PLANAR DISTRIBUTION OF DISLOCATION ARRAYS

by

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Abstract

It is intended in this paper to show the development of a dislocation approach to solving certain boundary value problems for an elastic medium. The application of this type of approach is demonstrated by (1) presenting problems previously solved by Head and Louat involving distributions of dislocations, (2) presenting the solutions of plane and anti-plane strain boundary value problems for a half space, previously solved by Lardner, and (3) deriving the solutions for certain axisymmetric boundary value problems for a half space, to which this approach has not been previously applied.

Introduction

Beginning with the field equations for a linear, elastic, isotropic body with prescribed boundary conditions an expression for the displacement field is found in terms of surface and volume integrals involving a Green's function. The Green's function is calculated for an infinite medium. By introducing the idea of a dislocation and a Burgers vector, the expression for the displacement field, known as Burgers formula is found. The expression for the stress field of a dislocation, known as the Peach-Koehler formula, is then found from the Burgers formula.

By using an energy consideration, the effects of external forces and of mutual interactions on dislocations are found, which leads to the investigation of problems of equilibrium for dislocations under various external and end conditions. Here the continuum approximation for the dislocation density suggested by Head and Louat [1] is introduced, and their solutions are presented.

The continuum approximation of Head and Louat, together with the Burgers and Peach-Koehler formulae are used to solve a plane and an anti-plane strain boundary value problem for a half space. The Peach-Koehler formula is used to evaluate on the boundary that particular component of stress that is prescribed, giving an integral equation for the unknown dislocation density function. The solution of this integral equation is well known and hence the rest of the stress field and displacement field can be found.

The axisymmetric boundary value problems are solved in much the same way. The first problem is one of the half space with an axisymmetric normal stress prescribed on the boundary. The Peach-Koehler formula is used to evaluate the expression for the normal stress on the boundary. This expression, which gives an integral equation for the unknown dislocation density function, involves elliptic integrals of the first and second kind. The solution is found by first writing the elliptic integrals in terms of Bessel functions, and then noting the similarity between the resulting expression and a Hankel transform.

The second axisymmetric boundary value problem considered is that of pure torsion with an axisymmetric shear stress prescribed on the boundary. It is not possible to begin with the Peach-Koehler formula since in its derivation it is assumed that the direction of the displacement discontinuity remains constant everywhere on the dislocation loop. It is necessary then to return to the original form of the displacement field equation. By differentiating this expression for the displacement, evaluating it at a point on the x-axis, and equating it with the prescribed boundary value of the shear stress, one obtains an integral equation for the unknown density function. The solution of this integral equation is found once again by writing the elliptic integrals in terms of Bessel functions and treating the resulting expression as a Hankel transform.

1. Elastic Boundary Value Problems

The derivation of the Burgers formula for the displacement field in an infinite linear elastic material subjected to certain deformations begins with the determination of the displacement field in terms of a Green's function.

Let a finite linear elastic body B be in equilibrium under given body forces whose components are f_i, under given tractions on a part S_1 of its boundary, and under given displacements over a part S_2 of its boundary. The field equations are then

$$\sigma_{ij,j} + f_i = 0 \tag{1.1}$$

Since the material is linear elastic,

$$\begin{aligned}\sigma_{ij} &= C_{ijkl}\, e_{kl} = \tfrac{1}{2} C_{ijkl} (u_{k,l} + u_{l,k}) \\ &= \tfrac{1}{2} C_{ijlk}\, u_{l,k} + \tfrac{1}{2} C_{ijkl}\, u_{k,l} \\ &= C_{ijkl}\, u_{l,k}\end{aligned}$$

The last step was made possible by $C_{ijkl} = C_{ijlk}$.

In terms of the displacements the field equations and the boundary values may be written as

$$C_{ijkl}\, u_{l,jk} + f_i = 0 \qquad \text{in } B \tag{1.2a}$$

$$C_{ijkl}\, u_{l,k}\, n_j = t_i \qquad \text{on } S_1 \tag{1.2b}$$

$$u_i = U_i \qquad \text{on } S_2 \tag{1.2c}$$

To construct a Green's function for this problem, let $\underline{r}'$

be a point in B. With $\underline{r}'$ as its centre, construct a sphere Σ_ϵ with radius ϵ. Let $B_\epsilon = B - \Sigma_\epsilon$ be the volume outside Σ_ϵ and inside B. Finally, let the Green's function $u_{ij}(\underline{r},\underline{r}')$ satisfy:

$$C_{ijkl}\, u_{lm,jk}(\underline{r},\underline{r}') = 0 \qquad \text{for } \underline{r} \text{ in } B_\epsilon \tag{1.3a}$$

$$C_{ijkl}\, u_{lm,k}(\underline{r},\underline{r}')\, n_j = 0 \qquad \text{for } \underline{r} \text{ on } S_1 \tag{1.3b}$$

$$u_{lm}(\underline{r},\underline{r}') = 0 \qquad \text{for } \underline{r} \text{ on } S_2 \tag{1.3c}$$

$$\int_{\partial\Sigma_\epsilon} C_{ijkl}\, u_{lm,k}(\underline{r},\underline{r}')\, n_j\, dS = \delta_{im} \qquad (\underline{n} \text{ here is the inward normal of } \Sigma_\epsilon) \tag{1.3d}$$

$$\epsilon^2 u_{lm}(\underline{r},\underline{r}') \to 0 \text{ as } \epsilon \to 0 \qquad \text{for } \underline{r} \text{ on } \Sigma_\epsilon \tag{1.3e}$$

The Green's function $u_{lm}(\underline{r},\underline{r}')$ is then the l^{th} component of the displacement field at $\underline{r}$ when the tractions on S_1 and displacements on S_2 are zero, and the only body force present is a unit force in the m direction applied at $\underline{r}'$.

Apply the Gauss divergence theorem on $u_i(\underline{r})$ and $u_{lm}(\underline{r},\underline{r}')$:

$$\int_{\partial B+\partial\Sigma_\epsilon} \left[u_i(\underline{r})\, u_{lm,k}(\underline{r},\underline{r}') - u_{im}(\underline{r},\underline{r}')\, u_{l,k}(\underline{r}) \right] n_j\, C_{ijkl}\, dS$$

$$= \int_{B_\epsilon} \left[u_i(\underline{r})\, u_{lm,k}(\underline{r},\underline{r}') - u_{im}(\underline{r},\underline{r}')\, u_{l,k}(\underline{r}) \right]_{,j} C_{ijkl}\, dV \tag{1.4}$$

The right hand side of (1.4) becomes

$$\int_{B_\epsilon} \Big[u_i(\underline{r})\, U_{\ell m,kj}(\underline{r},\underline{r}') + u_{i,j}(\underline{r})\, U_{\ell m,k}(\underline{r},\underline{r}') - U_{im}(\underline{r},\underline{r}')\, u_{\ell,kj}(\underline{r}) - U_{im,j}(\underline{r},\underline{r}')\, u_{\ell,k}(\underline{r}) \Big] C_{ijk\ell}\, dV .$$

From (1.2a) and (1.3a)

$$C_{ijk\ell}\, u_{\ell,jk}(\underline{r}) = -f_i \qquad \text{for } \underline{r} \text{ in } B$$

$$C_{ijk\ell}\, U_{\ell m,jk}(\underline{r},\underline{r}') = 0 \qquad \text{for } \underline{r} \text{ in } B_\epsilon$$

And from the symmetry of C_{ijkl}:

$$u_{i,j}(\underline{r})\, U_{\ell m,k}(\underline{r},\underline{r}')\, C_{ijk\ell} - U_{im,j}(\underline{r},\underline{r}')\, u_{\ell,k}(\underline{r})\, C_{ijk\ell} = 0 .$$

The right hand side of (1.4) can finally be written:

$$+\int_{B_\epsilon} U_{\ell m}(\underline{r},\underline{r}')\, f_\ell(\underline{r})\, dV \tag{1.5}$$

Since $\partial B = S_1+S_2$, part of the left hand side of (1.4) may be written

$$\int_{\partial B} = \int_{S_1+S_2} \Big[u_i(\underline{r})\, U_{\ell m,k}(\underline{r},\underline{r}') - U_{im}(\underline{r},\underline{r}')\, u_{\ell,k}(\underline{r}) \Big] n_j\, C_{ijk\ell}\, dS .$$

From (1.3b) and (1.3c):

$$\int_{S_1+S_2} = -\int_{S_1} u_{im}(\underline{r},\underline{r}')\, u_{l,k}(\underline{r})\, n_j C_{ijkl}\, dS$$

$$+\int_{S_2} u_i(\underline{r})\, u_{lm,k}(\underline{r},\underline{r}')\, n_j C_{ijkl}\, dS.$$

And from (1.2b) and (1.2c):

$$\int_{S_1+S_2} = -\int_{S_1} u_{lm}(\underline{r},\underline{r}')\, t_l\, dS$$

$$+\int_{S_2} U_i(\underline{r})\, C_{ijkl}\, u_{lm,k}(\underline{r},\underline{r}')\, n_j\, dS \qquad (1.6)$$

The integral remaining on the left hand side of (1.4) is over $\partial\Sigma_\epsilon$:

$$\int_{\partial\Sigma_\epsilon} = \int_{\Sigma_\epsilon} u_i(\underline{r})\, u_{lm,k}(\underline{r},\underline{r}')\, n_j C_{ijkl}\, dS$$

$$-\int_{\Omega} \epsilon^2 u_{im}(\underline{r},\underline{r}')\, u_{l,k}(\underline{r})\, n_j C_{ijkl}\, d\Omega.$$

Since $u_{i,k}(\underline{r})$ is bounded, the second integral over Ω tends to zero as ϵ tends to zero because of (1.3e). Also, since $u_i(\underline{r})$ are continuous, $u_i(\underline{r}) \rightarrow u_i(\underline{r}')$ as $\epsilon \rightarrow 0$. Using (1.3d), as $\epsilon \rightarrow 0$, the integral over $\partial\Sigma_\epsilon$ finally becomes

$$\int_{\partial\Sigma_\epsilon} \rightarrow \int_{\partial\Sigma_\epsilon} u_i(\underline{r}')\, u_{\ell m,k}(\underline{r},\underline{r}')\, C_{ijk\ell}\, n_j\, dS$$

$$= u_i(\underline{r}')\,\delta_{im} \quad = u_m(\underline{r}') . \tag{1.7}$$

Combining (1.5), (1.6) and (1.7)

$$u_m(\underline{r}') = \int_B u_{\ell m}(\underline{r},\underline{r}')\, f_\ell(\underline{r})\, dV$$
$$+ \int_{S_1} u_{\ell m}(\underline{r},\underline{r}')\, t_\ell\, dS$$
$$- \int_{S_2} U_i(\underline{r})\, u_{\ell m,k}(\underline{r},\underline{r}')\, n_j\, C_{ijk\ell}\, dS . \tag{1.8}$$

If we have a traction boundary value problem, S_2 is empty and $S_1 = \partial B$; then

$$u_m(\underline{r}') = \int_B u_{\ell m}(\underline{r},\underline{r}')\, f_\ell(\underline{r})\, dV$$
$$+ \int_{\partial B} u_{\ell m}(\underline{r},\underline{r}')\, t_\ell\, dS . \tag{1.9}$$

However, in deriving (1.9) the procedure must be modified. With S_2 empty there are no longer boundary tractions in equations (1.3a) - (1.3e) to balance the point force at $\underline{r}'$. To provide a balance, an equal and opposite force must be placed at some other point $\underline{r_o}$ as well, as a point couple which will cancel the torque created by the two point forces. See Appendix A for details.

If the medium extends to infinity, some restrictions on the displacements and the Green's function must be imposed. In addition to the requirement that $u_{lm}(\underline{r},\underline{r}')$ satisfy equations (1.3a) - (1.3e) we must also require that the integral of the tractions, $C_{ijkl}\, u_{lm,k}(\underline{r},\underline{r}')\, n_j$, over the surface of a large sphere of radius R must be bounded and must balance the unit force at $\underline{r}'$. It is sufficient if we suppose $u_{lm}(\underline{r},\underline{r}') \sim O\,(^1/r)$ and $u_{lm,k}(\underline{r},\underline{r}') \sim O\,(^1/r^2)$ as $r \to \infty$. Using again the Gauss divergence theorem applied to that part of B_ϵ which lies inside a large sphere Σ_R of radius R and centered at, say, the origin, (∂B is the part of the boundary inside the sphere Σ_R), we have that

$$\int_{\partial B + \partial\Sigma_R + \partial\Sigma_\epsilon} \left[u_i(\underline{r})\, u_{lm,k}(\underline{r},\underline{r}') - u_{im}(\underline{r},\underline{r}')\, u_{l,k}(\underline{r}') \right] C_{ijkl}\, n_j\, dS$$

$$= \int_{B_\epsilon} \left[u_i(\underline{r})\, u_{lm,k}(\underline{r},\underline{r}') - u_{im}(\underline{r},\underline{r}')\, u_{l,k}(\underline{r}) \right]_{,j} C_{ijkl}\, dV.$$

In order that $\int_{\partial \Sigma_R} \to 0$ as $R \to \infty$, we must add the following two conditions on $u_i(\underline{r})$ and $u_{l,k}(\underline{r})$:

1) $u_i(\underline{r}) \to 0$ as $\underline{r} \to \infty$

2) $r\, u_{l,k}(\underline{r}) \to 0$ as $\underline{r} \to \infty$

These two additional conditions are required since there is an r^2 contribution from dS. Then as $R \to \infty$, and as $\epsilon \to 0$ we have again equation (1.8).

An equation analagous to (1.8) may be derived for a subvolume V of the body B. Let the surface of the subvolume be ∂V. Then (1.8) becomes

$$u_m(\underline{r}') = \int_V u_{lm}(\underline{r},\underline{r}')\, f_l(\underline{r})\, dV + \int_{\partial V} u_{lm}(\underline{r},\underline{r}')\, \sigma_{lj}(\underline{r})\, n_j\, dS - \int_{\partial V} u_i(\underline{r})\, u_{lm,k}(\underline{r},\underline{r}')\, C_{ijkl}\, n_j\, dS\,, \qquad (1.10)$$

provided that $\underline{r}'$ is restricted to lie within V. In (1.10) $\sigma_{lj}(\underline{r})$ is the stress field corresponding to $u_i(\underline{r})$.

If in the case of an infinite body B, V also extends to infinity, the surface integral over $\partial \Sigma_R$ tends to zero because of the restrictions imposed before on $u_i(\underline{r})$ and $u_{lm}(\underline{r},\underline{r}')$.

The conditions, then, on $u_{lm}(\underline{r},\underline{r}')$ are that it is of order 1/r at infinity and satisfies equations (1.3a) and (1.3d). We may combine equations (1.3a) and (1.3d) into one equation in the following way.

Apply the divergence theorem to equation (1.3d):

$$\int_{\partial\Sigma_\epsilon} C_{ijkl}\, u_{lm,k}(\underline{r},\underline{r}')\, n_j\, dS$$

$$= -\int_{\Sigma_\epsilon} C_{ijkl}\, u_{lm,kj}(\underline{r},\underline{r}')\, dV = \delta_{im} \qquad , \qquad (1.11)$$

The minus sign enters since n_j is the inward normal of Σ_ϵ . Integrating the term $C_{ijkl}\, u_{lm,kj}(r,r')$ over $B_\epsilon + \Sigma_\epsilon$ and using equation (1.3a) gives:

$$\int_{B_\epsilon+\Sigma_\epsilon} C_{ijkl}\, u_{lm,kj}(\underline{r},\underline{r}')\, dV$$

$$= \int_{\Sigma_\epsilon} C_{ijkl}\, u_{lm,kj}(\underline{r},\underline{r}')\, dV \qquad (1.12)$$

Equations (1.11) and (1.12) combine to give

$$\int_{B_\epsilon+\Sigma_\epsilon} C_{ijkl}\, u_{lm,kj}(\underline{r},\underline{r}')\, dV = -\delta_{im}$$

$$= \int_{B_\epsilon+\Sigma_\epsilon} -\delta_{im}\, \delta(\underline{r}-\underline{r}')\, dV \quad ,$$

where $\delta(\underline{r} - \underline{r}')$ is the three dimensional delta function. So

$$C_{ijkl}\, u_{lm,kj}(\underline{r},\underline{r}') = -\delta_{im}\, \delta(\underline{r}-\underline{r}') \, . \qquad (1.13)$$

The Green's function must then be of order 1/r at infinity and satisfy equation (1.13).

To find the Green's function satisfying these conditions, following deWit,[2] we begin by taking a Fourier Transform on $\underline{r}$ defined by

$$\overline{U}_{\ell m}(\underline{\xi},\underline{r}') = \int U_{\ell m}(\underline{r},\underline{r}')\, e^{-i\underline{\xi}\cdot\underline{r}}\, \overline{dr} \ ,$$

$$\text{where } \overline{dr} = dr_1\, dr_2\, dr_3 \ . \tag{1.14}$$

The inverse is defined by

$$U_{\ell m}(\underline{r},\underline{r}') = \tfrac{1}{8\pi^3}\int \overline{U}_{\ell m}(\underline{\xi},\underline{r}')\, e^{i\underline{\xi}\cdot\underline{r}}\, \overline{d\xi} \ ,$$

$$\text{where } \overline{d\xi} = d\xi_1\, d\xi_2\, d\xi_3 \ . \tag{1.15}$$

The integration here extends over the whole space.

