THE SOLUTION OF MIXED BOUNDARY VALUE PROBLEMS IN LINEAR VISCOELASTICITY THAT INVOLVE TIME-DEPENDENT BOUNDARY REGIONS

by

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ABSTRACT

The classical method of solution to problems in linear viscoelasticity is to apply the Laplace transform to the time-dependent field equations and boundary conditions. If a solution to the transformed problem can be found then the solution to the original problem is reduced to a transform inversion. However, if the shape of the body under consideration or the type of boundary conditions specified at a point, or both, vary with time then this method no longer works.

In this thesis methods of solution to these problems are investigated. It is shown that, with the help of an alternate form of the Laplace transform inversion theorem, a method closely paralleling the above procedure solves problems for regions which ablate. This method reduces viscoelastic boundary value problems to the determination of elastic solutions which satisfy certain conditions at the boundary of the ablating region. For the case when the region does not vary with time (which is considered as a special case of an ablating body), but subregions of the boundary may be monotonically increasing or decreasing with time, the above conditions which the elastic solutions must satisfy are simplified. Examples of both cases are given, whose results agree with known solutions.

In the latter part of the thesis two problems are considered where although the region occupied by the body remains constant with time, subregions of the boundary may vary. Here it is shown that a solution given by Graham for the contact problem in viscoelasticity, where the contact region varies with any number of maxima or minima, can be extended and that the extended solution is equivalent to one by Ting. The equivalence is not at all obvious since the form of the two solutions differs greatly. Also a solution is given to the problem of a plane axisymmetrical crack in an infinite viscoelastic medium which is opened by normal pressure acting on its surface where initially the crack is extending and after a period of time contracts. It is found that while the crack is growing the normal pressure and surface area of the crack can be prescribed independently, but when contraction begins only one of these can be given, in order to keep the normal displacement continuous and null on the boundary of the crack area. An equation is found which relates the crack surface area and the normal pressure for times when it is contracting so that if one is specified then the other is determined. The solution while the crack is extending agrees with one given by Graham, but for the case when it is extending and then contracting there are no results which can be referred to.

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INTRODUCTION

The purpose of this thesis is to look at the methods of solution of boundary value problems in the linear quasi-static theory of viscoelasticity and extend these if possible. The problems considered include ones in which the viscoelastic body ablates, that is material may actually be removed from the body as time progresses. In general it is found that, in all the problems considered, the viscoelastic solution was related to some particular one-parameter family of solutions of corresponding problems in the linear theory of elasticity. Specific results, which have been found, are discussed in the following outline which begins with the first two sections.

Section 1 deals with the standard mathematical definitions and results used in the theory of viscoelasticity with one exception. It is a result which is similar to one derived by Ting [1] and gives an alternative form of the Laplace transform inversion theorem (for more detailed discussion see page 8). The next section gives the formulation of boundary value problems in quasi-static viscoelasticity for bodies which ablate. (The case when the region occupied by the body remains constant for all time is considered as a special case of an ablating body.) Here the fundamental system of field equations relevant to the linear theory of thermo-viscoelasticity is stated as well as mixed boundary conditions which prescribe the surface displacement and surface traction on complementary subsets of a boundary. These subregions of the boundary vary with time so that even when the boundary remains constant, the subregions may still be functions of time. In both of these sections, reference has been made to the work of Gurtin and Sternberg [2].

The classical method of solving boundary value problems in the linear quasi-static theory of viscoelasticity is to apply an integral transform (with respect to time) to the time-dependent field equations and boundary conditions. The transformed field equations then have the same form as the field equations of elasticity theory and if a solution to these, which is compatible with the transformed boundary conditions, can be found then the solution to the original problem is reduced to a transform inversion. This method of solving viscoelastic stress analysis problems is referred to as the "correspondence principle" (for reference see Lee [3]).

If the region which the body occupies does vary with time and ablates then the correspondence principle fails since, for problems of this type, there will be points of the (ablating) body for which the field quantities will not be defined for all time. Hence, for this case, methods of solution have concentrated on special approaches to specific problems with the exception of Graham [4], who gave a general method of solution, which was an extension of the Papkovitch-Neuber solution for viscoelasticity given by Gurtin and Sternberg [2]. However in section 3 it is shown that, with the help of the result concerning the Laplace transform inversion theorem, a method closely paralleling the classical correspondence principle solves problems for regions which ablate. This method reduces viscoelastic boundary value problems to the determination of elastic solutions which satisfy certain conditions at the boundary of the ablating region. Two applications of this method are given. One recovers Graham's extension, and the other solves the problem of a large sphere with a growing spherical cavity at its center. This solution agrees with one given by Williams [5].

In the case when the region occupied by the body does not vary with time, this correspondence principle is applicable whenever the type of boundary condition prescribed is the same at all points of the boundary. For mixed boundary value problems, the method is still applicable provided the regions over which different types of boundary conditions are given do not vary with time. However, for those mixed problems where the regions do vary with time (particular examples are indentation and crack propagation problems), the classical correspondence principle is not applicable since for these problems there will be points of the boundary at which only partial histories of some field quantities will be prescribed and the transforms of these quantities are not directly obtainable. To meet this case, both Graham [6] and Ting [1] gave "extended" correspondence principles which cover those mixed boundary value problems where subregions of the boundary may vary with time but must be either monotonically increasing or decreasing. For the more general case, Hunter [7], Graham [8], [9] and Ting [10], [11] have given solutions to the contact problem where the subregions may have one or more maximum or minimum. None of these used a formal application of any correspondence principle although Graham has shown that his solutions are related to a one-parameter family of elastic solutions for the same problem in elasticity which, in fact, is the spirit of the correspondence principle.

Section 4 considers mixed boundary value problems where the region occupied by the body is constant and subregions of the boundary may vary with time as long as they are either monotonically increasing or decreasing. The solutions given here depend on the findings of Ting [1] and Graham [6]. In these papers Ting extends Graham's results by removing two conditions that the latter assumes must hold. This section generalizes and orders these results such that they link with the previous section. Also an example of this method is given of a contact problem where the contact region is monotonically increasing. This is a simple example of the problem considered in the next section.

One of the contact problems which is most often studied in visco-elasticity is the boundary value problem arising when a rigid indentor, of smooth profile, (i.e. a profile which has a unique tangent plane at each of its points) is pressed against the, initially plane, surface of a linear viscoelastic half-space. The normal surface traction is assumed zero outside the contact area, while inside the contact area the normal surface displacements must conform to the surface geometry of the indentor. In general, the contact area will vary with time. This contact problem provides a meaningful example on which to estimate the value of any method of solution to the difficult mixed boundary value problems where the boundary has subregions which are functions of time. As a result, many authors have considered this problem and the fifth section gives a short history of these studies. It also looks at the most difficult case when the contact area varies with any number of maxima or minima. To do this, three papers are discussed, by Graham [8] and Ting [10], [11]. Their methods differ so greatly that only for simple cases (i.e. one or two maxima) can their results easily be seen to be equivalent. In section 5 it is shown that Graham's solution can be extended and that the extended solution is equivalent to one of Ting's solutions even for the most general case of "n" maxima and minima.

In the last section a solution is given to the problem of a plane axisymmetrical crack, in an infinite viscoelastic medium, which is opened by a normal pressure acting on its surface. Initially the crack is extending and after a period of time contracts. There are not many results concerning crack problems in viscoelasticity theory. Willis [12] has investigated a steady state dynamic viscoelastic crack propagation problem. He considered the specific case of an extending crack in anti-plane strain and solves this problem directly without resort to any correspondence principle. On the other hand, Graham [13] uses his extended correspondence principle to solve the extending crack problem for the symmetrically loaded two dimensional and the axisymmetrically loaded three dimensional quasi-static cases. The latter is treated as a specific example in section 6.

The case when the crack extends and then contracts has not been looked at previously. Thus no reference can be made to the results in section 6 for this case. It is found that while the crack is growing the normal pressure and the surface area of the crack can be prescribed independently, but when contraction begins only one of these can be given, in order to keep the normal displacement continuous and null on the boundary of the crack area. An equation is found which relates the crack surface area and the normal pressure for times when it is contracting so that if one is specified then the other is determined. In conclusion, it is noted that, while it might be possible to extend this solution to the case where the crack surface has "n" maxima and minima, along the lines of Graham's solution of the contact problem, it would be necessary to show that the solution generated normal displacements over the crack surface which were positive, for the solution to have any meaning.

1. Mathematical Preliminaries

Throughout all sections H(t) denotes the Heaviside unit step-function, of time alone, which is defined through

$$H(t) = 0 \quad -\infty < t < 0, \quad H(t) = 1 \quad 0 \leq t < \infty. \tag{1.1}$$

In this section f, g and h are sufficiently smooth functions of the position vector $\underline{x}$ and time t. Then the Stieltjes convolution [f*dg] stands for the function defined by

$$[f{*}dg](\underline{x},t) = f(\underline{x},t)\; g(\underline{x},0) + \int_0^t f(\underline{x},t-\tau)\,\frac{\partial g}{\partial \tau}\,(\underline{x},\tau)d\tau, \; 0 \leq t < \infty, \tag{1.2}$$

provided the integral is meaningful. Some properties of the convolution (1.2), which will be needed later, are listed below:

$$\begin{aligned} f{*}dg &= g{*}df \\ f{*}d(g{*}dh) &= (f{*}dg){*}dh = f{*}dg{*}dh \\ f{*}d(g+h) &= f{*}dg + f{*}dh \\ f{*}dH &= f. \end{aligned} \tag{1.3}$$

If $f(\underline{x},0)$ does not vanish then f has a unique Stieltjes inverse, f^{-1}, such that

$$f{*}df^{-1} = f^{-1}{*}df = H. \tag{1.4}$$

(Proofs of these properties, (1.3) and (1.4), are contained in Gurtin and Sternberg [2].)

We shall use the notation

$$\begin{aligned} \bar{f}(\underline{x},s) &= L[f(\underline{x},t);\; t \to s] = \int_0^\infty f(\underline{x},t)\; e^{-st}\, dt, \\ f(\underline{x},t) &= L^{-1}[\bar{f}(\underline{x},s);\; s \to t], \end{aligned} \tag{1.5}$$

for the Laplace transform with respect to time t, of a function $f(\underline{x},t)$ and for its inverse Laplace transform. The same notation will be used to denote the transform (inverse transform) of a vector valued function, with the agreement that it represents the vector whose components are the transforms (inverse transforms) of the components of the original vector. Two relations which are consequences of (1.5) are stated now (for proofs see e.g. Sneddon [14]). The Convolution Theorem states

$$L^{-1}[\bar{f}(\underline{x},s)\ \bar{g}(\underline{x},s);\ s \to t] = \int_0^t f(\underline{x},t-\tau)\ g(\underline{x},\tau)d\tau \tag{1.6}$$

where the roles of f and g on the right hand side of (1.6) may be reversed. The transform of the time derivative of a function is given by

$$\left(\overline{\frac{\partial f}{\partial t}}\right)(\underline{x},s) = s\ \bar{f}(\underline{x},s) - f(\underline{x},0). \tag{1.7}$$

Further results and their proofs are now given. If we take the Laplace transform of the convolution (1.2) and note (1.5) and (1.6) we obtain

$$L[(f*dg)\ (\underline{x},t);\ t \to s] = \bar{f}(\underline{x},s)\ g(\underline{x},0) + \bar{f}(\underline{x},s)\left(\overline{\frac{\partial g}{\partial t}}\right)(\underline{x},s)$$

which, on using equation (1.7), becomes

$$L[(f*dg)\ (\underline{x},t);\ t \to s] = s\ \bar{f}(\underline{x},s)\ \bar{g}(\underline{x},s). \tag{1.8}$$

By taking the inverse Laplace transform of (1.8) we get

$$[f*dg]\ (\underline{x},t) = L^{-1}[s\ \bar{f}(\underline{x},s)\ \bar{g}(\underline{x},s);\ s \to t]. \tag{1.9}$$

Next, let p be a function of $\underline{x}$, τ and s. We define P, a function of $\underline{x}$ and t, by

$$P(\underline{x},t) = L^{-1}[\int_0^\infty p(\underline{x},\tau,s)\ e^{-s\tau}d\tau;\ s \to t],\ t \geq 0. \tag{1.10}$$

Then, if P is a continuous function of t, we will show that

$$P(\underline{x},t) = \lim_{a \to t^-} L^{-1}[\int_0^t p(\underline{x},\tau,s)\ e^{-\tau s} d\ ;\ s \to a]. \tag{1.11}$$

This result corrects an expression which was derived by Ting [1]. In order to verify (1.11), we have from (1.10) for any non-negative "a"

$$\begin{aligned} P(\underline{x},a) &= L^{-1}[\int_0^t p(\underline{x},\tau,s)\ e^{-s\tau} d\tau;\ s \to a] \\ &\quad + L^{-1}[\int_t^\infty p(\underline{x},\tau,s)\ e^{-s\tau} d\tau;\ s \to a],\ a \geq 0, \end{aligned} \tag{1.12}$$

where t is any non-negative time. Now if we make the change of variable

$$\tau = \tau' + t$$

then

$$\int_t^\infty p(\underline{x},\tau,s)\ e^{-s\tau} d\tau = e^{-st} \int_0^\infty p(\underline{x},\tau'+t,s)\ e^{-\tau' s} d\tau'. \tag{1.13}$$

Since the Laplace transform with respect to τ of $H(\tau - t)$ is

$$L[H(\tau - t);\ \tau \to s] = \frac{e^{-st}}{s}, \tag{1.14}$$

equation (1.13) may be written as

$$\int_t^\infty p(\underline{x},\tau,s)\ e^{-s\tau} d\tau = sL[H(\tau - t);\ \tau \to s]\ \bar{F}(\underline{x},s)$$

where

$$\bar{F}(\underline{x},s) = \int_0^\infty p(\underline{x},\tau'+t,s)\ e^{-s\tau'} d\tau'.$$

Taking the inverse Laplace transform and using (1.9) and (1.2) we get

$$L^{-1}[\int_t^{\infty} p(\underline{x},\tau,s)\ e^{-s\tau}d\tau;\ s \to a]$$

$$= H(a-t)\ F(\underline{x},0) + \int_0^a H(a-\tau-t)\ \frac{\partial}{\partial\tau} F(\underline{x},\tau)d\tau, \qquad (1.15)$$

$$= \begin{cases} 0 & a < t, \\ F(\underline{x},a-t) & a \geq t, \end{cases}$$

where

$$F(\underline{x},a) = L^{-1}[\bar{F}(\underline{x},s);\ s \to a].$$

If we combine equations (1.12) and (1.15) then we obtain the following result,

$$L^{-1}[\int_t^{\infty} p(\underline{x},\tau,s)\ e^{-s\tau}d\tau;\ s \to a] = 0, \quad 0 \leq a < t,$$

and (1.16)

$$P(\underline{x},a) = L^{-1}[\int_0^t p(\underline{x},\tau,s)\ e^{-s\tau}d\tau;\ s \to a], \quad 0 \leq a < t.$$

