A CHARACTERIZATION OF THE HYPERARITHMETICAL SETS IN THE LANGUAGE OF RAMIFIED ANALYSIS

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ABSTRACT

This thesis shows the equivalence of the hyperarithmetical sets and the truth sets defined in a special language, the language of Ramified Analysis. In the Introduction, the basic relevant notions, such as formulas, recursivity, and ordinals, are discussed. Chapter I covers recursive ordinals and $\mathcal{O}$, a set of notations for recursive ordinals. In Chapter II, we apply knowledge of recursive ordinals, notations, and the language of Ramified Analysis to the definition of ordinal rank, validity, and sets $\mathcal{M}_\alpha$, which will be shown in the last two chapters to contain precisely the hyperarithmetical sets. The sets in each $\mathcal{M}_\alpha$ are reduced, in Chapter III, to truth sets T_α which are then shown to be hyperarithmetical. The converse is shown in Chapter IV, and depends upon the Recursion Theorem. Finally, Chapter V provides a full and a partial relativization of the preceding results.

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INTRODUCTION

The purpose of this paper is to characterize the hyperarithmetical sets in terms of the language of Ramified Analysis. The Language of Ramified Analysis has the following symbols:

(a) variables - $x, y, z, \ldots$

$x^a, y^a, z^a, \ldots \quad (a \in N)$

$X, Y, Z, \ldots$

$f, g, h, j, k, \ldots$

Variables may be subscripted, primed, or modified in any reasonable way for notational convenience. However, there are only a denumerable number of variables. Capital letter variables are called set variables; lower case variables are called number variables.

(b) predicate symbols - p_i for $i \in \{0, 1, \ldots\}$. Among these symbols are $\in$, $=$, and $<$. It is implicit that entries in predicate symbols are restricted to certain sorts of terms; for example, $x \in X$ is allowed, but $X \in x$ is not.

(c) function symbols - f_i for $i \in \{0, 1, \ldots\}$. Among these symbols are $+$ and $\cdot$. 0-ary function symbols are called constants and we explicitly refer to them as $\overline{0}, \overline{1}, \overline{2}, \ldots$. When x is a natural number, we write $\overline{x}$ for the corresponding constant.

(d) logical symbols - $\neg$, $\vee$, and $\exists$ (and symbols definable in terms of $\neg$, $\vee$, and $\exists$) and punctuation - (, $\cdot$,) .

The concept of inductive definition is very useful. Generally, an inductive definition of a set S is a collection of laws each of which states that an object x is an element of S if x meets certain criteria. Additionally, no object may be an element of S unless it follows from the laws that it is in S. In practice the laws may be interrelated. For example,

(i) $1 \in S$

(ii) $x \in S$ if $x = 2 \cdot y$ and $y \in S$.

This constitutes an inductive definition of the set of non-negative integral powers of 2. Later we will use the structure of inductive definitions as a base for proof by <u>induction</u>, either on the natural numbers or the ordinals.

We define <u>terms</u> and <u>formulas</u> by the following inductive definitions:

(i) a variable is a term

(ii) if $u_1, \ldots, u_n$ are terms and f is an n-ary function symbol, then $f(u_1, \ldots, u_n)$ is a term.

By (ii), a constant is a term.

If p is an n-ary predicate symbol and $u_1, \ldots, u_n$ are terms, then $p(u_1, \ldots, u_n)$ is an <u>atomic formula</u>.

The following four rules are an inductive definition of a <u>formula</u>:

(i) an atomic formula is a formula.

(ii) if u is a formula, then $\neg u$ is a formula.

(iii) if u and v are formulas, then $u \vee v$ is a formula.

(iv) if u is a formula, and v a (set or number) variable, then $\exists v\, u$ is a formula.

The concept of an <u>ordinal number</u> is fundamental to this thesis;

our exposition here is based on [Su., CH. V]. Ordinal numbers (or <u>ordinals</u>) have the following inductive definition:

1.) 0, the empty set, is an ordinal

2.) if x is an ordinal so is $x \cup \{x\}$

3.) if S is a set of ordinals, then $\bigcup S$ $(= \{x \mid (\exists B)(x \in B \ \& \ B \in S)\})$ is an ordinal.

Thus, the first three ordinals are $0, \{0\}, \{0, \{0\}\}$.

A set A is <u>well-ordered</u> by a relation R (called a <u>well-ordering</u> of A) if

(i) $(\forall x)(\forall y)(x, y \in A \ \& \ x \neq y \rightarrow R(x, y) \vee R(y, x))$

(ii) $(\forall x)(\neg R(x, x))$

and (iii) $(\forall B)(B \subseteq A \ \& \ B \neq 0 \rightarrow B$ has an R-least element)

We interpret "x is an R-least element of B" to mean "$x \in B \ \& \ (\forall y)(y \in B \rightarrow \neg R(y, x))$".

By the Axiom of Choice [Su, Chapter 8], the above conditions are equivalent to there not existing any infinite descending chains for R in A. By [Su., §3.2, Thm. 62] these conditions also suffice to insure that R is asymmetric and transitive in A.

For any two ordinals α and β either $\alpha \in \beta$, $\alpha = \beta$ or $\beta \in \alpha$; and if β is an ordinal and $\alpha \in \beta$ then α is an ordinal. We have the following two important facts about ordinals and well-orderings: (we shall henceforth say $\alpha < \beta$ when $\alpha \in \beta$):

1.) Each ordinal is well-ordered by $<$.

2.) [Su., §7.4, Thm. 81] Each well-ordering of a set may be represented by an ordinal which we shall call the <u>order type</u> of the well-ordering.

We can state 2.) more clearly with the help of two definitions:

Definition: $\langle A, R \rangle$, where A is a set and R is a relation, is a simple order structure if

$$(\forall x)(\forall y)(\forall z)[x, y, z \in A \to \{(R(x, y) \to \neg R(y, x)) \\ \& \ (R(x, y) \ \& \ R(y, z) \ .\!\to\!. \ R(x, z)) \\ \& \ (R(x, y) \vee R(y, x) \vee x = y)\}]$$

Definition: $\langle A, R \rangle$ is similar to $\langle B, S \rangle$ if there is a function f such that

(1) f is one - one

(2) domain $(f) = A$ and $f(A) = B$

(3) $(\forall x)(\forall y)(x \in A \ \& \ y \in A \to R(x, y) \leftrightarrow S(f(x), f(y)))$.

Then we may restate (2) as "If R well-orders A, then there is a unique ordinal α such that $\langle A, R \rangle$ is similar to $\langle \alpha, < \rangle$".

Our concept of induction may be extended to ordinals [Su., §7.1].

"If, for every ordinal α, $(\forall \beta)(\beta < \alpha \to \phi(\beta)) \to \phi(\alpha)$,
then for every α, $\phi(\alpha)$."

Using this "transfinite induction", or equivalent statements, and recursion schemas it is possible to define addition and multiplication of ordinals and, indeed, quite a number of other operations analogous to those of ordinary arithmetic. For example, ordinal addition may be defined by the following recursion scheme:

(i) (i) $\alpha + 0 = \alpha$

(ii) $\alpha + S\beta = S(\alpha + \beta)$

(iii) if β is a limit ordinal

$$\alpha + \beta = \bigcup_{\gamma<\beta} (\alpha + \gamma)$$

where $S\beta$ is the successor of β, and a limit ordinal is an ordinal, $\neq 0$, which is not the successor of any ordinal. The successor operation is defined by $S\beta = \beta \cup \{\beta\}$. Note that an <u>infinite ordinal</u> is simply one with an infinite number of elements.

This thesis is concerned with a particular sort of ordinal, the <u>recursive</u> ordinal. Thus we need the important notion of recursivity. We accept as basic the Turing machine characterization of recursive functions [R, §1.5]. That is, a partial recursive function is one which is defined by a list of instructions for a Turing machine. When a partial recursive function is total (i.e., when the Turing machine executes only a finite number of steps when supplied with any integer argument) then we simply call the function <u>recursive</u>.

Logicians assert that there is an intuitive concept of calculability of functions and have labelled a connection between the intuitive concept and recursion, called Church's Thesis. In effect, Church's Thesis states that calculable, or computable, functions are recursive, and vice versa. We will use this principle, usually without reference, to assert that obviously computable functions are indeed recursive.

A relation P is <u>recursive</u> when there is a Turing machine T which will compute the representing function of P. That is: $P(x) \leftrightarrow T$, when supplied with x, yields 0.

One can list all the possible sets of instructions for unary

Turing machines [cf. R., §1.8]. Thus we may list all the unary partial recursive functions, which we shall call ϕ_i $(i = 0, 1, \ldots)$. A common alternative notation for ϕ_i is $\{i\}$.

We can also list the ranges of the partial recursive functions. Define W_x = range ϕ_x (or $y \in W_x \leftrightarrow (\exists z)(\phi_x(z) = y)$). The W_x's are called the recursively enumerable sets. These comprise a natural class, as shown by [R., 5.2, Cor V(b)]:

"There exist recursive functions f and g such that

$$\text{range } \phi_{f(x)} = \text{domain } \phi_x$$

$$\text{and domain } \phi_{g(x)} = \text{range } \phi_x."$$

It is useful to be able to effectively code sequences of integers into single integers, and to be able to effectively decode them. A simple example of such a coding is $\tau(x, y)$ (found in [R., §5.3]), where $\tau(x, y) = 1/2(x^2 + 2xy + y^2 + 3x + y)$. τ is known to be a one - one recursive map of $N \times N$ onto N. ($N \times N = \{(v, w) \mid v \in N, w \in N\}$. There are two recursive functions, π_1 and π_2, which serve as inverses to τ: i.e., $\tau(\pi_1(z), \pi_2(z)) = z$. More powerful coding is described in [Sh. §6.4]. Suppose $(a_1, \ldots, a_n)$ is a sequence. Then there is a function $\langle\rangle$ which has the following two interesting properties:

(i) for each fixed n, $\langle a_1, \ldots, a_n\rangle$ is a recursive function of $a_1, \ldots, a_n$.

(ii) $\langle a_1, \ldots, a_n\rangle$ determines n and $a_1, \ldots, a_n$ via recursive functions. That is, there are two recursive functions $f(x)$ and $g(x, y)$ such that if $a = (a_1, \ldots, a_n)$ then $f(a) = n$ and for

$1 \le i \le n$, $g(a, i) = a_i$. We write $\ell h(a)$ for $f(a)$ and $(a)_i$ for $g(a, i)$. The definition of recursive relations may be extended to n-ary relations either directly or via codings; [R., §5.3] shows these two approaches to yield the same class of recursive relations.

Similarly, a <u>set A is recursive</u> if there is a recursive function f such that

$$x \in A \rightarrow f(x) = 0$$

and

$$x \notin A \rightarrow f(x) = 1 .$$

In other words, f allows us to decide, for any x, whether $x \in A$.

Finally, we define <u>reducibility</u> between sets:

A is <u>many - one reducible</u> to B ($A \le_m B$)

if there is a recursive function f such that $(\forall x)(x \in A \leftrightarrow f(x) \in B)$. In the case where f is one - one, we say A is <u>one - one</u> reducible to B ($A \le_1 B$).

We may extend the notion of recursive functions to "functions recursive in a set X, of integers". Several equivalent definitions of <u>relative recursiveness</u> are given in [R., §9.2]. Intuitively, f is recursive in X if, in addition to the usual components of a Turing machine, there may also be steps which "interrogate" X, i.e., which inquire if some integer is in X. A unary relation R is recursive in X if and only if there is a recursive relation S such that $R(a) \leftrightarrow S(a, X)$. Or, (as in [R., §9.2]), R is recursive in X if and only if the representing function for R is recursive in X.

Using the concepts of recursiveness, or computability, and well-orderings we will develop some basic properties of recursive ordinals and hyperarithmetical sets. Then we will inductively define, for each recursive ordinal α, a limited notion of validity, $\models_\alpha \mathcal{J}$, for a restricted set of formulas in the Language of Ramified Analysis. Simultaneously, we will define sets $\mathcal{M}_\alpha = \{S \mid S = \{x \mid \models_\alpha \mathcal{B}(\bar{x})\}$ for suitable formulas $\mathcal{B}\}$. Then the main result of this thesis can be stated simply:

"The class of hyperarithmetical sets (HYP) equals $\bigcup \mathcal{M}_\alpha$ where α ranges over the recursive ordinals."

In practice we will consider a set $\mathcal{O}_1$ which consists of unique notations for ordinals; that is, for each recursive ordinal α there is a unique $a \in \mathcal{O}_1$ and vice versa. Then we can write the main result as "HYP = $\bigcup_{a \in \mathcal{O}_1} \mathcal{M}_a$".

CHAPTER I

RECURSIVE ORDINALS AND ORDINAL NOTATIONS

1.1 BASIC PROPERTIES OF RECURSIVE ORDINALS

Definition: An ordinal α is a recursive ordinal if

(i) α is finite

or (ii) there exists a recursive relation $R(x, y)$ such that R well-orders N and such that α = the order type of $\langle N, R\rangle$; that is, α is similar to $\langle N, R\rangle$. R is called a recursive relation for α.

The following proposition indicates some basic facts about recursive ordinals.

PROPOSITION I

I(a). If α and β are recursive ordinals, then $\alpha + \beta$ is a recursive ordinal.

Proof: When both α and β are finite, then $\alpha + \beta$ are finite and thus recursive.

When α is finite and β is not, $\alpha + \beta = \beta$ and thus $\alpha + \beta$ is recursive.

Suppose α is not finite and β is finite. We define R':

$$R'(x, y) \leftrightarrow x \geq n \ \& \ y \geq n \ \& \ R(x - n, y - n)$$

where R is a recursive relation for α, and n is the natural number corresponding to β.

We may now define S:

$$S(x, y) \leftrightarrow [x < n \;\&\; y < n \;\&\; x < y]$$
$$\vee\; R'(x, y)$$
$$\vee\; [x \geq n \;\&\; y < n].$$

S is recursive, since R, $<$, and $\leq$ are recursive. S well-orders N: For $A \neq \emptyset$, if $A \subseteq \{0, 1, \ldots, n - 1\}$ then A is finite and of course has a $<$-first member. Otherwise, $A \cap \{x \mid x \geq n\}$ has an R-first member which is also clearly an S-first member for A. And $\alpha + \beta$ = order type of $\langle N, S\rangle$, by inspection.

Suppose neither α nor β is finite. Let R_1 be a recursive relation for α and R_2 a recursive relation for β. We define $S(x, y)$:

$$S(x, y) \leftrightarrow [(\exists z)(\exists w)(x = 2z \;\&\; y = 2w \;\&\; R_1(z, w)]$$
$$\vee\; [(\exists z)(\exists w)(x = 2z + 1 \;\&\; y = 2w + 1 \;\&\; R_2(z, w)]$$
$$\vee\; [(\exists z)(\exists w)(x = 2z \;\&\; y = 2w + 1)]$$

S is recursive since R_1 and R_2 are recursive and our quantifiers are implicitly bounded. For example, $(\exists z)(x = 2z)$ may be replaced by $(\exists z)_{z \leq x}(x = 2z)$. S well-orders N; if $A \neq \emptyset$, then either $A \subseteq \{x \mid (\exists z)(x = 2z + 1)\}$ and A has an R_2-first member which must also be an S-first member, or otherwise $A \cap \{x \mid (\exists z)(x = 2z)\}$ has an R_1-first member which must also be an S-first member of A. Again $\alpha + \beta$ = order type of S. ||

I(b). If α and β are recursive ordinals, then $\alpha \cdot \beta$ is a recursive ordinal.