Take the Fourier transform on (1.13) according to (1.14):

$$C_{ijk\ell}\,\overline{U}_{\ell m}(\underline{\xi},\underline{r}')\,\xi_j\,\xi_k = \delta_{im}\, e^{-i\underline{\xi}\cdot\underline{r}'} \ . \tag{1.16}$$

At this point we assume the material to be isotropic. Then

$$C_{ijk\ell} = \lambda\,\delta_{ij}\,\delta_{k\ell} + \mu\,(\delta_{ik}\,\delta_{j\ell} + \delta_{i\ell}\,\delta_{jk}) \ . \tag{1.17}$$

Equation (1.16) becomes

$$(\lambda+\mu)\,\xi_i\xi_l\,\bar{U}_{lm}(\underline{\xi},\underline{r}')$$

$$+\mu\,\xi^2\,\bar{U}_{im}(\underline{\xi},\underline{r}') = \delta_{im}\,e^{-i\underline{\xi}\cdot\underline{r}'},$$

$$\text{where } \xi^2 = \xi_p\xi_p, \text{ summed over } p. \tag{1.18}$$

Multiply (1.18) by ξ_i and sum over i:

$$(\lambda+\mu)\,\xi_i\,\xi_i\,\xi_l\,\bar{U}_{lm}(\underline{\xi},\underline{r}')$$

$$+\mu\,\xi^2\xi_i\,\bar{U}_{im}(\underline{\xi},\underline{r}') = \xi_i\,\delta_{im}\,e^{-i\underline{\xi}\cdot\underline{r}'}$$

$$\text{or} \quad \xi_l\,\bar{U}_{lm}(\underline{\xi},\underline{r}') = \frac{\xi_m\,e^{-i\underline{\xi}\cdot\underline{r}'}}{(\lambda+2\mu)\,\xi^2}. \tag{1.19}$$

Substituting (1.19) into (1.18)

$$\frac{(\lambda+\mu)\,\xi_i\,\xi_m\,e^{-i\underline{\xi}\cdot\underline{r}'}}{(\lambda+2\mu)\,\xi^2} + \mu\,\xi^2\,\bar{U}_{im}(\underline{\xi},\underline{r}') = \delta_{im}\,e^{-i\underline{\xi}\cdot\underline{r}'}.$$

$$\text{So} \quad \bar{U}_{im}(\underline{\xi},\underline{r}') = 1/\mu\,\Bigg[\frac{\delta_{im}\,e^{-i\underline{\xi}\cdot\underline{r}'}}{\xi^2}$$

$$-\frac{(\lambda+\mu)}{(\lambda+2\mu)}\,\frac{\xi_i\,\xi_m}{\xi^4}\,e^{-i\underline{\xi}\cdot\underline{r}'}\Bigg]. \tag{1.20}$$

Invert (1.20) according to (1.15)

$$\mathcal{U}_{im}(\underline{r},\underline{r}') = \frac{1}{8\pi^3\mu} \int \left[\frac{\delta_{im}}{\xi^2} - \frac{\lambda+\mu}{\lambda+2\mu} \frac{\xi_i \xi_m}{\xi^4} \right] \cdot e^{i\xi\cdot(\underline{r}-\underline{r}')} \, d\bar{\xi} \tag{1.21}$$

Now consider the integral

$$\int \frac{e^{i\underline{\xi}\cdot(\underline{r}-\underline{r}')}}{\xi^4} \, d\bar{\xi} = -\pi^2 |\underline{r}-\underline{r}'| = -\pi^2 R \, . \tag{1.22}$$

We write $R = |\underline{r}-\underline{r}'| = \left[(x_1-x_1')^2 + (x_2-x_2')^2 + (x_3-x_3')^2 \right]^{1/2}.$

Differentiate (1.22) twice to get

$$\frac{\partial^2}{\partial x_l \partial x_m} \int \frac{e^{i\underline{\xi}\cdot(\underline{r}-\underline{r}')}}{\xi^4} \, d\bar{\xi} = \int \frac{\xi_l \xi_m e^{i\underline{\xi}\cdot(\underline{r}-\underline{r}')}}{\xi^4} \, d\bar{\xi} = \pi^2 R_{,lm} \, . \tag{1.23}$$

In (1.23) let l = m. Then

$$\int \frac{e^{i\underline{\xi}\cdot(\underline{r}-\underline{r}')}}{\xi^2} \, d\bar{\xi} = \pi^2 R_{,ll} \, . \tag{1.24}$$

Using (1.23) and (1.24), equation (1.21) becomes

$$U_{im}(\underline{r},\underline{r}') = \frac{1}{8\pi\mu} \left[\delta_{im} R_{,ll} - \frac{(\lambda+\mu)}{(\lambda+2\mu)} R_{,im} \right] \tag{1.25}$$

2. Dislocation Loops

Having found an expression for the displacement field $u_m(\underline{r}')$ in terms of a Green's function, we are ready to define a dislocation loop and to calculate the displacement field due to this loop.

Consider a body B with boundary ∂B. Let Γ be a closed curve in B, and let A be a surface spanning this closed curve. Make a cut on the surface A and displace the two faces. Let the vector $\underline{b}$ be the displacement relative to each other of two corresponding points on the two faces of the cut. That is, if A_{-}^{+} are the two faces of the cut, then $\underline{u}(\underline{r}'_+) - \underline{u}(\underline{r}'_-) = \underline{b}(\underline{r}') = \Delta \underline{u}(\underline{r}')$. A direction is imposed on Γ in a right handed sense with respect to the unit vectors pointing from A_+ to A_- . The tractions on the two faces at corresponding points are required to be equal and opposite. The deformation so found is said to be a dislocation loop with Burgers vector $\underline{b}$ $(\underline{r}')$. The curve Γ is called the line of the dislocation.[3]

Now let V be the whole volume of B except for the cut A. Then the boundary of V, ∂V consists of $\partial V = \partial B + A_+ + A_-$. Assume that the body forces $\underline{f}$ $(\underline{r}')$ are zero and that the tractions $\underline{t}(\underline{r}')$ on ∂B vanish. The Green's function satisfies both equation (6a) of the Appendix A and the equation

$$C_{ijkl}\, U_{lm,k}(\underline{r},\underline{r}')\, n_j = 0 \quad \text{for } \underline{r} \text{ on } \partial B, \tag{2.1}$$

The displacement field, $u_i(\underline{r}')$, is given by equation (1.10) with $f_i(\underline{r}') = 0$:

$$U_m(\underline{r}) = \int\limits_{\partial V = \partial B + A_+ + A_-}\left[u_{\ell m}(\underline{r},\underline{r}')\, t_\ell(\underline{r}) - u_i(\underline{r})\, u_{\ell m,k}(\underline{r},\underline{r}')\, C_{ijk\ell}\, n_j\right] dS.$$

The integral over ∂B vanishes since the tractions there are assumed to be zero and by (2.1). So

$$U_m(\underline{r}') = \int\limits_{A_+ + A_-}\left[u_{\ell m}(\underline{r},\underline{r}')\,\sigma_{\ell j} - u_i(\underline{r})\, u_{\ell m,k}(\underline{r},\underline{r}')\, C_{ijk\ell}\, n_j\right] dS. \qquad (2.2)$$

By hypothesis,

$$\left.\begin{aligned} \sigma_{\ell j} n_j \Big|_{A_+} &= -\sigma_{\ell j} n_j \Big|_{A_-} \,, \\ \text{and} \quad u_i(\underline{r})\Big|_{A_+} - u_i(\underline{r})\Big|_{A_-} &= b_i(\underline{r}) = \Delta u_i(\underline{r}) \end{aligned}\right\} \qquad (2.3)$$

Then, since (2.2) may be written

$$U_m(\underline{r}') = \int\limits_{A_+} u_{\ell m}(\underline{r},\underline{r}')\,\sigma_{\ell j}(\underline{r})\, n_j\, dS$$

$$+ \int\limits_{A_-} u_{\ell m}(\underline{r},\underline{r}')\,\sigma_{\ell j}(\underline{r})\, n_j\, dS - \int\limits_{A_+} u_i(\underline{r})\, u_{\ell m,k}(\underline{r},\underline{r}')\, C_{ijk\ell}\, n_j\, dS$$

$$- \int\limits_{A_-} u_i(\underline{r})\, u_{\ell m,k}(\underline{r},\underline{r}')\, C_{ijk\ell}\, n_j\, dS \,,$$

by equations (2.3), we have

$$u_m(\underline{r}') = -\int_{A^+} \Delta u_i(\underline{r})\, u_{lm,k}(\underline{r},\underline{r}')\, C_{ijkl}\, n_j\, dS. \tag{2.4}$$

It should be remarked that at this point $\Delta \underline{u}(\underline{r})$ is not yet assumed constant.

We now specialize the above consideration of a dislocation loop by assuming the infinite body to be isotropic and linearly elastic. We also assume that in making a cut on the surface A, the following displacement $\underline{b}$ of the two faces relative to one another will be constant. Again we denote by Γ the curve enclosing the surface A.

Recall that the Green's function for an infinite isotropic medium was given by (1.25). Substituting the Green's function given by (1.25) into the expression for the displacement given by (2.4) and making use of (1.17) we have

$$\begin{aligned} u_m(\underline{r}') &= \frac{-b_i}{8\pi\mu} \int_{A_+} \Big[\lambda \delta_{ij} R_{,ppm} - \frac{\lambda+\mu}{\lambda+2\mu} \lambda \delta_{ij} R_{,ppm} \\ &\qquad + \mu \Big(\delta_{jm} R_{,ppi} + \delta_{im} R_{,ppj} - 2 \frac{\lambda+\mu}{\lambda+2\mu} R_{,jmi} \Big) \Big] n_j\, dS \\ &= \frac{-b_i}{8\pi} \int_{A_+} \Big[-\delta_{ij} R_{,ppm} + \delta_{jm} R_{,ppi} + \delta_{im} R_{,ppj} \Big] n_j\, dS \\ &\quad \frac{-b_i(\lambda+\mu)}{4\pi(\lambda+2\mu)} \int_{A_+} \Big[\delta_{ij} R_{,ppm} - R_{,jim} \Big] n_j\, dS \end{aligned} \tag{2.5}$$

We may write the right hand side of (2.5) in terms of a line integral around Γ by using Stokes theorem, of the form:

$$\int_A \varepsilon_{ijk} T_{,j} n_i \, dS = \oint_\Gamma T \, dl_k \quad , \tag{2.6}$$

where dl_k is an element of arc on Γ and T is any differentiable function. Since $\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$, (2.6) may be written

$$\begin{aligned}\oint_\Gamma \varepsilon_{klm} T \, dl_k &= \int_A \varepsilon_{ijk}\varepsilon_{klm} T_{,j} n_i \, dS \\ &= \int_A (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) T_{,j} n_i \, dS \\ &= \int_A (\delta_{il} T_{,m} - \delta_{im} T_{,l}) n_i \, dS . \end{aligned} \tag{2.7}$$

Now the second integral in (2.5) was of the form

$$\int_{A_+} [\delta_{ij} R_{,ppm} - R_{,ijm}] n_j \, dS \quad ,$$

and by (2.7) this becomes

$$\begin{aligned}&\int_{A_+} [\delta_{lp}\delta_{ij} R_{,mpl} - \delta_{li}\delta_{jp} R_{,mpl}] n_j \, dS \\ &= \int_{A_+} \varepsilon_{ljk}\varepsilon_{kpi} R_{,mpl} n_j \, dS \\ &= -\varepsilon_{kpi} \int_{A_+} \varepsilon_{jlk} R_{,mpl} n_j \, dS = -\oint \varepsilon_{kpi} R_{,mp} \, dl_k . \end{aligned} \tag{2.8}$$

The first two terms of the first integral in (2.5) are of the form

$$\int_{A^+} \left[-\delta_{ij} R_{,ppm} + \delta_{jm} R_{,ppi} \right] n_j \, dS$$

$$= \int_{A_+} \left[\delta_{li} \delta_{jm} R_{,ppl} - \delta_{jm} \delta_{ij} R_{,ppl} \right] n_j \, dS$$

$$= \int_{A_+} \epsilon_{ljk} \, \varepsilon_{kim} R_{,ppl} \, n_j \, dS = -\varepsilon_{kim} \int_{A_+} \varepsilon_{jlk} R_{,ppl} n_j dS$$

$$= -\oint_{\Gamma} \varepsilon_{kim} R_{,pp} \, dl_k \, . . \qquad (2.9)$$

Finally, the last term of the first integral in (2.5) is of the form

$$\int_{A_+} \delta_{im} R_{,ppj} \, n_j \, dS \qquad (2.10)$$

Now

$$R_{,p} = \tfrac{1}{2} R^{-1} \cdot 2 \, (x_p - x_p') = \frac{(x_p - x_p')}{R}$$

$$R_{,pp} = 2 R^{-1}$$

$$R_{,ppj} = \frac{-2 \, (x_j - x_j')}{R^2}$$

So (2.10) becomes

$$-2 \int_{A_+} \frac{(x_j - x_j')}{R^3} \delta_{im} n_j dS = 2 \delta_{im} \, \Omega(\underline{r}') \, . \qquad (2.11)$$

where $\Omega(\underline{r}')$ is the solid angle subtended at $\underline{r}'$ by the surface A_+.

which is bounded by the curve Γ . Then combining equations (2.8), (2.9) and (2.11), equation (2.5) becomes

$$u_m(\underline{r}') = \frac{-b_m \Omega(\underline{r}')}{4\pi} + \frac{b_i}{8\pi} \oint_\Gamma \varepsilon_{kim} R_{,pp} \, dl_k + \frac{b_i}{4\pi} \oint_\Gamma \frac{\lambda+\mu}{\lambda+2\mu} \varepsilon_{kpi} R_{,mp} \, dl_k . \tag{2.12}$$

It may be noticed from (2.12) that the displacement field depends only upon the curve Γ . So for a given curve Γ, the displacement field is unaffected by the particular surface A on which the cut is made. The equation (2.12) is known as the Burgers formula.[4]

To derive the stress and strain fields, the derivative of $\Omega(\underline{r}')$ must first be found.[2] From (2.11),

$$\Omega(\underline{r}') = \int_{A_+} R^{-1}_{,j} \, n_j \, dS \quad ,$$

So

$$\Omega_{,n'}(\underline{r}') = \left[\int_{A_+} R^{-1}_{,j} \, n_j \, dS \right]_{,n'} ,$$

where $,n'$ denotes $\frac{\partial}{\partial x_n'}$.

But $\frac{\partial}{\partial x_n} = -\frac{\partial}{\partial x_n'}$, so

$$\Omega_{,n'}(\underline{r}') = -\left[\int_{A_+} R^{-1}_{,j} \, n_j \, dS \right]_{,n} = -\int_{A_+} R^{-1}_{,nj} \, n_j \, dS$$

Since $R^{-1}_{pp} = 0$ as long as $r = r'$,

$$\Omega_{,n'}(\underline{r}') = +\int_{A_+} \left(R^{-1}_{,pp}\,\delta_{nj}\,n_j - R^{-1}_{,nj}\,n_j \right) dS$$

$$= +\int_{A_+} \left(\delta_{\ell m}\,\delta_{nj}\,R^{-1}_{,\ell m} - \delta_{\ell n}\,\delta_{mj}\,R^{-1}_{,\ell m} \right) n_j\, dS$$

$$= +\int_{A_+} \varepsilon_{\ell jk}\,\varepsilon_{kmn}\,R^{-1}_{,\ell m}\,n_j\, dS$$

$$= -\varepsilon_{kmn}\int_{A_+} \varepsilon_{j\ell k}\,R^{-1}_{,m\ell}\,n_j\, dS$$

$$= -\oint_{\Gamma} \varepsilon_{kmn}\,R^{-1}_{,m}\,d\ell_k$$

$$= -\tfrac{1}{2}\oint_{\Gamma} \varepsilon_{nkm}\,R_{,mpp}\,d\ell_k\ ,$$

so
$$U_{m,n'}(\underline{r}') = \frac{1}{8\pi}\oint \Big\{ \varepsilon_{nk\ell}\,b_m\,R_{,\ell pp} - \varepsilon_{kim}\,b_i\,R_{,ppn} - \frac{2(\lambda+\mu)}{\lambda+2\mu}\,b_i\,\varepsilon_{kpi}\,R_{,mpn} \Big\} d\ell_k. \tag{2.13}$$

From (2.13):

$$U_{n,n'}(\underline{r}') = \tfrac{1}{8\pi}\oint \Big\{ \varepsilon_{1k}\,b_n\,R_{,\ell pp} - \varepsilon_{kin}\,b_i\,R_{,ppn} - \frac{2(\lambda+\mu)}{(\lambda+2\mu)}\,b_i\,\varepsilon_{kpi}\,R_{,nnp} \Big\} d\ell_k,$$

$$u_{n,n'}(\underline{r}') = 1/8\pi \oint_\Gamma \Big\{ \varepsilon_{ikl} b_i R_{,lpp} - \varepsilon_{kil} b_i R_{,lpp}$$

$$- \frac{2(\lambda+\mu)}{\lambda+2\mu} b_i \varepsilon_{kli} R_{lpp} \Big\} dl_k$$

$$= 1/8\pi \oint_\Gamma \varepsilon_{ikl} b_i R_{,lpp} \Big\{ 1+1-\frac{2(\lambda+\mu)}{(\lambda+2\mu)} \Big\} dl_k$$

$$= \frac{1}{8\pi} \frac{1-2\nu}{1-\nu} \oint_\Gamma \varepsilon_{ikl} b_i R_{,lpp} \, dl_k, \qquad (2.14)$$

where $\nu = \dfrac{\lambda}{2(\lambda+\mu)}$.