Now for any particular positive t these identities are valid for every a, $0 \leq a < t$. Therefore, by letting a approach t from the left we find that

$$\lim_{a\to t^-} L^{-1}[\int_t^{\infty} p(\underline{x},\tau,s)\ e^{-s\tau}d\tau;\ s \to a] = 0,$$

and (1.17)

$$P(\underline{x},t^-) = \lim_{a\to t^-} L^{-1}[\int_0^t p(\underline{x},\tau,s)\ e^{-s\tau}d\tau;\ s \to a]$$

where

$$P(\underline{x},t^-) = \lim_{a\to t^-} P(\underline{x},a)\ .$$

If the function P defined by equation (1.10) is continuous at t then

$$P(\underline{x},t^-) = P(\underline{x},t)\ . \qquad (1.18)$$

Thus by (1.10), (1.17) and (1.18) we have

$$P(\underline{x},t) = L^{-1}[\int_0^{\infty} p(\underline{x},\tau,s)e^{-s\tau}d\tau;\ s \to t] = \lim_{a\to t^-} L^{-1}[\int_0^{t} p(\underline{x},\tau,s)e^{-s\tau}d\tau;\ s \to a], \tag{1.19}$$

which verifies equation (1.11). In the case when p is independent of s,

$$p(\underline{x},\tau,s) = p(\underline{x},\tau),$$

equation (1.19) reduces to

$$P(\underline{x},t) = p(\underline{x},t) = \lim_{a\to t^-} L^{-1}[\int_0^{t} p(\underline{x},\tau)e^{-s\tau}d\tau;\quad s \to a]. \tag{1.20}$$

Another result needed is derived below. Suppose that

$$p(\underline{x},\tau,s) = sy(s)\ q(\underline{x},\tau,s),$$

where y is some function of s alone, then from (1.10) and (1.9) we see that

$$P(\underline{x},t) = L^{-1}[sy(s)\int_0^{\infty} q(\underline{x},\tau,s)e^{-s\tau}d\tau;\ s \to t] = [Y{*}dQ](\underline{x},t)$$

where (1.21)

$$Y(t) = L^{-1}[y(s);\ s \to t],$$
$$Q(\underline{x},t) = L^{-1}[\int_0^{\infty} q(\underline{x},\tau,s)e^{-s\tau}d\tau;\ s \to t].$$

If P is continuous at t and Q at all times τ, $\tau \leq t$ then, by using the result (1.19), equations (1.21) simplify to give that

$$\lim_{a\to t^-} L^{-1}[sy(s)\int_0^{t} q(\underline{x},\tau,s)e^{-s\tau}d\tau;\ s \to a] = [Y{*}dQ](\underline{x},t) \tag{1.22}$$

where Y is given as above and

$$Q(\underline{x},t) = \lim_{a\to t^-} L^{-1}[\int_0^{t} q(\underline{x},\tau,s)e^{-s\tau}d\tau;\ s \to a]\ . \tag{1.23}$$

In conclusion we observe that

$$\lim_{a\to t^-} L^{-1}\left[\int_0^t \frac{\partial p}{\partial x_i}(\underline{x},\tau,s)e^{-s\tau}d\tau;\ s\to a\right] = \frac{\partial}{\partial x_i}\left\{\lim_{a\to t^-} L^{-1}\left[\int_0^t p(\underline{x},\tau,s)e^{-s\tau}d\tau;\ s\to a\right]\right\}. \tag{1.24}$$

2. The Formulation of Boundary Value Problems in Viscoelasticity

Suppose a time-dependent region represented by $R(t)$, with boundary $\partial R(t)$, $0 \leq t < \infty$, is occupied by a homogenous and isotropic linear thermo-viscoelastic solid. We will assume that as time progresses the region ablates in the sense that $R(t_2)$ is contained in $R(t_1)$ whenever $t_2 \geq t_1$, $(R(t_2) \subseteq R(t_1),\ t_2 \geq t_1)$. Thus material may actually be removed from the body as time progresses. The case when $R(t)$ remains constant for all time is therefore a special case of an ablating body.

Let u_i, e_{ij}, σ_{ij}, each of which is to represent a function of the position vector $\underline{x}$ and time t where $\underline{x}$ is a point of $R(t)$ and $0 \leq t < \infty$, denote the Cartesian components of displacement, strain and stress respectively. Then the fundamental system of field equations relevant to the linear theory of thermo-viscoelasticity may be written as follows (e.g. see Gurtin and Sternberg [2]):

$$2e_{ij}(\underline{x},t) = u_{i,j}(\underline{x},t) + u_{j,i}(\underline{x},t), \quad \underline{x} \text{ in } R(t), \tag{2.1}$$

$$\sigma_{ij,j}(\underline{x},t) + F_i(\underline{x},t) = 0, \quad \sigma_{ij}(\underline{x},t) = \sigma_{ji}(\underline{x},t), \quad \underline{x} \text{ in } R(t). \tag{2.2}$$

Here (2.2) are the stress equations of equilibrium where F_i denotes the components of the body force. In stating the accompanying constitutive equations we first define the following quantities. Let T represent the temperature which is a function of position $\underline{x}$ and time t and define the pseudo-temperature "θ" by

$$\theta(\underline{x},t) = \frac{1}{\alpha_0} \int_{T_0}^{T(\underline{x},t)} \alpha(T')dT', \quad \alpha_0 = \alpha(T_0),$$

where T_0 is the constant base temperature and α is the temperature dependent coefficient of thermal expansion. We denote by G_1 and G_2 the relaxation functions in shear and isotropic compression respectively. These are functions of time t, $0 \leq t < \infty$. Then the stress-strain relation is, using the notation of (1.2),

$$\sigma_{ij}(\underline{x},t) = [G_1 * de_{ij}](\underline{x},t) + \delta_{ij}\left(\frac{(G_2-G_1)}{3} * de_{kk} - \alpha_0(G_2 * d\theta)\right)(\underline{x},t), \quad \underline{x} \text{ in } R(t), \tag{2.3}$$

where δ_{ij} is Kronecker's delta.

To complete the formulation of any boundary value problem we need to specify certain boundary conditions. If we prescribe the surface displacement and traction, respectively, on complementary subsets $\partial R_1(t)$, $\partial R_2(t)$ of the boundary $\partial R(t)$ then the boundary conditions take the form

$$\begin{aligned} u_i(\underline{x},t) &= U_i(\underline{x},t) \quad &, \quad \underline{x} \text{ on } \partial R_1(t), \\ \sigma_{ij}(\underline{x},t)n_j(\underline{x},t) &= T_i(\underline{x},t), \quad & \underline{x} \text{ on } \partial R_2(t), \end{aligned} \tag{2.4}$$

where $n_j(\underline{x},t)$ are the components of the outward unit normal to $\partial R(t)$ and U and T are given vector valued functions. Equations (2.1), (2.2), (2.3) and (2.4) represent a complete formulation of a boundary value problem in thermo-viscoelasticity.

In the case that we have

$$G_1(t) = 2\mu H(t) \text{ and } G_2(t) = 3kH(t),$$

then, by using equation (1.3), we find (2.3) reduces to

$$\sigma_{ij}(\underline{x},t) = 2\mu e_{ij}(\underline{x},t) + \delta_{ij}\left(\frac{(3k-2\mu)}{3} e_{kk} - 3\alpha_0 k\theta\right)(\underline{x},t), \ \underline{x} \text{ in } R(t), \tag{2.5}$$

the stress-strain relation of linear thermo-elasticity. Here μ and k are constants standing for the shear and bulk modules respectively. Equation (2.5) with (2.1) and (2.2) represent the field equations of thermo-elasticity. These equations along with the boundary conditions (2.4) give the complete formulation of a one-parameter family of thermo-elastic boundary value problems (the parameter is t).

3. The Correspondence Principle

From section 2 we have for an ablating region, $R(t)$, which is occupied by a homogeneous and isotropic linear thermo-viscoelastic body, that the field equations, (2.1), (2.2) and (2.3) and the boundary conditions, (2.4), represent a complete formulation of a boundary value problem in thermo-viscoelasticity. To find solutions of such problems we first show how a solution to the field equations in viscoelasticity may be obtained from a certain family of elastic solutions. Then we state what conditions must be satisfied in order to meet the boundary conditions. Finally we solve several problems involving ablating bodies.

We will denote by

$$[u_i^{\varepsilon}(\underline{x},t,s),\; e_{ij}^{\varepsilon}(\underline{x},t,s),\; \sigma_{ij}^{\varepsilon}(\underline{x},t,s)],\quad \underline{x} \text{ in } R(t), \tag{3.1}$$

solutions valid in $R(t)$ of the system of equations, (2.1), (2.2) and (2.5) in the last of which we have set $2\mu = s\bar{G}(s)$ and $3k = s\bar{G}(s)$. Then we shall see that

$$\begin{aligned} u_i(\underline{x},t) &= \lim_{a\to t^-} L^{-1}[\int_0^t u_i^{\varepsilon}(\underline{x},\tau,s)e^{-s\tau}d\tau;\; s \to a], \\ e_{ij}(\underline{x},t) &= \lim_{a\to t^-} L^{-1}[\int_0^t e_{ij}^{\varepsilon}(\underline{x},\tau,s)e^{-s\tau}d\tau;\; s \to a], \\ \sigma_{ij}(\underline{x},t) &= \lim_{a\to t^-} L^{-1}[\int_0^t \sigma_{ij}^{\varepsilon}(\underline{x},\tau,s)e^{-s\tau}d\tau;\; s \to a], \end{aligned} \tag{3.2}$$

is always a solution to the viscoelastic field equations valid for $\underline{x}$ in $R(t)$.

To prove this we will show that the equations (2.1), (2.2) and (2.3) are satisfied by (3.2). Since (2.1), (2.2) and (2.5) are satisfied by

(3.1) for any time t and position $\underline{x}$ in R(t), then they hold at $\underline{x}$ for every time τ, $\tau \leq t$, as R(t) is ablating. Then by (3.2), (1.24) and (1.20) we have

$$[u_{i,j}+u_{j,i}](\underline{x},t) = \lim_{a\to t^-} L^{-1}[\int_0^t [u_{i,j}^{\varepsilon}+u_{j,i}^{\varepsilon}](\underline{x},\tau,s)e^{-s\tau}d\tau;\ s \to a]$$

and

$$\sigma_{ij,j}(\underline{x},t)+F_i(\underline{x},t) = \lim_{a\to t^-} L^{-1}[\int_0^t \sigma_{ij,j}^{\varepsilon}(\underline{x},\tau,s)e^{-s\tau}d\tau;\ s \to a]$$
$$+ \lim_{a\to t^-} L^{-1}[\int_0^t F_i(\underline{x},\tau)e^{-s\tau}d\tau;\ s \to a],$$

$$= \lim_{a\to t^-} L^{-1}[\int_0^t [\sigma_{ij,j}^{\varepsilon}(\underline{x},\tau,s)+F_i(\underline{x},\tau)]e^{-s\tau}d\tau;\ s \to a]$$

which reduce to (2.1) and (2.2) respectively, when $\underline{x}$ lies in R(t). Next, using the fact that (3.1) satisfies (2.5), we have the identity

$$0 = \lim_{a\to t^-} L^{-1}[\int_0^t \{\sigma_{ij}^{\varepsilon}(\underline{x},\tau,s)-s\bar{G}_1(s)e_{ij}^{\varepsilon}(\underline{x},\tau,s) - \delta_{ij}\left[s\frac{(\bar{G}_2(s)-\bar{G}_1(s))}{3}e_{kk}^{\varepsilon}(\underline{x},\tau,s)-\alpha_0 s\bar{G}_2(s)\theta(\underline{x},\tau)\right]\}e^{-s\tau}d\tau;\ s\to a] \tag{3.3}$$

when $\underline{x}$ is in R(t). Breaking up (3.3) we get

$$\lim_{a\to t^-} L^{-1}[\int_0^t \sigma_{ij}^{\varepsilon}(\underline{x},\tau,s)e^{-s\tau}d\tau;\ s \to a] = \lim_{a\to t^-} L^{-1}[\int_0^t \{e_{ij}^{\varepsilon}(\underline{x},\tau,s)s\bar{G}_1(s) - \delta_{ij}\left[s\frac{(\bar{G}_2(s)-\bar{G}_1(s))}{3}e_{kk}^{\varepsilon}(\underline{x},\tau,s)-\alpha_0 s\bar{G}_2(s)\theta(\underline{x},\tau)\right]\}e^{-s\tau}d\tau;\ s\to a]. \tag{3.4}$$

With the help of first (1.22) and then (1.5), (3.2) and (1.20), equation (3.4) reduces to (2.3). Thus the field equations of viscoelasticity are satisfied by the quantities defined through (3.2).

Now in order that (3.2) meets the boundary conditions (2.4) we must have the following conditions satisfied

$$\lim_{a\to t^-} L^{-1}\left[\int_0^t u_i^\varepsilon(\underline{x},\tau,s)e^{-s\tau}d\tau;\ s\to a\right] = U_i(\underline{x},t),\quad \underline{x}\ \text{on}\ \partial R_1(t),$$
$$\lim_{a\to t^-} L^{-1}\left[\int_0^t \sigma_{ij}^\varepsilon(\underline{x},\tau,s)e^{-s\tau}d\tau;\ s\to a\right]n_j(\underline{x},t) = T_i(\underline{x},t),\quad \underline{x}\ \text{on}\ \partial R_2(t), \tag{3.5}$$

where $\partial R_1(t)$ and $\partial R_2(t)$ are complementary subsets of $\partial R(t)$ and $n_j(\underline{x},t)$ is the unit normal to $\partial R(t)$. Thus if the family of elastic solutions (3.1) is chosen to satisfy (3.5) then (3.2) is a solution to a boundary value problem in viscoelasticity represented by the field equations, (2.1), (2.2), (2.3) and the boundary conditions (2.4).

