

A SURVEY OF COMPLETE SOLUTIONS
TO LIMIT DESIGN PROBLEMS IN
THE MATHEMATICAL THEORY
OF PERFECT PLASTICITY

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ABSTRACT

The purpose of this paper is to present the basic theory of limit design and to survey according to the technique of solution all the problems in plane strain, axial symmetry and plane stress which have been solved. The paper is divided into five chapters: the first presenting the basic theory and the other four presenting complete solutions.

The second chapter is devoted to plane strain problems and is presented in three parts. The first part deals with a detailed solution to the classical punch problem. The second part is devoted to problems whose solutions are derivable from the punch solution with minor modifications and it is the modifications which are emphasized. The remainder of the chapter deals with problems which are not derivable from the punch solution.

The third chapter deals with problems in axial symmetry and after a brief presentation of the basic assumptions, solutions are given for two types of problems. The first type are statically determinate problems and the second type are kinematically determinate.

Chapter four deals with plane stress problems and the solutions are presented for problems which utilize constant state stress fields separated by lines of stress discontinuity. In this chapter a solution for a thin wedge acted on by a normal load along the base and shear

tractions along the sides is presented.

The final chapter is devoted to some miscellaneous problems in which a fully plastic state of stress is assumed and sufficient symmetry exists so as to reduce the problem to an essentially one dimensional case.

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INTRODUCTION

The mathematical theory of plasticity treats the inelastic behaviour of material bodies. In contrast to the theory of elasticity which admits unbounded stresses and recoverable strains, and in which the stresses and strains at any given time are determined completely by the current values of external actions on the body, the theory of plasticity attempts to present a more realistic model of the physical response of ductile bodies to applied actions by incorporating the phenomena of yielding or plastic flow at bounded stresses, and by allowing non-recoverable plastic strains. Furthermore, (at least in the incremental, non-viscous theory) the constitutive equations are homogeneous in stress-rates and strain-rates, and in general the stresses and strains depend upon the entire history of the response of a body to the applied actions.

Although much work has been done on the foundations of the theory ([1], [2], [3], [4] and [5]),⁽¹⁾ many difficulties are encountered in attempting to obtain solutions to boundary value problems. In addition to the usual field equations, the field variables must satisfy the following two inequalities: the yield inequality, $f(\sigma_{ij}) \leq 0$, and the inequality of positive plastic work, $\sigma_{ij} \dot{e}_{ij}^P \geq 0$, where repeated indices denote summation. Another major difficulty is the determination of surfaces which separate non-plastic regions where the strict inequality, $f(\sigma_{ij}) < 0$, is satisfied from the plastic region where

(1) Numbers appearing in square brackets (eg. [1]) refer to the numbered references in the bibliography.

equality, $f(\sigma_{ij}) = 0$, holds. The determination of such surfaces forms an integral part of the solution and cannot be ignored.

One of the most important boundary value problems of plasticity concerns the quasi-static "collapse" or "impending flow"⁽¹⁾ of bodies under constant loads: the limit load problem. It is assumed that prior to collapse, plastic flow is contained and all deformations are of elastic order of magnitude. Consequently, changes in geometry are neglected and all equations are referred to the initial configuration. Furthermore, the solution to the limit load problem is the same for an "elastic perfectly-plastic" material as for a "rigid perfectly-plastic" material. Hence, it is sufficient to consider rigid-plastic bodies.

Purported solutions to many problems were given prior to 1951 when Prager and Hodge [3] set forth the requirements for a "complete solution". Bishop [7] reiterated these requirements in 1953 and concluded that in some incomplete solutions where the field equations were satisfied only in a sub-region of a body where a velocity field could be found, there was no possible extension of the known fields into the remainder of the body without violating yield. Bishop also developed a technique for extending incomplete solutions to some plane strain problems. Since then, many previously incomplete solutions have been extended and new solutions have been found. However, even now, few solutions to important problems are known.

The purpose of this thesis is to survey the techniques of finding complete solutions and to catalogue the problems for which

(1) Terms appearing in quotation marks are defined in Chapter 1.

complete solutions have been found. This thesis is restricted to quasi-static problems in plane strain, axial symmetry, plane stress and some miscellaneous problems. Problems in generalized forces in plates and shells ([7], [8]) are ignored, and only one problem with a body force term is considered.

CHAPTER 1. Mathematical Preliminaries and Basic Theory

As stated in the introduction, the assumption of small deformation and neglect of changes in geometry lead to the usual rate of strain-velocity equations and equilibrium equations referred to the initial configuration. These equations are given in cartesian coordinates by:

$$\dot{\epsilon}_{ij} = \frac{1}{2}(\dot{u}_{i,j} + \dot{u}_{j,i}) \quad (1.1)$$

$$\sigma_{ij,j} + F_i = 0$$

$$\sigma_{ij}n_j = T_i$$

} (1.2)

where $\dot{\epsilon}_{ij}$, \dot{u}_i , σ_{ij} , F_i , n_j and T_i denote strain-rates, material velocities, stresses, body forces per unit volume, unit outward normal and tractions respectively. ",j" denotes differentiation with respect to the j^{th} coordinate.

The material dealt with is assumed to satisfy Drucker's Stability Postulate [1] which states that if an external agency A slowly applies and removes additional stresses to some initial state of stress in a material element, and produces non-zero displacements, then the net work done in application of the additional stresses, or in the cycle of addition and removal of additional stresses is non-negative. That is, useful net energy cannot be extracted from the material element and the initial stresses.

The state of the continuum is defined to be elastic perfectly-plastic if for a given stress field, σ_{ij} , there exists a function of the stresses, $f(\sigma_{ij})$, called the yield function, or surface, such that:

the material behaves elastically if $f(\sigma_{ij}) < 0$ or if $f(\sigma_{ij}) = 0$ and $\dot{f}(\sigma_{ij}) < 0$;

the material behaves plastically if $f(\sigma_{ij}) = 0$ and $\dot{f}(\sigma_{ij}) = 0$;

and conditions for which $f(\sigma_{ij}) > 0$ or $f(\sigma_{ij}) = 0$ and $\dot{f}(\sigma_{ij}) > 0$ are inadmissible.

For an isotropic material, the yield function depends only on the principal invariants of the stress tensor, J_1 , J_2 , and J_3 defined by:

$$J_1 = \sigma_{11} + \sigma_{22} + \sigma_{33}$$

$$J_2 = \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{vmatrix} + \begin{vmatrix} \sigma_{22} & \sigma_{23} \\ \sigma_{32} & \sigma_{33} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \sigma_{13} \\ \sigma_{31} & \sigma_{33} \end{vmatrix}$$

$$J_3 = \begin{vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{vmatrix}$$

The total strain-rates, \dot{e}_{ij} , are assumed to be expressible as the sum of the elastic strain rates, \dot{e}_{ij}^e , and the plastic strain-rates, \dot{e}_{ij}^p , where the elastic strain-rates are defined by Hooke's Law:

$$\dot{e}_{ij}^e = \frac{1 + \nu}{E} \dot{\sigma}_{ij} - \frac{\nu \delta_{ij}}{E} \dot{\sigma}_{kk}$$

Any state of stress σ_{ij} for which $f(\sigma_{ij}) < 0$ shall be called safe and be denoted σ_{ij}^s , while any state of stress σ_{ij} for which $f(\sigma_{ij}) \leq 0$ shall be called neutrally-safe or allowable and be denoted σ_{ij}^a . With these definitions the stability postulates imply the following consequences for an elastic, perfectly-plastic material [9]:

(a) If σ_{ij} is a state of stress on the yield surface in which non-zero plastic strain-rates, \dot{e}_{ij}^P , occur, then:

$$(\sigma_{ij} - \sigma_{ij}^S) \dot{e}_{ij}^P \geq 0 \quad \text{for all safe states } \sigma_{ij}^S, \quad (1.3a)$$

$$(\sigma_{ij} - \sigma_{ij}^a) \dot{e}_{ij}^P \geq 0 \quad \text{for all allowable states } \sigma_{ij}^a.$$

(b) If $\dot{\sigma}_{ij}$ are stress-rates corresponding to plastic strain-rates \dot{e}_{ij}^P , then:

$$\dot{\sigma}_{ij} \dot{e}_{ij}^P \geq 0. \quad (1.3b)$$

(c) In virtue of (a) and (b) the yield surface is convex and the plastic strain-rate vector is normal to the yield surface at "smooth" points and between adjacent normals at a corner of the yield surface. (The yield surface is said to be smooth at a point if it has a continuously turning tangent there.)

In consequence of condition (c), the plastic strain-rates, \dot{e}_{ij}^P , are given by:

$$\dot{e}_{ij}^P = \lambda \frac{\partial f}{\partial \sigma_{ij}} \quad \text{if } f(\sigma_{ij}) = 0 \text{ and } \dot{f}(\sigma_{ij}) = 0 \quad (1.4 a)$$

at smooth points of the yield surface where λ is a positive function or by:

$$\dot{e}_{ij}^P = \sum_{\alpha \in \Gamma'} \lambda_{\alpha} \frac{\partial f_{\alpha}}{\partial \sigma_{ij}}, \quad \Gamma' = \{\alpha \mid f_{\alpha}(\sigma_{ij}) = 0, \dot{f}_{\alpha}(\sigma_{ij}) = 0\} \quad (1.4 b)$$

at a corner of the yield surface where λ_{α} are positive functions of position.

Thus, the total strain-rates are expressible as:

$$\dot{\epsilon}_{ij} = \begin{cases} \frac{1+\nu}{E} \dot{\sigma}_{ij} - \frac{\nu}{E} \delta_{ij} \dot{\sigma}_{kk} & \text{if } f < 0 \text{ or } f=0 \text{ and } \dot{f} < 0 \\ \frac{1+\nu}{E} \dot{\sigma}_{ij} - \frac{\nu}{E} \delta_{ij} \dot{\sigma}_{kk} + \lambda \frac{\partial f}{\partial \sigma_{ij}} & \text{if } f=0 \text{ and } \dot{f}=0 \end{cases}$$

at smooth points and by similar expressions at a corner of the yield surface.

The yield functions in most common usage are the von Mises and Tresca functions. The von Mises yield function is given by:

$$f(\sigma_{ij}) = J_2' - K^2 = s_{ij}s_{ij} - K^2 \quad (1.5)$$

where J_2' is the second invariant of the stress deviator tensor whose components s_{ij} are defined by:

$$s_{ij} = \sigma_{ij} - \frac{1}{3} \delta_{ij} \sigma_{kk}$$

and K is the yield stress in simple shear. The Tresca function is given in principal stress space by:

$$f(\sigma_{ij}) = \max(|\sigma_1 - \sigma_2|, |\sigma_2 - \sigma_3|, |\sigma_3 - \sigma_1|) - 2K \quad (1.6)$$

where σ_1 , σ_2 , and σ_3 denote the principal components of the stress tensor and K is the yield stress in simple shear.

As stated previously, it follows that the loading parameter for the special class of problems considered, limit load problems, is the same for elastic perfectly-plastic materials as for rigid perfectly-plastic materials and, consequently, the material is henceforth assumed

to be rigid perfectly plastic and, thus, the plastic strain-rates and the total strain-rates coincide and will be denoted by $\dot{\epsilon}_{ij}$. The material is further assumed to be incompressible in which case

$$\dot{\epsilon}_{ii} = \frac{\partial \dot{u}_i}{\partial x_i} = 0 \quad (1.7)$$

This inequality is identically satisfied for both the von Mises and Tresca yield functions.

So far it has been implicitly assumed that all field quantities are continuously differentiable. However, discontinuities in both stress and velocity are mathematically admissible and are utilized in determining solutions. They are usually justified by limiting physical arguments and certain restrictions must be imposed on both stresses and velocities at surfaces of discontinuity to satisfy physical conservation laws. These restrictions will be dealt with as they are needed. (1)

The boundary conditions for problems in plasticity may be of three types: stress type, velocity type, or mixed type. The boundary conditions are said to be of stress type if at each point of the boundary, ∂B , of the body, the surface tractions are prescribed or if the corresponding component of velocity is prescribed to be zero. If the components of velocity are prescribed at each point on the boundary of the body, the boundary conditions are said to be of velocity type. If over some part of the boundary, ∂B_T , the surface tractions are prescribed, over another portion, ∂B_V , the velocity components are prescribed and over the remainder of the boundary, ∂B_{TV} , of the body, B , some components of the surface tractions are prescribed and the remaining components

(1) For a detailed account of discontinuities see [10], [11], or [12].

of the velocity are prescribed then the problem is said to be of mixed type.

For the limit load problem, the boundary conditions are prescribed in a special way. If the problem is stress type, the distribution of the surface tractions, T_i , is prescribed but the magnitude is determined by a monotonically increasing parameter, m , where the surface tractions are mT_i . A solution to equations (1.1) through (1.6) is sought for which m is maximized under the condition that a state of unrestricted plastic deformation or flow should exist for a constant value m^* of m , assuming changes in geometry are neglected. Such a state of impending unrestricted flow is called a collapse state and m^* is called the limit load or the limiting value of the loading parameter.

If the problem is velocity type or mixed type, a solution to the differential equations which satisfies the boundary conditions and for which a collapse state exists is sought. The limit load is determined by calculating the surface tractions on the boundary from the stresses inside and equation (1.2b). All mixed type problems considered are indenter problems and the limit load is interpreted as a critical or limiting indenter load P^* .

A state of stress, σ_{ij} , which satisfies the equilibrium equations throughout the body and the stress boundary conditions is called statically admissible. A velocity field which satisfies the velocity boundary conditions and the incompressibility condition $\dot{u}_{i,i} = 0$ is called kinematically admissible. If the stresses and velocities are related through the appropriate flow rule they are called associated.

With these definitions (and the definitions on page 4) a particularly simple definition of complete solution can be given. Corresponding to a given set of boundary conditions, a complete solution to the limit load problem consists of a neutrally-safe, statically admissible stress field and an associated kinematically admissible velocity field.

In attempting to find complete solutions it is often possible to find statically admissible stress fields or kinematically admissible velocity fields for which no associated field can be found. In such cases, the following theorems which are stated without proof provide means of finding upper and lower bounds on the limit load. The theorems apply only to bounded regions and proof of the theorems can be found in [1], [4] and [5].

Theorem 1.1 (Lower Bound Theorem)

If a safe statically admissible state of stress can be found at any stage of loading, collapse will not occur under the given loading schedule.

Theorem 1.2 (Upper Bound Theorem)

If a kinematically admissible velocity field for which the rate at which the prescribed external forces do work equals or exceeds the rate of internal energy dissipation can be found, collapse must impend or have taken place previously.

Theorem 1.3

Increasing the size of a weightless body by adding weightless material

or moving a motion free surface outward without change in the position of applied loads cannot increase the limit load.

Using Theorem 1.1 and Theorem 1.2 the following uniqueness theorem and corollary can be proven easily:

Theorem 1.4

The limit load calculated for stress type, velocity type or mixed type problems is unique.

Corollary 1.1

The limit load is the same for an elastic perfectly plastic material as for a rigid perfectly-plastic material.

For problems specified in unbounded regions, Shoemaker and Chen [13] have proven restricted uniqueness theorems and have given some examples of non-uniqueness. Let m^* or P^* be the limit load corresponding to a bounded region of plastic deformation within an infinite medium for which a complete solution can be found and let m^∞ or P^∞ be the limit load corresponding to an infinite flow region for which a complete solution with the same boundary conditions can be found. Then the following theorems and corollaries given in [13] can be proven:

Theorem 1.5

For both the stress type and mixed type limit load problems m^* and P^* are unique provided only that rigid body motion vanishes outside some bounded subregion of the infinite region.

Theorem 1.6

For the stress type limit load problem, provided that rigid body motion vanishes outside a bounded subregion, $|m^*| \geq |m^\infty|$.

Theorem 1.7

For the mixed type frictionless indenter problem $P^* \geq P^\infty$, provided that rigid body motion vanishes outside some bounded region. Here P is positive if the load acts in the direction of indenter motion.

Corollary 1.2

If the body forces F_i vanish throughout the body and the surface tractions T_i are zero on ∂B_T for the indenter problem, then the limit load in tension P^{*-} equals minus the limit load in compression P^{*+} and $P^{*-} \leq P^\infty \leq P^{*+}$ where $P^{*-} = -P^{*+}$ are unique.

A more thorough treatment of the material presented here can be found in the papers and texts listed in the bibliography. Hopefully, some insight into the problems of limit analysis may be gained and some new techniques of solving problems may be developed by studying the techniques presented in the following chapters.

CHAPTER 2. Plane Strain Problems

In this section, plane strain problems for the Tresca and von Mises criteria are considered. These problems are characterized by the existence of a system of rectangular cartesian coordinates (x, y, z) such that \dot{u}_x and \dot{u}_y are independent of z and $\dot{u}_z = 0$. Under the assumption of plane strain the components of the strain-rate tensor are independent of z and $\dot{e}_{xz} = \dot{e}_{yz} = \dot{e}_{zz} = 0$.

It is clear that for these problems the von Mises and Tresca criteria are equivalent and have the same representation:

$$(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2 \leq 4K^2. \quad (2.1)$$

Furthermore, Geiringer [14] has shown that any yield function $g(\sigma_{ij})$ which is defined for an incompressible, isotropic material can be reduced to the form $A(\sigma_1 - \sigma_2) = 0$, where σ_1 and σ_2 are principal stresses in the (x, y) plane and A is a function of the indicated variable.