Proof: If α and β are finite, then $\alpha \cdot \beta$ is also finite and hence recursive.

If α is finite and β is not then $\alpha \cdot \beta = \beta$. Hence, $\alpha \cdot \beta$ is recursive.

Suppose β is finite and α is not. Let n be the integer corresponding to β. Intuitively we separate N into n classes of integers according to their remainder upon division by n. To effect this, we define $S(x, y)$:

$$S(x, y) \leftrightarrow (\exists k)(\exists p)(\exists m)(\exists \ell)$$
$$(x = kn + m \;\&\; y = \ell n + p \;\&\; m < n \;\&\; p < n \;\&\; m < p)$$
$$\vee\; (x = kn + m \;\&\; y = \ell n + m \;\&\; m < n \;\&\; R(x, y))$$

where R is a recursive relation for α.

S is recursive since $<$ and division are recursive, and since the quantifiers are implicitly bounded. Let $S_i = \{x \mid (\exists k)(x = kn + i)\}$. Clearly $N = \bigcup_{i=0}^{n-1} S_i$, and $i < j \rightarrow (x \in S_i \;\&\; y \in S_j \;.\!\rightarrow.\; S(x, y))$. If $A \subseteq N$, either $A \cap S_1 \neq \emptyset$ so that A has an S-first element in S_1, or if $A \cap S_1 = \emptyset$, then we try again with S_2, etc. In any case it is sure that $A \cap S_i \neq \emptyset$ for some $i < n$. Thus S must well-order N.

Suppose both α and β are infinite. When R_1 is a recursive relation for α and R_2 is a recursive relation for β we define $S(x, y)$:

$$S(x, y) \leftrightarrow (\exists a)(\exists b)(\exists c)(\exists d)$$
$$((x = \tau(a, b) \;\&\; y = \tau(c, d)$$
$$\&\; (R_1(a, c) \vee (a = c \;\&\; R_2(b, d)))\;.$$

Let $S_n = \{x \mid (\exists b)(x = \tau(n, b))\}$. We know $N = \bigcup S_n$, $n = 0, 1, \ldots$ and $m < n \rightarrow (x_m \in S_m \;\&\; x_n \in S_n \;\&\; R_1(m, n) \;.\!\rightarrow\!.\; S(x_m, x_n))$. For any $A \subseteq N$ there is an R_1-least n such that $A \cap S_n \neq \emptyset$, and $A \cap S_n$ has an R_2-least element which is an S-least element for A. Thus S well-orders N, and $\alpha \bullet \beta$ has the order type of $\langle N, S\rangle$.

I(c). If β is a recursive ordinal then any ordinal α less than β is recursive.

<u>Proof</u>: If α is finite, it is clearly recursive. Suppose α and β are infinite and $\alpha < \beta$ and R is the recursive relation for β. Then $\alpha < \beta$ means that there is an n such that α is similar to β restricted to n; that is, $\{m \mid R(m, n)\}$. This set is clearly recursively enumerable. Thus it is the range of a recursive function f which, as we remarked in the Introduction, we may choose to be one - one. Then we define $S(x, y)$:

$$S(x, y) \leftrightarrow R(f(x), f(y)).$$

Clearly S is recursive, and must well-order N since R does; S has the order type of α . ||

Finally we arrive at the most interesting property of recursive ordinals.

We need two notions to properly state I(d). First, we need the Kleene T-predicate [Sh., §7.4]; in particular, the fact that there is a recursive relation T_2 such that given F, $\exists f \ni F(x, y) = (z)_0$ where z is the least such that $T_2(f, x, y, z)$ holds. We say, in this case, that f is an index for F.

We also need the notion of the supremum of a set of ordinals. The supremum of a set of ordinals is, naively, the least ordinal greater than all the ordinals in the given set. For certain sets of recursive ordinals we can show that the supremum exists and is also a recursive ordinal.

I(d). If $\alpha_0, \alpha_1, \ldots,$ are recursive ordinals corresponding to recursively enumerable relations $R_0, R_1, \ldots$ where $R_n(x, y) \leftrightarrow (\exists z)\, T_2(f(n), x, y, z)$ for some recursive f, then supremum $\{\alpha_n \mid n \in N\}$ is a recursive ordinal.

Proof: By Prop. I(c), it suffices to show that there is a recursive ordinal β such that for all n, $\alpha_n \leq \beta$. Define $S(n, x, y) \leftrightarrow R_n(x, y)$. S is a recursively enumerable relation since, given n, we can effectively compute $f(n)$, and hence determine the (recursively enumerable) relation R_n. We now define $S_1(u, v)$:

$$S_1(u, v) \leftrightarrow \{n < m \vee (n = m \;\&\; S(n, x, y)\}$$

where $u = \tau(n, x)$ and $v = \tau(m, y)$.

S_1 is recursively enumerable and is a well-ordering of N since it breaks N into segments which are well-ordered with respect to each other and each of which is well-ordered similar to the well-ordering of some of the R_n's. In particular, for each n we have an order-preserving injection of $\langle N, R_n \rangle$ into $\langle N, S_1 \rangle$ - namely $\psi_n(x) = \tau(n, x)$. Now, a recursively enumerable well-ordering of N must be a recursive well-ordering. For it must be true that either $S_1(x, y)$ or $S_1(y, x)$, if $x \neq y$, and not both; so by simultaneously computing $S_1(x, y)$ and $S_1(y, x)$ we will find which one holds, and hence compute whether $S_1(x, y)$ holds or not. Then if S_1 is the recursive relation for some ordinal β, we have $\alpha_n \leq \beta$, as required. ||

An apparently stronger definition of recursive ordinal is possible:

Definition 2: An ordinal α is a recursive ordinal if

(i) α is finite

or (ii) there exists a recursive relation $R(x, y)$ such that R well-orders some recursively enumerable subset S of N and such that α = the order type of $\langle S, R \mid S \rangle$, where $R \mid S$ is the restriction of R to S.

It is clear that if an ordinal is recursive by definition 1 then it is recursive by definition 2. However, the converse implication also holds. Since S is recursively enumerable, there is a one - one recursive function such that $S = f(N)$. Define $R'(x, y)$:

$$R'(x, y) \leftrightarrow R(f(x), f(y)).$$

Clearly R' is recursive and well-orders N and $\langle N, R' \rangle \cong \langle S, R \mid S \rangle$. Thus α is recursive by definition 1.

1.2 NOTATIONS FOR RECURSIVE ORDINALS

It is necessary to have a system to refer to recursive ordinals. The motivation for this is discussed in detail in [R., §11.7]. We inductively define a set $\mathcal{O}$ (the set of ordinal notations) and a relation on $\mathcal{O}$ called $<_{\mathcal{O}}$:

(i) 0 receives notation 1

(ii) assume all ordinals $< \gamma$ have received their notations.

Then

(a) if $\gamma = \beta + 1$, then if x is any notation for β, 2^x is a notation for γ, and we say that

$$z <_{\mathcal{O}} x \vee z = x \to z <_{\mathcal{O}} 2^x$$

(b) if γ is a limit ordinal, then if $\phi_y(0), \phi_y(1), \ldots$ are notations for an increasing sequence of ordinals with limit γ, and $\phi_y(i) <_{\mathcal{O}} \phi_y(j)$ for $i < j$, then $3 \cdot 5^y$ is a notation for γ, and $z <_{\mathcal{O}} 3 \cdot 5^y$ whenever $z <_{\mathcal{O}} \phi_y(m)$ for some m.

$\mathcal{O}$ is thus a partially ordered set with ordering $<_{\mathcal{O}}$. Each infinite ordinal may have denumerably many notations. If $x \in \mathcal{O}$, then we say $|x|_{\mathcal{O}}$ is the ordinal for which x is a notation. $\mathcal{O}$ has several interesting properties. The following two are among the most significant.

(i) $\{x \mid x <_{\mathcal{O}} y\}$ is uniformly recursively enumerable. That is, there is a recursive f such that

$$(\forall y)(y \in \mathcal{O} \to \{x \mid x <_{\mathcal{O}} y\} = \{z \mid (\exists w)\ \phi_{f(y)}(w) = z\})$$

(ii) there exists a binary recursive function $+_{\mathcal{O}}$ such that for all x and y in $\mathcal{O}$

(a) $x +_{\mathcal{O}} y \in \mathcal{O}$

(b) $|x +_{\mathcal{O}} y|_{\mathcal{O}} = |x|_{\mathcal{O}} + |y|_{\mathcal{O}}$

(c) $y \neq 1 \Rightarrow x <_{\mathcal{O}} x +_{\mathcal{O}} y$.

1.3 Π^1_1 SETS

Both $\mathcal{O}$ and $\mathcal{O}_1$ (a special set to be defined in the next section) belong to a special class of sets known as the Π^1_1 sets. It is desirable to examine the definition of Π^1_1 sets and a few of their properties.

Definition: An <u>arithmetical relation</u> is the result of prefixing a recursive relation by a finite number of number quantifiers. Simple examples are:

$$(\exists y)(y > x)$$

or

$$(\exists z)(\exists y)(y > \phi_z(x))$$

Definition 1: A set P is Π^1_1 if there is a recursive relation S such that $x \in P \leftrightarrow (\forall f)(\exists y)S(f, x, y)$, for unary functions f.

An alternative definition is the following:

Definition 2: A set P is Π^1_1 if there is an arithmetical relation S such that $x \in P \leftrightarrow (\forall f)S(f, x)$, for unary functions f.

Assume P is Π^1_1 by definition 1. Define $S'(f, x) \leftrightarrow (\exists y)S(f, x, y)$. Then S' is arithmetical, so P is Π^1_1 by definition 2.

Conversely, assume P is Π^1_1 by definition 2. From [Sh., §7.8] we have the following equivalences:

(E1) $(\forall y)T(y) \leftrightarrow (\forall f)T(f(0))$

(E2) $(\exists x)(\forall f)T(f, x) \leftrightarrow (\forall f)(\exists x)T((f)_x, x)$

(E3) $(\forall f)(\forall g)T(f, g) \leftrightarrow (\forall h)T((h)_0, (h)_1)$

(E4) $(\exists y)(\exists z)T(y, z) \leftrightarrow (\exists w)T((w)_0, (w)_1)$.

In E2 and E3, $(f)_x = \lambda y[f(y))_x]$. ($\lambda$ is Church's Lambda Notation [Sh., §7.3]). E2 appears more obvious when negated:

$$(\forall x)(\exists f) \neg T(f, x) \leftrightarrow (\exists f)(\forall x) \neg T((f)_x, x) .$$

There must be an arithmetic S' and a recursive S'' such that

$$x \in P \leftrightarrow (\forall f)\ S'(f, x) \quad \text{since } P \text{ is } \Pi^1_1$$
$$\leftrightarrow (\forall f)(Q_1 x_1) \ldots (Q_n x_n)\ S''(f, x, x_1, \ldots, x_n)$$

by the definition of arithmetical relations. Q_i may be either $\exists$ or $\forall$.

By E1 we convert all the $(\forall x_i)$'s to $(\forall f_i)$'s. By E2 we move all the $(\forall f_i)$'s to the left of all the $(\exists x_i)$'s and we contract them with $(\forall f)$ by E3. Finally we contract all the $(\exists x_i)$'s together by E4. If S' contains no existential quantifiers then we insert a dummy quantifier. We then get $x \in P \leftrightarrow (\forall f)(\exists y)\ S''(\ldots)$. The contents of the brackets is some composition of recursive functions on each entry. Thus, since the composition of computable functions is computable, we know there is a recursive relation S''' such that

$$x \in P \leftrightarrow (\forall f)(\exists y)\ S'''(f, x, y)$$

Thus, the two definitions of Π^1_1 set turn out to be equivalent.

We can imitate the above definitions of arithmetical relations and Π^1_1 sets by using set variables. A result similar to the above clearly holds for the two definitions of Π^1_1 using set variables. We wish to show that the definitions using set and function variables are equivalent.

Definition: $\text{GRAPH } f = \{z \mid z = \langle x, y\rangle \;\&\; f(x) = y\}$.

PROPOSITION 2

Let $S(f, x)$ be an arithmetical relation. Then there is an arithmetical relation $S'(X, x)$ such that

$$X = \text{GRAPH } f \rightarrow (\forall x)(S(f, x) \leftrightarrow S'(X, x)) \; .$$

Proof: Suppose p is an n-ary predicate symbol, and $t, t_1, \ldots, t_n$ are terms and that $p(t_1, \ldots, t_n)$ occurs in S and moreover that $t_i = f(t)$. First replace $p(t_1, \ldots, t_n)$ by $(\exists z)(z = f(t) \;\&\; p(t_1, \ldots, t_{i-1}, z, t_{i+1}, \ldots, t_n))$. Then in this last expression, replace $z = f(t)$ by $\langle t, z\rangle \in X$ to get $(\exists z)(\langle t, z\rangle \in X \;\&\; p(t_1, \ldots, t_{i-1}, z, t_{i+1}, \ldots, t_n))$. As a simple example, $f(y) = z$ is transformed to $\langle y, z\rangle \in X$. Of course, at each stage in the elimination of f it is necessary to introduce distinct variables. But then it is clear that if $X = \text{GRAPH } f$ then $(\forall x)(S(f, x) \leftrightarrow S'(X, x))$. $\|$

Suppose we have a Π^1_1 relation $(\forall f)S(f, x)$. Then we have

$$(1) \quad (\forall f)S(f, x) \leftrightarrow (\forall X) \quad (X \text{ is a graph} \rightarrow S'(X, x))$$

where S' is the relation described in Prop. 2.

$$\text{"}X \text{ is a graph"} \leftrightarrow (\forall z)(\exists x)(\exists y)(\forall w)$$

$$\{(z \in X \;.\!\rightarrow.\; z = \langle x, y\rangle \;\&\; [\langle x, w\rangle \in X \rightarrow w = y])$$

$$\& \; ((\forall x)(\exists y)(\langle x, y\rangle \in X))\}.$$

So "X is a graph" is arithmetical.

Thus the right hand side of (1) is a Π^1_1 relation.

Conversely, suppose $(\forall X)S(X, x)$ is Π^1_1 via the new definition. Replace all occurrences of $y \in X$ by $(\exists z)(y = f(z))$ to get a new relation $S'(f, x)$ such that $(\forall f)S'(f, x)$ is Π^1_1 and $(\forall f)S'(f, x) \leftrightarrow (\forall X)S(X, x)$. Thus we see that the definitions of Π^1_1 sets in terms of functions and sets are equivalent.

PROPOSITION 3

If A is many - one reducible to B and B is Π^1_1, then A is Π^1_1.

Proof: Let f be a recursive function such that $x \in A \leftrightarrow f(x) \in B$. Since B is Π^1_1 there is a recursive relation S such that $x \in B \leftrightarrow (\forall g)(\exists y)S(g, y, x)$. Then clearly, if $S'(g, y, x) \leftrightarrow S(g, y, f(x))$, S' is recursive and $x \in A \leftrightarrow (\forall g)(\exists y)S'(g, y, x)$. Thus A is Π^1_1. $||$

We can give an example of a Π^1_1 - set.