If $\partial R(t)$, $\partial R_1(t)$, $\partial R_2(t)$ do not vary with time, then the conditions (3.5) reduce to

$$u_i^\varepsilon(\underline{x},t) = U_i(\underline{x},t),\quad \underline{x}\ \text{on}\ \partial R_1,$$
$$\sigma_{ij}^\varepsilon(\underline{x},t)n;(\underline{x},t) = T_i(\underline{x},t),\quad \underline{x}\ \text{on}\ \partial R_2. \tag{3.6}$$

Further, in this case we see, by using (1.19), that equations (3.2) can be written as

$$u_i(\underline{x},t) = L^{-1}\left[\int_0^\infty u_i^\varepsilon(\underline{x},\tau,s)e^{-s\tau}d\tau;\ s\to t\right],$$

$$e_{ij}(\underline{x},t) = L^{-1}\left[\int_0^\infty e_{ij}^\varepsilon(\underline{x},\tau,s)e^{-s\tau}d\tau;\ s\to t\right], \tag{3.7}$$

$$\sigma_{ij}^\varepsilon(\underline{x},t) = L^{-1}\left[\int_0^\infty \sigma_{ij}^\varepsilon(\underline{x},\tau,s)e^{-s\tau}d\tau;\ s\to t\right],$$

where $\underline{x}$ lies in R. Equations (3.2), (3.6) or (3.7), (3.6) reduce the solution of this class of linear viscoelastic boundary value problems to the solution of linear elastic boundary value problems. The relation given by (3.7), (3.6) has been known as the "correspondence principle". Note that we cannot use equations (3.7) in place of (3.2) when $R(t)$ is decreasing since, for example, $u_i(\underline{x},\tau,s)$ will not be defined for some $\underline{x}$ in $R(t)$ and all τ, $0 \leq \tau < \infty$. (For further discussion on the "correspondence principle", see the Introduction and Lee [3]).

One well known general solution to the field equations of elasticity is the Papkovich-Neuber solution (for a derivation, see Sokolnikoff [15]),

$$
\begin{aligned}
u_i^{\varepsilon}(\underline{x},t,s) &= [\Phi^{\varepsilon}_{,i}(\underline{x},t)+x_k\Psi^{\varepsilon}_{k,i}(\underline{x},t)]s(\bar{G}_1(s)+2\bar{G}_2(s))-\Psi_i^{\varepsilon}(\underline{x},t)s(7\bar{G}_1(s)+2\bar{G}_2(s)),\\
\sigma^{\varepsilon}_{ij}(\underline{x},t,s) &= [\Phi^{\varepsilon}_{,ij}(\underline{x},t)+x_k\Psi^{\varepsilon}_{k,ij}(\underline{x},t)]s^2(\bar{G}_1(s)+2\bar{G}_2(s))\bar{G}_1(s)\\
&\quad - 6s^2\bar{G}_1^2(s)\Psi_{i,j}(\underline{x},t)+\delta_{ij}s^2\bar{G}_1(s)(\bar{G}_1(s)-\bar{G}_2(s))\Psi^{\varepsilon}_{k,k}(\underline{x},t),\\
\Phi^{\varepsilon}_{,jj}(\underline{x},t) &= \Psi^{\varepsilon}_{i,jj}(\underline{x},t) = 0, \quad \underline{x} \text{ in } R(t),
\end{aligned}
\tag{3.8}
$$

where we have replaced $2\mu, 3k$ by $s\bar{G}_1(s), s\bar{G}_2(s)$ respectively and taken the body force F_i and the pseudo-temperature "θ" as being zero. Here the functions Φ^{ε} and Ψ_i^{ε} are unknown functions to be determined through the boundary conditions. We find, from above, that if (3.8) is substituted into (3.2), then (3.2) is the solution to the boundary value problem given by (2.1), (2.2), (2.3) and (2.4), provided we satisfy the following

$$
\begin{aligned}
&(\Phi_{,i}+x_k\Psi_{k,i})*d(G_1+2G_2)(\underline{x},t)-\Psi_i*d(7G_1+2G_2)(\underline{x},t) = U_i(\underline{x},t), \quad \underline{x} \text{ in } \partial R_1(t),\\
&G_1*d[(\Phi_{,ij}+x_k\Psi_{k,ij})*d(G_1+2G_2)-6\Psi_{i,j}*dG_1\\
&\quad + 2\delta_{ij}\Psi_{k,k}*d(G_1-G_2)](\underline{x},t)n_j(\underline{x},t) = T_i(\underline{x},t), \quad \underline{x} \text{ in } \partial R_2(t),
\end{aligned}
\tag{3.9}
$$

where

$$\Phi(\underline{x},t) = \lim_{a\to t^-} L^{-1}[\int_0^t \Phi^{\varepsilon}(\underline{x},\tau,s)e^{-s\tau}d\tau;\ s \to a],$$

and

$$\Psi(\underline{x},t) = \lim_{a\to t^-} L^{-1}\left[\int_0^t \Psi^{\varepsilon}(\underline{x},\tau,s)e^{-s\tau}d\tau;\ s \to a\right]. \tag{3.10}$$

Thus the solution of the viscoelastic boundary value problem reduces to finding harmonic functions Φ and Ψ_i which satisfy the equations (3.9). This result, which was previously derived by Graham [4], is an extension of an earlier result by Gurtin and Sternberg [2], where R(t) was constant.

In the same way as the Papkovitch-Neuber solution of elasticity was extended, we could extend the general solutions of the field equations in plane elasticity given by complex variable theory to cover the case of an ablating viscoelastic body. This idea will not be expanded any further here.

Now consider the case when the viscoelastic problem has stress boundary conditions only. Then (2.4) can be written as

$$\sigma_{ij}(\underline{x},t)n_j(\underline{x},t) = T_i(\underline{x},t), \quad \underline{x} \text{ on } \partial R(t). \tag{3.11}$$

In this case, if we can find elastic solutions, (3.1), where the stress field meets the boundary conditions (3.11) and is independent of s (i.e. elastic constants), then by (1.18) we see that the condition (3.5) is satisfied immediately since it reduces to

$$\sigma_{ij}^{\varepsilon}(\underline{x},t)n_j(\underline{x},t) = T_i(\underline{x},t), \quad \underline{x} \text{ on } \partial R(t). \tag{3.12}$$

Therefore if these elastic solutions are substituted into (3.2), then (3.2) is the solution to the viscoelastic boundary value problem represented by (2.1), (2.2), (2.3) and (3.11).

If instead we have displacement boundary conditions, only then we write (2.4) as

$$u_i(\underset{\sim}{x},t) = U_i(\underset{\sim}{x},t), \quad \underset{\sim}{x} \text{ on } \partial R(t). \tag{3.13}$$

Now if we can find elastic solutions, (3.1), that have a displacement field which both meets (3.13) and is independent of s, then the same argument as above follows and these elastic solutions provide, through (3.2), the viscoelastic solution to the problem.

As an example of the above cases, consider the problem of a large viscoelastic sphere with a growing spherical cavity at its center. In terms of spherical co-ordinates, (r,ϕ,θ), the boundary conditions are given as:

$$\begin{aligned} &\sigma_{rr}(a(t),t) = f_1(t), \quad \sigma_{r\phi}(a(t),t) = \sigma_{r\theta}(a(t),t) = 0, \\ &\sigma_{rr}(b,t) = f_2(t), \qquad \sigma_{r\phi}(b,t) = \sigma_{r\theta}(b,t) = 0, \end{aligned} \tag{3.14}$$

where the field quantities are independent of θ and ϕ, and f_1, f_2 are arbitrary prescribed functions. The inner and outer radii are represented by $a(t)$ and b, respectively, where a is an increasing function of time. Here the body force, F_i, is zero and the pseudo-temperature θ is a function of time only,

$$\theta = \theta(t).$$

If we solve the elastic problems which have the boundary conditions (3.14), then we get the following stress and displacement fields:

$$\sigma_{rr}^{\varepsilon}(r,t) = \frac{f_2(t)-(a^3(t)/b^3)f_1(t)}{[1-a^3(t)/b^3]} - \frac{[f_2(t)-f_1(t)]a^3(t)}{[1-a^3(t)/b^3]r^3} \quad ,$$

$$\sigma_{\phi\phi}^{\varepsilon}(r,t) = \sigma_{\theta\theta}^{\varepsilon}(r,t) = \frac{f_2(t)-(a^3(t)/b^3)f_1(t)}{[1-a^3(t)/b^3]} + \frac{[f_2(t)-f_1(t)]a^3(t)}{[1-a^3(t)/b^3]2r^3} \quad , \quad (3.15)^*$$

$$u_r(r,t,s) = \frac{[f_2(t)-(a^3(t)/b^3)f_1(t)]r}{[1-a^3(t)/b^3]s\bar{G}_2(s)} + \frac{[f_2(t)-f_1(t)]a^3(t)}{[1-a^3(t)/b^3]2s\bar{G}_1(s)r} + \alpha_0\theta r,$$

$$u_\phi = u_\theta = 0. \quad (3.16)$$

Since the equations (3.15) satisfy the boundary conditions (3.14) and are independent of s, then if (3.15) and (3.16) are substituted into (3.2) the resulting equations represent the solution to the viscoelastic problem which has the boundary conditions (3.14). The viscoelastic stresses are given by (3.15) while the only non-zero displacement is

$$u_r(r,t) = r\left[\left(\frac{f_2-(a^3/b^3)f_1}{1-a^3/b^3}\right) * dG_1^{-1}\right](t) + \frac{1}{2r^2}\left[\left(\frac{(f_2-f_1)a^3}{1-a^3/b^3}\right) * dG_2^{-1}\right](t) + \alpha_0\theta r. \quad (3.17)$$

This result was obtained by Williams [5] (when $\theta(t)$ is null for all time t) but he obtained it by formally using equations (3.7) which are not valid for ablating bodies.

Now if we had specified displacement boundary conditions only, instead of (3.14), then we would find that the displacement field which met the new boundary conditions was also free of elastic constants. An analysis analogous to that outlined above gives the viscoelastic solution, with the viscoelastic displacements the same as the elastic.

* $\left[\sigma_{r\phi} = \sigma_{r\theta} = \sigma_{\phi\theta} = 0\right]$

4. The Special Case When R(t) is Constant

In this section we will consider the particular case when R(t) (in future just R) remains constant. Here it will be shown that boundary conditions which the elastic solutions (3.1) satisfy are related to the boundary conditions of the viscoelastic problem. Then a problem will be solved using these relations.

First, we rewrite the boundary conditions (2.4). We will denote by u_n and u_s (σ_n and σ_s) the vector components of the displacement vector (traction vector) normal and tengential to ∂R respectively. In this way u_n, u_s, σ_n and σ_s are vector valued functions of both $\underline{x}$ and t. Then boundary conditions which prescribe the normal (tangential) components of the displacement and traction vectors on complementary subsets of ∂R are given by,

$$\begin{aligned} u_s(\underline{x},t) &= A(\underline{x},t)\ , \quad \underline{x} \text{ on } \partial R_1(t), \\ u_n(\underline{x},t) &= B(\underline{x},t)\ , \quad \underline{x} \text{ on } \partial R_2(t), \\ \sigma_s(\underline{x},t) &= C(\underline{x},t)\ , \quad \underline{x} \text{ on } \partial R-\partial R_1(t), \\ \sigma_n(\underline{x},t) &= D(\underline{x},t)\ , \quad \underline{x} \text{ on } \partial R-\partial R_2(t), \end{aligned} \tag{4.1}$$

where A,B,C and D are prescribed vector valued functions and ∂R_1, ∂R_2 vary with time t.

Now the analysis of the last section is still valid and (3.2) satisfies the field equations of viscoelasticity, (2.1), (2.2), (2.3), but since R is constant the equations (3.2) can be rewritten as

$$\begin{aligned} u_i(\underline{x},t) &= L^{-1}[\int_0^{\infty} u_i^{\varepsilon}(\underline{x},\tau,s)e^{-s\tau}d\tau;\ s \to t],\quad \underline{x} \text{ in } R, \\ e_{ij}(\underline{x},t) &= L^{-1}[\int_0^{\infty} e_{ij}^{\varepsilon}(\underline{x},\tau,s)e^{-s\tau}d\tau;\ s \to t],\quad \underline{x} \text{ in } R, \end{aligned} \tag{4.2}$$

$$\sigma_{ij}(\underline{x},t) = L^{-1}[\int_0^\infty \sigma_{ij}^{\varepsilon}(\underline{x},\tau,s)e^{-s\tau}d\tau;\ s \to t],\quad \underline{x} \text{ in } R, \tag{4.2}$$

by equation (1.19). We will assume ∂R_1 and ∂R_2 are monotonically increasing functions of time. Then for each $\underline{x}$ on ∂R we define t_i, (a function of $\underline{x}$), the time at which the particle at position $\underline{x}$ changes from $\partial R-\partial R_i(t)$ to $\partial R_i(t)$ where $i = 1,2$. Thus we have

$$\begin{aligned} &\underline{x} \text{ on } \partial R_i(t) \quad \text{if } t > t_i, \\ &\underline{x} \text{ on } \partial R-\partial R_i(t) \text{ if } t > t_i, \quad i = 1,2. \end{aligned} \tag{4.3}$$

Next suppose we have elastic solutions (3.1) which satisfy the following boundary conditions,

$$\begin{aligned} u_s^{\varepsilon}(\underline{x},t) &= a_1(\underline{x},t)\ , \quad \underline{x} \text{ on } \partial R_1(t), \\ u_n^{\varepsilon}(\underline{x},t) &= b_1(\underline{x},t)\ , \quad \underline{x} \text{ on } \partial R_2(t), \\ \sigma_s^{\varepsilon}(\underline{x},t) &= C(\underline{x},t)\ , \quad \underline{x} \text{ on } \partial R-\partial R_1(t), \\ \sigma_n^{\varepsilon}(\underline{x},t) &= D(\underline{x},t)\ , \quad \underline{x} \text{ on } \partial R-\partial R_2(t), \end{aligned} \tag{4.4}$$

where C,D are given in (4.1) and a_1,b_1 are certain prescribed vector valued functions. Then by a_2,b_2, c_2 and d_2 we shall denote the functions which result from the above elastic solutions such that

$$\begin{aligned} u_s^{\varepsilon}(\underline{x},t,s) &= a_2(\underline{x},t,s)\ , \quad \underline{x} \text{ on } \partial R-\partial R_1(t), \\ u_n^{\varepsilon}(\underline{x},t,s) &= b_2(\underline{x},t,s)\ , \quad \underline{x} \text{ on } \partial R-\partial R_2(t), \\ \sigma_s^{\varepsilon}(\underline{x},t,s) &= c_2(\underline{x},t,s)\ , \quad \underline{x} \text{ on } \partial R_1(t), \\ \sigma_n^{\varepsilon}(\underline{x},t,s) &= d_2(\underline{x},t,s)\ , \quad \underline{x} \text{ on } \partial R_2(t). \end{aligned} \tag{4.5}$$

Then, in order that (4.2) meets the boundary conditions (4.1), we require that the elastic solutions (3.1) must satisfy the boundary conditions (4.4) where a_1 and b_1 are given through the solution of

$$A(\underline{x},t) = a_1(\underline{x},t) + L^{-1}[\int_0^\infty a_2(\underline{x},\tau,s)H(t_1-\tau)e^{-s\tau}d\tau;\ s\to t],\ \underline{x} \text{ on } \partial R_1(t),$$

$$B(\underline{x},t) = b_1(\underline{x},t) + L^{-1}[\int_0^\infty b_2(\underline{x},\tau,s)H(t_2-\tau)e^{-s\tau}d\tau;\ s\to t],\ \underline{x} \text{ on } \partial R_2(t). \tag{4.6}$$

Since a_2, b_2 are related to a_1, b_1, (4.6) represents integral equations in a_1 and b_1.

