THE REPRESENTATION OF A LATTICE-ORDERED GROUP AS A GROUP OF AUTOMORPHISMS

by

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ABSTRACT

The purpose of this study is to review some of the developments in the theory of lattice-ordered groups closely related to the Holland representation for lattice ordered groups. In Chapter 0, basic definitions and results required throughout this study are reviewed. Chapter 1 contains a study of regular and prime subgroups of a lattice-ordered group and concludes with the very important Holland representation theorem. In Chapter 2, the Holland representation is used to derive the very nice result: "Every lattice-ordered group can be embedded in a divisible lattice-ordered group. Finally, Chapter 3 contains a study of transitive lattice ordered groups of order preserving permutations on a totally ordered set and also a discussion of a class of simple lattice ordered groups.

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INTRODUCTION

Lattices and groups seem to provide two of the most basic tools in the study of Universal Algebra. Also, algebraic systems endowed with a partial or total order are important in many branches of Mathematics. It is therefore not surprising that there should be increasing interest in the study of lattice-ordered groups. Prior to 1941, only lattice-ordered groups which are abelian or totally ordered had been studied; the most notable contribution made up to that time was probably due to Hahn in 1907 who developed an embedding for abelian totally ordered groups. This embedding was later extended to include abelian lattice-ordered groups by P. F. Conrad, J. Harvey and C. Holland in 1963. In 1941, Garrett Birkhoff published a paper which appeared in the Annals of Mathematics (1942) and in which he investigated properties of non-abelian lattice-ordered groups. This, no doubt, formed the basis for further investigation and, since then, many of the problems and conjectures listed in the conclusion of that paper have been resolved.

In the study which follows, some of the more recent developments in the theory of lattice-ordered groups have been reviewed, and an attempt has been made to make this presentation self-contained as far as possible. However, a basic knowledge of group theory has been assumed. Chapter 0 contains basic results and definitions which are used throughout the study with references being given whenever proofs are omitted. These results, and a general basic theory of lattice-ordered groups can be found in Birkhoff's book on "Lattice theory" and in Fuch's book on "Partially ordered algebraic systems".

In Chapter 1, a detailed discussion of the properties of regular and prime subgroups of an l-group is presented. Prime subgroups are of particular importance in obtaining representations of lattice-ordered groups. For, if M is a prime subgroup of a lattice-ordered group G, then the set of cosets of M can be endowed with a natural total order. It follows that if M is both prime and normal, then the set of cosets of M is a totally ordered group. This property was utilised by Holland in his representation theorem which is discussed at the end of this chapter. The Holland representation of a lattice-ordered group is a representation of a lattice-ordered group as a subdirect sum of ΠK_β where each K_β is a transitive l-subgroup of the lattice-ordered group of all order-preserving permutations on some totally ordered set. This answered a problem originally posed by Birkhoff in the second edition of his book on lattice theory. Though this representation throws little light on the internal structure of a lattice-ordered group, it is an invaluable tool in the study of the nature and occurrence of lattice ordered groups. An example of this is given in Chapter 2 when an application of the Holland representation is used to obtain an elegant embedding theorem: "Every lattice-ordered group can be embedded in a divisible lattice-ordered group".

Finally, Chapter 3 contains a study of the lattice-ordered group of order-preserving permutations on a totally ordered set. This chapter is divided into two sections. Section I contains a study of lattice-ordered groups of order-preserving permutations on a totally ordered set, which are transitive on that set, while, in Section 2, a class of simple lattice-ordered groups is discussed. That knowledge of the properties of the lattice-ordered group of order-preserving permutations on a totally ordered set yields important information about lattice-ordered groups in general, is clear from the Holland representation theorem.

NOTATION

$x \vee y$	lub$\{x,y\}$
$x \wedge y$	glb$\{x,y\}$
$x \perp y$	x and y are disjoint
$x \parallel y$	x and y are incomparable
G^+	The positive cone of a partially ordered group G
$\mathcal{C}(G)$	The set of all convex l-subgroups of an l-group G
$\mathcal{L}(G)$	The set of all l-ideals of an l-group G
$R(C)$	The set of all right cosets of C
$A(S)$	The set of all o-preserving permutations on the totally ordered set S
$C(g)$	The convex l-subgroup generated by g
$C(M,a)$	The convex l-subgroup generated by M and a
$G \boxplus H$	The cardinal sum of l-groups G and H
$G \overrightarrow{\times} H$	The lexicographic sum of G and H ordered from the left
$G \overset{l}{\cong} H$	G is l-isomorphic to H
$N_G(M)$	The normalizer of M in G

CHAPTER 0

For a general theory of lattice-ordered groups, the reader is referred to Birkhoff (1) and Fuchs (8). Included here are some basic definitions and results which are used throughout this study. In general, additive notation is used unless otherwise mentioned.

Definition 0.1

A partially ordered group (p.o. group), G, is a set G such that

(a) G is a group;

(b) G is a partially ordered set (p.o. set) under a relation $\leq$;

(c) If $a, b \in G$ and $a \leq b$, then $c+a \leq c+b$ and $a+c \leq b+c$ for every $c \in G$.

A p.o. group which is totally ordered is an o-group.

Definition 0.2

(a) A lattice-ordered group (l-group) is a p.o. group which is also a lattice under the relation $\leq$.

(b) If G is an l-group, then a subset H of G is an l-subgroup of G if and only if H is a subgroup of G and H is also a sublattice of G.

In a lattice, L, if $x, y \in L$, then denote $\text{glb}\{x,y\}$ by $x \wedge y$ and $\text{lub}\{x,y\}$ by $x \vee y$.

Definition 0.3

(a) Let G and H be p.o. groups. Then an o-homomorphism θ from G into H is an isotone group homomorphism. That is to say θ is a group homomorphism such that for any $x, y \in G$ if $x \leq y$, then $x\theta \leq y\theta$.

(b) θ is an o-isomorphism if θ is a 1-1 o-homomorphism.

(c) If G and H are l-groups, then an l-homomorphism θ from G into H is an o-homomorphism such that for any $x, y \in G$,

$$(x \vee y)\theta = x\theta \vee y\theta \qquad \text{(i)}$$

$$(x \wedge y)\theta = x\theta \wedge y\theta \qquad \text{(ii)}$$

(d) An l-isomorphism θ from an l-group G into an l-group H is an o-isomorphism from G into H such that (i) and (ii) of (c) hold. When G and H are l-isomorphic l-groups we write $G \overset{\ell}{\cong} H$.

Remark: Clearly if G is an l-group and if H is a subgroup of G such that for each $x \in H$, $x \vee o \in H$ then it follows easily that H is an l-subgroup of G.

Some elementary properties of l-groups.

L(1) In any l-group G, addition is distributive on meets and joins. That is, if $a, x, y, b \in G$, then,

$$a+(x \vee y) = (a+x) \vee (a+y), \quad (x \vee y)+b = (x+b) \vee (y+b) \qquad (1)$$

$$a+(x \wedge y) = (a+x) \wedge (a+y), \quad (x \wedge y)+b = (x+b) \wedge (y+b) \qquad (2)$$

L(2) In an l-group G, if $a, b \in G$, then

$$a \wedge b = -(-a \vee -b) \qquad (3)$$

L(3) As a result of L(2) the mapping $\theta: G \to G$, defined on an l-group G such that $x\theta = a-x+b$, a, b, $x \in G$ is a 1-1 mapping such that for x, $y \in G$ with $x \leq y$ then $x\theta \geq y\theta$. Also if $x\theta \leq y\theta$ then this implies $x \geq y$. Such a mapping is sometimes termed a dual isomorphism.

In any l-group G, $a-(x \vee y)+b = (a-x+b) \wedge (a-y+b)$ (4)

this follows from the dual of L(2).

L(4) In any l-group G, the generalisations of (1), (2) and (4) hold. That is to say

$$a+(\bigvee x_\sigma)+b = \bigvee (a+x_\sigma+b) \quad (5)$$

$$a+(\bigwedge x_\sigma)+b = \bigwedge (a+x_\sigma+b) \quad (6)$$

$$a-(\bigvee x_\sigma)+b = \bigwedge (a-x_\sigma+b) \text{ and dually} \quad (7)$$

where σ ranges over some finite arbitrary index set.

<u>Definition 0.4</u>

If a is an element of an l-group, G, then $a^+ = a \vee o$ and $a^- = a \wedge o$; a^+ is called the <u>positive part of a</u> and a^- is called the <u>negative part of a</u>.

<u>Lemma 0.5</u>

In any l-group G, $\forall a, b \in G$, $a-(a \wedge b)+b = b \vee a$.

<u>Proof:</u> In any l-group, $x-(a \vee b)+y = (x-a+y) \wedge (x-b+y)$ for every a, x, y, $b \in G$. Therefore setting $x = a$ and $y = b$, the result follows.

<u>Corollary 0.6</u>

In a commutative l-group, G, $a+b = a \vee b + a \wedge b$ $\forall a, b \in G$.

Corollary 0.7

In any l-group G, $\forall a \in G$, $a = a^+ + a^-$.

Proof: For any $a \in G$, by substituting o for b in lemma 0.5, the result follows.

Definition 0.8

In an l-group G, if $a \in G$, then $|a|$ = absolute value of $a = a \vee -a$.

Theorem 0.9

In an l-group G, $\forall a \in G$, (i) $|a| \geq 0$, moreover $|a| > o$ unless $a = o$.

(ii) $a^+ \wedge (-a)^+ = o$

(iii) $|a| = a^+ - a^- = a \vee o - a \wedge o = a \vee o + (-a) \vee o$.

Proof: See Birkhoff (1).

Lemma 0.10

If G is an l-group, let $G^+ = \{x \in G : x \geq o\}$. Then G^+ is the positive cone (or partial order) on G.

(i) If $u, v, w \in G^+$ then $u \wedge (v+w) \leq u \wedge v + u \wedge w$.

(ii) If $u \in G^+$ then for any $v, w \in G$, $u \vee (v+w) \leq u \vee v + u \vee w$.

Proof:

(i) Since $u, v, w \in G^+$, then $u \wedge (v+w) \in G^+$. Applying (2) of L(1), $u \wedge v + u \wedge w = (u \wedge v + u) \wedge (u \wedge v + w)$

$$= 2u \wedge (u+v) \wedge (u+w) \wedge (v+w)$$

Now $u \wedge (v+w) \leq u, (v+w)$. Hence clearly

$u \wedge (v+w) \leq 2u, u+v, u+w, v+w$.

Thus $u\wedge(v+w)\leq(u\wedge v)+(u+w)$.

(ii) Applying (1) of L(1), $u\vee v+u\vee w = (u\vee v+u)\vee(u\vee v+w)$

$$= 2u\vee(u+v)\vee(u+w)\vee(v+w)$$

Since $u\in G^+$ and $2u,(v+w)\leq(u\vee v)+(u\vee w)$, then $u,v+w\leq(u\vee v)+(u\vee w)$. Since G is an l-group, the result follows.

Lemma 0.11

In an l-group G, $|a+b|\leq|a|+|b|+|a|$ for every, a, $b\in G$.

Proof: In any l-group G, $|a| = |-a|$ for each $a\in G$. Hence

$$\begin{aligned} |a|+|b|+|a| &= |-a|+|b|+|-a| \\ &= a\vee o+a\vee o+b\vee o+-b\vee o+-a\vee o+a\vee o \\ &\geq -a\vee o+(a+b)\vee o+(-b-a)\vee o+a\vee o \text{ by lemma 0.10} \\ &= -a\vee o+|a+b|+a\vee o \\ &\geq |a+b|. \end{aligned}$$

Similarly it follows that $|a-b|\leq|a|+|b|+|a|$.

Definition 0.12

Two positive elements a and b in an l-group G are called disjoint (denoted $a\perp b$) if and only if $a\wedge b = o$.

Lemma 0.13

In any l-group G, disjoint elements are permutable.

Proof: If a, $b\in G$ such that $a\wedge b = o$, then clearly $a+b = a-a\wedge b+b = b\vee a$ from lemma 0.5. But $b\vee a=a\vee b=b-b\wedge a+a=b+a$.

Thus $a+b = b+a$.

Lemma 0.14

Let G be an l-group. If a, b∈G such that $a \perp b$, then $a \vee b = a+b$.

Proof: If $a \perp b$, then $a \wedge b = o$. By lemma 0.5, $a \vee b = b \vee a = a - a \wedge b + b = a+b$.

Definition 0.15

A p.o. group, G, is Archimedean if for a, b∈G, $na \leq b$ for every integer n implies $a = o$.

The next theorem which is stated without proof is due to Holder. The proof can be found in Fuchs (8), P.45.

Theorem 0.16 (Holder)

An o-group is archimedean if and only if it is o-isomorphic to a subgroup of the additive group of real numbers with the natural ordering. Thus, all totally ordered archimedean groups are commutative.

Definition 0.17

Let S be a totally ordered set. Then a 1-1 order-preserving mapping from S onto S is called an o-permutation (automorphism).

Now consider the set of all o-permutations on a totally ordered set S. This set is denoted by A(S) and is an l-group under the following order:

For $f \in A(S)$, let $f \geq 1 \Leftrightarrow xf \geq x \ \forall x \in S$, where 1 = identity mapping on S. Verification that A(S) is an l-group under this order is routine.

Definition 0.18

Suppose G and H are l-groups:

(a) The cardinal sum of G and H denoted by $G \boxplus H$ is the direct sum of G and H with the partial order defined by $(g,h) \geq o \Leftrightarrow g \geq o$ and $h \geq o$ for $g \in G$, $h \in H$. To verify that $G \boxplus H$ is an l-group is routine.

(b) The lexicographic sum of G and H is the direct sum of G and H with the lexicographic order defined by

$$(g,h) \geq o \Leftrightarrow \text{either } h > o \text{ or } h = o \text{ and } g \geq o.$$

Then GxH is ordered lexicographically from the right and the lexicographic sum is denoted by $\overleftarrow{G \times H}$. Similarly if ordered lexicographically from the left we denote this by $\overrightarrow{G \times H}$. Again the verification that $\overleftarrow{G \times H}$ is an l-group is routine.

CHAPTER I

In this chapter, the basic results related to regular and prime subgroups of a lattice-ordered group (l-group) are stated and proved. Finally, using the fact that if C is a prime subgroup of an l-group, G, then G/C is totally ordered, we discuss representations of l-groups as groups of order-preserving permutations on a totally ordered set; the main result being the Holland representation. Unless otherwise mentioned, the results are due to Conrad (5) and (6).

Definition 1.1

A subgroup C of an l-group G is convex if for any $0<a\in C$ and $0\leq x\leq a$ this implies $x\in C$.

Definition 1.2

(a) A subgroup C of an l-group G is upward directed if for every a, $b\in C$ there exists $c\in C$ such that $a\leq c$ and $b\leq c$.

(b) A subgroup C of an l-group G is downward directed if for every a, $b\in C$ there exists $c\in C$ such that $c\leq a$, $c\leq b$.

(c) A subgroup C of an l-group G is directed if it is both upward directed and downward directed.

Lemma 1.3

For a subgroup C of an l-group G, the following are equivalent:

(1) C is a convex l-subgroup;

(2) C is a directed convex subgroup of G;

(3) C is convex and $c \vee 0 \in C$ for each $c \in C$;

(4) Let $R(C) = \{C+g : g \in G\}$, the set of right cosets of C in G. If we define $C+g \leq C+h$ to mean there exists $c \in C$ with $c+g \leq h$, then this defines a partial order on R(C) which is a lattice with $(C+x) \vee (C+y) = C+(x \vee y)$ for $x, y \in G$ and dually;

(5) If $c \in C$, $g \in G$ and $|g| \leq |c|$ then $g \in C$.

<u>Proof</u>:

(1)⇒(2). Since C is a convex l-subgroup of G, then for every $a, b \in C$, $a \vee b \in C$ and $a \wedge b \in C$. Hence C is directed and convex.

(2)⇒(3). Since $0 \in C$, (3) is trivially implied by (2).

(3)⇒(4). First we show that the order defined on R(C) is a partial order. Since $0 \in C$, and for each $C+g \in R(C)$, $0+g \leq g$, then it follows that $C+g \leq C+g$ and $\leq$ is <u>reflexive</u>. Consider $C+g \leq C+h$ and $C+h \leq C+g$ for $C+g, C+h \in R(C)$. Then there exist $c_1, c_2 \in C$ such that $c_1+g \leq h$ and $c_2+h \leq g$. Therefore $c_2 \leq g-h \leq -c_1$. Since C is a convex l-subgroup, then $g-h \in C$. Thus, $C+g = C+h$ and so $\leq$ is <u>antisymmetric</u>. If now $C+x \leq C+y$ and $C+y \leq C+z$, with $C+x, C+y, C+z \in R(C)$, then there exist $c_1, c_2 \in C$ such that $c_1+x \leq y$ and $c_2+y \leq z$. Therefore,

$$c_2+c_1+x \leq c_2+y \leq z$$

But $c_2+c_1 \in C$ and so $C+x \leq C+z$ and $\leq$ is <u>transitive</u>. Hence $\leq$ as defined is a partial order on R(C). Clearly $C+(x \vee y)$ is an upper bound for C+x and C+y. Suppose now that $C+g \geq C+x$, C+y. Then there exist $c_1, c_2 \in C$ such that $c_1+x \leq g$ and $c_2+y \leq g$.