PROPOSITION 4

Define $W = \{e \mid \lambda xy(\exists z)T_2(e, x, y, z)$ is a well-ordering of $N\}$. W is a Π^1_1 set.

Proof:

$$e \in W \leftrightarrow (\forall x)\ \neg(\exists z)T_2(e, x, x, z)$$

(non-reflexivity)

$$\& \ (\forall x)(\forall y)[x \neq y \rightarrow \{(\exists z)T_2(e, x, y, z) \leftrightarrow \neg(\exists z)T_2(e, y, x, z)\}]$$

(asymmetry and totality)

$$\& \ (\forall x)(\forall y)(\forall w)\{[(\exists z)T_2(e, x, y, z) \ \& \ (\exists z)T_2(e, y, w, z)] \rightarrow (\exists z)T_2(e, x, w, z)\}$$

(transitivity)

$$\& \ (\forall f)(\exists y)(\forall w)[f(y) = f(w) \vee (\exists z)T_2(e, f(y), f(w), z)].$$

(no infinite descending sequences)

As f does not appear in the first three clauses, we can add the dummy quantifier $\forall f$ to each of them. Then since for relations $(\forall f)P_1(f, e)$ and $(\forall f)P_2(f, e)$, $(\forall f)(P_1(f, e) \ \& \ P_2(f, e)) \ .\leftrightarrow. \ (\forall f)P_1(f, e) \ \& \ (\forall f)P_2(f, e)$, we see that W is a Π^1_1 set. ||

In fact in [R., ex. 11 - 61] it is noted that W is one - one reducible to $\mathcal{O}$ and $\mathcal{O}$ is one - one reducible to W.

1.4 UNIQUE ORDINAL NOTATIONS

The set $\mathcal{O}$ is called "the set of ordinal notations" since to each ordinal we may assign a member of $\mathcal{O}$. There may be many such members for each ordinal; in fact, $\mathcal{O}$ is "designed" so that each non-finite ordinal will receive an infinite number of notations, many of which will be $<_{\mathcal{O}}$-incomparable. It is possible to define a path through $\mathcal{O}$ which we shall call $\mathcal{O}_1$, the <u>set of unique notations</u> for ordinals. That is, each ordinal can be uniquely associated with a member of $\mathcal{O}_1$, and vice versa. $\mathcal{O}_1$ shares some of the properties of $\mathcal{O}$; in particular $\mathcal{O}_1$ is a Π^1_1 set. Also:

PROPOSITION 5

There is a recursively enumerable relation $R(x, y)$ such that $R \mid \mathcal{O}_1$ is a well-ordering of $\mathcal{O}_1$ similar to $<\{\alpha : \alpha$ is a recursive ordinal$\}, <>$ and such that $x \in \mathcal{O}_1$ and $R(y, x) \rightarrow y \in \mathcal{O}_1$.

Proof: This is noted in [G] and shown in [G2]. In [G] the term "constructive ordinal" is used. [R., §11.8] shows that constructive ordinals and recursive ordinals are the same.

Note 1. Since R is a recursively enumerable relation, we can effectively enumerate $\{y \mid y <_R x\}$.

Note 2. If $a \in \mathcal{O}_1$, then $|a|$ is the ordinal corresponding to a, by definition.

Note 3. Without loss of generality, we can assume

(i) $0 \notin \mathcal{O}_1$

(ii) $1 \in \mathcal{O}_1$ and $|1| = 0$.

Note 4. In all further discussions any reference to a relation R will mean the R of Proposition 5, and for typographical convenience we will replace $<_{\mathcal{O}_1}$ by $<_R$ and $\leq_{\mathcal{O}_1}$ by $\leq_R$.

CHAPTER II

ORDINAL RANK AND VALIDITY

2.1 GÖDEL NUMBERS

As our language $\mathcal{L}$ has only denumerably many symbols, we can associate a unique prime integer with each symbol, and vice versa. Assume this is done in an effective manner; that is, given any symbol, we can compute the corresponding prime, and given any prime we can effectively determine the corresponding symbol.

We can assign, to each sequence of symbols of $\mathcal{L}$, a unique gödel number. Each symbol sequence is mapped to an integer $2^{k_1} \cdot 3^{k_2} \cdots p_n^{k_n}$ where p_n is the n^{th} prime and k_n is the prime corresponding to the n^{th} symbol in the sequence. Also, since each integer has a unique decomposition into primes, we have an evident map from the integers onto the set of sequences of symbols. If $\mathcal{J}$ is a formula we define $GN(\mathcal{J})$ to be the gödel number for the symbol sequence $\mathcal{J}$.

Definitions:

(i) A set variable has rank b if it is a set variable with superscript b.

(ii) A set variable is unranked if it is an unsuperscripted set variable.

(iii) A variable is a proper variable if it is a number variable, or has rank b for some $b \in \mathcal{O}_1$.

(iv) If all the set variables of $\mathcal{J}$ are ranked, then $\mathcal{J}$ is a ranked formula. If all the variables of $\mathcal{J}$ are proper, then $\mathcal{J}$ is a proper formula.

We can define a function $OR(n)$, which is loosely termed the ordinal rank of formulas. The definition is by cases.

(I) If n is not the gödel number of a formula, then $OR(n)$ is not defined.

(II) Suppose n is the gödel number of a formula $\mathcal{J}$; then

(1) If $\mathcal{J}$ has unranked set variables, then $OR(n)$ is not defined.

(2) If $\mathcal{J}$ has no set variables, $OR(n) = 0$.

(3) If $\mathcal{J}$ has only one superscript, a, then if a is the superscript only of free variables, $OR(n) = a$; else $OR(n) = 2^a$.

(4) If $\mathcal{J}$ has $A = \{a_1, \ldots, a_m\}$, $m > 1$, as distinct superscripts, we perform the following procedure.

At each stage in the procedure we perform one computation in the calculation of $R(a_i, a_j)$, for all i, j, $1 \leq i, j \leq m$. If, at any stage, there is some permutation of A, $\{b_1, \ldots, b_m\}$ such that $R(b_i, b_{i+1})$ holds for all i, $1 \leq i < m$, we say that "A has an apparent R-linear ordering with b_m as the greatest element". If b_m is the superscript of free variables only, then $OR(n) = b_m$; else $OR(n) = 2^{b_m}$. Of course, if we find no such permutation at any stage, OR(n) will remain undefined.

There are several points to note about the above definition. First, since by Proposition 5, R is recursively enumerable, and permutation generation is effective, OR is a partial recursive function. Second, if $\mathcal{J}$ is a proper ranked formula then OR($\mathcal{J}$) is always defined (here OR($\mathcal{J}$) is a convenient abbreviation for "OR(n) where n = GN($\mathcal{J}$)") since $R \mid \mathcal{O}_1$ is a well-ordering of $\mathcal{O}_1$. But the converse does not hold. Suppose $c \notin \mathcal{O}_1$; "$x \in X^c$" has rank c and by case (II) (3) of the definition of OR, $OR(x \in X^c) = c$.

Let $|a|$ be the ordinal associated with a. We reserve the right to say OR($\mathcal{J}$) = $|a|$ instead of OR($\mathcal{J}$) = a whenever this is convenient; that is usually in cases where we are given an ordinal α.

PROPOSITION 6

There is an effective enumeration of all formulas of ordinal rank $\leq_R |a|$, uniformly in a.

Proof. The statement of this proposition is equivalent to "the set of gödel numbers of formulas of ordinal rank $\leq_R |a|$ is recursively enumerable uniformly in a". Using the relation R, we can effectively enumerate the predecessors of a. At the n^{th} stage, we do the next computation for each of R(0, a), ..., R(n, a) and we check the numbers {0, ..., n} to see if they encode formulas involving only number variables and known predecessors of a as superscripts for set variables. We add all such gödel numbers to our enumeration.

Remark. Suppose we have an effective method for deciding if a symbol sequence of $\mathcal{L}$ has a certain form. Then we can effectively enumerate all such formulas of ordinal rank $\leq_R$ a, uniformly in a. This is clear from Proposition 6 and the fact that composition of recursive functions or relations yields another recursive function or relation.

2.2 PRENEX NORMAL FORMS

Given any formula $\mathcal{J}$ we may, using certain operations, obtain a new formula (perhaps the same as $\mathcal{J}$) called the prenex normal form. Intuitively, the prenex normal form of $\mathcal{J}$ has the same meaning as $\mathcal{J}$. A formula is in prenex normal form only if it has the form $(Q_1 x_1) \ldots (Q_n x_n)$ B where Q_i is either $\forall$n $\exists$ and x_i is a (set or number) variable, and B contains no quantifiers. The basic method for computing prenex normal forms is described in [Sh., §3.5], but we require that set function variables be handled the same way as number variables and that the method be completely specified so that each formula has only one corresponding prenex normal form. Finally, we define a recursive function PNF:

PNF(n) = the godel number of the prenex normal form of $\mathcal{J}$ where GN($\mathcal{J}$) = n, if n is the godel number of a formula.

= 0 otherwise .

We say a formula is a prenex formula if it is in prenex normal form.

As an abuse of notation, we may also write $PNF(\mathcal{J})$, where $PNF(\mathcal{J}) = PNF(n)$ when $n = GN(\mathcal{J})$.

2.3 DEFINITION OF THE CLASSES $\mathfrak{M}_\alpha$

We now wish to define certain sets $\mathfrak{M}_\alpha$ and validity, $\models_\alpha$ for recursive ordinals α. First we will devise an intuitive notion of validity, $\models'_\alpha$. We assume for this paragraph that $\mathcal{S}(x)$ represents a prenex formula of $\mathcal{L}$ with just x free. If $OR(\mathcal{S}(x)) = 0$, then $\models'_0 \mathcal{S}(\bar{n})$ means $\mathcal{S}(\bar{n})$ is true as a statement about integers. We define $\mathfrak{M}'_0 = \{S \mid$ There exists a formula $\mathcal{S}(x)$ such that $OR(\mathcal{S}(x)) = 0$ and $S = \{x \mid \models_0 \mathcal{S}(\bar{x})\}\}$. Now, suppose we have defined $\models'_\beta$ and $\mathfrak{M}'_\beta$ $\forall\beta < \alpha$. Let $|a| = \alpha$. Then, if $\mathcal{J}$ is a prenex sentence and $OR(\mathcal{J}) \leq_R a$, $\models'_\alpha \mathcal{J}$ if, when we interpret each superscripted set variable of rank b as ranking over $\mathfrak{M}'_{|b|}$ and number variables as ranging over the natural numbers, $\mathcal{J}$ is true. Finally, we define $\mathfrak{M}'_\alpha = \{S \mid$ there exists a formula $\mathcal{S}(x)$ such that $OR(\mathcal{S}(x)) \leq_R a$ and $S = \{x \mid \models'_\alpha \mathcal{S}(\bar{x})\}\}$.

We also need a more formal definition of validity. Some set-theoretic preliminaries are necessary, however.

Let A and B be sets well-ordered by S_A and S_B, respectively. We define the <u>natural well-ordering</u> $S_{A\times B}$ of $A \times B$ (relative to S_A and S_B) by

$$S_{A\times B}(x, y) \leftrightarrow \{S_A(a_1, a_2) \vee (a_1 = a_2 \;\&\; S_B(b_1, b_2))\}$$

where $x = (a_1, b_1)$ and $y = (a_2, b_2)$ are arbitrary elements of $A \times B$.

Now, let P be a subset of $A \times B$. Then $\{a \mid (a, b) \in P\}$ has

S_A-first element, a_0, since S_A well-orders A. And similarly $\{b \mid (a_0, b) \in P\}$ has an S_B-first element b_0. Then, clearly (a_0, b_0) is an $S_{A\times B}$-first element of P.

By induction we can define a natural well-ordering of $A_1 \times (A_2 \times \ldots \times A_n)$. Assume $S_{n-1}(x, y)$ is a natural well-ordering of $A_2 \times \ldots \times A_n$. Then define $S_n(x, y)$ by $S_n(x, y) \leftrightarrow x = (a_1, \ldots, a_n)$ & $y = (b_1, \ldots, b_n)$ & $[S(a_1, b_1) \vee (a_1 = b_1$ & $S_{n-1}((a_2, \ldots, a_n), (b_2, \ldots, b_n)))]$, where S is the well-ordering for A_1. It is easy to see that S_n is indeed a well-ordering of $A_1 \times (A_1 \times \cdots \times A_n)$.

Definition: To each prenex sentence $\mathcal{J}$ of $\mathcal{L}$ we assign an ordered triple called the <u>R-greatest ordinal triple for $\mathcal{J}$</u> (also written $TRP(\mathcal{J})$ or $TRP(n)$ if $n = GN(\mathcal{J})$). The triple is $(OR(n), \rho(n), \psi(n))$ where ρ and ψ are defined as follows:

(a) If $OR(n) = 0$, $\rho(n)$ = the number of (number) quantifiers in $\mathcal{J}$, and $\psi(n) = 0$.

(b) If $OR(n)$ is defined and $\neq 0$ then, since $\mathcal{J}$ is a sentence, there is a c such that $OR(n) = 2^c$. $\rho(n)$ is the number of occurrences of quantifiers associated with superscript c, and $\psi(n)$ is the number of quantifiers to the left of the leftmost occurrence of a quantifier associated with superscript c.

(c) For all other n, $\rho(n)$ and $\psi(n)$ are undefined.

Clearly, ρ and ψ are partial recursive.

We wish to see that the proper prenex sentences' triples are well-ordered. When we define $\mathcal{O}_1^+ : x \in \mathcal{O}_1^+ \leftrightarrow x \in \mathcal{O}_1 \vee x = 0$, the triples are well-ordered by the natural well-ordering, Rl, for $\mathcal{O}_1^+ \times (N \times N)$.

($\mathcal{O}_1^+$ is well-ordered by $R' \mid \mathcal{O}_1^+$, where $R'(x, y) \leftrightarrow (x = 0 \;\&\; y \neq 0) \vee R(x, y)$).

In short, we are ordering proper prenex sentences first by ordinal rank, and then by the contents of their prefixes: i.e., by number of occurrences of an R-greatest superscript and finally by the number of occurrences of R-lesser quantifiers to the left of the leftmost occurrence of the R-greatest superscript.

We now define $\mathcal{M}_\alpha$ and $\models_\alpha$ for recursive ordinals α.