To prove this, we assume that we have elastic solutions (3.1) which satisfy (4.4) and (4.6). Now if we use (4.4), (4.5) and (4.3) and substitute the result into (4.2), we obtain, with the help of (1.5), (1.16), (4.3), (4.6), the equations (4.1) and

$$\begin{aligned}
u_s(\underline{x},t) &= L^{-1}[\int_0^{t_1} a_2(\underline{x},\tau,s)e^{-s\tau}d\tau;\ s \to t],\ \underline{x} \text{ on } \partial R-\partial R_1(t),\\
u_n(\underline{x},t) &= L^{-1}[\int_0^{t_2} b_2(\underline{x},\tau,s)e^{-s\tau}d\tau;\ s \to t],\ \underline{x} \text{ on } \partial R-\partial R_2(t),\\
\sigma_s(\underline{x},t) &= L^{-1}[\int_{t_1}^{\infty} c_2(\underline{x},\tau,s)e^{-s\tau}d\tau;\ s \to t],\ \underline{x} \text{ on } \partial R_1(t),\\
\sigma_n(\underline{x},t) &= L^{-1}[\int_{t_2}^{\infty} d_2(\underline{x},\tau,s)e^{-s\tau}d\tau;\ s \to t],\ \underline{x} \text{ on } \partial R_2(t).
\end{aligned} \tag{4.7}$$

The condition (4.6) simplifies for two particular cases. The first is, if a_2 is independent of s,

$$a_2(\underline{x},\tau,s) = a_2(\underline{x},\tau), \tag{4.8}$$

then the equation containing a_2 in (4.6) reduces to

$$A(\underline{x},t) = a_1(\underline{x},t)\ ,\ \ \underline{x} \text{ on } \partial R_1(t). \tag{4.9}$$

Next, if a_2 can be written as follows,

$$a_2(\underline{x},\tau,s) = f(s)\ g(\underline{x},\tau) \tag{4.10}$$

then, by (1.6), the equation containing a_2 in (4.6) simplifies to

$$A(\underline{x},t) = a_1(\underline{x},t) + \int_0^{t_1} F(t-\tau)g(\underline{x},\tau)d\tau, \tag{4.11}$$

where

$$F(t) = L^{-1}[f(s);\ s \rightarrow t]. \tag{4.12}$$

In both of these cases, the same relationships hold for b_2.

Our choice that both $\partial R_1(t)$ and $\partial R_2(t)$ are monotonically increasing is arbitrary in that we can consider different cases where $\partial R_1(t)$ and $\partial R_2(t)$ are, independently or together, either monotonically increasing or decreasing. To find the solutions for these other cases we need only replace the elements of the group $(u_s, u_n, \sigma_s, \sigma_n)$ on the left hand side of (4.1), (4.4), (4.5) and (4.7) by the elements of one of these groups, $(\sigma_s, \sigma_n, u_s, u_n)$, $(\sigma_s, u_n, u_s, \sigma_n)$, or $(u_s, \sigma_n, \sigma_s, u_n)$, and keep our assumption that $\partial R_1(t)$ and $\partial R_2(t)$ are monotonically increasing functions of time.

These results can be extended to thermo-rhelogically simple visco-elastic media if the temperature field is either purely position-dependent or time-dependent. A generalization to anisotropic and inhomogenous materials is also possible.

This method of solution of boundary value problems in viscoelasticity was given by Ting [1] in the case that the pseudo-temperature $\theta(\underline{x},t)$ is zero and

$$\partial R_1(t) = 0. \tag{4.13}$$

One problem with his result was that he used a form of equation (1.11) which is not valid.

If, along with (4.13), we assume that

$$D(\underset{\sim}{x},t) = 0$$

and (4.14)

$$b_2(\underset{\sim}{x},t,s) = b_2(\underset{\sim}{x},t)$$

in (4.1) and (4.5), then this result is the same as the one given by Graham [6]. He also proved that his result held when the material is thermo-rheologically simple, [16].

As an example of this method, suppose that the region R is the half-space $z \geq 0$ with the boundary ∂R given by the plane $z = 0$. We consider, in terms of circular cylindrical co-ordinates (ρ,θ,z), the axisymmetric problem governed by the following boundary conditions:

$$\begin{aligned}
&\sigma_{\rho z}(\rho,0,t) = \sigma_{\theta z}(\rho,0,t) = 0 \,, && \rho \geq 0, \\
&u_z(\rho,0,t) = D(t) - \beta(\rho) \,, && 0 \leq \rho \leq a(t), \\
&\sigma_{zz}(\rho,0,t) = 0 \,, && \rho > a(t),
\end{aligned} \qquad (4.15)$$

where the field quantities are independent of θ and a is an increasing function of time. We assume that the body force F_i and the pseudo-temperature θ are zero. Equation (4.15) is a particular case of (4.1) when $\partial R_1(t) = 0$ and, as a result of this and (4.15), (4.6) takes the form

$$D(t) - \beta(\rho) = b_1(\rho,t) + L^{-1}[\int_0^\infty H(t_2-\tau)b_2(\rho,\tau,s)e^{-s\tau}d\tau;\ s \to t], \qquad 0 \leq \rho \leq a(t). \qquad (4.16)$$

If we assume that b_1 is of the form

$$b_1(\rho,t) = D'(t) - \beta'(\rho) \qquad (4.17)$$

where $D'(t)$ and $\beta'(\rho)$ are as yet unknown functions, then (4.4), for this case, is given by

$$\begin{aligned} \sigma^{\varepsilon}_{\rho z}(\rho,0,t) &= \sigma^{\varepsilon}_{\theta z}(\rho,0,t) = 0 \ , && \rho \geq 0, \\ u^{\varepsilon}_{z}(\rho,0,t) &= D'(t) - \beta'(\rho) \ , && 0 \leq \rho \leq a(t), \\ \sigma^{\varepsilon}_{zz}(\rho,0,t) &= 0 \ , && \rho \geq a(t). \end{aligned} \tag{4.18}$$

In this instance, the elastic solutions (3.1) which satisfy (4.18) are given by Sneddon [17], in particular,

$$u^{\varepsilon}_{z}(\rho,0,t) = \int_{0}^{a(t)} \frac{g(y,t)}{(\rho^2-y^2)^{\frac{1}{2}}}\, dy \ , \quad \rho > a(t), \tag{4.19}$$

where

$$g(y,t) = \frac{2}{\pi}\left\{D'(t) - y\int_{0}^{y} \frac{d\beta'/d\rho}{(y^2-\rho^2)^{\frac{1}{2}}}\, d\rho\right\} \tag{4.20}$$

and

$$\sigma^{\varepsilon}_{zz}(\rho,0,t) = \frac{s\bar{G}_1(s)(\bar{G}_1(s)+2\bar{G}_2(s))}{2(\bar{G}_2(s)+2\bar{G}_1(s))}\, \frac{1}{\rho}\, \frac{d}{d\rho}\int_{\rho}^{a(t)} \frac{y g(y,t) dy}{(y^2-\rho^2)^{\frac{1}{2}}} \ , \quad 0 \leq \rho \leq a(t). \tag{4.21}$$

Then from (4.19), (4.20), (4.8), (4.17) and (4.16) we find that

$$D'(t) = D(t)$$

and (4.22)

$$\beta'(\rho) = \beta(\rho)$$

Thus (4.2) is the solution to the viscoelastic boundary value problem represented by (2.1), (2.2), (2.3) and (4.15) if the elastic solutions (3.1) satisfy the boundary conditions (4.18) with (4.22). In particular, the normal stress for $\rho \leq a(t)$ is given, with the help of (4.20), (4.21), (4.22), (4.7), (1.9), (1.3) and (1.4), as

$$\sigma_{zz}(\rho,0,t) = (K*dM)(\rho,t) \ , \quad 0 \leq \rho \leq a(t),$$

where

$$K(t) = [G_1*d(G_1+2G_2)*d(2G_1+G_2)^{-1}](t) \tag{4.23}$$

and

$$M(\rho,t) = \frac{H(t-t_2)}{\rho} \frac{\partial}{\partial\rho} \int_{\rho}^{a(t)} \frac{yg(y,t)dy}{(y^2-\rho^2)^{\frac{1}{2}}} \ .$$

Note that with (4.3) we have that

$$H(t-t_2) = H(a(t)-\rho). \tag{4.24}$$

The above solution agrees with the one given by Graham [6]. This problem is known as the contact problem and a discussion of it follows in the next section.

5. The Contact Problem

In this section we will consider the contact problem of a rigid indentor pressed against the surface of a viscoelastic half-space where the contact area varies with time such that it may have any number of maxima or minima. To do this we will examine three papers, by Graham [8] and Ting [10], [11]. It will be shown that Graham's solution can be extended and that the extended solution is equivalent to either of Ting's solutions. But first we will give a short history of the contact problem in viscoelasticity.

Lee and Radok [18] solved the contact problem of a rigid sphere pressed into a viscoelastic half-space when the contact area increases monotonically with time. They used a technique due to Radok [19] which has also been used by Al-Khozaie and Lee [20] to study the two dimensional problem of the contact of a rigid cylinder and viscoelastic half-space. A theory, which includes ageing effects, of the contact of two axisymmetric viscoelastic bodies, was given by Predeleanu [21] who recovered Lee and Radok's solution (for non-ageing materials) as a special case. The corresponding plane problem was studied by Prokopovici [22]. Hunter [7] subsequently rederived Lee and Radok's solution, using dual integral equations, and extended it to the case where the contact area increases to a single maximum and then decreases. The results of Hunter's solution were recovered by Graham [9] using a simpler analysis. Graham also considered an indentor of asymmetric surface and the Hertz contact problem of two viscoelastic bodies with quadratic surfaces. The latter was also studied by Yang [23] with the restriction that the contact area is only monotonically increasing.

Expanding on these earlier solutions, Ting [10] studied the problem of an axisymmetric rigid indentor on a viscoelastic half-space where the contact area has any number of maxima or minima. In this paper, the continuity of the solutions at the time when the contact area is a maximum or minimum is not clear and the procedure for obtaining these solutions becomes unwieldy as the number of maxima and minima increases. More recently, Graham [8] extended his earlier results to the case considered by Ting with the restriction that minima of the contact area are null and each new maximum must contain all previous maxima. In this case the continuity of the solutions is satisfied. Following this, Ting [11] gave a solution to the problem considered by Graham with the restrictions on the contact area removed and which meets all continuity requirements. In this last paper Ting points out that the current pressure and displacement distributions, over the contact area, are not dependent on the entire loading history. The pressure is independent of those previous intervals of time when the associated contact regions contain the current region, whereas the displacement is independent of those intervals of time when the associated regions are contained by the current region. One question that remains in these last three papers is whether the normal pressure over the contact area remains positive for n maxima and minima. If it does not, then the problem solved might not be realistic.

In detail, the problem considered in these last three papers is that of determining the displacement and stress fields set up in a viscoelastic half-space, occupying the region $z \geqslant 0$, where boundary $z = 0(B)$ is deformed by a rigid indentor. It is assumed that over the contact area, $\Omega(t)$, the normal surface displacement must conform to the surface geometry of the indentor. The boundary conditions to this problem are given by:

$$\begin{aligned} \sigma_{yz}(x,y,0,t) &= \sigma_{xz}(x,y,0,t) = 0 \ , \quad (x,y) \text{ on } B , \\ u_z(x,y,0,t) &= D(t) - \beta(x,y)H(t), \quad (x,y) \text{ on } \Omega(t), \\ \sigma_{zz}(x,y,0,t) &= 0 \qquad\qquad , \quad (x,y) \text{ on } B - \Omega(t), \end{aligned} \tag{5.1}$$

where β is prescribed by the surface of the indentor and $D(t)$ is the depth of penetration, at time t, of its tip into the half-space. It turns out that this problem reduces to finding the distribution of normal surface traction, $p(x,y,t)$, acting over $\Omega(t)$ (p is zero outside of $\Omega(t)$) and the relationship between $D(t)$ and $\Omega(t)$.

We will denote by t^n_{min} the time when the contact area is at a minimum and has already had $(n-1)$ maxima occurring at times t^r_{max} $(r = 1,2,\ldots,n-1)$ and $(n-2)$ minima occurring at times t^r_{min} $(r = 2,3,\ldots,n-1)$. The times t^r_{max} and t^r_{min} are labelled in order of increasing magnitude and we define $t^1_{min} = -\infty$. We therefore have

$$t^{r-1}_{min} < t^r_{max} < t^r_{min}, \quad r = 2,3,\ldots,n. \tag{5.2}$$

Graham and Ting consider the above problem for two cases, $t \geq t^n_{min}$ $(t \geq t^n_{max})$ where the contact area, $\Omega(t)$, is monotonically increasing (decreasing) with time and it is assumed that the history of displacement and stress fields is known for times up till t^n_{min} (t^n_{max}).