From equations (1.2), (1.4), (1.5) and (2.1) with equality it follows that the differential equations of plane strain plastic flow reduce to:

$$\left. \begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} &= 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} &= 0 \\ (\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2 &= 4K^2 \\ \dot{e}_{xx} = -\dot{e}_{yy} &= \lambda(\sigma_x - \sigma_y) \end{aligned} \right\} (2.2)$$

$$\dot{\epsilon}_{xy} = 2\lambda\tau_{xy}$$

where λ is defined by equations (1.3).

Introducing the parameters ω and θ defined by:

$$\left. \begin{aligned} \omega &= \frac{\sigma_x + \sigma_y}{4K} \\ \theta &= \frac{1}{2} \text{Arctan} \frac{2\tau_{xy}}{\sigma_x - \sigma_y} + \frac{\pi}{4} \end{aligned} \right\} (2.3)$$

and eliminating λ from equations (2.2), the system reduces to:

$$\frac{\partial \omega}{\partial x} + \frac{\partial \theta}{\partial x} \cos 2\theta + \frac{\partial \theta}{\partial y} \sin 2\theta = 0 \quad (2.4a)$$

$$\frac{\partial \omega}{\partial y} + \frac{\partial \theta}{\partial x} \sin 2\theta - \frac{\partial \theta}{\partial y} \cos 2\theta = 0$$

$$\frac{\partial \dot{u}}{\partial x} + \frac{\partial \dot{u}}{\partial y} = 0 \quad (2.4b)$$

$$\frac{\frac{\partial \dot{u}}{\partial y} + \frac{\partial \dot{u}}{\partial x}}{\frac{\partial \dot{u}}{\partial x} - \frac{\partial \dot{u}}{\partial y}} = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

It can be shown that the system (2.4) is ultra-hyperbolic with characteristic directions determined by:

$$\frac{dy}{dx} = \tan \theta \quad (2.5)$$

$$\frac{dy}{dx} = -\text{ctn} \theta$$

where each characteristic is of multiplicity two.

Note that ω is a measure of the mean pressure at a point and that θ is the orientation of the characteristics measured counter-clockwise from the (-y) axis. The characteristics for which $\frac{dy}{dx} = \tan\theta$ are called first shear-lines and those for which $\frac{dy}{dx} = -\cot\theta$ are called second shear lines. Physically, they are the lines across which the maximum shear stress intensity, K , is transmitted.

Furthermore, the characteristics determined by (2.5) are the characteristics for the two hyperbolic systems (2.4a) and (2.4b). When the boundary conditions are such that the stress equations (2.4a) can be solved in the entire flow region, the problem is called statically determinate. If, in the statically determinate case, a solution to equations (2.4a) can be found and if this solution corresponds to the stress field of the complete solution, then the characteristics of the velocity field are known and the velocities are easily determined.

Introducing the parameters α and β along the first and second shear-lines respectively, the following simple integrals for the equilibrium equations are obtained:

$$\begin{aligned} \omega - \theta &= g_1(\beta) \quad \text{along a first shear-line} \\ \omega + \theta &= g_2(\alpha) \quad \text{along a second shear-line} \end{aligned} \quad (2.6)$$

where g_1 and g_2 are arbitrary functions of the respective parameters α and β .

Before proceeding to problems, the restrictions imposed on the stresses and velocities at surfaces of discontinuity will be discussed

for plane strain. Let Γ in figure 2.1 be a line of discontinuity separating the regions 1, and 2 in which the stress and velocity fields are given by $\sigma_{ij}^1, \dot{u}_i^1$ and $\sigma_{ij}^2, \dot{u}_i^2$ respectively.

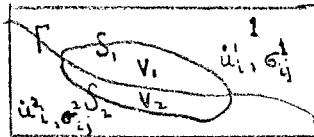


Figure 2.1 Surfaces of discontinuity.

First, the discontinuities in stress are examined. If the fields in 1 and 2 are fully plastic the components of traction acting on a line element at angle α counter-clockwise from the $(-y)$ axis are given respectively by:

$$N_{\alpha} = 2K\omega + K \sin 2(\theta - \alpha)$$

$$T_{\alpha} = K \cos 2(\theta - \alpha).$$

Since the field is assumed to be in quasi-static equilibrium, it follows that the traction must be continuous across Γ , i.e.:

$$(\sigma_{ij}^1 - \sigma_{ij}^2)n_j = 0$$

or

$$2\omega_2 + \sin 2(\theta_2 - \alpha) = 2\omega_1 + \sin 2(\theta_1 - \alpha)$$

$$\cos 2(\theta_2 - \alpha) = \cos 2(\theta_1 - \alpha)$$

} (2.7)

From the second of equations (2.7), it follows that:

$$\theta_2 - \alpha = \theta_1 - \alpha \pm n\pi$$

$$n = 1, 2, \dots$$

$$\theta_2 - \alpha = -(\theta_1 - \alpha \pm n\pi)$$

} (2.8)

The first of (2.8) gives:

$$\theta_2 = \theta_1 \pm n\pi, \quad n = 1, 2, \dots$$

which, when substituted into the first of (2.7), gives:

$$\omega_1 = \omega_2.$$

These last two conditions imply that the stress components are continuous and are ignored.

The second of equations (2.8) implies:

$$\theta_2 = -\theta_1 + 2\alpha \pm n\pi, \quad n = 0, 1, 2, \dots \quad (2.9)$$

and substitution for θ_2 from this equation into the first of equations (2.7) gives:

$$\omega_2 = \omega_1 + \sin 2(\theta_1 - \alpha). \quad (2.10)$$

Equation (2.9) implies that:

$$\alpha = \frac{\theta_2 + \theta_1}{2} \pm \frac{n\pi}{2}, \quad n = 0, 1, 2, \dots$$

and hence, the discontinuity is necessarily non-characteristic and bisects the angle between the first shear lines or their extensions which meet at the discontinuity. (2.10) gives the jump in pressure across the discontinuity.

Consider now a discontinuity in velocity across Γ . (The following proof is due to Thomas [11].) Assume that Γ is moving with normal velocity G . Then from consideration of conservation of mass it follows that:

$$\int_V \frac{\partial \rho}{\partial t} dV - \int_{S_1} \rho_1 \dot{u}_n^1 dS_1 + \int_{S_2} \rho_2 \dot{u}_n^2 dS_2 + \int_{\Gamma} (\rho_1 - \rho_2) G dS = 0$$

where ρ denotes density and ρ_1, ρ_2 , denote the density in the regions V_1 and V_2 bounded by S_1 and S_2 and separated by Γ . "n" denotes the unit normal directed from side 1 to side 2 across Γ . Taking the limit as S_1 and S_2 approach Γ , it follows that:

$$\dot{u}_n^1 = \dot{u}_n^2 \quad (2.11)$$

since the density is constant throughout the body and since the extent of Γ over which the integration is carried out is arbitrary.

Furthermore, it can be shown that discontinuities in velocity can occur only across characteristics. Consider a finite jump in velocity across a thin layer of thickness δV containing the median curve Γ , throughout which the velocity varies continuously. Assume that the velocity \dot{u}_i and its first partial derivatives are continuous on the boundaries S_1 and S_2 and outside the transition region. Let \dot{u}_i^1 and \dot{u}_i^2 be the velocity components on S_1 and S_2 respectively and let n and t be coordinates normal and tangential to Γ . Then the first partial derivatives on Γ can be represented by:

$$\frac{\partial \dot{u}_i}{\partial n} \approx \frac{\dot{u}_i^1 - \dot{u}_i^2}{\delta V} = \frac{[\dot{u}_i]}{\delta V}$$

$$\left| \frac{\partial \dot{u}_i}{\partial t} \right| < M \quad \text{for some } M \text{ independent of } \delta V. \quad \text{Taking the limit}$$

as S_1 and S_2 approach Γ and assuming that $[\dot{u}_i]$ is fixed, it follows that the strain rates on Γ are given by:

$$\dot{e}_{tt} = t_i \frac{\partial \dot{u}_i}{\partial t}, \quad (|\dot{e}_{tt}| < 2M)$$

$$\dot{\epsilon}_{nn} = -\dot{\epsilon}_{tt} = \lim_{\delta V \rightarrow 0} \frac{n_i [\dot{u}_i]}{\delta V} \quad \text{where incompressibility has been used.}$$

$$\text{and } \dot{\epsilon}_{nt} = \lim_{\delta V \rightarrow 0} \frac{1}{2} \left[n_i \frac{\partial \dot{u}_i}{\partial t} + t_i \frac{[\dot{u}_i]}{\delta V} \right]$$

where n_i and t_i are the cartesian components of n and t .

Since $\dot{\epsilon}_{nn} = -\dot{\epsilon}_{tt}$ is bounded it follows that $n_i [\dot{u}_i] = 0$ while $t_i [\dot{u}_i] \neq 0$. Consequently, $\dot{\epsilon}_{nt}$ becomes infinite. Thus Γ is a surface of maximum shear strain-rate. Furthermore, since surfaces of maximum shear strain-rate must coincide with surfaces of maximum shear stress, it follows that Γ must be a characteristic and the jump in velocity must be a jump in the tangential component since the normal component is continuous from equation 2.11.

Considering the line of discontinuity to lie between a plastic region and a rigid region, the first restriction (2.7) must still be satisfied. However, no further restriction is placed on stresses along the discontinuity. The development of conditions on velocities at discontinuities is not altered at a rigid-plastic interface and exactly the same restrictions can be derived in this case.

2.1 Classical Punch Problem for a Layer or Half-Space.

The half-space problem had been considered by several researchers, notably Prandtl [15] and Hill [2], but a complete solution was not given until 1953 when Bishop [6] extended Prandtl's solution. In 1954 Shield [16] gave another extension in the context of weightless soils with cohesion whose angles of internal friction are less than 75 degrees. This extension will not be presented here however. Hill [2] also considered the problem of indentation of a layer, but a complete solution was not given until 1967 when Salencon [17] presented his solution. In the same paper, he provided a proof that yield was not violated anywhere in Bishop's extension [6] of Prandtl's solution to the half-space problem.

Consider an infinite half-space acted upon over a central portion of width 2 units by a uniform pressure P per unit length with the remainder of the boundary traction free and take (x,y) axes as shown in figure 2.2.

The solution is assumed to be symmetric about the y -axis. The incomplete solution given by Prandtl covers region ABOCDA and its symmetric image about the y -axis. The fields in region ABOCDA are fully plastic and since the tractions are constant along the straight lines OB and BA, the stress fields in OBC and BDA are constant state fields with slip lines at $\pm 45^\circ$ to the x -axis. The stress field in region BCD is then determined from the boundary conditions on BC and BD and is found to be a centered fan field. (See [3] for a description.) Thus the fields in regions OBC, BCD and BDA are given respectively by:

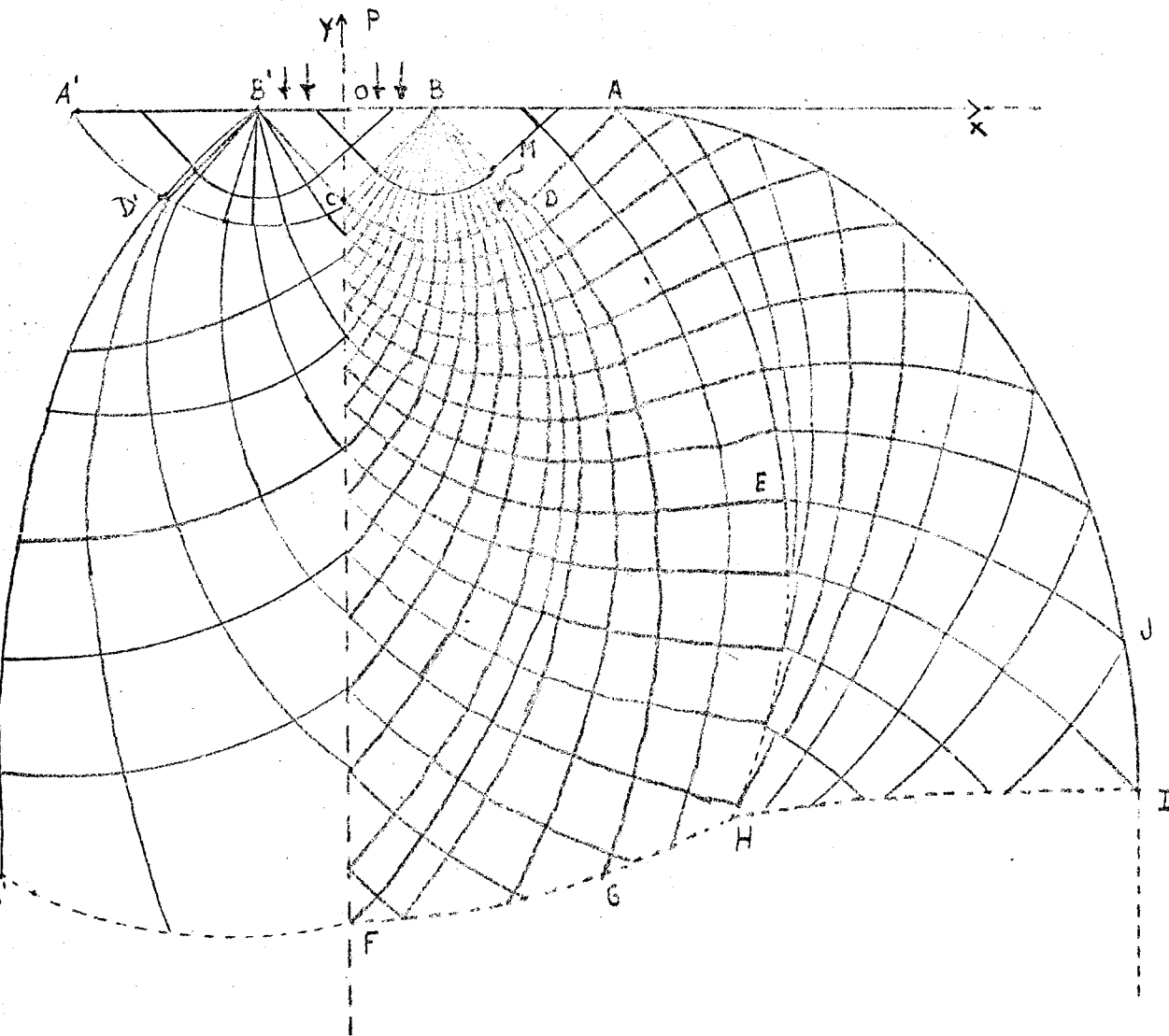


Figure 2.2 Stress field for the classical punch problem.

$$\begin{aligned}
 \sigma_x &= -\pi K & \sigma_r &= -2K\left(\frac{1}{2} + \frac{1}{4}\pi + \theta\right) & \sigma_x &= -2K \\
 \sigma_y &= -(2+\pi)K & \sigma_\theta &= -2K\left(\frac{1}{2} + \frac{1}{4}\pi + \theta\right) & \text{and } \sigma_y &= 0 \\
 \tau_{xy} &= 0 & \tau_{r\theta} &= K & \tau_{xy} &= 0
 \end{aligned}$$

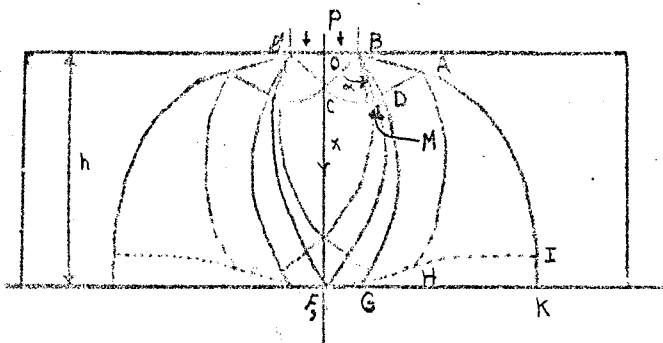
where the value of P has been determined by the fully plastic condition to be $P = (2 + \pi)K$.

Bishop's extension utilizes fully plastic fields in CDAIFH and uniaxial compression below FGHI which is below yield except at I. Since CD and CD' are characteristics along which the surface tractions are known, a type II boundary value problem is specified to determine the stress solution in region D'CDGFG' and has been described analytically by Hill [2]. Then, since DG and AD are also characteristics along which the surface tractions are known, another type II boundary value problem is specified for which Ewing [18] has developed a series method for constructing the stress fields analytically. Extending the field to the right of AE is accomplished by utilizing the characteristic AE and by constructing the stress free surface AIJ which intersects the x-axis at an angle of 0° at A. (Hill [2] has given a description of how to develop the stress free surface and Ewing [18] has given an analytic solution.) In this process the characteristics begin to run together at E and necessitate the introduction of a discontinuity EH. In order to construct the field in EJIH, the characteristic, EJ, and the stress jump across the discontinuity EH are used and the solution is again extended to the stress free surface JI. The discontinuity FGIH is constructed by requiring that the stress field below should be one of uniaxial compression. Salencon [17] showed that the magnitude of the compressive stress increases monotonically with x from the y-axis to the free surface at I.