Let $\mathcal{B}(x)$ be a prenex formula of $\mathcal{L}$ of ordinal rank 0 with just x free. We say $\models_0 \mathcal{B}(\bar{n})$ means $\mathcal{B}(\bar{n})$ is true as a statement about integers. Then we define

$$\mathcal{M}_0 = \{S \mid \text{there exists a prenex formula } \mathcal{B}(x) \text{ of } \mathcal{L} \text{ with just } x \text{ free such that } OR(\mathcal{B}(x)) = 0 \text{ and } S = \{x \mid \models_0 \mathcal{B}(\bar{x})\}\}$$

Suppose we have defined $\models_\beta$ for all $\beta < \alpha$ and have defined $\models_\alpha \mathcal{B}$ for all prenex sentences $\mathcal{B}$ of $\mathcal{L}$ which have smaller (i.e., in the ordering of triples) ordinal triples than $\mathcal{J}$ has, where $\mathcal{J}$ is a prenex sentence of $\mathcal{L}$ such that $OR(\mathcal{J}) \leq_R a$, for $|a| = \alpha$, then $\models_\alpha \mathcal{J}$ if and only if one of the following five clauses holds: (assume $a \in \mathcal{O}_1 \ni |a| = \alpha$):

1./ $(\exists b)(R(b, a) \;\&\models_{|b|} \mathcal{J})$

2./ $\mathcal{J} = (\exists x)\mathcal{B}(x) \;\&\; OR(\mathcal{B}(x)) \leq_R a \;\&\models_\alpha \mathcal{B}(\bar{n})$ for some $n \in N$

3./ $\mathcal{J} = (\forall x)\mathcal{B}(x) \;\&\; OR(\mathcal{B}(x)) \leq_R a \;\&\models_\alpha \mathcal{B}(\bar{n})$ for all $n \in N$

4./ $\mathcal{J} = (\exists x^b)\mathcal{B}(x^b) \;\&\; OR(\mathcal{B}(x^b)) \leq_R a \;\&\; R(b, a) \;\&$ for some prenex formula $\mathcal{X}(x)$, with just x free and $OR(\mathcal{X}(x)) \leq_R b$,

$\models_\alpha \mathrm{PNF}(\mathscr{B}(\hat{x}\,\mathscr{N}(x)))$.

5./ $\mathscr{T} = (\forall X^b)\mathscr{B}(X^b)$ & $\mathrm{OR}(\mathscr{B}(X^b)) \leq_R a$ & $R(b, a)$ & for all prenex formulas $\mathscr{N}(x)$ with just x free and $\mathrm{OR}(\mathscr{N}(x)) \leq_R b$, $\models_\alpha \mathrm{PNF}(\mathscr{B}(\hat{x}\,\mathscr{N}(x)))$.

We wish to see that this is a proper inductive definition. As an illustration, we will examine clause 4. If $\mathrm{OR}(\mathrm{PNF}(\mathscr{B}(\hat{x}\,\mathscr{N}(x)))) <_R \mathrm{OR}(\mathscr{T})$, then $\mathrm{TRP}(\mathrm{PNF}(\mathscr{B}(\hat{x}\,\mathscr{N}(x)))) <_{R1} \mathrm{TRP}(\mathscr{T})$. If the ordinal ranks are equal then either

(i) $|b| + 1 = \alpha$.

In this case, b is the R-maximum superscript, and $\mathrm{PNF}(\mathscr{B}(\hat{x}\,\mathscr{N}(x)))$ has one less occurrence of it than $\mathscr{T}$ has.

or

(ii) $|b| + 1 < \alpha$.

In this case, b is not the R-maximum superscript, and thus the number of occurrences of the R-maximum superscript, c, is the same in $\mathscr{T}$ and $\mathrm{PNF}(\mathscr{B}(\hat{x}\,\mathscr{N}(x)))$. Note that $\mathscr{B}(\hat{x}\,\mathscr{N}(x))$ is the result of replacing all occurrences of $t \in X^b$ (for terms t) in $\mathscr{B}(X^b)$ by $\mathscr{N}(t)$. When we take $\mathrm{PNF}(\mathscr{B}(\hat{x}\,\mathscr{N}(x)))$ note that no quantifiers from $\mathscr{N}$ are moved into or to the left of the prefix of $\mathscr{B}$. We have one less quantifier to the left of the leftmost occurrence of the R-maximum superscript in $\mathrm{PNF}(\mathscr{B}(\hat{x}\,\mathscr{N}(x)))$ than in $\mathscr{T}$.

Thus, $\mathrm{PNF}(\mathscr{B}(\hat{x}\,\mathscr{N}(x)))$ has indeed a lesser ordinal triple than $\mathscr{T}$.

Finally, we define for $\alpha = |a|$, $\mathfrak{M}_\alpha = \{S \mid$ there exists a prenex formula $\mathscr{B}(x)$ with just x free such that $\mathrm{OR}(\mathscr{B}(x)) \leq_R a$ and $S = \{x \mid \models_\alpha \mathscr{B}(\bar{x})\}\}$.

<u>Remark</u>: We wish to assure ourselves that the formal and intuitive notions of validity do indeed coincide for proper prenex sentences. The argument is really by induction on ordinal triples. If, for example, $\mathcal{J} = (\exists X^{c})\mathcal{B}(X^{c})$, then $\models_{\alpha}\mathcal{J} \leftrightarrow \mathrm{OR}(\mathcal{B}(X^{c})) \leq_{R} a$, where $\alpha = |a|$ and there is a prenex formula $\mathcal{K}(x)$ with just x free, such that $\mathrm{OR}(\mathcal{K}(x)) \leq_{R} c$ and $\models_{\alpha}$ $\mathrm{PNF}(\mathcal{B}(\hat{x}\ \mathcal{K}(x)))$. Since $\mathrm{TRP}(\mathrm{PNF}(\mathcal{B}(\hat{x}\ \mathcal{K}(x)))) <_{R1} \mathrm{TRP}(\mathcal{J})$, $\models_{\alpha} \mathrm{PNF}(\mathcal{B}(\hat{x}\ \mathcal{K}(x))) \leftrightarrow \models'_{\alpha} \mathrm{PNF}(\mathcal{B}(\hat{x}\ \mathcal{K}(x)))$. It only remains to see that "there exists a formula $\mathcal{K}(x)$ of ordinal rank $\leq_{R} c$ such that $\models'_{\alpha} \mathcal{B}(\hat{x}\ \mathcal{K}(x))$" if and only if $\models'_{\alpha}$ $(\exists X^{c})\mathcal{B}(X^{c})$. Assume the left-hand side; clearly $X^{c} = \{x \mid \models'_{|c|}\mathcal{K}(\bar{x})\}$. The other direction holds from our intuitive interpretation of the ranked set variables. Arguments for the other inductive clauses (of the definition of $\models_{\alpha}$) are analogous.

In view of the coincidence of both definitions of validity, we will use only the symbol $\models_{\alpha}$. When the meaning is clear (i.e., when it is obvious what α is) and when notational convenience is desired, we may write $\models$, and the correct subscript will be assumed. We will employ usual properties of truth definitions: for example, "It is not possible that $\models \mathcal{J}$ and $\models \mathrm{PNF}(\neg\mathcal{J})$, for a proper prenex sentence $\mathcal{J}$." Also, for an arbitrary proper sentence $\mathcal{J}$ we have that $\models\mathcal{J} \leftrightarrow \models \mathrm{PNF}(\mathcal{J})$, so that our theory does not really depend on prenex sentences. This is clear from the intuitive definition of validity.

CHAPTER III

EACH SET IN $\mathcal{M}_\alpha$ IS A Δ_1^1 SET

3.1 TRUTH SETS

Definition: For each recursive ordinal α, we define $T_\alpha = \{GN(\mathcal{J}) \mid \mathcal{J}$ is a prenex formula with no free variables & $OR(\mathcal{J}) \leq_R a$ & $\models_\alpha \mathcal{J}\}$, where $\alpha = |a|$. T_α is called the "truth set for α".

<u>PROPOSITION 7</u>

$T_\alpha \notin \mathcal{M}_\alpha$ for all recursive ordinals α.

<u>Proof:</u> Assume $a \in N \ni |a| = \alpha$. Let $\langle \mathcal{J}_n(x) \rangle$ be an effective enumeration of prenex formulas of ordinal rank $\leq_R a$, with just x free. (The existence of such an enumeration is guaranteed by the remark following Proposition 6.)

Assume $T_\alpha \in \mathcal{M}_\alpha$. Then there is a prenex formula $\mathcal{K}(x)$ with just x free such that $OR(\mathcal{K}(x)) \leq_R a$, and such that $T_\alpha = \{n \mid \models_\alpha \mathcal{J}(\overline{n})\}$. Define $S = \{n \mid \models_\alpha PNF(\neg \mathcal{J}_n(\overline{n}))\}$. Then

$$\begin{aligned} n \in S &\leftrightarrow \models_\alpha PNF(\neg \mathcal{J}_n(\overline{n})) \\ &\leftrightarrow GN(PNF(\neg \mathcal{J}_n(\overline{n})) \in T_\alpha \\ &\leftrightarrow \models_\alpha \mathcal{K}\,(\overline{GN(PNF(\neg \mathcal{J}_n(\overline{n}))})) \\ &\leftrightarrow \models_\alpha \mathcal{K}'(\overline{n}) \end{aligned}$$

where $\mathcal{K}'(x) = \mathcal{K}(g(x))$, and g is a recursive function defined by

$$g(n) = GN(PNF(\neg \mathcal{J}_n(\overline{n})).$$

So we see that $S \in \mathcal{M}_\alpha$. But if $S \in \mathcal{M}_\alpha$ then there is an m_0 such that $S = \{n \mid \models_\alpha \mathcal{I}_{m_0}(\bar{n})\}$, and then $m_0 \in S \leftrightarrow \models_\alpha \mathcal{I}_{m_0}(\bar{m}_0)$ by the above but $m_0 \in S \leftrightarrow \models_\alpha \mathrm{PNF}(\neg \mathcal{I}_{m_0}(\bar{m}_0))$ by the definition of S. Thus we have a contradiction, and it must be that $T_\alpha \notin \mathcal{M}_\alpha$. ||

3.2 THE RELATION BETWEEN T_α AND $\mathcal{M}_\alpha$.

PROPOSITION 8

(i) $S \in \mathcal{M}_0 \leftrightarrow S$ is arithmetical.

(ii) $S \in \mathcal{M}_{\alpha+1} \leftrightarrow S$ is arithmetical in T_α.

Proof:

(i) This is obvious, since a formula of ordinal rank 0 has only number quantifiers.

(ii) Let us assume the left-hand side of the equivalence, i.e., $S \in \mathcal{M}_{\alpha+1}$. Then $S = \{n \mid \models_{\alpha+1} (Q_1W_1) \ldots (Q_mW_m)\ T(W_1, \ldots, W_m, \bar{n})\}$ for a quantifier-free formula T of ordinal rank $\leq_R a'$ where $|a'| = \alpha + 1$. For each b, fix $\langle \mathcal{I}_i^b(x) \rangle$ as an effective enumeration of all formulas, with just x free, of ordinal rank $\leq_R b$ $(|b| \leq \alpha)$. We may replace number variables by set variables by an analogue to E1 of Section 1.3; so we may consider all the W_i to be set variables such that rank $(W_i) = b_i$ and $|b_i| \leq \alpha$.

By the remark following Proposition 6, we may define a recursive function f by $f(\ell_1, \ldots, \ell_m, n) = \mathrm{GN}(\mathrm{PNF}(T(\hat{x}\mathcal{I}_{\ell_1}^{b_1}(x), \ldots, \hat{x}\mathcal{I}_{\ell_m}^{b_m}(x), \bar{n})))$. Then

$$n \in S \leftrightarrow \models_{\alpha+1} (Q_1W_1) \ldots (Q_mW_m)\ T(W_1, \ldots, W_m, \bar{n})$$

$$\leftrightarrow (Q_1\ell_1) \ldots (Q_m\ell_m) \models_{\alpha+1} \mathrm{PNF}(T(\hat{x}\,\mathcal{J}_{\ell_1}^{b_1}(x), \ldots, \hat{x}\,\mathcal{J}_{\ell_m}^{b_m}(x), \bar{n}))$$

$$\leftrightarrow (Q_1\ell_1) \ldots (Q_m\ell_m) \models_{\alpha} \mathrm{PNF}(T(\hat{x}\,\mathcal{J}_{\ell_1}^{b_1}(x), \ldots, \hat{x}\,\mathcal{J}_{\ell_m}^{b_m}(x), \bar{n}))$$

$$\leftrightarrow (Q_1\ell_1) \ldots (Q_m\ell_m)\ [f(\ell_1, \ldots, \ell_m, n) \in T_\alpha]$$

or, S is arithmetical in T_α.

The various enumerations are needed; for if $|b| < \alpha$ and $\mathcal{J}$ has only x^b free, then $\models_\alpha (\exists x^b)\mathcal{J}(x^b)$ just if $(\exists\ell_1)\ [\mathrm{OR}(\mathcal{J}_{\ell_1}(x)) \leq_R b\ \&$ $\models_{|b|} \mathcal{J}(\hat{x}\,\mathcal{J}_{\ell_1}(x))$.

We will now show that if we assume $T_\alpha \in \mathcal{M}_{\alpha+1}$ then we can prove the right-to-left implication of Proposition 8 - (ii). If $T_\alpha \in \mathcal{M}_{\alpha+1}$ then there is a formula $\mathcal{J}(x)$ with just x free such that $\mathrm{OR}(\mathcal{J}(x)) \leq_R a'$ and $T_\alpha = \{n \mid \models_{\alpha+1} \mathcal{J}(\bar{n})\}$. If S is arithmetical in T_α then there is an arithmetical relation A such that $n \in S \leftrightarrow A(n, T_\alpha)$. But then there is a formula $\mathcal{B}(x, X)$ with no ranked set variables such that $A(n, T_\alpha) \leftrightarrow \models_{\alpha+1} \mathcal{B}(\bar{n}, \hat{z}\,\mathcal{J}(z))$. Then $m \in S \leftrightarrow \models \mathrm{PNF}(\mathcal{B}(\bar{m}, \hat{z}\,\mathcal{J}(z)))$. This last formula has ordinal rank $\leq \alpha + 1$ since $\mathrm{OR}(\mathcal{J}(x)) \leq_R a'$.

It remains to prove $T_\alpha \in \mathcal{M}_{\alpha+1}$. We define the following relations Δ_1, Δ_2, Δ_3, and Δ_4:

$$\Delta_1(g, X) \leftrightarrow g = \mathrm{GN}(\mathrm{PNF}((\exists x)\mathcal{B}(x)))\ \&$$
$$(\exists n)(\mathrm{GN}(\mathrm{PNF}(\mathcal{B}(\bar{n}))) \in X).$$

$$\Delta_2(g, X) \leftrightarrow g = \mathrm{GN}(\mathrm{PNF}((\forall x)\mathcal{B}(x)))\ \&$$
$$(\forall n)(\mathrm{GN}(\mathrm{PNF}(\mathcal{B}(\bar{n}))) \in X).$$

$$\Delta_3(g, x, X) \leftrightarrow g = \mathrm{GN}(\mathrm{PNF}((\exists x^a)\mathcal{B}(x^a))) \text{ where } R(a, x),$$

and there is a prenex formula $\mathcal{K}(z)$ with just z free such that

$OR(\mathscr{K}(z)) \leq_R a$, where $|a| = \alpha$, and $GN(PNF(\mathscr{G}(\hat{z}\, \mathscr{K}(z))) \in X]$.

and

$$\Delta_4(g, x, X) \leftrightarrow [g = GN(PNF((\forall X^a)\mathscr{G}(X^a))) \text{ where } R(a, x),$$

and for all prenex formulas $\mathscr{K}(z)$ with just z free such that $OR(\mathscr{K}(z)) \leq_R a$, then $GN(PNF(\mathscr{G}(\hat{z}\, \mathscr{K}(z)))) \in X]$.