That is $x \leq -c_1+g$ and $y \leq -c_2+g$. By hypothesis and from remark on page 2, $\exists c \in C$ such that $-c_1 \leq c$ and $-c_2 \leq c$. Then $x, y \leq c+g$. Thus, $x \vee y \leq c+g$. Hence $-c+x \vee y \leq g$ and since $-c \in C$, we get $C+x \vee y \leq C+g$ and so $C+x \vee y = (C+x) \vee (C+y)$. The dual can be shown similarly. Hence $R(C)$ is a lattice.

(4)⇒(5). If $c \in C$ and $g \in G$ and $|g| \leq |c|$, then, $-|c| = c \wedge -c \leq -|g| = g \wedge -g \leq g \leq |g| \leq |c|$. Thus $c \wedge -c \leq g \leq c \vee -c$. From (4), $C = C \wedge C = (C+c) \wedge (C-c) = C+c \wedge -c \leq C+g$. But $C+g \leq C+c \vee -c = (C+c) \vee (C-c) = C \vee C = C$. Thus $C+g = C$ and $g \in C$.

(5)⇒(1). If $0 < a \in C$ and $0 \leq x \leq a$ with $x \in G$, then, $x = |x|$, $a = |a|$ and $|x| \leq |a|$. From (5), we have $x \in C$ and so C is a convex subgroup. If $x \in C$, then $0 \leq |x^+| = x^+ \leq |x|$ and so $x^+ \in C$. Therefore C is a convex l-subgroup of G.

<u>Corollary 1.4</u>

If $A, B \in \mathcal{C}(G)$, where $\mathcal{C}(G)$ is the collection of all convex l-subgroups of an l-group G and if $A \subseteq B$, then the mapping $A+x \to B+x$ for $x \in G$ defines an lattice homomorphism from $R(A)$ onto $R(B)$.

From the definition of the partial order on $R(A)$ and $R(B)$, such a mapping is well defined. The surjectiveness follows since $A \subseteq B$.

<u>Definition 1.5</u>

A convex l-subgroup C of an l-group G is an <u>l-ideal</u> if C is also a normal subgroup of G.

Corollary 1.6

Let $M \in \mathcal{L}(G)$, where $\mathcal{L}(G)$ is the collection of all l-deals of G, then the canonical mapping of the subgroups of G containing M onto the subgroups of G/M induces a bijective correspondence between the convex l-subgroups (l-ideals) of G containing M and $\mathcal{C}(G/M)$ (respectively $\mathcal{L}(G/M)$).

Remark: If in particular $C \in \mathcal{L}(G)$, then in (5) of lemma 1.3, R(C) is an l-group.

Notation: Consider $a \in G^+$ and S a sub-semi-group of G^+ such that $0 \in S$. Then we denote the sub-semi-group of G^+ generated by S and a by $\langle S,a \rangle$. Thus $\langle S,a \rangle$ consists of all elements of the form $u_1+a+u_2+a+\ldots+u_{n-1}+a+u_n$, $u_i \in S$ for $1 \leq i \leq n$.

Lemma 1.7 (Clifford)

If M is a convex l-subgroup of an l-group G and if $a \in G^+ \backslash M$, then,

$$C(M,a) = \{x \in G : |x| \leq p \text{ for some } p \in \langle M^+,a \rangle\}$$

is the smallest convex l-subgroup of G containing M and a. If a, $b \in G^+ \backslash M$, then $C(M,a) \wedge C(M,b) = C(M,a \wedge b)$. In particular, when M = 0, $C(a) = \{x \in G : |x| \leq na \text{ for some positive integer } n\}$.

Proof: If x, $y \in C(M,a)$ then $|x| \leq p$ and $|y| \leq q$ for some p, $q \in \langle M^+,a \rangle$. Applying lemma 0.11, we get $|x-y| \leq |x|+|y|+|x| \leq p+q+p \in \langle M^+,a \rangle$. Therefore $x-y \in C(M,a)$ and so C(M,a) is a group. Now if $|g| \leq |c|$ for $g \in G$ and $c \in C(M,a)$, then there exists $r\in\langle M^+,a\rangle$ such that $|c|\leq r$. Thus $|g|\leq r$ for $r\in\langle M^+,a\rangle$. Hence $g\in C(M,a)$ and so $C(M,a)$ is a convex l-subgroup containing M and a and must be the smallest such. Now, consider $0<x\in C(M,a)\cap C(M,b)$. Then $x\leq m_1+a+m_2+a+\ldots+m_{h-1}+a+m_h$, $m_i\in M^+$, $1\leq i\leq h$ and $x\leq n_1+b+n_2+b\ldots+n_{k-1}+b+n_k$, $n_j\in M^+$, $1\leq j\leq k$. Thus $x\leq(m_1+a+\ldots+m_h)\wedge(n_1+b+\ldots+n_k)$. From Lemma 0.10, for any u, v, $w\in G^+$, $u\wedge(v+w)\leq u\wedge v+u\wedge w$. Hence x is less than or equal to a sum of positive elements of the form $m_i\wedge n_j$, $m_i\wedge b$, $a\wedge n_j$, $a\wedge b$. But all such elements belong to $C(M,a\wedge b)$ hence $C(M,a)\cap C(M,b)\subseteq C(M,a\wedge b)$. To obtain the other inclusion, consider $x\in C(M,a\wedge b)$. Then $x\leq m_1+a\wedge b+m_2+a\wedge b+\ldots+m_{r-1}+a\wedge b+m_r$, where $m_i\in M^+$, $1\leq i\leq r$. For each u, v, w, $\in G^+$, $u+v\wedge w = (u+v)\wedge(u+w)$. Therefore we get,

$$x\leq(m_1+a)\wedge(m_1+b)+(m_2+a)\wedge(m_2+b)+\ldots+(m_{r-1}+a)\wedge(m_{r-1}+b)+m_r.$$

Therefore $x\leq m_1+a+m_2+a+\ldots+m_{r-1}+a+m_r\in\langle M^+,a\rangle$ and also $x\leq m_1+b+m_2+b+\ldots+m_{r-1}+b+m_r\in\langle M^+,b\rangle$. Therefore $x\in C(M,a)\cap C(M,b)$. This completes the proof.

<u>Corollary 1.8</u>

Let $K = \cap\{C\in\mathcal{C}(G):C\neq\{0\}\}$. If $K\neq\{0\}$, then G is an o-group and K is the convex l-subgroup of G that covers zero.

<u>Proof</u>: If $K\neq\{0\}$ and G is not an o-group, then there exists a, $b\in G$ such that a and b are strictly positive elements and $a\wedge b = 0$. Since $C(a)$, $C(b)\in\mathcal{C}(G)$, then $K\subseteq C(a)\cap C(b) = C(a\wedge b) = C(0) = \{0\}$. This contradicts the

hypothesis. Therefore G must be an o-group.

Corollary 1.9

If G does not have a proper convex l-subgroup, then G is o-isomorphic to a subgroup of the reals, R.

Proof: By corollary 1.8, G is an o-group with no proper convex l-subgroups. Consider a, b$\in$G such that na$\leq$b for n = 0, $\pm$1, $\pm$2, ... Then b$\notin$C(a). Hence C(a) is a convex l-subgroup of G and C(a) $\neq$ G. Thus C(a) = {0} which implies that a = 0. Thus G is archimedean. Hence by Holder's theorem (Fuch's p.45), G is o-isomorphic to an additive subgroup of the reals.

Definition 1.10

A convex l-subgroup M of G is called regular if there exists g$\in$G such that M is maximal with respect to not containing g, and in this case M is said to be a value of g.

Lemma 1.11

Each convex l-subgroup of an l-group G is the intersection of regular convex l-subgroups of G. Each 0 $\neq$ g$\in$G has at least one value.

Proof: Let C$\in\mathcal{C}$(G). Let g$\in$G\C. By Zorn's lemma, $\exists$M$\in\mathcal{C}$(G) such that M is maximal with respect to containing C and not containing g. For, consider $\mathcal{A}$ = {S:S$\in\mathcal{C}$(G), g$\notin$S$\supseteq$C}. Let $\mathcal{L}\subseteq\mathcal{A}$, $\mathcal{L}$ = {S_i:i$\in$I} linearly ordered by set inclusion. Then, $g \notin \bigcup_{i\in I} S_i \supseteq C$ and $\bigcup_{i\in I} S_i \in \mathcal{C}(G)$ by the linear ordering. Therefore $\bigcup_{i\in I} S_i$ is an upper bound of $\mathcal{L}$ in $\mathcal{A}$ and so by Zorn's lemma, $\mathcal{A}$ has a maximal element. Let M_g be a maximal element of $\mathcal{A}$. Then M_g is regular and is a value of $g\in G\setminus C$. Now consider the set of all such M_g, that is $\{M_g : g\in G\setminus C\}$. Then, $C\subseteq M_g$ for each $g\in G\setminus C$ implies that $C\subseteq\bigcap\{M_g : g\in G\setminus C\}$. If $x\in\bigcap\{M_g : g\in G\setminus C\}$, then $x\in M_g$ for each $g\in G\setminus C$. Thus $x\notin G\setminus C$ and so $x\in C$. Hence, $\bigcap\{M_g : g\in G\setminus C\}\subseteq C$ and so $\bigcap\{M_g : g\in G\setminus C\} = C$.

<u>Definition 1.12</u>

An element a of a lattice L is called <u>meet irreducible</u> if a is not the greatest element in L and if $a < \wedge b$ ($b\in L$ and $b>a$). This is more restrictive than the usual concept of finite meet irreducible ($b,c\in L$, $b>a$, $c>a \Rightarrow b\wedge c>a$).

<u>Theorem 1.13</u>

Let $M\in\mathcal{C}(G)$, then the following conditions are equivalent:

(1) M is regular;

(2) There exists $M^*\in\mathcal{C}(G)$ such that $M\subset M^*$ and M^* is contained in every convex l-subgroup of G that properly contains M;

(3) M is meet irreducible in $\mathcal{C}(G)$ which is a lattice under set inclusion.

If $M\triangleleft G$, then each of the above conditions is equivalent to

(4) G/M is an o-group with a convex l-subgroup that covers zero.

Proof:

(1)⇒(2). Suppose M is a regular convex l-subgroup and let M be a value of $g \in G$. Let $M^* = \cap\{C \in \mathcal{C}(G) : M \subset C\}$. Then $M^* \in \mathcal{C}(G)$ and $M^* \subseteq C$ for every $C \in \{C \in \mathcal{C}(G) : M \subset C\}$. Since M is regular, for every $C \in \{C \in \mathcal{C}(G) : M \subset C\}$, $g \in C$. Thus $g \in M^* \setminus M$ and so $M \subset M^*$.

(2)⇒(3). We have that $M^* \in \mathcal{C}(G)$ such that $M \subset M^*$ where $M^* = \cap\{C \in \mathcal{C}(G) : M \subset C\}$. It follows immediately from Definition 1.13 that M is meet irreducible in $\mathcal{C}(G)$.

(3)⇒(1). By lemma 1.11, M is the intersection of regular convex l-subgroups of G. So, if M is meet irreducible, then M must be regular.

(4)⇒(2). If $M \vartriangleleft G$, then $M \in \mathcal{L}(G)$. Assuming (4), let $K = M^*/M$ be the convex l-subgroup of G/M that covers zero. Then, $M^* \in \mathcal{C}(G)$ such that $M \subset M^*$ and $\nexists C \in \mathcal{C}(G)$ such that $M \subset C \subset M^*$. Thus, M^* is contained in every convex l-subgroup of G that contains M.

(2)⇒(4). If M satisfies (2), then $M^*/M = \cap\{C/M : C \in \mathcal{C}(G), M \subset C\}$. Since $M \subset M^*$, $M^*/M \neq M/M$. By Corollary 1.8, G/M is an o-group and M^*/M is the convex l-subgroup which covers the zero.

Corollary 1.14

If M is a regular convex l-subgroup of G and $a, b \in G^+ \setminus M$ then $a \wedge b \in G^+ \setminus M$.

Proof: From lemma 1.7, $C(M, a \wedge b) = C(M,a) \cap C(M,b) \subset M$. Since M is regular, $\exists M^* \in \mathcal{C}(G)$ such that $M \subset M^*$ and $M^* \subseteq C$ for every $C \in \{C \in \mathcal{C}(G) : M \subset C\}$. Thus, $C(M, a \wedge b) = C(M,a) \cap C(M,b) \supseteq M^*$. If

$a \wedge b \in M$, then $M = C(M, a\wedge b) \supseteq M^*$ which implies that $M = M^*$. This is impossible. So $a\wedge b \notin M$ and $a \wedge b \in G^* \setminus M$.

In the following theorem, the notion of a prime l-subgroup is introduced and a number of equivalences proved. The equivalences (4), (5), and (6) have been proved by Holland while those of (1), (3), (4) and (8) have been proved by Johnson and Kist (12).

<u>Theorem 1.15</u>

Let $M \in \mathcal{C}(G)$ then the following are equivalent:

(1) If $A \cap B \subseteq M$ where $A, B \in \mathcal{C}(G)$ then $A \subseteq M$ or $B \subseteq M$;

(2) If $M \subset A$ and $M \subset B$ where $A, B \in \mathcal{C}(G)$ then $M \subset A \cap B$;

(3) If $a, b \in G^+ \setminus M$, then $a \wedge b \in G^+ \setminus M$;

(4) If $a, b \in G^+ \setminus M$, then $a \wedge b > 0$;

(5) The lattice $\mathcal{R}(M)$ of right cosets of M is totally ordered;

(6) The convex l-subgroups of G that contain M form a chain;

(7) M is the intersection of a chain of regular convex l-subgroups.

If $M \triangleleft G$, then each of the above is equivalent to

(8) G/M is an o-group.

<u>Proof</u>:

(1)⇒(2). If $M \subset A$ and $M \subset B$ for $A, B \in \mathcal{C}(G)$, then $M \subseteq A \cap B$. If $M = A \cap B$, then from (1), $A = M$ or $B = M$. This contradicts the hypothesis and so $M \subset A \cap B$.

(2)⇒(3). If $a, b \in G^+ \setminus M$, then $C(M, a \wedge b) = C(M,a) \cap C(M,b) \supset M$ since $M \subset C(M,a)$ and $M \subset C(M,b)$. Thus $a \wedge b \notin M$ and so $a \wedge b \in G^+ \setminus M$.

(3)⇒(4). This follows trivially.

(4)⇒(5). Consider $M+a$, $M+b\in\mathcal{R}(M)$ with $a, b\in G\setminus M$. Then $a = \bar{a}+a\wedge b$, $b = \bar{b}+a\wedge b$ where $\bar{a}\wedge\bar{b} = o$ for then, $a\wedge b = (\bar{a}+a\wedge b)\wedge(\bar{b}+a\wedge b) = \bar{a}\wedge\bar{b}+a\wedge b = a\wedge b$. Since $\bar{a}\wedge\bar{b} = o$, from (4), $\bar{a}\in M$ or $\bar{b}\in M$. Suppose $\bar{a}\in M$. Then $M+a = M+a\wedge b\leq M+b$. Similarly, if $\bar{b}\in M$, then we get $M+b\leq M+a$ and so it follows that $\mathcal{R}(M)$ is totally ordered.

(5)⇒(6). Assume (5) then suppose that the convex l-subgroups of G containing H do not form a chain. Then, $\exists A,B\in\mathcal{C}(G)$ such that $M\subset B$ and $A\|B$. Consider $o<a\in A\setminus B$ and $o<b\in B\setminus A$. Then we can write $a = \bar{a}+a\wedge b$, $b = \bar{b}+a\wedge b$ where $\bar{a}\wedge\bar{b} = o$. Since $\mathcal{R}(M)$ is totally ordered, we have say $M+\bar{a}\leq M+\bar{b}$. Thus $M = M+\bar{a}\wedge\bar{b} = (M+\bar{a})\wedge(M+\bar{b}) = M+\bar{a}$ and so $\bar{a}\in M\subset B$. But $B\in\mathcal{C}(G)$ and $o<a\wedge b\in B$. Therefore $a = \bar{a}+a\wedge b\in B$. This contradicts the hypothesis so $A\not\| B$.

