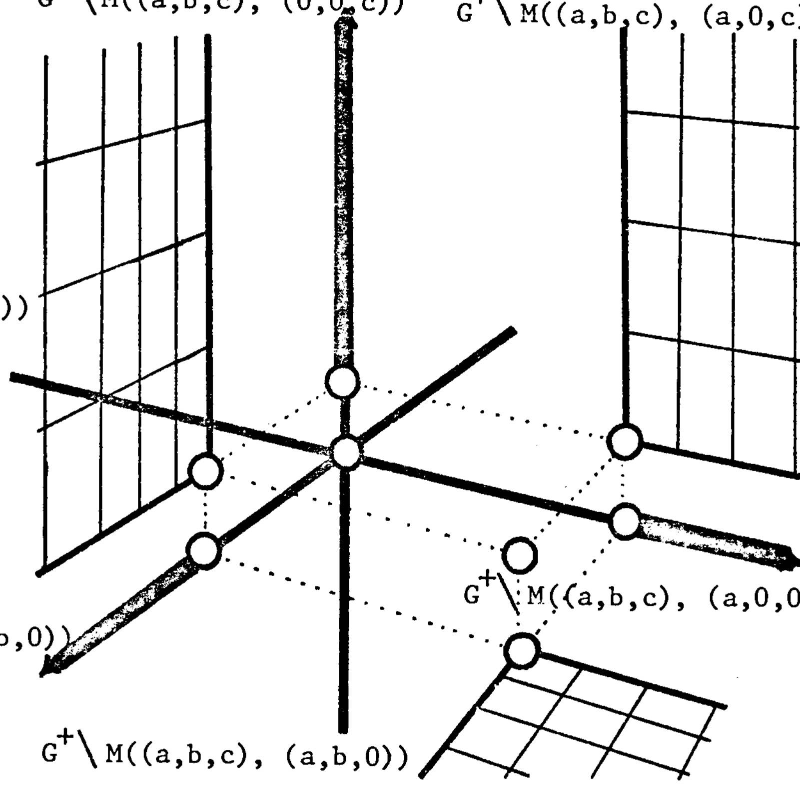
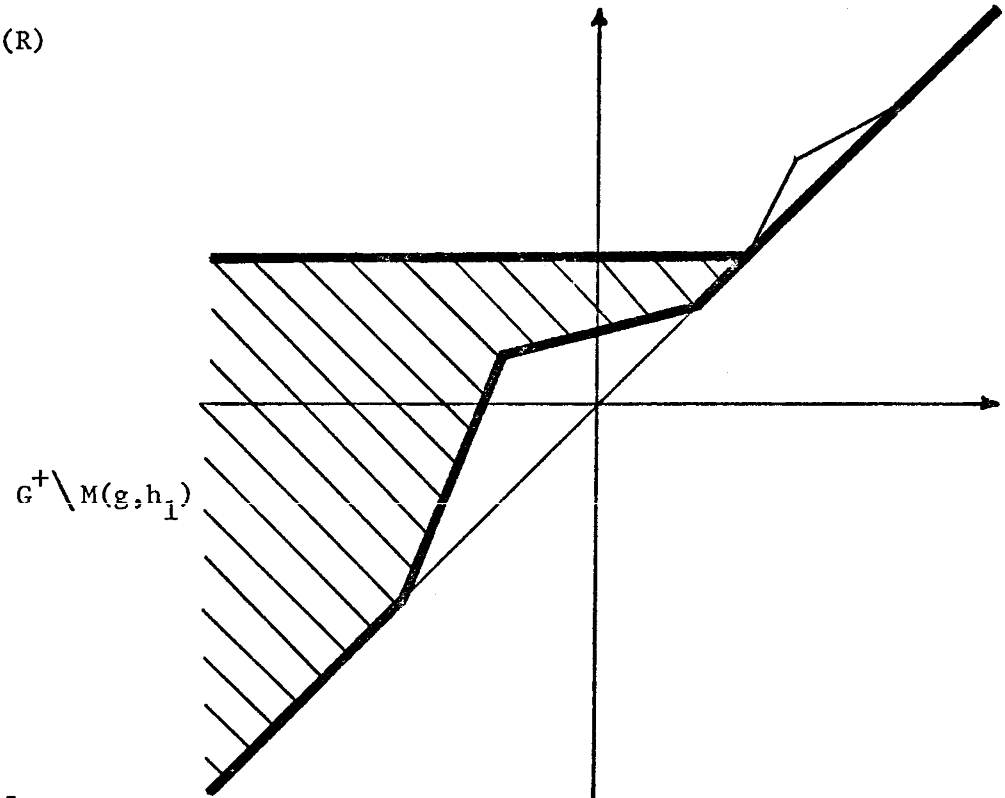


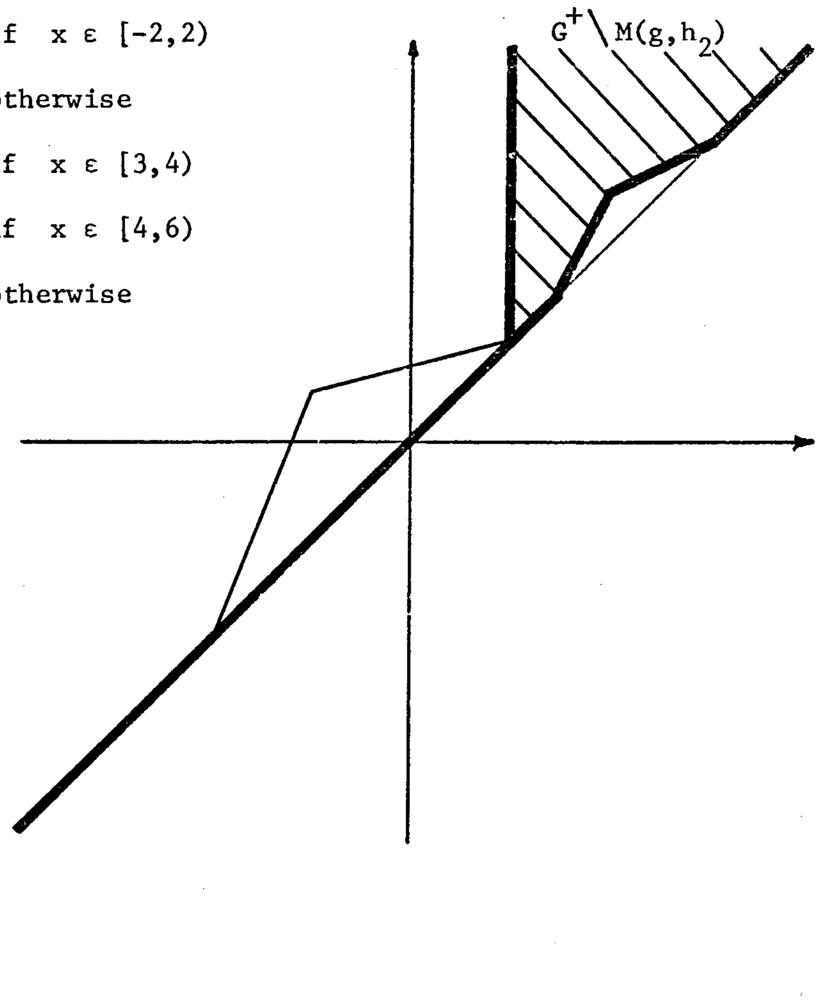
figure XX

$G = A(R)$



$$xh_1 = \begin{cases} \frac{5}{2}x + 6 & \text{if } x \in [-4,-2) \\ \frac{1}{4}x + \frac{3}{2} & \text{if } x \in [-2,2) \\ x & \text{otherwise} \end{cases}$$

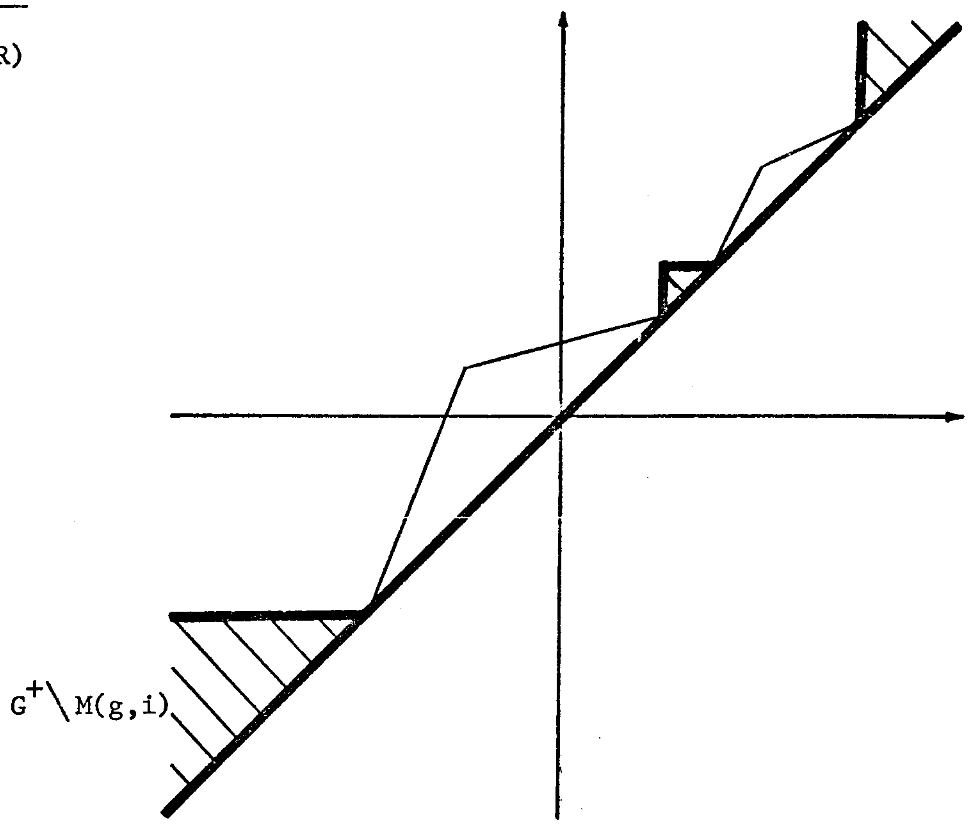
$$xh_2 = \begin{cases} 2x - 3 & \text{if } x \in [3,4) \\ \frac{1}{2}x + 3 & \text{if } x \in [4,6) \\ x & \text{otherwise} \end{cases}$$