Finally, define $P(g, x)$:

(*) $P(g, x) \leftrightarrow (\exists y)$ (g is the gödel number of a prenex sentence of ordinal rank $\leq_R y$ where $R(y, x)$).

The statement within parentheses on the right-hand side of the equivalence is recursively enumerable, from the definitions of OR and R.

We can now define Γ:

$$\begin{aligned} \Gamma(x, X, Y) \leftrightarrow (\forall g)\{g \in Y \;.\!\leftrightarrow\!.\; & P(g, x) \\ & \& \; [g \in X \vee \Delta_1(g, X) \vee \Delta_2(g, X) \\ & \qquad \vee \Delta_3(g, x, X) \vee \Delta_4(g, x, X)] \end{aligned}$$

Lemma 9: Γ is an arithmetical relation.

<u>Proof</u>: We shall examine only Δ_3, since the proofs for the other Δ's are quite analogous.

We define:

$$\begin{aligned} SUB(a, b, c) &= GN(PNF(\mathscr{G}(\hat{z}\, \mathscr{K}(z)))) \text{ if } a = GN(PNF(\mathscr{K}(z))) \\ &\qquad \text{and } b = GN(PNF((\exists X^c)\mathscr{G}(X^c))) \\ &= 0 \text{ if } a \text{ and } b \text{ are not gödel numbers} \\ &\qquad \text{of syntactically suitable formulas.} \end{aligned}$$

$G(a, c) \leftrightarrow a$ is the gödel number of a prenex formula which begins with $\exists x^c$.

SUB and G are easily computed by gödel number decoding and symbol manipulation, and are evidently recursive.

Also, we define a recursive relation Q

$$Q(g, a) \leftrightarrow g \text{ is the gödel number of a prenex formula of ordinal rank } \leq_R a, \text{ with just } z \text{ free.}$$

Q is clearly recursively enumerable.

Then, $$\Delta_3(g, x, X) \leftrightarrow (\exists a)(R(a, x) \mathbin{\&} G(g, a) \mathbin{\&} (\exists h)(Q(h, a) \mathbin{\&} SUB(h, g, a) \in X)).$$

Q is recursively enumerable, since by Proposition 6 we have an effective enumeration of all formulas of ordinal rank $\leq_R a$, uniformly in a.

Thus, by inspection Δ_3 is arithmetical. $||$

Definition: For any set S we define $S[x_1, \ldots, x_n] = \{y \mid \langle x_1, \ldots, x_n, y\rangle \in S\}$.

This definition can be expressed by a formula with only number quantifiers, so we may write it in the language of Ramified Analysis as well. We now proceed to prove $T_\alpha \in \mathcal{M}_{\alpha+1}$ by induction on recursive ordinals α. (Subscripts are ordinals and superscripts are ordinal notations, unless otherwise indicated.)

Case I: $\alpha = 0$.

Recall that $|1| = 0$. We wish to show

(1) $x \in T_0 \leftrightarrow \models_1 (\exists Y^1)(\exists n)[\{Y^1[0]$ = the set of gödel numbers of true

quantifier-free sentences} & $(\forall y)_{y<n}\ \Gamma(1, Y^1[y], Y^1[y+1])$ & $x \in Y^1[n]]$.

Proof: First, we define Y such that $Y(x)$ = the set of gödel numbers of true prenex sentences with $\leq x$ quantifiers, all of which are number quantifiers. Note that $Y(0)$ is recursive, since it just contains true variable-free arithmetic statements, e.g., $\overline{0} = \overline{0}$, $\overline{1} + \overline{1} = \overline{2}$. Now

$$\begin{aligned} g \in Y(n+1) \leftrightarrow {} & [g = GN((\exists x)S(x)) \ \&\ (\exists m) \\ & \quad GN(S(\overline{m})) \in Y(n)] \\ & \vee\ [g = GN((\forall x)S(x)) \ \&\ (\forall m) \\ & \quad GN(S(\overline{m})) \in Y(n)] \ . \end{aligned}$$

Thus $Y(n)$ is arithmetical for all n by induction.

It is clear from the definition of $\Gamma(\Delta_1$ and $\Delta_2)$ and $Y(\ )$ that $Y(n+1)$ is the unique Z such that $\Gamma(1, Y(n), Z)$.

Let $x \in T_0$. Suppose the sentence whose gödel number is x has m quantifiers. Then the right hand side of (1) is satisfied for x by taking $Y^1 = \{\langle v, w\rangle \mid v \leq m \ \&\ w \in Y(v)\}$ and $n = m$, since Y^1 ranges over $\mathfrak{M}_{|1|} = \mathfrak{M}_0$.

Now suppose x satisfies the right hand side of (1). Clearly $Y^1[0] = Y(0)$. We know $\Gamma(1, Y(y), Y(y+1))$ for all y and by hypothesis $\Gamma(1, Y^1[y], Y^1[y+1])$ for all $y < n$, so by induction $Y^1[y] = Y(y)$ for all $y \leq n$. But then $x \in Y^1[n] \rightarrow x \in Y(n)$ so $x \in T_0$. We conclude $\{x \mid x \in T_0\} \in \mathfrak{M}_1$.

Case II: $|b| = |c| + 1$.

We wish to show

(2) $x \in T_{|b|} \leftrightarrow \models_{|b|+1} (\exists Y^b)(\exists n)[\{Y^b[0] = T_{|c|}\}$

$\& (\forall y)_{y<n} \Gamma(b, Y^b[y], Y^b[y + 1]) \& x \in Y^b[n]]$.

Proof: This case is very similar to Case I. We define $Y(x) =$ set of gödel numbers of true prenex sentences obtained from formulas of ordinal rank $\leq |c|$ by adding at most x quantifiers - these quantifiers are either number or set quantifiers on X^a where $R(a, c)$. Clearly $Y(0) = T_{|c|}$. By induction we see that $Y(n)$ is arithmetical in $T_{|c|}$ for all n. As before $Y(n + 1)$ is the unique Z such that $\Gamma(b, Y(n), Z)$.

The proof is exactly as for Case I: replace all superscript 1's by b's, 1 by b in Γ, and 0 (= $|1|$) by $\leq_R b$ in such statements as "x is a formula of ordinal rank 0". We conclude $\{x \mid x \in T_{|b|}\} \in \mathfrak{M}_{|b|+1}$.

Case III. $|b|$ is a limit ordinal.

(3) $x \in T_{|b|} \leftrightarrow \models_{|b|+1} (\exists X^b)(\exists m)(m <_R b) \& \{X^b[1, 0] =$ the set of true quantifier-free sentences$\} \& (\forall y)(y \neq 1 \& y \leq_R m .\rightarrow. [X^b[y, 0] = \bigcup\{X^b[z, w] \mid z <_R y \& w \in N\}]) \& (\forall y)(\forall z)(y \leq_R m .\rightarrow. \Gamma(y, X^b[y, z], X^b[y, z + 1])) \& x \in \bigcup\{X^b[m, z] \mid z \in N\})$.

Proof: Let us define X such that $X(d, x) =$ the set of gödel numbers of true prenex sentences of ordinal rank $\leq_R d$, obtained by prefixing at most x quantifiers, either number or of the form QX^a where $R(a, d)$ or $a = d$, to a formula of ordinal rank $<_R d$.

Assume $x \in T_b$. x is the gödel number of some true prenex sentence of ordinal rank $\leq_R b$, say $\mathcal{J}$. Since $\mathcal{J}$ is a sentence and $|b|$ is a limit ordinal, $OR(\mathcal{F}) = c$ for some c such that $c <_R b$. Choose $X^b = \{\langle x, y, z\rangle \mid x \leq_R c \ \& \ y \in N \ \& \ z \in X(x, y)\}$. Note that $w \in X^b[y, z] \leftrightarrow \langle y, z, w\rangle \in X^b \leftrightarrow y \leq_R c \ \& \ z \in N \ \& \ w \in X(y, z)$.

Thus $X^b[y, z] = X(y, z)\ \forall y \leq_R c$ and $z \in N$. In particular, $X^b[1, 0] = X(1, 0)$, so $X^b[1, 0]$ is indeed the set of gödel numbers of true quantifier-free sentences.

Assume $m \leq_R c$. Then

$$(\forall y)_{1 <_R y \leq_R m} [X^b[y, 0] = \cup\{X^b[z, w] \mid z <_R y \ \& \ w \in N\}]$$

is the same as

$$(\forall y)_{1 <_R y \leq_R m} [X(y, 0) = \cup\{X(z, w) \mid z <_R y \ \& \ w \in N\}] .$$

But $x \in X(y, 0) \leftrightarrow x$ is the gödel number of a true prenex sentence of ordinal rank $\leq_R y$, obtained by prefixing at most 0 quantifiers to formula $\mathcal{J}$ of ordinal rank $<_R y$.

$\leftrightarrow x$ is the gödel number of a true sentence of ordinal rank v where $R(v, y)$, which has w quantifiers which are number quantifiers or of the form QX^s, where $R(s, v)$.

$\leftrightarrow (\exists v)(\exists w)(x \in X(v, w) \ \& \ R(v, y) \ \& \ w \in N)$

$\leftrightarrow x \in \cup\{X(z, w) \mid z <_R y \ \& \ w \in N\}$.

Finally, fix $m = c$. Then $x \in \bigcup_{z \in N} X^b[m, z]$, since $x \in X(c, w)$ where w is the number of quantifiers of the formula whose gödel number is x.

From the definitions of Γ and $X(\ ,\ )$ it is clear that $\Gamma(y, X(y, x), X(y, x+1))$ for all x and y (and also that $X(y, x+1)$ is the unique Z such that $\Gamma(y, X(y, x), Z)$). And since $\forall y \leq_R c$ and all x, $X(y, x) = X^b(y, x)$, we get for all $y \leq_R c$ and all x, $\Gamma(y, X^b[y, z], X^b[y, z+1])$.

Thus all the clauses of the right hand side of (3) are satisfied.

To prove the equivalence from right to left, assume we have a suitable X^b. Clearly $X^b[1, 0] = X(1, 0)$. We have $\Gamma(1, X^b[1, x], X^b[1, x+1])$ and $\Gamma(1, X(1, x), X(1, x+1))$ for all $x \in N$. So $X^b[0, x] = X(0, x)$. Assume $1 <_R y$ and $y \leq_R m$ and $X^b[z, w] = X(z, w)$ for all w, and all z such that $z <_R y$. Now

$$\begin{aligned} X^b[y, 0] &= \bigcup_{z <_R y} \bigcup_{w \in N} X^b[z, w] \\ &= \bigcup_{z <_R y} \bigcup_{w \in N} X(z, w) \\ &= X(y, 0) \quad \text{from the definition of } X(y, 0)\ . \end{aligned}$$

But since $\forall y \leq_R m$ and all x, $\Gamma(y, X^b[y, x], X^b[y, x+1])$ and $\Gamma(y, X(y, x), X(y, x+1))$ for all x, we obtain $X^b[y, x] = X(y, x)\ \forall y \leq_R m$ and all x. Therefore $x \in \bigcup_{z \in N} X^b[m, z]$ implies that there exists a w such that $x \in X^b[m, w]$ so $x \in X(m, w)$. But this just says $x \in T_m$ so $x \in T_b$.

This ends the proof of Lemma 9. ||

Note that we could combine our cases into one comprehensive case:

(4) $x \in T_{|b|} \leftrightarrow \models_{|b|+1} (\exists X^b)(\exists Y^b)(\exists m)(\exists n)[R(m, b)\ \&\ (X^b[1, 0] =$

the set of true quantifier-free sentences)

$\& \ (\forall y)_{1 <_R y \leq_R m}(X^b[y, 0] = \cup\{X^b[z, w] \mid z <_R y \ \& \ w \in N\})$

$\& \ (\forall y)_{y <_R m} (\forall z) \ \Gamma(y, X^b[y, z], X^b[y, z + 1])$

$\& \ Y^b[0] = \cup\{X^b[m, z] \mid z \in N\}$

$\& \ x \in Y^b[n].$

When $|b|$ is a limit ordinal, $n = 0$.

Note that the right hand side of (4) easily yields a ranked formula $\mathcal{J}(x)$ such that $T_{|b|} = \{x \mid \models_{|b|+1} \mathcal{J}(\bar{x})\}$.

At this point we digress slightly to make an interesting observation about the union of truth sets. Define $T = \{GN(\mathcal{J}) \mid \mathcal{J}$ is a proper prenex sentence and $\exists a$ such that $a \in \mathcal{O}_1$ and $\models_{|a|} \mathcal{J}\}$. Clearly, $T = \bigcup_{a \in \mathcal{O}_1} T_{|a|}$. We can show that T is a Π^1_1 set. $x \in T \leftrightarrow [x$ is the gödel number of a proper prenex sentence $\mathcal{J}]$ & $(\exists y)[OR(\mathcal{J}) = y \ \& \ (\forall X)(X[1, 0] =$ the set of gödel numbers of all variable-free sentences & $((\forall z)_{z \leq_R y}(\forall n) \ \Gamma(z, X[z, u], X[z, u + 1]) \rightarrow x \in \cup\{X[u, w] \mid u <_R y \ \& \ w \in N\})]$. Now, "$x$ is the gödel number of a proper prenex sentence of $\mathcal{J}$" is Π^1_1 since it involves examining the superscripts of the set variables in $\mathcal{J}$ for membership in $\mathcal{O}_1$. Since OR is a partial recursive function and Γ is arithmetical, T must be a Π^1_1 set.

3.3 $\mathcal{M}$ AND Δ^1_1 SETS

Definition: $\mathcal{M} = \cup \mathcal{M}_\alpha$ for $\alpha \in$ recursive ordinals. We may alternately write $\mathcal{M} = \bigcup_{a \in \mathcal{O}_1} \mathcal{M}_{|a|}$.

Definition: A set P is $\underline{\Sigma_1^1}$ if there is an arithmetical relation S such that $x \in P \leftrightarrow (\exists X) S(x, X)$. We may make alternate definitions of Σ_1^1 sets, as was done for Π_1^1 sets in Section 1.3, and the analogous propositions will hold.

Note that if a set is Π_1^1 then its complement, $\overline{P}$, is Σ_1^1, and vice versa: i.e.

$$\begin{aligned} x \in \overline{P} &\leftrightarrow \neg (\exists X) S(x, X) \\ &\leftrightarrow (\forall X) \neg S(x, X) \\ &\leftrightarrow (\forall X) S^1(x, X) \end{aligned}$$

where $S^1(x, X) \equiv \neg S(x, X)$ is clearly also arithmetical.

Definition: A set P is Δ_1^1 if it is both Σ_1^1 and Π_1^1.

PROPOSITION 10

$T_{|b|}$ is a Σ_1^1 set.

Proof: In (4) above it was shown that

$$x \in T_{|b|} \leftrightarrow (\exists Z^b)\ \Delta(Z^b, x)$$

where $(\exists Z^b)\ \Delta(Z^b, x)$ is the result of collapsing the existential quantifiers in the right hand side of (4). Note that Δ is arithmetical; also, from the proof of (3), that in fact $(\exists Z)\ \Delta(Z, x) \leftrightarrow (\exists Z^b)\ \Delta(Z^b, x)$. ||

PROPOSITION 11

If $S \in \mathfrak{M}$, then S is a Δ_1^1 set.