Now

$$e_{ij}(\underline{r}') = 1/2 \left(u_{i,j'}(\underline{r}') + u_{j,i'}(\underline{r}') \right) \qquad (2.15)$$

ubstituting (2.13) into (2.15) gives

$$e_{mn}(\underline{r}') = 1/8\pi \oint_\Gamma \Big\{ 1/2 \big(\varepsilon_{nkl} b_m R_{,lpp} - \varepsilon_{kim} b_i R_{,ppn}$$

$$+ \varepsilon_{mkl} b_n R_{,lpp} - \varepsilon_{kin} b_i R_{,mpp} \big)$$

$$- \frac{2(\lambda+\mu)}{\lambda+2\mu} b_i \varepsilon_{kpi} R_{,mpn} \Big\} dl_k . \qquad (2.16)$$

But

$$\varepsilon_{nkl}\, b_m R_{,lpp} - \varepsilon_{kim}\, b_i R_{,ppn} + \varepsilon_{mkl}\, b_n R_{,lpp} - \varepsilon_{kin}\, b_i R_{,mpp}$$

$$= \left(\varepsilon_{nki}\, b_m R_{,i} - \varepsilon_{kin}\, b_i R_{,m} + \varepsilon_{mki}\, b_n R_{,i} - \varepsilon_{kim}\, b_i R_{,n}\right)_{,pp}$$

$$= \left[b_l R_{,j} \left(\varepsilon_{kim}\,\delta_{ml}\,\delta_{ij} - \varepsilon_{kin}\,\delta_{il}\,\delta_{mj} + \varepsilon_{kim}\,\delta_{nl}\,\delta_{ij} - \varepsilon_{kim}\,\delta_{il}\,\delta_{nj}\right)\right]_{,pp}$$

$$= \left[b_l R_{,j} \left(\varepsilon_{kim}\,\varepsilon_{miq}\,\varepsilon_{qlj} + \varepsilon_{kim}\,\varepsilon_{niq}\,\varepsilon_{qlj}\right)\right]_{,pp}$$

$$= \left[b_l R_{,j} \left\{\varepsilon_{qlj}\left(\delta_{nq}\,\delta_{km} - \delta_{nm}\,\delta_{kq}\right) + \varepsilon_{qlj}\left(\delta_{mq}\,\delta_{kn} - \delta_{mn}\,\delta_{kq}\right)\right\}\right]_{,pp}$$

$$= \left[b_l R_{,j} \left(-2\varepsilon_{klj}\,\delta_{nm} + \varepsilon_{nlj}\,\delta_{km} + \varepsilon_{mlj}\,\delta_{kn}\right)\right]_{,pp}.$$

o (2.16) becomes

$$e_{mn}(\underline{r}') = \tfrac{1}{8\pi} \oint_{\Gamma} b_l \left[R_{,jpp}\left(-\varepsilon_{klj}\,\delta_{nm} + \frac{\varepsilon_{nlj}}{2}\,\delta_{km} + \frac{\varepsilon_{mlj}}{2}\,\delta_{kn}\right) - 2\frac{(\lambda+\mu)}{\lambda+2\mu}\,\varepsilon_{kpl}\, R_{,mpn}\right] dl_k,$$

and

$$e_{mm}(\underline{r}') = \frac{1}{8\pi} \oint_\Gamma b_\ell \left[\frac{-2\mu}{\lambda+2\mu}\right] \varepsilon_{k\ell j} R_{,ipp}\, d\ell_k \quad .$$

Since

$$\sigma_{mn}(\underline{r}') = \lambda\, e_{kk}(\underline{r}')\, \delta_{mn} + 2\mu\, e_{mn}(\underline{r}') \quad ,$$

then

$$\sigma_{mn}(\underline{r}') = \frac{-2\mu\lambda}{8\pi(\lambda+2\mu)} \oint_\Gamma \delta_{mn} b_\ell \varepsilon_{k\ell j} R_{,ipp}\, d\ell_k$$

$$+\frac{2\mu}{8\pi} \oint_\Gamma b_\ell \Big[R_{,jpp}\Big(-\varepsilon_{k\ell j}\delta_{nm} + \frac{\varepsilon_{n\ell j}}{2}\delta_{km}$$

$$+\frac{\varepsilon_{m\ell j}}{2}\delta_{kn}\Big) - \frac{2(\lambda+\mu)}{\lambda+2\mu}\varepsilon_{kp\ell} R_{,mpn}\Big] d\ell_k$$

$$= \frac{\mu b_\ell}{4\pi} \oint_\Gamma \Big\{ -\delta_{mn} \varepsilon_{k\ell j} R_{,ipp}\left(\frac{\lambda}{\lambda+2\mu}+1\right) d\ell_k$$

$$+ \frac{1}{2} R_{,ipp}\left(\varepsilon_{n\ell j}\, d\ell_m + \varepsilon_{m\ell j}\, d\ell_n\right)$$

$$- \frac{2(\lambda+\mu)}{\lambda+2\mu} \varepsilon_{kp\ell} R_{,mnp}\, d\ell_k \Big\}$$

$$= \frac{\mu b_\ell}{4\pi} \oint_\Gamma \Big\{ \frac{-2(\lambda+\mu)}{\lambda+2\mu}\left(\delta_{mn} \varepsilon_{k\ell j} R_{,jpp} + \varepsilon_{kj\ell} R_{,mnj}\right) d\ell_k$$

$$+ \frac{1}{2} R_{,ipp}\left(\varepsilon_{n\ell j}\, d\ell_m + \varepsilon_{m\ell j}\, d\ell_n\right) \Big\} \quad ,$$

or

$$\sigma_{mn}(\underline{r}') = \frac{-\mu b_\ell}{4\pi} \oint_\Gamma \Big\{ \tfrac{1}{2} R_{,ipp} \left(\varepsilon_{nj\ell}\, dl_m + \varepsilon_{mj\ell}\, dl_n \right)$$

$$+ \frac{1}{1-\nu} \varepsilon_{kj\ell} \left(R_{,mnj} - \delta_{mn} R_{,jpp} \right) dl_k \Big\}. \tag{2.17}$$

This expression for the stress field is known as the Peach Koehler formula.[5]

3. Energy And Forces of Interaction[6],[7]

It would be possible at this stage, with the addition of a few assumptions, to investigate some boundary value problems. However, a deeper insight may be gained if we were to follow historically the development of the dislocation considerations that lead to the solution of certain boundary value problems.

In this chapter we will investigate the effect of one dislocation on another, or more specifically, the forces of interaction of one dislocation on another.

Consider first an infinite body B with boundary ∂B. This body is in equilibrium under body forces f_i and surface tractions t_i. Let Σ_R be a sphere of large radius R, and B_R be that part of the body contained within Σ_R. The strain energy density for B_R is defined as:

$$
\begin{aligned}
U &= \tfrac{1}{2}\int_{B_R} C_{ijkl}\, e_{ij}\, e_{kl}\, dV \\
&= \tfrac{1}{2}\int_{B_R} e_{ij}\,\sigma_{ij}\, dV = \tfrac{1}{2}\int_{B} u_{i,j}\,\sigma_{ij}\, dV \\
&= \tfrac{1}{2}\int_{B_R} \left\{ (u_i\,\sigma_{ij})_{,j} - u_i\,\sigma_{ij,j} \right\} dV \qquad (3.1)
\end{aligned}
$$

If we are to use the divergence theorem, there would be surface integrals over the whole or part of Σ_R and an integral over that part of ∂B_R of the boundary of the body contained within Σ_R. Taking the limit as $R \to \infty$ gives, if the integral over Σ_R goes to zero and the integrals over B_R and

∂B_R converge,

$$U = \tfrac{1}{2}\int_{\partial B} u_i \sigma_{ij} n_j \, dS - \tfrac{1}{2}\int_{B} u_i \sigma_{ij,j} \, dV \,..$$

However, since $t_i = \sigma_{ij} n_j$ and $\sigma_{ij,j} = -f_i$,

$$U = \tfrac{1}{2}\int_{\partial B} t_i u_i \, dS + \tfrac{1}{2}\int_{B} f_i u_i \, dV \qquad (3.2)$$

This result will hold if, as $R \to \infty$, $\sigma_{ij} = O(R^{-3/2-\alpha})$, $u_i = O(R^{-1/2-\alpha})$, and $f_i = O(R^{-5/2-\alpha})$, for some $\alpha > 0$.

We now calculate the total energy of a system with tractions applied to its outer surface and in which a dislocation is created.

Let a body B have an external boundary S. On this external boundary place tractions $\underline{t}^{(e)}$, and call the corresponding displacement field $\underline{u}^{(e)}$. In this body B make a cut and call the two faces of the cut A_+ and A_-. With the application of equal and opposite tractions $\underline{t}_+^{(d)} = -\underline{t}_-^{(d)}$, on the two faces, A_+ and A_- are deformed relative to one another by an amount $\underline{u}_+^{(d)} - \underline{u}_-^{(d)} = \Delta\underline{u}^{(d)}$. If we let ΔU^{ext} be the work done on B by the external agent on S in causing the deformation field $\underline{u}^{(e)}$ and during the creation of the dislocation, then the total energy of the system is

$$U_{tot} = U - \Delta U^{ext} , \qquad (3.3)$$

where

$$\Delta U^{ext} = \tfrac{1}{2}\int_S \underline{t}^{(e)} \cdot \underline{u}^{(e)} \, dS + \int_S \underline{t}^{(e)} \cdot \underline{u}^{(d)} \, dS , \qquad (3.4)$$

and

$$\mathcal{U} = \tfrac{1}{2}\int_B C_{ijkl}\left(e_{ij}^{(e)} + e_{ij}^{(d)}\right)\left(e_{kl}^{(e)} + e_{kl}^{(d)}\right) dV$$

U is the strain energy of the body. Then,

$$\mathcal{U} = \tfrac{1}{2}\int_B \left(e_{ij}^{(e)}\sigma_{ij}^{(e)} + e_{ij}^{(d)}\sigma_{ij}^{(d)} + 2e_{ij}^{(e)}\sigma_{ij}^{(d)}\right) dV$$

$$= \tfrac{1}{2}\int_B \Big\{\left(u_i^{(e)}\sigma_{ij}^{(e)}\right)_{,j} + \left(u_i^{(d)}\sigma_{ij}^{(d)}\right)_{,j} + 2\left(u_i^{(e)}\sigma_{ij}^{(d)}\right)_{,j}$$

$$- u_i^{(e)}\sigma_{ij,j}^{(e)} - u_i^{(d)}\sigma_{ij,j}^{(d)} - 2u_i^{(e)}\sigma_{ij,j}^{(d)}\Big\} dV.$$

In the absence of body forces, we finally have that

$$\mathcal{U} = \tfrac{1}{2}\int_{\partial B}\Big(u_i^{(e)}\sigma_{ij}^{(e)} + u_i^{(d)}\sigma_{ij}^{(d)}$$

$$+ 2u_i^{(e)}\sigma_{ij}^{(d)}\Big) n_j\, dS, \qquad (3.5)$$

where $\partial B = S + A_+ + A_-$.

Before substituting (3.4) and (3.5) into (3.3), we may simplify equation (3.5) further. The first term in the integral becomes:

$$\tfrac{1}{2}\int_{\partial B} u_i^{(e)} t_i^{(e)}\, dS = \tfrac{1}{2}\int_S \underline{u}^{(e)}\cdot\underline{t}^{(e)}\, dS \qquad (3.6)$$

since $\underline{u}^{(e)}$ and $\underline{\sigma}^{(e)}$ are continuous across $A\pm$. The second term in the integral is

$$\frac{1}{2}\int_{\partial B} u_i^{(d)} \sigma_{ij}^{(d)} n_j \, dS = \frac{1}{2}\int_{A_+ + A_-} u_i^{(d)} t_i^{(d)} \, dS$$

$$= \frac{1}{2}\left[\int_{A_+} \underline{u}_+^{(d)} \cdot \underline{t}_+^{(d)} dS + \int_{A_-} \underline{u}_-^{(d} \cdot \underline{t}_-^{(d)} dS \right]$$

$$= \frac{1}{2}\left[\int_{A_+} \underline{u}_+^{(d)} \cdot \underline{t}_+^{(d)} dS - \int_{A_+} \underline{u}_-^{(d)} \cdot \underline{t}_+^{(d)} dS \right]$$

$$= \frac{1}{2}\int_{A_+} \underline{t}_+^{(d)} \cdot \Delta u^{(d)} \, dS \qquad , \qquad (3.7)$$

since $\underline{t}_+^{(d)} = -\underline{t}_-^{(d)}$ and $\underline{u}_+^{(d)} - \underline{u}_-^{(d)} = \Delta \underline{u}^{(d)}$.

Also since $\underline{t}^{(d)} = 0$ on S, and $\underline{u}^{(e)}$ is continuous across $A\pm$,

$$2\int_{\partial B} u_i^{(e)} \sigma_{ij}^{(d)} n_j \, dS = 0 \quad ,$$

$$\text{or} \quad \int_{A_+ + A_-} \underline{u}^{(e)} \cdot \underline{t}^{(d)} = 0 \; . \qquad (3.8)$$

It is evident from the Betti reciprocal theorem and from (3.8) that

$$\int_{\partial B} \underline{u}^{(e)} \cdot \underline{t}^{(d)} dS = \int_{\partial B} \underline{u}^{(d)} \cdot \underline{t}^{(e)} dS = 0 \, . \qquad (3.9)$$

Then from the second of (3.9)

$$\int_{S} \underline{u}^{(d)} \cdot \underline{t}^{(e)} dS + \int_{A_+ + A_-} \underline{u}^{(d)} \cdot t^{(e)} dS = 0 \quad ,$$

$$\text{or} \quad \int_S \underline{u}^{(d)} \cdot \underline{t}^{(e)} dS = - \int_{A_+ + A_-} \underline{u}^{(d)} \cdot \underline{t}^{(e)} dS$$

$$= - \int_{A_+} \underline{t}^{(e)} \cdot \Delta \underline{u}^{(d)} dS \,. \qquad (3.10)$$

With the help of (3.4), (3.6), (3.7), and (3.8) equation (3.3) becomes

$$U_{tot} = \tfrac{1}{2} \int_S \underline{u}^{(e)} \cdot \underline{t}^{(e)} dS + \tfrac{1}{2} \int_{A_+} \underline{t}_+^{(d)} \cdot \Delta \underline{u}^{(d)} dS$$
$$- \tfrac{1}{2} \int_S \underline{t}^{(e)} \cdot \underline{u}^{(e)} dS - \int_S \underline{t}^{(e)} \cdot \underline{u}^{(d)} dS \,.$$

Finally, using (3.10) we have that

$$U_{tot} = \tfrac{1}{2} \int_{A_+} \underline{t}_+^{(d)} \cdot \Delta \underline{u}^{(d)} dS + \int_{A_+} \underline{t}^{(e)} \cdot \Delta \underline{u}^{(d)} dS \qquad (3.11)$$

Having found the expression for the total energy, given by (3.11), we may now calculate an expression for the force on a dislocation line as the line moves a small distance.[8] This force is just minus the rate of change of the total energy of the system with respect to the dislocation position.

Let us consider again an infinite body B with a cut A. The surface A is bounded by the curve Γ, and the faces of the cut are displaced relative to one another by a constant amount $\Delta \underline{u}^{(d)} = \underline{b}$.

From (3.11) when Γ moves through an amount $\underline{\delta r}$, the change in U_{tot} is:

$$\delta U_{tot} = \delta U^{(d)} + \delta \left[\underline{b} \cdot \int_{A_+} \underline{t}^{(e)} dS \right] , \qquad (3.12)$$

where

$$U^{(d)} = \frac{1}{2} \int_{A_+} \underline{t}_+^{(d)} \cdot \Delta \underline{u}^{(d)} dS = \frac{1}{2} \underline{b} \cdot \int_{A_+} \underline{t}_+^{(d)} dS .$$

However for an infinite medium, the dislocation energy $U^{(d)}$ is independent of the dislocation position. Hence $\delta U^{(d)} = 0$. Now as Γ moves through $\underline{\delta r}$, it sweeps out a new cut δA_+, so (3.12) becomes:

$$\delta U_{tot} = \underline{b} \cdot \int_{\delta A_+} \underline{t}^{(e)} dS \qquad . \qquad (3.13)$$

Let $\underline{\sigma}^{(e)}$ again be an external stress tensor. Let $\underline{n}$ be the unit normal to δA_+ ($\underline{n}$ is pointing from A_+ to A_-). Then on δA_+, $\underline{t}^{(e)} = \underline{\sigma}^{(e)} \cdot \underline{n}$. Let dl be an element of arc on Γ and $\underline{\xi}$ be a unit vector along Γ. See Figure (3.1). Then the

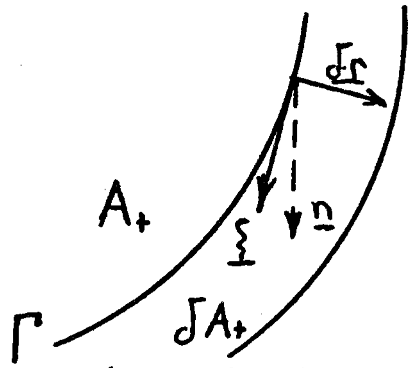


Figure (3.1)

element of area, $ds = dl\,|\underline{\delta r}|$, and $(-\underline{\xi}\times\underline{\delta r})/|\underline{\delta r}| = \underline{n}$.
So on δA_+, $\underline{t}^{(e)} = \underline{\sigma}^{(e)}\cdot\underline{n} = \sigma^{(e)}\cdot\dfrac{(\xi\times\underline{\delta r})}{|\underline{\delta r}|}$, and (3.13) becomes

$$\delta U_{tot} = -\underline{b}\cdot\int_{\delta A_+}\underline{\sigma}^{(e)}\cdot\frac{(\xi\times\underline{\delta r})}{|\underline{\delta r}|}\,dl\,|\underline{\delta r}|$$

$$= -\int_{\Gamma}\underline{b}\cdot\underline{\sigma}^{(e)}\cdot(\xi\times\underline{\delta r})\,dl\,,$$

or

$$\delta U_{tot} = -\int_{\Gamma}(\underline{b}\cdot\underline{\sigma}^{(e)}\times\underline{\xi})\cdot\underline{\delta r}\,dl \quad . \qquad (3.14)$$

We specialise now to the case where Γ is straight, and $\underline{\sigma}^{(e)}$ is constant along the length of Γ. Then $\underline{\delta r}$ is constant, and (3.14) becomes:

$$\delta U_{tot} = -(\underline{b}\cdot\underline{\sigma}^{(e)}\times\underline{\xi})\cdot\delta r\cdot L\,,$$

or

$$\delta U_{tot}/L = -(\underline{b}\cdot\underline{\sigma}^{(e)}\times\underline{\xi})\cdot\underline{\delta r} \quad . \qquad (3.15)$$

The force per unit length of Γ, written $\underline{F}/L$, on a dislocation line as stated before is minus the rate of change of energy with respect to the dislocation position, or

$$\delta U_{tot}/L = -(\underline{F}/L)\cdot\underline{\delta r} \quad . \qquad (3.16)$$

Comparing (3.16) with (3.15) gives

$$\underline{F}/L = \underline{b} \cdot \underline{\sigma}^{(e)} \times \underline{\xi} \quad . \tag{3.17}$$

Equations (3.11) and (3.17) may be used to calculate the effects of two dislocations on one another. A system of two dislocations can be pictured in the following way.