(6)⇒(7). This follows immediately from lemma 1.11.

(7)⇒(1). Assume (7) and suppose that $\exists A$, $B\in\mathcal{C}(G)$ such that $A\cap B\subseteq M$, $A\nsubseteq M$ and $B\nsubseteq M$. Let $\{M_i : i\in I\}$ be a chain of regular convex l-subgroups of G such that $M = \bigcap_{i\in I} M_i$. Choose $a\in A^+\setminus M$ and $b\in B^+\setminus M$. Then $\exists j\in I$ such that $a, b\notin M_j$, that is, $a, b\in G^+\setminus M_j$. By Corollary 1.14, $a\wedge b\in G^+\setminus M_j$, but $A\cap B\in\mathcal{C}(G)$ and so $o<a\wedge b\in A\cap B\subseteq M\subseteq M_j$. This yields a contradiction. Hence $A\cap B\subseteq M$ implies $A\subseteq B$ or $B\subseteq M$. Finally, if $M\lhd G$, (5) and (8) are obviously equivalent.

<u>Definition 1.16</u>

A convex l-subgroup of an l-group G which satisfies any of the conditions (1) through (7) in the preceding theorem is called prime.

The above definition is certainly equivalent to the following: An l-subgroup M of G is prime if whenever $a \wedge b = o$ where $a, b \in G$ then $a \in M$ or $b \in M$. This definition is analogous to the ring theoretic definition of prime ideals "An ideal I of a ring R is prime if $ab \in I \Rightarrow a \in I$ or $b \in I$".

Remarks (1) It follows immediately from corollary 1.14 and (3) that every regular convex l-subgroup is prime.

(2) From condition (6), it follows that the partially ordered set of prime subgroups of G is a root system. [A p.o. set Δ is a root system if for each $\delta \in \Delta$, $\{\alpha \in \Delta : \alpha \geq \delta\}$ is totally ordered].

(3) From condition (7), the intersection of a maximal chain of regular subgroups is a minimal prime subgroup. Thus every prime subgroup contains a minimal prime subgroup.

(4) Every l-automorphism π of G induces an l-automorphism on the lattice $\mathcal{C}(G)$ and so also on $\mathcal{L}(G)$. If M is a prime subgroup (respectively regular), then $M\pi$ is also prime (respectively regular).

(5) The prime convex l-subgroups of an l-group G can be used to represent G as a group of o-permutations of a totally ordered set. The representation of l-groups in this manner is due to C. Holland (10) and is discussed later.

Definition 1.17

A subgroup H of a direct sum of groups πG_λ is a subdirect sum of πG_λ if for any $x_\lambda \in G_\lambda \exists c \in H$ having x_λ for its component in G_λ. That is to say, the projection $\pi_\lambda : H \to G_\lambda$ is surjective.

Definition 1.18

An l-group G is representable if there exists an l-isomorphism σ of G onto a subdirect sum of πG_λ where each G_λ is an o-group. The pair $(\sigma, \pi G_\lambda)$ is called the representation of G.

Definition 1.19

A group G is an O-group if G admits at least one total order.

Example: All free groups are O-groups (Neumann (17)).

Elementary properties of representable groups

P(1) If σ is an l-isomorphism of G into πK_λ, where each K_λ is an o-group, then $(\sigma, \pi G_\lambda)$ is a representation of G with $G_\lambda = pr_\lambda G\sigma$. [$pr_\lambda G\sigma$ denotes the λth projection of $G\sigma$].

Proof: Since σ is an l-isomorphism and each K_λ is an o-group for each λ, then $pr_\lambda G\sigma = G_\lambda$ is an l-subgroup of K_λ for each λ and hence is an o-group. Also, πG_λ is an l-subgroup of πK_λ and hence $(\sigma, \pi G_\lambda)$ is a representation of G.

P(2) Every l-subgroup of a representable l-group is representable. Every cardinal sum of representable l-groups is representable.

Proof: Let G be a representable l-group. Let $(\sigma, \pi G_\lambda)$ be a representation of G. Let H be any l-subgroup of G. Consider $j: H \to G$, the inclusion mapping. Then j is an l-isomorphism of H into G. Hence, $j\sigma: H \to \pi G_\lambda$ is an l-isomorphism into πG_λ. By P(1), since each G_λ is an o-group, then $(j\sigma, \pi H_\lambda)$ is a representation of H with $H_\lambda = pr_\lambda H(j\sigma)$. It is easily shown that a cardinal sum of representable l-groups is representable, for if A and B are representable l-groups with representations $(\sigma_a, \pi A_\lambda)$, $(\sigma_b, \pi B_\lambda)$ respectively, then $(\sigma_a \boxplus \sigma_b, \pi A_\lambda \boxplus \pi B_\lambda)$ is a representation of $A \boxplus B$, the cardinal sum of A and B.

(P3) A group G (not ordered) admits the structure of a representable group if and only if it is an O-group.

Proof: (⇐) If G is an O-group, G can be totally ordered and so admits the structure of a representable group.

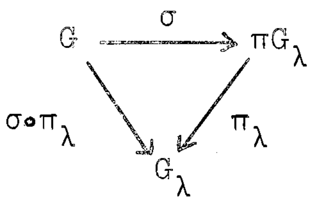
(⇒) If G is representable, let $(\sigma, \pi G_\lambda)$ be a representation of G. Define a well-order on Λ and then define a lexicographic order on πG_λ as follows: $g = (\ldots, g_\lambda, \ldots) > 0$ if $g_\lambda > 0$ where λ is the smallest index in the well-ordering of Λ. This defines a total order on πG_λ and so on $G\sigma$ and since σ is an isomorphism, then G must be an O-group.

P(4) G is representable if and only if it admits a class of prime normal subgroups whose intersection is $\{o\}$.

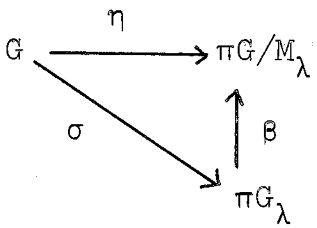
Proof: (⇐) Let $\{M_\lambda : \lambda \in \Lambda\}$ be a class of prime normal subgroups whose intersection is $\{o\}$. Then $(\sigma, \pi_{\lambda \in \Lambda} G/M_\lambda)$ is a representation of G where $\sigma: G \to \pi_{\lambda \in \Lambda} G/M_\lambda$ is defined by $g\sigma = (\ldots, M_\lambda + g, \ldots)$.

Indeed, from theorem 1.15, each G/M_λ is an o-group. If $g\sigma = \bar{o}$ where $\bar{o}$ denotes the zero for $\pi_{\lambda\in\Lambda} G/M_\lambda$, then $M_\lambda+g = M_\lambda$ for each $\lambda\in\Lambda$. That is $g\in M_\lambda$ for every $\lambda\in\Lambda$. But $\bigcap_{\lambda\in\Lambda} M_\lambda = \{o\}$ by hypothesis and so $g = o$. Thus <u>σ is injective</u>. Therefore $(\sigma,\pi G/M_\lambda)$ is a representation of G.

($\Rightarrow$) Let $(\sigma,\pi G_\lambda)$ be a representation of G. If π_λ is the projection of πG_λ onto G_λ, then, $\sigma\circ\pi_\lambda$ is an l-homomorphism of G onto G_λ.



Let Ker $\sigma\circ\pi_\lambda = M_\lambda$. Then $M_\lambda \trianglelefteq G$. If $a, b\in G\backslash M_\lambda$ then $a(\sigma\circ\pi_\lambda)$, $b(\sigma\circ\pi_\lambda)\in G_\lambda$. But G_λ is an o-group and so $a(\sigma\circ\pi_\lambda)$ and $b(\sigma\circ\pi_\lambda)$ are comparable. Thus, $M_\lambda+a$ and $M_\lambda+b$ are comparable for every $a, b\in G$. Therefore G/M_λ is an o-group and so M_λ is prime. Also the diagram below commutes.



where $\eta = \pi_{\lambda\in\Lambda} \pi_\lambda$ is canonical, and $\beta = \pi_{\lambda\in\Lambda} (\sigma\circ\pi_\lambda)$

Hence $\bigcap_{\lambda\in\Lambda} M_\lambda \subset \text{Ker}\,\sigma = \{o\}$ since σ is an isomorphism. Thus $\{M_\lambda : \lambda\in\Lambda\}$ is the required family of prime normal subgroups of G.

<u>Theorem 1.20</u>

For an l-group the following are equivalent:

(1) G is representable;

(2) G admits a family of normal prime subgroups whose

intersection is $\{o\}$;

(3) $a \wedge (-x+a+x) = o \Rightarrow a = o \quad \forall\, a,\ x \in G$;

(4) $a,\ b,\ x \in G,\ a \wedge b = o \Rightarrow a \wedge (-x+b+x) = o$;

(5) The intersection of all conjugates of a prime subgroup is prime;

(6) Every minimal prime subgroup is normal;

(7) The conjugates of a prime subgroup of G are comparable;

(8) Every regular l-ideal of G is a prime subgroup.

(An l-ideal L of G is regular, if there exists $g \in G$ such that L is maximal among those elements of $\mathcal{L}(G)$ which do not contain g).

Remark; The equivalences of (1), (3) and (4) are due to Lorenzen, the equivalence of (1) and (6) due to Sik, (1), (6) and (7) due to Byrd and (1), (8) due to Conrad.

Before proceeding with the proof of the theorem, two lemmas on conjugate subgroups of l-groups are stated. The proofs are routine and are not included.

Lemma 1.21

If $M \in \mathcal{C}(G)$ and if, for $x \in G$, M^x denotes the conjugate subgroup of M with respect to x, then $M^x \in \mathcal{C}(G)$.

Lemma 1.22

If M is a prime l-subgroup of G, then for any $x \in G$, M^x is a prime l-subgroup of G.

Proof of theorem 1.20:

(1)⇔(2) has already been established (see P(4) preceding the theorem).

(1)⇒(3). If G is an o-group, (3) follows immediately since if $a\wedge(-x+a+x) = 0$, either $a = o$ or $-x+a+x = o$ which implies that $a = o$. Thus, $a\wedge(-x+a+x) = o \Rightarrow a = o\ \forall a, x\in G$. If G is a cardinal sum of o-groups, the result follows easily by considering components. Similarly, the result follows if G is a subdirect sum of o-groups. Hence, if G is representable, let $(\sigma, \pi G_{\lambda})$ be a representation, and suppose $a\wedge(-x+a+x) = o$ for $a, x\in G$. Then, $a\sigma\wedge(-x+a+x)\sigma = o \Rightarrow a\sigma = o$ because $a\sigma$ is an element of the subdirect sum of πG_{λ}. Therefore $a = o$ since σ is an l-isomorphism.

(3)⇒(4). Suppose $a\wedge b = o$. Then $a, b\geq o$. Hence, $-x+b+x$, $x+a-x\geq o\ \forall x\in G$. Thus $a\wedge(-x+b+x)\wedge(x+a-x)\wedge b = o$. Let $g = a\wedge -x+b+x$. Then $x+g-x = x+[a\wedge(-x+b+x)]-x$

$$= (x+a-x)\wedge b.$$

Therefore $g\wedge x+g-x = a\wedge(-x+b+x)\wedge(x+a-x)\wedge b = o$. From (3) this implies that $g = a\wedge(-x+b+x) = o$.

(4)⇒(5). Let M be a prime subgroup of G. Let $J = \bigcap_{x\in G} M^x =$ intersection of all conjugates of M. Then $J \trianglelefteq G$ and $J\in\mathcal{C}(G)$, for, if $M\in\mathcal{C}(G)$ then, for all $x\in G$, $M^x\in\mathcal{C}(G)$ by lemma 1.21. Thus $J\in\mathcal{L}(G)$. To show that J is prime, we argue by contradiction. Suppose J is not prime. Then $\exists\ a, b\in G^+\setminus J$ such that $a\wedge b = o$. Consider that for some M^x, $b\notin M^x$. Then since $a\wedge b = o$, $a\wedge -g+b+g = o$ for all $g\in G$ (from (4)). By lemma 1.22, M^x is prime since M is prime. Therefore $(M^x)^g = -g+M^x+g$ is also a prime l-subgroup and, since $b \notin M^x$, $-g+b+g \notin M^{x+g}$. Therefore $a \in M^{x+g}$. Thus $a \in \bigcap_{g \in G} M^{x+g} = J$. This is a contradiction. Therefore J is a prime l-subgroup of G.

(5)⇒(6). From (5), every minimal prime subgroup must be the intersection of all its conjugates and hence must be normal.

(6)⇒(7). Let N be a prime subgroup of G. Let M be a minimal prime subgroup contained in N. Then, $\forall g \in G$, $M = -g+M+g \subseteq -g+N+g$. Since M is prime, the convex l-subgroups containing M form a chain. Hence N and $-g+N+g$ must be comparable.

(7)⇒(8). Let $M \in \mathcal{L}(G)$ such that M is maximal among the elements of $\mathcal{L}(G)$ which do not contain $a \in G$. (M is a regular l-ideal of G.). Then $\exists$ a value N of a such that $M \subseteq N$. Let $T = \bigcap_{x \in G} N^x$. Then $M \subseteq T$. But $T \in \mathcal{L}(G)$ and $a \notin N \Rightarrow a \notin T$. Since M is maximal in $\mathcal{L}(G)$ with respect to not containing a, then $T \subseteq M$. Thus $M = T$. But, from (7), every conjugate of a prime subgroup of G is comparable. Therefore T is the intersection of a chain of regular subgroups and so is prime. Hence M is prime.

(8)⇒(2). Let $\mathcal{M}$ be the family of all regular l-ideals of G. Then by (8), each $M \in \mathcal{M}$ is a prime subgroup. Also, $\bigcap_{M \in \mathcal{M}} M = \{0\}$ and so (2) is satisfied.

<u>Corollary 1.23</u>

Every commutative l-group is representable.

<u>Proof</u>: This follows immediately from conditions (4) of theorem 1.20. For, if G is a commutative l-group, then,

$\forall a, b, x \in G$, $a\wedge b = o \Rightarrow a\wedge(b+x-x) = o \Rightarrow a\wedge -x+b+x = o$. Hence G is representable.

Corollary 1.24

If G is representable and $C \in \mathcal{L}(G)$ then G/C is representable. (That is a homomorphic image of a representable l-group is representable).

Proof: Using condition (4) of the theorem, suppose $C+a, C+b \in G/C$ such that $C+a\wedge C+b = C+a\wedge b = C$. We can write $a = a\wedge b+\bar{a}$, $b = a\wedge b+\bar{b}$ where $\bar{a}\wedge\bar{b} = o$. Then,

$$\begin{aligned} C+a\wedge -(C+x)+(C+b)+(C+x) &= (C+a\wedge b+\bar{a})\wedge(C-x+b+x) \\ &= (C+a\wedge b+\bar{a})\wedge(C-x+a\wedge b+\bar{b}+x) \\ &= C+\bar{a}\wedge C-x+\bar{b}+x \text{ (since } a\wedge b \in C \trianglelefteq G) \\ &= C+(\bar{a}\wedge -x+\bar{b}+x) = C. \end{aligned}$$

This follows from (4) since $\bar{a}\wedge\bar{b} = o \Rightarrow \bar{a}\wedge -x+\bar{b}+x = o$, G being representable. Thus G/C is representable.

Corollary 1.25

Let G be a representable group, M a regular subgroup of G, M* the convex l-subgroup of G which covers M. Then the normaliser of M in G is also that of M*(in particular, $M \trianglelefteq M^*$ and M*/M is o-isomorphic to a subgroup of the real numbers).

Notation: For a group G and M or subgroup of G, denote the normaliser of M in G as $N_G(M)$. Then $N_G(M) = \{a \in G : a+M-a \subseteq M\}$.

Proof: Since M^* covers M, then $N_G(M) \subseteq N_G(M^*)$. Consider $a \in N_G(M^*)$. Then $-a+M^*+a = M^*$, for $M^* \trianglelefteq N_G(M^*)$. By condition (7) of theorem 1.20, every conjugate of a prime subgroup is comparable. Hence since M is regular, M is prime and so either $-a+M+a \subseteq M$ or $M \subseteq -a+M+a$. If $-a+M+a \subseteq M$, then $a \in N_G(M)$. Now considering the second case, we get $M \subseteq -a+M+a \subset -a+M^*+a = M^*$. Since M^* covers M, then $M = -a+M+a$ and so $a \in N_G(M)$. Hence $N_G(M) = N_G(M^*)$ as required.