The existence of a statically admissible stress field which satisfies the boundary conditions has now been exhibited. This field is also valid for a layer with zero normal velocity and shear traction on the lower edge replacing the conditions at infinity provided $h \geq 8.74$. The proof

that yield is nowhere violated will be left until after a discussion of the modifications necessary if $h \leq 8.74$.

Figure 2.3 Stress field for a layer under pressure.



In this case the centered fan does not extend through 90° (see figure 2.3) but through some lesser angle α , where $1.02 \approx \alpha \approx \frac{\pi}{2}$. This restricts the distance OF to at least 5.2 units. The relationship between α and the thickness of the layer is $h = A_0(2\alpha) + I_0(2\alpha)$ and the limit load, p , is

$$P = 2K \frac{\{I_0(2\alpha) + 2\alpha[I_0(2\alpha) + I_1(2\alpha)]\}}{A_0(2\alpha) + (2\alpha) + I_1(2\alpha)}$$

where I_0 and I_1 are modified Bessel functions and $A_0(x) = \int_0^x I_0(t) dt$, according to Salençon. The extension of the field is analogous to the extension when $h \geq 8.74$ and hence will not be described again.

It is obvious from the construction of the field that yield is not violated anywhere above the discontinuity FGHI. Assume that line FGHI has the form $y=f(x)$. Then, the jump conditions across the line are:

$$\begin{aligned}
 (\sigma_x^A - 0)(-y') + (\tau_{xy}^A - 0) &= 0 \\
 (\tau_{xy}^A - 0)(-y') + (\sigma_y^A - \sigma_y^B) &= 0
 \end{aligned}
 \tag{2.12}$$

where superscripts A and B denote above and below FGHI respectively.

Furthermore, the material above the line is at yield so that

$$(\sigma_x^A - \sigma_y^A)^2 + 4\tau_{xy}^A{}^2 = 4K^2
 \tag{2.13}$$

From the jump conditions (2.12) and the yield conditions, (2.13)

$$\sigma_y^B = \frac{\sigma_x^A{}^2 + \sigma_y^A{}^2}{2\sigma_x^A} + \frac{2\tau_{xy}^A}{\sigma_x^A} - 4K^2
 \tag{2.14}$$

Considering the Mohr's circle, and redefining σ_x^A , σ_y^A and τ_{xy}^A in terms of ω and θ :

$$\begin{aligned}
 \sigma_x^A &= 2\omega K + K \sin 2\theta \\
 \sigma_y^A &= 2\omega K - K \sin 2\theta \\
 \tau_{xy}^A &= K \cos 2\theta
 \end{aligned}
 \tag{2.15}$$

and substituting from equations (2.15) into equation (2.14) gives:

$$\sigma_y^B = -K \left(\frac{1 - 2\omega^2}{2\omega + \sin 2\theta} \right)$$

It is easy to show that σ_y^B decreases monotonically from F to H and from H to I by calculating ω at points on the arc CD and the change in θ along the slip lines and utilizing equations (2.6). The stress

σ_y^B decreases from a value of $\sigma_y^B = -1.157K$ at F to $\sigma_y^B = -2K$ at I.

Furthermore, at each point along FI the absolute value of σ_y^B is less than $2K$. The jump in σ_y^B across the vertical extension of the discontinuity EH below H is also readily found to be

$$\sigma_y^B = [\omega] \left\{ \omega + K \frac{(\sin 2\theta - \sin [\theta])}{1 - \sin(\theta + \theta')} \right\}^2 \frac{1 - \sin(\theta + \theta')}{(\omega' + \sin 2\theta')(\omega + \sin 2\theta)}$$

where θ and θ' give the directions of the first principal directions on the two sides of the vertical extension below H, $[\omega]$ denotes the jump in ω across the discontinuity, and yield is not violated.

Thus since yield is nowhere violated, and the equilibrium equations and the stress boundary conditions are all satisfied, the stress field described is statically admissible. Furthermore, for $\alpha = \frac{\pi}{2}$ the results are still valid and in that case, the solution is the same as Bishop's solution. Hence, yield is nowhere violated in Bishop's solution.

The determination of a kinematically admissible velocity field associated with the stress fields described above is quite simple. However, it is very different for the two cases: the layer of thickness less than or equal to 8.74 and the layer of thickness greater than or equal to 8.74 .

In the first case, with the velocity field given by Hill [2], the entire plastic bulb OFMB and its symmetric image deforms while

the remainder of the layer moves in the x-direction as a rigid body. It is easily shown that this provides an associated kinematically admissible velocity field for which the plastic work done is positive. In the second case, the velocity field developed by Prandtl [5] is used. The region OBC moves as a rigid body downward with unit velocity. In the fan region, BCD, the velocity components in polar coordinates are $u_r = 0, \dot{u}_\theta = \frac{\sqrt{2}}{2} r$ while the region ABD moves as a rigid body with speed $\frac{\sqrt{2}}{2}$ in the direction of DA. The remainder of the layer or half-space remains undeformed. This velocity field contains three lines of discontinuity: BC, CD and DA.

Calculating the plastic work, the energy dissipation E_0 in the material is obtained from the deforming region BCD and the three velocity discontinuities and is given by

$$E_0 = K[\dot{u}] = \frac{\sqrt{2}}{2} K \text{ at points of a discontinuity line, while}$$

$$\sigma_{ij} \dot{\epsilon}_{ij} = \frac{\sqrt{2}}{2} K \text{ at points in the deforming region BCD.}$$

Since the plastic work done is positive, the solution is complete and the limit load is $P = (2 + \pi) K$.

Note that if the problem had been considered as a mixed type problem where the half-space or layer was indented by a flat rigid lubricated punch with unit downward velocity replacing the uniform pressure across BB' in figures 2.2 or 2.3 and retaining

the other boundary conditions as shown in the figures, the same stress and velocity fields as obtained for the stress type problem would be valid for the mixed type problem. Hence the same value would be obtained for the limit load and the punch would have to be subjected to a pressure $P = (2 + \pi) K$ per unit width to provide a unit downward velocity.

It is easily shown that for the half-space problem the velocity field is not unique. Hill [2] has given another velocity field, and consequently a different flow region, which also provides a complete solution. Hodge, [3], has shown that any velocity field which is obtained by a combination of Hill's solution and Prandtl's solution and lies between them is also acceptable. All of these complete solutions yield the same limit load however.

In considering the layer and half-space problems some conclusions regarding uniqueness can be obtained. From Theorem 1.5 we conclude that Bishop's solution provides a unique limit load over all possible solutions for which the flow region is bounded. However, it has not been shown that an infinite flow region does not exist and thus there may exist a different limit load which corresponds to an infinite flow region. Salencon's solution does not satisfy all the conditions of Theorem 1.5 but by imposing the condition of zero stresses at infinity Theorem 1.5 applies and the limit load found is unique over all solutions corresponding to a finite flow region. Again, an infinite flow region may exist and a smaller limit load could exist.

2.2 Problems Directly Solvable by Half-Space Solution

Referring to figure (2.3), it is obvious that the same solution obtained for the layer is valid for any truncated wedge subjected to uniform pressure across BOB' for which the angle OBA is greater than or equal to $(\frac{\pi}{2} + 1.02)$ radians, since the material to the right of BA and its extension is stress free, provided that the wedge rests on a rigid smooth foundation. Furthermore, by modifying Bishop's solution by decreasing the fan angle to lie between 1.02 and $\frac{\pi}{2}$, the solution is valid for wedges of any depth h greater than 8.74. The limit load for these wedges is dependent upon the fan angle α and is given by $P = 2K(1 + \alpha)$.

These same solutions are valid for V-notched bars pulled in tension provided that $\beta = \frac{\pi}{2} - \gamma \leq \frac{\pi}{2} - 1.02$, $\frac{D}{d}$ is sufficiently large and the boundary conditions are specified as mixed type with zero shear on A'A and the bar above A'A has unit upward speed. This reverses the signs of both stresses and velocities and thus the

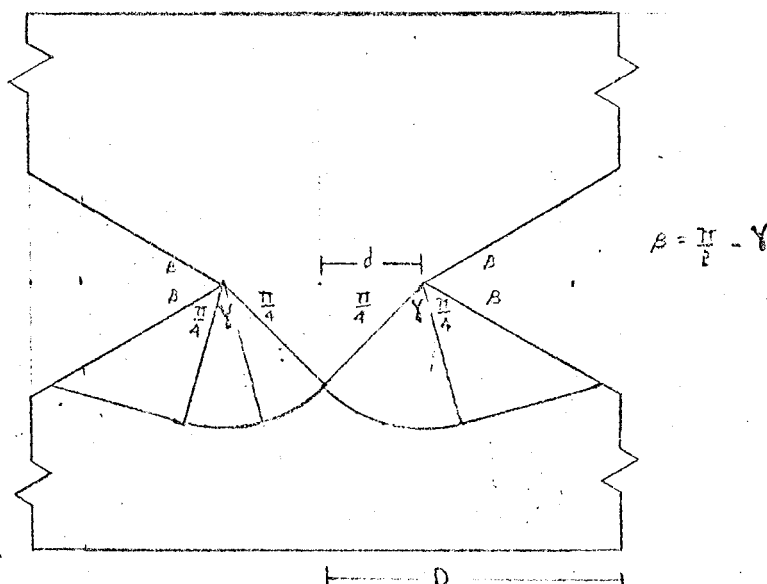
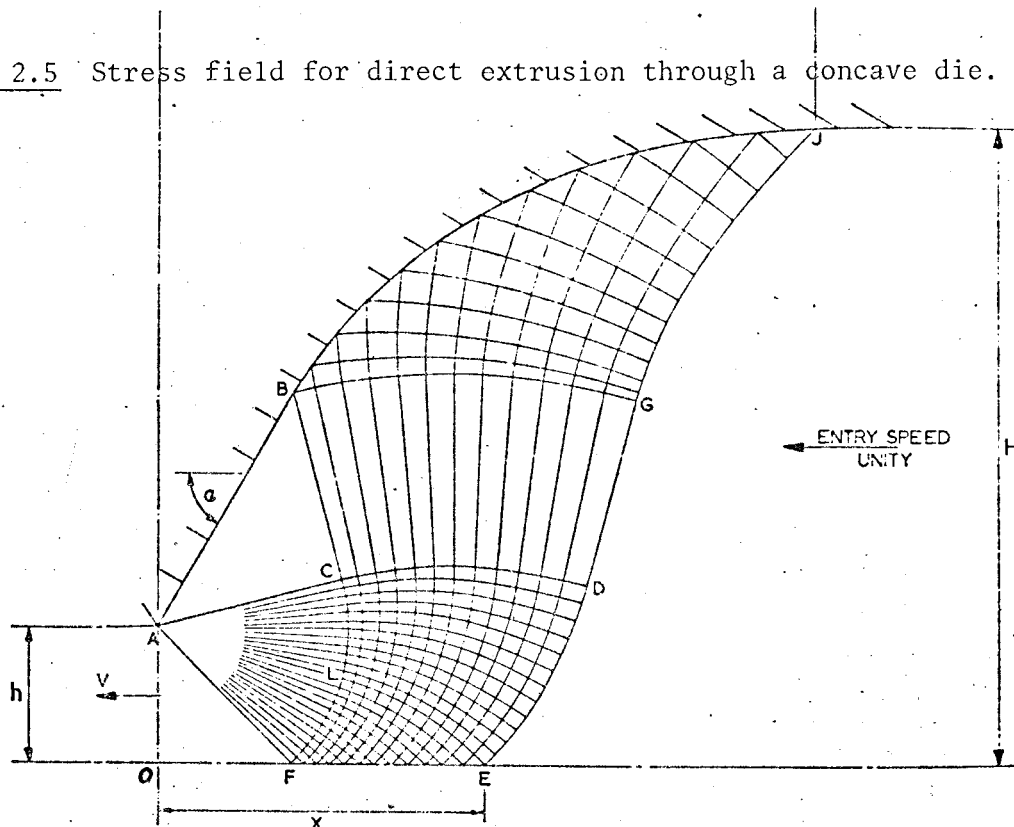


Figure 2.4 Stress field for notched bar pulled in tension.

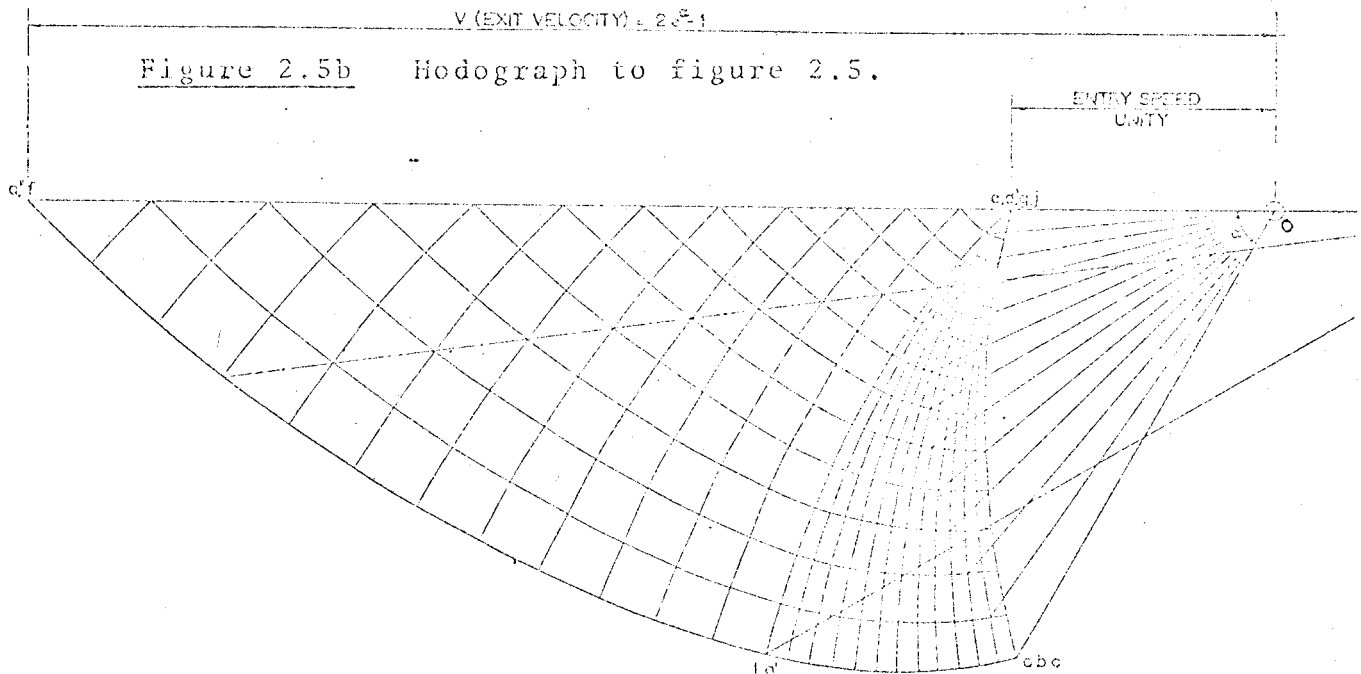
plastic work is still positive at each point. Bishop [6] has shown that $\frac{D}{d}$ must be at least 8.67. (For a further discussion of notched tension specimens see [19] and [20]).

Another problem which can be solved directly utilizing Bishop's stress field is that of direct extrusion through a smooth concave die whose shape coincides with the traction free surface of figure 2.3 as shown in figure 2.5. Assume that the material enters the die with unit speed and exits with speed v . A kinematically admissible velocity field has been given by Sowerby, Johnson and Samanta [21]. The material enters the die and moves as a rigid body to the curve EDGJ in figure 2.5. Then since this curve is a characteristic and since $\dot{u}_n = 0$ across FE and JB, the velocity field can be determined in FCBJGDE. Region ABC moves as a rigid body in the direction of BC. The field in the fan region is then determined from the boundary conditions on AC and FC. Finally the region AOF moves as a rigid body in the direction of FO as indicated.

Figure 2.5 Stress field for direct extrusion through a concave die.



To show that the plastic work is positive, the results presented by Prager [39] are utilized. Consider the hodograph of figure 2.5b. The plastic work is positive if the velocity increases in magnitude as one proceeds in the direction of the velocity. In the hodograph, velocity is measured by the distance from 0. Clearly, as one proceeds from EDGJ to FLCB in the stress plane or from e,d,g,j to abc or a''f in the hodograph plane velocity increases and hence the plastic work is positive.



2.3 Plastic Bending of Notched Bars

The solutions to bending of single and double v-notched bars and to bars with single or double circular notches have been given by Dietrich and Szczépiniski [23] who utilized a modification of Bishop's solution to the wedge problem. The single v-notch problem is considered here and the solutions for other notches can be found in [23].

Consider a v-notched bar which is subjected to surface tractions which are equivalent to a couple M acting over the ends as shown in figure 2.6.