In the first paper we shall look at, Graham makes the restrictions that all minima of the contact area are null and each new maximum must contain all previous maxima, that is,

$$\Omega(t^r_{min}) = 0 \text{ and } \Omega(t^{r-1}_{max}) \subseteq \Omega(t^r_{max}), \quad r = 1,2,\ldots,n. \tag{5.3}$$

For the case when $t \geq t^n_{min}$, $n \geq 2$, and $\Omega(t)$ is monotonically increasing, his method of solution is to note that from an earlier work [9] the normal surface displacement is related to the pressure distribution $p(x,y,t)$ acting over $\Omega(t)$ through equation,

$$u_z(x,y,0,t) = \int_{-\infty}^{t} K(t-\theta)\frac{\partial}{\partial\theta} \iint_{\Omega(\theta)} \frac{p(\xi,\eta,\theta)d\xi d\eta}{[(x-\xi)^2+(y-\eta)^2]^{\frac{1}{2}}}\, d\theta, \tag{5.4}$$

where K is an auxiliary response function defined by

$$K(t) = \left[\frac{1}{\pi}(2G_1 + G_2)*d(G_1 + 2G_2)^{-1}*dG_1^{-1} \right](t). \tag{5.5}$$

He then considers the one-parameter family of corresponding elastic problems. For this case, the corresponding equation to (5.4) is written as

$$u^{\varepsilon}_z(x,y,0,t) = \iint_{\Omega(t)} \frac{\kappa\ p(\xi,\eta,t)d\xi d\eta}{[(x-\xi)^2+(y-\eta)^2]^{\frac{1}{2}}}\ , \tag{5.6}$$

where

$$\kappa = \frac{1}{4\pi}\frac{(4\mu+3k)}{\mu(\mu+3k)}\ .$$

He supposes the pressure distribution which generates through equation (5.6) normal surface displacements consistent with (5.1) is given by

$$p(x,y,t) = q(x,y,t), \quad (x,y) \text{ on } \omega(t), \tag{5.7}$$

with D related to ω through the equation

$$D(t) = f(\omega(t)), \tag{5.8}$$

where f is a given function of positive slope. The solution given through equations (5.7), (5.8) involves the time t as a parameter only and $\omega(t)$ is always a member of the one-parameter family of elastic contact areas.

Now he lets the viscoelastic pressure distribution over $\omega(t)$, $t \leq t^n_{min}$ be given by

$$p(x,y,t) = r(x,y,t), \quad (x,y) \text{ on } \omega(t), \quad t \leq t^n_{min}, \tag{5.9}$$

where r is extended so that

$$r(x,y,t) = 0, \quad (x,y) \text{ on } \omega(t), \quad t \leq t^n_{min}. \tag{5.10}$$

Similarly, the depth of penetration for times $t \leq t^n_{min}$ is

$$D(t) = \alpha(t), \quad t \leq t^n_{min}. \tag{5.11}$$

He then writes (5.4) as

$$u_z(x,y,0,t) - u'_z(x,y,t) = \int_{t^n_{min}}^{t} K(t-\theta)\frac{\partial}{\partial\theta} \iint_{\omega(\theta)} \frac{p(\xi,\eta,\theta)d\xi d\eta}{[(x-\xi)^2+(y-\eta)^2]^{\frac{1}{2}}}\, d\theta, \tag{5.12}$$

where

$$u'(x,y,t) = \int_{-\infty}^{t^n_{min}} K(t-\theta)\frac{\partial}{\partial\theta} R(x,y,\theta)d\theta, \tag{5.13}$$

$$R(x,y,t) = \iint_{\omega(t)} \frac{r(\xi,\eta,t)d\xi d\eta}{[(x-\xi)^2+(y-\eta)^2]^{\frac{1}{2}}}. \tag{5.14}$$

Next, for times $t \geq t^n_{min}$, he defines $t^-_0(t)$, $t^+_r(t)$, $t^-_r(t)$, $(r = 1,2,\ldots,n-1)$ so that

$$t^-_0(t) = -\infty,$$

$$t^+_r(t) = t^-_r(t) = t^r_{max} \text{ if } \omega(t) \supseteq \omega(t^r_{max}),$$

$$t^-_{r-1}(t) = t^+_r(t) = t^r_{min} \text{ if } \omega(t^r_{min}) \supsetneq \omega(t), \tag{5.15}$$

$$\omega(t^+_r(t)) = \omega(t^-_r(t)) = \omega(t),\ t^r_{min} \leqslant t^+_r(t) \leqslant t^r_{max} \leqslant t^-_r(t) \leqslant t^{r+1}_{min}\ ,$$

$$r = 1,2,\ldots,n-1,$$

where for times $t > t^n_{min}$ ω is prescribed and at first strictly monotonically increasing. It is immediate from (5.15) that

$$t^-_{r-1}(t) \leqslant t^+_r(t) \leqslant t^-_r(t), \quad r = 1,2,\ldots,(n-1). \tag{5.16}$$

In future we shall refer to $t^-_0(t)$, $t^+_r(t)$, $t^-_r(t)$ as t^-_0, t^+_r, t^-_r, keeping in mind that they depend on t.

With the aid of the quantities introduced so far, and the identity,

$$R(t) = \int_{-\infty}^{t} K(t-\theta)d \int_{-\infty}^{\theta} K^{-1}(\theta-\tau)dR(\tau) \tag{5.17}$$

where for simplicity we write

$$R(x,y,t) = R(t), \tag{5.18}$$

the author succeeds in reducing (5.13) to

$$u'(x,y,\theta) - N(x,y,\theta,t) = \iint_B \frac{m(\xi,\eta,\theta;t)d\xi d\eta}{[(x-\xi)^2+(y-\eta)^2]^{\frac{1}{2}}} \tag{5.19}$$

where N is given by the recurrence relation

$$N(\theta;t) = \sum_{r_1=1}^{n-1} \int_{t^+_{r_1}}^{t^-_{r_1}} K(\theta-\theta_2)d\left\{\sum_{r_2=1}^{r_1} \int_{t^+_{r_2}}^{t^-_{r_2}} K^{-1}(\theta_2-\theta_1)d[K^*dR](\theta_1)\right.$$

$$\left. + \sum_{r_2=1}^{r_1} \int_{t^-_{r_2-1}}^{t^+_{r_2}} K^{-1}(\theta_2-\theta_1)dN_{r_2}(\theta_1;t)\right\},$$

$$N_{r_m}(\theta;t) = \sum_{r_{2m+1}=1}^{r_{2m}-1} \int_{t^+_{r_{2m+1}}}^{t^-_{r_{2m+1}}} K(\theta-\theta_2)\,d\left\{\sum_{r_{2m+2}=1}^{r_{2m+1}} \int_{t^+_{r_{2m+2}}}^{t^-_{r_{2m+2}}} K^{-1}(\theta_2-\theta_1)\,d[K^*dR](\theta_1)\right.$$

$$\left. + \sum_{r_{2m+2}=1}^{r_{2m+1}} \int_{t^-_{(r_{2m+2}-1)}}^{t^+_{r_{2m+2}}} K^{-1}(\theta_2-\theta_1)\,dN_{r_{2m+2}}(\theta_1;t)\right\}, \tag{5.20}$$

$$m = 1,2,\ldots,(n-3),$$

$$N_{2n-4}(\theta;t) = \sum_{r_{2n-3}=1}^{r_{2n-4}-1} \int_{t^+_{r_{2n-3}}}^{t^-_{r_{2n-3}}} K(\theta-\theta_2)\,d\left\{\sum_{r_{2n-2}=1}^{r_{2n-3}} \int_{t^+_{r_{2n-2}}}^{t^-_{r_{2n-2}}} K^{-1}(\theta_2-\theta_1)\,d[K^*dR](\theta_1)\right\}.$$

Similarly $m(\theta,t)$ is given by the recurrence relation,

$$m(\theta;t) = \int_{t^-_{(n-1)}}^{t^n_{min}} K(\theta-\theta_1)\,dr(\theta_1) + \sum_{r_1=1}^{n-1} \int_{t^-_{(r_1-1)}}^{t^+_{r_1}} K(\theta-\theta_1)\,dr(\theta_1)$$

$$+ \sum_{r_1=1}^{n-1} \int_{t^+_{r_1}}^{t^-_{r_1}} K(\theta-\theta_1)\,dM_{r_1}(\theta_1;t),$$

$$M_{r_{2m-1}}(\theta;t) = \sum_{r_{2m}=1}^{r_{2m-1}} \int_{t^-_{(r_{2m}-1)}}^{t^+_{r_{2m}}} K^{-1}(\theta-\theta_2)\,d\left\{\sum_{r_{2m+1}=1}^{r_{2m}} \int_{t^-_{(r_{2m+1}-1)}}^{t^+_{r_{2m+1}}} K(\theta_2-\theta_1)\,dr(\theta_1)\right.$$

$$\left. + \sum_{r_{2m+1}=1}^{r_{2m}-1} \int_{t^+_{r_{2m+1}}}^{t^-_{r_{2m+1}}} K(\theta_2-\theta_1)\,dM_{r_{2m+1}}(\theta_1;t)\right\}, \tag{5.21}$$

$$m = 1,2,\ldots,(n-2),$$

$$M_{r_{2n-3}}(\theta;t) = \sum_{r_{2n-2}=1}^{r_{2n-3}} \int_{t^-_{(r_{2n-2}-1)}}^{t^+_{r_{2n-2}}} K^{-1}(\theta-\theta_2)\,d\left\{\sum_{r_{2n-1}=1}^{r_{2n-1}} \int_{t^-_{(r_{2n-1}-1)}}^{t^+_{r_{2n-1}}} K(\theta_2-\theta_1)\,dr(\theta_1).\right.$$

From (5.19) he finds the depth of penetration D(t) and the pressure p(x,y,t) acting over $\omega(t)$ while using the assumption

$$\omega(t^r_{min}) = 0, \quad r = 1,2,\ldots,n.$$

It is from this point that we can diverge from Graham's solution and find D(t) and p(x,y,t) without the above assumption.

Now if we use (5.19), (5.21), (5.10), (5.6), (5.7) and the facts that $\omega(t)$ is monotonically increasing and the pressure p is zero outside the contact area, then equation (5.12) becomes

$$u_z(x,y,0,t) - u^\varepsilon_z(x,y,0,t) - N(x,y,t;t) = \iint_{\omega(t)} \left\{ \frac{m(\xi,\eta,t;t) - \kappa q(\xi,\eta,t) + \int_{t^n_{min}}^{t} K(t-\theta)dp(\xi,\eta,\theta)}{[(x-\xi)^2+(y-\eta)^2]^{\frac{1}{2}}} \right\} d\xi d\eta. \quad (5.22)$$

If (x,y) lie on $\omega(t)$ then from (5.1), (5.8) and (5.10), (5.4), (5.1) and (5.11) the left hand side of (5.22) becomes

$$D(t) - f(\omega(t)) - n(t;t), \quad (5.23)$$

a function of t alone, where n(t;t) is given by (5.20) if we replace [K*dR](t) by $\alpha(t)$. Now if we consider (5.6) when u^ε_z is a constant (or a function of time) over the contact area, then this refers to the case when a flat-ended cylindrical punch presses into an elastic half-space and the pressure will be discontinuous at the edge of the contact area. If the pressure is continuous then the displacement must vary over the contact area or be zero. Combining (5.22) and (5.23) we see that the result takes the same form as (5.6) and here, what refers to u^ε_z is a function of time and what refers to the pressure is continuous at the edge of $\omega(t)$. Thus we deduce that

$$D(t) = f(\omega(t)) + n(t;t) \tag{5.24}$$

and

$$\int_{t^n_{min}}^{t} K(t-\theta)\frac{\partial}{\partial\theta} p(x,y,\theta)d\theta = \kappa q(x,y,t) - m(x,y,t;t), \tag{5.25}$$

$$(x,y) \text{ on } \omega(t).$$

We can solve (5.25) for $p(x,y,t)$ and get

$$p(x,y,t) = p(x,y,t^n_{min}) + \int_{t^n_{min}}^{t} K^{-1}(t-\theta)d[\kappa q(x,y,\theta) - m(x,y,\theta;t)], \tag{5.26}$$

$$(x,y) \text{ on } \omega(t).$$

These results give the same answer as Graham's when we use the assumption (5.3).

Next Graham looks at the case when $t \geq t^n_{max}$ and $\omega(t)$ is monotonically decreasing. In solving this problem he invokes his second assumption,

$$\omega(t^{r-1}_{max}) \subseteq \omega(t^r_{max}), \quad r = 1,2,\ldots,n,$$

in order to meet the continuity requirements of $p(x,y,t)$ at time $t = t^n_{max}$. This assumption is not necessary if we use the following analysis.

We first note that the author defines the function P by the equation

$$P(x,y,t) = \int_{-\infty}^{t^n_{max}} K^{-1}(t-\theta)d[K*dR](\theta), \quad (x,y) \text{ on } B. \tag{5.27}$$

From (1.4) and (5.14) we see that this simplifies at $t = t^n_{max}$, to

$$P(x,y,t^n_{max}) = \iint_{\omega(t^n_{max})} \frac{r(\xi,\eta,t^n_{max})d\xi d\eta}{[(x-\xi)^2+(y-\eta)^2]^{\frac{1}{2}}}, \quad (x,y) \text{ on } B. \tag{5.28}$$

Graham then writes the equation

$$P(x,y,t) + \int_{t^n_{max}}^{t} K^{-1}(t-\theta)\frac{\partial}{\partial\theta} u_z(x,y,0,\theta)d\theta$$
$$= \iint_{\omega(t)} \frac{w(\xi,\eta,t)d\xi d\eta}{[(x-\xi)^2+(y-\eta)^2]^{\frac{1}{2}}}, \quad (x,y) \text{ on } B, \quad t > t^n_{max}, \tag{5.29}$$

where $w(x,y,t)$ is the viscoelastic pressure at time t acting over $\omega(t)$. But if we let $t \to t^n_{max}$ in (5.29) then, from (5.28) and the fact that u_z is continuous at t^n_{max}, we find

$$r(x,y,t^n_{max}) = \omega(x,y,t^n_{max}), \quad (x,y) \text{ on } B.$$

Next we will give a recurrence relation that we will need later. From (5.20) we note that, when (x,y) lies on $\omega(t)$,

$$N_n(\theta;t) = n(\theta;t). \tag{5.30}$$

With (5.30) and equations (5.20) we obtain the relation

$$N_n(\theta;t) = N_{n-1}(\theta;t) + \int_{t^+_{n-1}}^{t^-_{n-1}} K(\theta-\theta_2)d\Big\{\sum_{r=1}^{n-1}\Big[\int_{t^+_r}^{t^-_r} K^{-1}(\theta_2-\theta_1)d\alpha(\theta_1)$$
$$+ \int_{t^-_{r-1}}^{t^+_r} K^{-1}(\theta_2-\theta_1)dN_r(\theta_1;t)\Big]\Big\}, \quad n \geq 2. \tag{5.31}$$