Remark: It follows from corollary 1.25 above that if M is a maximal convex l-subgroup of a representable group G, then $M \trianglelefteq G$. To see this, observe that by corollary 1.25, $N_G(M) = N_G(G) = G$.

The Holland representation of an l-group as a group of permutations

Lemma 1.26

Let G be an l-group and C a prime convex l-subgroup of G. If we define a mapping π from G into the group of all permutations of $\mathcal{R}(C)$ such that $(C+x)g\pi = C+x+g$ where $x,g \in G$, then for each $g \in G$, $g\pi \in A(\mathcal{R}(C))$. Also π is an l-group homomorphism from G into $A(\mathcal{R}(C))$ = the l-group of o-preserving permutations of $\mathcal{R}(C)$.

Proof: By theorem 1.15, since C is a prime convex l-subgroup of G, $\mathcal{R}(C)$ is totally ordered. To show that, for each $g \in G$, $g\pi \in A(\mathcal{R}(C))$, consider any $C+x$, $C+y \in R(C)$. If

$(C+x)g\pi=(C+y)g\pi$, then $C+x+g=C+y+g$. Hence $C+x = C+y$ and $g\pi$ is injective. For any $C+x\in R(C)$, $C+x = C+x-g+g = (C+x-g)g\pi$. But $C+x-g\in R(C)$ and so $g\pi$ is surjective. $g\pi$ is order-preserving since if $C+x\leq C+y$ for $C+x$, $C+y\in\mathcal{R}(C)$, then $\exists c\in C$ such that $c+x\leq y$. Hence $c+x+g\leq y+g$ and so $C+x+g\leq C+y+g$. That is $(C+x)g\pi\leq(C+y)g\pi$. Thus $g\pi\in A(R(C))$. To show that π is an l-group homomorphism, consider g, $h\in G$. For any $C+x\in\mathcal{R}(C)$, $(C+x)(g+h)\pi = C+x+g+h = (C+x+g)h\pi = (C+x)g\pi h\pi$. Thus $(g+h)\pi = g\pi h\pi$ and π is a group homomorphism. Now, if 1 is the identity in $A(\mathcal{R}(C))$, we must show that for any $g\in G$, $g\pi\vee 1 = (g\vee o)\pi$. Take any $(C+x)\in\mathcal{R}(C)$. Then $(C+x)(g\pi\vee 1) = C+x+g\vee C+x = C+(x+g)\vee x = C+x+(g\vee o) = (C+x)(g\vee o)\pi$. Thus $g\pi\vee 1 = (g\vee o)\pi$. The dual is shown similarly. Hence π is an l-group homomorphism.

Remark (1): $G\pi$ is transitive on $\mathcal{R}(C)$. To see this, consider any $C+x$, $C+y\in\mathcal{R}(C)$ and notice that

$$C+y = C+x-x+y = (C+x)(-x+y)\pi.$$

Thus $G\pi$ is transitive on $\mathcal{R}(C)$.

(2) $\operatorname{Ker}\pi = \bigcap_{x\in G} -x+C+x$. To see this, consider $g\in\operatorname{Ker}\pi$. Then $(C+x)g\pi = C+x$ $\forall C+x\in\mathcal{R}(C)$. Then $C+x+g = C+x$ and so $x+g-x\in C$. Therefore, $g\in -x+C+x$ $\forall x\in G$ and so $\operatorname{Ker}\pi\subset \bigcap_{x\in G} -x+C+x$. If $g\in \bigcap_{x\in G} -x+C+x$, then, $\forall x\in G$, $\exists c_x\in C$ such that $g = -x+c_x+x$, and so $(C+y)g\pi = (C+y)(-y+c_y+y)\pi$ for any $C+y\in G$. Therefore, $(C+y)g\pi = C+c_y+y = C+y$. Thus $g\in\operatorname{Ker}\pi$ and $\operatorname{Ker}\pi = \bigcap_{x\in G} -x+C+x$.

(3) If in addition $C\triangleleft G$, then $C\in\mathcal{L}(G)$ and the diagram below commutes:

$$\begin{array}{ccc} G/C & \xrightarrow{\pi^*} & A(R(c)) \\ \delta \uparrow & & \uparrow i \\ G & \xrightarrow[\pi]{} & A(R(c)) \end{array}$$

where δ is the canonical l-homomorphism, i the identity mapping, $G/C \approx G\pi$ and π^* is defined by $(C+g)\pi^* = g\pi$. Clearly, if $C \vartriangleleft G$, G/C is an o-group. Also, for any x, $g \in G$, $(C+x)(C+g)\pi^* = (C+x)g\delta\pi^* = (C+x)g\pi$ and the diagram commutes.

<u>Theorem 1.27 (Holland) The main embedding theorem</u>.

An l-group G is l-isomorphic to a subdirect sum of πK_λ where each K_λ is a transitive l-subgroup of the l-group of all o-permutations of a totally ordered set T_λ.

<u>Proof</u>: Let $\mathcal{M} = \{M \in \mathcal{C}(G) : M \text{ a minimal prime subgroup of } G\}$. Then, for $M \in \mathcal{M}$, $G/M = \mathcal{R}(M)$ is totally ordered. From lemma 1.26, $\pi_M : G \to A(R(M))$ is an l-homomorphism and $G\pi_M$ is transitive on R(M). So, define $\varphi : G \to \pi_{M \in \mathcal{M}} A(R(M))$ such that $g\varphi = (\ldots, g\pi_M, \ldots)$. Then φ is an l-homomorphism. φ is also injective, for since each $M \in \mathcal{M}$ is a minimal prime subgroup, then $M = \bigcap_{x \in G} M^x$. Thus $M = \text{Ker } \pi_M$. Also, for any g, $h \in G$, $g\varphi = h\varphi \Leftrightarrow g\pi_M = h\pi_M, \forall M \in \mathcal{M}$

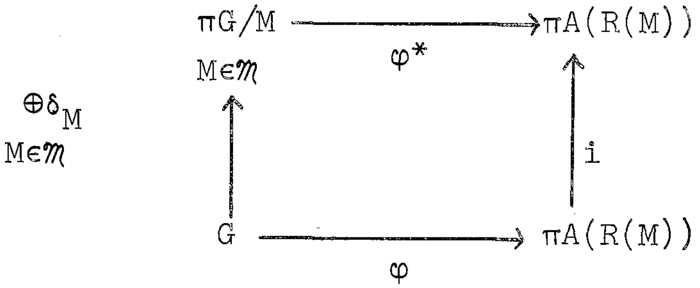
$$\Leftrightarrow (g-h) \in \text{Ker } \pi_M = M, \forall M \in \mathcal{M}$$

$$\Leftrightarrow (g-h) \in \bigcap_{M \in \mathcal{M}} M$$

But $\bigcap_{M \in \mathcal{M}} M = \{o\}$ = intersection of minimal prime subgroups of G. Therefore $g\varphi = h\varphi \Leftrightarrow g = h$. Thus φ is an l-isomorphism of G into $\pi_{M\in\mathcal{M}} A(R(M))$ and therefore fulfills the conditions of the theorem.

Remark (1): For the class $\mathcal{M}$ of minimal prime subgroups of G one can take any class of prime subgroups of G having as their intersection $\{o\}$.

(2) If G is representable, then every minimal prime subgroup of G is normal. Hence the diagram below commutes.



where $\delta_M: G\to G/M$ is the canonical l-homomorphism, i the identity map, φ the mapping defined in theorem 1.27, and φ^* defined by $(\ldots, M+g, \ldots)\varphi^* = (\ldots, (M+g)\pi^*_M, \ldots)$ where $\pi^*_M: G/M\to A(R(M))$ is defined by $(M+x)((M+g)\pi^*_M) = M+x+g$ as in remark 3 following lemma 1.26. It is immediate from that remark that the diagram commutes.

Theorem 1.28 (Holland)

Every l-group is l-isomorphic to an l-subgroup of A(T) where T is a totally ordered set.

Proof: Let $\mathcal{M} = \{M\in\mathcal{C}(G): M$ a minimal prime subgroup of G$\}$. Define a well order $\prec$ on $\mathcal{M}$. Let $T = \{M+x: M\in\mathcal{M}, x\in G\}$. Define a partial order $\leq$ on T such that $M+x \leq N+y$ if $M<N$ or if $M = N$ and $M+x \leq M+y$. Then $(T,\leq)$ is a totally ordered set. Now, for each $g \in G$, define $g\sigma$ as follows:

$$(M+x)g\sigma = M+x+g.$$

Then as in lemma 1.26, it is easily verified that σ is an l-isomorphism of G into A(T).

The following theorem, a reformulation of theorem 3 in Holland (10) answered the question "What l-groups are transitive groups of automorphisms of totally ordered sets?"

Theorem 1.29

For an l-group G, the following conditions are equivalent:

(1) $\exists$ an l-isomorphism $\sigma: G \to A(T)$ where T is a totally ordered set such that $G\sigma$ is transitive on T;

(2) $\exists$ a prime l-subgroup C of G such that $\{o\}$ is the only normal subgroup of G contained in C;

(3) $\exists$ a prime subgroup C of G such that $\{o\}$ is the only l-ideal contained in C;

Proof:

(2)$\Rightarrow$(3). Assuming (2), if $\exists H \neq \{o\}$ such that $C \supset H \in \mathcal{L}(G)$, then $H \triangleleft G$ and this contradicts (2).

(3)$\Rightarrow$(1). Assume (3). Let $\pi: G \to A(R(C))$ be the l-homomorphism described in lemma 1.26. Then Ker $\pi \subset \bigcap_{x \in G} C^x = \{o\}$ (by (3)). Thus Ker $\pi = \{o\}$ and π is an l-isomorphism of G into A(R(C)) and R(C) is totally ordered since C is prime. Also by lemma 1.26, $G\pi$ is transitive on $\mathcal{R}(C)$. Thus we have (1).

(1)⇒(2). Let $\sigma: G \to A(T)$ be an l-isomorphism such that $G\sigma$ is transitive on T where T is a totally ordered set. For $t \in T$, let $C_t = \{g \in G : t(g\sigma) = t\}$. Then C_t is a prime convex l-subgroup. To see this notice that $o\sigma = 1$ and so $o \in C_t$. Also if a, $b \in C_t$, then $t(a-b)\sigma = t(a\sigma)(b^{-1}\sigma) = t$. So $a - b \in C_t$. Therefore C_t is a <u>subgroup of G</u>. If $c \in C_t$, then $t(c \vee o)\sigma = t(c\sigma \vee 1) = t(c\sigma) \vee t = t$. Thus $c \vee o \in C_t$ and C_t is an <u>l-subgroup</u> of G. C_t is convex: Consider $o < a \in C_t$. If $o < x < a$, then $1 < x\sigma < a\sigma$ and so $t \le t(x\sigma) \le t(a\sigma) = t$. Thus $t(x\sigma) = t$ and $x \in C_t$. C_t is prime: If a, $b \in G^+ \setminus C_t$, then $t(a\sigma) \ne t \ne t(b\sigma)$. Also $t < t(a\sigma)$, $t(b\sigma)$. Therefore, since T is totally ordered, $t(a \wedge b)\sigma) = t(a\sigma \wedge b\sigma) = t(a\sigma) \wedge t(b\sigma) \ne t$. Thus $a \wedge b \notin C_t$ and so $a \wedge b \in G^+ \setminus C_t$ and C_t is prime. To show that $\{o\}$ is the only normal subgroup of G contained in C_t, suppose $H \lhd G$ and $H \subseteq C_t$. Then $\forall g \in G$, $H^g = H \subseteq C_t$. Therefore $H \subseteq \bigcap_{g \in G} C_t^{\,g}$. Thus it is sufficient to show that $\bigcap_{g \in G} C^g = \{o\}$: To do this, proceed as follows: Consider $o \ne c \in G$. Then $c\sigma \ne 1$ and so $\exists s \in T$ such that $s(c\sigma) \ne s$. Since $G\sigma$ is transitive on T, $\exists g \in G$ such that $t(g\sigma) = s$. Then,

$t(y+c-g)\sigma = t(g\sigma)(c\sigma)(g\sigma)^{-1} = s(c\sigma)(g\sigma)^{-1} \ne s(g\sigma^{-1}) = t$, therefore $g+c-g \notin C_t$ and $c \notin -g+C_t+g = C_t^{\,g}$. Hence $\bigcap_{g \in G} C_t^{\,g} = \{o\}$. This completes the proof.

<u>Corollary 1.30</u>

Let L be the intersection of all non-zero l-deals of G. If $L \ne 0$, then condition (1) holds. In particular, a simple l-group is l-isomorphic to a transitive l-subgroup of the group of all o-permutations of a totally ordered set.

Proof: $L = \bigcap_{o \neq S \in \mathcal{L}(G)} S \neq \{o\}$. Consider $o \neq a \in L$ and let C be a value of a. Then, $a \notin \bigcap_{g \in G} C^g = \{o\}$ since this is an l-ideal which does not contain L. Also, C being regular is also prime and hence $\mathcal{R}(C)$ is totally ordered and $\sigma: G \to A(\mathcal{R}(C))$ is an l-isomorphism with $G\sigma$ being transitive on $\mathcal{R}(C)$.

Corollary 1.31

Let G be an l-subgroup of A(T) which is transitive on T. If G is representable, then G is an o-group.

Proof: From theorem 1.29, $\exists$ a prime l-subgroup C of G such that {o} is the only normal subgroup of G contained in C. But C contains a minimal prime subgroup M of G and since G is representable, then $M \triangleleft G$. Hence $M = \{o\}$ and $G = G/M$ is an o-group.

Remark: Conversely, we have that if G is an o-group, then G is l-isomorphic to a transitive l-subgroup of A(T) where T is a totally ordered set. This is clear for if ρ_G denotes the group of right translations on G, then ρ_G is an l-subgroup of A(G). Also, $\rho_G \overset{l}{\cong} G$ and ρ_G is transitive on G.

CHAPTER 2

Definition 2.1

A group G is divisible if, for each $g\in G$ and each positive integer n, there exists $h\in G$ such that $nh = g$.

B. H. Neumann (16) has proved that every group can be embedded in a divisible group. As an application of his representation theorem for l-groups, Holland (10) proved the analogue to this, namely that "every l-group can be embedded in a divisible l-group". However, his proof depended on the existence, for arbitrarily large ordinals α, of η_α-sets of power $\aleph_\alpha$, a set-theoretic restriction. (There exists an η_α-set of power $\aleph_\alpha$ exactly when $\aleph_\alpha$ is a regular cardinal such that $2^{\aleph_\beta}\leq\aleph_\alpha$ whenever $\beta<\alpha$. For $\alpha = \beta+1$, this is the form of the generalised continuum hypothesis; $2^{\aleph_\beta} = \aleph_{\beta+1}$).
The defect was first noticed by Lloyd who, in (13), described several classes of l-groups which could be embedded in divisible l-groups. The proof which is presented here is due to E. C. Weinberg (20) who proves the existence of such embeddings using nothing more than the axiom of choice.

Definition 2.2

For a totally ordered set L, and for A(L), the l-group of o-preserving permutations of L, let $g\in A(L)$. Then for any $x\in L$, an interval of g, $I_g(x)$ is given by
$I_g(x) = \{y\in L: \exists m,n \text{ integers such that } xg^n\leq y\leq xg^m\}$.
An interval containing more than one point is a supporting interval of g and the union of supporting intervals is the

support of g.

From this definition, it is clear that for $g\in A(L)$ and any $x\in L$, $I_g(x)$ is a convex subset of L and that $I_g(x)g = I_g(x)$. Also, if $y\in I_g(x)$, then $I_g(x) = I_g(y)$. Hence these intervals determine an equivalence relation on L defined by:

$$x\sim y \Leftrightarrow I_g(x) = I_g(y).$$

Lemma 2.3 (Holland)

Let S be a totally ordered set in which any two non-trivial closed intervals are isomorphic. Then A(S) is divisible.