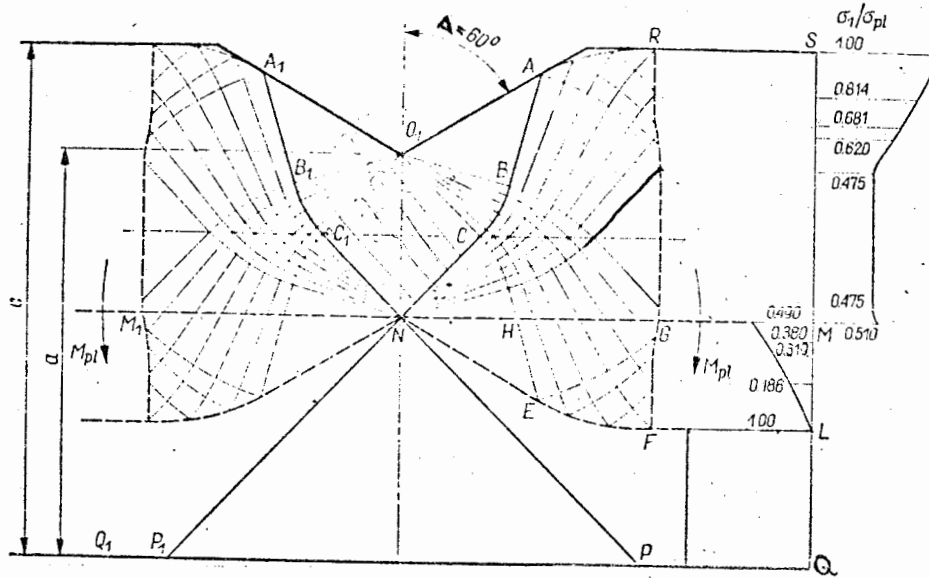


Figure 2.6a Stress field for bending of notched bars.

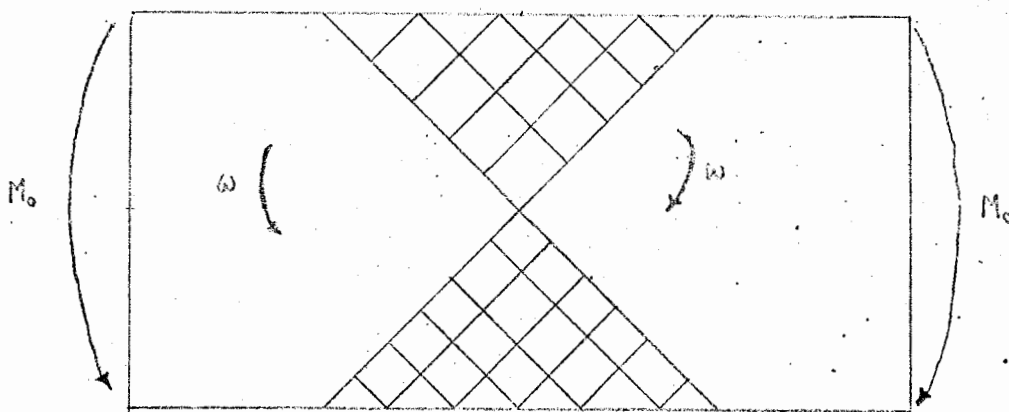


Figure 2.6b Stress field when notch angle is $\frac{\pi}{2}$.

The regions of plastic deformation are $ABNB_1A_1O_1$ and PNP_1 . The stresses in OARGFEN and its symmetric image are determined by Bishop's solution to the wedge [6]. Onto this field is superimposed the strip M_1Q_1QM of uniaxial compressive stress $\sigma_x = -2K$. This provides a statically admissible stress field for which yield is nowhere violated in RGNO since yield is nowhere violated in Bishop's solution. Furthermore, it can be shown that yield is not violated in NEFGH provided that the angle Δ , which is shown for 60° , is greater than or equal to 60° . If Δ is less than 60° , yield is violated in region NHE. An associated velocity field can be constructed by considering a rigid rotation of the two ends of the bar about the point N in the same direction as the acting moments. With this velocity field, the authors [23] have shown that the plastic work is positive. The limit load has been worked out by Green [24] and is given by $M = \frac{Ka^2}{2} \left(1 + \frac{\pi - 2\Delta}{4 + \pi - 2\Delta} \right)$ for $\Delta \geq \frac{\pi}{3}$. For the case $\Delta = \frac{\pi}{2}$, (figure 2.6b) the limit load is $M = \frac{Ka^2}{2}$ and for $\Delta = \frac{\pi}{3}$ the limit load is $M = .6075 Ka^2$.

Green [25] has also considered the bending of notched bars when $\Delta < \frac{\pi}{3}$. However, he has not shown that yield is not violated and thus the solution can not be called complete.

The remainder of this chapter is devoted to problems which have been solved utilizing fields which are not derivable by modifying the punch solution. The problems considered include acute angle wedges, bending of beams under uniform loads, and the bending of truncated wedges under uniform loads.

2.4 Acute Angle Wedges with Normal Pressure on one Edge.

Consider a symmetric wedge of angle $2\psi \leq \frac{\pi}{2}$ with uniform normal pressure, P , acting along one surface as in figure 2.7 and traction free elsewhere.

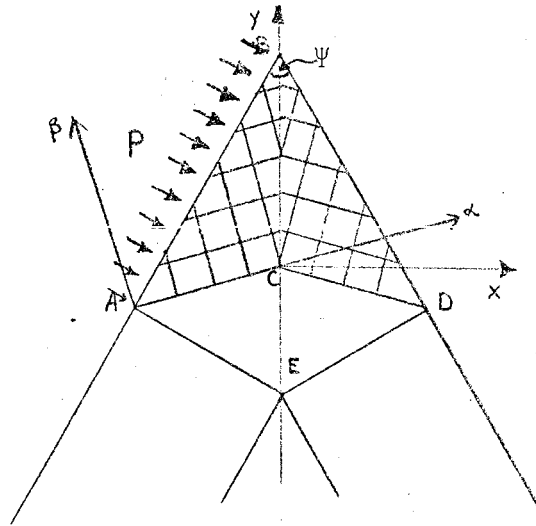


Figure 2.7 Stress field for acute angle loaded along one side.

This problem was first considered by Lee [26] who derived an incomplete solution which was later extended by Chen and Shoemaker [27]. From the normal pressure P acting on AB and the stress free surface BD two constant state plastic stress fields can be constructed which overlap along BC . This overlapping necessitates the introduction of the stress discontinuity BC and from the jump conditions 2.6 together with the boundary conditions, the stress fields can be derived and are given by

$$\sigma_x = -K (1 - \cos 2\psi)$$

$$\sigma_x = -K (1 - \cos 2\psi)$$

$$\sigma_y = -K (1 - 3\cos 2\psi) \quad \text{in } ABC \quad \text{and} \quad \sigma_y = -K (1 + \cos 2\psi) \quad \text{In } BCD$$

$$\tau_{xy} = K \sin 2\psi$$

$$\tau_{xy} = K \sin 2\psi$$

$$\text{and } P = 2K \left[1 + \sin \left(2\Psi - \frac{\pi}{2} \right) \right]$$

An extension of this field can be found by extending the constant state fields to AE and ED which are respectively perpendicular to AB and BD and taking a uniaxial stress parallel to AB and BD respectively below AE and ED.

An associated velocity field can be determined in ABC by assuming a velocity distribution with zero tangential component and a normal component along AB, \dot{u}_n , which increases from A to B. Taking coordinates α and β in the first and second shear directions and η a parameter along AB where $\alpha = \beta = \eta$ the normal velocity can be written as a function of η , $\dot{u} = f(\eta)$. Lee has shown that this function must take the form

$$\sqrt{2} f(\eta) = \phi \left(\frac{\eta}{\cos \psi} \right) \cos \psi + \phi \left(\frac{\eta - 1}{\sin \psi} \right) \sin \psi H(\eta - 1)$$

where $\phi(\xi)$ is a continuous function which gives the velocity of an elastic filament at a distance ξ from C along CB, $\phi(0) = 0$, $\phi'(\xi) \geq 0$, and $\phi'(\xi)$ is a non-decreasing function and where $H(\xi)$ is the unit step function.

The components in ABC are then given by

$$\dot{u}_\alpha = \phi \left(\frac{\beta}{\cos \psi} \right) \cos \psi$$

$$\dot{u}_\beta = -\phi \left(\frac{\alpha - 1}{\sin \psi} \right) \sin \psi$$

By symmetry, the velocity field in BCD can be adapted from that in ABC leading to the same normal velocity on AB and EB.

The condition of positive plastic work has been verified by Lee and is not repeated here. Considering uniqueness, the limit load obtained is unique over all solutions obtained for a finite flow region. There is no guarantee that an infinite flow region does not exist, however, and thus there may exist a smaller limit load corresponding to an infinite flow region.

2.5a Bending of Perfectly-Plastic Beams.

Consider a simply supported beam of depth h and length 2ℓ which is subjected to a uniform pressure P applied over a central portion of the beam of length $2C$. Anderson and Shield [28] have given a complete solution for $\frac{C}{h} > \frac{1}{2}$ with limit load

$$P = \left\{ \frac{h^2}{2C(2\ell - C)} + \left(1 + \left[\frac{h^2}{2C(2\ell - C)} \right]^{2\frac{1}{2}} - 1 \right) \right\} 2K$$

and in the same paper presented an incorrect solution for $\frac{C}{h} < \frac{1}{2}$.

For $\frac{C}{h} > \frac{1}{2}$, the stress field is shown in figure 2.8 where uniform shear is distributed over the ends of the beam of magnitude $\frac{PC}{h}$. The stress fields in regions 1 through 5 are separated by discontinuities AB, AD, ODBE and OD'B'G. Regions OCAE and OFG are assumed to be fully plastic. Then from the boundary conditions on CE, FG and EG

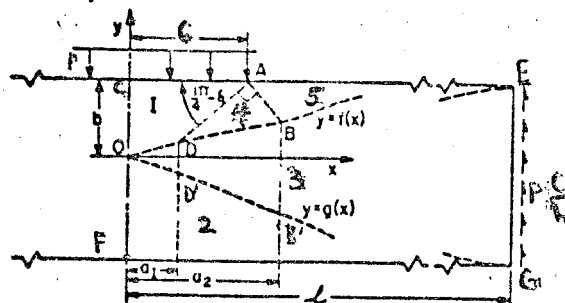


Figure 2.8 Stress field for bending of beams.

together with the yield condition and the jump conditions across OE and OG it is a simple matter to determine the stress fields and the discontinuities. No details of the analysis are presented here but the stress fields in the various regions are given respectively by

$$\sigma_x = -P - 2K \quad \sigma_y = -P \quad \tau_{xy} = 0 \quad \text{in region 1}$$

$$\sigma_x = 2K \quad \sigma_y = 0 \quad \tau_{xy} = 0 \quad \text{in region 2}$$

$$\sigma_x = 0 \quad \sigma_y = -[g'(x)]^2 \quad \tau_{xy} = -g'(x) \quad \text{in 3}$$

$$\sigma_x = \left(\frac{p^2}{4} - \frac{P-1}{2}\right)2K \quad \sigma_y = -\left(\frac{P^2+P}{4} \frac{2K}{2}\right) \quad \tau_{xy} = \frac{P(1-P^2)}{2} 2K \quad \text{in 4}$$

$$\sigma_x = -2K \quad \sigma_y = 0 \quad \tau_{xy} = 0 \quad \text{in region 5}$$

where

$$f(x) = \begin{cases} \left\{ \frac{P'}{[1+P'][2+P']} \right\}^{\frac{1}{2}} x & 0 \leq x \leq a_1 \\ \left(\frac{1}{4} h^2 - P'c(\ell - x) \right)^{\frac{1}{2}} - \frac{P'h}{2+P'} & a_2 \leq x \leq \ell \end{cases}$$

and

$$g(x) = \begin{cases} -(1+P')f(x) & 0 \leq x \leq a_1 \\ -f(x) - \frac{P'h}{2+P'} & a_2 \leq x \leq \ell \end{cases}$$

and

$$P' = \frac{P}{2K}$$

while for $a_1 \leq x \leq a_2$, $f(x)$ and $g(x)$ can be determined from:

$$xf(x) + g(x) = - (x - a_1)\tau_{xy}(a_1) - f(a_1)\sigma_y(a_1)$$

$$xf^2(x) + g^2(x) = (x - a_1)^2\sigma_y(a_1) + 2(x - a_1)A + B$$

where $A = \sigma_x(a_1)f(a_1)f'(a_1 + 0) + g(a_1)g'(a_1)$

$$B = \sigma_x(a_1)f^2(a_1) + g^2(a_1).$$

This stress field is statically admissible provided that:

$$P \leq .808K$$

$$\text{and } C \geq \frac{h}{2 + P} \text{ctn } \left(\frac{\pi}{4} - \delta \right)$$

An associated kinematically admissible velocity field can be constructed by considering the plastic yield hinge where deformation takes place, bounded by the lines $y = \pm x$ passing through 0 with the remainder of the bar rotating rigidly as in the bending of notched bars for the case $\Delta = \frac{\pi}{2}$. With this velocity field it is easily shown that the plastic work is positive.

For $\frac{C}{h} < \frac{1}{2}$ the solution given by Anderson and Shield [25] is incorrect since the plastic work is negative.

2.5b Cantilever Beams under End Shear.

Chen and Shoemaker [29] have considered the problem of cantilever beams under end shear and have found a complete solution, figure 2.9, when the length L over the minimum thickness t of the beam, satisfies

$$\frac{L}{t} \geq \frac{1}{2} (\sin 2\theta - \cos 2\theta) \tan 2\theta, \quad \frac{5\pi}{12} \leq \theta \leq \frac{\pi}{2}$$

When $\frac{L}{t}$ is less than this value, they have found close upper and lower bounds.

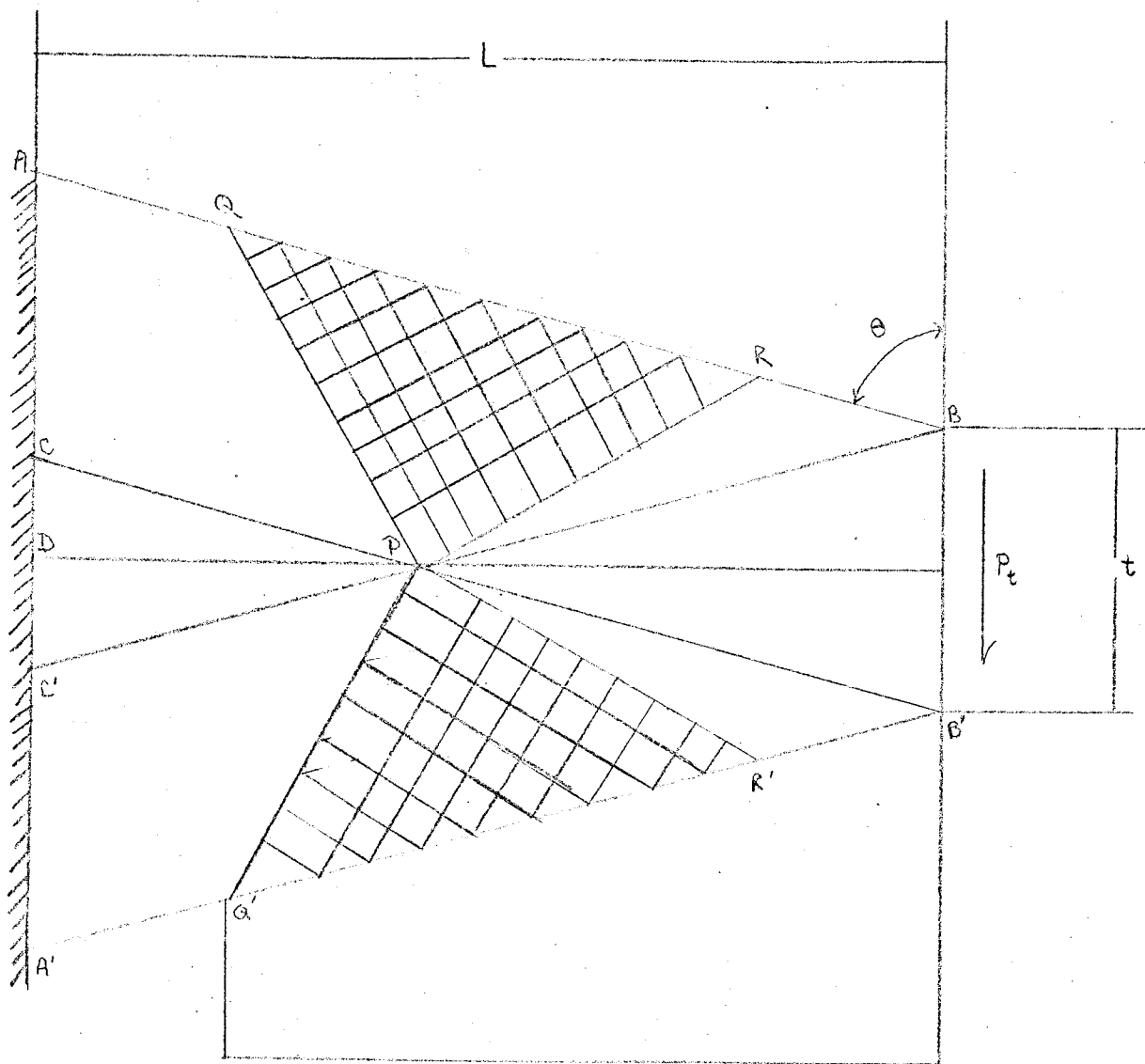


Figure 2.9 Stress field for bending of cantilever beams.