Proof: Say $S \in \mathfrak{M}$. Then there is an a such that $S \in \mathfrak{M}_{|a|}$. Thus there is a formula $\mathcal{J}(x)$ of ordinal rank $\leq_R a$ such that $S = \{x \mid \models_{|a|} \mathcal{J}(\bar{x})\}$. Then $x \in S \leftrightarrow \models_{|a|} \mathcal{J}(\bar{x})$

$$\leftrightarrow GN(\mathcal{J}(\bar{x})) \in T_{|a|}.$$

So S is $1-1$ reducible to $T_{|a|}$. By the obvious analogue to Proposition 3, S is Σ_1^1. But since $S \in \mathfrak{M}_{|a|}$, $\bar{S} \in \mathfrak{M}_{|a|}$ and similarly $\bar{S}$ is Σ_1^1. Thus S is Δ_1^1. ||

The ultimate significance of Δ_1^1 sets in this thesis lies in the Characterization Theorem - [Sh., 7.10] -

A set is Δ_1^1 if and only if it is hyperarithmetical.

Thus, we have shown that each set in $\mathfrak{M}_\alpha$ is a hyperarithmetical set. It remains to show the converse.

CHAPTER IV

EACH HYPERARITHMETICAL SET IS IN $\mathfrak{M}$

4.1 HYPERARITHMETICAL SETS

Let $\langle \phi_i \rangle$ be a standard enumeration of all unary partial recursive functions. We define W_i = range ϕ_i, as in the Introduction.

We define <u>H-index</u> and <u>hyperarithmetical set</u> as in [Sh., §7.9].

Definition: The following three rules constitute an inductive definition of <u>H-index</u>:

1. For all e, $\langle 0, e \rangle$ is an H-index.
2. If e is an H-index, $\langle 1, e \rangle$ is an H-index.
3. If each $d \in W_e$ is an H-index, then $\langle 2, e \rangle$ is an H-index.

Definition: We define the hyperarithmetical sets J_i for H-indices i, as follows:

1.) if $i = \langle 0, e \rangle$, $J_i = W_e$

2.) if $i = \langle 1, e \rangle$, $J_i = \overline{J}_e$

3.) if $i = \langle 2, e \rangle$, $J_i = \bigcup J_x$, $x \in W_e$.

Now, let $*$ be a concatenation operator on sequence numbers such that if $\sigma_0 = \langle a_1, \ldots, a_m \rangle$ and $\sigma_1 = \langle b_1, \ldots, b_n \rangle$ then $\sigma_0 * \sigma_1 = \langle a_1, \ldots, a_m, b_1, \ldots, b_n \rangle$. This operator has the obvious properties that $(\sigma_1 * \sigma_2) * \sigma_3 = \sigma_1 * (\sigma_2 * \sigma_3)$ and, if $\pi = \langle \emptyset \rangle$, that $\sigma * \pi = \pi * \sigma = \sigma$ for sequence numbers σ, σ_1, σ_2, and σ_3. Given any $i \in N$ we may define a set $\mathcal{S}_i$ thusly:

Rule 1: $\langle i\rangle \in \mathcal{S}_i$

Rule 2: if $\sigma * \langle j\rangle \in \mathcal{S}_i$ and $j = \langle i, e\rangle$ then $\sigma * \langle j, e\rangle \in \mathcal{S}_i$

and

Rule 3: if $\sigma * \langle j\rangle \in \mathcal{S}_i$ and $j = \langle 2, e\rangle$ then, for each $k \in W_e$, $\sigma * \langle j, k\rangle \in \mathcal{S}_i$.

If i is not an H-index, then $\mathcal{S}_i$ will not have a structure of any interest to us; but there is no reason to restrict the definition of $\mathcal{S}_i$ to H-indices.

We can show several interesting facts about the $\mathcal{S}_i$'s and about their inter-relationships.

Note 1. For a given i, inspection of the definition of $\mathcal{S}_i$ shows that each member of $\mathcal{S}_i$ is of the form $\langle i\rangle * \rho$ where ρ is some sequence number (perhaps that of the null sequence).

Note 2. If $i = \langle 1, e\rangle$ and $\rho \in \mathcal{S}_e$ then $\langle i\rangle * \rho \in \mathcal{S}_i$. To see this, consider $\rho = \langle a_1, \ldots, a_n\rangle$ where $n \geq 1$. Now since $\rho \in \mathcal{S}_e$, $a_1 = e$ and $\langle i\rangle * \langle e\rangle \in \mathcal{S}_i$ by Rule 2. Then $\langle i, e\rangle * \langle a_2\rangle \in \mathcal{S}_i$ by application of the same rule (either Rule 2 or 3) which insured that $\langle e\rangle * \langle a_2\rangle \in \mathcal{S}_e$. By induction, $\langle i\rangle * \rho = \langle i\rangle * \langle a_1\rangle * \ldots * \langle a_n\rangle \in \mathcal{S}_i$.

Note 3. If $i = \langle 2, e\rangle$ and $\rho \in \mathcal{S}_f$ where $f \in W_e$ then $\langle i\rangle * \rho \in \mathcal{S}_i$. The proof is as in Note 2 except that since $\rho \in \mathcal{S}_f$,

$a_1 = f$ and $<i> * <f> \in \mathcal{S}_i$ by Rule 3.

Note 4. There are converses of a sort to Notes 2 and 3. If $i = <1, e>$ and $<i> * \rho \in \mathcal{S}_i$, where $\rho \neq <\emptyset>$, then $\rho \in \mathcal{S}_e$. This is easy to see since $i = <1, e>$, $\rho = <e, a_1, \ldots, a_n>$. Now $<e> \in \mathcal{S}_e$. $<e> * <a_1> \in \mathcal{S}_e$ is in $\mathcal{S}_e$ by application of the same rule that put $<i> * <e> * <a_1>$ in $\mathcal{S}_i$. By induction we obtain $\rho \in \mathcal{S}_e$. Again, the "limit ordinal" case is very similar. When $i = <2, e>$, the statement of the Note is: "if $<i> * \rho \in \mathcal{S}_i$, where $\rho \neq <\emptyset>$, then $\exists f \ni f \in W_e$ and $\rho \in \mathcal{S}_f$". The proof is the same; since $i = <2, e>$, $\rho = <f, a_1, \ldots, a_n>$ and we can show $\rho \in \mathcal{S}_f$ by induction as above in this note.

Note 5. The previous notes imply the following:

$$\text{If } i = <1, e> \text{ then } \mathcal{S}_i = \{\sigma \mid \sigma = <i> * \rho,\ \rho \in \mathcal{S}_e\} \cup \{<i>\}.$$

$$\text{If } i = <2, e> \text{ then } \mathcal{S}_i = \{\sigma = <i> * \rho,\ (\exists f)(f \in W_e \ \& \ \rho \in \mathcal{S}_f)\} \cup \{<i>\}.$$

Let us show only the second of the two equations, for it is slightly more complex. Note that $<i>$ is in both sets. All elements of $\mathcal{S}_i$, other than $<i>$ are of the form $<i> * \rho$ where $\rho \neq <\emptyset>$. By Note 4 there is an f such that $f \in W_e$ and $\rho \in \mathcal{S}_f$. Note 3 shows the relation in the other direction.

We now define an ordering of the set of all sequence numbers. First we select a simultaneous effective enumeration of all the W_j's with the properties that at each stage in the computation at most one W_j gains a member and that W_j gets only one new member at any stage.

Let σ and τ be sequence numbers. We say $\sigma \subset \tau$ if there is a sequence number ρ such that $\rho \neq \langle\emptyset\rangle$ and $\sigma * \rho = \tau$. Then $\sigma \subseteq \tau$ if $\sigma \subset \tau$ or $\sigma = \tau$.

Then we define a relation $<$ on $N \times N$:

For $\sigma_0, \sigma_1 \in N$,

$\sigma_0 < \sigma_1 \leftrightarrow \sigma_0$ and σ_1 are sequence numbers &

$[\sigma_1 \subset \sigma_0 \ .\mathrm{v}.\ (\exists\rho)(\exists j, e)(\exists k_0, k_1)\{\rho * \langle j\rangle$ is the number of the longest sequence common to σ_0 and σ_1 & $j = \langle 2, e\rangle$ & $\rho * \langle j, k_0\rangle \subseteq \sigma_0$ & $\rho * \langle j, k_1\rangle \subseteq \sigma_1$ & k_0 enters W_e before $k_1]$.

$<$ is a recursively enumerable relation since the set of sequence numbers is recursive, as we have a special effective enumeration of the W_e's, and since we can effectively examine and compare the sequence associated with each sequence number.

Let us define $<_i$ as the restriction of $<$ to $\mathcal{S}_i$. Again, it is clear that $<_i$ is a recursively enumerable relation, since each $\mathcal{S}_i$ is a recursively enumerable set. We now wish to show that, when i is an H-index, $<_i$ well-orders $\mathcal{S}_i$.

We are able to show the relation between $<_i$ and $<_m$ where m is a "lower" H-index.

<u>Note 6</u>. Assume $\sigma_1 \neq \langle i\rangle$.

If $i = \langle 1, e\rangle$, $\sigma_0 <_i \sigma_1$, and $\sigma_0 = \langle i\rangle * \rho_0$, $\sigma_1 = \langle i\rangle * \rho_1$, then $\rho_0 <_e \rho_1$.

If $i = \langle 2, e\rangle$, $\sigma_0 <_i \sigma_1$, $\sigma_0 = \langle i\rangle * \rho_0$, $\sigma_1 = \langle i\rangle * \rho_1$, then

$(\exists m)(m \in W_e,\ \rho_0, \rho_1 \in \mathcal{S}_m,\ \rho_0 <_m \rho_1) \vee (\exists k_0, \exists\, k_1)(k_0, k_1 \in W_e$ such that $\rho_0 \in \mathcal{S}_{k_0}, \rho_1 \in \mathcal{S}_{k_1}$ and k_0 enters W_e before k_1).

Only the proof for the case $i = \langle 2, e\rangle$ will be shown. The proof for the case $i = \langle 1, e\rangle$ is similar but much simpler. If $\sigma_1 \subset \sigma_0$, then $\sigma_1 = \langle a_1, \ldots, a_m\rangle$ and $\sigma_0 = \langle a_1, \ldots, a_n\rangle$ where $2 \leq m < n$ so $\rho_1 \subset \rho_0$ and thus $\rho_0 <_{a_2} \rho_1$. If $\sigma_1 \not\subset \sigma_0$ it may be that just $\langle i\rangle$ is common to σ_0 and σ_1. Then since $i = \langle 2, e\rangle$ there exist $k_0, k_1 \in W_e$ such that $\langle i, k_0\rangle \subseteq \sigma_1$ and k_0 enters W_e before k_1. Clearly $\rho_0 \in \mathcal{S}_{k_0}$ and $\rho_1 \in \mathcal{S}_{k_1}$ by Note 4 and thus we obtain the last clause. Let $\langle i, a_1, \ldots, a_n\rangle$ be the number of the greatest sequence common to σ_0 and σ_1. Now $a_1 \in W_e$, since $i = \langle 2, e\rangle$; so, $\rho_0, \rho_1 \in \mathcal{S}_{a_1}$. Since $\sigma_0 <_i \sigma_1$ it must be that there are $\ell_0, \ell_1 \in W_e$ such that $\langle i, a_1, \ldots, a_n, \ell_0\rangle \subseteq \sigma_0$ and $\langle i, a_1, \ldots, a_n, \ell_1\rangle \subseteq \sigma_1$ and ℓ_0 enters W_{a_n} before ℓ_1. But this suffices to insure $\rho_0 <_{a_1} \rho_1$, and so clause 1 holds.

We have converses to Note 6.

<u>Note 7</u>. If $i = \langle 1, e\rangle$ and $\rho_0 <_e \rho_1$ then $\langle i\rangle * \rho_0 <_i \langle i\rangle * \rho_1$.

If $i = \langle 2, e\rangle$ and $\exists f \in W_e \ni \rho_0, \rho_1 \in \mathcal{S}_f$ and $\rho_0 <_f \rho_1$, then $\langle i\rangle * \rho_0 <_i \langle i\rangle * \rho_1$.

If $i = \langle 2, e\rangle$, and $\exists f, g \in W_e \ni \rho_0 \in \mathcal{S}_f, \rho_1 \in \mathcal{S}_g$, and f enters W_e before g then $\langle i\rangle * \rho_0 <_i \langle i\rangle * \rho_1$.

The first and second case of Note 7 are much the same. Let $i = \langle 2, e\rangle$ and $f \in W_e$. If $\rho_1 \subset \rho_0$ then the proof is obvious.

If $\rho_1 \not\leq \rho_0$ then $\langle i\rangle * \rho_0 <_i \langle i\rangle * \rho_1$ is true by the same application of Rule 2 or 3 which insured that $\rho_0 <_f \rho_1$.

Let us consider the last case. Note that $\langle i\rangle$ is all that $\langle i\rangle * \rho_0$ and $\langle i\rangle * \rho_1$ have in common since $\langle f\rangle$ is the first element of ρ_0 and $\langle g\rangle$ is the first element of ρ_1. Then Rule 3 immediately gives the result.

<u>Note 8</u>. By combining the results of Notes 6 and 7 we get the following:

I./ If $i = \langle 1, e\rangle$ then $\sigma_0 <_i \sigma_1$

$$\leftrightarrow [\sigma_1 = \langle i\rangle \text{ and } \sigma_0 \neq \langle i\rangle]$$

$$\vee\ [\sigma_0 = \langle i\rangle * \rho_0,\ \sigma_1 = \langle i\rangle * \rho_1,\ \rho_0, \rho_1 \in \mathcal{S}_e \ \&\ \rho_0 <_e \rho_1]$$

II./ If $i = \langle 2, e\rangle$ then $\sigma_0 <_i \sigma_1$

$$\leftrightarrow [\sigma_1 = \langle i\rangle \text{ and } \sigma_0 \neq \langle i\rangle]$$

$$\vee\ [(\exists m)(m \in W_e \ \&\ \sigma_0 = \langle i\rangle * \rho_0,\ \sigma_1 = \langle i\rangle * \rho_1, \ \&\ \rho_0, \rho_1 \in \mathcal{S}_m \ \&\ \rho_0 <_m \rho_1]$$

$$\vee\ [(\exists m, n)(m, n \in W_e \ \&\ \sigma_0 = \langle i\rangle * \rho_0,\ \sigma_1 = \langle i\rangle * \rho_1, \ \&\ \rho_0 \in \mathcal{S}_m \ \&\ \rho_1 \in \mathcal{S}_n \text{ and } m \text{ enters } W_e \text{ before } n].$$

<u>PROPOSITION 12</u>

If i is an H-index, let $o(i)$ = the order type of $\langle \mathcal{S}_i, <_i\rangle$. Then each $o(i)$ is a recursive ordinal.