Proof: Let $g\in A(S)$ and consider n an integer such that $n>o$. Without loss of generality, we may assume $g>1$ and also that g has only one supporting interval. By the hypothesis, if $x, y\in S$ and if $x<y$, then $\exists z\in S$ such that $x<z<y$. Let $a_o<a_og$ for some $a_o\in S$. Choose

$$a_o<a_1<a_2<\ldots a_{n-1}<a_n = a_og<a_1g = a_{n+1}<a_2g\ldots$$

Since any two non-trivial closed intervals in S are isomorphic, $\exists$ isomorphisms

$$\varphi_i:(a_{i-1},a_i] \longrightarrow (a_i,a_{i+1}] \ , \ 1\leq i\leq n-1.$$

Define

$$\varphi_n:(a_{n-1},a_n] \longrightarrow (a_n,a_1g] \text{ by}$$
$$x\varphi_n = x\varphi_{n-1}^{-1}\ \varphi_{n-2}^{-1}\cdots\varphi_1^{-1}g.$$

Now, let $\varphi^*:(a_o,a_n] \longrightarrow (a_1,a_1g]$ be the extension of all the φ_i for $1\leq i\leq n$. Then φ^* is an isomorphism. By assumption, the support of $g = I_g(a_o)$. Define f as follows:

$$xf = \begin{cases} x \text{ if } x\notin I_g(a_o) \\ xg^{-m(x)}\varphi^*g^{m(x)} \text{ if } x\in I_g(a_o), \end{cases}$$

where if $x \in I_g(a_o)$, then $m(x)$ is the unique integer not necessarily positive such that $a_o g^{m(x)} < x \leq a_o g^{m(x)+1} = a_n g^{m(x)}$ namely the greatest integer such that $x > a_o g^{m(x)}$. Then, $f \in A(S)$ and for $x \notin I_g(a_o)$, $xf^n = x = xg$. Also, for $x \in I_g(a_o)$, it can be shown by routine computation that

$xf^n = xg^{-m(x)} \varphi^{*n} g^{m(x)} = xg$. Hence $f^n = g$ and so $A(S)$ is divisible and the proof is complete.

Remark: If in a totally ordered set S any two non-trivial closed intervals are isomorphic, then this is equivalent to saying that A(S) is doubly transitive (o-2-transitive) on S. (For the definition of o-2-transitive see Chapter 3, Definition 3.1.3(b)). Hence we have that if A(S) is doubly transitive on a totally ordered set, S, then A(S) is divisible.

Lemma 2.4

If F is a totally ordered field, then A(F), the o-group of o-preserving permutations, is doubly transitive on F.

Proof: Consider any a, b, c, $d \in F$ with $a < b$ and $c < d$. Then $b-a, d-c > o$. Define a mapping α such that

$$x\alpha = (x-a)(d-c)(b-a)^{-1}+c.$$

Then $\alpha \in A(F)$ for if $x, y \in F$ and $x\alpha = y\alpha$, this gives $(x-a)(d-c)(b-a)^{-1}+c = (y-a)(d-c)(b-a)^{-1}+c$. Therefore $x = y$ and so α is 1-1. If $x \leq y$, then $x-a \leq y-a$. Hence, $(x-a)(d-c)(b-a)^{-1} \leq (y-a)(d-c)(b-a)^{-1}$ since $d-c, b-a > o$. Therefore, $x\alpha \leq y\alpha$ and so α is o-preserving. Also, $a\alpha = c$ and $b\alpha = d$. Thus, $\exists \alpha \in A(F)$ such that $a\alpha = c$, $b\alpha = d$ and so

A(F) is doubly transitive on F.

To complete the proof of the main theorem, we need only show that every totally ordered set S can be embedded in a totally ordered field F in such a way that A(S) is l-isomorphic to an l-subgroup of A(F). Then, since A(F) is doubly transitive by lemma 2.4, lemma 2.3 yields that A(F) is divisible. An application of the Holland embedding theorem then completes the proof. The next lemma yields the required embedding.

Definition 2.5

For a partially ordered set S, the subset c of S is an ideal of S if and only if $x \in c$, $t \leq x \Rightarrow t \in c$.

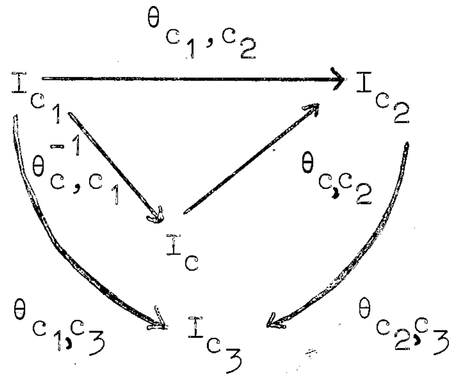
Remark: If S is a totally ordered set, then the set of all ideals is complete and is totally ordered by set inclusion since for $c_1, c_2 \in C(S)$ = the set of all ideals of S, then either $c_1 \leq c_2$ or $c_2 \leq c_1$.

Lemma 2.6

Let S be a subset of a totally ordered set T. If every o-permutation of S can be extended to an o-permutation of T, then A(S) is l-isomorphic to an l-subgroup of A(T).

Proof: Let C(S) be the complete totally ordered set of ideals of S partially ordered by inclusion. Identify the elements of S with the principal ideals of S (for any $a \in S$, $\hat{a} = \{s \in S : s \leq a\}$ is the principal ideal generated by a). If $c \in C(S)$, define

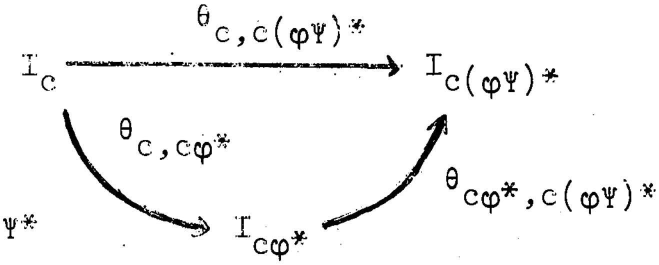
$I_c = \{t \in T : s_1 < t \le s_2$ whenever $s_1 \in c$ and $s_2 \notin c$, $s_i \in S, 1 \le i \le 2\}$. Now, for any $t \in T$, $t \in I_c$ where $c = \{s \in S : s < t\}$, then consider any $c \ne c_1 \in C(S)$. Since $C(S)$ is totally ordered, we have $c \subset c_1$ or $c_1 \subset c$. It is easily seen that $t \notin I_{c_1}$. Hence, $\{I_c : c \in C(S)\}$ forms a decomposition of T. Define the relation ρ on $C(S)$ as follows: for c_1, $c_2 \in C(S)$, $c_1 \rho c_2 \Leftrightarrow I_{c_1}$ is isomorphic to I_{c_2}. Then ρ defines an equivalence relation on $C(S)$. From each equivalence class E, we choose a representative c say. Let $\theta_{c,c}$ denote the identity map on I_c and let θ_{c,c_1} be an isomorphism of I_c onto I_{c_1}. If $c_1, c_2 \in E$ define $\theta_{c_1,c_2} : I_{c_1} \longrightarrow I_{c_2}$ in the natural way such that the upper section of the diagram below commutes, that is,



$\theta_{c_1,c_2} = \theta^{-1}_{c,c_1}\theta_{c,c_2}$. Then if $c_1 \rho c_2$ and $c_2 \rho c_3$, it follows naturally that $\theta_{c_1,c_3} = \theta_{c_1,c_2}\theta_{c_2,c_3}$. Let $\varphi \in A(S)$ and let φ^* be the unique automorphism of $C(S)$ which extends φ such that $c\varphi^* = \bigvee\{s\varphi : s \in c\} = \{s\varphi : s \in c\}$. Then the mapping $\Phi : A(S) \longrightarrow A(C(S))$ defined by $\varphi\Phi = \varphi^*$ is an l-isomorphism of $A(S)$ into $A(C(S))$. For each $c \in C(S)$, we have $c\rho(c\varphi^*)$, for clearly I_c and $I_{c\varphi^*}$ are isomorphic since φ can be extended to an o-isomorphism of T. In fact, if Ψ is any automorphism of T which extends φ, then Ψ maps I_c onto $I_{c\varphi^*}$. To see this, consider $t \in I_c$ and suppose that $s_1 \in c\varphi^*$, $s_2 \notin c\varphi^*$. Then $s_1\varphi^{-1} \in c$ and $s_2\varphi^{-1} \notin c$. Therefore, $s_1\varphi^{-1} < t \le s_2\varphi^{-1}$. Hence,

$s_1 < t\Psi \le s_2$ which gives $t\Psi \in I_{c\Psi^*}$. Therefore Ψ maps I_c into $I_{c\varphi^*}$ and since $c \rho c\varphi^*$, it follows that Ψ maps I_c onto $I_{c\varphi^*}$. Define $\varphi' \in A(T)$ such that $t\varphi' = t\theta_{c,c\varphi^*}$ if $t \in I_c$. Then the mapping $\alpha: A(S) \to A(T)$ defined by $\varphi\alpha = \varphi'$ is the required l-isomorphism of $A(S)$ into $A(T)$. To show this, consider $\varphi, \Psi \in A(S)$. If $x \in I_c$,

$$x((\varphi\Psi)\alpha) = x(\varphi\Psi)' = x\theta_{c,c(\varphi\Psi)^*}$$
$$= x\theta_{c,c\varphi^*\Psi^*}$$
$$= x\theta_{c,c\varphi^*}\theta_{c\varphi^*,c\varphi^*\Psi^*}$$
$$= x\varphi'\Psi' = x(\varphi\alpha)(\Psi\alpha)$$



Thus α is a homomorphism. If now, $\varphi \in \text{Ker}\,\alpha$, then, $\forall x \in S$, $x(\varphi\alpha) = x\varphi' = x$. But if $x \in I_c$, then, $x\varphi' = x\theta_{c,c\varphi^*} = x$. Hence $c\varphi^* = c$ and so $x\varphi = x$. Thus α is a group isomorphism.

Now, let e be the identity automorphism on S. Consider $\varphi \in A(S)$. We must show that $(\varphi \vee e)\alpha = \varphi' \vee e'$. Take $x \in I_c$. Note that $x\varphi' \le x \Leftrightarrow c\varphi^* \le c$. Thus, if $x\varphi' \le x$, then, $x(\varphi \vee e)' = x\theta_{c,c(\varphi\vee e)^*} = x\theta_{c,c} = x = x(\varphi' \vee e')$. If $x > x\varphi'$, then, $x(\varphi\vee e)' = x\theta_{c,c(\varphi\vee e)^*} = x\theta_{c,c\varphi^*} = x\varphi' = x(\varphi' \vee e')$. Thus, $(\varphi \vee e)' = \varphi' \vee e'$. Hence α is an l-isomorphism and the proof is complete.

<u>Notation</u>: If G is a totally ordered group and F is a field, the group algebra of G over F is given by the set of all formal sums $\sum_{g\in G} c_g g$ where $c_g \in F$ and $c_g \ne 0$ for at most a finite number of $g \in G$. Define addition such that

$\sum_{g\in G} c_g g + \sum_{g\in G} d_g g = \sum_{g\in G} (c_g + d_g) g$. Also, if $\lambda \in F$, then

$\lambda \sum_{g\in G} c_g g = \sum_{g\in G} (\lambda c_g) g$. Multiplication is defined such that

$(\Sigma c_g g)(\Sigma d_h h) = \Sigma c_g d_h gh$. Then F(G), the group algebra of G over F is an integral domain. To see this, suppose $\Sigma c_g g, \Sigma d_h h \in F(G)$ such that $\Sigma c_g g \neq o \neq \Sigma d_h h$. Then let $G_1 = \{g \in G; c_g \neq o\}$ and $G_2 = \{h \in G: d_h \neq o\}$. Pick $g' \in G_1$ and $h' \in G_2$ such that for every $g \in G_1$, $g \leq g'$ and for every $h \in G_2$, $h \leq h'$. Then, for every $g \in G_1$ with $g < g'$ and every $h \in G_2$ with $h < h'$, $gh < g'h'$. Also $c_{g'} d_{h'} g'h' \neq o$. Hence $\Sigma c_g d_h gh \neq o$. Thus F(G) is an integral domain.

Lemma 2.7

Any totally ordered set S may be embedded in a totally ordered field F in such a way that each automorphism of S may be extended to an o-preserving automorphism of F.

Proof: Let G(S) denote the free abelian group on S as the set of free generators. The order on S may be extended to G(S) as follows:

$\Sigma n_i s_i > o$ if $n_j > o$ and at most finite number of $n_i \neq o$ where $s_j = \vee\{s_i : n_i \neq o\}$. Let F be the quotient field of the group algebra Q[G(S)] of G(S) over the field of rational numbers. Order Q[G(S)] lexicographically such that:

$\sum_{g \in G(S)} q_g x^g > o$ if $q_h > o$ when $h = \vee\{g \in G(S) : q_g \neq o\}$. This order has a unique extension to an order on F (Fuchs P.109 Theorem 3). Then the canonical extensions of an automorphism φ of S first to G(S), then to Q[G(S)] and finally to F are certainly o-preserving whenever φ is.

Theorem 2.8

Every l-group can be embedded in a divisible l-group.

Proof: Applying the Holland representation theorem, every l-group can be embedded in an l-group A(T) of o-permutations of some totally ordered set T. By lemmas 2.3, 2.5 and 2.6, we may assume that T is doubly transitive. Then by lemma 2.2 due to Holland, A(T) is a divisible l-group. This completes the proof.

Remark: The above construction can also be used to embed any l-group in the l-subgroup of bounded o-permutations of a totally order field. Hence the result: 'Every l-group can be embedded in a simple l-group.'

CHAPTER III

As mentioned in the introduction, the set A(S) of order preserving permutations (o-permutations or automorphisms) of a totally ordered set S becomes an l-group if the group operation is taken as composition and the partial order is defined as follows:

for $f \in A(S)$, $f \geq 1$ if and only if $xf \geq x \ \forall x \in S$.

Then, for $f, g \in A(S)$ and $\forall x \in S$,

$x(f \vee g) = xf \vee xg$ and $x(f \wedge g) = xf \wedge xg$.

In this chapter, a theory of transitive o-permutation groups is first developed. This was due to C. Holland (11) and the theory is somewhat analogous to the general theory of permutation groups. In section 2, a class of simple l-groups each containing an insular element (defined later) is shown to be just the simple l-groups which can be represented as o-permutations of a totally ordered set with bounded support. In conclusion, examples of such groups are given. The theory here is due to C. Holland (9).

Section I.

In this section, unless otherwise stated, G is an l-subgroup of A(S), the l-group of o-permutations on a totally order set S. Multiplicative notation is used in discussions of l-permutation groups.

Definition 3.1.1

Let G be an l-subgroup of A(S). A convex congruence on S (with respect to G is an equivalence relation $\sim$ on S such that

(i) $x\sim y \Rightarrow xg\sim yg \quad \forall g\in G$

(ii) if $x\leq y\leq z$ and $x\sim z$, then $x\sim y$.

A convex congruence is non-trivial if one of the equivalence classes contains more than one element and is not all of S.

Lemma 3.1.2

If $\sim$ is a convex congruence on S, then $S/\sim$ is totally ordered by letting $(x\sim)\leq(y\sim)$ if $x\leq y$ or $x\sim y$. There is a natural l-homomorphism of G into $A(S/\sim)$ such that for $g\in G$, $g\to g'\in A(S/\sim)$ with $(x\sim)g' = xg\sim$.

Proof: $\leq$ as defined above is reflexive for clearly $x\sim x$ and so $(x\sim)\leq(x\sim)$. Now, if $(x\sim)\leq(y\sim)$ and $(y\sim)\leq(x\sim)$, then, either $x\leq y$ and $y\leq x$ or $x\sim y$ or $y\sim x$. In any of these cases, it follows that $(x\sim) = (y\sim)$ and so $\leq$ is antisymmetric. If $(x\sim)\leq(y\sim)$ and $(y\sim)\leq(z\sim)$, then consider the following cases:

Case (1): $x<y$ and $y<z$ which implies $x<z$ and so $(x\sim)\leq(z\sim)$.

Case (2): $x<y$ and $y\sim z$ which implies $(y\sim) = (z\sim)$ and so $(x\sim)\leq(z\sim)$.

Case (3): $x\sim y$ and $y<z$ which implies $(x\sim) = (y\sim)$ and so $(x\sim)\leq(z\sim)$.

Case (4): $(x\sim y)$ and $y\sim z$ which implies $x\sim z$ and so $(x\sim)\leq(z\sim)$.

Hence $\leq$ is transitive and therefore a partial order. Since S is totally ordered, for any $(x\sim)$, $(y\sim)\in S/\sim$, either $x<y$, $y<x$ or $x = y$ and so either $(x\sim)\leq(y\sim)$, $(y\sim)\leq(x\sim)$ or $(x\sim) = (y\sim)$. Thus $S/\sim$ is totally ordered by $\leq$ as defined in

the lemma. Now, consider $\theta: G \to A(S/\sim)$ defined such that $g\theta = g'$ where $(x\sim)g' = (xg)\sim$. θ is a homomorphism, for, consider any $g,h \in G$ then $(x\sim)g^{\theta}h^{\theta} = ((xg)\sim)h^{\theta} = (xgh)\sim = (x\sim)(gh)\theta$. Therefore $(gh)\theta = g\theta h\theta$. Consider the identity permutation $1 \in G$ and take any $g \in G$. Then,

$$(x\sim)(g \vee 1)\theta = x(g \vee 1)\sim = (xg \vee x)\sim = (xg\sim) \vee (x\sim) = (x\sim)g^{\theta} \vee (x\sim)1'$$
$$= (x\sim)g\theta \vee 1'$$

Therefore $(g \vee 1)\theta = g\theta \vee 1' = g' \vee 1'$ where $1'$ is the identity in $A(S/\sim)$. Similarly $(g \wedge 1)\theta = g' \wedge 1'$ and so θ is an l-homomorphism.