When $\frac{L}{t} \geq \frac{1}{2} (\sin 2\theta - \cos 2\theta) \tan 2\theta$ the stress solution is split into four regions, CPC', ABPC, BPB', and A'C'PB'. Region CPC' is stress free. Region ABPC is a region of uniform tensile stress parallel to AB and of magnitude $2K$. Region BPB' is in a state of uniform shear of

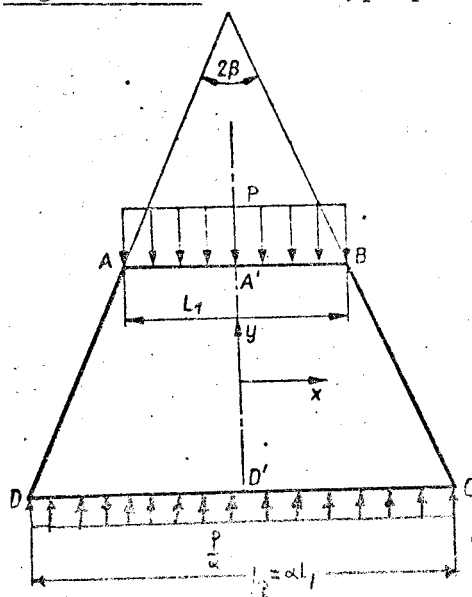
magnitude $2K \sin 2\theta$. Finally, region $A'C'PB'$ is a region of uniform compressive stress parallel to $A'B'$ of uniform compressive stress parallel to $A'B'$ of magnitude $2K$. The limit load $P_t = 2K \sin 2\theta$.

An associated kinematically admissible velocity field is constructed by considering the rigid rotation of $BRPR'B'$ about P in the direction of the applied end shear and a yield hinge of shearing deformation in region $QRPQ'R'$ and the plastic work is positive.

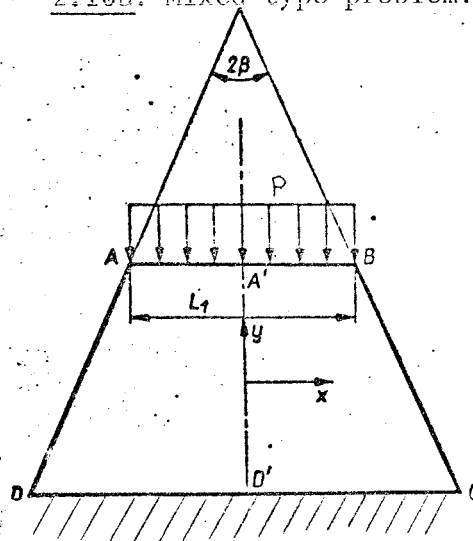
2.6 Truncated Wedges. [30]

Complete solutions are obtained for the family of symmetric wedges of arbitrary geometry ($0 \leq \beta \leq \frac{\pi}{2}$, $1 \leq \alpha = \frac{L_2}{L_1} < \infty$) for stress type and mixed type boundary conditions shown in figure 2.16.

Figure 2.10a Stress type problem.



2.10b. Mixed type problem.



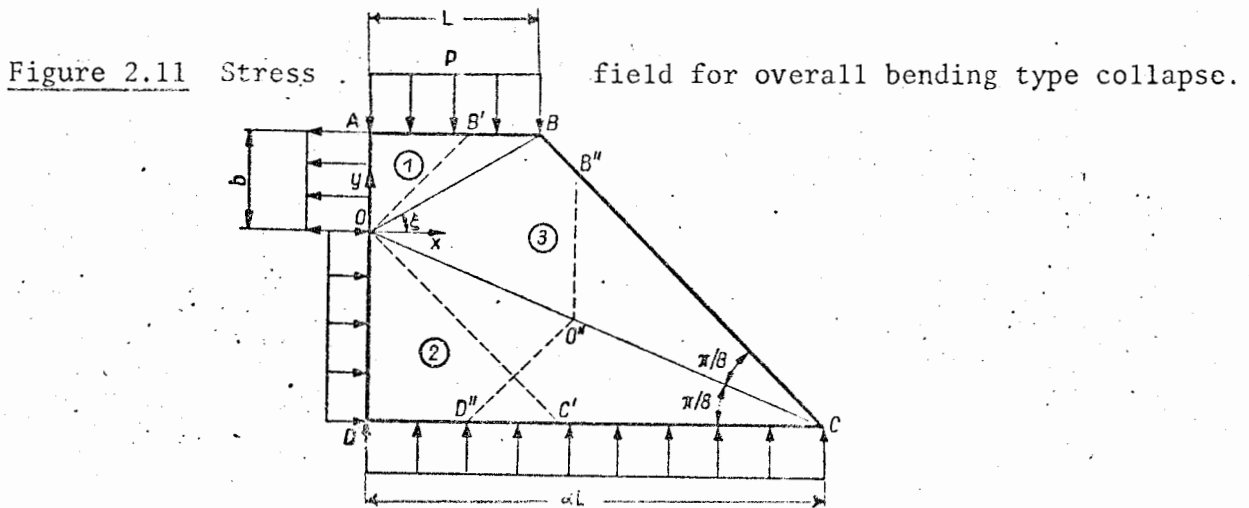
For the stress type problem, three modes of collapse are necessary while for the mixed problem, one mode is sufficient to obtain solutions

spanning the entire range of the parameters α and β .

For the stress type problem the three modes of collapse are

- (a) overall bending type collapse,
- (b) local acute wedge type collapse,
- and (c) local obtuse wedge type collapse.

(a) Overall Bending Type Collapse.



One expects collapse to be associated with rigid rotation of some segment $BB'OCC$ about some point O on AD shown in figure 2.11, and the superposition of constant shear flows in the regions $AB'O$ and ODC' . Any solution must satisfy two conditions: the cross section of maximum moment should be AD , and the net shear force on AD should be zero. An incomplete stress field can be determined in regions 1 and 2 by introducing constant state stress fields in both regions where σ_x is compressive in ABO and tensile in OCD . The location of O and the pressures on AB and CD are obtained from the conditions of overall equilibrium of the right half of the wedge. Then, from the jump conditions (2.7), the stresses in region 3 can be determined.

In what follows assume that θ is 45° . From overall equilibrium in the y-direction and from the boundary conditions prescribed on AB and CD, the stresses in region 1 and region 2 are given respectively by:

$$\left. \begin{aligned} \sigma_x &= -P - 2K, \quad \sigma_y = -P, \quad \tau_{xy} = 0 \\ \sigma_x &= \frac{-P}{\alpha} + 2K, \quad \sigma_y = -\frac{P}{\alpha}, \quad \tau_{xy} = 0 \end{aligned} \right\} (2.16)$$

From overall equilibrium in the x-direction,

$$-b(P + 2K) + \{(\alpha-1)L - b\}2K - \frac{P}{\alpha} = 0 \quad (2.17)$$

and from the moment condition, taking moments about O,

$$-b^2(P + 2K) + \{(\alpha-1)L - b\}^2 \left(2K - \frac{P}{\alpha}\right) + L^2 \alpha^2 P - L^2 P = 0. \quad (2.18)$$

Solving equations (2.17) and (2.18) for the unknowns α and b gives the position of O and the critical pressure in terms of α :

$$\left. \begin{aligned} P &= (\alpha - 1)K \\ b &= \frac{2K\alpha(\alpha - 1) - P(\alpha - 1)}{4K\alpha + P(\alpha - 1)} \end{aligned} \right\} (2.19)$$

where negative values of P have been neglected. Substituting from (2.19) into (2.16) now gives the stresses:

$$\sigma_x = -(\alpha+1)K, \quad \sigma_y = -(\alpha-1)K, \quad \tau_{xy} = 0 \quad \text{in region 1} \quad (2.20)$$

$$\sigma_x = \frac{1+\alpha}{\alpha} K, \quad \sigma_y = \frac{1-\alpha}{\alpha}, \quad \tau_{xy} = 0 \quad \text{in region 2}$$

To find the stresses in region 3 assume that the discontinuity lines OB and OC are given respectively by the functions $y = f(x)$ and $y = g(x)$ and that the stresses in region 3 have the form

$$\sigma_x = \sigma_o, \quad \sigma_y = -y \tau(x) + \sigma(x), \quad \tau_{xy} = \tau(x) \quad (2.21)$$

From the jump conditions (2.7) across OB and OC and the boundary conditions on BC, there are 6 equations to determine the five unknowns $f(x)$, $g(x)$, σ_o , $\sigma(x)$, and $\tau(x)$. However, the six equations are not independent since overall equilibrium has already been satisfied and the system reduces to five independent equations in five unknowns. The solutions to these equations are given by:

$$f(x) = -g(x) = \frac{\alpha-1}{\alpha+1} x$$

$$\tau(x) = (\alpha-1) \frac{(\alpha+1)}{\alpha} K$$

$$\sigma(x) = \frac{2(1-\alpha)}{(1+\alpha)} K$$

$$\sigma_o = \frac{1-\alpha^2}{2\alpha} K$$

(2.22)

Substituting from (2.22) into (2.21) gives the stresses in region 3:

$$\sigma_x = \sigma_y = -\tau_{xy} = \frac{1-\alpha^2}{2} K \quad (2.23)$$

Finally, substituting from (2.23) into the yield inequality (2.1) gives

$$\left| \frac{\alpha^2 - 1}{2\alpha} \right| \leq 1 \text{ but } \alpha \geq 1 \quad \text{so that}$$

yield is not violated provided that $1 \leq \alpha \leq 1 + \sqrt{2}$ (2.24).

An associated kinematically admissible velocity field for which the plastic work is positive is given by:

$$\dot{u}_x = \begin{cases} -\Omega x & y > 0 \\ x\Omega & y < 0 \end{cases} \quad \dot{u}_y = \begin{cases} \Omega y & y > 0 \\ -\Omega y & y < 0 \end{cases} \quad (2.25)$$

where $\Omega > 0$.

(b) Local Acute Wedge Collapse.

For local acute wedge type collapse the velocity field of

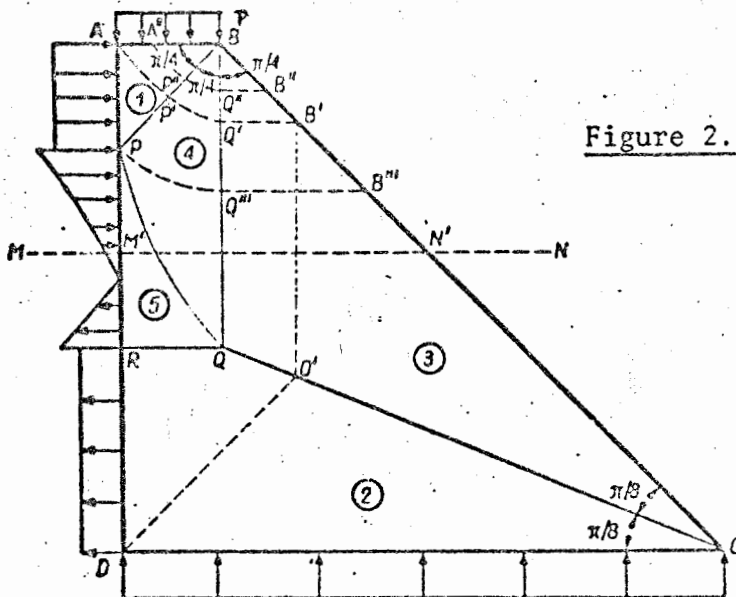


Figure 2.12 Stress field for acute wedge collapse.

Lee [24] for acute angle wedges given in section 2.4 is utilized together with the following stress fields to give a complete solution for $\tan \left(\frac{\pi}{4} + \frac{\beta}{2} \right) \leq \alpha \leq \tan^2 \left(\frac{\pi}{4} + \frac{\beta}{2} \right)$ with limit load $P = 2 \alpha (1 - \sin \beta) K$.

The stress field is calculated for $\beta = \frac{\pi}{4}$:

$$\text{Region 1 } \sigma_x = -2K \left(\sin \left(\frac{\pi}{4} - 2\delta \right) + 2\delta \sin \left(\frac{\pi}{4} + 2\delta \right) \right) \quad \sigma_y = \sigma_x - 2K \quad \tau_{xy} = 0$$

$$\text{Region 2 } \sigma_x = 2K \quad \sigma_y = -(2\sqrt{2})K \quad \tau_{xy} = 0$$

$$\text{Region 3 } \sigma_x = -K \quad \sigma_y = -K \quad \tau_{xy} = K$$

$$\begin{aligned} \text{Region 4 } \sigma_x &= -K \left\{ 1 - \sin \left(\frac{\pi}{4} - 2\delta \right) + \frac{(3\pi + 2\delta - 2\theta - \sin 2\theta)}{4} \sin \left(\frac{\pi}{4} + 2\delta \right) \right\} \\ \sigma_y &= -K \left\{ 1 + \sin \left(\frac{\pi}{4} - 2\delta \right) + \frac{(3\pi + 2\delta - 2\theta + \sin 2\theta)}{4} \sin \left(\frac{\pi}{4} + 2\delta \right) \right\} \\ \tau_{xy} &= -K \sin \left(\frac{\pi}{4} + 2\delta \right) \cos 2\theta \\ \sigma_x &= K F_1(\theta) \end{aligned}$$

$$\text{Region 5 } \sigma_y = (\sqrt{2} - 2) K$$

$$\tau_{xy} = 0$$

$$\text{where } F_1(\theta) = \frac{\sin^2 \left(\frac{\pi}{4} + 2\delta \right) \cos^2 \theta}{\sqrt{2} - 2 + H(\theta, \delta) + \sin 2\theta \sin \left(\frac{\pi}{4} + 2\delta \right)} - H(\theta, \delta) + \sin 2\theta \sin \left(\frac{\pi}{4} + 2\delta \right)$$

The lines of stress discontinuity separating the various regions

are given by:

$$\text{RQ: } y = L \sec \theta \sin \left(\frac{3\pi}{8} + \delta \right)$$

$$\text{OC: } y = -\tan \frac{\pi}{4} x$$

$$\text{PQ: } r_1(\theta) = \frac{J(\delta) - \frac{3\pi}{4} + 2\delta + \tan \left(\frac{3\pi}{4} - \delta \right)}{J(\delta) - 2\theta + \tan \theta} L \sec \theta$$

$$\text{where } J(\delta) = \frac{\sqrt{2} - 1 + \sin \left(\frac{\pi}{4} - 2\delta \right)}{\sin \left(\frac{\pi}{4} + 2\delta \right)} + \frac{3\pi + 2\delta}{4}$$

(c) Local Obtuse Wedge Type Collapse.

For local obtuse wedge type collapse the stress fields are given in figure 2.13. The velocity field is the same as that for

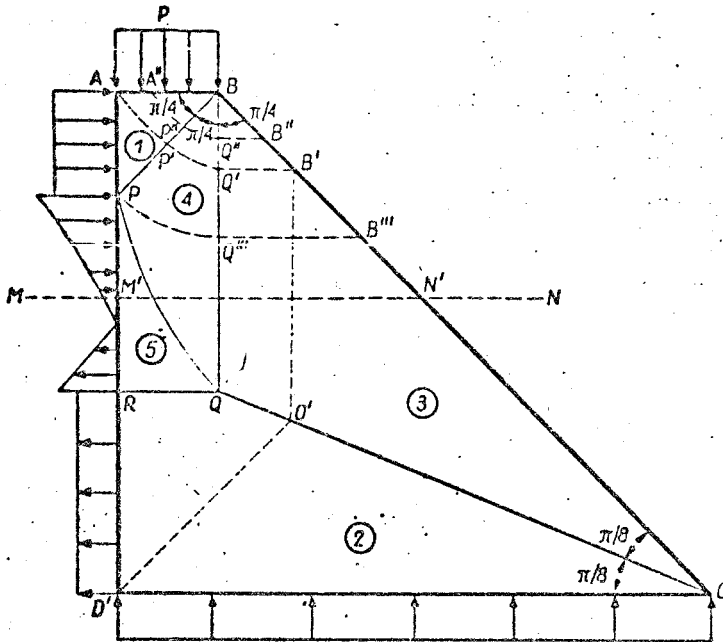


Figure 2.13 Stress field for obtuse wedge type collapse.

local obtuse wedge type collapse and can be found in [3]. The stress field is given in the respective regions for $\beta = \frac{\pi}{4}$ by:

$$\text{Region 1} \quad \sigma_x = -\frac{\pi}{2} K, \quad \sigma_y = -2K(1+\frac{\pi}{2}), \quad \tau_{xy} = 0$$

$$\text{Region 2} \quad \sigma_x = \frac{2K(1 + \pi/4)}{\alpha - 2(1 + \pi/4)}, \quad \sigma_y = \frac{-2K(1+\pi/4)}{\alpha}, \quad \tau_{xy} = 0$$

$$\text{Region 3} \quad \sigma_x = 2K, \quad \sigma_y = -K, \quad \tau_{xy} = 0$$

$$\text{Region 4} \quad \sigma_x = -K(1 + \pi - 2\theta - \sin 2\theta)$$

$$\tau_{xy} = -K \cos 2\theta$$

$$\sigma_y = -K(1 + \pi - 2\theta + \sin 2\theta)$$

Region 5 $\sigma_x = KM(\theta)$, $\sigma_y = \frac{-2K(1 + b/4)}{\alpha}$, $\tau_{xy} = 0$

where $M(\theta) = \frac{\cos^2 \theta}{\frac{2(1 + \pi/4)}{\alpha} + 1 + \pi - 2\theta + \sin 2\theta} - 1 + \pi - 2\theta - \sin 2\theta$

The lines of stress discontinuity PQ, RQ and CQ are given as follows:

PQ: $r(\theta) = \frac{(\sigma_4 + 2 + \pi/2)L \sec \theta}{\sigma_4 + 1 + \pi - 2\theta + \tan \theta}$ where $\sigma_4 = \frac{-2(1 + \pi/4)}{\alpha}$

RQ: $y = r(\pi/2)$

CQ: $y = -x \tan \eta + r(\pi/2)$

where $\eta = \tan^{-1} \left(\frac{\alpha - 2(1 + \pi/4)}{\alpha} \right)$

This solution is valid for $\frac{2(1 + \pi/4)}{2 - \sqrt{2}} \leq \alpha < \infty$ and the limit load is $P = 2(1 + \pi/4)K$.