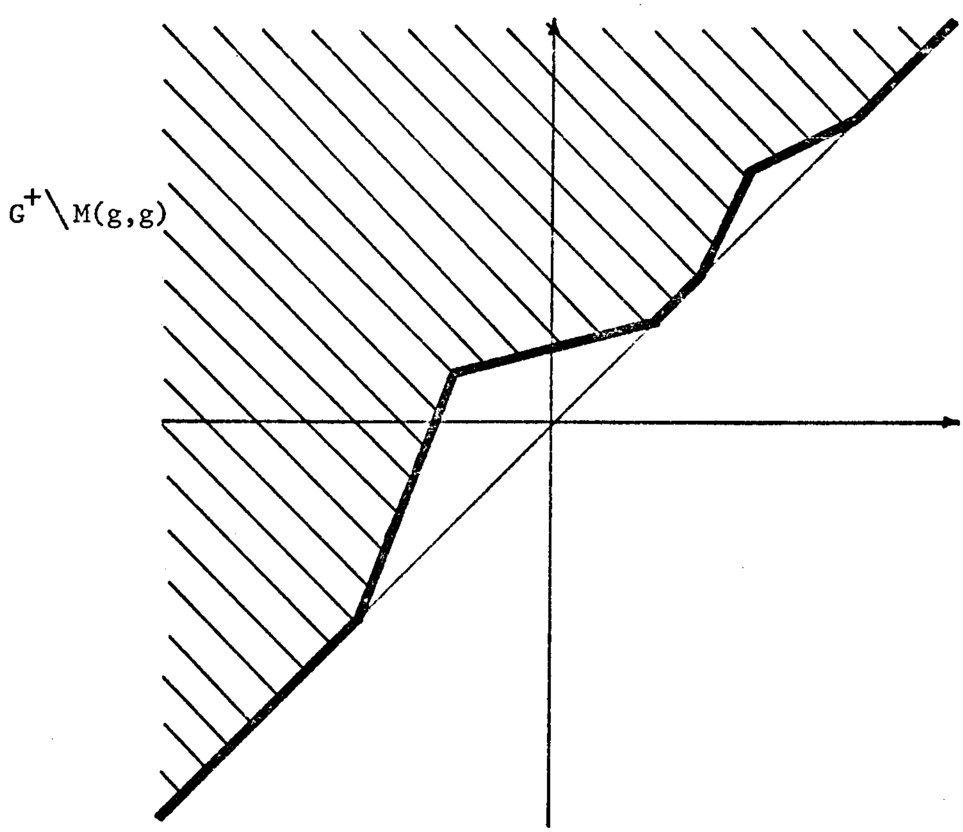


$g = h_1 \vee h_2$

figure XXI

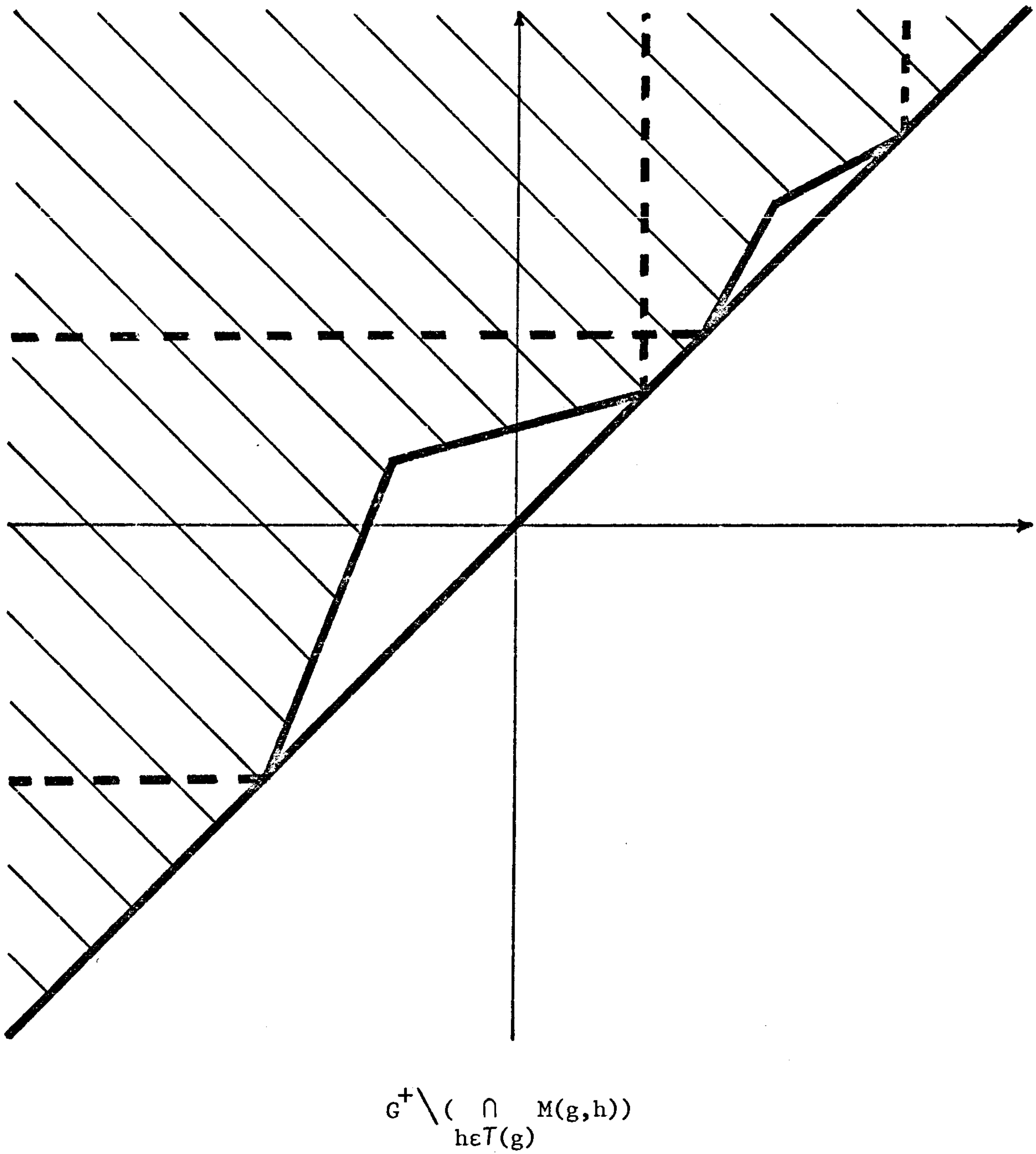
G = A(R)





<u>figure XXII</u>

G = A(R)



$G^+ \setminus (\bigcap_{h \in T(g)} M(g,h))$

<u>Lemma 6.9</u>: Let $g \in G^+ \setminus \{0\}$. If $\mathcal{D}^* \cap (\bigcap_{h \in \mathcal{T}(g)} M(g,h)) \neq \emptyset$, then there exists $H \in \mathcal{N}_3(0)$ such that $g \notin H$.

<u>Proof</u>: Let $\ell \in \mathcal{D}^* \cap (\bigcap_{h \in \mathcal{T}(g)} M(g,h))$. If $\ell \in \mathcal{D}_1$, let $h \in \mathcal{D}_1$ be such that $h + h \leqq \ell$ and $h^\perp = \ell^\perp$. Let $H = N(0,h) \in \mathcal{N}_1(0)$. If $\ell \in \mathcal{D}_2$, let $h = \ell$ and let $H = D(\ell) + \ell^\perp \in \mathcal{N}_2(0)$. Suppose $g \in H$. By Lemma 2.2(b), and since $h^\perp = \ell^\perp$, $H \subseteq [-\ell,\ell] + \ell^\perp$. By Lemma 2.13(a), since $g > 0$, $g = a + b$ for $a \in [0,\ell] \cap H$ and $b \in (\ell^\perp)^+ \cap H$. Since $a \wedge b = 0$, $a \in \mathcal{T}(g)$ and $a' = b$. Since by construction of H $\ell \notin H$, $a < \ell$, i.e. $\ell \in (a,\infty)$. Since $b \in \ell^\perp$, $\ell \in b^\perp$. Thus $\ell \in (a,\infty) \cap (a'^\perp)$, i.e. $\ell \notin M(g,a)$. This implies that $\ell \notin \bigcap_{h \in \mathcal{T}(g)} M(g,h)$, which contradicts our choice of ℓ. |☒|

<u>Lemma 6.10</u>: Let $\ell \in \mathcal{D}_2$. Then for all $k \in \ell + D(\ell)$, $k \in \mathcal{D}_2$ and $D(k) = D(\ell)$.

<u>Proof</u>: By Lemma 2.2(a), $k > d$ for all $d \in D(\ell)$. Suppose that $a < t < b$ for $a, -k + b \in D(\ell)$. Since $k \in \ell + D(\ell)$, $k = \ell + d$ for $d \in D(\ell)$. Then $-k + b = -d - \ell + b$ and thus, since $-k + b \in D(\ell)$, $-\ell + b \in d + D(\ell)$. Thus $t \in D(\ell) \cup (\ell + D(\ell))$. But $k + D(\ell) = \ell + d + D(\ell) = \ell + D(\ell)$. Thus $t \in D(\ell) \cup (k + D(\ell))$. Thus $k \in \mathcal{D}_2$ and $D(k) = D(\ell)$. |☒|

<u>Lemma 6.11</u>: Let $\ell, h \in \mathcal{D}_2$ and suppose that $h < \ell$. Then $D(h) \subseteq D(\ell)$.

<u>Proof</u>: Since $0 < h < \ell$, $h \in D(\ell) \cup (\ell + D(\ell))$. If $h \in \ell + D(\ell)$, by Lemma 6.10 $D(h) = D(\ell)$. Thus suppose $h \in D(\ell)$. By Lemma 2.2(b), $D(h) \subseteq [-h,h]$. Since $D(\ell)$ is convex, $D(h) \subseteq D(\ell)$. |☒|

Lemma 6.12: Let G have Hausdorff $\mathfrak{T}$-topology. If $\ell \in \mathcal{D}_2 \setminus \mathcal{D}_2^{\P}$, then $\mathcal{D}^* \cap ([G \setminus (G(\ell) \cup \ell^{\perp})] \cup D(\ell)) \neq \emptyset$.

Proof: Let $g \in D(\ell)^+ \setminus \{0\}$. Since G has Hausdorff $\mathfrak{T}$-topology, by Theorem 6.7 there exists $L \in N_3(0)$ such that $g \notin L$. If $L \in N_1(0)$, then $L = N(0,h)$ for $h \in \mathcal{D}_1$. If $h \in \ell^{\perp}$, then $\ell \in h^{\perp} \subseteq N(0,h) = L$. Since $g \in D(\ell)$, $0 < g < \ell$ and thus $g \in L$, which contradicts our choice of L. Suppose that $h \in G(\ell)$. If $h \wedge \ell \notin D(\ell)$, then $h \wedge \ell \in \ell + D(\ell)$ and by Lemma 2.2(a) $h \wedge \ell > g$. Hence $g \in [0, h \wedge \ell] \subseteq N(0,h) = L$, which contradicts our choice of L. Thus $h \wedge \ell \in D(\ell)$ and by Lemma 2.9, $h \in D(\ell)$.

Suppose that $L \in N_2(0)$. Then $L = D(h) + h^{\perp}$ for $h \in \mathcal{D}_2$. If $h \in \ell^{\perp}$, then $\ell \in h^{\perp} \subseteq L$ and as above this implies that $g \in L$, which contradicts our choice of L. Thus suppose that $h \in G(\ell)$. If $h \wedge \ell \notin D(\ell)$, then $h \wedge \ell \in \ell + D(\ell)$ and by Lemma 6.10, $h \wedge \ell \in \mathcal{D}_2$ and $D(h \wedge \ell) = D(\ell)$. Since $h \wedge \ell$ and $h \in \mathcal{D}_2$, by Lemma 6.11 $D(h \wedge \ell) \subseteq D(h)$. Thus $g \in D(\ell) \subseteq D(h) \subseteq L$, which contradicts our choice of L. Hence $h \wedge \ell \in D(\ell)$ and by Lemma 2.9 $h \in D(\ell)$. |▮|

Lemma 6.13: Let $g \in G^+ \setminus \{0\}$. If $h \in \mathfrak{U} \cap \mathcal{T}(g)$, then $(0,h) \cap \mathcal{T}(g) = \emptyset$.

Proof: Suppose that $k \in (0,h) \cap \mathcal{T}(g)$. Then there exists $k' \in \mathcal{T}(g)$ such that $k \wedge k' = 0$ and $k \vee k' = g$. Thus $(h \wedge k') \wedge k = 0$ and $(h \wedge k') \vee k = (h \vee k) \wedge (k' \vee k) = h \wedge g = h$. Hence $k \in \mathcal{T}(h)$, which contradicts the fact that $h \in \mathfrak{U}$. |▮|

<u>Lemma 6.14</u>: Let $g \in G^+ \setminus \{0\}$. If $a, b \in \mathfrak{A} \cap \mathcal{T}(g)$ and $a \neq b$, then $a \wedge b = 0$.

<u>Proof</u>: Let $a', b' \in G$ be such that $a \wedge a' = 0 = b \wedge b'$ and $a \vee a' = g = b \vee b'$. Then

$$(a \wedge b) \wedge (a' \vee b') = (a \wedge b \wedge a') \vee (a \wedge b \wedge b') = 0,$$

and

$$(a \wedge b) \vee (a' \vee b') = (a \vee a' \vee b') \wedge (b \vee a' \vee b') = g.$$

Thus $a \wedge b \in \mathcal{T}(g)$. Since $a \in \mathfrak{A} \cap \mathcal{T}(g)$, by Lemma 6.13 $a \wedge b = 0$ or $a \wedge b = a$. If $a \wedge b = a$, then $a \wedge b \neq 0$ and applying Lemma 6.13 to $b \in \mathfrak{A} \cap \mathcal{T}(g)$, we have that $a = a \wedge b = b$, which contradicts our choice of a and b. |X|

<u>Theorem 6.15</u>: An ℓ-group G has Hausdorff $\mathfrak{T}$-topology if and only if for all $g \in G^+ \setminus \{0\}$, $\mathcal{D}^* \cap \left(\bigcap_{h \in \mathcal{T}(g)} M(g,h) \right) \neq \emptyset$.