Proof: First, we wish to note that the definition (definition 2, Ch. I, see 1.1) of a recursive ordinal may be weakened by requiring only that the well-ordering R be a recursively enumerable well-ordering of a subset of N. We shall refer to the new definition as definition 3. It is obvious that an ordinal recursive by definition 2 is recursive by definition 3. Conversely, suppose a recursively enumerable relation $R(x, y)$ well-orders a recursively enumerable subset S of N, Then either S is finite (and $\langle R, S\rangle$ is the order type of a finite, and hence recursive, ordinal), or there is a one - one recursive function such that $S = f(N)$, and we may define $R'(x, y) \leftrightarrow R(f(x), f(y))$. Now R' can be seen to be a recursively enumerable well-ordering of N, so we know that for any $x, y \in N$ $x \neq y \rightarrow R'(x, y) \vee R'(y, x)$, so that R' must indeed be recursive. And since $\langle R, S\rangle \simeq \langle R', N\rangle$, $\langle R, S\rangle$ must be the order type of an ordinal recursive under definition 2.

This is precisely the case we have; $<_i$ is a recursively enumerable relation, and each $\mathcal{S}_i$ is a recursively enumerable set. It suffices now to show that $<_i$ well-orders $\mathcal{S}_i$. We proceed by induction on H-indices.

If $i = <0, e>$, then $\mathcal{S}_i = \{<i>\}$ and the result is obvious.

If $i = <1, e>$, let α be the recursive ordinal for which $O(e)$ is the order type, by induction hypothesis. Part I of Note 8 shows that the ordinal associated with $\mathcal{S}_i$ is the successor of that associated with $\mathcal{S}_e$ since $<i> * \rho <_i <i>$ for all $\rho \in \mathcal{S}_e$.

Suppose $i = <2, e>$. Let $f_1, f_2, \ldots$ be the members of W_e as they arise in computation. Then, by Part II of Note 8, $O(i) = (O(f_1) + O(f_2) + \ldots) + 1$, and this ordinal is again recursive.

We now state a general lemma concerning the embedding of recursively enumerable sets in $\mathcal{O}$.

LEMMA 13. If M is a recursively enumerable set well-ordered by a recursively enumerable relation T, then there exists a partial recursive function f such that

$$\text{(i)} \quad x \in M \rightarrow f(x) \in \mathcal{O}$$

and

$$\text{(ii)} \quad x, y \in M \rightarrow [T(x, y) \rightarrow f(x) <_{\mathcal{O}} f(y)].$$

Proof: We first consider the special case where $M = N$. Note that we define $\sum_{m=0}^{i} a_m$, where the addition operator is $+_{\mathcal{O}}$, as follows (for the purpose of this proof):

$$\sum_{m=0}^{0} a_m = a_0$$

and

$$\sum_{m=0}^{n+1} a_m = \sum_{m=0}^{n} a_m +_{\mathcal{O}} a_{n+1} .$$

Then, there exist functions k and ℓ such that

$$\{k(d)\}(n) \simeq \sum_{m=0}^{n} (\{d\}(m) +_{\mathcal{O}} a)$$

where a is the notation for the ordinal 1, and

$$\begin{aligned} \{\ell(e, x)\}(n) &\simeq 1 \quad \text{if} \quad x = n \vee T(x, n) \\ &\simeq \{e\}(n) \quad \text{if} \quad T(n, x) . \end{aligned}$$

In particular, we may fix k and ℓ to be unique recursive functions since their definitions consist of instructions for the effective computation of indices of recursive functions.

Now, by the Recursion Theorem, there is an integer b such that $\{b\}(x) = k(\ell(b, x))$. Let us define $f(x) = 3 \cdot 5^{\{b\}(x)}$; we wish to show that f is the desired function. First, we redefine ℓ: $\ell(x) = \ell(b, x)$; so $f(x) = 3 \cdot 5^{k(\ell(x))}$.

Fix s, and let $P = \{t \mid T(t, s)\}$; assume that we know $t \in P \to f(t) \in \mathcal{O}$, and $t_1, t_2 \in P \to [T(t_1, t_2) \to f(t_1) <_{\mathcal{O}} f(t_2)]$. From the definitions of $\mathcal{O}$ and $<_{\mathcal{O}}$ we have that $f(s) \in \mathcal{O} \leftrightarrow (\forall n)[\{k(\ell(s))\}(n) \in \mathcal{O} \;\&\; \{k(\ell(s))\}(n) <_{\mathcal{O}} \{k(\ell(s))\}(n+1)]$.

But it is clear from the definition of k, and from the property of $+_{\mathcal{O}}$ that $y \neq 1 \to x <_{\mathcal{O}} x +_{\mathcal{O}} y$ (provided $x, y \in \mathcal{O}$) that $\{k(\ell(s))\}(n) <_{\mathcal{O}} \{k(\ell(s))\}(n+1)$ (provided they are $\in \mathcal{O}$). It remains to show $\{k(\ell(s))\}(n) \in \mathcal{O}$, $\forall n$. Now $\{k(\ell(s))\}(n) = \sum_{m=0}^{n} (\{\ell(s)\}(m) +_{\mathcal{O}} a)$. Either $\{\ell(s)\}(m) \in \mathcal{O}$ since $T(s, m) \vee s = m$ and thus $\{\ell(s)\}(m) = 1 \in \mathcal{O}$, or $T(m, s)$ and $\{\ell(s)\}(m) = f(m)$, and $f(m) \in \mathcal{O}$ by the induction hypothesis. Note that at this point we used the assumption that $M = N$.

It is clear from the definitions of $\mathcal{O}$, $<_{\mathcal{O}}$, and $+_{\mathcal{O}}$ that $\{z\}(n) <_{\mathcal{O}} 3 \cdot 5^{k(z)}$, when $k(z) \in \mathcal{O}$. For $3 \cdot 5^{k(z)} \in \mathcal{O}$ only if $\{k(z)\}(n) \in \mathcal{O}$ for $n = 0, 1, \ldots,$ and $\{k(z)\}(n) <_{\mathcal{O}} 3 \cdot 5^{k(z)}$ from the definitions of $\mathcal{O}$ and $<_{\mathcal{O}}$. But $\{k(z)\}(n) = \sum_{m=0}^{n}(\{z\}(m) +_{\mathcal{O}} a)$ and then $\{z\}(n) <_{\mathcal{O}} \{k(z)\}(n)$, since for $x, y \in \mathcal{O}$, $y \neq 1 \to x <_{\mathcal{O}} x +_{\mathcal{O}} y$. Now, if $T(t, s)$, then $f(t) = \{\ell(s)\}(t)$, and then letting $n = t$ and $z = \ell(s)$ we get:

$$f(t) = \{\ell(s)\}(t) <_{\mathcal{O}} 3 \cdot 5^{k(z)} = g(s).$$

That is, $T(t, s) \to g(t) <_{\mathcal{O}} g(s)$.

We now prove the lemma for the case $M \neq N$. If M is finite, the result is evident, so let us assume M is infinite. Since M is recursively enumerable, there is a one - one recursive function h such that $h(N) = M$. We can define an inverse, in a sense, to h: to calculate $h^{-1}(x)$, we compute $h(0), h(1), \ldots$ until we find a y such that $h(y) = x$. Of course if $x \notin M$, then no such y will be found; clearly, h^{-1} is partial recursive. Now define $T'(x, y) \leftrightarrow T(h(x), h(y))$; T' is a recursively enumerable relation since T is recursively enumerable and h is partial recursive. We then have $x, y \in M \to [T(x, y) \leftrightarrow T'(h^{-1}(x), h^{-1}(y))]$: By the case when $M = N$, we have an f such that $(\forall x)(f(x) \in \mathcal{O})$ and $(\forall x)(\forall y)[T'(x, y) \to f(x) <_{\mathcal{O}} f(y)]$. Then, for $x, y \in M$,

$$\begin{aligned} T'(x, y) &\to T'(h^{-1}(x), h^{-1}(y)) \\ &\to fh^{-1}(x) <_{\mathcal{O}} fh^{-1}(y) \end{aligned}$$

Define f^1 as fh^{-1}, and we have

$$x, y \in M \to [T'(x, y) \to f^1(x) <_{\mathcal{O}} f^1(y)]$$

and of course $x \in M \to f^1(x) \in \mathcal{O}$, since $f(N) \subset \mathcal{O}$. ||

With the aid of Lemma 13 we can prove the following important proposition. Assume g satisfies Lemma 13 for $\langle S_i, <_i \rangle$ for some H-index.

PROPOSITION 14

There exists a partial recursive $k : \mathcal{S}_i \to N$ such that for each $\sigma * \langle j\rangle \in \mathcal{S}_i, k(\sigma * \langle j\rangle)$ = the gödel number of a formula $\mathcal{J}(x)$ with only x free such that $J_j = \{n \mid \models \mathcal{J}(\overline{n})\}$ and $OR(\mathcal{J}) \leq_R w$, where w is the notation for $|g(\langle i\rangle)| + 1$.

Proof: Let $\mathcal{S}$ be a formula which expresses membership in the range of unary partial recursive functions. That is, $\mathcal{S}$ is a formula of ordinal rank 0 with just x and y free such that for all m and $n \in N$,

$$m \in W_n \leftrightarrow \models \mathcal{S}(\overline{m}, \overline{n}).$$

Note that, by the remarks following Lemma 9, we have an explicit form for the T_α's. In particular we can display a formula $\mathcal{K}(x)$ such that

$$n \in T_{g(\langle i\rangle)} \leftrightarrow \models \mathcal{K}(\overline{n}).$$

$\mathcal{K}(\overline{n})$ has ordinal rank $= |g(\langle i\rangle)| + 1$.

We now define a partial recursive function χ :

$$\chi(p, y, n) = GN(\mathcal{L}(\overline{n})) \quad \text{if } \phi_p(\sigma * \langle y\rangle) = GN(\mathcal{L}(x))$$

divergent otherwise (i.e., if $\phi_p(\sigma * \langle y\rangle)$ is not the gödel number of a suitable formula) .

Then there is a formula $\mathcal{X}$, of rank 0, such that $\chi(p, y, n) = m \leftrightarrow \models \mathcal{X}(\overline{p}, \overline{y}, \overline{n}, \overline{m})$.

Finally we define k_0:

$k_0(p, q)$ diverges if q is not a sequence number, or if $q = \sigma * \langle j\rangle$ and j is neither $\langle 0, e\rangle$, $\langle 1, e\rangle$, nor $\langle 2, e\rangle$ for some $e \in N$.

(a) if $j = \langle 0, e\rangle$, then $k_0(p, \sigma * \langle j\rangle) = GN(\mathcal{S}(x, e))$

(b) if $j = \langle 1, e\rangle$, then $k_0(p, \sigma * \langle j\rangle) = GN(\neg \mathcal{B}(x))$
where $\emptyset_p(\sigma * \langle j, e\rangle) = GN(\mathcal{B}(x))$

(c) if $j = \langle 2, e\rangle$, then $k_0(p, \sigma * \langle j\rangle) =$
$GN(\exists y)(\exists z)(\mathcal{K}(p, y, x, z) \; \& \; \mathcal{N}(z) \; \& \; \mathcal{S}(y, e))$.

k_0 is evidently partial recursive. By the Recursion Theorem there is an f such that $\emptyset_f = \lambda x k_0(f, x)$. It remains to show that we can indeed take $k = \lambda x\, \emptyset_f(x)$. We define a new χ : $\chi(y, n) = \chi(f, y, n)$. That is,

$$\chi(y, n) = \begin{cases} GN(\mathcal{B}(\overline{n})) & \text{if } k(\sigma * \langle y\rangle) = GN(\mathcal{B}(x)) \\ \text{divergent otherwise} . \end{cases}$$

Then we define a formula $\mathcal{K}$ of rank 0 such that $\chi(y, n) = m \leftrightarrow \; \models \mathcal{K}(\overline{y}, \overline{n}, \overline{m})$.

If $j = \langle 0, e\rangle$, $J_j = W_e$ and $W_e = \{x \mid \; \models \mathcal{S}(\overline{x}, \overline{e})\}$ and $k(\sigma * \langle j\rangle) = GN(\mathcal{S}(x, e))$.

If $j = \langle 1, e\rangle$, $J_j = \overline{J}_e$ and $k(\sigma * \langle j\rangle)$ is the gödel number of the negation of the formula $\mathcal{B}(x)$ whose gödel number is $k(\sigma * \langle j, e\rangle)$, and by the induction hypothesis we know $J_e = \{x \mid \; \models \mathcal{B}(\overline{x})\}$.

If $j = \langle 2, e\rangle$, let $Z(x) = (\exists y)(\exists z)(\mathcal{K}(y, x, z) \; \& \; \mathcal{N}(z) \; \& \; \mathcal{S}(y, e))$, where $Z(x)$ is the formula with gödel number $k(\sigma * \langle j\rangle)$. We must show $Z(x)$ is such that $J_j = \{x \mid \models Z(\bar{x})\}$.

$$\models Z(\bar{n}) \leftrightarrow \models (\exists y)(\exists z)(\mathcal{K}(y, n, z) \ \& \ \mathcal{N}(z) \ \& \ \mathcal{S}(y, e))$$

$\leftrightarrow (\exists y)(\exists z)$ (there exists a prenex formula $\mathcal{B}(x)$ with just x free whose gödel number is $z = k(\sigma * \langle j, y\rangle)$ and $\models \mathcal{B}(\bar{n})$ and $y \in W_e$)

$\leftrightarrow (\exists y)(y \in W_e \ \& \ n \in J_y)$ by the induction hypothesis

$\leftrightarrow n \in J_j$ since $J_j = \bigcup_{y \in W_e} J_y$.

$\mathcal{N}(z)$ insures that $Z(x)$ has the appropriate ordinal rank. ||

PROPOSITION 15

$\mathcal{M}$ includes the class of hyperarithmetical sets.

Proof: If $i = \langle 0, e\rangle$, then $J_i = W_e$ and we know that $W_e \in \mathcal{M}_0$.

If $i = \langle 1, e\rangle$, then $J_i = \bar{J}_e$ and $J_e \in \mathcal{M}_\alpha \to \bar{J}_e \in \mathcal{M}_\alpha$.

If $i = \langle 2, e\rangle$, then $J_i = \bigcup_{x \in W_e} J_x$. When $J_x \in \mathcal{M}$ for each $x \in W_e$, we apply Proposition 15 with $j = i$ and get a ranked formula $\mathcal{J}(x)$ such that $J_i = \{n \mid \models_\alpha \mathcal{J}(\bar{n})\}$ for some α. So $J_i \in \mathcal{M}_\alpha$, i.e., $J_i \in \mathcal{M}$. ||

COROLLARY 16. $\mathcal{M}$ = the class of hyperarithmetical sets.

Proof: This follows immediately from Proposition 12 (and the Characterization Theorem) and Proposition 16. ||

CHAPTER V

THE RELATIVIZED THEORY

This final chapter presents a relativization of the results of previous chapters and will provide a clearer relationship between this thesis and [G]. As these results are not central to this thesis, a more loose approach is taken to definitions and proofs.

5.1 BASIC ELEMENTS OF RELATIVIZATION

Let us fix X as a set of natural numbers. In the Introduction, we defined the notions of a function partial recursive in X and a relation recursive in X. Relative recursiveness is further discussed in [R., §9.2]. We continue with more definitions.

1. W_n^X = range ϕ_n^X, where $\langle \phi_n^X \rangle$ is an enumeration of the unary functions partial recursive in X. The sets W_n^X are called the sets <u>recursively enumerable in</u> X.

2. A set A is <u>recursive in</u> X if there is a function f recursive in X such that $x \in A \rightarrow f(x) = 0$ and $x \notin A \rightarrow f(x) = 1$.