For wedges of arbitrary angle β the complete solutions corresponding to the three different modes of collapse and the associated ranges of α are given by the following:

for overall bending collapse:

$$P = \frac{(2 - (1-\alpha)^2 \cot^2 \beta) - (4\alpha^2 + (1-\alpha) \cot^4 \beta)^{1/2}}{(1 + \csc^2 \beta)} K \quad 0 \leq \beta \leq \frac{\pi}{2}, \quad 1 \leq \alpha \leq \tan\left(\frac{\pi + \beta}{4}\right)$$

for acute wedge type collapse:

$$P = 2\alpha(1 - \sin \beta) K \quad 0 \leq \beta \leq \frac{\pi}{2}, \quad \tan\left(\frac{\pi + \beta}{4}\right) \leq \alpha \leq \frac{1 + \beta}{1 - \sin \beta}$$

and for obtuse wedge type collapse:

$$P = 2(1 + \beta) K \quad 0 \leq \beta \leq \frac{\pi}{2}, \quad \frac{1 + \beta}{1 - \sin \beta} \leq \alpha \leq \infty$$

For the mixed type boundary value problem for wedges, the obtuse wedge type mode of collapse is sufficient to cover the entire range of β , $0 \leq \beta \leq \frac{\pi}{2}$ and α , $1 \leq \alpha < \infty$ and the collapse pressure is $P = 2K\alpha(1 + \beta)$. (For justification of this statement see [30].)

CHAPTER 3. Problems in Axial Symmetry

In this section, cylindrical polar coordinates (r, θ, z) will be considered. The material body and the boundary conditions are assumed to be symmetric about the z -axis. It can be shown that if the von Mises criterion is used, the differential equations governing stress and velocity are elliptic and considerable difficulty is encountered in determining solutions. (Hill [2], Parsons [31]) In contrast, by using the Tresca criterion and associated flow rule the equations governing stress and velocity are hyperbolic and are either statically or kinematically determinate. (Shield [32], Lippmann [33])

The statically determinate case arises when the Haar and von Karman hypothesis is utilized. This states that the circumferential stress, σ_{θ} , which is a principal stress, is equal to one of the principal stresses in the meridional plane. The statically determinate case is dealt with first. The case of kinematically determinate problems is dealt with later in the chapter.

Denoting the principal stresses by σ_1, σ_2 and σ_3 where, for the statically determinate case, σ_3 is assumed to correspond to σ_{θ} and $\sigma_{\theta} = \sigma_1$, $\sigma_1 \geq \sigma_2$, the Tresca yield condition and associated flow rule simplify to:

$$\sigma_1 = \sigma_3 = \sigma_2 + 2K \quad \text{at yield}$$

$$\dot{e}_1 = \mu$$

(3.1)

$$\dot{e}_2 = -\mu - \lambda \quad \text{where } \lambda, \mu \text{ are positive functions of position}$$

$$\dot{e}_3 = \lambda$$

Here σ_θ is assumed to be equal to the maximum principal stress in the meridional plane. Similar equations result if σ_θ is the minimum principal stress.

In cylindrical polar coordinates with $\sigma_\theta = \sigma_3$, a stress distribution which is radially symmetric involves only four non-zero stresses. These are σ_r , σ_z and τ_{rz} . Calculating the principal stress components in the (r, z) plane, σ_1 and σ_2 are given by:

$$\begin{aligned}\sigma_1 &= \frac{\sigma_r + \sigma_z}{2} + \frac{\{(\sigma_r - \sigma_z)^2 + 4\tau_{rz}^2\}^{1/2}}{2} \\ \sigma_2 &= \frac{\sigma_r + \sigma_z}{2} - \frac{\{(\sigma_r - \sigma_z)^2 + 4\tau_{rz}^2\}^{1/2}}{2} \\ \sigma_3 &= \sigma_\theta\end{aligned}\quad (3.2)$$

The equilibrium equations (1.2) reduce in cylindrical polar coordinates to:

$$\begin{aligned}\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} &= 0 \\ \frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} &= 0\end{aligned}\quad (3.3)$$

The strain-rate velocity equations (1.1) become:

$$\begin{aligned}\dot{\epsilon}_r &= \frac{\partial \dot{u}_r}{\partial r} \\ \dot{\epsilon}_\theta &= \frac{\dot{u}_r}{r} \\ \dot{\epsilon}_z &= \frac{\partial \dot{u}_z}{\partial z} \\ \dot{\epsilon}_{rz} &= \frac{1}{2} \left(\frac{\partial \dot{u}_r}{\partial z} + \frac{\partial \dot{u}_z}{\partial r} \right)\end{aligned}\quad (3.4)$$

and the principal strain-rates are given by:

$$\begin{aligned} \dot{e}_1 &= \frac{1}{2}(\dot{e}_r + \dot{e}_z) + \frac{1}{2}\{(\dot{e}_r - \dot{e}_z)^2 + 4\dot{e}_{rz}^2\}^{\frac{1}{2}} \\ \dot{e}_2 &= \frac{1}{2}(\dot{e}_r + \dot{e}_z) - \frac{1}{2}\{(\dot{e}_r - \dot{e}_z)^2 + 4\dot{e}_{rz}^2\}^{\frac{1}{2}} \\ \dot{e}_3 &= \dot{e}_\theta \end{aligned} \quad (3.5)$$

Equations (3.1) and (3.3) provide two algebraic equations and two differential equations to determine the four unknown stresses. After substituting from (3.1) into (3.3), there remain two differential equations in two unknowns: The system is hyperbolic and has two orthogonal sets of characteristics. Denoting by $-P$ and K the normal and shear components of stress at a point, it follows that:

$$\sigma_r = -P - K \sin 2\phi$$

$$\sigma_z = -P + K \sin 2\phi$$

$$\tau_{rz} = K \cos 2\phi$$

$$\text{and } \sigma_\theta = -P + K$$

where ϕ is the inclination of a first shear line to the r -axis.

Substitution into the equilibrium equations referred to the characteristic directions s_1 and s_2 yields:

$$dP + 2K d\phi + K(\sin \phi + \cos \phi) \frac{ds_1}{r} = 0 \quad \text{on a 1-line.}$$

$$\text{and } dP - 2K d\phi - K(\sin \phi + \cos \phi) \frac{ds_2}{r} = 0 \quad \text{on a 2-line.}$$

The velocity components are determined by the incompressibility condition and isotropy condition given by:

$$\begin{aligned} \frac{\partial \dot{u}_r}{\partial r} + \frac{\dot{u}_r}{r} + \frac{\partial \dot{u}_z}{\partial z} &= 0 \\ \frac{\partial \dot{u}_r}{\partial z} + \frac{\partial \dot{u}_z}{\partial r} &= -\text{ctn}2\phi \\ \frac{\partial \dot{u}_r}{\partial z} - \frac{\partial \dot{u}_z}{\partial r} &= 0 \end{aligned} \quad (3.6)$$

The restrictions $\dot{e}_2 \leq 0$, $\dot{e}_3 \geq 0$ require that:

$$\dot{u}_r \geq 0, \quad \left(\frac{\partial \dot{u}_r}{\partial r} - \frac{\partial \dot{u}_z}{\partial z} \right)^2 + \left(\frac{\partial \dot{u}_r}{\partial z} + \frac{\partial \dot{u}_z}{\partial r} \right)^2 \geq \frac{\dot{u}_z^2}{r^2} \quad (3.7)$$

The system of equations (3.6) is also hyperbolic and the characteristics coincide with the stress characteristics. Equations (3.6) can be written:

$$\begin{aligned} \cos \phi \, d\dot{u}_r + \sin \phi \, d\dot{u}_z + \dot{u}_r \frac{ds_1}{2r} &= 0 \quad \text{on a 1-line} \\ \sin \phi \, d\dot{u}_r - \cos \phi \, d\dot{u}_z - \dot{u}_r \frac{ds_2}{2r} &= 0 \quad \text{on a 2-line.} \end{aligned} \quad (3.8)$$

The maximum shearing strain-rate, Γ , in the (r,z) plane is given by:

$$\Gamma = \left(-\frac{\partial \dot{u}_r}{\partial s_1} + \frac{\partial \dot{u}_z}{\partial s_2} \right) \sin \phi + \left(\frac{\partial \dot{u}_z}{\partial s_1} + \frac{\partial \dot{u}_r}{\partial s_2} \right) \cos \phi$$

and the condition (3.7) can be replaced by:

$$\Gamma \geq \frac{\dot{u}_z}{r} \geq 0 \quad (3.9)$$

So far all field quantities have been assumed to be continuous.

However, discontinuous states are possible and are utilized in the circular punch problem. The restrictions on stress and velocity discontinuities are determined just as in plane strain and will not be repeated here.

Considering now the case of kinematically determinate problems, the governing equations are given by (Lippmann [33]):

$$\max_i \sigma_i - \min_i \sigma_i = 2K \quad (3.1)'$$

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} = 0 \quad (3.3)'$$

$$\dot{e}_r = \frac{\partial \dot{u}_r}{\partial r}$$

$$\dot{e}_\theta = \frac{\dot{u}_r}{r}$$

$$\dot{e}_z = \frac{\partial \dot{u}_z}{\partial z} \quad (3.4)'$$

$$\dot{e}_{rz} = \frac{1}{2} \left(\frac{\partial \dot{u}_r}{\partial z} + \frac{\partial \dot{u}_z}{\partial r} \right)$$

The transformation equations from (r, θ, z) coordinates to the principal coordinates where e_i denotes a unit vector in the i -direction are:

$$e_1 = \cos \psi e_r - \sin \psi e_z$$

$$e_2 = \sin \psi e_r + \cos \psi e_z \quad (3.10)$$

$$e_3 = e_\theta$$

If the three principal stresses are pairwise different and $\sigma_1 > \sigma_2 > \sigma_3$ or $\sigma_1 < \sigma_2 < \sigma_3$, then the principal strain-rate corresponding to σ_2 is identically zero. This, together with (3.10) and (3.4)' gives the following system of equations for the velocity:

$$\frac{\partial \dot{u}_r}{\partial r} = -\cos^2 \psi \frac{\dot{u}_r}{r}$$

$$\frac{\partial \dot{u}_z}{\partial z} = -\sin^2 \psi \frac{\dot{u}_r}{r}$$

$$\frac{\partial \dot{u}_r}{\partial z} + \frac{\partial \dot{u}_z}{\partial r} = \sin 2\psi \frac{\dot{u}_r}{r}$$

(3.11)

Since σ_2 is the intermediate stress, the yield condition (3.3)' reduces to either:

$$\sigma_3 > \sigma_2 > \sigma_1, \sigma_3 - \sigma_1 = 2K \quad \text{if } \dot{u}_r > 0$$

$$\text{or } \sigma_1 < \sigma_2 < \sigma_3, \sigma_1 - \sigma_3 = 2K \quad \text{if } \dot{u}_r < 0.$$

Introducing coordinates ξ and η in the principal directions corresponding to σ_1 and σ_2 leads to the characteristic system:

$$\frac{\partial \dot{u}_r}{\partial \xi} + q \cot \psi \frac{\partial r}{\partial \xi} = 0$$

$$\dot{u}_r \frac{\partial \psi}{\partial \xi} + r \frac{\partial q}{\partial \xi} = 0$$

$$\frac{\partial \dot{u}_r}{\partial \eta} + \left(\frac{\dot{u}_r}{r} - q \tan \psi \right) \frac{\partial r}{\partial \eta} = 0$$

$$\frac{\partial \dot{u}_r}{\partial \eta} - r \frac{\partial q}{\partial \eta} - \left(\frac{\dot{u}_r}{r} \tan \psi + q \right) \frac{\partial r}{\partial \eta} = 0$$

(3.12)

$$\text{where } q = \frac{1}{2} \left(\frac{\partial \dot{u}_z}{\partial r} - \frac{\partial \dot{u}_r}{\partial z} \right)$$

Note that there are only five unknowns, \dot{u}_r , \dot{u}_z , r , q , and ψ , in this system of equations. Equations (3.11) provide three equations to

determine three unknowns, \dot{u}_r , \dot{u}_z , and ψ , in terms of the dependent variables r and z . Redefining r and z in terms of ξ and η introduces two equations and the resulting system of five equations in five unknowns reduces to (3.12) with q replacing z as the fifth unknown.

Lippmann [33] has investigated several special cases of equations (3.12) in order to find simple prototypes of principal line fields by assuming each of the following conditions in turn:

$$\frac{\partial \psi}{\partial \xi} = 0, \quad \frac{\partial \psi}{\partial \eta} = 0, \quad \frac{\partial \dot{u}_r}{\partial \xi} = 0, \quad \frac{\partial q}{\partial \xi} = 0, \quad \text{and} \quad \frac{\partial q}{\partial \eta} = 0$$

and has found that only the first one seems to give a reasonable solution.

In the first case, $\frac{\partial \psi}{\partial \xi} = 0$, the solution is a "fan" field shown in figure 3.1 where $\psi = \eta$, $r_0 = 0$. Then, r , q , \dot{u}_r , and \dot{u}_z are given by:

$$r = \xi \sin \eta$$

$$q = \frac{B}{\sin \eta}$$

$$\dot{u}_r = \frac{(C - B\xi \cos \eta)}{\sin \eta}$$

$$\dot{u}_z = D + B\xi + C \ln(\tanh \frac{\eta}{2})$$

(3.13)

where B , C , and D are constants of integration.

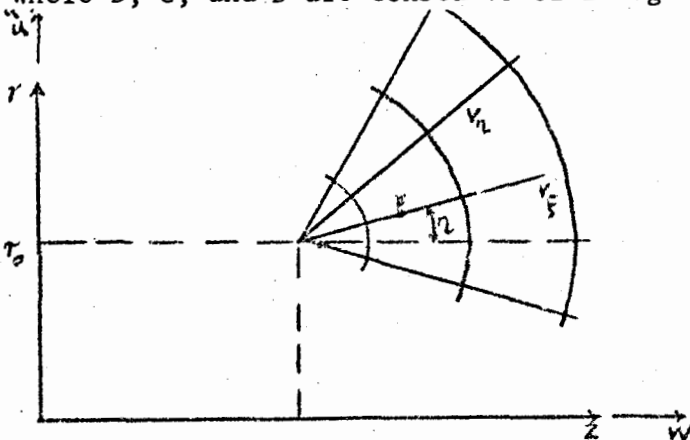


Figure 3.1 Fan field solution in axial symmetry.

Taking B and D equal to zero in (3.13), the material velocities in the directions ξ and η then simplify to:

$$\left. \begin{aligned} \dot{u}_\xi &= \dot{u}_r \sin \eta + \dot{u}_z \cos \eta = 0 \\ \dot{u}_\eta &= \dot{u}_r \cos \eta - \dot{u}_z \sin \eta = \frac{-B}{\sin \eta} \end{aligned} \right\} (3.14)$$

This characterizes a circular flow in the (r,z) plane and will be utilized later.

Considering now the problems of axial symmetry, the solutions will be given for the statically determinate problems of tension of a circular bar, indentation of a half-space by a circular punch. Only one solution to a kinematically determinate problem has been developed; the problem of tube nosing.

3.1 Tension of a Cylindrical Bar [32]

Consider a circular cylinder stressed by uniaxial compression to the yielding point. Then the stress throughout the cylinder is $\sigma_z = -2K$ with all other stresses zero. The slip lines are straight lines inclined at 45° to the axes. The boundary conditions may be specified as stress type with uniform tension over the ends or as mixed type with \dot{u}_n given.

The velocity equations (3.6) simplify to:

$$\frac{\partial \dot{u}_r}{\partial r} + \frac{\dot{u}_r}{r} + \frac{\partial \dot{u}_z}{\partial z} = 0$$

$$\frac{\partial \dot{u}_r}{\partial z} + \frac{\partial \dot{u}_z}{\partial r} = 0$$

A simple solution to these equations is given by:

$$\dot{u}_r = \frac{1}{2} \frac{r}{h}$$

$$\dot{u}_z = 1 - \frac{z}{h}$$

where h is an arbitrary constant.

In order to satisfy the condition (3.7) it is sufficient to require that h be positive. Thus a complete solution to the problem of compression of a circular cylinder has been found.