Make two cuts, $A_{\pm}^{(1)}$ and $A_{\pm}^{(2)}$ displacing the faces so that on $A^{(1)}$: $\Delta\underline{u}^{(1)} = \underline{u}_+^{(1)} - \underline{u}_-^{(1)}$, and $\Delta\underline{u}^{(2)} = 0$. On $A^{(2)}$: $\Delta\underline{u}^{(2)} = \underline{u}_+^{(2)} - \underline{u}_-^{(2)}$, and $\Delta\underline{u}^{(1)} = 0$. Also on both $A^{(1)}$ and $A^{(2)}$, $\underline{t}_+^{(1)} = -\underline{t}_-^{(1)}$, and $\underline{t}_+^{(2)} = -\underline{t}_-^{(2)}$. The boundary ∂B of the infinite medium consists of $S + A^{(1)} + A^{(2)}$. Then equation (3.5) becomes

$$U = \tfrac{1}{2}\int_{\partial B} \left(u_i^{(1)} \sigma_{ij}^{(1)} n_j + u_i^{(2)} \sigma_{ij}^{(2)} n_j + 2 u_i^{(1)} \sigma_{ij}^{(2)} n_j \right) dS.$$

From the above equation an expression for the total energy is found which is similar to equation (3.11):

$$U_{tot} = \tfrac{1}{2}\int_{A_+^{(1)}} \underline{t}^{(1)} \cdot \Delta\underline{u}^{(1)}\, dS + \int_{A_+^{(2)}} \underline{t}^{(1)} \cdot \Delta\underline{u}^{(2)}\, dS$$

$$= \tfrac{1}{2}\int_{A_+^{(1)}} \underline{t}^{(1)} \cdot \Delta\underline{u}^{(1)}\, dS + \int_{A_+^{(1)}} \underline{t}^{(2)} \cdot \Delta\underline{u}^{(1)}\, dS. \tag{3.18}$$

From (3.18) it can be seen that the effect on one dislocation by another is the same as if it were acted on by an external stress. For example, if dislocation (2) were fixed, the second term in equation (3.18) gives the effect on dislocation (1). Then the force on one dislocation by another is given by equation (3.17) with $\underset{\sim}{\sigma}^{(e)}$ being the stress field due to both external tractions and any other dislocations.[9]

4. Head-Louat Approximation

Let us now examine a simple application of Burgers formula (2.12). In an infinite medium, beginning on the positive side of the origin, we make a cut parallel to the z-axis in the positive direction of the x axis. The upper surface of the cut is denoted by A_+ and the lower by A_-. All points on A_+ except those close to the origin are displaced a distance $\frac{1}{2}b$ in the negative z direction, and all points on A_-, except again those close to the origin are displaced a distance $\frac{1}{2}b$ in the positive z direction. See Figure (4.1).

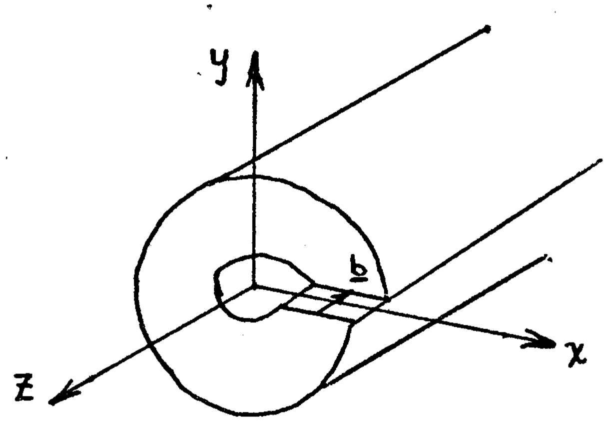


Figure (4.1)

This type of dislocation is called a screw dislocation, and has a Burgers vector of $-\frac{1}{2}b-\frac{1}{2}b = -b$ in the z direction. So $\underline{b} = (0,0,-b)$. Now the curve Γ, which encloses the cut will consist of (1) a straight line lying in the z-x plane, parallel to the z axis, running from $z=\infty$ to $z=-\infty$; and (2) a semicircle in the z-x plane, centered at the origin and having the large radius, a. The path along the semicircle will be taken in the clockwise direction. See Figure (4.2). Notice that the positive unit vectors are pointing from A_+ to A_-, hence the curve Γ is right handed with respect to these unit vectors.

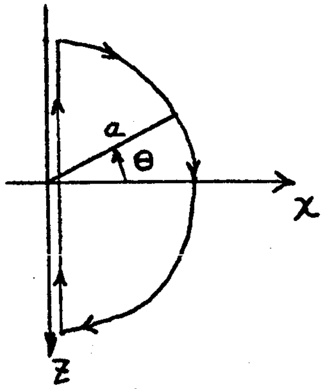


Figure (4.2)

Recalling Burgers formula (2.12) for the displacement field, we have, for the displacement in the z direction:

$$u_3(\underline{r}') = \frac{b\,\Omega(\underline{r}')}{4\pi} - b\int_\Gamma \varepsilon_{k33} R_{,pp}\, dl_k$$

$$-\frac{b(\lambda+\mu)}{4\pi(\lambda+2\mu)} \int_\Gamma \varepsilon_{kp3} R_{,3p}\, dl_k$$

$$= \frac{b\,\Omega(\underline{r}')}{4\pi} - \frac{b(\lambda+\mu)}{(\lambda+2\mu)} \int_C \frac{x_2'(x_3-x_3')}{R_o^3}\, dx_1 \quad ,$$

where C is the semicircle of radius a as shown in Figure (4.2), and $R_o = \left[(x_1-x_1')^2 + x_2'^2 + (x_3-x_3')^2 \right]^{1/2}$. We want to show that the integral over C vanishes as the radius a goes to infinity. Now on C we have $x_1 = a\cos\theta$, $dx_1 = -a\sin\theta d\theta$, and $x_3 = a\sin\theta$, so

$$\int_C \frac{x_2'(x_3-x_3')}{R_o^3}\, dx_1 = -x_2' a \int_{\pi}^{-\pi} \frac{(a\sin\alpha - x_3')}{R_o^3} \sin\alpha\, d\alpha$$

$$= \frac{-x_2'}{a}\int_{\pi}^{-\pi} \frac{\left(\sin^2\alpha - \frac{x_3'}{a}\sin\alpha\right)\, d\alpha}{\left[1 + \frac{c^2}{a^2} \cdot \frac{2}{a}\left(x_1'\cos\alpha + x_3'\sin\alpha\right)\right]^{3/2}} .$$

Now for large a, the integrand is finite, and the factor 1/a outside the integral causes the whole term to tend to zero. So we have that

$$U_3(\underline{r}') = + \frac{b\,\Omega(\underline{r}')}{4\pi} .$$

Now $\Omega(\underline{r}')$ is the solid angle subtended by Γ at $\underline{r}'$. If θ is the angle at $\underline{r}'$ between the vector $\underline{r}'$ and the z-x plane, then $\theta = \tan^{-1} y'/x'$, and $\Omega(\underline{r}') = 2\tan^{-1} y'/x'$. So

$$U_3(\underline{r}') = + \frac{b\tan^{-1} y'/x'}{2\pi} = \frac{b\,\theta}{2\pi} ,$$

and this is the only non zero component of the displacement, since u_1 and u_2 involve only vanishing integrals around C as above. Then the non vanishing strains are (dropping the primes)

$$e_{zx}(\underline{r}) = \tfrac{1}{2}\frac{\partial U_3}{\partial x_1} = \frac{-b}{4\pi}\,\frac{y}{x^2+y^2} ,$$

$$e_{zy}(\underline{r}) = \tfrac{1}{2}\frac{\partial U_3}{\partial x_2} = \frac{b}{4\pi}\,\frac{x}{x^2+y^2} .$$

And the stress field is

$$\left.\begin{aligned} \sigma_{zx} &= 2\mu e_{zx} = \frac{\mu b}{2\pi} \frac{y}{x^2+y^2} \\ \sigma_{zy} &= 2\mu e_{zy} = \frac{\mu b}{2\pi} \frac{x}{x^2+y^2} \end{aligned}\right\} . \tag{4.1}$$

Equations (4.1) then give the stress field caused by one screw dislocation. We now consider a system containing an array of dislocations.

Let there be a cut in the x-z plane as described in the first part of this chapter. Now instead of displacing the whole of the two faces of the cut by an amount $\underline{b}$ relative to eachother, let there be an array of displacements in the z direction, each displacement having a magnitude b_i in the z direction, and each having a position on the x axis as x_i. Let there be n of these displacements. Each of these dislocations produces a force of repulsion on all the other dislocations. (see Chapter 3)

We shall now calculate the force (per unit length) on the k^{th} dislocation located at x_k due to all the other dislocations. Let $\sigma_{ij}^{(k)}$ be the stress field due to all the other dislocations. The force on the k^{th} dislocation, $\underline{F}^{(k)}/L$, due to $\sigma_{ij}^{(k)}$ is given by equation (3.17) as

$$\underline{F}^{(k)}/L = \underline{b}^{(k)} \cdot \underline{\sigma}^{(k)} \times \underline{\xi} \; ,$$

where $\underline{b}^{(k)} = (0,0,-b_k)$, $\underline{\xi} = (0,0,-1)$.

Then

$$\underline{F}^{(k)}/L = -b_k\left(\sigma_{xz}^{(k)}, \sigma_{yz}^{(k)} \sigma_{zx}^{(k)}\right) \times (0, 0, -1)$$

$$= \left(b_k \sigma_{yz}^{(k)}, -b_k \sigma_{xz}^{(k)}, 0\right) \quad .$$

So the force in the x direction on the k^{th} dislocation is $b_k \sigma_{yz}^{(k)}$. But from (4.1)

$$\sigma_{yz}^{(k)} = \frac{\mu}{2\pi} \sum_{\ell \neq k} \frac{b_\ell (x_k - x_\ell)}{(x_k - x_\ell)^2}$$

$$= \frac{\mu}{2\pi} \sum_{\ell \neq k} \frac{b_\ell}{x_k - x_\ell} \quad , \qquad (4.2)$$

which is just the sum of all the other stress fields due to all the other dislocations.

If $b_k P_k$ represents any additional forces on the k^{th} dislocation (such as external forces), then for equilibrium:

$$b_k \sum_{\ell \neq k} \frac{\mu}{2\pi} \frac{b_\ell}{x_k - x_\ell} + b_k P_k = 0$$

$$k = 1, 2, \ldots n$$

or

$$\frac{\mu}{2\pi} \sum_{\ell \neq k} \frac{b_\ell}{x_k - x_\ell} + P_k = 0 \quad . \qquad (4.3)$$

The problem of finding the equilibrium positions, x_k, for an array of dislocations under the influence of certain prescribed external shear stresses taking into account the mutual interactions of the dislocations in the array was considered by Eshelby, Frank, and Nabarro.[10] Their method of solution of equation (4.3) is exact, However, Head and Louat[1] proposed an approximate method for the solution of the problem of Eshelby, Frank, and Nabarro. This method is of special interest since it has many other useful applications as we shall later see.

The approximation of Head and Louat is to replace the distribution of finite dislocations by a continuous distribution of infinitesimal dislocations. Then for a small distance along the x axis there is a displacement of $bf(x)dx$ in the z direction. The shear stress given by equation (4.2) for example can be written as an integral rather than as a sum. The only difficulty in making this approximation is when x_k is in the neighbourhood of x_1. To solve this, we exclude the interval between $x-\epsilon$ and $x+\epsilon$ and then take the limit as $\epsilon \rightarrow 0$. So the shear stress at some point x due to all the dislocations except those in the neighbourhood of x is:

$$\lim_{\epsilon \rightarrow 0} \int_{-\infty}^{x-\epsilon} \frac{\mu b}{2\pi} \frac{f(x')\,dx'}{x-x'} + \lim_{\epsilon \rightarrow 0} \int_{x+\epsilon}^{\infty} \frac{\mu b}{2\pi} \frac{f(x')\,dx'}{x-x'} .$$

And this is just a Cauchy Principal Value integral,

$$\sigma_{yz}(x) = P\int_{-\infty}^{\infty} \frac{\mu b}{2\pi} \frac{f(x')\,dx'}{x-x'} . \tag{4.4}$$

If the external shear stress at the point x is given by P(x), then for equilibrium

$$P\int_{-\infty}^{\infty}\frac{\mu b}{2\pi}\,\frac{f(x')\,dx'}{x-x'} + P(x) = 0 \tag{4.5}$$

or

$$P\int_{-\infty}^{\infty}\frac{f(x')\,dx'}{x-x'} \equiv F(x)\ , \tag{4.6}$$

where $F(x) \equiv \dfrac{-2\pi\,P(x)}{\mu b}$

Equation (4.6) is then the approximate equivalent of equation (4.3). It is from (4.6) that an integral equation for the dislocation density, f(x), may be obtained. The solution of equation (4.6) is known to be, for finite limits of integration, say (a,b), [11], [12]

$$f(x) = \frac{1}{\sqrt{(x-a)(x-b)}}\cdot\left\{P\int_a^b -\frac{1}{\pi^2}\,\frac{\sqrt{(x'-a)(x'-b)}}{x'-x}\,F(x')\,dx' + D\right\}. \tag{4.7}$$

The constant, D, is often determined by the conditions on f(x) at the end points. If f(x) is to be bounded at x = a, we must have

$$\frac{-1}{\pi^2}\,P\int_a^b \frac{\sqrt{(x'-a)(x'-b)}}{x'-a}\,F(x')\,dx' + D = 0\ .$$

The above equation determines the constant, D, so that (4.7) becomes

$$f(x) = \frac{1}{\sqrt{(x-a)(x-b)}} \left\{ \frac{-1}{\pi^2} P \int_a^b \frac{\sqrt{(x'-a)(x'-b)}}{x'-x} F(x')dx' \right.$$

$$\left. + \frac{1}{\pi^2} P \int_a^b \frac{\sqrt{(x'-a)(x'-b)}}{x'-a} F(x')\,dx' \right\}$$

$$= \frac{1}{\sqrt{(x-a)(x-b)}} \left\{ \frac{1}{\pi^2} P \int_a^b \left(\frac{\sqrt{x'-b}}{x'-a} \right. \right.$$

$$\left. \left. - \frac{\sqrt{(x'-a)(x'-b)}}{x'-x} \right) F(x')dx' \right\}$$

$$= - \frac{\sqrt{x-a}}{\sqrt{x-b}} \cdot \frac{1}{\pi^2} P \int_a^b \sqrt{\frac{x'-b}{x'-a}} \, \frac{F(x')\,dx'}{x'-x} \; . \qquad (4.8)$$

If $f(x)$ is to be bounded at $x = b$, then

$$f(x) = - \frac{\sqrt{x-b}}{\sqrt{x-a}} \, \frac{1}{\pi^2} P \int_a^b \frac{\sqrt{x'-a}}{\sqrt{x'-b}} \, \frac{F(x')dx'}{x'-x} \; . \qquad (4.9)$$

If $f(x)$ is to be bounded at both $x = a$ and $x = b$, then from (4.8) we have first that

$$\int_a^b \sqrt{\frac{x'-b}{x'-a}} \, \frac{F(x')\,dx'}{x'-b} = 0 \, ,$$

or $$\int_a^b \frac{F(x')dx'}{\sqrt{(x'-a)(x'-b)}} = 0 \quad .$$

Then (4.8) may be written as:

$$f(x) = -\sqrt{\frac{x-a}{x-b}}\,\frac{1}{\pi^2}\left\{P\int_a^b \sqrt{\frac{x'-b}{x'-a}}\;\frac{F(x')dx'}{x'-x} - \int_a^b \frac{F(x')dx'}{\sqrt{(x'-a)(x'-b)}}\right\}$$

$$= -\frac{1}{\pi^2}\sqrt{(x-a)(x-b)}\;P\int_a^b \frac{F(x')\;dx'}{\sqrt{(x'-a)(x'-b)}(x'-x)} \quad .$$

(4.10)

It is now possible to sumarize some particular examples considered by Head and Louat.[1]

(i) Let there be n positive dislocations under a shear stress $P(x)=-Cx$. Assume $f(x)$ to be symmetrical about $x=0$, and is zero at $x=\pm a$, where a depends upon n. Since $f(x)$ is bounded at both end points, the solution of equation (4.6) is given by equation (4.10), with $F(x)=\dfrac{2\pi Cx}{\mu b}$:

$$f(x) = -\frac{1}{\pi^2}\sqrt{a^2-x^2}\;P\int_{-a}^{a} \frac{2\pi Cx'dx'}{\mu b\sqrt{a^2-x'^2}\;(x'-x)} \quad . \tag{4.11}$$

To evaluate the integral in (4.11) we consider the integral,

$$\oint \frac{z\,dz}{(a^2-z^2)^{1/2}(z-x)} \quad ,$$

over the contour shown in Figure (4.3). A branch cut is made between the branch points z=-a and z=a. Because of this branch cut, the term $(a^2-z^2)^{1/2}$ will be positive above the branch and negative below the branch. The large circle DE is centered at the origin, and has radius R.