Definition 3.1.3

(a) G is transitive on S if for each $x,y \in S$ there exists $g \in G$ such that $xg = y$.

(b) G is o-2-transitive on S if for each $x,y,z,w \in S$, if $x<y$ and $z<w$ then there exists $g \in G$ such that $xg = z$ and $yg = w$.

Example: (a) If G is a totally ordered group, the group ρ_G of right translations of G is an l-subgroup of A(G) and is transitive on G. Notice that for any $x,y \in G$, $x\rho_{-x+y} = x-x+y = y$. Notice also that G is not o-2-transitive

(b) An o-2-transitive l-group: If C is a totally ordered field, then A(F) is o-2-transitive (Lemma 2.4).

Definition 3.1.4

G is o-primitive on S if there exist no non-trivial convex congruences on S.

<u>Remark</u>: It is shown later that any o-2-transitive l-group is o-primitive.

<u>Definition 3.1.5</u>

G is <u>weakly o-primitive</u> on S if, whenever $\sim$ is a non-trivial convex congruence on S, $\exists$ $1 \neq g \in G$ such that $x \sim xg$ $\forall x \in S$ (i.e. the natural l-homomorphism θ of lemma 3.1.2 fails to be injective). In this case S is said to be <u>minimal</u> for G.

<u>Remark</u>: If G is <u>not</u> weakly o-primitive. Then for some convex congruence θ is injective and we say <u>S can be reduced to $S/\sim$</u>.

<u>Definition 3.1.6</u>

C is a <u>representing subgroup</u> of G if C is a convex prime l-subgroup of G which contains no l-ideal of G other than $\{1\}$.

At this point recall Theorem 1.29 which can be restated as follows: an l-group G has a representing subgroup if and only if $\exists$ an l-isomorphism σ of G into A(T) where T is a totally ordered set and such that $G\sigma$ acts transitively on T.

<u>Lemma 3.1.7</u>

There exists a one-to-one l-homomorphism $i_s : A(S) \rightarrow A(\overline{S})$ where $\overline{S}$ denotes the completion of S by Dedekind cuts (without end-points).

<u>Proof</u>: For $g\epsilon A(S)$ and $a\epsilon\overline{S}$, define i_s such that $a(gi_s) = \vee\{xg : x\epsilon S,\ x\leq a\}$. Then clearly $gi_s\epsilon A(\overline{S})$. Also, any two elements of $A(\overline{S})$ which agree on S must agree on $\overline{S}$ also (from the definition above) and so must be equal.

i_s is injective for consider $g\epsilon \text{Ker } i_s$. Then $gi_s = 1_{\overline{S}}$.
For every $a\epsilon S$, $agi_s = a = \vee\{xg : x\epsilon S,\ x\leq a\}$

$= ag$ since g preserves order.

Hence $g = 1_s$ and so i_s is one-to-one. To show that i_s is an l-homomorphism, take any $x\epsilon S$ and $g,h\epsilon A(S)$. Then, $x(gi_s)(hi_s) = x\,gh = x((gh)i_s)$. Therefore $(gi_s)(hi_s)$ and $(gh)i_s$ agree on S and hence must be equal. Also, $x((g\vee h)i_s) = x(g\vee h) = xg\vee xh = x(gi_s)\vee x(hi_s) = x(gi_s\vee hi_s)$. Again $(g\vee h)i_s$ and $(gi_s\vee hi_s)$ agree on S and therefore must be equal. Similarly, $(g\wedge h)i_s = gi_s\wedge hi_s$ and so i_s is a 1-1 l-homomorphism.

<u>Notation</u>: For $x\epsilon\overline{S}$, $g\epsilon A(S)$ instead of $x(gi_s)$ one usually writes xg and it is assumed that $A(S)\subseteq A(\overline{S})$.

<u>Remark</u>: It is clear that for every $\alpha\epsilon A(S)$ $\exists$ $\overline{\alpha}\epsilon A(\overline{S})$ such that $\overline{\alpha}|_S = \alpha$. However, the converse of the lemma does not hold. That is to say there does not exist a 1-1 function mapping $A(\overline{S})\rightarrow A(S)$. To see this, consider the totally ordered set of all rational numbers Q and the dedekind completion of the rationals to the reals, $\overline{Q}$. Then $\exists\alpha\epsilon A(\overline{Q})$ defined such that

$$x\alpha = \begin{cases} \sqrt{x} & \forall\, x\geq 0 \\ x & \forall\, x<0 \end{cases}$$

However, α cannot be restricted to Q since all positive rational numbers which are not perfect squares are mapped to irrational numbers.

Lemma 3.1.8

Let G be transitive on S, and E be a convex subset of S. If $a\in E$ is such that if $g\in G$, $ag\in E$ then $Eg = E$, then E determines a convex congruence $\sim$ on S defined by $x\sim y$ if for some $g\in G$, $x,y\in Eg$.

Proof: For each $g\in G$, Eg is convex since E is convex. Since G is transitive, $S = \bigcup_{g\in G} Eg$. Now, if $x\in Eg\cap Ef$, then $\exists e\in E$ such that $eg = x$. But G is transitive on S. Hence $\exists h\in G$ such that $ah = e$. Then, $ahgf^{-1} = egf^{-1} = xf^{-1}\in E$ since $x\in Ef$. Thus, $Ehgf^{-1} = E$ which gives $Ehg = Ef$. But since $ah = e\in E$, then $Eh = E$. Therefore $Ehg = Eg = Ef$. So $\sim$ is an equivalence relation on S.

Finally, if $x,y\in Eg$, then for any $f\in G$, $xf, yf\in Egf$. Since for every $g\in G$, Eg is convex, then $\sim$ is a convex congruence on S.

Notation: Consider G an l-subgroup of A(S) for some totally ordered set S. Then for any $a\in S$, $G_a = \{g\in A(S): ag = a\}$.

Lemma 3.1.9

If $\sim$ is a convex congruence on S, a totally ordered set, if $a\in S$, and if $C = \{g\in G: ag\sim a\}$, then C is a convex prime l-subgroup of G and $G_a \subseteq C$. Conversely, if G is transitive on S and C is a convex l-subgroup of G containing G_a, then the relation $x \sim y$ if for some $g \in G$, $xg, yg \in aC$ is a convex congruence on S.

Proof: Clearly $C \subseteq G$. For any $g,h \in C_1$ then $ag \sim a \sim ah$. Since $\sim$ is a convex congruence, then $agh^{-1} \sim a$ and so $gh^{-1} \in C$. Therefore C is a subgroup of G. For any $g \in C$, $a(g \vee 1) = ag \vee a \sim a$ and so $g \vee 1 \in C$. Similarly, $g \wedge 1 \in C$ and so C is an l-subgroup of G. C is convex, for consider $1 < g \in C$. If $h \in G$ such that $1 < h < g$, then $a < ah < ag$ and since $\sim$ is a convex congruence and since $g \in C$ implies $ag \sim a$, then $a \sim ah$. Therefore $h \in C$. C is a prime l-subgroup for, suppose $f,g \in G$ such that $f \wedge g = 1$, then $af \wedge ag = a$ and since S is totally ordered, either $af = a$ or $ag = a$. Thus either $f \in C$ or $g \in C$ and C is prime. Since $G_a = \{g \in G : ag = a\}$ then clearly $G_a \subseteq C$.

Conversely, let $E = aC$. Then, for $f, g \in C$, $af, ag \in aC$. Therefore, if for some $x \in S$, $af \leq x \leq ag$, then since G is transitive, $\exists h \in G$ such that $x = ah$. Thus $a((f \vee h) \wedge g) = a((f \wedge g) \vee (g \wedge h)) = x$. Also, $g \wedge f \leq (g \wedge f) \vee (g \wedge h) \leq g$. Since C is convex, then $(f \vee h) \wedge g \in C$ and so $x \in aC$. Therefore E is a convex subset of S. To show that $\sim$ is a convex congruence, we apply lemma 3.1.8. Hence it must be shown that if for some $g \in G$, $ag \in E$ then $Eg = E$. Suppose for some $g \in G$, $ag \in aC = E$. Then $\exists h \in C$ such that $ag = ah$. Therefore $agh^{-1} = a$ and so $gh^{-1} \in G_a \subseteq C$. Since $h \in C$, then $g \in C$. Hence $Eg = aCg = aC = E$. Therefore E determines a convex congruence $\sim$ on S defined as stated in the lemma. Also, $C = \{g \in G : ag \sim a\}$. This is easily verified.

Remark: It follows easily from the fact that the convex l-subgroups of G containing a prime l-subgroup C form a chain (Theorem 1.15), that if G is transitive on S, then the convex congruences on S form a tower. This is proved in detail later.

Corollary 3.1.10

For each $x \in \overline{S}$, G_x is a convex prime l-subgroup of G.

Proof: Replace S by $\overline{S}$ and assume as earlier mentioned that $A(S) \subset A(\overline{S})$. Then define $\sim$ on $\overline{S}$ such that $x \sim y$ if and only if $x = y$. Clearly $\sim$ is a convex congruence on $\overline{S}$. But $G_x = \{ g \in G : xg = x \}$. Hence by lemma 3.1.9, G_x is a convex prime l-subgroup of G.

Lemma 3.1.11

Let $a \in S$, and let K be a convex subgroup of G containing G_a and such that for any $x \in S$ $\exists f \in K$ with $x \leq af$. Then $K = G$.

Proof: We have $K \subseteq G$. Let $1 < g \in G$ and let $f \in K$ such that $ag \leq af$. If $h = (g \vee f)f^{-1}$, then $ah = aff^{-1} = a$ and so $h \in G_a \subseteq K$. Thus $g \vee f = ((g \vee f)f^{-1})f = hf \in K$ and also $1 \leq g \leq g \vee f \in K$. Since K is convex, then $g \in K$. Hence $K = G$.

In the following definitions, S is a totally ordered set and T is a subset of S.

Definition 3.1.12

For $x,y \in S$, x and y are T-connected if for every $g \in G$ either $xg, yg \in T$ or $xg, yg \notin T$.

Definition 3.1.13

T is bounded if $\exists a,b \in S$ such that $a \leq t \leq b$ for all $t \in T$.

Definition 3.1.14

T is dense in S if whenever $a<b<c$ with $a,b,c \in S$ $\exists t \in T$ such that $a<t<b$.

Theorem 3.1.15

If G is transitive on S, the following are equivalent:

(1) G is o-primitive on S;

(2) For each $a \in \overline{S}$, G_a is a maximal convex l-subgroup of G;

(3) For each $a \in S$, G_a is a maximal convex l-subgroup of G;

(4) If $x \in \overline{S}$, xG is dense in $\overline{S}$;

(5) If T is a convex bounded subset of S, then no two different elements of S are T-connected.

Proof:

(1)$\Rightarrow$(2). Let $a \in \overline{S}$. By Corollary 3.1.10, G_a is a convex prime l-subgroup of G. Let C be a convex l-subgroup of G such that $G_a \subset C$. Let $1<g \in C \setminus G_a$. Then $a<ag$. Hence $\exists b \in S$ such that $a \leq b \leq ag$. Let $1 \leq f \in G_b$. Then $af \leq bf = b \leq ag$. So $afg^{-1} \leq a$. Hence $(fg^{-1} \vee 1) \in G_a$. Therefore $1 \leq f \leq f \vee g = (fg^{-1} \vee 1)g \in G_a C \subseteq C$. Since C is convex, then $f \in C$ and so $G_b \subseteq C$. By lemma 3.1.9, C determines a

convex congruence $\sim$ on S such that $C = \{g\in G: bg\sim b\}$. Since $b<bg\sim b$ and since G is o-primitive on S, then for every $y\in S$, $b\sim y$. Therefore, for every $y\in S$, $\exists h\in C$ such that $bh = y$. Thus, by lemma 3.1.11, $C = G$. Hence G_a is maximal.

(2)$\Rightarrow$(3). This is immediate.

(3)$\Rightarrow$(4). Let $x\in\overline{S}$. Define $\sim$ on S by $z\sim y$ if $\nexists g\in G$ such that xg lies between z and y. Then $\sim$ is a convex congruence on S, for it is easily verified that $\sim$ is an equivalence relation and, if $z\sim y$, suppose for some $h\in G$ $zh\not\sim yh$, then $\exists g\in G$ such that xg lies between zh and yh. We may assume without loss of generality that $zh<xg<yh$. Then $z<xgh^{-1}<y$ and $gh^{-1}\in G$ which contradicts $z\sim y$. Hence for every $h\in G$, $zh\sim yh$. Also, if for y_1, y_2, $y_3\in S$, $y_1\leq y_2\leq y_3$ and $y_1\sim y_3$ then $\nexists g\in G$ such that xg lies between y_1 and y_3. If $y_1\not\sim y_2$, then $\exists h\in G$ such that $y_1\leq xh\leq y_2$. But $y_2\leq y_3$ hence $y_1\leq xh\leq y_3$ and $y_1\not\sim y_3$, a contradiction. Therefore $y_1\sim y_2$ and $\sim$ is a convex congruence.

Now, let $a\in S$. By lemma 3.1.9, $C = \{g\in G: ag\sim a\}$ is a convex l-subgroup of G containing G_a. If $G = C$, then since G is transitive on S, $aC = S$ and all elements of S are equivalent. But $x\in\overline{S}$ so this is impossible. Hence $G\neq C$ and so $G_a = C$. If $a\sim b\in S$, then since G is transitive, $\exists g\in G$ such that $ag = b$ and so $g\in C = G_a$. Hence $a = b$. Therefore xG is dense in $\overline{S}$.

(4)$\Rightarrow$(5). Let T be a convex bounded subset of S. Let $x = \mathrm{glb}T\in\overline{S}$. If a, $b\in S$ and the open interval $(a,b)\neq\emptyset$, then, since xG is dense in $\overline{S}$, $\exists g\in G$ such that $a<xg<b$. Then $ag^{-1}<x<bg^{-1}$. Hence $\exists t\in T$ such that $t<bg^{-1}$. Since G is

transitive, $\exists f\in G$ such that $f\leq 1$ and $t = bg^{-1}f$. Thus $ag^{-1}f\leq ag^{-1}<x\leq t = bg^{-1}f<bg^{-1}$. Since $x = \text{glb } T$, then $ag^{-1}f\notin T$ but $bg^{-1}f = t\in T$ and so a and b are not T-connected. If now $a<b$ and $\nexists x\in S$ strictly between a and b, then, since G is transitive on S $\exists c\in S$ such that $a<b<c$ and $\nexists y\in S$ strictly between b and c. Since xG is dense in $\overline{S}$, for some $g\in G$, $xg = b$. Since no element of S lies strictly between $x = bg^{-1}$ and ag^{-1}, then $x\in T$. But $ag^{-1}<x = bg^{-1}$. So $ag^{-1}\notin T$ and again a and b are not T-connected.

(5)$\Rightarrow$(1). Suppose G is not o-primitive on S. Then $\exists$ a non-trivial congruence on S, and there are at least two congruence classes say T and Q where $q<t$ $\forall g\in Q$ and $t\in T$. Thus, any $q\in Q$ is a lower bound for T. Since G is transitive, for some $q\in Q$ and $g\in G$, $qg\in T$. Hence qg^2 is an upper bound for T. Thus T is a bounded convex subset of S. Since $\exists$ some congruence classes with more than one element, by transitivity $\exists$ $a,b\in T$ with $a\neq b$. But, for every $f\in G$, either $af, bf\in T$ or $af, bf\notin T$. Hence a and b are T-connected and this contradicts (5). Therefore G must be o-primitive. This completes the proof.

<u>Corollary 3.1.16</u>

If G is o-2-transitive on S, then G is o-primitive on S.

<u>Proof</u>: Suppose G is not o-primitive on S. Then from condition (5) of theorem 3.1.15, if T is a convex bounded subset of S such that for $x,y\in S\setminus T$, $x<t<y$ for each $t\in T$, then $\exists$ some $a,b\in S$ with $a<b$ such that a and b are T-connected.