Then if we define D_i such that,

$$D_{n-1}(\theta;t) = N_n(\theta;t) + f(\omega(t)), \quad n \geq 2, \tag{5.32}$$

we have from (5.31) that

$$D_{n-1}(\theta;t) = D_{n-2}(\theta;t) + \int_{t_{n-1}^+}^{t_{n-1}^-} K(\theta-\theta_2)d\left\{ \sum_{r=1}^{n-1} \left[\int_{t_r^+}^{t_r^-} K^{-1}(\theta_2-\theta_1)d\alpha(\theta_1) + \int_{t_{r-1}^-}^{t_r^+} K^{-1}(\theta_2-\theta_1)dD_{r-1}(\theta_1;t) \right] \right\}, \quad n \geq 2, \tag{5.33}$$

where from (5.31) and (5.30),

$$D_0(\theta;t) = f(\omega(t)). \tag{5.34}$$

Since the viscoelastic and elastic displacements are the same for time $t \leq t_{max}^1$, and as a result of equations (5.11), (5.15), (5.8), we have

$$\begin{aligned} \alpha(t_1^+) &= f(\omega(t_1^+)) \\ &= f(\omega(t)) . \end{aligned} \tag{5.35}$$

The last point to note here is that from (5.32), (5.30) and (5.24),

$$D_{n-1}(t;t) = D(t), \quad t \geq t_{max}^n. \tag{5.36}$$

In discussing Ting's paper, [11], we note his assumptions do not include (5.3). Following a straightforward procedure, using the boundary conditions (5.1) and an identity, Ting soon reduces (5.4) so that he may deduce the depth of penetration D'(t) and the pressure distribution

$p'(x,y,t)$ in the contact area, to be

$$D'(t) = \alpha(t_{n-1}^-) - \sum_{r=1}^{n-1} \int_{t_r^+}^{t_r^-} F_{n-1,2r}(t;\theta)\frac{d}{d\theta}\alpha(\theta)d\theta \tag{5.37}$$

and

$$p'(x,y,t) = r(x,y,t_{n-1}^-) - \int_{t_{n-1}^-}^{t} K^{-1}(t-\theta)\frac{\partial}{\partial\theta}\left\{\sum_{r=1}^{n-1} \int_{t_{r-1}^-}^{t_r^+} F_{n-1,2r-1}(t;\theta_1)dr(x,y,\theta_1)\right\}, \tag{5.38}$$

where $t \geq t_{min}^n$, $n \geq 2$ and we define F by the equations

$$F_{n-1,2n-2}(t;\theta) = \int_{t_{n-1}^-}^{t} K(t-\theta_1)\frac{\partial}{\partial\theta_1}K^{-1}(\theta_1-\theta)d\theta_1,$$

$$F_{n-1,2n-3}(t;\theta) = \int_{t_{n-1}^-}^{t} K(t-\theta_1)\frac{\partial}{\partial\theta_1}\int_{t_{n-1}^+}^{\theta_1} K^{-1}(\theta_1-\theta_2) \cdot$$

$$\cdot \frac{\partial}{\partial\theta_2}K(\theta_2-\theta)d\theta_2 d\theta_1, \tag{5.39}$$

$$F_{n-1,2n-4}(t;\theta) = \int_{t_{n-1}^-}^{t} K(t-\theta_1)\frac{\partial}{\partial\theta_1}\int_{t_{n-1}^+}^{\theta_1} K^{-1}(\theta_1-\theta_2) \cdot$$

$$\cdot \frac{\partial}{\partial\theta_2}\int_{t_{n-2}^-}^{\theta_2} K(\theta_2-\theta_3)\frac{\partial}{\partial\theta_3}K^{-1}(\theta_3-\theta)d\theta_3 d\theta_2 d\theta_1,$$

....

[Note that for the sake of clarity, here and throughout this discussion, we will transform Ting's notation into the one already introduced.]

In showing that this result is the same as Graham's extended version, we will make the following simplification of the above notation such that

$$\int_{t_{n-1}^-}^{\tau} K(\tau-\theta_1)\frac{\partial}{\partial\theta_1}\int_{t_{n-1}^+}^{\theta_1} K^{-1}(\theta_1-\theta_2)\frac{\partial}{\partial\theta_2}\int_{t_{n-2}^-}^{\theta_2} K(\theta_2-\theta_3)\,\cdot$$

$$\cdot\frac{\partial}{\partial\theta_3}\int_{t_{n-2}^+}^{\theta_3} K^{-1}(\theta_3-\theta_4)\frac{\partial}{\partial\theta_4}\cdots\frac{\partial}{\partial\theta_{2(n-1-r)}}\int_{t_r^-}^{\theta_{2(n-1-r)}} K(\theta_{2(n-1-r)}-\theta_{2(n-2-r)})$$

$$\cdot\frac{\partial}{\partial\theta_{2(n-2-r)}}\int_{t_r^+}^{t_r^-} K^{-1}(\theta_{2(n-2-r)}-\theta)\frac{\partial}{\partial\theta}\alpha(\theta)\,d\theta\, d\theta_{2(n-2-r)}\cdots d\theta_2 d\theta_1$$

$$=\left[\int_{t_{n-1}^-}^{\tau}\cdots\int_{t_r^-}^{\theta_{2(n-1-r)}}\int_{t_r^+}^{t_r^-}\right](\tau;t) \tag{5.40}$$

where $r \leqslant n-2$. If we use (5.40) then we can define D_K' and W_K as

$$D_{n-1}'(\tau;t) = \alpha(t_{n-1}^-) + W_{n-1}(\tau;t), \quad n \geqslant 2, \tag{5.41}$$

and

$$W_{n-1}(\tau;t) = -\int_{t_{n-1}^-}^{\tau} K(\tau-\theta_1)\frac{\partial}{\partial\theta_1}\int_{t_{n-1}^+}^{t_{n-1}^-} K^{-1}(\theta_1-\theta)\frac{d}{d\theta}\alpha(\theta)\,d\theta$$
$$-\sum_{r=1}^{n-2}\left[\int_{t_{n-1}^-}^{\tau}\cdots\int_{t_r^-}^{\theta_{2(n-1-r)}}\int_{t_r^+}^{t_r^-}\right](\tau;t), \ n \geqslant 2. \tag{5.42}$$

We can see that, with (5.37), (5.41) and (5.42),

$$D_{n-1}'(t;t) = D'(t), \quad t \geqslant t_{min}^n. \tag{5.43}$$

Next we give two identities which are similar to the one used by Ting,

$$v(t) = v(b) + \int_b^t K(t-\theta)\frac{\partial}{\partial\theta}\int_b^{\theta} K^{-1}(\theta-\tau)\frac{\partial}{\partial\tau}v(\tau)\,d\tau\, d\theta, \tag{5.44}$$

and

$$\int_a^b K(t-\theta)\frac{\partial}{\partial\theta}\int_a^\theta K^{-1}(\theta-\tau)\frac{\partial}{\partial\tau}v(\tau)d\tau\,d\theta$$
$$= v(b) - v(a) - \int_b^t K(t-\theta)\frac{\partial}{\partial\theta}\int_a^b K^{-1}(\theta-\tau)\frac{\partial}{\partial\tau}v(\tau)d\tau\,d\theta, \qquad (5.45)$$
$$a \leqslant b \leqslant t.$$

We will now rewrite W_{n-1} in a form similar to (5.31). If we add and subtract the same quantity to (5.40), we can find, after some rearranging,

$$W_{n-1}(\tau;t) = -\int_{t_{n-1}^-}^{\tau} K(\tau-\theta_1)\frac{\partial}{\partial\theta_1}\int_{t_{n-1}^+}^{t_{n-1}^-} K^{-1}(\theta_1-\theta)\frac{\partial}{\partial\theta}\alpha(\theta)d\theta\,d\theta_1$$
$$-\int_{t_{n-1}^+}^{\tau}\int_{t_{n+1}^+}^{\theta_1}\left\{\sum_{r=1}^{n-3}\int_{t_{n-2}^-}^{\theta_2}\cdots\int_{t_r^-}^{\theta_{2(n-1-r)}}\int_{t_r^+}^{t_r^-} + \int_{t_{n-2}^-}^{\theta_2}\int_{t_{n-2}^+}^{t_{n-2}^-}\right\}(\tau;t) \qquad (5.46)$$
$$+\sum_{r=1}^{n-2}\left\{\int_{t_{n-1}^+}^{t_{n-1}^-}\int_{t_{n-1}^+}^{\theta_1}\int_{t_{n-2}^-}^{\theta_2}\cdots\int_{t_r^-}^{\theta_{2(n-1-r)}}\int_{t_r^+}^{t_r^-}\right\}(\tau;t)$$

Using (5.45), the first integrals can be written as

$$\alpha(t_{n-1}^+) - \alpha(t_{n-1}^-) + \int_{t_{n-1}^+}^{t_{n-1}^-} K(\tau-\theta_1)\frac{\partial}{\partial\theta_1}\int_{t_{n-1}^+}^{\theta_1} K^{-1}(\theta_1-\theta)\frac{\partial\alpha(\theta)}{\partial\theta}d\theta\,d\theta_1. \qquad (5.47)$$

Comparing the first series of integrals in (5.46) and equation (5.42), we can see that they are equal to

$$-\int_{t_{n-1}^+}^{\tau} K(\tau-\theta_1)\frac{\partial}{\partial\theta_1}\int_{t_{n-1}^+}^{\theta_1} K^{-1}(\theta_1-\theta_2)\frac{\partial}{\partial\theta_2}W_{n-2}(\theta_2;t)d\theta_2 d\theta_1, \qquad (5.48)$$

which by using (5.44) reduces to

$$- W_{n-2}(\tau;t) + W(t^+_{n-1};t). \tag{5.49}$$

For convenience we will refer to the last series of integrals as I. If we add and subtract the same quantity to I we can get

$$I = \sum_{r=1}^{n-2} \left\{ \int_{t^+_{n-1}}^{t^-_{n-1}} \int_{t^-_{n-2}}^{\theta_1} \int_{t^-_{n-2}}^{\theta_2} \cdots \int_{t^-_r}^{\theta_{2(n-1-r)}} \int_{t^+_r}^{t^-_r} \right\} (\tau;t)$$

$$- \int_{t^+_{n-1}}^{t^-_{n-1}} \int_{t^-_{n-2}}^{t^+_{n-1}} \left\{ \sum_{r=1}^{n-3} \int_{t^-_{n-2}}^{\theta_2} \cdots \int_{t^-_r}^{\theta_{2(n-1-r)}} \int_{t^+_r}^{t^-_r} + \int_{t^-_{n-2}}^{\theta_2} \int_{t^+_{n-2}}^{t^-_{n-2}} \right\} (\tau;t).$$

Here if we use (5.44) on the first series and note (5.42) when considering the second, then I simplifies to

$$I = \sum_{r=1}^{n-3} \left\{ \int_{t^+_{n-1}}^{t^-_{n-1}} \int_{t^+_{n-2}}^{\theta_3} \cdots \int_{t^-_r}^{\theta_{2(n-1-r)}} \int_{t^+_r}^{t^-_r} \right\} (\tau;t)$$

$$+ \int_{t^+_{n-1}}^{t^-_{n-1}} K(\tau-\theta_1)\frac{\partial}{\partial\theta_1} \int_{t^+_{n-2}}^{t^-_{n-2}} K^{-1}(\theta_1-\theta)\frac{\partial}{\partial\theta}\alpha(\theta)\,d\theta\,d\theta_1 \tag{5.50}$$

$$+ \int_{t^+_{n-1}}^{t^-_{n-1}} K(\tau-\theta_1)\frac{\partial}{\partial\theta_1} \int_{t^-_{n-2}}^{t^+_{n-1}} K^{-1}(\theta_1-\theta)\frac{\partial}{\partial\theta}W_{n-2}(\theta;t)\,d\theta_1\,d\theta_2$$

The same procedure used on I can be applied to the series in (5.50) and this process can be repeated (n-4) times finally giving,

$$I = \sum_{r=1}^{n-2} \int_{t_{n-1}^+}^{t_{n-1}^-} K(\tau-\theta_1)\frac{\partial}{\partial\theta_1} \int_{t_r^+}^{t_r^-} K^{-1}(\theta_1-\theta)\frac{\partial}{\partial\theta}\alpha(\theta)d\theta d\theta_1$$
$$+ \sum_{r=2}^{n-1} \int_{t_{n-1}^+}^{t_{n-1}^-} K(\tau-\theta_1)\frac{\partial}{\partial\theta_1} \int_{t_{r-1}^-}^{t_r^+} K^{-1}(\theta_1-\theta)\frac{\partial}{\partial\theta}W_{r-1}(\theta;t)d\theta d\theta_1 \quad (5.51)$$

Substituting (5.47), (5.49) and (5.51) into (5.46) we obtain

$$W_{n-1}(\tau;t) = W_{n-2}(\tau;t) - W_{n-2}(t_{n-1}^+;t) + \alpha(t_{n-1}^+) - \alpha(t_{n-1}^-)$$
$$+ \sum_{r=1}^{n-1} \int_{t_{n-1}^+}^{t_{n-1}^-} K(\tau-\theta_1)\frac{\partial}{\partial\theta_1}\Big\{ \int_{t_r^+}^{t_r^-} K^{-1}(\theta_1-\theta)\frac{\partial}{\partial\theta}\alpha(\theta)d\theta \quad (5.52)$$
$$+ \int_{t_{r-1}^-}^{t_r^+} K^{-1}(\theta_1-\theta)\frac{\partial}{\partial\theta}W_{r-1}(\theta;t)d\theta\Big\}d\theta_1, \quad n \geq 2,$$

where $W_0(\tau;t)$ is independent of τ. It follows directly from (5.41) that (5.52) implies that

$$D'_{n-1}(\tau;t) = D'_{n-2}(\tau;t) + \sum_{r=1}^{n-1} \int_{t_{n-1}^+}^{t_{n-1}^-} K(\tau-\theta_1)\frac{\partial}{\partial\theta_1} \cdot$$
$$\cdot \Big\{ \int_{t_r^+}^{t_r^-} K^{-1}(\theta_1-\theta)\frac{\partial}{\partial\theta}\alpha(\theta)d\theta + \int_{t_{r-1}^-}^{t_r^+} K^{-1}(\theta_1-\theta)\frac{\partial}{\partial\theta}D'_{r-1}(\theta;t)d\theta\Big\}d\theta_1, \quad n \geq 2, \quad (5.53)$$

where we have used the fact that from (5.41), (5.42), (5.37) and (5.11),

$$D'_{n-2}(t_{n-1}^+;t) = \alpha(t_{n-1}^+).$$

From (5.51), (5.45), (5.41) and (5.42) it is also seen that

$$D_0'(\tau;t) = W_0(\tau;t) = \alpha(t_1^+). \tag{5.54}$$

Now for both solutions, (5.24) and (5.37), we assume that $\alpha(t)$ is the same for $t \leq t_{min}^n$. Thus, after considering (5.33), (5.36) and (5.53), (5.43), we can see that it is only necessary to show that

$$D_1'(\tau;t) = D_1(\tau;t),$$

in order to prove that the depths of penetration for the two solutions are the same. From (5.41), (5.42) and (5.45) we have

$$D_1'(\tau;t) = \alpha(t_1^+) + \int_{t_1^+}^{t_1^-} K(\tau-\theta)\frac{\partial}{\partial\theta_1} \int_{t_1^+}^{\theta} K^{-1}(\theta_1-\theta)\frac{\partial}{\partial\theta}\alpha(\theta)d\theta d\theta_1.$$

The same expression is found for $D_1(\tau;t)$ if we combine equations (5.32), (5.31) and (5.35). Therefore since the displacements over the contact region are the same for both solutions and they both use (5.4), then the pressure distributions over the contact area must be the same also.