Other velocity fields are discussed by Shield [32] and will not be given here. They involve limiting deformation to small areas of the cylinder and can be used to solve the problem of necking of circular cylinders in tension.

3.2 Indentation of a Half-Space by a Circular Punch

As in the case of plane strain, the problem to be considered can be stated as a stress type problem or as a mixed type problem. It will be stated here as a mixed type problem.

Assume that a flat lubricated circular punch indents the surface of a half-space with the origin of a cylindrical polar coordinate system situated on the surface of the half-space at the center of the punch. (See figure 3.2) Then the boundary conditions are $\dot{u}_y = -1$ and the shear tractions are zero under the punch and the remainder of the boundary is traction free.

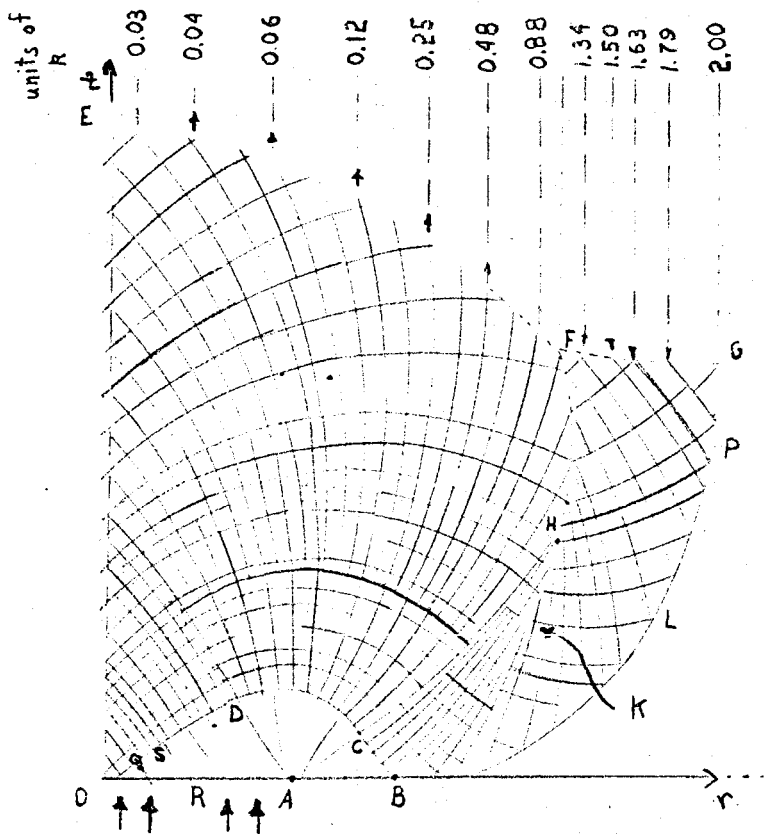


Figure 3.2 Stress field under a smooth punch in axial symmetry

The entire region OABCD is assumed to be fully plastic. Then the field in region ABC is generated by the stress free surface AB. The normal stress P and the first shear direction are determined by the equations:

$$\chi = 2\phi - 2\psi \quad \text{where } \psi = \tan^{-1} \frac{z}{r}, \quad \frac{\pi}{2} \leq \psi \leq \frac{\pi}{2}$$

$$\chi' \sin \chi + 1 + 3\sin \chi + \cos \chi \tan \psi = 0$$

$$\frac{P}{K} = A + \sin \chi - \ln(\cos \chi) + \int (3\cos \chi - \sin \chi \tan \psi) d\psi$$

where A is an arbitrary constant of integration.

Shield [32] has given a solution to these equations in his paper and the lines AC and BC are first and second shear lines respectively. The field in OAD is determined by the condition of zero shear on OA. Then the field in ACD is determined as the solution of a type I boundary value

problem with the stresses prescribed on AC and AD and is given as a fan field of angle $\frac{\pi}{2}$. The extension into the rest of the region OABGFE is determined in the same way as was Bishop's extension in the classical punch problem. The description of how this extension is developed is not repeated here. Shield has determined the distribution of normal pressure acting on OA and has found that the average value of the pressure at collapse is:

$$P = 5.69 K$$

and the ratio of OA to AB is 1.58 units.

Consider now the velocity field. The field of flow is assumed to be confined to OABCD. An associated velocity field has been given by Shield by assuming that the velocity of the punch is unity. The normal velocity across ODCB is zero and a solution to equations (3.6) on ODCB yield:

$$\dot{u}_r \cos \phi + \dot{u}_z \sin \phi = 0$$

$$-\dot{u}_r \sin \phi + \dot{u}_z \cos \phi = \frac{A}{r^2}$$

where A is arbitrary.

However, since this implies that \dot{u}_r and \dot{u}_z are infinite at the origin for $A \neq 0$, A is taken to be zero. The velocity field near the origin, in ORS, can be determined from the requirement that the characteristics are at 45° near the origin, the normal velocity on OA is one and the velocity is identically zero across OS. The solution at a point Q is given by:

$$\dot{u}_r = \frac{2 \cos^{\frac{1}{2}} 2\psi}{\pi \cos \chi}$$

$$\dot{u}_z = \frac{2}{\pi} \tan^{-1} \left(\frac{\cos^{\frac{1}{2}} 2\psi}{\sin \psi} \right)$$

where ψ = angle between the
r-axis and the line through OQ.

The velocity is now known on SD and SR and there is a type one problem defined for equations (3.8). Solving this problem then yields another type one problem for the determination of the velocities in SDAR for which the solution is easily obtained. The boundary condition on BCD and the known velocity on AD then defines the solution to another type one problem in the region ABCD. Shield has given numerical solutions for the velocity fields and has shown that conditions (3.7) are satisfied at a sufficient number of points. The velocity field is an associated kinematically admissible field and hence the solution is complete and the limit load is an average value of 5.69K.

Eason and Shield [34] have considered the indentation of a half-space by a rough punch and have derived a complete solution with a limit load of average pressure $P = 6.05K$. This solution is valid provided that the coefficient of friction between the punch and the half-space is greater than .159. The technique used in the solution is similar to the technique used in the smooth punch problem. However, the boundary conditions on OA are changed. The solution is not presented here but the slip line field is shown in figure 3.3 for the left half of the half-space. The solution can be found in Eason and Shield [34].

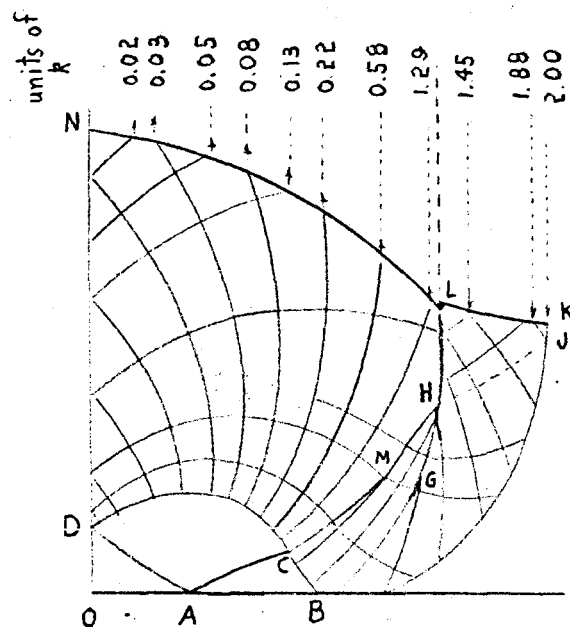


Figure 3.3 Stress field under a rough punch in axial symmetry.

Considering uniqueness of the solution, the same conclusion applies here as for the punch problem in plane strain. There is no guarantee that an infinite flow region cannot exist. Hence the solution can only be considered as giving a unique value for the limit load over all solutions with finite flow field.

As in the punch problem in plane strain, by modifying the fields, it should be possible to obtain solutions to other problems. Certainly, by reducing the fan angle, the solution to an axially symmetric wedge problem can be found, and the solution to the problem of uniformly notched circular bars can be found. Since the basic fields for the plane strain punch problem and the axially symmetric punch problem are so similar, it is expected that wire drawing problems can also be solved by utilizing a similar velocity field to that used in plane strain extrusion although no such solution has been presented.

3.3 Tube Nosing [33]

Consider a cylindrical tube of outer radius R and inner radius $R-S$ acted upon over the end by pressure P which forces the tube into a smooth spherical die as shown in figure 3.4. The interior surface of the tube is also acted upon by uniform pressure P_s below the surface of the die. The solution, and hence the boundary conditions are determined by an inverse technique

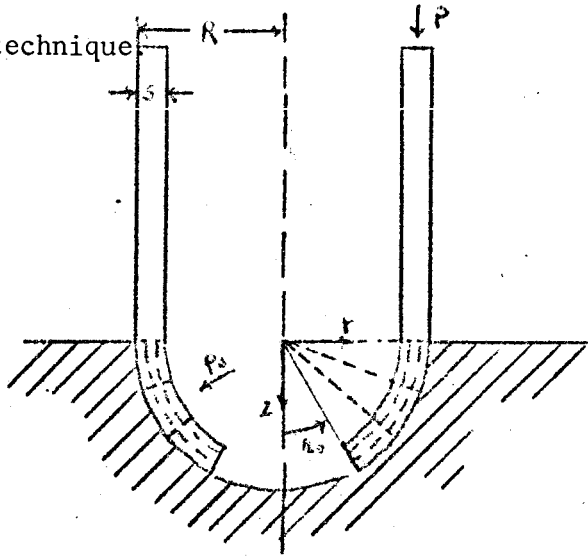


Figure 3.4 Stress field for tube nosing.

The velocity field is given by equations (3.14). Taking $\sigma_1 = \sigma_\eta$ and $\sigma_2 = \sigma_\xi$ and $\sigma_3 = \sigma_\theta$ where σ_ξ is the intermediate stress, the Yield condition (3.1)' becomes:

$$\sigma_\eta - \sigma_\theta = 2K.$$

The stress solution is given by:

$$\sigma_\eta = 2K \ln \left(\frac{\sin \eta_0}{\sin \eta} \right)$$

$$\sigma_\xi = -2K \left\{ \left(\frac{R-S}{\xi} \right)^2 P_s + \left[1 - \left(\frac{R-S}{\xi} \right)^2 \right] \left[\frac{1}{2} + \ln \left(\frac{\sin \eta_0}{\sin \eta} \right) \right] \right\}$$

$$\sigma_{\theta} = 2K \left(\ln \left(\frac{\sin \eta_0}{\sin \eta} \right) + 1 \right).$$

The limit load for this problem is:

$$P = -2K \ln (\sin \eta_0).$$

For ξ such that $R-S \leq \xi \leq R$, σ_{ξ} is indeed the intermediate stress provided that P_S is bounded by the following inequalities for equal thickness nosing:

$$\ln \left(\frac{\sin \eta}{\sin \eta_0} \right) < \frac{P_S}{2K} < \left(1 + \ln \left(\frac{\sin \eta}{\sin \eta_0} \right) \right). \quad (3.15)$$

An extension of this field for $z > 0$ is given by considering uniaxial compression $\sigma_z = -2K \ln (\sin \eta_0)$ which is below yield. Thus, a complete solution has been given for tube nosing with limit load $P = -2K \ln (\sin \eta_0)$ and pressure acting upon the inner surface of the deformed nose P_S which is bounded by equation (3.15) to guarantee that σ_{ξ} is indeed the intermediate stress.

CHAPTER 4. PLANE STRESS PROBLEMS

In this chapter only the Tresca criterion is considered. The z-axis is taken perpendicular to the plane of the material body. The generalized plane stress problem is characterized by taking $\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0$ while in the plane stresses are averaged through the thickness. Thus, σ_{xx} , σ_{xy} , and σ_{yy} are independent of z. The problems considered involve thin sheets and, hence the assumption of generalized plane stress is justified. (Hodge[35])

In all problems considered, the boundary conditions and geometry are such that all stress fields are constant state fields for which equilibrium is automatically satisfied. Utilizing superscripts to denote the stresses on opposite sides of the discontinuity, the stress intensities are determined by the yield condition and the stress jump conditions across discontinuities given by:

$$(\sigma_{ij}^1 - \sigma_{ij}^2)n_j = 0 \quad (4.1)$$

Once a stress distribution is determined, the velocity distribution is determined by assuming that there is sliding out of the plane along discontinuities within plastic regions with rigid body motion elsewhere. Finally, the plastic power dissipation is shown to be non-negative throughout the flow region.

4.1 Square Plate with a Slit Under Uniaxial Tension [35]

Consider a thin square plate with sides of length 2a and thickness

h, with a slit centered in the square parallel to the sides pulled in tension. The slit is of length $2ab$, $0 < b < 1$, and having zero width as shown in figure 4.1.

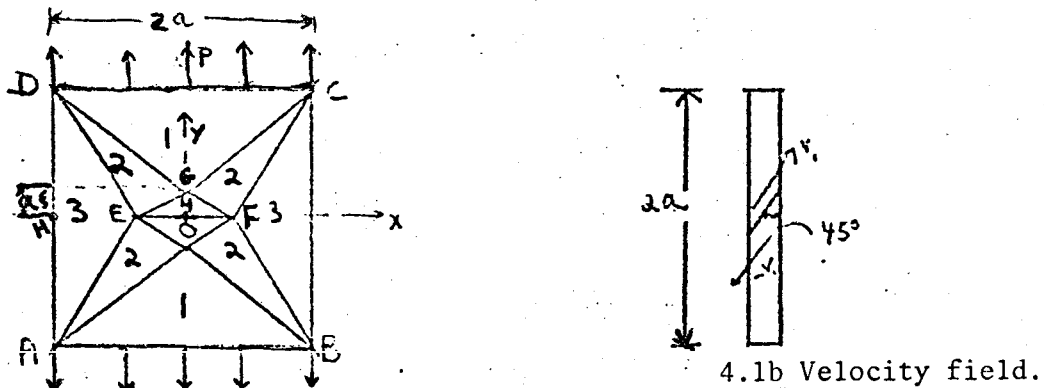


Figure 4.1 Stress field for a square plate pulled in tension.

Assuming that regions 3 will become plastic while the remainder of the plate remains rigid, and introducing regions 4 where distance OG is called $a\xi$ to isolate the stress free slit from the loaded edges AB and CD , the stresses in each region can be determined from the yield condition, the boundary conditions and the jump conditions across the boundaries separating the regions. Since the edges AD and BC of the plate are stress free, the yield criterion in regions 3 is given by:

$$|\sigma_y - \sigma_z| = |\sigma_y - \sigma_x| = 2K$$

The stress fields in the four regions are given as follows utilizing a numerical superscript to denote in which region the stresses are defined:

$$\left. \begin{aligned} \sigma_x^1 &= \frac{Pb}{1-\xi} & , & & \sigma_y^1 &= P & , & & \tau_{xy}^1 &= 0 \\ \sigma_x^2 &= \frac{\xi P}{b + \xi(1-b)} & , & & \sigma_y^2 &= \frac{Pb(1-b)}{b + \xi(1-b)} & , & & \tau_{xy}^2 &= \frac{Pb}{b + \xi(1-b)} \\ \sigma_x^3 &= 0 & , & & \sigma_y^3 &= 2K & , & & \tau_{xy}^3 &= 0 \\ \sigma_x^4 &= -\frac{Pb}{\xi} & , & & \sigma_y^4 &= 0 & , & & \tau_{xy}^4 &= 0 \end{aligned} \right\} (4.2)$$

and the critical value of P is:

$$P = 2K(1-b).$$

Calculating the principal stresses in region 2 gives:

$$\sigma_1^2 = \frac{P[\xi - b(1-b) + \{[\xi + b(1-b)]^2 + 4b^2\}^{1/2}]}{2[b + \xi(1-b)]}$$

$$\sigma_1^2 = \frac{P[\xi - b(1-b) - \{[\xi + b(1-b)]^2 + 4b^2\}^{1/2}]}{2[b + \xi(1-b)]} .$$

It can easily be shown that yield is not violated in regions 1,2 and 4 provided that the following three inequalities are satisfied:

$$b(1-b) \leq 1 - \xi$$

$$(1-b)\{[\xi + b(1-b)]^2 + 4b^2\} \leq b + \xi(1-b)$$

$$b(1-b) \leq \xi.$$

All three inequalities can be satisfied simultaneously by taking:

$$\xi = 1 - b + b^2. \quad (4.3)$$

An associated velocity field can be determined by assuming that there is sliding out of the plane along HE and FK at 45° to the plane as shown in figure 4.1b. If the block is pulled in tension, sides DC and AB move apart with relative velocity v . Calculating the internal energy dissipation across the discontinuity gives:

$$\frac{2Kv}{2} \cos \frac{\pi}{4} h \csc \frac{\pi}{4} = Kvh$$

which is positive provided that v is positive.