Case (1) If for every $g \in G$, ag, $bg \in T$, then $x < ag < bg < y$ $\forall y \in G$. But G is o-2-transitive on S and so $\exists h \in G$ such that $xh = ag$ and $bgh = y$. But then, $x < xh = ag < agh < bgh = y$ and so $agh \in T$ but $bgh \notin T$. This is a contradiction. Now, if for every $g \in G$. ag. $bg \notin T$, consider the following:

Case (2) $ag < bg < x < t < y$ for each $g \in G$ and $t \in T$. Then $\exists h \in H$ such that $agh = t$ and $bgh = y$ (since G is o-2-transitive). Thus $agh \in T$, $bgh \notin T$ and again this is a contradiction.

Case (3) $ag < x < t < y < bg$ for each $g \in G$ and $t \in T$. Then since G is o-2-transitive, $\exists h \in H$ such that $agh = t$ and $yh = bg$. Then $ag < x < t = agh < y < bg = yh < bgh$ and again $agh \in T$ but $bgh \notin T$. Thus this contradicts the hypothesis. Hence G is o-primitive on S.

Remark: The converse of this corollary is false as the example following the next corollary shows.

Corollary 3.1.17

If $S = \overline{S}$ and G is transitive on S, then G is o-primitive on S.

Proof: G is transitive on $\overline{S}$ and so for $x \in \overline{S}$, $xG = \overline{S}$. Hence, xG is dense on $\overline{S}$. It follows immediately from theorem 3.1.15 that G is o-primitive on S.

Remark: G is o-primitive on S $\not\Rightarrow$ G is transitive on S. To see this, consider the following example: Consider the totally ordered group of real numbers R under addition. Let $G = \{ \rho_q \in A(R) : x\rho_q = x+q,\ x \in R$ and q a rational number$\}$. Then G is o-primitive. However, G is not transitive on R for given $x,y \in R$ with x rational and y irrational, there exists no $\rho_q \in G$ such that $x\rho_q = y$. Clearly since G is not transitive, then G is not doubly transitive. Hence this example also shows that G is o-primitive $\not\Rightarrow$ G is doubly transitive.

Corollary 3.1.18

If G is transitive and o-primitive on S, then for every $x \in \overline{S}$, G_x is a representing subgroup of G.

Proof: By corollary 3.1.10 for each $x \in \overline{S}$, G_x is a convex prime l-subgroup of G. Therefore we need only show that $\nexists$ l-ideal of G_x except $\{1\}$. From Theorem 3.1.15, for each $x \in \overline{S}$, xG is dense in $\overline{S}$. Now, if $1 \neq g \in G_x$, then $\exists a \in S$ such that $a \neq ag$. Since xG is dense in $\overline{S}$, for some $f \in G$, xf lies between a and ag, say $a \leq xf \leq ag$. Then $a \leq xf \leq ag \leq xfg$ and so $xf \neq xfg$ or $x \neq xfgf^{-1}$. Hence $fgf^{-1} \notin G_x$. Therefore G_x contains no normal subgroup of G except $\{1\}$ and G_x is a representing subgroup of G.

Lemma 3.1.19

Let $\sim$ be a convex congruence on S and let

$$H = \{g \in G : x \sim xg\ \forall x \in S\}.$$

Then H is an l-ideal of G.

Proof: Clearly H is a subgroup of G. For any $h \in H$, $x \sim xh \; \forall x \in S$. In particular, since $xg \in S \; \forall g \in G$, then $xg \sim xgh$. Thus $x \sim xghg^{-1}$ and so $ghg^{-1} \in H$ for every $g \in G$ and $h \in H$. Hence $H \triangleleft G$. Also for all $x \in S$, $x(h \vee 1) = xh \vee x \sim x$ since $h \in H$. Therefore $h \vee 1 \in H$. Similarly, $h \wedge 1 \in H$ and so H is a sublattice of G. H is convex, for if $1 < h \in H$ and $g \in H$ with $1 \leq g \leq h$, then $x \leq xg \leq xh$. But $\sim$ is convex and $x \sim xh$. Therefore $x \sim xg$. Hence $g \in H$. Thus H is an l-ideal of G.

Theorem 3.1.20

Let G be transitive on S. Then the following are equivalent:

(1) G is weakly o-primitive;

(2) For each $a \in S$, G_a is a maximal representing subgroup of G;

(3) If $x \in \bar{S}$ and xG is not dense in $\bar{S}$, then $\exists 1 \neq g \in G$ such that $y \in xG$, $yg = y$;

(4) If T is a convex bounded subset of S, and if $\exists\, a, b \in S$, $a \neq b$, with a and b T-connected, then $\exists\, 1 \neq g \in G \ni \forall\, x \in S$, x and xg are T-connected.

Proof:

(1)$\Rightarrow$(2). Since G is transitive on S, and $G \subseteq A(S)$, then by Theorem 1.29, G contains a representing subgroup. Also, from the proof of that theorem, $G_a = \{g : ag = a\}$ is a representing subgroup of G. It remains only to show that G_a is maximal. Let C be a convex l-subgroup of G and $G_a \subseteq C$, and let $\sim$ be the convex congruence on S determined by C as in lemma 3.1.9. If $\sim$ is trivial, then as in the proof of theorem 3.1.15, either $C = G_a$ or $C = G$. If $\sim$ is non-trivial, then $\exists 1 \neq g \in G$ such that $x \sim xg$ for every $x \in S$. Therefore $\{1\} \neq H = \{g \in G : x \sim xg \ \forall x \in S\}$. But by lemma 3.1.19, H is an l-ideal of G. Moreover $H \subseteq C$. Thus C is not a representing subgroup of G. Therefore G_a is a maximal representing subgroup.

(2)$\Rightarrow$(3). If $x \in \overline{S}$ and xG is not dense in $\overline{S}$, let $a \in S$ and define $\sim$ on S such that $z \sim y$ if $\nexists g \in G$ such that xg lies between z and y. Then, as in part (3) of the proof of 3.1.15, $\sim$ is a convex congruence on S and $G_a \subset C = \{g \in G : ag \sim a\}$ where C is a convex prime l-subgroup of G. Since G_a is a maximal representing subgroup, C contains an l-ideal $H \neq \{1\}$ of G. Consider $1 < g \in H$. If for some $xf \in xG$, $xfg \neq xf$, then $\exists b \in S$ such that $b < xf < bg$. Since G is transitive on S, $\exists k \in G$ such that $bk = a$. Then, $ak^{-1}gk = bgk > xfk > bk = a$. Hence $k^{-1}gk \notin C$. This contradicts the existence of $H \neq \{1\}$. Therefore $xfg = xf \ \forall xf \in xG$.

(3)$\Rightarrow$(4). Let $x = \text{glb}\, T \in \overline{S}$ where T is a convex bounded subset of S. Suppose $a, b \in S$, $a < b$ and a and b are T-connected. Since G is transitive, $\exists f \in G$ such that $af = b$. Then b and bf are T-connected and so a and bf are T-connected. Also, $a < b = af < bf$. By Theorem 3.1.15, $\nexists xg \in xG$ such that $a < xg < bf$. In particular xG is not dense in $\overline{S}$. Therefore from (3) $\exists 1 \neq g \in G$ such that for all $y \in xG$, $yg = y$. Without loss of generality, $g > 1$. Let $z \in S$. Clearly $\nexists f \in G$ such that $z < xf < zg$, for if there exists such an $f \in G$, then since $xfg = xf$, this gives $z = zg$. Hence if $zgh \in T$ for some $h \in G$, then $zh \in T$ for otherwise $zh \leq x \leq zgh$ which gives $z \leq xh^{-1} \leq zg$ and $xh^{-1} \in xG$.

Similarly if $x' = \text{lub}T$, a similar argument shows that $zgh \in T$ whenever $zh \in T$. Hence z and zg are T-connected.

(4)⇒(1). Let ~ be a non-trivial convex congruence on S. As in the proof of part (5) theorem 3.1.15, there are at least two congruence classes say Q and T where $q<t$ $\forall q \in Q$ and $t \in T$. Thus T is bounded below by some $q \in Q$ and since G is transitive, for some $q \in Q$ and $g \in G$, $qg \in T$. Hence qg^2 is an upper bound for T. Thus T is a convex bounded subset of S and since ~ is non-trivial $\exists a, b \in T$ with $a \neq b$. Thus a and b are T-connected. From (4), $\exists 1 \neq g \in G$ such that $\forall x \in S$, xg and x are T-connected. Now, since G is transitive, for any $x \in S$, $\exists f \in G$ such that $xf \in T$. Hence $xfg \in T$ and so $xf \sim xgf$. Therefore $x \sim xg$. Thus G is weakly o-primitive.

<u>Definition 3.1.21</u>

The support of an element $h \in A(S)$, denoted $\sigma(h)$, is given by $\sigma(h) = \{ x \in S : xh \neq x \}$.

The lemma which follows gives a necessary condition for the l-group of all o-permutations on a totally ordered set S if transitive on S, not to be totally ordered.

<u>Lemma 3.1.22</u>

If A(S) is transitive on S, a totally ordered set, and not totally ordered, then A(S) contains an element $g \neq 1$ of bounded support.

<u>Proof</u>: Suppose $\exists f \in A(S)$ and a, b, $c \in \bar{S}$ with $a<b<c$ and

$af = a$, $bf \neq b$, and $cf = c$. Define a mapping g such that $xg = xf$ if $a \leq x \leq c$ and $xg = x$ otherwise. Then $1 \neq g \in A(S)$ and g has bounded support. If no such f exists, then since $A(S)$ is not totally ordered, $\exists f_1, f_2 \in A(S)$ such that $f_1 \wedge f_2 = 1$ and $f_1 \neq 1 \neq f_2$. Then, $\exists a \in \bar{S}$ such that $a(f_1 \vee f_2) = a$. In fact, the fixed points of $f_1 \vee f_2$ form a closed bounded interval with respect to the interval topology say $[y, y'] \subseteq \bar{S}$. Suppose $\sigma(f_1) \subseteq \{x \in \bar{S} : x < y\}$ and $\sigma(f_2) \subseteq \{x \in \bar{S} ; y < x\}$. As $A(S)$ is transitive on S, $\exists h \in A(S) \ni y'h < y$. Let $f_2' = h^{-2} f_2 h^2$. Then $\sigma(f_2') = \{x \in \bar{S} : y'h^2 < x\}$. Therefore $\sigma(f_1 \wedge f_2') = \{x \in \bar{S} : y'h^2 < x < y\}$. Clearly $y'h \in \sigma(f_1 \wedge f_2')$. Hence $\sigma(f_1 \wedge f_2') \neq \emptyset$. Thus $1 \neq f_1 \wedge f_2'$ has bounded support.

The next theorem generalises the following results:

(1) "If S is a Dedekind complete totally ordered set, if S is not discrete, and if A(S) is transitive on S, then A(S) is o-2-transitive on S". This result is due to Treybig.

(2) "If S is a totally ordered set and if A(S) is transitive on S and if $A(S)_x$ is a maximal convex l-subgroup of A(S), then A(S) is totally ordered or A(S) is doubly transitive on S". This result is due to Lloyd (15).

<u>Theorem 3.1.23</u>

If G is an l-subgroup of A(S), if G is transitive and o-primitive on S, and if G contains an element $1 < g$ whose support is bounded below (or above) then G is o-2-transitive on S.

<u>Proof</u>: Consider $1<g\in G$ such that $\sigma(g)$ is bounded below. Let $a = \text{glb}\,\sigma(g) \in \bar{S}$. First it is shown that if x, c, $d\in S$ and $x<c\leq d$, then $\exists f'\in G_x$ such that $cf'\geq d$. It follows that since G is transitive on S, $\exists 1<f''\in G$ such that $cf'' = d$. Now, let $f = f'\wedge f''$. Then $cf = cf'\wedge cf'' = d$ and also $f\in G_x$. Hence, for x, c, $d\in S$ and $x<c\leq d$, then $\exists f\in G_x$ such that $cf = d$. Since G is transitive on S, it follows easily that G is o-2-transitive on S.

To prove the assertion that "if x, c, $d\in S$ and $x<c\leq d$, then $\exists f'\in G_x$ such that $cf'\geq d$", we proceed as follows: Suppose $y\in\bar{S}$ and $x<y$. By theorem 3.1.15, aG is dense in $\bar{S}$. Therefore, $\exists f\in G$ such that $x\leq af<y$. Now, $af = \text{glb}\,\sigma(f^{-1}gf)$, for, if not, then $\exists b\in S$ such that $af<b<s$ for all $s\in\sigma(f^{-1}gf)$. But $b\notin\sigma(f^{-1}gf)$ implies that $bf^{-1}gf = b$. Therefore $aff^{-1} = a<bf^{-1} = bf^{-1}g$. Thus $bf^{-1}\notin\sigma(g)$ which contradicts $a = \text{glb}\,\sigma(g)$. Hence $af = \text{glb}\,\sigma(f^{-1}gf)$. Since $x\leq af<y$, then $xf^{-1}gf = x \notin \sigma(f^{-1}gf)$ and also $\exists w\in S$, $af<w<y$ such that $w<wf^{-1}gf$. Now $y\in\bar{S}$ and so yG is dense in $\bar{S}$. Hence $\exists h\in G$, $1\geq h$ such that $w<yh<wf^{-1}gf$. Let $k = hf^{-1}gfh^{-1}$. Then it follows that $wh^{-1}<y<wf^{-1}gfh^{-1} = wh^{-1}k$. Since $xh\leq x\leq af$, then $xk = xhf^{-1}gfh^{-1} = xhh^{-1} = x$. In particular, $y<wh^{-1}k\leq yk$.

Now, let $U = cG_x$. If U has an upper bound, let $y = \text{lub}\,cG_x$. As before, $\exists k\in G_x$ such that $y<yk$. Then, $x<yk^{-1}<y$. Since $y = \text{lub}\,cG_x$, $\exists r\in G_x$ such that $yk^{-1}<cr$. Then, $rk\in G_x$ and $y<crk\in cG_x$, a contradiction. Thus cG_x has no upper bound and so the proof is complete.

Remark: If $\sim$ and $\approx$ are convex congruences on S with respect to G and if we define $\sim\leq\approx$ if $a\sim b\Rightarrow a\approx b$, then the set $\mathcal{C}$ of all convex congruences on S forms a lattice in which the maximal and minimal elements o and I respectively are the two trivial congruences on S.

The proof of this is routine and is not done here. It follows from the above remark that G is o-primitive on S exactly when the cardinality of $\mathcal{C}$ is 2.

Definition 3.1.24

G is locally o-primitive on S when there is a unique minimal non-0 element of $\mathcal{C}$, the lattice of all convex congruences on S with respect to G.

Definition 3.1.25

The congruence classes of a unique minimal non-0 convex congruence are called the primitive segments of S.

Lemma 3.1.26

If G is transitive on S, the convex congruences on S form a tower.

Proof: If $a\in S$, by Corollary 3.1.10, G_a is a convex prime l-subgroup of G. Therefore by Theorem 1.15, the set of convex l-subgroups of G containing G_a form a tower under inclusion. Hence it is necessary only to show that the correspondence established in lemma 3.1.9 between the convex congruences on S and the convex l-subgroups of G containing G_a is 1-1 and order preserving.

Let σ and ρ be any two convex congruences on S with respect to G. Let C_σ and C_ρ be convex l-subgroups of G each containing G_a and determined by G_a. If $\sigma \neq \rho$, suppose $C_\sigma = C_\rho$. Then $\exists x,y \in S$ such that $x\sigma y$ let $x \not\rho y$. But then $\exists g \in G$ such that $xg, yg \in aC_\sigma = aC_\rho$. Hence $x\rho y$. This yields a contradiction. Therefore $C_\sigma \neq C_\rho$.

On the other hand, if C and C' are convex l-subgroups of G containing G_a and such that C and C' determine the same congruence, then, for each $g \in C$, $\exists f \in C'$ such that $af = ag$. Therefore $gf^{-1} \in G_a \subseteq C'$ and so $g \in C'$. Thus $C \subseteq C'$. Similarly it follows that $C' \subseteq C$ and so $C = C'$. That the correspondence preserves order is clear. For, if $\sigma, \rho \in \mathcal{C}$ and if $C_\sigma \subseteq C_\rho$, then $\sigma \leq \rho$ and conversely.

<u>Definition 3.1.27</u>

A convex l-subgroup K of G is said to <u>cover</u> the convex l-subgroup H of G if K properly contains H and there is no other convex l-subgroup of G between K and H.

<u>Theorem 3.1.28</u>

Let G be transitive on S. The following are equivalent:

(1) G is locally o-primitive on S;

(2) For each $a \in S$, G_a is covered by a convex l-subgroup K(a) of G;

(3) $\exists$ $a \in S$ such that G_a is covered by a convex l-subgroup K(a)

of G.