<u>Proof</u>: Suppose that for all $g \in G^+ \setminus \{0\}$, $\mathcal{D}^* \cap \left(\bigcap_{h \in \mathcal{T}(g)} M(g,h) \right) \neq \emptyset$. By Lemma 6.9 and Theorem 6.7 G has Hausdorff $\mathfrak{T}$-topology.

Conversely, suppose that G has Hausdorff $\mathfrak{T}$-topology. Let $g \in G^+ \setminus \{0\}$. Then by Theorem 6.7 there exists $L \in \mathcal{N}_3(0)$ such that $g \notin L$. If $L \in \mathcal{N}_1(0)$, then $L = N(0,\ell)$ for $\ell \in \mathcal{D}_1$. If $\ell \notin \bigcap_{h \in \mathcal{T}(g)} M(g,h)$, then $\ell \notin M(g,h)$ for some $h \in \mathcal{T}(g)$. Thus $\ell \in (h,\infty) \cap (h'^{\perp})$. Since $0 < h < \ell$, $h \in [-\ell,\ell]$. Since $\ell \in h'^{\perp}$, $h' \in \ell^{\perp}$. Thus

$$g = h + h' \in [-\ell,\ell] + \ell^{\perp} = L,$$

which contradicts our choice of L. Thus $\ell \in \bigcap_{h \in \mathcal{T}(g)} M(g,h)$.

If $L \in \mathcal{N}_2(0)$, then $L = D(\ell) + \ell^{\perp}$ for $\ell \in \mathcal{D}_2$. Suppose that $\ell \in \mathcal{D}_2^{\P}$ and suppose further that $\ell \notin \bigcap_{h \in T(g)} M(g,h)$. Then $\ell \notin M(g,h)$ for some $h \in T(g)$. Thus $\ell \in (h,\infty) \cap (h'^{\perp})$. Since $\ell \in \mathcal{D}_2^{\P}$ and since $h < \ell$, then $h = 0$ and thus $h' = g$. Since $\ell \in h'^{\perp}$,

$$g = h' \in \ell^{\perp} = \{0\} + \ell^{\perp} = L,$$

which contradicts our choice of L. Hence $\ell \in \bigcap_{h \in T(g)} M(g,h)$.

Suppose that $\ell \in \mathcal{D}_2 \setminus \mathcal{D}_2^{\P}$. If $\ell \in \bigcap_{h \in T(g)} M(g,h)$, we are done. Thus suppose that $\ell \notin \bigcap_{h \in T(g)} M(g,h)$. Then $\ell \notin M(g,p)$ for some $p \in T(g)$, and hence $\ell \in (p,\infty) \cap (p'^{\perp})$. Since $0 \leqq p < \ell$, $p \in D(\ell) \cup (\ell + D(\ell))$. Since $\ell \in p'^{\perp}$, $p' \in \ell^{\perp}$. Thus if $p \in D(\ell)$,

$$g = p + p' \in D(\ell) + \ell^{\perp} = L,$$

which contradicts our choice of L. Thus $p \in \ell + D(\ell)$. By Lemma 6.10, $p \in \mathcal{D}_2$ and $D(p) = D(\ell)$. Hence by Proposition 2.3, $p \in \mathfrak{A}$. By Lemma 6.12, there exists $k \in \mathcal{D}^* \cap ([G \setminus (G(\ell) \cup \ell^{\perp})] \cup D(\ell))$. Suppose that $k \in D(\ell)$. Then $k \in D(p)$ and $k \in (0,p) \subseteq [0,g]$. If $k \notin M(g,h)$ for some $h \in T(g)$, then $k \in (h,\infty) \cap (h'^{\perp})$ and thus

$$g = h + h' < k + h' = k \vee h' \leqq g \vee g = g,$$

which is a contradiction. Thus $k \in \bigcap_{h \in T(g)} M(g,h)$. Suppose that $k \in G \setminus (G(\ell) \cup \ell^{\perp})$. If $k \notin \bigcap_{h \in T(g)} M(g,h)$, then $k \notin M(g,h)$ for some $h \in T(g)$. By the arguments above we must have $k \in \mathcal{D}_2 \setminus \mathcal{D}_2^{\P}$ and $h \in k + D(k)$. By Lemma 6.10 and Proposition 2.3, $h \in \mathfrak{A}$. Thus by Lemma 6.14, since $h \neq p$, $h \wedge p = 0$. By Lemma 2.8, $h^{\perp} = k^{\perp}$ and $p^{\perp} = \ell^{\perp}$. Thus $k \wedge \ell = 0$, i.e. $k \in \ell^{\perp}$, which contradicts our choice of k. Thus $k \in \bigcap_{h \in T(g)} M(g,h)$. |⌛|

Let $T_2(G)$ be the convex ℓ-subgroup of G generated by

$$\{g \in G^+\setminus\{0\} \mid \mathcal{D}^* \cap (\bigcap_{h\in T(g)} M(g,h)) = \emptyset\} \cup \{0\}.$$

Then clearly Theorem 6.15 implies the following:

Corollary 6.16: An ℓ-group G has Hausdorff $\mathfrak{T}$-topology if and only if $T_2(G) = \{0\}$. |☒|

The next three propositions give some basic information about $T_2(G)$.

Proposition 6.17: $T_2(G) \supseteq \bigcap N(0)$.

Proof: By Lemmas 2.12 and 2.23, each $H \in N(0)$ is a symmetric sublattice of G. Hence $\bigcap N(0)$ is a symmetric sublattice of G and thus $g \in \bigcap N(0)$ if and only if $|g| \in \bigcap N(0)$. Since $0 \in T_2(G)$, it therefore suffices to show that $\bigcap N(0)^+\setminus\{0\} \subseteq T_2(G)$. Let $g \in \bigcap N(0)^+\setminus\{0\}$. If $\ell \in \mathcal{D}_1$, then there exists $\ell^* \in \mathcal{D}_1$ such that $\ell^* + \ell^* \leqq \ell$ and $\ell^{*\perp} = \ell^\perp$. Since $g \in \bigcap N(0)$, $g \in N(0,\ell^*)$ and hence by Lemma 2.13(a) $g = a + b$ for $a \in [0,\ell^*]$ and $b \in \ell^{*\perp}$. Since $\ell^* < \ell$, $\ell \in (a,\infty)$. Since $\ell^{*\perp} = \ell^\perp$, $b \in \ell^\perp$, and hence $\ell \in b^\perp$. Since $a \wedge b = 0$, $a \in T(g)$ and $b = a'$. Thus $\ell \in (a,\infty) \cap (a'^\perp)$, i.e. $\ell \notin M(g,a)$. Suppose that $k \in \mathcal{D}_2$. Since $g \in \bigcap N(0)$, $g \in D(k) + k^\perp$. Then by Lemmas 2.13(a) and 2.2(b), $g = a + b$ for $a \in D(k)^+$ and $b \in (k^\perp)^+$. Since $a \in D(k)$, $k \in (a,\infty)$ and since $b \in k^\perp$, $k \in b^\perp$. Since $a \wedge b = 0$, $a \in T(g)$ and $b = a'$. Thus $k \in (a,\infty) \cap (a'^\perp)$, i.e. $k \notin M(g,a)$. Therefore

$$\mathcal{D}^* \cap (\bigcap_{h\in T(g)} M(g,h)) = \emptyset,$$

and hence $g \in \{g \in G^+\setminus\{0\} \mid \mathcal{D}^* \cap (\bigcap_{h\in T(g)} M(g,h)) = \emptyset\} \subseteq T_2(G)$. |☒|

We note that Proposition 6.17 could be used in place of Lemma 6.9 in the proof of Theorem 6.15. That is, if for all $g \in G^+ \setminus \{0\}$, $\mathcal{D}^* \cap (\bigcap_{h \in \mathcal{T}(g)} M(g,h)) \neq \emptyset$, then $T_2(G) = \{0\}$. By Proposition 6.17, $\bigcap \mathcal{N}(0) \subseteq T_2(G)$ and hence $\bigcap \mathcal{N}(0) = \{0\}$. Thus by Theorem 6.7, G has Hausdorff $\mathfrak{T}$-topology.

Let N be a convex ℓ-subgroup of G. Conrad [17] defines N to be <u>regular</u> if N is a value of some $g \in G$, i.e. if there exists $g \in G$ such that N is maximal with respect to not containing g (cf. Chapter 2). Let

$$T_2^*(G) = \{N \mid N \text{ is a regular convex } \ell\text{-subgroup of } G \text{ and for all } g \in G^+ \setminus N,\ \mathcal{D}^* \cap (\bigcap_{h \in \mathcal{T}(g)} M(g,h)) \neq \emptyset\}.$$

Similarly to [17, Proposition 3.5], we have the following:

<u>Proposition 6.18</u>: $T_2(G) = \bigcap_{N \in T_2^*(G)} N.$

<u>Proof</u>: Clearly $\bigcap_{N \in T_2^*(G)} N$ is a convex ℓ-subgroup of G which contains $\{g \in G^+ \setminus \{0\} \mid \mathcal{D}^* \cap (\bigcap_{h \in \mathcal{T}(g)} M(g,h)) = \emptyset\}$. Hence $T_2(G) \subseteq \bigcap_{N \in T_2^*(G)} N$. On the other hand, suppose that $h \notin T_2(G)$. Then by Zorn's Lemma, there exists a convex ℓ-subgroup N which is maximal with respect to containing $T_2(G)$ but not containing h. Then clearly N is regular and since $T_2(G) \subseteq N$, $N \in T_2^*(G)$. Thus $T_2(G) \supseteq \bigcap_{N \in T_2^*(G)} N$. |X|

<u>Proposition 6.19</u>: $T_2(G)$ is an ℓ-ideal of G.

Proof: It suffices to show that

$$\{g \in G^+ \backslash \{0\} \mid \mathcal{D}^* \cap (\bigcap_{h \in \mathcal{T}(g)} M(g,h)) = \emptyset\}$$

is a normal subset of G. By Lemmas 2.15 and 2.16 and the proof of Lemma 2.12(c), $\mathcal{D}^*$ is a normal subset of G. Let $a \in G$ and suppose that $g \in G^+ \backslash \{0\}$ is such that $\mathcal{D}^* \cap (\bigcap_{h \in \mathcal{T}(g)} M(g,h)) = \emptyset$. Let $p \in \mathcal{D}^*$. Since $\mathcal{D}^*$ is normal, $-a + p + a \in \mathcal{D}^*$. Thus there exists $h \in \mathcal{T}(g)$ such that $-a + p + a \notin M(g,h)$, i.e. $-a + p + a \in (h,\infty) \cap (h'^{\perp})$. Then clearly $p \in (a + h - a, \infty)$. By Lemma 2.14(b), since $p \in a + h'^{\perp} - a$, $p \in (a + h' - a)^{\perp}$. By Lemma 2.15, $a + h - a \in \mathcal{T}(a + g - a)$. Clearly $(a + h - a)' = a + h' - a$. Thus $p \in (a + h - a, \infty) \cap ((a + h - a)')^{\perp}$, i.e. $p \notin M(a + g - a, a + h - a)$. Since p was an arbitrary element of $\mathcal{D}^*$, we conclude that $\mathcal{D}^* \cap (\bigcap_{k \in \mathcal{T}(a+g-a)} M(a + g - a, k)) = \emptyset$. Clearly $a + g - a \in G^+ \backslash \{0\}$, and hence

$$\{g \in G^+ \backslash \{0\} \mid \mathcal{D}^* \cap (\bigcap_{h \in \mathcal{T}(g)} M(g,h)) = \emptyset\}$$

is a normal subset of G. |X|

7. MORE EXAMPLES

In this chapter we look at some examples of the $\mathfrak{T}$-topology with regard to the Hausdorff separation axiom.

Example 7.1: In this example we show that $A(R)$ has Hausdorff $\mathfrak{T}$-topology. As in Example 3.1, this result follows from the same result for $A(\Omega)$ where Ω is a totally ordered set and $A(\Omega)$ is doubly transitive.

Proposition 7.2: Let Ω be a totally ordered set. If $A(\Omega)$ is doubly transitive, then $A(\Omega)$ has Hausdorff $\mathfrak{T}$-topology.