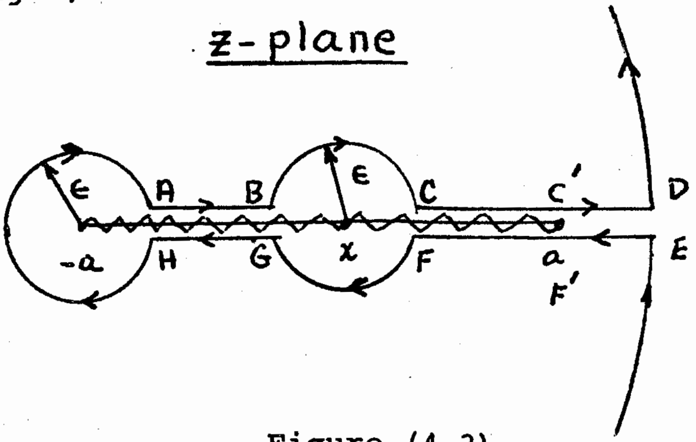


Figure (4.3)

Then since there are no poles enclosed by the contour,

$$\oint \frac{z\,dz}{(a^2-z^2)(z-x)} = 0$$

Now the integral evaluated along AB and CC' is just the principal value integral

$$P\int_{-a}^{a} \frac{x'dx'}{\sqrt{a^2-x'^2}\;(x'-x)} \quad .$$

Integrating along FF' and GH gives

$$-P\int_{a}^{-a}\frac{x'dx'}{\sqrt{a^2-x'^2}\,(x'-x)} = P\int_{-a}^{a}\frac{x'dx'}{\sqrt{a^2-x'^2}\,(x'-x)}\,.$$

The integrals along C'D and EF' cancel each other. Also, the integrals along the arcs BC and FG cancel each other, since on these arcs $z = \epsilon e^{i\theta} + x$, and therefore

$$\int_{BC} = \int_{\pi}^{0}\frac{(\epsilon e^{i\theta}+x)\,\epsilon i e^{i\theta}\,d\theta}{\sqrt{a^2-(\epsilon e^{i\theta}+x)^2}\;\epsilon e^{i\theta}} \longrightarrow -\frac{i\pi x}{\sqrt{a^2-x^2}} \text{ as } \epsilon \to 0,$$

$$\int_{FG} = -\int_{0}^{-\pi}\frac{(\epsilon e^{i\theta}+x)\,\epsilon i e^{i\theta}\,d\theta}{\sqrt{a^2-(\epsilon e^{i\theta}+x)^2}\;\epsilon e^{i\theta}} \longrightarrow \frac{i\pi x}{\sqrt{a^2-x^2}} \text{ as } \epsilon \to 0.$$

There is no contribution from the small circle HA as $\epsilon \to 0$, since

$$\int_{HA} = \int_{2\pi}^{0}\frac{(\epsilon e^{i\theta}-a)\,\epsilon i e^{i\theta}\,d\theta}{\sqrt{(a-\epsilon e^{i\theta}+a)(a+\epsilon e^{i\theta}-a)}\,(\epsilon e^{i\theta}-a+x)}$$

$$= \int_{2\pi}^{0}\frac{(\epsilon e^{i\theta}-a)\sqrt{\epsilon e^{i\theta}}\; i\,d\theta}{\sqrt{2a-\epsilon e^{i\theta}}\;(\epsilon e^{i\theta}-a+x)} \to 0 \text{ as } \epsilon \to 0.$$

Finally, the integral around DE is

$$\int_{DE} = \int_{0}^{2\pi}\frac{Re^{i\theta}\,Ri e^{i\theta}\,d\theta}{\sqrt{a^2-R^2e^{2i\theta}}\;(Re^{i\theta}-x)}$$

$$= \int_0^{2\pi} \frac{i e^{2i\theta} d\theta}{\sqrt{\frac{a^2}{R^2} - e^{2i\theta}} \left(e^{i\theta} - \frac{x}{R}\right)} ,$$

and as $R \to \infty$ this becomes

$$\int_0^{2\pi} \frac{i e^{2i\theta} d\theta}{i e^{2i\theta}} = 2\pi .$$

Combining these results, we have that

$$\oint \frac{z\,dz}{\sqrt{a^2 - z^2}\,(z - x)} = 0$$

$$= 2P \int_{-a}^{a} \frac{x'dx'}{\sqrt{a^2 - x'^2}\,(x' - x)} + 2\pi ,$$

or

$$P \int_{-a}^{a} \frac{x'dx'}{\sqrt{a^2 - x'^2}\,(x' - x)} = -\pi .$$

Then equation (4.11) becomes

$$f(x) = -\frac{1}{\pi^2} \sqrt{a^2 - x^2} \cdot \frac{2\pi C}{\mu b} (-\pi)$$

$$= 2 \sqrt{a^2 - x^2} \frac{C}{\mu b} .$$

Since f(x) is the density of dislocations between x and x+dx the total number of dislocations, n, is equal to the integral of this density over the whole interval [-a,a]:

$$\int_{-a}^{a} f(x)dx = n \quad ,$$

or

$$\int_{-a}^{a} \frac{2C}{\mu b} \sqrt{a^2-x^2}\, dx = n \quad ,$$

which gives

$$\frac{2C}{\mu b} \cdot \frac{1}{2} a^2 = n \quad .$$

Then we have the relation between the number of dislocations, n, and the distance from the origin of the last dislocation in the pileup:

$$a = \sqrt{\frac{\mu b n}{C}} \quad .$$

(ii) Let there be n positive dislocations between blocks at $x=\pm a$, with no applied shear stress. Since $P(x)=0$, $F(x)=0$. Requiring $f(x)$ to be bounded at either or both $x=\pm a$ would give a trivial solution. The only non zero solution arises if $f(x)$ is unbounded at both $x=\pm a$. Then from equation (4.7),

$$f(x) = \frac{1}{\sqrt{a^2-x^2}} D \quad ,$$

where D is an arbitrary constant.

However,

$$\int_{-a}^{a} f(x)\, dx = n ,$$

and this gives

$$D = n/\pi \ .$$

Then

$$f(x) = n/\pi \cdot \frac{1}{\sqrt{a^2 - x^2}} \ .$$

(iii) Let there be n positive dislocations between unit positive dislocations locked at $x = \pm a$. These n dislocations are in equilibrium under their own mutual repulsions and the repulsion of the two locked dislocations at $x = \pm a$. We can treat the shear stress of the two locked dislocations as an external stress, so

$$P(x) = \frac{\mu b}{2\pi}\left[\frac{1}{x-a} + \frac{1}{x+a}\right] ,$$

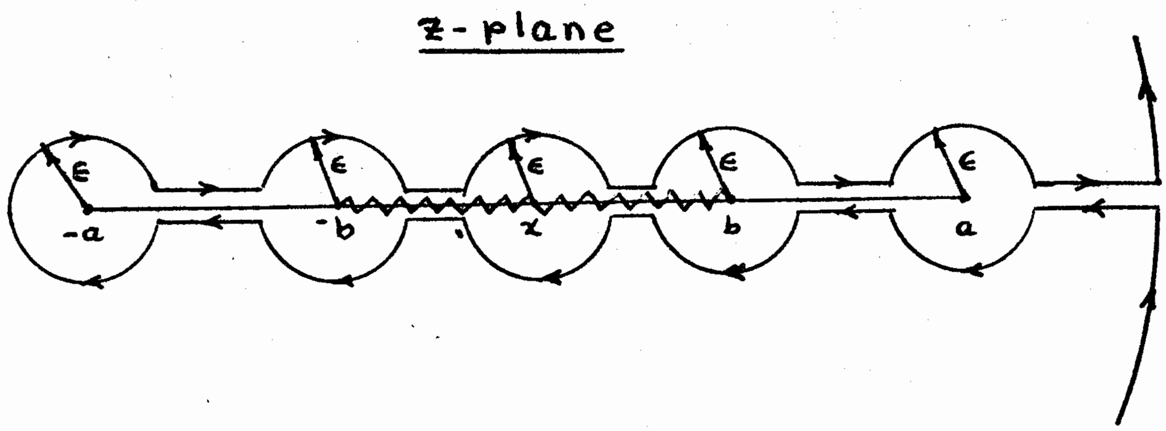
then

$$F(x) = -\left[\frac{1}{x-a} + \frac{1}{x+a}\right]$$

Now somewhere close to $x = \pm a$, because of the repulsion of the two fixed dislocations, $f(x) = 0$. Let $f(x) = 0$ at points $x = \pm b$, where $b < a$. The solution from (4.10) is

$$f(x) = + \frac{1}{\pi^2}\sqrt{b^2 - x^2}\; P\int_{-b}^{b} \frac{\left[\frac{1}{x'-a} + \frac{1}{x'+a}\right] dx'}{\sqrt{b^2 - x'^2}\;\;(x'-x)}$$

$$= \frac{\sqrt{b^2 - x^2}}{\pi^2}\left\{ P\int_{-b}^{b} \frac{dx'}{\sqrt{b^2 - x'^2}\,(x'-a)(x'-x)} + P\int_{-b}^{b} \frac{dx'}{\sqrt{b^2 - x'^2}\,(x'+a)(x'-x)} \right\} .$$



We evaluate the integrals by considering integrals of the type

$$\oint \frac{dz}{(b^2-z^2)^{1/2}\,(z \pm a)(z-x)}$$

around the contour as shown in Figure (4.4). The first integral gives

$$P\int_{-b}^{b} \frac{dx'}{\sqrt{b^2-x'^2}\;(x'-a)(x'-x)} = \frac{\pi}{\sqrt{a^2-b^2}\;(x-a)} \quad .$$

The contributions here were from integrals around x=+a. The second integral becomes

$$P\int_{-b}^{b} \frac{dx'}{\sqrt{b^2-x'^2}\;(x'+a)(x'-x)} = -\frac{\pi}{\sqrt{a^2-b^2}\;(x+a)} \quad .$$

The contributions here were from integrals around x=−a. So we have for the dislocation density

$$f(x) = \frac{1}{\pi^2}\sqrt{b^2-x^2}\left[\frac{\pi}{\sqrt{a^2-b^2}\;(x-a)} - \frac{\pi}{\sqrt{a^2-b^2}\;(x+a)}\right] ,$$

or $$f(x) = \frac{\sqrt{b^2-x^2}}{\pi\sqrt{a^2-b^2}}\left[\frac{2a}{a^2-x^2}\right].$$

The distance b is determined from

$$\int_{-b}^{b} f(x)dx = n .$$

Evaluating the integral gives

$$\frac{2a}{\pi\sqrt{a^2-b^2}}\int_{-b}^{b}\frac{\sqrt{b^2-x^2}}{a^2-x^2}\,dx = n ,$$

so

$$b = a\sqrt{\frac{n(n+4)}{(n+2)}} .$$

(iv) Let there be n positive dislocations in the region $x>0$. These n dislocations are forced against a block at x=0 by a uniform stress $P(x)=-\sigma$. So $F(x)=\frac{2\pi}{\mu b}\sigma$. It is reasonable to assume that f(x) is unbounded at x=0 and decreases with x to a bounded value at some point x=a. The solution for this case is from (4.7), with the limits of integration from x=0 to x=a:

$$f(x) = \frac{1}{\sqrt{x}\sqrt{a-x}}\left\{-\frac{1}{\pi^2}\,P\int_0^a\frac{\sqrt{x'}\sqrt{a-x}}{x'-x}F(x')\,dx' + D\right\}.$$

For the solution to be bounded at x=a,

$$-\frac{1}{\pi^2}\, P\int_0^a \frac{\sqrt{x'}\,\sqrt{a-x'}}{x'-x}\, F(x')\, dx' + D = 0 \;.$$

Then,

$$f(x) = \frac{1}{\sqrt{x}\,\sqrt{a-x}}\;\frac{1}{\pi^2}\left\{ -P\int_0^a \frac{\sqrt{x'}\,\sqrt{a-x'}}{x'-x}\, F(x')\, dx' \right.$$

$$\left. + P\int_0^a \frac{\sqrt{x'}\,\sqrt{a-x'}}{x'-a}\, F(x')\, dx' \right.$$

$$= \frac{-1}{\pi^2}\,\frac{\sqrt{a-x}}{\sqrt{x}}\; P\int_0^a \frac{\sqrt{x'}\; F(x')\, dx'}{\sqrt{a-x'}\;(x'-x)} \;.$$

To evaluate the above integral we consider the integral

$$\oint \frac{z^{1/2}\, dz}{\sqrt{a-z}\;(z-x)} = 0$$

over the contour shown in Figure (4.5).

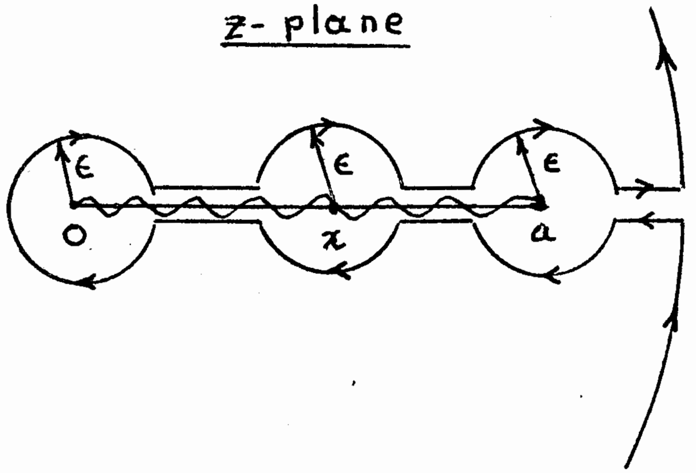


Figure (4.5)

There are no contributions around z=0, around z=a above or below the branch cut. The contributions around z=x above and below the cut cancel eachother, and the remaining integrals above and below the cut give twice the principal value integral:

$$2P\int_0^a \frac{(x')^{1/2}\,dx'}{\sqrt{a-x'}\;(x'-x)} \quad .$$

The integral over the large circle, $z=Re^{i\theta}$ gives

$$\int_0^{2\pi} \frac{R^{1/2}e^{i\theta/2}\,Rie^{i\theta}\,d\theta}{\sqrt{a-Re^{i\theta}}\;(Re^{i\theta}-x)} \xrightarrow[R\to\infty]{} \int_0^{2\pi} d\theta = 2\pi$$

So $$\oint \frac{z^{1/2}\,dz}{\sqrt{a-z}\;(z-x)} = 0 = 2P\int_0^a \frac{\sqrt{x'}\,dx'}{\sqrt{a-x'}\;(x'-x)} + 2\pi \; ,$$

or $$P\int_0^a \frac{\sqrt{x'}\,dx'}{\sqrt{a-x'}\;(x'-x)} = -\pi \; .$$

Then the expression for the dislocation density becomes

$$f(x) = -\frac{1}{\pi^2}\frac{\sqrt{a-x}}{\sqrt{x}}\left(\frac{2\pi\sigma}{\mu b}\right)(-\pi) = \frac{2\sigma}{\mu b}\frac{\sqrt{a-x}}{\sqrt{x}} \; .$$

Again the length a is determined by

$$\int_0^a f(x)\,dx = n \; ,$$

or $$\frac{2\sigma}{\mu b}\int_0^a \frac{\sqrt{a-x}}{\sqrt{x}}\,dx = \frac{2\sigma}{\mu b}\int_0^{a^{1/2}} 2\sqrt{a-u^2}\,du \; .$$

So the distance a is given by

$$a = \frac{\mu b n}{\sigma \pi}$$

(v) Let there be blocks at $x=\pm a$ and a dislocation source at $x=0$. A uniform stress $P(x)=\sigma$ causes the source to generate equal numbers of positive and negative dislocations which move off in opposite directions until held up by the blocks. The source continues to generate dislocations until the net stress at the source is reduced to zero. Since $P(x)=\sigma$, then $F(x)=-\,2\pi/\mu b \cdot \sigma$. The dislocation density, $f(x)$, will be unbounded at $x=\pm a$. The solution is given by equation (4.7):

$$f(x) = \frac{1}{\sqrt{a^2-x^2}}\left\{+\frac{1}{\pi^2}\, P\int_{-a}^{a} \frac{\sqrt{a^2-x'^2}}{x'-x}\cdot\frac{2\pi\sigma}{\mu b}\,dx' + D\right\}$$

The principal value integral turns out to be:

$$P\int_{-a}^{a} \frac{\sqrt{a^2-x'^2}}{x'-x}\,dx' = \pi x \quad .$$

So the dislocation density becomes

$$f(x) = \frac{1}{\sqrt{a^2-x^2}}\cdot\frac{1}{\pi^2}\,\pi x\cdot\frac{2\pi\sigma}{\mu b} + \frac{D}{\sqrt{a^2-x^2}}$$

$$f(x) = \frac{2\sigma}{\mu b}\,\frac{x}{\sqrt{a^2-x^2}} + \frac{D}{\sqrt{a^2-x^2}} \quad .$$

However since the dislocation source generates the same number of positive and negative dislocations, f(x) should be odd about x=0. This would require that D=0. So finally,

$$f(x) = \frac{2\sigma}{\mu b} \frac{x}{\sqrt{a^2-x^2}} .$$

Again the number of positive dislocations is given by

$$n = \int_0^a f(x)\,dx$$

$$= \frac{2\sigma}{\mu b} \int_0^a \frac{x}{\sqrt{a^2-x^2}}\,dx$$

$$= \frac{2\sigma a}{\mu b} .$$

5. Boundary Value Problems For a Half Space:

The continuum approximation of Head & Louat can be used to solve certain plane and anti-plane strain boundary value problems for a half space[10].

We consider an anti-plane strain problem first. Let there be a half space $y > 0$ with a shear stress, applied to the boundary $y = 0$, given by $\sigma_{zy}(x,0) = t(x)$. The only non zero displacement is $u_z(x,y)$. Let $u_z(x,0) \triangleq u(x)$.

To begin this problem, we first note that the straight screw dislocation shown in Fig. (4.1), with a Burgers vector $\underline{b} = (0,0,-b)$ will give a displacement field similar to that of an anti plane strain deformation. We then find the shear stress, σ_{zy} , caused by one straight screw dislocation loop. Then by using the continuum approximation for the dislocation density, $f(x)$, we may develop an integral equation for $f(x)$.

For the straight screw dislocation shown in Fig. (4.1) we again take the curve Γ to consist of a large semicircle C and of a line in the z-x plane, parallel to the z-axis, running from $z = \infty$ to $z = -\infty$. (See Fig. 4.2).