If we consider the case when $t \geq t_{max}^n$ and $\omega(t)$ is monotonically decreasing then we can also show that the two solutions are equivalent. To do this, we follow the same analysis as above, only considering the pressure distributions first and then show that as a result of these being the same, the depths of penetration are also equivalent.

In looking at Ting's other paper [10], which is an earlier paper to the one discussed, we first notice that the indentor is axisymmetric. For the case when $t \geq t_{max}^n$ and $\omega(t)$ is monotonically increasing, he finds the depth of penetration $D(t)$ to be

$$D(t) = f(\omega(t)) + \int_{-\infty}^{t_{n-1}^-} K(t-\theta_1)\frac{\partial}{\partial\theta_1}\int_0^{\omega(\theta_1)} J(0,\zeta)\zeta p(\zeta,\theta_1)d\zeta d\theta_1$$

$$+ \int_0^{\omega(t)} \frac{\omega(t)}{[\omega^2(t)-r^2]}\frac{\partial}{\partial r}\int_{-\infty}^{t_{n-1}^-} K(t-\theta_1)\frac{\partial}{\partial\theta_1}\int_0^{\omega(\theta_1)} J(r,\zeta)\zeta p(\zeta,\theta_1)d\zeta d\theta_1 dr$$

where $p(r,t)$ is the pressure acting over $\omega(t)$. After quite a lengthy discussion this expression can be reduced to Graham's. Any further discussion of this solution would have little value since either of the two papers previously mentioned presents a clearer and more general solution.

6. A Three Dimensional Crack Problem Which Extends and Then Contracts

In this section we will give a solution to the problem of a plane axisymmetric crack in an infinite viscoelastic medium which is opened by a normal pressure acting on its surface. The distribution of stress and displacement for this problem is the same as that in a semi-infinite body, $z \geq 0$, when its surface, B, is subject to the boundary conditions:

$$\begin{aligned} &\sigma_{xz}(x,y,0,t) = \sigma_{yz}(x,y,o,t) = 0 \ , && (x,y) \text{ on } B, \\ &\sigma_{zz}(x,y,0,t) = - P(t) && , \quad (x,y) \text{ on } \omega(t), \\ &u_z(x,y,0,t) = 0 && , \quad (x,y) \text{ on } B\text{-}\omega(t), \end{aligned} \tag{6.1}$$

and the conditions at infinity,

$$\sigma_{ij}(\underline{x},t) \to 0 \text{ as } x_K x_K \to \infty, \text{ for all } i,j. \tag{6.2}$$

Initially the crack surface, $\omega(t)$, is monotonically increasing till some time t_m when it begins to decrease. Here we assume that the body force, F_i, and the pseudo-temperature, θ, are both zero. With the above conditions, the problem is to solve the field equations (2.1), (2.2) and (2.3) with the boundary conditions (6.1). In order to do this we first solve the viscoelastic problem with the following boundary conditions replacing equations (6.1):

$$\begin{aligned} &\sigma_{zx}(x,y,0,t) = \sigma_{zy}(x,y,0,t) = 0 \ , && (x,y) \text{ on } B, \\ &u_z(x,y,0,t) = u(x,y,t) && , \quad (x,y) \text{ on } \omega(t), \\ &u_z(x,y,0,t) = 0 && , \quad (x,y) \text{ on } B\text{-}\omega(t). \end{aligned} \tag{6.3}$$

From the results of section 4 we can see immediately that the elastic solutions (3.1), which satisfy the boundary conditions (6.3), substituted into (4.2) give the solution to the equations (2.1), (2.2), (2.3) and (6.3). The viscoelastic displacements and stresses are given as follows,

$$u_x(\underline{x},t) = \frac{(2G_1+G_2)^{-1}}{4\pi} *d\left\{3G_1*d\left(\frac{\partial U}{\partial x}\right) + z(G_1+2G_2)*d\left(\frac{\partial^2 U}{\partial x\partial z}\right)\right\}(\underline{x},t),$$

$$u_y(\underline{x},t) = \frac{(2G_1+G_2)^{-1}}{4\pi} *d\left\{3G_1*d\left(\frac{\partial U}{\partial y}\right) + z(G_1+2G_2)*d\left(\frac{\partial^2 U}{\partial y\partial z}\right)\right\}(\underline{x},t),$$

$$u_z(\underline{x},t) = \frac{1}{4\pi}\left\{-2\frac{\partial U}{\partial z} + z(2G_1+G_2)^{-1}*d(G_1+2G_2)*d\left(\frac{\partial^2 U}{\partial z^2}\right)\right\}(\underline{x},t),$$

$$\sigma_{xx}(\underline{x},t) = \frac{G_1}{4\pi}*d(2G_1+G_2)^{-1}*d\left\{3G_1*d\left(\frac{\partial^2 U}{\partial x^2}\right) + 2(G_1-G_2)*d\left(\frac{\partial^2 U}{\partial z^2}\right)\right.$$

$$\left. + z(G_1+2G_2)*d\left(\frac{\partial^3 U}{\partial z\partial x^2}\right)\right\}(\underline{x},t), \tag{6.4}$$

$$\sigma_{yy}(\underline{x},t) = \frac{G_1}{4\pi}*d(2G_1+G_2)^{-1}*d\left\{3G_1*d\left(\frac{\partial^2 U}{\partial y^2}\right) + 2(G_1-G_2)*d\left(\frac{\partial^2 U}{\partial z^2}\right)\right.$$

$$\left. + z(G_1+2G_2)*d\left(\frac{\partial^3 U}{\partial z\partial y^2}\right)\right\}(\underline{x},t),$$

$$\sigma_{zz}(\underline{x},t) = \frac{G_1}{4\pi}*d(2G_1+G_2)^{-1}*d(G_1+2G_2)*d\left\{-\frac{\partial^2 U}{\partial z^2} + z\frac{\partial^3 U}{\partial z^3}\right\}(\underline{x},t),$$

$$\sigma_{xy}(\underline{x},t) = \frac{G_1}{4\pi}*d(2G_1+G_2)^{-1}*d\left\{3G_1*d\left(\frac{\partial^2 U}{\partial x\partial y}\right) + z(G_1+2G_2)*d\left(\frac{\partial^3 U}{\partial x\partial y\partial z}\right)\right\}(\underline{x},t),$$

$$\sigma_{xz}(\underline{x},t) = \frac{G_1}{4\pi}*d(2G_1+G_2)^{-1}*d(G_1+2G_2)*d\left\{z\frac{\partial^3 U}{\partial x\partial z^2}\right\}(\underline{x},t),$$

$$\sigma_{yz}(\underline{x},t) = \frac{G_1}{4\pi}*d(2G_1+G_2)^{-1}*d(G_1+2G_2)*d\left\{z\frac{\partial^3 U}{\partial y\partial z^2}\right\}(\underline{x},t),$$

where U is given by

$$U(\underline{x},t) = \iint_{\omega(t)} \frac{u(\xi,\eta,t)d\xi d\eta}{[(x-\xi)^2+(y-\eta)^2+z^2]^{\frac{1}{2}}} \quad . \tag{6.5}$$

A result which can be used here is that

$$\begin{aligned} \frac{\partial U}{\partial z}(x,y,0,t) &= -2\pi u(x,y,t) \;, \quad (x,y) \text{ on } \omega(t), \\ &= 0 \qquad\qquad\qquad , \quad (x,y) \text{ on } B-\omega(t). \end{aligned} \tag{6.6}$$

[The solutions to these elastic problems are based on the Papkovitch-Neuber stress function solution.]

Now we compare our original problem with the second and see, that if we can find a displacement distribution u(x,y,t), (x,y) on $\omega(t)$, such that $\sigma_{zz}(x,y,0,t)$ given by (6.4) satisfies (6.1) then we will have a solution to the first problem.

From (6.4) we note that the normal stress on B is related to the normal displacement acting over $\omega(t)$ through the equation

$$\sigma_{zz}(x,y,0,t) = \left[\frac{K}{2\pi} * d\left\{\iint_{\omega} \frac{u(\xi,\eta)}{\rho^3} d\xi d\eta\right\}\right](x,y,t), \tag{6.7}$$

where K and ρ are given as

$$2K(t) = [G_1 * d(G_1+2G_2) * d(2G_1+G_2)^{-1}](t) \tag{6.8}$$

and

$$\rho^2 = (x-\xi)^2 + (y-\eta)^2.$$

For the corresponding elastic problems, (6.7) is written as

$$\sigma_{zz}^{\varepsilon}(x,y,0,t) = \frac{\kappa}{2\pi} \iint_{\omega(t)} \frac{u(\xi,\eta,t)d\xi d\eta}{\rho^3} \tag{6.9}$$

where

$$\kappa = \frac{\mu(2\mu+6k)}{(4\mu+3k)} . \tag{6.10}$$

Now we prescribe for the elastic case the normal stress acting over $\omega(t)$ to be given as

$$\sigma_{zz}^{\varepsilon}(x,y,0,t) = -P^{\varepsilon}(t), \quad (x,y) \text{ on } \omega(t), \tag{6.11}$$

where $P^{\varepsilon}(t)$ will be determined later in terms of the known function $P(t)$. We can combine equations (6.9) and (6.11), and invert the result to find the elastic normal displacement acting over $\omega(t)$. Suppose, therefore, that at any time t the elastic displacements which through (6.9) satisfy (6.11) are given by

$$u(x,y,t) = v(x,y,t), \quad (x,y) \text{ on } \omega(t). \tag{6.12}$$

At this point we define the function $t_1(t)$ through

$$t_1(t) = t \quad \text{if} \quad t \leq t_m,$$
$$\omega(t_1(t)) = \omega(t), \; t_1(t) < t_m \quad \text{if} \quad t > t_m. \tag{6.13}$$

We also extend the domain of $v(x,y,t)$ by defining that

$$v(x,y,t) = 0, \; (x,y) \text{ on } B-\omega(t). \tag{6.14}$$

Then by (6.9), (6.12) and (6.14) we find that

$$\sigma_{zz}^{\varepsilon}(x,y,0,t) = \frac{\kappa}{2\pi} \iint_{B} \frac{v(\xi,\eta,t)d\xi d\eta}{\rho^3} \tag{6.15}$$

which by (1.3) and (1.4) can be written as

$$\sigma_{zz}^{\varepsilon}(x,y,0,t) = \frac{K}{2\pi} * d\left\{\iint_B \frac{\kappa K^{-1} * dv(\xi,\eta) d\xi d\eta}{\rho^3}\right\}(x,y,t). \tag{6.16}$$

Breaking up (6.16) we get

$$\sigma_{zz}^{\varepsilon}(x,y,0,t) - \int_{0^-}^{t} K(t-\theta) d \int_{t_1(\theta)}^{\theta} K^{-1}(\theta-\tau) d\left\{\frac{\kappa}{2\pi} \iint_B \frac{v(\xi,\eta,\tau) d\xi d\eta}{\rho^3}\right\}(x,y,t)$$
$$= \frac{1}{2\pi} \int_{0^-}^{t} K(t- \ ) d\left\{\iint_B \frac{\kappa \int_{0^-}^{t_1(\theta)} K^{-1}(\theta-\tau) dv(\xi,\eta,\tau) d\xi d\eta}{\rho^3}\right\}. \tag{6.17}$$

(In (6.17) we make a notational change from equation (1.2).) Since $\omega(\theta) \subseteq \omega(t)$ whenever $\theta \leq t_1(t)$, we have, by (6.14), that

$$\int_{0^-}^{t_1(t)} K^{-1}(t-\theta) dv(x,y,\theta) = 0, \quad (x,y) \text{ on } B-\omega(t). \tag{6.18}$$

Then by using (6.13), (6.15) and (6.18) we obtain

$$\sigma_{zz}^{\varepsilon}(x,y,0,t) - \int_{t_m}^{t} K(t-\theta) d \int_{t_1(\theta)}^{\theta} K^{-1}(\theta-\tau) d\sigma_{zz}^{\varepsilon}(x,y,0,\tau)$$
$$= \frac{1}{2\pi} \int_{0^-}^{t} K(t-\theta) d\left\{\iint_{\omega(\theta)} \frac{\kappa \int_{-0}^{t_1(\theta)} K^{-1}(\theta-\tau) dv(\xi,\eta,\tau) d\xi d\eta}{\rho^3}\right\}. \tag{6.19}$$

However if (x,y) belongs to $\omega(t)$ then (x,y) also belongs to $\omega(\tau)$ whenever $t_1(\theta) \leq \tau \leq \theta$, $t_m \leq \theta \leq t$. Thus by (6.11) we see

$$\sigma_{zz}^{\varepsilon}(x,y,0,t) - \int_{t_m}^{t} K(t-\theta)d \int_{t_1(\theta)}^{\theta} K^{-1}(\theta-\tau)d\sigma_{zz}^{\varepsilon}(x,y,0,\tau)$$
$$= - P^{\varepsilon}(t) + \int_{t_m}^{t} K(t-\theta)d \int_{t_1(\theta)}^{\theta} K^{-1}(\theta-\tau)dP^{\varepsilon}(\tau), \quad (x,y) \text{ on } \omega(t). \tag{6.20}$$

Now from (6.19) and (6.20) it is seen that the normal viscoelastic displacement u defined by

$$u(x,y,t) = \kappa \int_{0^-}^{t_1(t)} K^{-1}(t-\tau)dv(x,y,\tau), \quad (x,y) \text{ on } \omega(t), \tag{6.21}$$

acting over the crack surface $\omega(t)$ generates through (6.7) the normal pressure $P(t)$ related to the function $P^{\varepsilon}(t)$ through the equation

$$P(t) = P^{\varepsilon}(t) - \int_{t_m}^{t} K(t-\theta)d \int_{t_1(\theta)}^{\theta} K^{-1}(\theta-\tau)dP^{\varepsilon}(\tau) \tag{6.22}$$

$$= \int_{0^-}^{t} K(t-\theta)d \int_{0^-}^{t_1(\theta)} K^{-1}(\theta-\tau)dP^{\varepsilon}(\tau). \tag{6.23}$$

From (6.21) and (6.23) we see that u and the viscoelastic solution are determined by a knowledge of $P^{\varepsilon}(\tau)$ (and hence $v(x,y,\tau)$) for times $\tau \leq t_1(t)$. From (6.22), it is clear that if $t \leq t_m$ then

$$P^{\varepsilon}(t) = P(t), \quad t \leq t_m. \tag{6.24}$$

In summary, if we can find a one-parameter family of elastic solutions which meets (6.1) for all times $\tau \leq t_1(t)$ then through (6.21) and (6.4) we have a solution to the viscoelastic problem represented by (2.1), (2.2), (2.3) and (6.1) at time t. The basics for this technique of solution were developed by Graham [9] for the contact problem.