Proof: Let $g \in A(\Omega)^{+}\backslash\{i\}$. Similarly to Example 3.1, for $\omega \in \Omega$, we let

$$I(\omega,g) = \{\tau \in \Omega \mid \text{there exist integers } m,n \text{ such that } \tau g^{n} \leq \tau \leq \tau g^{m}\}.$$

Clearly $I(\omega,g)$ is convex for any $\omega \in \Omega$. Since $g > i$, there exists $\tau \in \Omega$ such that $\tau g > \tau$. Let $\omega \in I(\tau,g)$. Since $\tau g > \tau$, $\omega g > \omega$. Clearly $\vee I(\tau,g), \wedge I(\tau,g) \in \hat{\Omega}\backslash I(\tau,g)$. Thus, since $I(\tau,g)$ is convex,

$$(\wedge I(\tau,g), \vee I(\tau,g)) \cap \Omega = I(\tau,g).$$

Define a function $f:\Omega \to \Omega$ by

$$\omega f = \begin{cases} \omega g & \text{if } \omega \in I(\tau,g) \\ \omega & \text{otherwise.} \end{cases}$$

By Lemma 3.3, $f \in A(\Omega)^{+}$. Since $\tau g > \tau$, $f > i$. Since $(\wedge I(\tau,g), \vee I(\tau,g)) \cap \Omega = I(\tau,g)$, we have that

$S(\bar{f}) = (\wedge I(\tau,g), \vee I(\tau,g))$. Thus by Proposition 3.2, $f \varepsilon \mathfrak{A}$. By Proposition 3.5, since $A(\Omega)$ is doubly transitive, $\mathfrak{A} = \mathcal{D}_1$. Clearly $f \varepsilon [i,g]$, and thus $\mathcal{D}_1 \cap [i,g] \neq \emptyset$. Since g was an arbitrary element of $A(\Omega)^+ \setminus \{i\}$, we have by Corollary 6.8 that $A(\Omega)$ has Hausdorff $\mathfrak{T}$-topology. Alternately, we note that if $\ell \varepsilon [i,g] \setminus M(g,h)$ for some $h \varepsilon \mathcal{T}(g)$, then $\ell > h$ and $\ell \wedge h' = 0$ so that, as in the proof of Theorem 6.15,

$$g = h + h' < \ell + h' = \ell \vee h' \leqq g \vee g = g.$$

This is a contadiction, and hence $[i,g] \subseteq \bigcap_{h \varepsilon \mathcal{T}(g)} M(g,h)$. Thus by Theorem 6.15, $A(\Omega)$ has Hausdorff $\mathfrak{T}$-topology. |⊠|

Thus $A(R)$ has Hausdorff $\mathfrak{T}$-topology.

<u>Corollary 7.3</u>: Every ℓ-group is ℓ-isomorphic to an ℓ-subgroup of an ℓ-group with Hausdorff $\mathfrak{T}$-topology.

<u>Proof</u>: The corollary follows from Proposition 7.2 and the Holland representation of an ℓ-group [33]. |⊠|

<u>Example 7.4</u>: In this example we show that $C(R)$ has Hausdorff $\mathfrak{T}$-topology.

Let Y be a connected topological space with topology $\mathcal{U}$. Let $C(Y)$ be the set of all continuous functions from Y to R. Recall (Chapter 1) that $C(Y)$ may be considered as an abelian ℓ-group with addition defined pointwise, i.e. by

$$x(f + g) = (xf) + (xg),$$

and a partial order defined by

$$f \leqq g \text{ if and only if } xf \leqq xg \text{ for all } x \varepsilon Y.$$

Then $x(f \wedge g) = (xf) \wedge (xg)$ and $x(f \vee g) = (xf) \vee (xg)$.

Let

$$Q^* = \{f \in C(Y) \mid xf = q \text{ for all } x \in Y \text{ and for } q \in Q^+ \setminus \{0\}\}.$$

Lemma 7.5: $Q^* \subseteq \mathcal{D}_1$.

Proof: Let $f \in Q^*$. Let $q \in Q^+ \setminus \{0\}$ be such that $xf = q$. Suppose that $h, \ell \in C(Y)$ are such that $h \vee \ell = f$ and $h \wedge \ell = 0$. If $x \in Y$ is such that $xh = 0 = x\ell$, then $q = xf = x(h \vee \ell) = 0$, which contradicts our choice of q. Thus $(0)h^{-1} \cap (0)\ell^{-1} = \phi$. If $x \in Y \setminus ((0)h^{-1} \cup (0)\ell^{-1})$, then $xh > 0$ and $x\ell > 0$. Thus $0 < (xh) \wedge (x\ell) = x(h \wedge \ell) = 0$, which is a contradiction. Hence $Y = (0)h^{-1} \cup (0)\ell^{-1}$. Since h and ℓ are continuous and since $\{0\}$ is closed in the interval topology of R, $(0)h^{-1}$ and $(0)\ell^{-1}$ are closed with respect to $\mathcal{U}$. Thus, since Y is connected, either $(0)\ell^{-1} = \phi$ or $(0)h^{-1} = \phi$, i.e. either $\ell = f$ or $h = f$. Therefore $f \in \mathfrak{A}$, and hence $Q^* \subseteq \mathfrak{A}$.

Let $f \in C(Y)$. For $n \in N$, let $\frac{f}{n} : Y \to R$ be defined by $x(\frac{f}{n}) = (\frac{1}{n})(xf)$. Clearly $\frac{f}{n} \in C(Y)$ and $n(\frac{f}{n}) = f$. Thus $C(Y)$ is divisible and by Proposition 3.4, $\mathfrak{A} = \mathcal{D}_1$. Therefore $Q^* \subseteq \mathcal{D}_1$. |⊠|

Proposition 7.6: Let Y be a connected topological space. Then $C(Y)$ has Hausdorff $\mathfrak{T}$-topology.

Proof: Let $g \in C(Y)^+ \setminus \{0\}$. Let $r \in R^+ \setminus \{0\}$ be such that $ug = r$ for some $u \in Y$. Since $r > 0$, there exists $q \in Q$ such that $0 < q < r$. Define $q^* \in C(Y)$ by $xq^* = q$ for all $x \in Y$. Clearly $q^* \in Q^*$, and hence by Lemma 7.5, $q^* \in \mathcal{D}_1$. Suppose that

$q^* \notin M(g,h)$ for some $h \in \mathcal{T}(g)$. Then $q^* \in (h,\infty) \cap (h'^{\perp})$. Clearly $q^{*\perp} = \{0\}$ and thus $h' = 0$, i.e. $h = g$. Since $uq^* = q < r = ug$, $q^* \not> g$. This contradicts our choice of h. Thus $q^* \in \bigcap_{h\,\mathcal{T}(g)} M(g,h)$, and hence $\mathcal{D}^* \cap (\bigcap_{h\,\mathcal{T}(g)} M(g,h)) \neq \emptyset$. Therefore by Theorem 6.15, $C(Y)$ has Hausdorff $\mathfrak{T}$-topology. Alternately, since $q^{*\perp} = \{0\}$, $N(0,q^*) = [-q^*,q^*]$ and thus since $q^* \notin (g,\infty)$, $g \notin N(0,q^*)$. Thus $C(Y)$ has Hausdorff $\mathfrak{T}$-topology by Theorem 6.7. |X|

<u>Corollary 7.7</u>: $C(R)$ has Hausdorff $\mathfrak{T}$-topology. |X|

<u>Example 7.8</u>: Let G be the ℓ-group constructed in Example 3.6. We show that G has Hausdorff $\mathfrak{T}$-topology; the method we use is similar to that used to prove Proposition 7.6.

In Example 3.6, we proved that $(P \cup F)^+\setminus\{0\} \subseteq \mathcal{D}_1$. Let $g \in G^+\setminus\{0\}$, let $r \in R$ be such that $rg > 0$, and let $q \in Q^+\setminus\{0\}$ be such that $rg > q > 0$. If $r \leqq -2$, let $k = g_q$; if $r > -2$, let $k = f_q$. Then $k \in \mathcal{D}_1$. Suppose that $h \in \mathcal{T}(g)$ and that $h' \in G$ is such that $h \wedge h' = 0$ and $h \vee h' = g$. If $k \in h'^{\perp}$, then $S(h') \cap S(k) = \emptyset$, and since $rk > 0$, $r \notin S(h')$. Thus $r \in S(h)$ and $rh = rg > q \geqq rk > 0$. Therefore $k \notin (h,\infty)$. Hence $k \in \bigcap_{h\in\mathcal{T}(g)} M(g,h)$, and since g was an arbitrary element of $G^+\setminus\{0\}$, by Theorem 6.15 G has Hausdorff $\mathfrak{T}$-topology. Alternately, we have the following: If $g \in N(0,k)$, then $g = h + h'$ for $h \in [-k,k]$ and $h' \in k^{\perp}$. Since $rk > 0$, $rh' = 0$ and thus $rh = rg$. But $rg > q \geqq rk$, which contradicts our choice of h. Therefore by Theorem 6.7, G has Hausdorff $\mathfrak{T}$-topology.

Example 7.9: In this example we construct an ℓ-group which is a (non-convex) ℓ-subgroup of a cardinal product of totally ordered groups and which has $\mathfrak{A} = \emptyset$.

Let $G = \left|\prod_{1}^{\infty}\right| R$. Let

$$L = \{f \in G \mid \text{there is an } n \in N \text{ such that for all } i \quad (i + n)f = (i)f\}.$$

Let $f, g \in L$ and let $n, m \in N$ be such that $(i + n)f = (i)f$ and $(i + m)g = (i)g$ for all i. Then clearly $(i + n)(-f) = (i)(-f)$, $(i + mn)(f + g) = (i)(f + g)$, $(i + mn)(f \vee g) = (i)(f \vee g)$, and $(i + mn)(f \wedge g) = (i)(f \wedge g)$ for all i. Thus L is an ℓ-subgroup of G. Let $f \in L^+ \setminus \{0\}$ and let $n \in N$ be such that $(i + n)f = (i)f$. Define $h, \ell \in G$ by

$$(i)h = \begin{cases} (i)f & \text{if } 2kn < i \leqq (2k + 1)n \text{ for some } k \in Z^+ \\ 0 & \text{if } (2k + 1)n < i \leqq (2k + 2)n \text{ for some } k \in Z^+, \end{cases}$$

$$(i)\ell = \begin{cases} 0 & \text{if } 2kn < i \leqq (2k + 1)n \text{ for some } k \in Z^+ \\ (i)f & \text{if } (2k + 1)n < i \leqq (2k + 2)n \text{ for some } k \in Z^+. \end{cases}$$

Since $f > 0$, we must have $h > 0$ and $\ell > 0$. Clearly $(i + 2n)h = (i)h$ and $(i + 2n)\ell = (i)\ell$. Thus $h, \ell \in L$. It is also clear that $h + \ell = f$ and $h \wedge \ell = 0$. Hence $f \notin \mathfrak{A}$. Therefore $\mathfrak{A} = \emptyset$ and hence L has indiscrete $\mathfrak{T}$-topology.

Example 7.10: The ℓ-group of this example has Hausdorff $\mathfrak{T}$-topology and an element g such that $\mathcal{D}^* \cap [0,g] = \emptyset$. It is similar to the ℓ-group constructed in Example 3.6.

Let L be the ℓ-group of Example 7.9 and let $C(R)$ be the ℓ-group of all continuous functions from R to R (cf. Example 7.4).