The stress field due to one straight screw dislocation is given by the Peach-Koehler formula equation (2.17). Using this formula, the expression for the shear stress, σ_{zy} , at any point $\underline{r}'$ in the half space is given by

$$\sigma_{zy}(\underline{r}') = \frac{\mu b}{4\pi}\oint_\Gamma \Big\{ \frac{1}{2} R_{,jpp}(\epsilon_{2j3}\, dl_3 + \epsilon_{3j3}\, dl_2)$$

$$+ \frac{1}{\gamma}\epsilon_{kj3} R_{,32j}\, dl_k \Big\}$$

$$= \frac{\mu b}{4\pi}\oint \Big\{ -\frac{1}{2} R_{,1pp}\, dl_3$$

$$+ \frac{1}{1-\gamma}(R_{,322}\, dl_1 - R_{,321}\, dl_2) \Big\} \ .$$

We now evaluate the shear on the boundary, $y' = 0$. Because the stress and displacement fields are independent of z', we write $\sigma_{zy}(x', 0)$ to mean $\sigma_{zy}(\underline{r}')$ evaluated on $y' = 0$.

$$\sigma_{zy}(x', 0) = \frac{+\mu b}{4\pi}\oint_\Gamma (x - x')R^{-3}dz - \frac{1}{1-\gamma}(z - z')R^{-3}dx \ .$$

Now, part of Γ consists of the semicircle of radius a centred at the origin. As $a \rightarrow \infty$, the integrals around $C \rightarrow 0$. The other part of Γ is parallel to the z axis from $z = +\infty$ to $z = -\infty$. The shear then becomes

$$\sigma_{zy}(x', 0) = \frac{\mu b}{4\pi}(x - x')\int_{+\infty}^{-\infty} \frac{dz}{[(x - x')^2 + (z - z')^2]^{3/2}}$$

$$= \frac{\mu b}{4\pi}(x - x')\left.\frac{(z - z')}{(x - x')^2\sqrt{(x - x')^2 + (z - z')^2}}\right]_{+\infty}^{-\infty}$$

$$= -\frac{\mu b}{2\pi}\cdot\frac{1}{x - x'} \ .$$

This is the shear stress, $\sigma_{zy}(x', 0)$ caused by one straight screw dislocation. Making the same continuum assumption as in chapter 4, the shear stress caused by a distributionoof dislocations along the x axis, of density f(x) can be written in integral form; (we switch the notation from primed to unprimed)

$$\sigma_{zy}(x,0) = -\frac{\mu b}{2\pi} P \int_{-\infty}^{\infty} \frac{f(x')\,dx'}{x'-x} . \tag{5.1}$$

Equation (5.1) is then an expression for the shear stress, $\sigma_{zy}(x,0)$ caused by a layer of dislocations distributed on the boundary y=0. Returning to the boundary value problem, we see that the shear stress, σ_{zy} (x,0) is given by t(x). So using equation (5.1) and this boundary value we have an integral equation for the unknown dislocation density, f(x):

$$-\frac{\mu b}{2\pi} P \int_{-\infty}^{\infty} \frac{f(x')\,dx'}{x'-x} = t(x) .$$

The solution of this integral equation is;[13]

$$bf(x) = \frac{2}{\pi\mu} P \int_{-\infty}^{\infty} \frac{t(x')\,dx'}{x'-x} . \tag{5.2}$$

Once the dislocation density, f(x) is known, we may write expressions for the stress components. We first use the Peach-Koehler formula (2.17) to find the stress due to one dislocation loop, and then knowing the dislocation density, we integrate the product. For example, the shear stress, σ_{zy}, at any point (x,y) in the half space is given by

$$\sigma_{zy}(x,y) = \frac{\mu b}{4\pi}\int_{-\infty}^{\infty} f(x')\oint_{\Gamma}\Big\{-\frac{1}{2}R_{,pp}\,dl_3 - \frac{1}{1-\nu}\left(R_{,322}\,dl_1 - R_{,321}\,dl_2\right)\Big\}\,dx'$$

$$= \frac{\mu b}{2\pi}\int_{-\infty}^{\infty} f(x')\,\frac{(x-x')}{(x-x')^2+y^2}\,dx' . \qquad (5.3)$$

Using (5.2) in (5.3) gives:

$$\sigma_{zy}(x,y) = \frac{1}{\pi^2}\int_{-\infty}^{\infty} P\int_{-\infty}^{\infty}\frac{t(x'')}{(x''-x')}\,\frac{x-x'}{(x-x')^2+y^2}\,dx''\,dx' .$$

The integration over x' is performed to give

$$\sigma_{zy}(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y\,t(x'')\,dx''}{\left[(x''-x)^2+y^2\right]} . \qquad (5.4)$$

The expression for $\sigma_{xz}(x,y)$ is found in a similar way, again using the Peach Koehler formula.

$$\sigma_{zx}(x,y) = \frac{\mu}{4\pi}\int_{-\infty}^{\infty} b\,f(x')\,dx'\oint_{\Gamma}\Big\{\frac{1}{2}R_{,2pp}\,dl_3 + \frac{1}{1-\nu}\,\epsilon_{kj3}\,R_{,31j}\,dl_k\Big\} .$$

The curve Γ is as defined before. The integrals around the large semicircle go to zero as the radius goes to infinity. So the only contributions remaining are from the integral parallel to the z axis from z = +∞ to z = −∞. So evaluating the integral around Γ first gives

$$\sigma_{zx}(x,y) = \frac{-\mu}{4\pi}\int_{-\infty}^{\infty} b\,f(x')\,dx' \int_{\infty}^{-\infty} \frac{y\,dz}{[(x-x')^2+y^2+(z-z')^2]^{3/2}}$$

$$= \frac{\mu}{2\pi}\int_{-\infty}^{\infty} \frac{b\,f(x')\,y\,dx'}{[(x-x')^2+y^2]} \quad . \tag{5.5}$$

Again using equation (5.2), this becomes

$$\sigma_{zx}(x,y) = \frac{-1}{\pi^2}\int_{-\infty}^{\infty} P\int_{-\infty}^{\infty} \frac{t(x'')}{x''-x'}\;\frac{y}{[(x-x')^2+y^2]}\,dx''\,dx'$$

Performing the x' integration gives

$$\sigma_{zx}(x,y) = \frac{-1}{\pi^2}\int_{-\infty}^{\infty} \frac{(x''-x)\,t(x'')\,dx''}{[(x''-x)^2+y^2]} \quad . \tag{5.6}$$

The displacement, $u_z(x,y)$, is found by using the Burgers formula, equation (2.12). The same contour Γ, is used. The only contribution is from the solid angle Ω ($\underline{r}$), so

$$u_z(x,y) = \frac{1}{4\pi}\int_{-\infty}^{\infty} b\,f(x')\,\Omega(\underline{r}')\,dx'$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} b\,f(x')\tan^{-1}\frac{y}{x-x'}\,dx' \tag{5.7}$$

Using equation (5.2), this becomes:

$$u_z(x,y) = \frac{1}{\pi^2\mu} \int_{-\infty}^{\infty} P \int_{-\infty}^{\infty} \frac{t(x'')}{x''-x'} \tan^{-1}\frac{y}{x-x'}\, dx''dx'.$$

Performing the integration over x' gives

$$u_z(x,y) = \frac{1}{2\pi\mu} \int_{-\infty}^{\infty} \ln\left[(x-x'')^2 + y^2\right] t(x'')\, dx'' . \tag{5.8}$$

This completes the solution of the traction boundary value problem for anti plane strain in a half space.

Notice that if the anti plane strain problem were one of given displacements on the boundary, i.e., $u_z(x,0)=u(x)$, then since $bf(x)=-2\frac{du(x)}{dx}$, equations (5.3), (5.5), and (5.7) would become:

$$\sigma_{zy}(x,y) = \frac{\mu b}{2\pi} \int_{-\infty}^{\infty} \frac{f(x')(x-x')dx'}{(x-x')^2+y^2}$$

$$= \frac{-\mu}{\pi} \int_{-\infty}^{\infty} \frac{(x-x')\frac{du(x')}{dx'}}{(x-x')^2+y^2}\, dx',$$

$$\sigma_{zx}(x,y) = \frac{-\mu b}{2\pi}\int_{-\infty}^{\infty} \frac{y\, f(x')\,dx'}{(x-x')^2+y^2}$$

$$= \frac{\mu}{\pi}\int_{-\infty}^{\infty} \frac{y\,\frac{du(x')}{dx'}}{(x-x')^2+y^2}\,dx' ,$$

$$u_z(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} b\, f(x')\tan^{-1}\frac{y}{x-x'}\,dx'$$

$$= \frac{-1}{\pi}\int_{-\infty}^{\infty} \frac{du(x')}{dx'}\tan^{-1}\frac{y}{x-x'}\,dx' .$$

These three equations then form the solution of the displacement boundary value problem for anti plane strain in a half space.

We consider next a plane strain problem for the half space $y > 0$. Let the boundary values be $\sigma_{yy}(x,0)=0$ and $\sigma_{xy}(x,0)=S(x)$. In order that we may apply directly the Peach-Koehler and Burgers formulae to this problem, we first define another type of dislocation. As in Chapter 4 we make a cut through the plane $y=0$. This cut again begins on the positive side of the origin and extends, parallel to the z axis, in the positive direction of the x axis. The two faces of the cut are denoted by A_+ and A_- as before. Now we displace the surface A_+ by an amount ½b in the negative x direction, and the surface A_- by an amount ½b in the positive x direction. Again, we exclude points close to the origin. The Burgers vector of the displacement will be $\underline{b} = (-b,0,0)$. See Fig. (5.1).

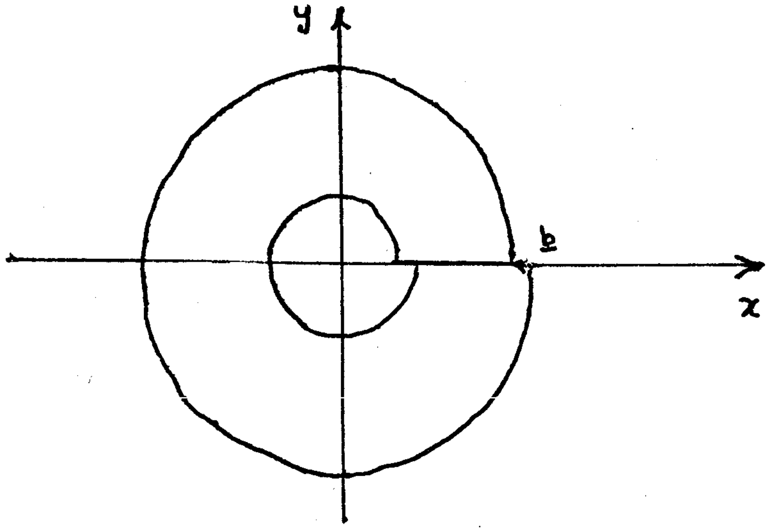


Figure (5.1)

The dislocation shown in Figure (5.1) is called a straight edge dislocation with Burgers vector $(-b,0,0)$. The curve, Γ , that encloses the surface A of the cut will be the same as that shown in Figure (4.2). We are now ready to solve the above plane strain problem.

Let there be a straight edge dislocation with Burgers vector $\underline{b} = (-b,0,0)$. Then from the Peach Koehler formula (2.17) the shear stress, σ_{xy}, for any point, $\underline{r}'$, in the half space due to one edge dislocation with Burgers vector $\underline{b} = (-b,0,0)$ is given by

$$\sigma_{xy}(\underline{r}) = \frac{\mu b}{4\pi} \oint_{\Gamma} \Big\{ \frac{1}{2} R_{,jpp} (\epsilon_{2j1}\, dl_1 + \epsilon_{1j1}\, dl_2)$$

$$+ \frac{1}{1-\nu} \epsilon_{kj1} (R_{,12j})\, dl_k \Big\} .$$

Again the integrals over the large semicircle vanish as the radius goes to infinity, so that only the integrals over z are left:

$$\sigma_{xy}(\underline{r}') = \frac{\mu b}{4\pi(1-\nu)} \int_{+\infty}^{-\infty} -R_{,122}\, dz$$

$$= \frac{\mu b\,(x-x')}{4\pi(1-\nu)} \left\{ \int_{\infty}^{-\infty} \frac{dz}{\left[(x-x')^2+y^2+(z-z')^2\right]^{3/2}} - 3y^2 \int_{\infty}^{-\infty} \frac{dz}{\left[(x-x')^2+y^2+(z-z')^2\right]^{5/2}} \right\}.$$

We switch from primed to unprimed notation on the left hand side and evaluate the integrals, giving

$$\sigma_{xy}(x,y) = \frac{\mu b}{4\pi(1-\nu)} (x-x') \left\{ \frac{2}{(x-x')^2+y^2} - \frac{3y^2}{\left[(x-x')^2+y^2\right]^2} \left(2-\tfrac{2}{3}\right) \right\}$$

$$= \frac{\mu b}{2\pi(1-\nu)} (x-x') \frac{\left[(x-x')^2-y^2\right]}{\left[(x-x')^2+y^2\right]^2} . \qquad (5.9)$$

Equation (5.9) gives the shear stress, $\sigma_{xy}(x,y)$, due to one straight edge dislocation at $x = x'$. At the boundary, $y = 0$, (5.9) becomes:

$$\sigma_{xy}(x,0) = \frac{\mu b}{2\pi(1-\nu)(x-x')} ,$$

Again making the continuum approximation, the shear, $\sigma_{xy}(x,0)$, due to a distribution of dislocations of density f(x) per unit length along the x axis can be written in integral form:

$$\sigma_{xy}(x,0) = \frac{\mu b}{2\pi(1-\nu)} P\int_{-\infty}^{\infty} \frac{f(x')dx'}{x-x'} = S(x) \; . \tag{5.10}$$

From equation (5.10) we may find the expression for bf(x)[13]

$$bf(x) = \frac{2(1-\nu)}{\pi\mu} P\int_{-\infty}^{\infty} \frac{S(x'')dx''}{x''-x} \; . \tag{5.11}$$

Now from equation (5.9) the shear, $\sigma_{xy}(x,y)$, due to a distribution of straight edge dislocations with the given Burgers vector is given by

$$\sigma_{xy}(x,y) = \frac{\mu}{2\pi(1-\nu)} \int_{-\infty}^{\infty} \frac{(x-x')\left[(x-x')^2 - y^2\right]}{\left[(x-x')^2 + y^2\right]^2} bf(x')dx' \tag{5.12}$$

Using equation (5.11) in equation (5.12) gives

$$\sigma_{xy}(x,y) = \frac{1}{\pi^2} \int_{-\infty}^{\infty} \frac{(x-x')\left[(x-x')^2 - y^2\right]}{\left[(x-x')^2 + y^2\right]} P\int_{-\infty}^{\infty} \frac{S(x'')dx''}{x''-x'} dx' \; .$$

And performing the x' integration gives

$$\sigma_{xy}(x,y) = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{y\,(x-x'')^2\, S(x'')\, dx''}{[(x-x'')^2+y^2]^2} \tag{5.13}$$

In a similar way the stresses $\sigma_{XX}(x,y)$ and $\sigma_{YY}(x,y)$, due to a distribution of straight edge dislocations of density $f(x)$ along the x axis may be found (numbered subscrips not summed)

$$\sigma_{xx}(x,y) = \frac{-\mu}{4\pi} \int_{-\infty}^{\infty} b\, f(x') \oint_{\Gamma} \Big\{ \tfrac{1}{2} R_{,jpp} (\varepsilon_{ij1}\, dl_1 + \varepsilon_{j1}\, dl_1)$$

$$+ \frac{1}{1-\nu}\, \varepsilon_{kj1} (R_{,11j} - R_{,jpp})\, dl_k \Big\}\, dx',$$

$$\sigma_{xx}(x,y) = \frac{-\mu}{4\pi} \int_{-\infty}^{\infty} b\, f(x') \int_{-\infty}^{\infty} \frac{1}{1-\nu} (R_{,222} + R_{,233})\, dz\, dx'$$

$$= \frac{-\mu}{2\pi(1-\nu)} \int_{-\infty}^{\infty} \frac{y\,[3(x-x')^2+y^2]}{[(x-x')^2+y^2]^2}\, b\, f(x')\, dx'. \tag{5.14}$$

And from equation (5.11) equation (5.14) becomes

$$\sigma_{xx}(x,y) = \frac{-1}{\pi^2} \int_{-\infty}^{\infty} \frac{y\,[3(x-x')^2+y^2]}{[(x-x')^2+y^2]^2}\; P\!\!\int_{-\infty}^{\infty} \frac{S(x'')\, dx''}{x''-x}\, dx'.$$

Integrating over x' gives

$$\sigma_{xx}(x,y) = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{(x-x'')^3}{[(x-x'')^2+y^2]^2} S(x'')\,dx'' \quad . \tag{5.15}$$

The expression for $\sigma_{yy}(x,y)$ is found in the same way,

$$\sigma_{yy}(x,y) = \frac{+\mu}{2\pi(1-\nu)}\int_{-\infty}^{\infty} \frac{y\,[(x-x')^2-y^2]}{[(x-x')^2+y^2]^2}\, b\,f(x')\,dx' \tag{5.16}$$

$$\sigma_{yy}(x,y) = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{(x-x'')\,y^2}{[(x-x'')^2+y^2]^2} S(x'')\,dx'' \quad . \tag{5.17}$$

The displacements can be found from the Burgers formula (2.12),

$$U_x(x,y) = \frac{1}{4\pi}\int_{-\infty}^{\infty} b\,f(x')\,\Omega(\underline{r}')\,dx' + \frac{1}{8\pi}\int_{-\infty}^{\infty} b\,f(x')\,dx' \int_{\infty}^{-\infty} R_{,12}\,dz \quad . \tag{5.18}$$

Again, $\Omega(\underline{r}')$ is defined as before in Chapter 2 as the solid angle subtended at $\underline{r}'$. Using equation (5.11) and integrating over x' gives

$$u_x(x,y) = \frac{1}{2\pi\mu}\int_{-\infty}^{\infty}\left\{(1-\nu)\,\ln\left[(x-x'')^2+y^2\right] + \frac{y^2}{\left[(x-x'')^2+y^2\right]}\right\}S(x'')dx''. \tag{5.19}$$

Applying this same procedure to the Burgers formula for the displacement, $u_y(x,y)$, gives,

$$u_y(x,y) = \frac{1}{4\pi}\int_{-\infty}^{\infty} b f(x')\int_{\infty}^{-\infty} R^{-1} dz\, dx' + \frac{(1-\nu)}{8\pi}\int_{-\infty}^{\infty} b f(x')\int_{\infty}^{-\infty}(R^{-1}-y^2R^{-3})dz\, dx', \tag{5.20}$$

$$u_y(x,y) = \frac{1}{2\pi\mu}\int_{-\infty}^{\infty}\left\{(1-2\nu)\tan^{-1}\frac{y}{x-x''} - \frac{(x-x'')y}{\left[(x-x'')^2+y^2\right]}\right\}S(x'')dx''. \tag{5.21}$$

So the equations (5.13), (5.15), (5.17), (5.19), and (5.21) give the complete solution for the plane shear boundary value problem for the half space $y > 0$.

If the plain strain problem had the boundary values given as displacement in the x direction, i.e., $u_x(x,0)=u(x)$

as well as $\sigma_{yy}(x,0)=0$, then since $bf(x)=-\frac{2du(x)}{dx}$, we would substitute this quantity into equations (5.12), (5.15), (5.16), and (5.20) to obtain the solutions.