Next we look at equation (6.23). This equation is a condition imposed on the viscoelastic solution when the crack surface is contracting, by the fact that we require the normal displacement to be null and continuous on the boundary of the crack area. From it, we can see that, for times $t > t_m$, if either the crack surface area, $\omega(t)$, or the normal pressure, $P(t)$, is specified, then the other is determined from this equation. [Recall that $t_1(\theta)$ is related to $\omega(\theta)$, $\theta > t_m$, by (6.13).] However if we choose to prescribe $P(t)$ then it can be seen that $a(t)$ will not begin to contract until $P(t)$ is negative. There exists a problem here since it may be possible that (6.23) can never generate negative $P(t)$. In this case, the above solution would only be valid if we prescribe $a(t)$ and determine $P(t)$.

We will now briefly consider the problem that arises when an infinite viscoelastic medium contains a crack in the plane $z = 0$, (x,y) on $\omega(t)$, which is opened up by the action of stresses at infinity. Consider the viscoelastic problem governed by the boundary conditions:

$$\begin{aligned}
&\sigma(x,y,0,t) = \sigma_{zx}(x,y,0,t) = \sigma_{zy}(x,y,0,t) = 0, \quad (x,y) \text{ on } \omega(t) \\
&\sigma_{zz}(\underline{x},t) \to P(t),\ \sigma_{xx}(\underline{x},t) \to Q(t),\ \sigma_{yy}(\underline{x},t) \to R(t) \qquad (6.25) \\
&\qquad\qquad \text{as } x_k x_k \to \infty, \\
&\sigma_{zx}(\underline{x},t) \to 0,\ \sigma_{zy}(\underline{x},t) \to 0,\quad \sigma_{xy}(\underline{x},t) \to 0 \text{ as } x_k x_k \to \infty
\end{aligned}$$

It is easy to verify that a solution to this problem is obtained by superimposing on the solution given by (6.4), (6.21) and (6.23), the following solution to the field equations (2.1), (2.2) and (2.3):

$$\sigma_{zz} = P(t),\ \sigma_{xx} = Q(t),\ \sigma_{yy} = R(t),$$
$$\sigma_{zx} = \sigma_{zy} = \sigma_{xy} = 0,$$
$$e_{zz} = \frac{1}{3}\,[P*d(G_2^{-1}+2G_1^{-1})](t) + \frac{1}{3}\,[(Q+R)*d(G_2^{-1}-G_1^{-1})](t),$$
$$e_{xx} = \frac{1}{3}[Q*d(G_2^{-1}+2G_1^{-1})](t) + \frac{1}{3}\,[(P+R)*d(G_2^{-1}-G_1^{-1})](t), \qquad (6.26)$$
$$e_{yy} = \frac{1}{3}\,[R*d(G_2^{-1}+2G_1^{-1})](t) + \frac{1}{3}\,[(P+Q)*d(G_2^{-1}-G_1^{-1})](t),$$
$$e_{zx} = e_{zy} = e_{xy} = 0,$$
$$u_z = z\,e_{zz},\ u_x = x\,e_{xx},\ u_y = y\,e_{yy}.$$

As an example of this method we will consider the problem of an infinite linear viscoelastic medium containing a plane circular crack which is opened by a normal pressure acting on its surface. In terms of circular cylindrical co-ordinates (ρ,θ,z) the boundary conditions are given as:

$$\begin{aligned}
&\sigma_{z\rho}(\rho,0,t) = \sigma_{z\theta}(\rho,0,t) = 0, && \rho \geq 0,\\
&\sigma_{zz}(\rho,0,t) = -\,P(t) &&, \quad 0 \leq \rho \leq a(t),\\
&u_z(\rho,0,t) = 0 &&, \quad \rho > a(t),
\end{aligned} \qquad (6.27)$$

and equation (6.2). Here $a(t)$, which gives the radius of the crack at time t, is initially monotonically increasing and after some time t_m decreasing. [Note that the field quantities are independent of θ.]

From Sneddon [25], we can write (for this problem) the equations corresponding to (6.4). In order to do this, we will use the notation

$$f_\nu^T(\rho,t) = \mathcal{H}_\nu[f(\rho,t)\ ;\ \rho \to \xi] = \int_0^\infty \rho f(\rho,t) J_\nu(\rho\xi)\,d\rho \qquad (6.28)$$

for the Hankel transform of order ν of the function f. The inverse is given by the inversion theorem

$$f(\rho,t) = \mathcal{H}_\nu[f_\nu^T(\xi,t)\ ;\ \xi \to \rho] = \int_0^\infty \xi f_\nu^T(\xi,t) J_\nu(\rho\xi) d\xi \tag{6.29}$$

(for reference see Sneddon [14]). If, for clarity, we make the further notation that

$$V = u_0^T(\xi,t) e^{-\xi z} \tag{6.30}$$

then the equations replacing (6.4) for this case are given as:

$$\begin{aligned}
u_\rho &= -\frac{(2G_1+G_2)^{-1}}{2} * d\Big\{3G_1 * d\mathcal{H}_1[V;\ \xi \to \rho] + (G_1+2G_2) * d\mathcal{H}_1[\xi V;\ \xi \to \rho]\Big\}, \\
u_z &= \mathcal{H}_0[V;\ \xi \to t] + \frac{z}{2}(G_1+2G_2) * d(2G_1+G_2)^{-1} * d\mathcal{H}_0[\xi V;\ \xi \to \rho], \\
\sigma_{\rho\rho} &= -K^* d\Big\{\mathcal{H}_0[(1-\xi z)\xi V;\ \xi \to \rho] + \frac{z}{\rho}\mathcal{H}_1[\xi V;\ \xi \to \rho] \\
&\qquad - \frac{3}{\rho} G_1 * d(G_1+2G_2)^{-1} * d\mathcal{H}_1[V;\ \xi \to \rho]\Big\}, \\
\sigma_{\theta\theta} &= K^* d\Big\{2(G_1-G_2) * d(G_1+2G_2)^{-1} * d\mathcal{H}_0[\xi V;\ \xi \to \rho] \\
&\qquad + \frac{z}{\rho}\mathcal{H}_1[\xi V;\ \xi \to \rho] - \frac{3}{\rho} G_1 * d(G_1+2G_2)^{-1} * d\mathcal{H}_1[V;\ \xi \to \rho]\Big\}, \\
\sigma_{zz} &= -K^* d\mathcal{H}_0[(1+\xi z)\xi V;\ \xi \to \rho], \\
\sigma_{\rho z} &= -zK^* d\mathcal{H}_1[\xi^2 V;\ \xi \to \rho], \\
u_\theta &= \sigma_{\rho\theta} = \sigma_{z\theta} = 0,
\end{aligned} \tag{6.31}$$

where we have made use of (6.8). [Note that from (6.29), (6.30), and (6.31), when $z = 0$,

$$u_z(\rho,0,t) = u(\rho,t).] \tag{6.32}$$

From the previous analysis we see that we must find a one-parameter family of elastic solutions which meet the boundary conditions (6.27) for times $\tau \leq t_1(t)$. Such a family of solutions is given by Sneddon [24].

In particular, two quantities of interest are

$$u_z^{\varepsilon}(\rho,0,t) = \frac{2(3k+4\mu)}{\pi\mu(6k+2\mu)} P(t)[a^2(t)-\rho^2]^{\frac{1}{2}} H(a(t)-\rho), \tag{6.33}$$

$$\sigma_{zz}^{\varepsilon}(\rho,0,t) = \frac{2}{\pi} P(t)\left\{\frac{a(t)}{[\rho^2-a^2(t)]^{\frac{1}{2}}} - \sin^{-1}\left(\frac{a(t)}{\rho}\right)\right\}, \quad \rho > a(t). \tag{6.34}$$

From (6.10), (6.21) and (6.34) we find that

$$u(\rho,t) = \frac{2}{\pi}\int_{0^-}^{t_1(t)} K^{-1}(t-\vartheta)d\left\{P(\theta)[a^2(\theta)-\rho^2]^{\frac{1}{2}} H(a(\theta)-\rho)\right\}. \tag{6.35}$$

With (6.35) it is possible to generate through (6.31) viscoelastic displacement and stress fields which satisfy the equations (2.1), (2.2), (2.3) and (6.27) where a(t) can increase to a maximum and then decrease. For example, the normal stress σ_{zz} is given by

$$\begin{aligned}\sigma_{zz}(\rho,z,t) &= -\left\{K*d\mathscr{H}_0[(1+\xi z)\xi u_0^T(\xi)e^{-\xi z};\ \xi \to \rho]\right\}(\rho,z,t) \\ &= -\frac{2}{\pi}\int_{0^-}^{t} K(t-\theta)d\int_{0^-}^{t_1(\theta)} K^{-1}(\theta-\tau)dP(\tau)\ \cdot \\ &\quad\cdot\left\{\mathscr{H}_0[1+\xi z)\xi f_0^T(\xi,\tau)e^{-\xi\tau};\ \xi \to \rho]\right\}(\rho,z,t)\end{aligned} \tag{6.36}$$

where

$$f(\rho,t) = [a^2(t)-\rho]^{\frac{1}{2}} H(a(t)-\rho), \tag{6.37}$$

$$f_0^T(\xi,t) = \xi^{-3}\sin(a\xi) - a\xi^{-2}\cos(a\xi). \tag{6.38}$$

Using the result,

$$\begin{aligned}&\mathscr{H}_0[\xi f_0^T(\xi,t)\ ;\ \xi \to \rho] \\ &= \frac{\pi}{2} H(a(t)-\rho) - \left\{\frac{a(t)}{[\rho^2-a^2(t)]^{\frac{1}{2}}} - \sin^{-1}\left(\frac{a(t)}{\rho}\right)\right\} H(\rho-a(t))\end{aligned} \tag{6.39}$$

we have that when $z = 0$, (6.36) reduces to

$$\sigma_{zz}(\rho,0,t) = -\int_{0^-}^{t} K(t-\theta)d\int_{0^-}^{t_1(\theta)} K^{-1}(\theta-\tau)dP(\tau)\cdot \tag{6.40}$$
$$\cdot\left\{H(a(\tau)-\rho) - \frac{2}{\pi}\left[\frac{a(\tau)}{[\rho^2-a^2(\tau)]^{\frac{1}{2}}}\right] - \sin^{-1}\left(\frac{a(\tau)}{\rho}\right)\right]H(\rho-a(\tau))\right\}$$

One quantity of interest in the analysis of crack problems is the stress intensity factor, $N(t)$, which is defined by

$$N(t) = \lim_{\rho\to a(t)}\{[\rho-a(t)]^{\frac{1}{2}}\,\sigma_{zz}(\rho,0,t)\} \tag{6.41}$$

Making use of (6.13) and the fact that a is increasing for times $t \leq t_m$ and strictly decreasing afterwards, we find that

$$N(t) = K(0)K^{-1}(t-t_1(t))P(t_1(t))\left\{\frac{2a(t)}{\pi^2}\right\}^{\frac{1}{2}}, \tag{6.42}$$

which, for times $t \leq t_m$, is the same as would be obtained from the analysis of the corresponding elastic problems. The fact that N is positive when $a(t)$ is decreasing seems unrealistic.

For times $t \leq t_m$ we have from (6.13), (6.35) and (6.40),

$$u_z(\rho,0,t) = \frac{2}{\pi}\int_{0^-}^{t} K^{-1}(t-\tau)d\left\{P(\tau)[a^2(\tau)-\rho^2]^{\frac{1}{2}}\,H(a(\tau)-\rho)\right\} \tag{6.43}$$

$$\sigma_{zz}(x,0,t) = \frac{2}{\pi}P(t)\left\{\frac{a(t)}{[\rho^2-a^2(t)]^{\frac{1}{2}}} - \sin^{-1}\left(\frac{a(t)}{\rho}\right)\right\},\ \rho > a(t). \tag{6.44}$$

These latter results agree with those given by Graham [13].

The above example, of course, can be extended to the case where the crack surface is free of normal tractions and stresses are applied at infinity. We can find a solution to this problem if we superimpose on the above solution the following solution to the field equations (2.1), (2.2), (2.3),

$$
\begin{aligned}
&\sigma_{zz} = P(t),\ \sigma_{\rho\rho} = \sigma_{\theta\theta} = Q(t),\ \sigma_{z\rho} = \sigma_{z\theta} = \sigma_{\rho\theta} = 0, \\
&e_{zz} = \frac{1}{3} P^{*}d(G_2^{-1}+2G_1^{-1}) + \frac{2}{3} Q^{*}d(G_2^{-1}-G_1^{-1}), \\
&e_{\rho\rho} = e_{\theta\theta} = \frac{1}{3} P^{*}d(G_2^{-1}-G_1^{-1}) + \frac{1}{3} Q^{*}d(2G_2^{-1}+G_1^{-1}), \\
&e_{z\rho} = e_{z\theta} = e_{\rho\theta} = 0, \\
&u_z = z\, e_{zz},\ u_\rho = \rho\, e_{\rho\rho},\ u_\theta = 0.
\end{aligned}
\tag{6.45}
$$

The possibility of extending the present case where the crack surface has only one maximum to the case where it can have "n" maxima and minima has been considered. While there exists a technique for doing this (similar to Graham's solution of the contact problem [8]), it is not evident that the normal displacements over the boundary B would always be positive or null. Thus, to solve this problem, a method must be found to show that the normal displacement on B is never negative.

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