Define a function $\pi : L \to C(R)$ as follows: for $\ell \in L$, $\ell\pi \in C(R)$ is the function defined by

$$x(\ell\pi) = \begin{cases} 2(i\ell)(x - i) & \text{if } x \in (i, i + \frac{1}{2}] \\ 2(i\ell)(-x + i + 1) & \text{if } x \in (i + \frac{1}{2}, i + 1] \\ 0 & \text{otherwise} \end{cases}$$

where $i = 1,2,3, \ldots$. See figure XXIII. If $\ell \in L$, then $-(\ell\pi) = (-\ell)\pi \in L$. Let $\ell, g \in L$, and suppose $\ell \neq g$. Then clearly $\ell\pi \neq g\pi$, since $(i + \frac{1}{2})(h\pi) = ih$ for all $h \in L$. Further

$$x(\ell\pi + g\pi)$$

$$= \begin{cases} 2(i\ell)(x - i) + 2(ig)(x - i) & \text{if } x \in (i, i+\frac{1}{2}] \\ 2(i\ell)(-x + i + 1) + 2(ig)(-x + i + 1) & \text{if } x \in (i+\frac{1}{2}, i+1] \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 2(i)(\ell + g)(x - i) & \text{if } x \in (i, i+\frac{1}{2}] \\ 2(i)(\ell + g)(-x + i + 1) & \text{if } x \in (i+\frac{1}{2}, i+1] \\ 0\ , & \text{otherwise} \end{cases}$$

$$= x[(\ell + g)\pi].$$

Thus L is group isomorphic to L . Also

$$x(\ell\pi \vee g\pi)$$

$$= \begin{cases} [2(i\ell)(x - i)] \vee [2(ig)(x - i)] & \text{if } x \in (i, i+\frac{1}{2}] \\ [2(i\ell)(-x + i + 1)] \vee [2(ig)(-x + i + 1)] & \text{if } x \in (i+\frac{1}{2}, i+1] \\ 0 & \text{otherwise.} \end{cases}$$

If $x \in (i, i + \frac{1}{2}]$, then $x - i > 0$, and thus

$$[2(i\ell)(x - i)] \vee [2(ig)(x - i)] = 2[(i\ell) \vee (ig)](x - i).$$

If $x \in (i + \frac{1}{2}, i + 1]$, then $-x + i + 1 > 0$, and thus

$$[2(i\ell)(-x + i + 1)] \vee [2(ig)(-x + i + 1)]$$

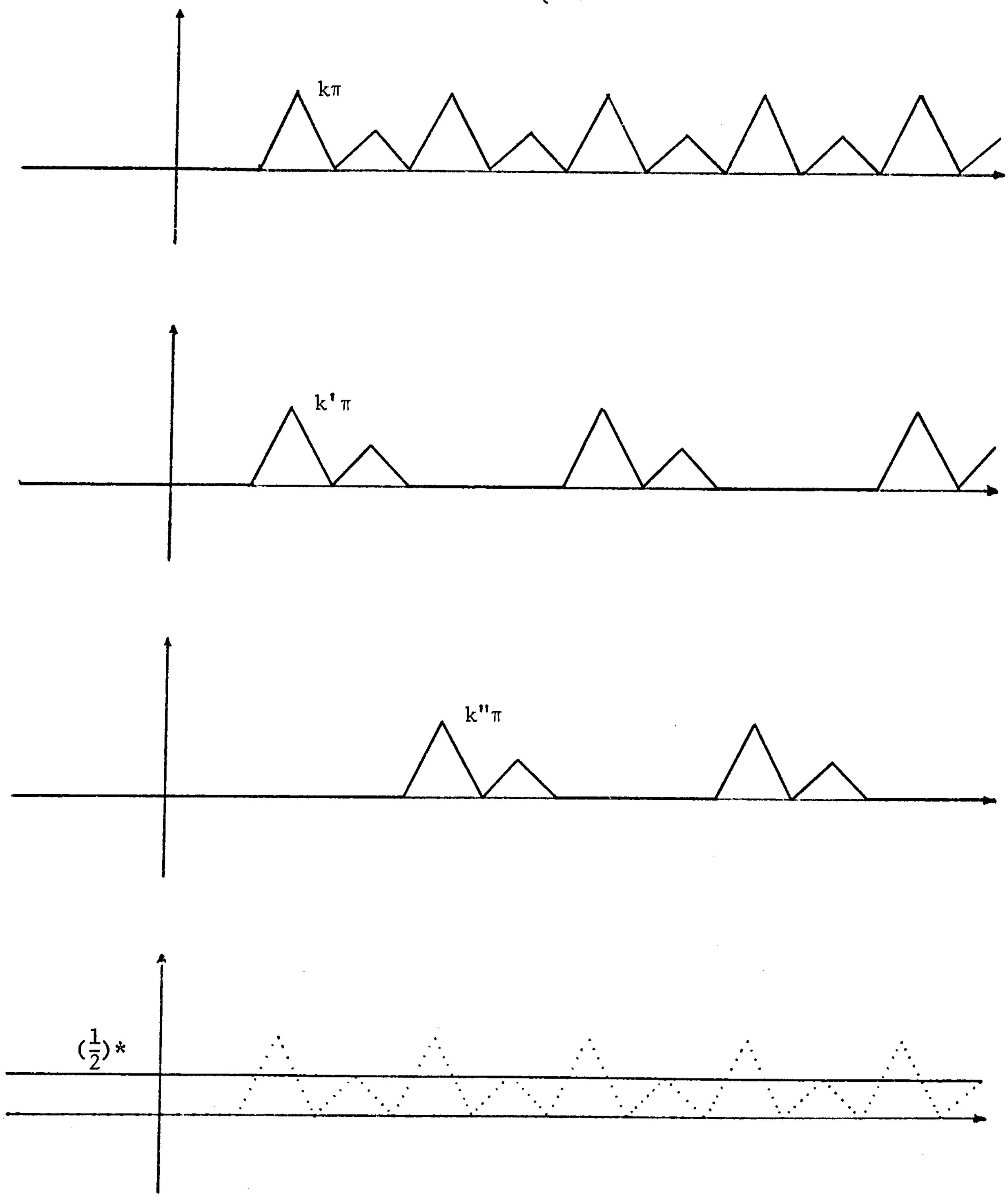
$$= 2[(i\ell) \vee (ig)](-x + i + 1).$$

figure XXIII

$G = C(R)$

$$k \in \left|\prod_{1}^{\infty}\right| R$$

$$(i)k = \begin{cases} 1 & \text{if } i = 2n - 1 \text{ for } n \in N \\ \frac{1}{2} & \text{if } i = 2n \text{ for } n \in N \end{cases}$$



Therefore

$$x(\ell\pi \vee g\pi) = \begin{cases} 2(i)(\ell \vee g)(x - i) & \text{if } x \in (i, i + \frac{1}{2}] \\ 2(i)(\ell \vee g)(-x + i + 1) & \text{if } x \in (i + \frac{1}{2}, i + 1] \\ 0 & \text{otherwise} \end{cases}$$

$$= x[(\ell \vee g)\pi]$$

Similarly $x(\ell\pi \wedge g\pi) = x[(\ell \wedge g)\pi]$. Thus L is lattice isomorphic to L .

As in Example 7.4, let

$$Q^* = \{f \in C(R) \mid tf = q \text{ for all } t \in R, \text{ for some } q \in Q^+\setminus\{0\}\}.$$

Let F be the ℓ-subgroup of $C(R)$ generated by $(L\pi) \cup (Q^*)$. Let $p \in L$ be defined by $(i)p = 1$ for all i. Let $g = p\pi$. For the remainder of this example, intervals refer to F rather than $C(R)$. If $f \in [0,g]$, then $1f = 0$. Since $f \in F$,

$$f = \bigvee_{\alpha=1}^{n} \bigwedge_{\beta=1}^{m} (h_{\alpha\beta} + \ell_{\alpha\beta}\pi)$$

where $h_{\alpha\beta} \in Q^*$, $\ell_{\alpha\beta} \in L$ for all α and β. If $x \in R$, then

$$xf = \bigvee_{\alpha=1}^{n} \bigwedge_{\beta=1}^{m} (xh_{\alpha\beta} + x(\ell_{\alpha\beta}\pi)).$$

Since $1f = 0$,

$$0 = \bigvee_{\alpha=1}^{n} \bigwedge_{\beta=1}^{m} (1h_{\alpha\beta} + 1(\ell_{\alpha\beta}\pi)) = \bigvee_{\alpha=1}^{n} \bigwedge_{\beta=1}^{m} 1h_{\alpha\beta}.$$

Since $\bigvee_{\alpha=1}^{n} \bigwedge_{\beta=1}^{m} h_{\alpha\beta} \in Q^*$, then for all $x \in R$, $x(\bigvee_{\alpha=1}^{n} \bigwedge_{\beta=1}^{m} h_{\alpha\beta}) = 0$. For $1 \leqq \alpha \leqq n$, $1 \leqq \beta \leqq m$, let $n_{\alpha\beta} \in N$ be such that $(i + n_{\alpha\beta})\ell_{\alpha\beta} = (i)\ell_{\alpha\beta}$. Let d be the least common multiple of $\{n_{\alpha\beta} \mid 1 \leqq \alpha \leqq n, 1 \leqq \beta \leqq m\}$. Let $\ell'_{\alpha\beta}, \ell''_{\alpha\beta} \in L$ be defined by

$$(i)\ell'_{\alpha\beta} = \begin{cases} (i)\ell_{\alpha\beta} & \text{if } 2kd < i \leqq (2k + 1)d \text{ for some } k \in Z^+ \\ 0 & \text{if } (2k + 1)d < i \leqq (2k + 2)d \text{ for some } k \in Z^+ \end{cases}$$

$$(i)\ell''_{\alpha\beta} = \begin{cases} 0 & \text{if } 2kd < i \leqq (2k+1)d \text{ for some } k \in Z^+ \\ (i)\ell_{\alpha\beta} & \text{if } (2k+1)d < i \leqq (2k+2)d \text{ for some } k \in Z^+. \end{cases}$$

Let

$$f' = \bigvee_{\alpha=1}^{n} \bigwedge_{\beta=1}^{m} (h_{\alpha\beta} + \ell'_{\alpha\beta}\pi),$$

$$f'' = \bigvee_{\alpha=1}^{n} \bigwedge_{\beta=1}^{m} (h_{\alpha\beta} + \ell''_{\alpha\beta}\pi).$$

If $x \in (1 \vee (2kd), (2k+1)d]$ for some $k \in Z^+$, then for all α, β, $x(\ell''_{\alpha\beta}\pi) = 0$, and hence

$$xf'' = x(\bigvee_{\alpha=1}^{n} \bigwedge_{\beta=1}^{m} h_{\alpha\beta}) = 0.$$

Also $x(\ell'_{\alpha\beta}\pi) = x(\ell_{\alpha\beta}\pi)$, and hence $xf' = xf \geqq 0$. If $x \in ((2k+1)d, (2k+2)d]$ for some $k \in Z^+$, then for all α, β, $x(\ell'_{\alpha\beta}\pi) = 0$, and hence

$$xf' = x(\bigvee_{\alpha=1}^{n} \bigwedge_{\beta=1}^{m} h_{\alpha\beta}) = 0.$$

Also $x(\ell''_{\alpha\beta}\pi) = x(\ell_{\alpha\beta}\pi)$, and hence $xf'' = xf \geqq 0$. If $x \leqq 1$, then $x(\ell'_{\alpha\beta}\pi) = x(\ell''_{\alpha\beta}\pi) = x(\ell_{\alpha\beta}\pi) = 0$, and hence

$$xf' = x(\bigvee_{\alpha=1}^{n} \bigwedge_{\beta=1}^{m} h_{\alpha\beta}) = xf'' = xf = 0.$$

Thus $f' \wedge f'' = 0$ and $f' \vee f'' = f$. Therefore $f \notin \mathfrak{A}$. Since f was an arbitrary element of $[0,g]$, $\mathfrak{A} \cap [0,g] = \phi$. Thus $\mathcal{D}^* \cap [0,g] = \phi$.

We now show that F has Hausdorff $\mathfrak{T}$-topology. Let $g \in F^+ \setminus \{0\}$. Let $r \in R$ be such that $rg > 0$. Let $q \in Q$ be such that $rg > q > 0$. Let $q^* \in (Q^*)^+ \setminus \{0\} \subseteq F^+ \setminus \{0\}$ be defined by $xq^* = q$ for all $x \in R$. Clearly $q^{*\perp} = \{0\}$. If $h' \in \mathcal{T}(g)$ and $q^* \in h'^{\perp}$, then $h' \in q^{*\perp}$ and hence $h' = 0$. Thus $q^* \in M(g,h)$ for all $h \in \mathcal{T}(g) \setminus \{g\}$. Since $rg > q = rq^*$, $q^* \notin (g,\infty)$ and thus $q^* \in M(g,g)$. Therefore $q^* \in \bigcap_{h \in T(g)} M(g,h)$. If $\mathfrak{A}(C(R))$ is the set $\mathfrak{A}$ for $C(R)$ and $\mathfrak{A}$ is the set $\mathfrak{A}$ for F, then $\mathfrak{A}(C(R)) \cap F \subseteq \mathfrak{A}$. By Lemma 7.5, $Q^* \subseteq \mathfrak{A}(C(R))$. Thus, since $Q^* \subseteq F$, $Q^* \subseteq \mathfrak{A}$. Clearly L is divisible and Q^* is divisible. Thus F is divisible. By Proposition 3.4, $\mathfrak{A} = \mathcal{D}_1$. Thus $q^* \in Q^* \subseteq \mathcal{D}_1$. Therefore $\mathcal{D}^* \cap (\bigcap_{h \in T(g)} M(g,h)) \neq \emptyset$. We conclude that, by Theorem 6.15, F has Hausdorff $\mathfrak{T}$-topology. Alternately, we can apply Theorem 6.7 as in the proof of Proposition 7.6.