In (2) and (3), the subgroup K(a) is unique.

<u>Proof</u>:

(1)⇒(2). By lemma 3.1.26, the convex congruences on S form a tower. From (1), ∃! minimal non-0 element of the lattice of all convex congruences on S with respect to G. From the proof of 3.1.26, it follows that for $a \in S$, G_a is covered by a convex l-subgroup K(a) of G.

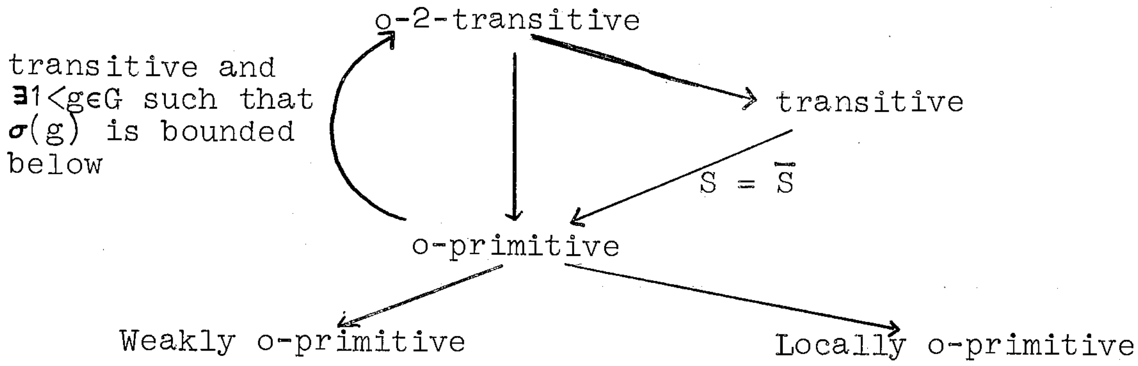
(2)⇒(3). This is immediate. Also K(a) must be unique as the proof of 3.1.26 indicates.

(3)⇒(1). If $\exists a \in S$ such that G_a is covered by a unique convex l-subgroup K(a) of G, then it is immediate from the proof of 3.1.26 that ∃! non-0 minimal convex congruence on S corresponding to K(a). Thus G is locally o-primitive.

<u>Remark</u>: If G is an l-group, let $1 \neq g \in G$. and let C be a regular l-subgroup of G not containing g. Then C is prime. If $K = \cap\{S: S \in \mathcal{C}(G)$ such that $C \cup \{g\} \subseteq S\}$, then K covers C. Theorem 1.27 can now be restated as follows:

"An l-group G is l-isomorphic to a subdirect sum of πK_β where each K_β is a transitive locally o-primitive l-subgroup of $A(S_\beta)$ where each $S_\beta = K_\beta / C_\beta$, C_β being a regular l-subgroup of K_β".

In conclusion, we present a diagram of implications together with some examples.



Examples

(1) An o-primitive l-group which is not o-2-transitive: The example which follows Corollary 3.1.17 is sufficient. Also the o-group of right translations of a totally ordered abelian group which is "full" in the sense of Cohn (4) is also o-primitive but not o-2-transitive.

(2) A weakly and locally o-primitive l-group which is not o-primitive: Let S be the totally ordered set of reals with the integers removed. Then A(S) has the desired properties.

(3) An l-group which is locally o-primitive but not weakly o-primitive: Let G be a non-archimedean totally ordered l-group without l-ideals. That such l-groups exists is clear from the example given by A. H. Clifford (3). Let $1 \neq g \in G$, C the regular l-subgroup not containing g, and K the minimal convex subgroup containing g. Then C and K are both representing subgroups of G, and K covers C. Hence by

theorem 3.1.28, G is locally o-primitive on the set of right cosets G/C, but by theorem 3.1.20, G is not weakly o-primitive.

(4) An l-group which is weakly o-primitive but not locally o-primitive: Let G be a totally ordered abelian group which has no smallest proper convex subgroup. Then $\{1\}$ is a representing subgroup, and by Theorem 3.1.20, G is weakly o-primitive on $G/\{1\}$, but by Theorem 3.1.28, G is not locally o-primitive.

(5) An l-group which is neither weakly nor locally o-primitive: Let G be the l-group of example 3. Every convex subgroup of G not equal to G is a representing subgroup, and $\exists$ no smallest proper convex subgroup. Hence by 3.1.20 and 3.1.28, G is neither weakly nor locally o-primitive on $G/\{1\}$.

Section II.

An l-group is simple if it has no non-trivial l-ideals.

Definition 3.2.1

Let G be an l-group. For $1 \leq f$, $g \in G$, f is right of g if for all $1 \leq h \in G$, $g \wedge h^{-1} f h = 1$.

Definition 3.2.2

An element $g \in G$ is insular if for some conjugate g^* of g, g^* is right of g.

Definition 3.2.3

If G is an l-group of o-permutations of a totally ordered set S, then for $f \in G$, f is bounded if its support, $\sigma(f)$ lies in a closed interval of S.

Lemma 3.2.4

Let G be a transitive l-group of o-permutations on a totally ordered set S. An element $1<g\in G$ is insular $\Leftrightarrow$ g is bounded.

Proof: ($\Leftarrow$) Suppose for $1<g\in G$, $\sigma(g)$ lies in the closed interval [a,b] of S. Since G is transitive on S, $\exists 1\leq h\in G$ such that $ah = b$. Let $g^* = h^{-1}gh$. Then, for every $x\in[a,b]$ and $1\leq k\in G$, $xk^{-1}h^{-1}\leq xh^{-1}\leq bh^{-1} = a$. Hence $xk^{-1}h^{-1}\notin\sigma(g)$ and so $xk^{-1}h^{-1}g = xk^{-1}h^{-1}$. Thus $xk^{-1}g^*k = xk^{-1}h^{-1}ghk = x$. Hence $\sigma(k^{-1}g^*k)$ lies completely outside [a,b] for every $1\leq K\in G$. Therefore $g\wedge k^{-1}g^*k = 1$ for all $1\leq k\in G$. Hence g is insular.

($\Rightarrow$) Conversely, suppose $1<g$ is an insular element. Let $g^* = k^{-1}gk$ be right of g. It may be assumed without loss of generality that $k\geq 1$. There exists $x\in S$ such that $x<xg^*$. If $\exists y\in S$ such that $x<y<yg$, then since G is transitive, $\exists 1<f\in G$ such that $xf = y$. It follows then that $yf^{-1}g^*f = xg^*f>xf = y$ which implies $y(g\wedge f^{-1}g^*f)>y$. Therefore, $g\wedge f^{-1}g^*f>1$. This contradicts the insularity of g. Hence $\nexists y\in S$ such that $x<y<yg$ and so $\sigma(g)$ is bounded above by x.

Similarly, $\exists z\in S$ such that $z<zg$. Let $w\leq zk^{-1}$. Then $wk\leq z$. So $\exists 1\leq h\in G$ such that $wkh = z$. Since $g\wedge h^{-1}g^*h = 1$, then,

$z = zh^{-1}g^*h = wgkh \geq wkh = z$. Hence $wgkh = wkh$ and so $w = wg$. Therefore $\sigma(g)$ is bounded below by zk^{-1}.

Remark: Clearly, if $\beta: G \to A(S)$ is an l-isomorphism of an l-group G onto a transitive l-subgroup of A(S), then $1 < g \in G$ is insular $\Leftrightarrow$ $g\beta \in G\beta$ is insular $\Leftrightarrow g\beta$ is bounded.

Lemma 3.2.5

If G is an l-group of o-permutations of a totally ordered set S, then the set $H = \{g \in G : g \text{ is bounded}\}$ is an l-ideal of G.

Proof: Clearly, if 1 = identity of G, then $\sigma(1) = \emptyset$ and so $1 \in H$. If $g, h \in H$, $gh \neq 1$, then suppose $\sigma(g)$ and $\sigma(h)$ are contained in the closed intervals $[a,b]$ and $[c,d]$ of S respectively. Then since $\sigma(gh) \subseteq \sigma(g) \cup \sigma(h)$, it follows that

$$\sigma(gh) \subseteq [a,b] \cup [c,d] \subseteq [a \wedge c, b \vee d] .$$

Therefore $gh \in H$. Clearly if $gh = 1$, then $gh \in H$. Thus H is a subgroup of G. For any $h \in H$, $\sigma(h \vee 1) = \{ x \in S : x(h \vee 1) \neq x \}$

$$= \{ x \in S : xh > x\} \subseteq \sigma(h)$$

Since h is bounded, then $h \vee 1$ is bounded and so $h \vee 1 \in H$. Similarly $\sigma(h \wedge 1) \subseteq \sigma(h)$ and so $h \wedge 1 \in H$. Therefore H is a sublattice of G. Suppose for $1 < h \in H$ $\exists g \in G$ such that $1 < g < h$. Then $\sigma(g) = \{ x \in S : xg \neq x \} = \{ x \in S : xg > x\} \subseteq \sigma(h)$. Therefore $g \in H$ and H is convex. For any $h \in H$ and $g \in G$, $\sigma(g^{-1}hg) = \{x \in S : xg^{-1}h \neq xg^{-1}\}$. Thus $x \in \sigma(g^{-1}hg) \Rightarrow xg^{-1} \in \sigma(h) \Rightarrow x \in \sigma(h)g$. Therefore, if $\sigma(h) \subseteq [a,b]$ a closed interval of S, then $\sigma(h)g \subseteq [ag,bg]$. Thus $\sigma(g^{-1}hg) \subseteq [ag,bg]$ and so $g^{-1}hg \in H$. Thus H is an l-ideal of G.

Theorem 3.2.6

G is a simple l-group containing an insular element $\Leftrightarrow$ G is a transitive o-primitive l-group of bounded o-permutations of a totally ordered set, S.

Proof: Let g be an insular element of the simple l-group G. By corollary 1.30, G is l-isomorphic to a transitive l-group of o-permutations of a totally ordered set. By lemma 3.2.4, since $g \in G$ is insular, then g is bounded. By lemma 3.2.5, if $H = \{g \in G : g \text{ is bounded}\}$, then H is an l-ideal of G. Since $1 \neq g \in H$ and since G is simple, then H = G. That is, every element of g is bounded and so G is a transitive l-group of bounded o-permutations of S. Now, G may not be o-primitive, so let $\sim$ be any convex congruence on S. Then from lemma 3.1.19, $H' = \{g \in G : x \sim xg \ \forall x \in S\}$ is an l-ideal of G. Since G is simple, the H = {1} or H = G. For $1 \neq f \in G$, let $\sigma(f)$ lie in the closed interval [a,b] of S. Then if for any non-trivial congruence $\sim$, $a \sim b$, then $f \in H$ and so H = G. But this is impossible since G is transitive on S. It follows that the union of any tower of proper convex congruences on S is a proper convex congruence. Therefore, by Zorn's lemma, there is a maximal proper convex congruence ρ on S. Also, S/ρ is totally ordered as in lemma 3.1.2. Let $\beta : G \; A(S/\rho)$ be the natural l-homomorphism of lemma 3.1.2. Then, for $g \in G$ and $x \in S$, $(xg)\rho = (x\rho)g\beta$. Also, since H = {1}, $\nexists 1 \neq f \in G$ such that $\forall x \in S$, $x \rho xf$ and so β is 1-1. It follows that G is o-primitive on S/ρ and so is a transitive o-primitive l-group of bounded o-permutations on S/ρ.

Conversely, let G be a transitive o-primitive l-group of bounded o-permutations on a totally ordered set S. Let $\{1\}\neq N$ be a l-ideal of G. Define $\sim$ on S such that for $x,y\in S$, $x\sim y \Leftrightarrow \exists 1<f\in N$ such that $x\leq yf$ and $y\leq xf$. Then it is easily seen that $\sim$ is an equivalence on S. Also $\sim$ is a convex congruence. To see this, notice that if $x\sim y$, let $1<f\in N$ be such that $x\leq yf$ and $y\leq xf$. Then, for any $g\in G$, $xg\leq yfg$ and $yg\leq xfg$. Take $f' = g^{-1}fg$. Then since N is an l-ideal, $1<f'\in N$. Also, $xg\leq yfg = ygf'$ and $yg\leq xfg = xgf'$. Thus $xg\sim yg$ for every $g\in G$. Now, if $x,y,z\in S$, $x\leq y\leq z$ and $x\sim z$, then $\exists 1<f\in N$ such that $x\leq zf$ and $z\leq xf$. Therefore, $y\leq z\leq xf$ and $x\leq y\leq yf$ since $f>1$. Thus $x\sim y$. Therefore $\sim$ is convex. It follows immediately from the definition of $\sim$ that $\forall f\in N$, $x\sim xf$ $\forall x\in S$. Since $N\neq\{1\}$, then $\exists$ at least one congruence class E containing more than one element. But G is o-primitive, therefore $E = S$ and must be the only congruence class. Let $1<g\in G$. By assumption g is bounded. So let $\sigma(g)$ lie in some closed interval $[a,b]$ of S. Since $a\sim b$, then $\exists 1<f\in N$ such that $b\leq af$. Hence, for any $x\in [a,b]$, $xg\leq b\leq af\leq xf$. For any $x\in S\setminus[a,b]$, $xg = x\leq xf$. Thus $g\leq f$. But N is convex and so $g\in N$. Therefore $G = N$, and G is simple. By lemma 3.2.4, every positive element of g is insular. This completes the proof.

<u>Corollary 3.2.7</u>

If G is a simple l-group with an insular element, then every positive element of G is insular.

Proof: This is immediate from the theorem.

Corollary 3.2.8

If G is a simple l-group with an insular element, then for every $1<g\in G$ there is an infinite collection of pairwise disjoint conjugates of g.

Proof: By Corollary 3.2.7, every $1<g\in G$ is insular. The result follows immediately from definitions 3.2.1 and 3.2.2.

In conclusion two examples of simple l-groups are given.

Examples

(1) If F is a totally ordered field, then A(F) is doubly transitive. Hence A(F) is o-primitive. Therefore if G is the l-group of bounded o-permutations of F, then by theorem 3.2.6, G is simple. In particular this is true if the field is the field of real numbers, a result obtained originally by Holland (10).

(2) A non-totally ordered simple l-group which does not contain an insular element: Let G be the l-subgroup of A(R), where R is the totally ordered field of real numbers, such that $G = \{\alpha\in A(R): x\alpha+1 = (x+1)\alpha \ \forall x\in R\}$. Then G is transitive on R for given any x, $y\in R$ with $x<y$, let ρ_{y-x} denote the right translation (additive) by y-x. Then

$x\rho_{y-x}=y$ and also $\rho_{y-x}\in G$. Then by Corollary 3.1.17, G is o-primitive. That G is not totally ordered is clear. Also, for each $g\in G$, $\sigma(g)$ is not a bounded subset of G hence G contains no insular elements. That G is simple follows from the following: For $g\in G$, $\sigma(g)$ is an open set with respect to the interval topology on R. Also, for any $1_R<g\in G$, $\sigma(g)\cap[o,1]\neq\emptyset$ where $[o,1]$ denotes the closed interval between o and $1\in R$. It is easily seen that if $x\in[o,1]$ and $xg=x$, then $\exists g^*$ some conjugate of G with $xg^*\neq x$ that is, $x\in\sigma(g^*)$. For each x select such a g^*. Then $[o,1]\subseteq\cup\sigma(g^*)$. Since $[o,1]$ is compact, then $\exists$ a finite number of conjugates $\{g_i\}_{i=1}^n$ of g such that $[o,1]\subset\bigcup_{i=1}^n\sigma(g_i)\subseteq\sigma(\bigvee_{i=1}^n g_i)$. Let $h=\bigvee_{i=1}^n g_i$. Then it follows that h has no fixed points in $[o,1]$ and so none in R by definition of G. Therefore, for any $1_R<f\in G$, $\exists$ some integer K such that $1f<o(h)^k$. Thus, for each $x\in[o,1]$, $xf\leq 1f<o(h)^k\leq x(h)^k$. Hence for every $y\in R$, $yf\leq y(h)^k$. Thus, $1_R<f\leq(h)^k$ and so any l-ideal of G which contains g must also contain any such f. Hence G is simple.

Using the results of section I, C. Holland also showed in (11) that under suitable but rather general conditions, if an l-group G is l-isomorphic to a transitive l-subgroup of A(S) and to a transitive l-subgroup of A(T), where S and T are totally ordered sets, then $T\subseteq\overline{S}$ where $\overline{S}$ is the completion of S by Dedekind cuts (without end-points) and the action of G on T is obtained by first extending the action of G on S to an action on $\overline{S}$ and then cutting back to T.

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