Another plane strain problem for the half space $y>0$ that we may consider is one where the normal traction on the boundary is given, and the shear there is zero. Then we have $\sigma_{yy}(x,0)=P(x)$ and $\sigma_{xy}(x,0)=0$. For this plane strain problem we consider an edge dislocation with Burgers vector $\underline{b}=(0,-b,0)$. This dislocation is formed by making the same cut in the y=0 plane, but displacing the two faces in the y direction. The curve, Γ, is again as described before (Figure (4.2)).

If we consider a distribution along the x axis of such edge dislocations, the Peach-Koehler formula gives an expression for the normal stress:

$$\sigma_{yy}(x,y) = \frac{\mu}{4\pi}\int_{-\infty}^{\infty}\int_{\infty}^{-\infty}\frac{1}{1-\nu}\,\varepsilon_{3j2}\left(R_{,22j}-R_{,jpp}\right)dz\,bf(x')dx'$$

$$= \frac{\mu}{4\pi(1-\nu)}\int_{-\infty}^{\infty} b\,f(x')dx'\int_{+\infty}^{\infty}\left\{\frac{4(x-x')}{R^3} - \frac{3(x-x')}{R^5\left[(x-x')^2+(z-z')^2\right]}\right\}dz$$

$$= \frac{-\mu}{2\pi(1-\nu)}\int_{-\infty}^{\infty}\frac{(x-x')\left[(x-x')^2+3y^2\right]}{\left[(x-x')^2+y^2\right]^2}\,b\,f(x')dx'. \tag{5.22}$$

Again an integral equation may be formed by evaluating σ_{yy} on the boundary, y=0, and equating the resulting expression to the given boundary value:

$$\sigma_{yy}(x,0) = \frac{+\mu}{2\pi(1-\nu)} P\int_{-\infty}^{\infty} \frac{b f(x')\,dx'}{x'-x} = p(x) .$$

The solution of the above integral equation is given by [13]

$$b f(x) = \frac{-2(1-\nu)}{\pi\mu} P\int_{-\infty}^{\infty} \frac{p(x'')\,dx''}{x''-x} . \tag{5.23}$$

So equation (5.22), with equation (5.23) becomes

$$\sigma_{yy}(x,y) = \frac{1}{\pi^2}\int_{-\infty}^{\infty} \frac{(x-x')\left[(x-x')^2+y^2 3\right]}{\left[(x-x')^2+y^2\right]^2}\; P\int_{-\infty}^{\infty} \frac{p(x'')\,dx''\,dx'}{x''-x'} .$$

Integration over x' gives

$$\sigma_{yy}(x,y) = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{y^3}{\left[(x-x'')^2+y^2\right]^2}\, p(x'')\,dx'' . \tag{5.24}$$

Once the dislocation density f(x) is known, then the remaining stresses can be found again from the Peach-Koehler formula:

$$\sigma_{xy}(x,y) = \frac{\mu}{4\pi}\int_{-\infty}^{\infty} b f(x')\,dx' \int_{\infty}^{-\infty} \frac{-1}{1-\nu}\left(R_{,122} - R_{,133}\right) dz$$

$$= \frac{\mu}{4\pi} \int_{-\infty}^{\infty} b\, f(x')\,dx' \int_{+\infty}^{-\infty} \frac{1}{1-\nu} \left[\frac{2(x-x')}{R^3} - \frac{3(x-x')\{y^2+(z-z')^2\}}{R^5} \right] dz$$

$$= \frac{-\mu}{2\pi(1-\nu)} \int_{-\infty}^{\infty} \frac{(x-x')\left[(x-x')^2-y^2\right]}{\left[(x-x')^2+y^2\right]^2}\, b f(x')\,dx' . \qquad (5.25)$$

And using equation (5.23) in equation (5.25), we have

$$\sigma_{xx}(x,y) = \frac{1}{\pi^2} \int_{-\infty}^{\infty} \frac{(x-x')\left[(x-x')^2-y^2\right]}{\left[(x-x')^2+y^2\right]^2}\, P\!\!\int_{-\infty}^{\infty} \frac{P(x'')\,dx''}{x''-x'}\, dx' .$$

Integrating over x' gives

$$\sigma_{xx}(x,y) = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{(x-x'')^2 y}{\left[(x-x'')^2+y^2\right]^2}\, P(x'')\,dx'' . \qquad (5.26)$$

Also,

$$\sigma_{xy}(x,y) = \frac{\mu}{4\pi} \int_{-\infty}^{\infty} b f(x')\,dx' \int_{\infty}^{-\infty} \frac{1}{1-\nu} R_{,112}\, dz$$

$$= \frac{\mu}{4\pi} \int_{-\infty}^{\infty} b f(x')\,dx' \int_{\infty}^{-\infty} \frac{1}{1-\nu} \left[\frac{-y}{R^3} + \frac{3(x-x')^2 y}{R^5} \right] dz$$

$$= \frac{-\mu}{2\pi(1-\nu)} \int_{-\infty}^{\infty} \frac{y\,[(x-x')^2-y^2]}{[(x-x')^2+y^2]^2}\, bf(x')\,dx' \qquad (5.27)$$

$$= \frac{1}{\pi^2} \int_{-\infty}^{\infty} \frac{y\,[(x-x')^2-y^2]}{[(x-x')^2+y^2]^2}\, dx'\; P\!\int_{-\infty}^{\infty} \frac{p(x'')}{x''-x'}\, dx''$$

$$\sigma_{xy}(x,y) = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{(x-x'')\,y^2}{[(x-x'')^2+y^2]^2}\, p(x'')\,dx'' \,. \qquad (5.28)$$

The displacements are found from the Burgers formula:

$$u_x(x,y) = \frac{1}{8\pi} \int_{-\infty}^{\infty} bf(x')\,dx' \int_{\infty}^{-\infty} \left[R_{,11} + R_{,22} + R_{,33} - \frac{1}{1-\nu} R_{,11}\right] dz$$

$$+ \frac{1}{8\pi} \int_{-\infty}^{\infty} bf(x')\,dx' \int_{C} R_{,13}\, dx$$

$$= \frac{1}{8\pi} \int_{-\infty}^{\infty} bf(x')\,dx' \int_{+\infty}^{-\infty} \frac{2}{R} - \frac{1}{1-\nu}\left[\frac{1}{R} - \frac{(x-x')^2}{R^3}\right] dz$$

$$= \frac{-(1-\nu)}{\pi^2 \mu}\, P\!\int_{-\infty}^{\infty} dx'\; P\!\int_{-\infty}^{\infty} \frac{p(x'')\,dx''}{x''-x'} \int_{\infty}^{-\infty} \frac{2}{R} - \frac{1}{1-\nu}\left[\frac{1}{R} - \frac{(x-x')^2}{R^3}\right] dz$$

$$= \frac{-1}{2\pi\mu} \int_{-\infty}^{\infty} \left\{(1-2\nu)\tan^{-1}\left(\frac{y}{x-x''}\right) + \frac{(x-x'')\,y}{[(x-x'')^2+y^2]}\right\} p(x'')\,dx'' \,. \qquad (5.29)$$

Also,

$$
\begin{aligned}
u_y(x,y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} b\,f(x')\tan^{-1}\left(\frac{y}{x-x'}\right)dx' \\
&\quad - \int_{-\infty}^{\infty}\frac{b\,f(x')dx'}{8\pi(1-\nu)}\left\{\int_{\infty}^{-\infty} R_{,21}\,dz - \oint_C \cancelto{0}{R_{,23}}\,dx\right\} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} b\,f(x')\tan^{-1}\left(\frac{y}{x-x'}\right)dx' \\
&\quad + \frac{1}{8\pi(1-\nu)}\int_{-\infty}^{\infty} b\,f(x')dx'\int_{\infty}^{-\infty}\frac{(x-x')y\,dz}{\left[(x-x')^2+y^2+(z-z')^2\right]^{3/2}} \\
&= \frac{1}{2\pi\mu}\int_{-\infty}^{\infty}\left\{(1-\nu)\ln\left[(x-x'')^2+y^2\right]\right. \\
&\quad \left. + \frac{(x-x'')^2}{\left[(x-x'')^2+y^2\right]}\right\} p(x'')\,dx'' .
\end{aligned}
\tag{5.30}
$$

Equations (5.24), (5.26), (5.28), (5.29), and (5.30) form the solution of the normal traction plane strain boundary value problem.

6. Axisymmetric Boundary Value Problems For a Half Space

We now consider two axisymmetric boundary value problems for the half space $z > 0$. The first is concerned with an axisymmetric normal stress given on the boundary; the second problem is one of torsion with an axisymmetric shear stress given on the boundary.

Let there be a half space $z > 0$ with the $z = 0$ plane as the boundary. Let the normal stress on the boundary be given as $\sigma_{zz}(r,\theta,0) = g(r)$. Since σ_{zz} is independent of θ, we shall omit writing "θ" and use the notation $\sigma_{zz}(r,z)$ to mean $\sigma_{zz}(r,\theta,z)$. So using this notation, $\sigma_{zz}(r,0) = g(r)$, where $r^2 = x^2 + y^2$. For this axisymmetric problem we consider the closed curve, Γ, to be a circle of radius r lying in the $z = 0$ plane. A cut is made throughout the inside of the circle Γ and the two faces are displaced by an amount b relative to each other in the z direction. This dislocation loop then has a Burgers vector of $\underline{b} = (0,0,b)$. We may find an expression for the normal stress due to one of these dislocation loops by making use of the Peach-Koehler formula (2.17):

$$\sigma_{zz}(r',z') = \frac{-\mu}{4\pi} \oint_{\Gamma} \frac{1}{1-\nu} \varepsilon_{kj3}\left(R_{,33j} - R_{,jpp}\right) dl_k \qquad (6.1)$$

If we then assume that there is a distribution of such dislocations in the z = 0 plane of density f(r), the normal stress, σ_{zz}, due to these dislocations will be, with the help of equation (6.1),

$$\sigma_{zz}(r',z') = \frac{-\mu}{4\pi(1-\nu)} \int_0^\infty b f(r)\, dr \oint_\Gamma \varepsilon_{kj3} (R_{,33j} - R_{,jpp})\, dl_k \qquad (6.2)$$

Equation (6.2) may be simplified further to

$$\sigma_{zz}(r',z') = \frac{-\mu}{4\pi(1-\nu)} \int_0^\infty b f(r)\, dr \oint_\Gamma \{(R_{,122} - R_{,111})\, dl_1 - (R_{,211} + R_{,222})\, dl_1\} \; .$$

We notice that on the curve, Γ ;

$x_1 = r \cos\theta$, $dx_1 = dl_1 = -r \sin\theta d\theta$,

$x_2 = r \sin\theta$, $dx_2 = dl_2 = r \cos\theta d\theta$,

$x_3 = 0$,

and that

$$R^2 = (x_1 - x_1')^2 + (x_2 - x_2')^2 + x_3'^2 \; .$$

So we have

$$\sigma_{zz}(r',z') = \frac{-\mu r}{4\pi(1-\nu)} \int_0^\infty b f(r)\, dr \oint_\Gamma \left[(R_{,122} + R_{,111}) \cos\theta + (R_{,122} + R_{,211}) \sin\theta\right] d\theta$$

$$= \frac{\mu r}{4\pi(1-\nu)} \int_0^\infty b f(r) dr \oint_\Gamma R^{-3} \left[(x_2 - x_2') \sin\theta + (x_1 - x_1') \cos\theta \right] d\theta \quad . \tag{6.3}$$

We may now evaluate the normal stress on the boundary, z'= 0:

$$\sigma_{zz}(r',0) = \frac{\mu r}{4\pi(1-\nu)} \int_0^\infty b f(r) dr \oint_\Gamma R_o^{-3} \left[(x_2 - x_2') \sin\theta + (x_1 - x_1') \cos\theta \right] d\theta \quad , \tag{6.4}$$

where $R_o^2 = (x_1 - x_1')^2 + (x_2 - x_2')^2$.

When $z' = 0$, $r'^2 = x_1'^2 + x_2'^2$, where $x_1' = r' \cos\phi$, $x_2' = r' \sin\phi$. Then if we let $\alpha = \theta - \phi$, $d\alpha = d\theta$, for a fixed angle ϕ , equation (6.3) becomes

$$\sigma_{zz}(r',0) = \frac{+\mu r}{4\pi(1-\nu)} \int_0^\infty b f(r) dr \oint_\Gamma R_o^{-3} \left\{ \left[r \sin(\alpha+\phi) - r' \sin\phi \right] \cdot \sin(\alpha+\phi) + \left[r\cos(\alpha+\phi) - r'\cos\phi \right] \cos(\alpha+\phi) \right\} d\alpha ,$$

So

$$\sigma_{zz}(r',0) = \frac{+\mu r}{4\pi(1-\nu)} \int_0^\infty b f(r) dr \oint_\Gamma R_o^{-3} (r - r'\cos\alpha) \, d\alpha \quad , \tag{6.5}$$

where R_o^2 can now be written as

$$R_o^2 = r^2 + r'^2 - 2rr'\cos\alpha \quad .$$

If we let $m = \frac{4\,rr'}{(r+r')^2}$, then (6.4) becomes

$$\sigma_{zz}(r',0) = \frac{+\mu r}{4\pi(1-\nu)} \int_0^\infty bf(r)dr \; \cdot$$

$$\int_0^{2\pi} \frac{(r-r'\cos\alpha)d\alpha}{\left(\frac{rr'}{m}\right)^{3/2}\left[(4-2m)-2m\cos\alpha\right]^{3/2}} \quad . \qquad (6.6)$$

Now elliptic integrals of the first and second kind are respectively:

$$K(m) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-m\sin^2\theta}}$$

$$= \int_0^{\pi} \frac{d\alpha}{\sqrt{(4-2m)-2m\cos\alpha}} \quad , \quad \text{where } 2\theta = \alpha$$

$$E(m) = \int_0^{\pi/2} (1-m\sin^2\theta)d\theta$$

$$= \tfrac{1}{4}\int_0^{\pi} \left[(4-2m)-2m\cos\alpha\right]^{1/2} d\alpha \quad .$$

Integrating (6.5) by parts over α gives,

$$\int_0^{2\pi} \frac{(r-r'\cos\alpha)d\alpha}{\left(\frac{rr'}{m}\right)^{3/2}\left[(4-2m)-2m\cos\alpha\right]^{3/2}}$$

$$= r\left(\frac{m}{rr'}\right)^{3/2} \frac{E(m)}{(1-m)} - r'\left\{\left(\frac{m}{rr'}\right)^{3/2}\left[\frac{(2-m)\,E(m)}{2m(1-m)} - \frac{1}{m}K(m)\right]\right\} \; .$$

So after combining terms, (6.5) becomes

$$\sigma_{zz}(r',0) = \frac{+\mu}{2\pi(1-\nu)} \int_0^\infty \left[E(m) + (1-m)^{1/2} K(m)\right] \cdot \frac{b\, f(r)\, dr}{r-r'}$$

$$= q(r') . \tag{6.7}$$

Notice that a small displacement on the boundary in the z direction, $du_z(r,0)$, is given by $\frac{1}{2}b\ f(r)dr$. Then, since $\sigma_{zz}(r',0) = g(r')$, equation (6.6) becomes

$$\sigma_{zz}(r',0) = \frac{\mu}{\pi(1-\nu)} \int_0^\infty \frac{du_z}{dr}(r,0) \left[E(m) + (1-m)^{1/2} K(m)\right] \cdot \frac{1}{r-r'}\, dr$$

$$= q(r') \tag{6.8}$$

The solution of (6.8) for $\frac{du_z}{dr}(r,0)$ and hence for b f(r) is not readily found as in the previous half space problems (chapter 5). However, the elliptic integrals of equation (6.8) may be written in terms of Bessel functions.[14]

$$\int_0^\infty \xi^{-1/2} J_0(r'\xi)\, J_0(r\xi)\, (r'\xi)^{1/2}\, d\xi$$

$$= \frac{r'^{1/2}}{\pi} \int_0^\pi \frac{d\phi}{\sqrt{r^2 - r'^2 \cos^2\phi}} ,$$

So $$\int_0^\infty J_0(r'\xi)\,J_0(r\xi)\,d\xi = \frac{2}{\pi}\int_0^{\pi/2}\frac{d\phi}{\sqrt{r^2-r'^2\cos^2\phi}} \quad .$$

Let $\beta = \pi/2 - \phi$, then

$$\frac{2}{\pi}\int_0^{\pi/2}\frac{d\phi}{\sqrt{r^2-r'^2\cos^2\phi}} = \frac{2}{\pi}\int_0^{\pi/2}\frac{d\beta}{r\sqrt{1-\frac{r'^2}{r^2}\sin^2\beta}}$$

$$= \frac{2\,K(p)}{\pi r} \quad , \text{ where } p = \frac{r'^2}{r^2} \quad .$$

By a Landen transformation:[15]

$$K(p) = \tfrac{1}{2}K(m)\left[1+(1-m)^{1/2}\right]$$

$$= \frac{r}{r+r'}\,K(m) \quad .$$

Then $$\int_0^\infty J_0(r'\xi)\,J_0(r\xi)\,d\xi = \frac{2\,K(m)}{\pi\,(r+r')} \quad . \qquad (6.9)$$

Differentiating (6.9) with respect to r gives

$$\int_0^\infty J_0(r'\xi)\,J_1(r\xi)\,\xi\,d\xi$$

$$= \frac{-2}{\pi}\left\{\frac{-K(m)}{(r+r')^2} + (r+r')^{-1}\frac{dK(m)}{dk}\,\frac{dk}{dm}\,\frac{dm}{dr}\right\} \quad ,$$

where $$\frac{dK(m)}{dk} = \frac{E(m)}{k\,k'^2} - \frac{K(m)}{k} \quad ,$$

$$k'^2 = 1 - k^2 \quad ,$$

$$\frac{dk}{dm} = \frac{1}{2m^{1/2}} \quad ,$$

$$m = k^2$$

$$\frac{dm}{dr} = \frac{4r'(r'-r)}{(r+r')^3} \quad .$$

So

$$\int_0^\infty J_0(r'\xi) J_1(r\xi)\xi \, d\xi$$

$$= \frac{1}{\pi r(r-r')} \left\{ (1-m)^{1/2} K(m) + E(m) \right\} ,$$

or

$$r\int_0^\infty J_0(r'\xi) J_1(r\xi)\xi \, d\xi$$

$$= \frac{1}{\pi(r-r')} \left\{ (1-m)^{1/2} K(m) + E(m) \right\} . \qquad (6.10)$$