Example 7.11: In this example we construct an ℓ-group with non-Hausdorff $\mathfrak{T}$-topology such that $\mathcal{D}^* = \mathcal{D}_2$ and for all $h \in \mathcal{D}_2$, $D(h)$ is closed with respect to the $\mathfrak{T}$-topology.

Let L be the ℓ-group constructed in Example 7.9. Let $G = L \overleftarrow{X} Z$. By Lemma 5.15, $\mathfrak{A} = L \times (Z^+ \setminus \{0\})$. By Lemma 5.16, $\mathcal{D}_1 = \emptyset$. Thus $\mathcal{D}^* = \mathcal{D}_2$. Let $h \in \mathcal{D}_2$. Then by Lemma 5.17, $h = (\ell,1)$ for some $\ell \in L$. Then clearly $D(h) = L \times \{0\}$. By Lemma 5.14, $h^{\perp} = \{0\}$. Thus $D(h) = D(h) + h^{\perp}$ which is closed with respect to the $\mathfrak{T}$-topology by Proposition 2.31.

Example 7.12: This is an example of another ℓ-group with indiscrete $\mathfrak{T}$-topology. Let Q be the rational numbers with usual topology and let $C(Q)$ be the ℓ-group of all continuous real-valued functions of Q. (We are indebted to Norman Reilly for pointing out that $\mathfrak{A} = \emptyset$ in this example.)

Let $g \in C(Q)^+ \setminus \{0\}$. Then there exists $q \in Q$ such that $qg > 0$. Since g is continuous and $\{0\}$ is a closed set of R,

$(0)g^{-1} = Q\setminus S(g)$ is a closed set of Q, i.e. $S(g)$ is an open set of Q. Hence there exist $x,y \in Q$ such that $q \in (x,y) \subsetneq S(g)$, i.e. there exists $p \in Q$ such that $p \neq q$ and $pg > 0$. Without loss of generality we may assume that $q < p$. Let $r \in R\setminus Q$ be such that $q < r < p$. Let $a,b : Q \to R$ be defined by

$$xa = \begin{cases} xg & \text{if } x > r \\ 0 & \text{if } x \leqq r, \end{cases}$$

$$xb = \begin{cases} 0 & \text{if } x \geqq r \\ xg & \text{if } x < r. \end{cases}$$

Let $u,v \in R$ be such that $u < v$. If $0 \geqq v$, then $(u,v)a^{-1} = (u,v)b^{-1} = \emptyset$. If $0 \in (u,v)$, then $(u,v)a^{-1} = (u,v)g^{-1} \cup (-\infty,r]$ and $(u,v)b^{-1} = (u,v)g^{-1} \cup [r,\infty)$. If $0 \leqq u$, then $(u,v)a^{-1} = S(g) \cap (r,\infty)$ and $(u,v)b^{-1} = S(g) \cap (-\infty,r)$. Since $r \in R\setminus Q$, $[r,\infty) = (r,\infty)$ and $(-\infty,r] = (-\infty,r)$ are open sets of Q. Since g is continuous, $(u,v)g^{-1}$ and (as above) $S(g)$ are open sets of Q. We conclude that $(u,v)a^{-1}$ and $(u,v)b^{-1}$ are open sets of Q, and hence that $a,b \in C(Q)$. Clearly $a \wedge b = 0$ and $a \vee b = g$. Thus $g \notin \mathfrak{U}$. Since g was an arbitrary element of $C(Q)^{+}\setminus\{0\}$, $\mathfrak{U} = \emptyset$. Therefore $C(Q)$ has indiscrete $\mathfrak{T}$-topology.

8. $\mathfrak{T}$-TOPOLOGY CONVERGENCE

In this chapter we investigate convergence with respect to the $\mathfrak{T}$-topology and the relationship between the $\mathfrak{T}$-topology and the topology derived from α-convergence.

Lemma 8.1: Let G be an ℓ-group. Let $g \in G^+ \setminus \{0\}$. Suppose that $A \subseteq [-g,g]$ is such that $A + g^{\perp}$ is a symmetric sublattice of G. If $h \in A + g^{\perp}$, then $g \wedge |h| \in T(|h|) \cap A$.

Proof: Let $h \in A + g^{\perp}$. Since $A + g^{\perp}$ is a symmetric sublattice of G, $|h| = h \vee (-h) \in A + g^{\perp}$. Since $A \subseteq [-g,g]$ and $|h| \in (A + g^{\perp})^+$, then by Lemma 2.13(a) $|h| = a + b$ for $a \in A^+$ and $b \in (g^{\perp})^+$. Thus, since $a \wedge b = 0$, $a \in T(|h|)$ and

$$|h| \wedge g = (a \vee b) \wedge g = (a \wedge g) \vee (b \wedge g) = a \wedge g = a.$$

Therefore $|h| \wedge g \in T(|h|) \cap A$. |X|

Theorem 8.2: Let $\{x_\beta \mid \beta \in B\}$ be a net in an ℓ-group G. Then $\{x_\beta\}$ converges to $x \in G$ with respect to the $\mathfrak{T}$-topology on G if and only if (a) for all $g \in \mathcal{D}^*$, there is an $\alpha \in B$ such that whenever $\beta \geqq \alpha$, $|-x + x_\beta| \wedge g \in T(|-x + x_\beta|)$, and (b) for all $g \in \mathcal{D}_2$, there is a $\gamma \in B$ such that whenever $\beta \geqq \gamma$, $|-x + x_\beta| \wedge g \in D(g)$.

Proof: Suppose that $\{x_\beta\}$ $\mathfrak{T}$-converges to $x \in G$. Let $g \in \mathcal{D}^*$ and if $g \in \mathcal{D}_1$, let $H = N(0,g)$; if $g \in \mathcal{D}_2$, let $H = D(g) + g^{\perp}$. Then there is an $\alpha \in B$ such that whenever

$\beta \geqq \alpha$, $x_\beta \in x + \mathrm{Int}(H) \subseteq x + H$. Hence $-x + x_\beta \in H$ for all $\beta \geqq \alpha$. By Lemmas 2.12 and 2.23, H is a symmetric sublattice of G. We wish to apply Lemma 8.1 to H: if $g \in \mathcal{D}_1$, we let $A = [-g,g]$ and conclude that $g \wedge |-x + x_\beta| \in T(|-x + x_\beta|)$; if $g \in \mathcal{D}_2$, by Lemma 2.2(b), $D(g) \subseteq [-g,g]$ so that letting $A = D(g)$, we may conclude that $g \wedge |-x + x_\beta| \in T(|-x + x_\beta|) \cap D(g)$. This proves (a) and (b).

Conversely, suppose that $\{x_\beta\}$ is a net in G such that there is an $x \in G$ satisfying (a) and (b). Let $H \in N_1(0)$. Then by (a) there is an $\alpha \in B$ such that whenever $\beta \geqq \alpha$, $|-x + x_\beta| \in H$. By Lemmas 2.12 and 2.23, H is a symmetric and convex. Hence $-x + x_\beta \in H$, i.e. $x_\beta \in x + H$ for all $\beta \geqq \alpha$. Let $H \in N_2(0)$. Then by (a) and (b), there is an $\alpha \in B$ such that wnenever $\beta \geqq \alpha$, $|-x + x_\beta| \in H$. As above, this implies that $x_\beta \in x + H$ for all $\beta \geqq \alpha$. Let $\mathfrak{T}$ be the $\mathfrak{T}$-topology on G and let $U \in \mathfrak{T}$ be such that $x \in U$. Then there is an $H \in N(0)$ such that $x \in x + H \subseteq U$. By definition of $N(0)$, $H = \bigcap_{i=1}^{n} H_i$ for $H_i \in N_3(0)$. As shown above, for each i there is an $\alpha_i \in B$ such that whenever $\beta \geqq \alpha_i$, $x_\beta \in x + H_i$. Let $\alpha \geqq \alpha_i$ for all i. Then whenever $\beta \geqq \alpha$, $x_\beta \in \bigcap_{i=1}^{n}(x + H_i) = x + H$. Therefore $\{x_\beta\}$ $\mathfrak{T}$-converges to $x \in G$. |X|

We shall need the following standard result from topology.

<u>Lemma 8.3</u>: Let Y be a set with topologies $\mathcal{U}_1$ and $\mathcal{U}_2$. Then $\mathcal{U}_1 \supseteq \mathcal{U}_2$ if and only if every net converging to x with respect to $\mathcal{U}_1$ converges to x with respect to $\mathcal{U}_2$. |X|

Let L be a lattice. Papangelou [49,50] defined α-convergence in L as follows: A net $\{x_\beta \mid \beta \in B\}$ is said to <u>α-converge</u> to to $x \in L$ if and only if x is the only element of L satisfying

$$x = \bigvee_{\beta \geq \alpha} (x_\beta \wedge x) = \bigwedge_{\beta \geq \alpha} (x_\beta \vee x)$$

for all $\alpha \in B$. Papangelou investigated α-convergence in abelian ℓ-groups. Ellis [22] showed that there is a topology $\mathcal{S}$ on an ℓ-group G such that convergence with respect to $\mathcal{S}$ is equivalent to α-convergence, if and only if G is completely distributive. When such a topology $\mathcal{S}$ exists, Ellis noted that it must be Hausdorff. Madell [43] proved that with respect to $\mathcal{S}$, any completely distributive ℓ-group is a topological group and a topological lattice. He also showed that any Hausdorff topology on a completely distributive ℓ-group G with respect to which G is both a topological group and a topological lattice lies between $\mathcal{S}$ and the discrete topology. Therefore we have the following result:

<u>Theorem 8.4</u>: Let G be a completely distributive ℓ-group. If $\mathcal{S}$ is the topology derived from α-convergence on G and if $\mathfrak{T}$ is the $\mathfrak{T}$-topology, then the following statements are equivalent:

(a) $\mathcal{S} \subseteq \mathfrak{T}$.

(b) $\mathfrak{T}$ is Hausdorff. |⊠|

Concerning inclusion the other way, we have the following theorem.

Theorem 8.5: Let G be a completely distributive ℓ-group with $\mathfrak{T}$-topology $\mathfrak{T}$ and topology S derived from α-convergence. Then $\mathfrak{T} \subseteq S$ if and only if for all nets $\{x_\beta \mid \beta \in B\} \subseteq G^+\setminus\{0\}$ such that $\bigwedge_{\delta\in\Delta} x_\delta = 0$ for all cofinal subsets Δ of B, and for all $g \in \mathcal{D}^*\setminus\mathcal{D}_2^{\P}$, there is an $\alpha \in B$ such that whenever $\beta \geqq \alpha$, $g \wedge x_\beta \in T(x_\beta)$.

Papangelou [50] proved the following lemma in the case when G is a completely distributive abelian ℓ-group. Madell [43] noted that the lemma remained true when the assumption of commutativity was removed.

Lemma 8.6: If G is a completely distributive ℓ-group and $\{x_\beta \mid \beta \in B\}$ is a net in G, then the following statements are equivalent:

(a) $\{x_\beta\}$ α-converges to 0.

(b) For each cofinal subset Δ of B, $\bigwedge_{\delta\in\Delta} |x_\delta| = 0$. |X|

Proof of Theorem 8.5: If $\mathfrak{T} \subseteq S$, then by Lemma 8.3 every net which α-converges to $x \in G$ converges to x with respect to $\mathfrak{T}$. Let $\{x_\beta \mid \beta \in B\} \subseteq G^+\setminus\{0\}$ be a net such that $\bigwedge_{\delta\in\Delta} x_\delta = 0$ for all cofinal subsets Δ of B. By Lemma 8.6 $\{x_\beta\}$ α-converges to 0. Hence $\{x_\beta\}$ converges to 0 with respect to $\mathfrak{T}$. By Theorem 8.2, for all $g \in \mathcal{D}^*\setminus\mathcal{D}_2^{\P}$, there is an $\alpha \in B$ such that whenever $\beta \geqq \alpha$, $g \wedge x_\beta \in T(x_\beta)$.