Hence, the complete solution is given by the stress field of (4.2) subject to the restriction on ξ given by (4.3) with the velocity field described above, and the limit load is:

$$P = 2K(1 - b)$$

4.2 Wedges in Plane Stress [36]

Consider a symmetric wedge with dimensions and loading as indicated in figure 4.2. Assume that the solution is symmetric about

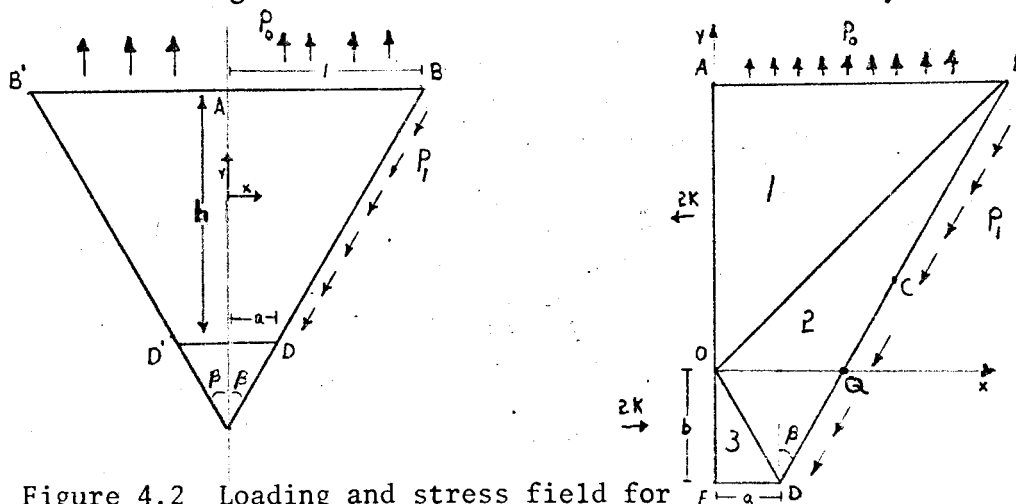


Figure 4.2 Loading and stress field for

wedge problem in plane stress.

the y -axis and that the right half of the wedge can be split into three regions with 1 and 3 plastic and 2 rigid. The position of the origin, 0, is determined by the parameter b . (See figure 4.2b.)

Calling the principal stresses in the plane σ_1 and σ_2 with the third principal stress $\sigma_3 = 0$, the Tresca criterion reduces to:

$$\max \{ |\sigma_1 - \sigma_2|, |\sigma_1|, |\sigma_2| \} \leq 2K.$$

Utilizing the boundary conditions and yield equality, the stresses in regions 1 and 3, denoted by superscripts, can be calculated immediately and are given respectively by:

$$\left. \begin{aligned} \sigma_x^1 &= 2K, \quad \sigma_y^1 = P_0, \quad \sigma_{xy}^1 = 0, \quad P_0 < 2K \\ \sigma_x^3 &= -2K, \quad \sigma_y^3 = 0, \quad \sigma_{xy}^3 = 0 \end{aligned} \right\} (4.5)$$

From overall equilibrium of the right half of the wedge, three equations can be derived to determine P_0 , P_1 and b . The solutions are:

$$\left. \begin{aligned} b &= \frac{\text{ctn } \beta}{2} \{1 - 2a + [(1 - a)^2 + a^2]^{1/2}\} \\ \frac{P_0}{2K} &= \text{ctn}^2 \beta \{-a + [(1 - a)^2 + a^2]^{1/2}\} \\ \frac{P_1}{2K} &= \frac{-\text{ctn } \beta \{a - [(1 - a)^2 + a^2]^{1/2}\}}{1 - a} \end{aligned} \right\} (4.6)$$

To find the stress in region 2, the jump conditions (4.1) across OB and OD are utilized and give four equations of which only three are independent:

$$\left. \begin{aligned} \sigma_x^2 &= -\tau_{xy}^2 \frac{a}{b} - 2K \\ \sigma_y^2 &= -\tau_{xy}^2 \frac{b}{a} \\ \tau_{xy}^2 &= \frac{-4Kb\{(1 - a) \text{ctn } \beta - b\}}{(1 - a)(a \text{ctn } \beta + b)} \\ \frac{P_D}{2K} &= \frac{2b}{a} \{(1 - a) \text{ctn } \beta - b\} \end{aligned} \right\} (4.7)$$

The fourth of equations (4.7) is the same as the moment equation used in finding (4.6) and is identically satisfied.

Since overall equilibrium has already been satisfied, the

solution to equations (4.7) automatically satisfies the boundary conditions on BD and the stresses in region 2 are given by:

$$\begin{aligned}\sigma_x^2 &= \frac{\{1 + a^2 - [(1-a)^2 + a^2]^{1/2}\}2K}{(1-a)^2} \\ \sigma_y^2 &= \text{ctn}^2\beta \frac{\{1 - 2a + 3a^2 - 2a[(1-a)^2 + a^2]^{1/2}\}2K}{(1-a)^2} \\ \tau_{xy}^2 &= \text{ctn}\beta \frac{\{1 - a + 2a^2 - (1+a)[(1-a)^2 + a^2]^{1/2}\}2K}{(1-a)^2}\end{aligned}\quad (4.8)$$

Having calculated the stress fields in the three regions, it is necessary to check that yield is nowhere violated. In equations (4.5a), P_0 was restricted to be less than $2K$. this implies that:

$$\tan \beta \leq -a + [(1-a)^2 + a^2]^{1/2}. \quad (4.9)$$

A further restriction is imposed by the condition that region 2 remain rigid. The principal stresses in region 2 are:

$$\begin{aligned}\sigma_1^2 &= \frac{\sigma_x^2 + \sigma_y^2}{2} + \frac{[(\sigma_x^2 - \sigma_y^2)^2 + 4\tau_{xy}^2]^{1/2}}{2} \\ \sigma_2^2 &= \frac{\sigma_x^2 + \sigma_y^2}{2} - \frac{[(\sigma_x^2 - \sigma_y^2)^2 + 4\tau_{xy}^2]^{1/2}}{2}\end{aligned}$$

but since $\sigma_x^2 \leq 0$ and $\sigma_y^2 \geq 0$ for $0 \leq a \leq 1$, the yield inequality simplifies to $|\sigma_1 - \sigma_2| < 2K$. Calculating $\frac{(\sigma_1 - \sigma_2)^2}{4K^2}$ gives the following implicit restriction on a and β :

$$\begin{aligned}\frac{(\sigma_1 - \sigma_2)^2}{4K^2} &= \{\text{ctn}^4\beta (1 - 2a + 3a^2 - 2ac)^2 + (2\text{ctn}^2\beta) \cdot \\ &\quad (3 - 6a + 12a^2 - 10a^3 + 9a^4 - 2c - 2ac + 2a^2c - 6a^3c) \\ &\quad + (1 + a^2 - 2c)^2\} / (1-a)^4 \leq 1 \quad (4.10)\end{aligned}$$

$$\text{where } c = [(1 - a)^2 + a^2]^{\frac{1}{2}}$$

By taking particular values of β , the restriction on a implied by the above inequality was calculated on an IBM 360 computer and the results are presented in table 4.1.

Table 4.1 The restriction on a for fixed values of β .

β°	3	6	15	30	45	60	75
$x \ni: x \leq a < 1$.9513	.9058	.7858	.6224	.4823	.3474	.2000

Furthermore, it was found that if a and β satisfied the restriction (4.10), the previous restriction (4.9) was automatically satisfied.

An associated velocity field is obtained by considering a rigid rotation with speed $\frac{\Omega}{2}$ within the plane of the right half of the wedge about the point of intersection of the x -axis and side BD , Q . The left half rotates counterclockwise about the symmetric point on side $B'D'$ at the same speed. The direction of the rotation has been chosen to make the plastic work positive if P_0 is tensile along $B'AB$ and must be reversed if P_0 is compressive. This rotation introduces a discontinuity in velocity across a plane through AE inclined at 45° to the (y, z) plane. For the purpose of computing the internal energy dissipation, only the relative jump in \dot{u}_x across the discontinuity is needed explicitly. This is given by:

$$[\dot{u}_x] = \Omega y.$$

Calculating the internal energy dissipation due to the discontinuity in velocity at a point P on AE a distance d from Q gives:

$$I = 2Kh\Omega dv$$

The internal energy dissipation is positive provided that Ω is positive; that is if Ω has the direction specified.

Thus the stress fields (4.5) and (4.7) subject to the restrictions (4.9) and (4.10) together with the velocity described above provide a complete solution with the limit load given by:

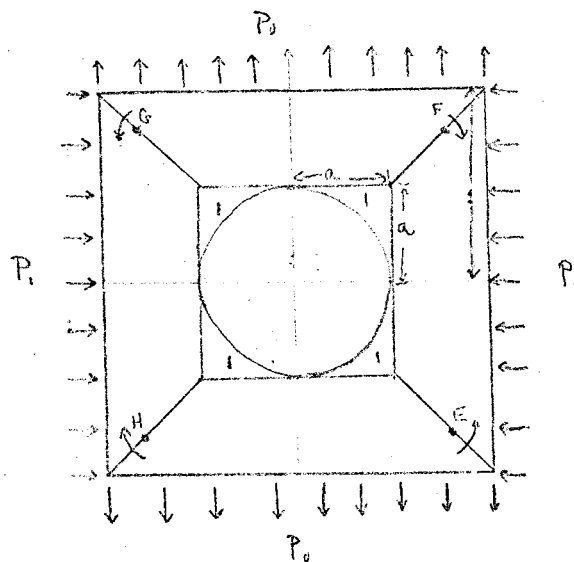
$$P_0 = 2K \left\{ a - \left[(1-a)^2 + a^{2\frac{1}{2}} \right] \right\} \text{ctn}^2 \beta$$

$$P_1 = -2K \frac{\left\{ a - \left[(1-a)^2 + a^{2\frac{1}{2}} \right] \right\} \text{ctn} \beta}{1-a}$$

4.3 Problems Solvable by the Wedge Solution

Consider a square plate with either a square hole or circular hole centered within the plate as shown in figure 4.3.

Figure 4.3 Breakdown of square plate with hole into several wedge problems.



The plate is loaded in tension in one direction and in compression in the other direction. By utilizing the stress fields obtained in the wedge problem for the four wedges shown in figure 4.3 and taking for a velocity field rigid rotations about the points E, F, G, and H which correspond to Q in figure 4.2, a complete solution is obtained provided that $a \geq .4823$ to satisfy the restrictions 4.9 and (4.10). If the hole is circular, then the regions marked 1 in the figure are all stress free and do not affect the solution. This problem was first solved by Gaydon [37] who split the square into four regions as shown by the vertical and horizontal lines passing through the middle of the plate. It was by studying Gaydon's solution that the author came by the idea for his solution to the wedge problem.

CHAPTER 5. Miscellaneous Problems

The problems considered in this section include those for which a fully plastic state exists at the limit load. This situation is restricted to problems where sufficient symmetry exists. In these cases, the problem becomes one dimensional and the solution can be found by substituting directly from the yield equality into the equilibrium equations and solving the resulting ordinary differential equations. Determination of an associated kinematically admissible velocity field is also quite simple in the one dimensional problems.

5.1 Spherical Shell under Internal Pressure [2]

Consider a spherical shell under internal pressure with inner radius a and outer radius b . Choosing spherical polar coordinates and assuming that the solution is independent of both angular coordinates the equilibrium equations and Tresca criterion reduce to

$$\frac{d\sigma_r}{dr} + 2 \frac{\sigma_r - \sigma_\theta}{r} = 0 \quad (5.1)$$

$$\sigma_\theta - \sigma_r \leq 2K \quad (5.2)$$

The stress strain-rate relations become:

$$\left. \begin{aligned} \dot{e}_r &= -\lambda = \frac{\dot{u}_r}{r} \\ \dot{e}_\theta &= \frac{\lambda}{2} = \frac{\dot{u}_r}{2r} \\ \dot{e}_\phi &= \dot{e}_\theta \end{aligned} \right\} (5.3)$$

and incompressibility implies:

$$\frac{d\dot{u}_r}{dr} + 2\frac{\dot{u}_r}{r} = 0. \quad (5.4)$$

The boundary conditions are:

$$\left. \begin{aligned} \sigma_r(a) &= -P \\ \sigma_r(b) &= 0. \end{aligned} \right\} (5.5)$$

The complete solution to the problem consists of the stresses:

$$\sigma_r = 4K \ln\left(\frac{r}{b}\right)$$

$$\sigma_\theta = 2K \left[1 - 2 \ln\left(\frac{b}{r}\right) \right]$$

the velocity field:

$$\dot{u}_r = \frac{A}{r^2}, \quad \text{where } A \text{ is taken positive to satisfy 1.3.}$$

and the limit load:

$$P = 4K \ln\left(\frac{b}{a}\right).$$

5.2 Hollow Cylinder under Internal Pressure [14]

Consider a cylindrical shell in cylindrical polar coordinates with inner radius a and outer radius b subjected to internal pressure, P . Assume a plane strain solution with the stresses independent of both z and θ .

The equations of equilibrium, yield and incompressibility then reduce to the same form as in the case of a spherical shell under internal pressure but with different constants:

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$

$$\sigma_\theta - \sigma_r = \sqrt{3} K$$

$$\frac{d\dot{u}_r}{dr} + \frac{\dot{u}_r}{r} = 0$$

The boundary conditions on stress are:

$$\sigma_r(b) = 0, \quad \sigma_r(a) = -P$$

Thus, the stress and velocity solutions are:

$$\sigma_r = -\sqrt{3} K \ln\left(\frac{b}{r}\right)$$

$$\sigma_\theta = K\left(1 - \sqrt{3} \ln\left(\frac{b}{r}\right)\right)$$

$$\dot{u}_r = \frac{A}{r}, \quad \text{where } A \text{ is taken positive to satisfy 1.3.}$$

and the limit load is:

$$P = \sqrt{3} K \ln\left(\frac{b}{a}\right)$$

Note that no consideration has been given of the boundary conditions at the end of the tube. However, the solution is valid for closed end condition and for an open end condition if it can be shown that $\sigma_z = \frac{\sigma_x + \sigma_y}{2}$ is always the intermediate stress, and if the Tresca criterion is used. If the von Mises criterion is used, the solution to the problem is dependent upon the end condition, but some results have been obtained for open, closed end and plane strain end conditions. (For comments and references see Geiringer [14].)

5.3 Critical Speed of a Rotating Solid Disc [38]

Consider a disc of radius b rotating at constant speed ω . The equilibrium equations must be modified for this problem since there is a body force term present here. Shoemaker [38] has shown that no complete solution exists if the Tresca criterion is used since no non-trivial associated velocity exists. Consequently, a solution is sought using a modified Tresca criterion.

Considering the problem in cylindrical polar coordinates, the field quantities are assumed to be independent of z and θ . The equilibrium equations, the modified Tresca criterion and the associated flow rule are given by:

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = -\rho r \omega^2 \quad \text{where } \rho \text{ denotes density and } \omega \text{ denotes angular velocity.}$$

$$\sigma_\theta = (\sqrt{3K} + \sigma) - \sigma \frac{\sigma_r}{\sqrt{3K}} \quad \text{where } \sigma > 0 \text{ is arbitrary and } 0 \leq \sigma_r \leq \sqrt{3K} \text{ where } K \text{ is the yield stress in simple tension.}$$

$$\dot{\epsilon}_{ij} = \lambda \frac{\partial f}{\partial \sigma_{ij}}$$

The boundary conditions are:

$$\sigma_r(b) = 0, \quad \sigma_r(0) \leq M \text{ for some finite constant } M.$$

Solving $f(\sigma_{ij}) = 0$ for σ_θ and substituting into the equilibrium equations gives:

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r}{r} \left(1 + \frac{\sigma}{\sqrt{3K}}\right) - \frac{3K + \sigma}{r} = -\rho \omega^2 r$$

for which the solution which is bounded at the origin is:

$$\sigma_r = \sqrt{3K} \left(1 - \frac{\rho \omega^2 r^2}{3\sqrt{3K} + \sigma} \right)$$

$$\sigma_\theta = \sqrt{3K} + \sigma - \sigma \left(1 - \frac{\rho \omega^2 r^2}{3\sqrt{3K} + \sigma} \right)$$

Applying the other boundary condition and solving for ω gives:

$$\omega = \frac{1}{b} \frac{(3\sqrt{3K} + \sigma)^{\frac{1}{2}}}{\rho}$$

From the flow rule, the strain-rates are found to be:

$$\dot{\epsilon}_r = \frac{\sigma}{\sqrt{3K}} \lambda = \frac{d\dot{u}_r}{dr}$$

$$\dot{\epsilon}_\theta = \lambda = \frac{\dot{u}_r}{r}$$

Hence, eliminating λ , the differential equation for \dot{u}_r becomes:

$$\frac{d\dot{u}_r}{dr} + \frac{\sigma}{\sqrt{3K}} \frac{\dot{u}_r}{r} = 0$$

Solving for the velocity \dot{u}_r gives:

$$\dot{u}_r = A(\omega) r^{\sigma/\sqrt{3K}}$$

which automatically satisfies $\dot{u}_r = 0$ and the energy dissipation is positive provided that $A(\omega) > 0$.

Thus a complete solution has been given with limit load, interpreted as the critical speed, given by:

$$\omega = \frac{1}{b} \frac{(3\sqrt{3K} + \sigma)}{\rho}$$

Note that as $\sigma \rightarrow 0$ the solution is still valid and that the resulting limit load agrees with the lower bound previously obtained using the Tresca criterion. Shoemaker [38] has also shown existence of a complete solution using the von Mises criterion and obtains the critical speed by a numerical method.

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