3. A relation P is <u>arithmetical in</u> X if it is the result of prefixing a relation recursive in X with a finite number of number quantifiers.

4. A set P is $\underline{\Pi_1^1 \text{ in}}$ X if there is a relation S recursive in X such that $x \in P \leftrightarrow (\forall f)(\exists y)S(f, x, y)$ for unary functions f.

5. A set P is $\underline{\Delta_1^1 \text{ in}}$ X if P and $\overline{P}$ are Π_1^1 in X.

It is somewhat more difficult to relativize the notion of a hyperarithmetical set.

First we define H, X-indices.

(a) For all e, <0, e> is an H, X-index.

(b) If e is an H, X-index, then <1, e> is an H, X-index.

(c) If each $n \in W_e^X$ is an H, X-index, then <2, e> is an H, X-index.

Then we define the sets hyperarithmetic in X as follows:

(a) If $i = \langle 0, e\rangle$, then $J_{i,X} = W_e^X$

(b) If $i = \langle 1, e\rangle$, then $J_{i,X} = \overline{J}_{e,X}$

(c) If $i = \langle 2, e\rangle$, then $J_{i,X} = \bigcup J_{f,X}$, $f \in W_e^X$.

By [Sh., §7.10] we are assured that a set is Δ_1^1 in X if and only if it is hyperarithmetical in X (that is, by the relativized Characterization Theorem).

5.2 RELATIVIZATION OF PREVIOUS RESULTS

Let us define the notion of an ordinal α recursive in X:

(1) α is finite

or

(2) there exists a relation $S(x, y)$ recursive in X such that S well-orders N and such that α = the order type of $\langle N, S\rangle$.

We let ω_1 be the least non-recursive ordinal. In [R., §11.7, Cor. XVI] it is noted that the recursive ordinals form a denumerable initial segment of ordinals, so ω_1 exists and is unique. Similarly,

ω_1^X is defined as the least ordinal not recursive in X.

Having described relativized ordinals, we can now meaningfully detail relativized notations. It is simple to copy the definitions of $\mathcal{O}$ and $\mathcal{O}_1$ to create the sets $\mathcal{O}^X$ and $\mathcal{O}_1^X$, and the relation RX (or $<_{\mathcal{O}_1^X}$). RX is an analogue of R; thus RX is recursively enumerable in X and we can effectively enumerate, relative to X, $\{y \mid y <_{RX} x\}$ uniformly in x. If $a \in \mathcal{O}_1^X$ then $|a|_X$ is the <u>ordinal corresponding to</u> a, by definition. Without loss of generality we may assume $0 \notin \mathcal{O}_1^X$, $1 \in \mathcal{O}_1^X$, and $|1|_X = 0$. Finally we define ORX, the <u>ordinal rank relative to</u> X, which is specified in the same way as OR, except RX is used in place of R. Clearly ORX is partial recursive in X. We can now properly define $X \models_\alpha$, truth relative to X, and sets $\mathcal{M}_\alpha(X)$. Let $\mathcal{L}(\underline{S})$ be the language of Ramified Analysis augmented with the set constant symbol $\underline{S}$. $\underline{S}$ participates in the construction of formulas in the same way as set variables, but cannot be quantified. The defintions of OR and ORX "ignore" the presence of $\underline{S}$ in a formula, so it is easy to modify our definition of $\models_\alpha$ to $X \models_\alpha$. First we define (noting that X is a fixed set) $X \models_0 \overline{n} \in \underline{S} \leftrightarrow n \in X$. Then, in general for $\mathcal{B}(x)$ a prenex formula of $\mathcal{L}(\underline{S})$ with just x free and $\text{ORX}(\mathcal{B}(x)) = 0$, $X \models_0 \mathcal{B}(\overline{n})$ means $\mathcal{B}(\overline{n})$ is true as a statement about integers and membership in X. Suppose we have defined $X \models_\beta$ for all $\beta < \alpha$ and have defined $X \models_\alpha \mathcal{B}$ for all prenex sentences which have "smaller" ordinal triples than $\mathcal{J}$, where $\mathcal{J}$ is a prenex sentence of $\mathcal{L}(\underline{S})$ such that $\text{ORX}(\mathcal{J}) \leq_{RX} a$, for $|a|_X = \alpha$, then $X \models_\alpha \mathcal{J}$ if and only if one of the following five clauses holds:

1. $(\exists b)(R(b, a)\ \&\ X \models_{|b|_X} \mathcal{J})$

2. $\mathcal{J} = (\exists x)\mathcal{G}(x)\ \&\ ORX(\mathcal{G}(x)) \leq_{RX} a\ \&\ X \models_\alpha \mathcal{G}(\bar{n})$ for some $n \in N$

3. $\mathcal{J} = (\forall x)\mathcal{G}(x)\ \&\ ORX(\mathcal{G}(x)) \leq_{RX} a\ \&\ X \models_\alpha \mathcal{G}(\bar{n})$ for all $n \in N$

4. $\mathcal{J} = (\exists X^b)\mathcal{G}(X^b)\ \&\ ORX(\mathcal{G}(X^b)) \leq_{RX} a\ \&\ RX(b, a)\ \&$ for some prenex formula $\mathcal{A}(x)$, with just x free and $ORX(\mathcal{A}(x)) \leq_{RX} b$, $X \models_\alpha PNF(\mathcal{G}(\hat{x}\ \mathcal{A}(x)))$

5. $\mathcal{J} = (\forall X^b)\mathcal{G}(X^b)\ \&\ OR(\mathcal{G}(X^b)) \leq_{RX} a\ \&\ ORX(b, a)\ \&$ for all prenex formulas $\mathcal{A}(x)$ with just x free and $ORX(\mathcal{A}(x)) \leq_{RX} b$, $X \models_\alpha PNF(\mathcal{G}(\hat{x}\ \mathcal{A}(x)))$.

Then the following formulation for the $\mathcal{M}_\alpha(X)$'s is evident:

$\mathcal{M}_{|a|}(X) = \{U \mid$ there exists a prenex formula $\mathcal{G}(x)$ of $\mathcal{L}(\underline{S})$ with just x free such that $ORX(\mathcal{G}(x)) \leq_{RX} a$ and $U = \{x \mid X \models_{|a|_X} (\mathcal{G}(\bar{x})\}\}$.

Let us proceed through the results of Chapters III and IV. Define $T_\alpha(X) = \{GN(\mathcal{J}) \mid \mathcal{J}$ is a prenex formula with no free variables & $ORX(\mathcal{J}) \leq_{RX} a\ \&\ X \models_\alpha \mathcal{J}\}$, where $\alpha = |a|_X$; T_α is the truth set for α relative to X.

PROPOSITION 17

(i) $T_\alpha(X) \notin \mathcal{M}_\alpha(X)$.

(ii) $U \in \mathcal{M}_0(X) \leftrightarrow U$ is arithmetical in X.

(iii) $U \in \mathcal{M}_{\alpha+1}(X) \leftrightarrow U$ is arithmetical in $T_\alpha(X)$.

(iv) $T_\alpha(X) \in \mathcal{M}_{\alpha+1}(X)$.

These proofs are direct relativizations of previous results; however, the "set of true quantifier-free sentences" is now those $\mathcal{J}$

of $\mathcal{L}(\underline{S})$ such that $X \models_0 \mathcal{T}$.

PROPOSITION 18

(a) $T(X)$ $(= \bigcup T_{|a|_X}(X),\ a \in \mathcal{O}_1^X)$ is a set Π_1^1 in X.

(b) Let $\mathcal{M}(X) = \bigcup \mathcal{M}_{|a|_X}(X),\ a \in \mathcal{O}_1^X$. Then if $S \in \mathcal{M}(X)$, S is a set Π_1^1 in X.

PROPOSITION 19

$T_\alpha(X)$ is a set Σ_1^1 in X.

This completes the relativization of Chapter III's results; Chapter IV is almost as simple. We use H, X-indices, the sets $\mathcal{S}_i$ are now recursively enumerable in X, and each $\mathcal{O}(i)$ is an ordinal recursive in X. Problems arise with the analogues to Lemma 13 and Proposition 14, which use the Recursion Theorem. The formulation of the Recursion Theorem used in this thesis is [R., §11.2, Thm. I] , and by inspection of the proof there the relativized Recursion Theorem can be proven. Its statement is

> "Let f be any function recursive in X. Then
> there exists an n such that $\phi_n^X = \phi_{f(n)}^X$"..

Then the analogue of Corollary 16 states that $\mathcal{M}(X)$ is exactly the class of sets hyperarithmetical in X.

5.3 RESTRICTED RELATIVIZATION

In this last section of the thesis we will assume that X is such that $\omega_1^{X} = \omega_1$. Under this assumption we will show that a partially relativized class, $\mathcal{M}^*_\alpha(X)$, is equal to $\mathcal{M}_\alpha(X)$. To define $X \models^*_\alpha \mathcal{J}$, for $\mathcal{J} \in \mathcal{L}(\underline{S})$, and $\mathcal{M}^*_\alpha(X)$ we copy the definitions of $X \models_\alpha \mathcal{J}$ and $\mathcal{M}_\alpha(X)$ but with OR instead of ORX and $\mathcal{O}_1$ instead of $\mathcal{O}_1^X$. Our goal is then to show that $\mathcal{M}_\alpha(X) = \mathcal{M}^*_\alpha(X)$ for all $\alpha < \omega_1$. Assume $U \in \mathcal{M}_\alpha(X)$ and define a function $f = \{(a, 0) \mid a \notin \mathcal{O}_1^X\} \cup \{(a, b) \mid a \in \mathcal{O}_1^X$ and $b \in \mathcal{O}_1$ and $|a|_X = |b|\}$. It is clear that $f(\mathcal{O}_1^X) = \mathcal{O}_1$, $f(\overline{\mathcal{O}_1^X}) = \{0\}$ and f is order-preserving (since $\alpha < \beta \leftrightarrow R(a, b)$, for $a, b \in \mathcal{O}_1$ and $\alpha < \beta \leftrightarrow RX(a^1, b^1)$ for $a^1, b^1 \in \mathcal{O}_1^X$, where $|a| = |a^1|_X = \alpha$ and $|b| = |b^1|_X = \beta$).

We employ f to map formulas in such a way that if $\mathcal{J}$ if a formula which is proper ranked relative to $\mathcal{O}_1^X$ (i.e., all its superscripts are members of $\mathcal{O}_1^X$) then $f\mathcal{J}$, the unique formula resulting from the replacement of each superscript a in $\mathcal{J}$ by $f(a)$, is a formula proper ranked relative to $\mathcal{O}_1$. Thus we need only show that if $U = \{x \mid X \models_\alpha \mathcal{J}(x)\}$ then $U = \{x \mid X \models^*_\alpha f(\mathcal{J}(\overline{x}))\}$; we will prove this by induction.

When $\alpha = 0$ we have $X \models_\alpha \mathcal{J} \leftrightarrow X \models^*_\alpha f\mathcal{J}$ since $\mathcal{J} = f\mathcal{J}$ (as $\mathcal{J}$ has no set variables). For $\alpha > 0$ we need an "inverse" function for f. Define $g = \{(a, 0) \mid a \notin \mathcal{O}_1\} \cup \{(a, b) \mid a \in \mathcal{O}_1$, $b \in \mathcal{O}_1^X$ and $|a| = |b|_X\}$. Then $g(\mathcal{O}_1) = \mathcal{O}_1^X$, $g(\overline{\mathcal{O}_1}) = \{0\}$ and g is order-preserving, and $a \in \mathcal{O}_1 \rightarrow f(g(a)) = a$ and $b \in \mathcal{O}_1^X \rightarrow g(f(b)) = b$. And, if all the superscripts of $\mathcal{J}^*$ are in $\mathcal{O}_1$, then we have that $g\mathcal{J}^*$ is a proper ranked formula relative to $\mathcal{O}_1^X$. The following note on substitution is important: $f\mathcal{B}(\hat{x}\,\mathcal{K}^*(x)) = f(\mathcal{B}(\hat{x}\ g(\mathcal{K}^*(\overline{x}))))$, in other words the result of substituting into a transformed formula is the same as transforming a formula into which a reverse-transformed formula has been substituted; this is easy to verify mentally. Our induction hypothesis is that for any formula $\mathcal{J}_2$ where TRPX($\mathcal{J}_2$) (the greatest ordinal triple, relative to X, of $\mathcal{J}_2$) is less than TRPX($\mathcal{J}$), then $X \models_\alpha \mathcal{J}_2 \to X \models_\alpha^* f\,\mathcal{J}_2$. Let $\mathcal{X}^*(x)$ be any proper prenex formula with just x free such that $\mathrm{OR}(\mathcal{X}(x)) \leq_R a$. Then clause 4 (of the definition of $X \models_\alpha^*$) states that $X \models_\alpha \mathcal{J} \to$ $X \models_\alpha \mathcal{L}(\hat{x}\ g\mathcal{X}^*(\bar{x})))$, and the induction hypothesis yields $X \models_\alpha^* f(\mathcal{L}(\hat{x}\ g\mathcal{X}^*(\bar{x}))))$. But, by our note on substitutions, this is $X \models_\alpha^* f\mathcal{L}(\hat{x}\ \mathcal{X}^*(\bar{x}))$ and thus $X \models_\alpha^* f\mathcal{J}_1$ since $f\mathcal{J} = (\forall X^{f(b)})\ f\,\mathcal{L}(X^b)$. The other clauses of the definition of $X \models_\alpha$ are similarly simple to prove and, in fact the whole proof may be obviously modified to show that $X \models_\alpha^* \mathcal{J} \to X \models_\alpha g(\mathcal{J})$, and we immediately obtain $\mathcal{M}_\alpha(X) = \mathcal{M}_\alpha^*(X)$ for all $\alpha < \omega_1$ ($\mathcal{M}_\alpha^*$ has no meaning for $\alpha \geq \omega_1$).

Define $\mathcal{M}^*(X) = \bigcup \mathcal{M}_\alpha^*(X)$, $\alpha < \omega_1$. We can summarize our results as follows:

$$\mathcal{M}^*(X) = \bigcup_{\alpha<\omega_1} \mathcal{M}_\alpha^*(X) = \bigcup_{\alpha<\omega_1} \mathcal{M}_\alpha(X)$$

$$\subseteq \bigcup_{\alpha<\omega_1^X} \mathcal{M}_\alpha(X) = \mathcal{M}(X) = \Delta_1^1(X),$$

the set of sets Δ_1^1 in X .

Note that $\omega_1^X = \omega_1 \Leftrightarrow \bigcup_{\alpha<\omega_1} \mathcal{M}_\alpha(X) = \bigcup_{\alpha<\omega_1^X} \mathcal{M}_\alpha(X)$, since $T_\alpha(X) \notin \mathcal{M}_\alpha(X)$.

and $T_\alpha(X) \in \mathcal{M}_{\alpha+1}(X)$.

CONCLUSION: We have obtained our main goal, that of showing that the hyperarithmetical sets have a formulation in the language of Ramified Analysis, a result that is almost as surprising as the Characterization Theorem. The last chapter relates this thesis to [G], since the sets $\mathcal{M}_{\alpha}(X)$ referred to in [G] are in fact the sets $\mathcal{M}_{\alpha}^{*}(X)$ in this thesis.

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