Conversely, suppose that for all nets $\{x_\beta \mid \beta \in B\} \subseteq G^+\setminus\{0\}$ such that $\bigwedge_{\delta\in\Delta} x_\delta = 0$ for all cofinal subsets Δ of B, and for all $g \in \mathcal{D}^*\setminus\mathcal{D}_2^{\P}$, there is an $\alpha \in B$ such that whenever $\beta \geqq \alpha$,

$g \wedge x_\beta \in \mathcal{T}(x_\beta)$. Suppose further that $\{x_\beta | \beta \in B\}$ is a net in G which α-converges to $x \in G$. Then $\{-x + x_\beta\}$ α-converges to 0, and by Lemma 8.6 $\bigwedge_{\delta \in \Delta} |-x + x_\delta| = 0$ for all cofinal subsets Δ of B. Hence, by our assumption, for all $g \in \mathcal{D}^* \setminus \mathcal{D}_2^\P$ there is an $\alpha \in B$ such that whenever $\beta \geqq \alpha$, $g \wedge |-x + x_\beta| \in \mathcal{T}(|-x + x_\beta|)$. Let $g \in \mathcal{D}_2 \setminus \mathcal{D}_2^\P$. Let $\ell \in D(g)^+ \setminus \{0\}$. Suppose that for all $\alpha \in B$, there is a $\beta \in B$ such that $\beta \geqq \alpha$ and $g \wedge |-x + x_\beta| \notin D(g)$. Since $0 \leqq g \wedge |-x + x_\beta| \leqq g$, $g \wedge |-x + x_\beta| \in g + D(g)$. Thus by Lemma 2.2(a), $g \wedge |-x + x_\beta| > \ell$. Let $\Delta = \{\beta \in B | g \wedge |-x + x_\beta| \notin D(g)\}$. Then Δ is a cofinal subset of B, but

$$\bigwedge_{\delta \in \Delta} |-x + x_\delta| \geqq \bigwedge_{\delta \in \Delta} (|-x + x_\delta| \wedge g) \geqq \ell > 0.$$

This contradicts Lemma 8.6 and thus there is an $\alpha \in B$ such that whenever $\beta \geqq \alpha$, $g \wedge |-x + x_\beta| \in D(g)$. Let $g \in \mathcal{D}_2^\P$. Suppose that for $\alpha \in B$, there is a $\beta \in B$ such that $\beta \geqq \alpha$ and $g \wedge |-x + x_\beta| > 0$. Then $g \wedge |-x + x_\beta| = g$, i.e. $|-x + x_\beta| \geqq g$. Let $\Delta = \{\beta \in B | g \wedge |-x + x_\beta| > 0\}$. Then Δ is a cofinal subset of B, but

$$\bigwedge_{\delta \in \Delta} |-x + x_\delta| \geqq \bigwedge_{\delta \in \Delta} (|-x + x_\delta| \wedge g) \geqq g > 0.$$

This contradicts Lemma 8.6 and thus there is an $\alpha \in B$ such that whenever $\beta \geqq \alpha$, $g \wedge |-x + x_\beta| = 0$. Hence for all $g \in \mathcal{D}_2^\P$, there is an $\alpha \in B$ such that whenever $\beta \geqq \alpha$, $g \wedge |-x + x_\beta| \in \mathcal{T}(|-x + x_\beta|)$. Further, for all $g \in \mathcal{D}_2$, there is an $\alpha \in B$ such that whenever $\beta \geqq \alpha$, $g \wedge |-x + x_\beta| \in D(g)$. Therefore, by Theorem 8.2, $\{x_\beta\}$ converges to x with respect to $\mathfrak{T}$, and hence by Lemma 8.3, $\mathfrak{T} \subseteq \mathcal{S}$. |X|

We now apply Theorems 8.4 and 8.5 to two particular situations. These applications show that, at least in certain circumstances, the criterion established in Theorem 8.5 is a convenient one to use.

<u>Proposition 8.7</u>: Let $\{T_\lambda \mid \lambda \in \Lambda\}$ be a collection of totally ordered groups. Suppose that G is a completely distributive ℓ-subgroup of $\underset{\lambda\in\Lambda}{|\Pi|} T_\lambda$ which contains $\underset{\lambda\in\Lambda}{|\Sigma|} T_\lambda$. Then $S = \mathfrak{T}$ on G.

<u>Proof</u>: By Theorem 4.5, the $\mathfrak{T}$-topology on each T_λ is the interval topology and hence is Hausdorff. By Theorem 5.1, the $\mathfrak{T}$-topology on G is the topology inherited from the product of the $\mathfrak{T}$-topologies on the T_λ. Therefore G has Hausdorff $\mathfrak{T}$-topology and hence by Theorem 8.4, $S \subseteq \mathfrak{T}$.

Let $\{x_\beta \mid \beta \in B\} \subseteq G^+ \setminus \{0\}$ be a net such that $\bigwedge_{\delta\in\Delta} x_\delta = 0$ whenever Δ is a cofinal subset of B. Let $g \in \mathcal{D}^* \setminus \mathcal{D}_2^\P$, and suppose that for all $\alpha \in B$, there exists $\beta \geqq \alpha$ such that $x_\beta > g$. If $\Delta = \{\beta \in B \mid x_\beta > g\}$, then Δ is a cofinal subset of B, but $\bigwedge_{\delta\in\Delta} x_\delta \geqq g > 0$. This contradicts our choice of $\{x_\beta\}$ and hence there is an $\alpha \in B$ such that whenever $\beta \geqq \alpha$, $x_\beta \not> g$. Since $g \in \mathcal{D}^* \setminus \mathcal{D}_2^\P$, $g \in \mathfrak{U}$ and hence by Lemma 5.8, there is a $\gamma \in \Lambda$ such that $\overline{\gamma g} = g$. Then $\lambda x_\beta \geqq 0 = \lambda g$ for all $\beta \in B$, for all $\lambda \neq \gamma$, and hence, since T_γ is totally ordered, $\gamma g \geqq \gamma x_\beta$ for all $\beta \geqq \alpha$. Thus $g \wedge x_\beta = \overline{\gamma x_\beta}$ for all $\beta \geqq \alpha$. Let $\delta_\beta \in \underset{\lambda\in\Lambda}{|\Pi|} T_\lambda$ be defined by

$$\lambda\delta_\beta = \begin{cases} \lambda x_\beta & \text{if } \lambda \neq \gamma \\ 0 & \text{if } \lambda = \gamma \end{cases}$$

Then $\delta_\beta = x_\beta - \overline{\gamma x_\beta}$, and hence, since $\overline{\gamma x_\beta} \in G$, we have $\delta_\beta \in G$. Clearly $\delta_\beta \vee \overline{\gamma x_\beta} = x_\beta$ and $\delta_\beta \wedge \overline{\gamma x_\beta} = 0$. Hence $g \wedge x_\beta = \overline{\gamma x_\beta} \in T(x_\beta)$. Thus by Theorem 8.5, $S \subseteq \mathfrak{T}$. Therefore $S = \mathfrak{T}$. |☒|

Example 8.8: The second situation that we investigate is A(R): we show that for A(R) the $\mathfrak{T}$-topology properly contains the topology derived from α-convergence.

In [39] Lloyd proves that $A(\Omega)$ is completely distributive for any totally ordered set Ω. Let S be the topology derived from α-convergence on A(R); let $\mathfrak{T}$ be the $\mathfrak{T}$-topology on A(R). By Example 7.1, $\mathfrak{T}$ is Hausdorff; thus Theorem 8.4 implies that $S \subseteq \mathfrak{T}$.

Let $g \in A(R)$ be defined by

$$xg = \begin{cases} 2x & \text{if } x \in [0,1) \\ \frac{1}{2}x + \frac{3}{2} & \text{if } x \in [1,3) \\ x & \text{otherwise.} \end{cases}$$

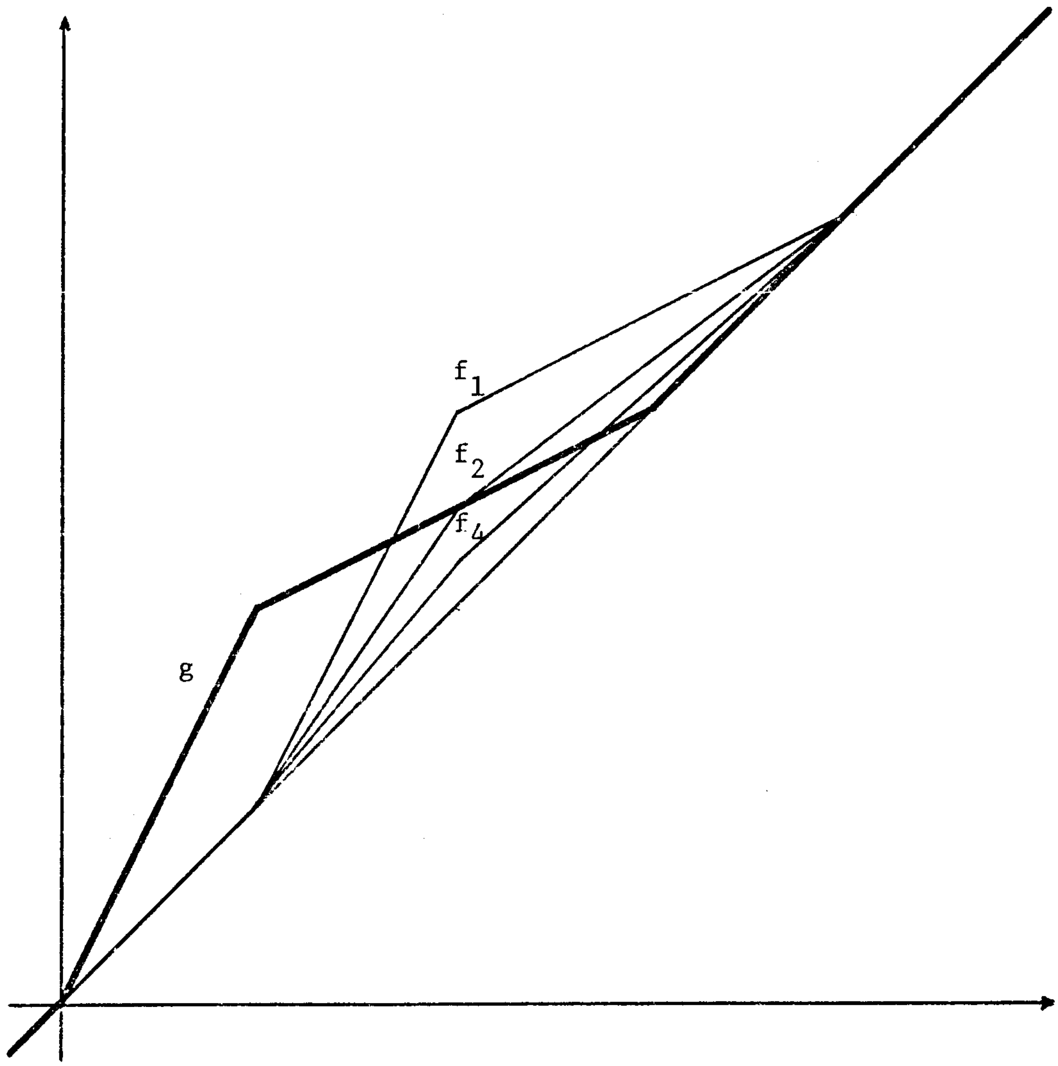
See figure XXIV. Clearly $g > i$ and $S(\bar{g}) = S(g) = (0,3)$. Hence by Proposition 3.2, $g \in \mathfrak{A}$, and by Proposition 3.5, $g \in \mathcal{D}_1$. Define a net $\{f_n \mid n \in N\}$ in $A(R)^+ \setminus \{0\}$ by

$$xf_n = \begin{cases} (1 + \frac{1}{n})x - \frac{1}{n} & \text{if } x \in [1,2) \\ (1 - \frac{1}{2n})x + \frac{2}{n} & \text{if } x \in [2,4) \\ x & \text{otherwise.} \end{cases}$$

Clearly $f_n > i$ and $S(\overline{f_n}) = S(f_n) = (1,4)$ for all n. Hence by Proposition 3.2, $f_n \in \mathfrak{A}$ for all n. Since $0 < f_n \wedge g < f_n$ for all n, $f_n \wedge g \notin T(f_n)$ for all n. Clearly $\bigwedge_{\delta\in\Delta} f_\delta = i$ for any cofinal subset Δ of N, and hence by Theorem 8.5, $S \not\supseteq \mathfrak{T}$.

figure XXIV

G = A(R)



$$xg = \begin{cases} 2x & \text{if } x \in [0,1) \\ \frac{1}{2}x + \frac{3}{2} & \text{if } x \in [1,3) \\ x & \text{otherwise} \end{cases} \qquad xf_n = \begin{cases} (1 + \frac{1}{n})x - \frac{1}{n} & \text{if } x \in [1,2) \\ (1 - \frac{1}{2n})x + \frac{2}{n} & \text{if } x \in [2,4) \\ x & \text{otherwise} \end{cases}$$

Using [12], Ellis [22] proves the following: (Madell [43] gives a somewhat different proof) Let G be a completely distributive ℓ-group and let $\{N_\beta \mid \beta \in B\}$ be any collection of L-closed prime convex ℓ-subgroups of G with $\bigcap_{\beta\in B} N_\beta = \{0\}$. For $\beta \in B$ let G/N_β denote the chain of left cosets of N_β and give G/N_β the interval topology. Let $|\Pi|_{\beta\in B} G/N_\beta$ be given the product topology. Then the topology derived from α-convergence is equivalent to the topology that G inherits from $|\Pi|_{\beta\in B} G/N_\beta$ via the natural lattice monomorphism $\pi: G \to |\Pi|_{\beta\in B} G/N_\beta$.

In A(R), letting $\{N_r \mid r \in R\}$ be the collection of prime subgroups

$$N_r = \{f \in A(R) \mid rf = r\},$$

we have typical S-neighborhoods of i depicted in figures XXV and XXVI. Compare these with figures XI and XII.

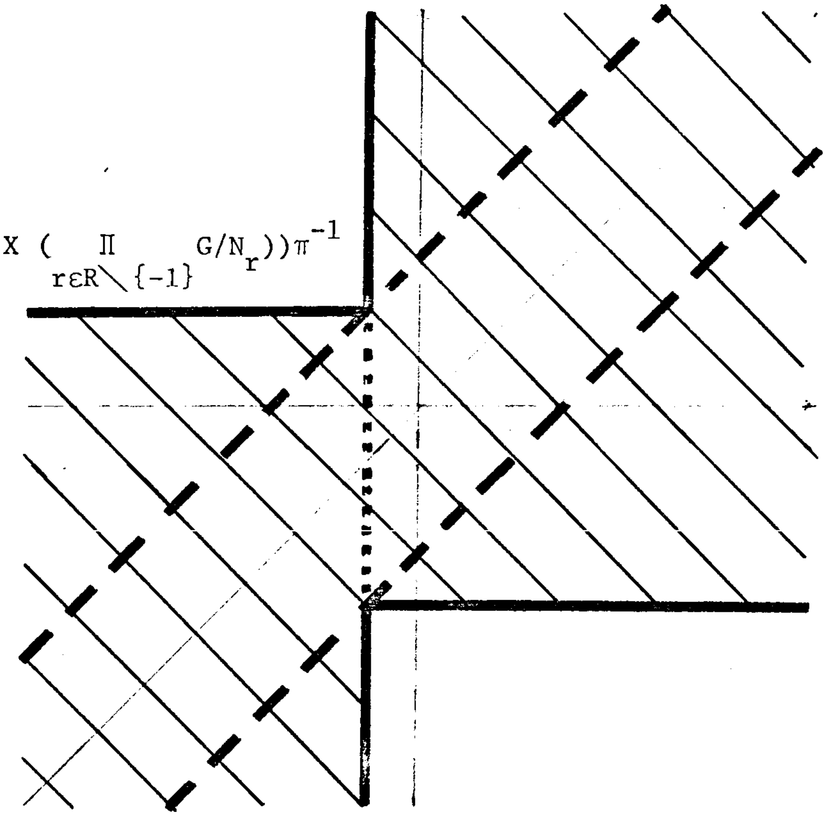
Example 8.9: Let Y be a completely regular connected topological space, and suppose that Y does not have a dense set of isolated points. Then Weinberg [60] has shown that C(Y), the ℓ-group of all continuous functions from Y to R, is not completely distributive. Thus C(Y) has no topology of α-convergence. However, by Proposition 7.6, C(Y) has Hausdorff $\mathfrak{T}$-topology.

figure XXV

$G = A(R)$

$xf = x + 3$

$$([f^{-1}N_{-1}, fN_{-1}] \times (\prod_{r \in R \setminus \{-1\}} G/N_r))\pi^{-1}$$



$$xg = \begin{cases} 2x + 3 & \text{if } x \in [-3,0) \\ \frac{1}{4}x + 3 & \text{if } x \in [0,4) \\ x & \text{otherwise} \end{cases}$$

$$([g^{-1}N_1, gN_1] \times (\prod_{r \in R \setminus \{1\}} G/N_r))\pi^{-1}$$

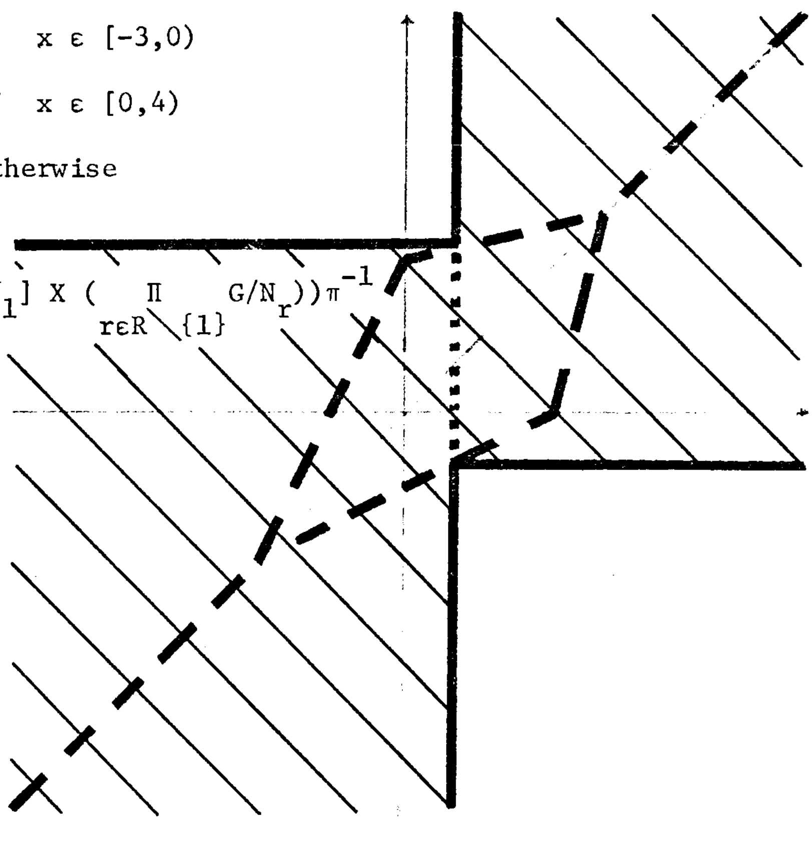
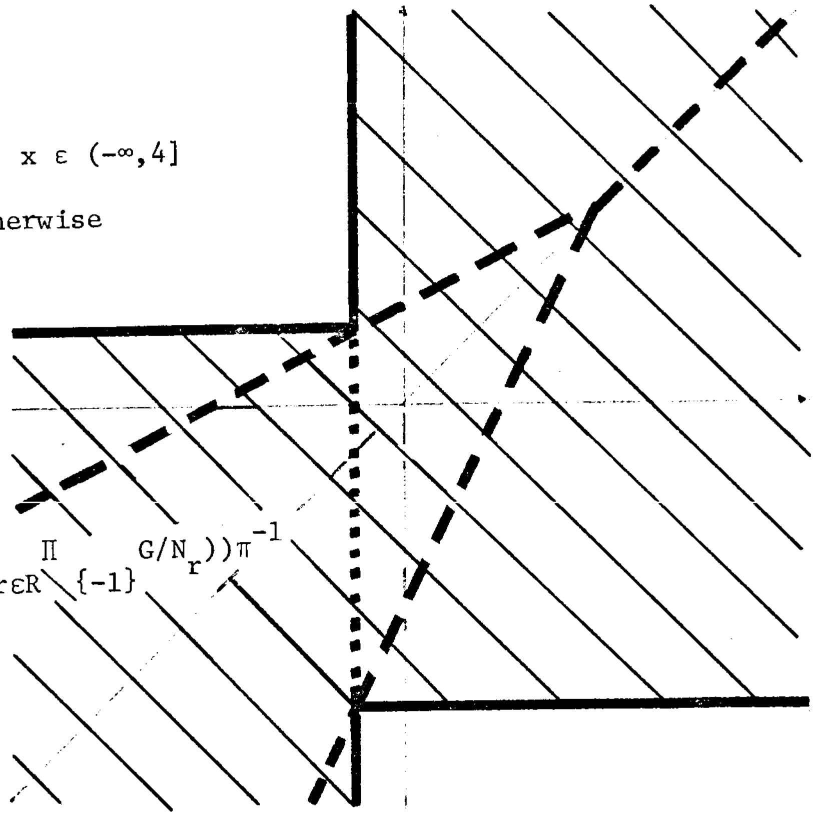


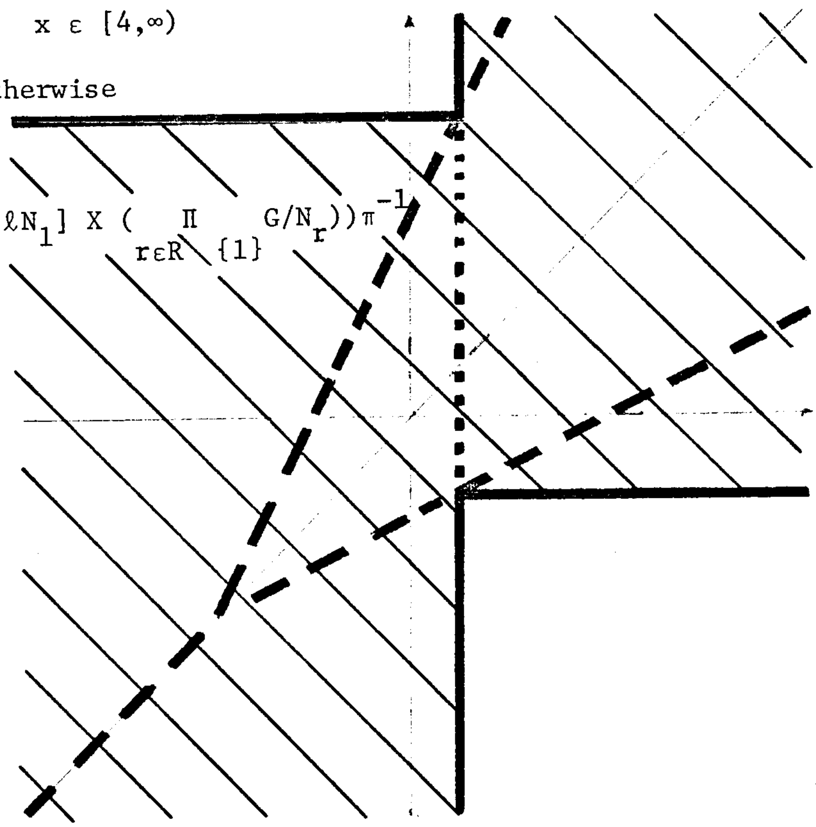
figure XXVI

$G = A(R)$

$$xh = \begin{cases} \frac{1}{2}x + 2 & \text{if } x \in (-\infty, 4] \\ x & \text{otherwise} \end{cases}$$



$$x\ell = \begin{cases} 2x + 4 & \text{if } x \in [4, \infty) \\ x & \text{otherwise} \end{cases}$$



|⊠| END |⊠|

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