With the help of (6.10), equation (6.8) may be written

$$\frac{\mu}{1-\nu} \int_0^\infty \int_0^\infty \frac{du_z}{dr}(r,0) \, J_0(r'\xi) J_1(r\xi) r \xi \, dr \, d\xi = q(r') \quad . \qquad (6.11)$$

If the Hankel transform of order n and its inverse are defined by

$$\overline{q(\xi)} = \int_0^\infty q(r') J_n(r'\xi) r' dr' \quad , \qquad (6.12)$$

$$q(r) = \int_0^\infty \overline{q(\xi)} J_n(r\xi)\xi \, d\xi \quad , \qquad (6.13)$$

then we may write equation (6.11) as

$$\frac{\mu}{1-\nu}\int_0^\infty\int_0^\infty \frac{du_z}{dr}(r,0)\, J_0(r'\xi)\, J_1(r,\xi)\, r\xi\, dr\, d\xi$$

$$= \int_0^\infty \overline{q}(\xi)\, J_0(r'\xi)\,\xi\, d\xi \quad .$$

Comparing terms on both sides of the above equation, we have that

$$\overline{q}(\xi) = \frac{\mu}{1-\nu}\int_0^\infty \frac{du_z}{dr}(r,0)\, J_1(r\xi)\, r\, dr \quad . \tag{6.14}$$

Inverting (6.14) according to equation (6.13) gives

$$\frac{du_z(r,0)}{dr} = \frac{1-\nu}{\mu}\int_0^\infty \overline{q}(\xi)\, J_1(r\xi)\,\xi\, d\xi \quad . \tag{6.15}$$

Equation (6.15) is then the solution of the integral equation (6.8). This solution may be simplified further by noting that the Hankel transform of g(r') is given by equation (6.12). Then (6.15) becomes

$$\frac{du_z(r,0)}{dr} = \frac{1-\nu}{\mu}\int_0^\infty\int_0^\infty g(r')\, J_0(r'\xi)\, J_1(r\xi)\, r'\xi\, dr'\, d\xi \quad .$$

Integration over ξ , with the help of (6.10) gives

$$\frac{du_z(r,0)}{dr} = \frac{1-\nu}{\mu r}\int_0^\infty \frac{r' g(r')}{\pi}\left[(1-m)^{1/2} K(m) + E(m)\right]\frac{dr'}{r-r'} \quad . \tag{6.16}$$

From (6.16) the dislocation density, f(r), is known, since

$$bf(r) = -\frac{2\,du_z(r,0)}{dr} \quad .$$

Then the complete stress and displacement fields may be found using the Peach-Koehler formula (2.17) and Burgers formula (2.12).

Next we consider an axisymmetric torsion problem for the half space, $z > 0$. Let there be an axisymmetric shear stress given on the boundary as $\sigma_{z\theta}(r,0) = S(r)$. (Again we have dropped writing "θ" in the argument). Let Γ be a circle of infinite radius in the $z = 0$ plane, and slice the material inside this circle. The two faces of the cut are then rotated relative to each other. We notice that here the relative displacement, $\Delta \underline{u}$ will not be constant around a closed curve, Γ. Since the Burgers formula, equation (2.12), was derived assuming the relative displacement to be constant around a closed curve Γ, we must go back to equation (2.4) to find the expression for the displacement field.

Recall that equation (2.4) was written

$$u_m(\underline{r}') = -\int_{A_+} \Delta u_i(\underline{r})\, u_{\ell m,k}(\underline{r},\underline{r}')\, C_{ijk\ell}\, n_j\, dS$$

where A_+ is the top surface of the cut, covering the whole $z = 0$ plane. Then the displacement in the y direction, $u_y(\underline{r}')$

is written as

$$u_y(\underline{r}') = \int_{A_+} C_{i3kl}\, \Delta u_i(\underline{r})\, u_{l2,k}(\underline{r},\underline{r}')\, dS \quad .$$

Notice that the positive unit normal as defined in Chapter 2 is pointing in the negative z direction.

To evaluate this displacement on the boundary of the half space, we set $z' = 0$. Also, if the relative displacement of the two faces at any point $(r,\theta,0)$ is denoted by $\Delta u(r)$, then the components of displacement in the x and y directions are:

$$\Delta u_1(r,\theta,0) = -\Delta u(r)\sin\theta,$$
$$\Delta u_2(r,\theta,0) = \Delta u(r)\cos\theta,$$
$$\Delta u_3(r,\theta,0) = 0,$$

where $r^2 = x_1^2 + x_2^2$. See Figure (6.1).

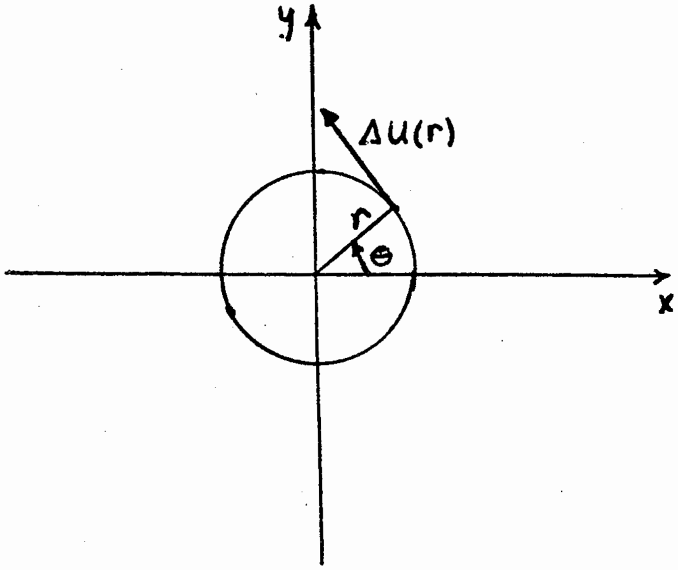


Figure (6.1)

So u_y, evaluated on the boundary can be written as

$$u_y(r',0) = \int_0^\infty\int_0^{2\pi} \Big[-C_{13kl}\,\Delta u(r)\sin\theta + C_{23kl}\,\Delta u(r)\cos\theta \Big] \cdot u_{l2,k}(r,r')\,r\,d\theta\,dr \quad . \tag{6.17}$$

We notice that on the x' axis, at the point (r',0,0) the shear stresses, $\sigma_{z\theta}$ and σ_{zy} are equivalent. Then,

$$\sigma_{z\theta}(r',0,0) = \sigma_{zy}(r',0,0)$$

$$= 2\mu \frac{\partial u_y}{\partial z'}(r',0,0) = S(r') \quad .$$

From equation (6.17) we have(writing $u_y(r',0)$ to mean $u_y(r',0,0)$),

$$\frac{\partial u_y(r',z')}{\partial z'} = -\int_0^\infty\int_0^{2\pi} \left(C_{13kl}\sin\theta - C_{23kl}\cos\theta \right) \Delta u(r)\, u_{l2,k3'}(r,r')\, r\,d\theta\,dr \quad ,$$

$$\text{where } u_{l2,k3'} = \frac{\partial u_{l2,k}}{\partial z'} \quad . \tag{6.18}$$

In order to perform the θ integration, we must first simplify equation (6.18). From equation (1.25) Green's function for an infinite medium is:

$$U_{\ell m}(r, r') = \frac{1}{8\pi\mu}\left[\delta_{\ell m} R_{,nn} - \frac{\lambda+\mu}{\lambda+2\mu} R_{,\ell m}\right],$$

$$U_{\ell m,k}(r, r') = \frac{1}{8\pi\mu}\left[\delta_{\ell m} R_{,nnk} - \frac{\lambda+\mu}{\lambda+2\mu} R_{,\ell m k}\right] \qquad (6.19)$$

The only non zero components possible with C_{13kl} and C_{23kl} (for a linear, elastic, isotropic material) are

$$C_{1313} = C_{1331} = C_{2323} = C_{2332} = \mu \quad .$$

So equation (6.19) becomes

$$\frac{\partial U_y}{\partial z'}(r', z') = -\int_0^\infty \int_0^{2\pi} \Big\{ \sin\theta \left(U_{32,13'}(r,r') + U_{12,33'}(r,r')\right) - \cos\theta \left(U_{32,23'}(r,r') + U_{22,33'}(r,r')\right) \Big\} \Delta U(r)\, r\, d\theta\, dr \qquad (6.20)$$

Evaluating equation (6.20) on the x' axis, with y'=z'=0, x'=r', $x_3=0$, $x_1=r\cos\theta$, and $x_2=r\sin\theta$, we have:

$$\frac{\partial u_y(r',0)}{\partial z'} = \frac{-1}{8\pi}\int_0^{\infty}\int_0^{2\pi}\left\{\frac{-2(\lambda+\mu)}{\lambda+2\mu}\sin\theta\left[-3(r\cos\theta-r')\,r\sin\theta\,R_0^{-5}\right]\right.$$

$$-\cos\theta\left[2R_0^{-3}-2\frac{(\lambda+\mu)}{\lambda+2\mu}\left(-3r^2\sin^2\theta\right.\right.$$

$$\left.\left.\left.+R_0^{-3}\right)\right]\right\}\Delta u(r)\,r\,d\theta\,dr\;,$$

where $\qquad R_0^2 = (x_1-x_1')^2+(x_2)^2$

$$= (r\cos\theta - r')^2 + r^2\sin^2\theta$$

$$= r^2+r'^2-2rr'\cos\theta\;.$$

Simplifying further gives

$$\frac{\partial u_y(r',0)}{\partial z'} = \frac{-1}{8\pi}\int_0^{\infty}\int_0^{2\pi}\left\{\frac{-2\mu}{\lambda+2\mu}R_0^{-3}\cos\theta\right.$$

$$\left.-\frac{6(\lambda+\mu)}{\lambda+2\mu}rr'\sin^2\theta\,R^{-5}\right\}\Delta u(r)\,r\,d\theta\,dr\;.$$

The integration over θ may now be performed. By integration by parts, and by using the results of the integrals evaluated in the first axisymmetric problem we have that:

$$\int_0^{2\pi} \frac{\cos\theta \, d\theta}{R^3} = \left(\frac{m}{rr'}\right)^{3/2} \left[\frac{(2-m)E(m)}{2m(1-m)} - \frac{K(m)}{m}\right] \equiv \left(\frac{m}{rr'}\right)^{3/2} I ,$$

and that:

$$\int_0^{2\pi} \frac{\sin^2\theta \, d\theta}{R^5} = \left(\frac{m}{rr'}\right)^{5/2} \cdot \frac{I}{3m} .$$

So

$$\frac{\partial u_y(r',0)}{\partial z'} = \frac{-1}{8\pi} \int_0^\infty \left\{ \frac{-2\mu}{\lambda+2\mu} \left(\frac{m}{rr'}\right)^{3/2} I - 6rr' \left(\frac{m}{rr'}\right)^{5/2} \frac{\lambda+\mu}{\lambda+2\mu} \frac{I}{3m} \right\} \Delta U(r) \, r \, dr .$$

Algebraically simplifying gives,

$$\frac{\partial u_y(r',0)}{\partial z'} = +\frac{1}{2\pi} \int_0^\infty \frac{1}{rr'(r+r')} \left[\frac{(r^2+r'^2)E(m)}{(r'-r)^2} - K(m) \right] \Delta U(r) \, r \, dr$$

$$= \frac{1}{2\mu} S(r') .$$

So we have an integral equation for $\Delta u(r)$:

$$S(r') = \frac{\mu}{\pi r'}\int_0^{\infty}\left[\frac{(r^2+r'^2)\,E(m)}{(r'-r)^2} - K(m)\right]\frac{\Delta U(r)}{r+r'}\,dr. \tag{6.21}$$

Again the solution of the integral equation (6.21) for $\Delta u(r)$ can be found by expressing the elliptic integrals in terms of Bessel functions. Recall that from equation (6.10)

$$r\int_0^{\infty} J_0(r'\xi)\,J_1(r\xi)\,\xi\,d\xi = \frac{1}{\pi(r-r')}\left\{(1-m)^{1/2}K(m) + E(m)\right\}.$$

Then differentiating equation (6.10) with respect to r' gives

$$r\int_0^{\infty} J_1(r'\xi)\,J_1(r\xi)\,\xi^2\,d\xi = \frac{-1}{\pi(r-r')^2}\left[\frac{r-r'}{r+r'}K(m) + E(m)\right]$$
$$-\frac{1}{\pi(r-r')}\left[\frac{-rK(m)}{r'(r+r')} + \frac{rE(m)}{r'(r+r')}\right]$$

$$= \frac{-1}{\pi r'(r+r')}\left[\frac{(r^2+r'^2)}{(r-r')^2}E(m) - K(m)\right]. \tag{6.22}$$

Then, using equation (6.22) in equation (6.21), we have

$$S(r') = -\mu \int_0^\infty \int_0^\infty r J_1(r'\xi) J_1(r\xi) \xi^2 \Delta u(r) d\xi dr$$

(6.23)

Recall that the inverse Hankel transform of $\overline{S(\xi)}$ is defined by equation (6.13). So from (6.13), equation (6.23) becomes:

$$-\mu \int_0^\infty \int_0^\infty r J_1(r'\xi) J_1(r\xi) \xi^2 \Delta u(r) d\xi dr$$

$$= \int_0^\infty \overline{S(\xi)} J_1(r'\xi) \xi d\xi \quad .$$

(6.24)

Comparing terms in equation (6.24) gives

$$\overline{S(\xi)} = -\mu \xi \int_0^\infty r J_1(r\xi) \Delta u(r) dr$$

(6.25)

Inverting equation (6.25) according to equation (6.13) gives

$$\Delta u(r) = -\frac{1}{\mu} \int_0^\infty \overline{S(\xi)} J_1(r\xi) \quad d\xi \quad .$$

(6.26)

Then, since the Hankel transform of S(r') of order one is defined by equation (6.12), equation (6.26) becomes

$$\Delta u(r) = -\frac{1}{\mu}\int_0^\infty\int_0^\infty S(r')\,J_1(r'\xi)\,J_1(r\xi)\,r'\,d\xi\,dr',$$

or,

$$\Delta u(r) = -\frac{1}{\mu}\int_0^\infty S(r')\,r'\,dr'\int_0^\infty J_1(r'\xi)\,J_1(r\xi)\,d\xi \qquad (6.27)$$

Equation (6.27) is then the solution of the integral equation (6.21). However by using the following result we may simplify equation (6.27) further.

$$\int_0^\infty J_1(r'\xi)\,J_1(r\xi)\,d\xi = \frac{rr'}{\pi}\int_0^\pi \frac{\sin^2\phi\,d\phi}{\left[r^2 - r'^2\cos^2\phi\right]^{3/2}}$$

$$= \frac{r'}{r^2\pi}\int_0^{\pi/2} \frac{\cos^2\alpha\,d\alpha}{\left[1 - \frac{r'^2}{r^2}\sin^2\alpha\right]^{3/2}},$$

where $\alpha = \phi/2$. Let $P^2 = r'^2/r^2$, then

$$\int_0^\infty J_1(r'\xi)\,J_1(r\xi)\,d\xi = \frac{1}{r'\pi}\left[K(P) - E(P)\right].$$

And equation (6.27) becomes

$$\Delta u(r) = -\frac{1}{\pi\mu}\int_0^\infty S(r')\left[K(P) - E(P)\right]dr'. \qquad (6.28)$$

Then with the relative displacement known from equation (6.28), the displacement field is known through equation (2.4), and hence the axisymmetric torsion boundary value problem for a half space is solved.

Appendix A

Suppose that we have a boundary value problem in which S_1 in equation (1.2b) is the whole boundary of B, so that equation (1.2c) is absent. Then with a point force at $\underline{r}'$ the Green's function satisfies the inhomogeneous field equation given by

$$C_{ijkl}\, U_{lm,jk}(\underline{r},\underline{r}') = \delta_{im}\, \delta(\underline{r}-\underline{r}') \quad \text{for } \underline{r} \text{ in } B\,, \tag{A.1}$$

and satisfies the boundary condition

$$C_{ijkl}\, U_{lm,k}(\underline{r},\underline{r}') = 0 \qquad \text{on } \partial B = S_1 \;.$$

To balance the point force at $\underline{r}'$ it is necessary to place another point force of magnitude -1 at $\underline{r}_o$. To balance the torque created by the point forces at $\underline{r}'$ and $\underline{r}_o$ it is necessary to place a force $\underline{F}$ at $\underline{r}_o+\underline{\epsilon}$, $-\underline{F}$ at $\underline{r}_o-\underline{\epsilon}$, $\underline{F}'$ at $\underline{r}_o+\underline{\epsilon}'$, and $-\underline{F}'$ at $\underline{r}_0-\underline{\epsilon}'$, making $\underline{F}$ parallel to $\underline{\epsilon}'$ and making $\underline{F}'$ parallel to $\underline{\epsilon}$. See Figure (A.1).

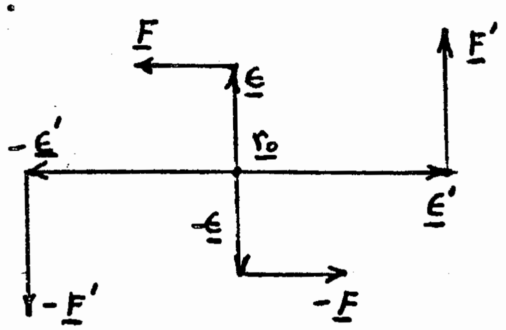


Figure (A.1)

The forces $\underline{F}$ and $\underline{F}'$ are chosen as to contribute equal torques. So,

$$\underline{\epsilon} \times \underline{F} = \underline{\epsilon}' \times \underline{F}' \quad ,$$

and

$$\underline{\epsilon} \times \underline{F} + \underline{\epsilon}' \times \underline{F}' = -\tfrac{1}{2} (\underline{r}' + \underline{r}_o) \times \underline{1}$$

or

$$\begin{aligned} \underline{\epsilon} \times \underline{F} &= \underline{\epsilon}' \times \underline{F}' \\ &= \tfrac{1}{4} (\underline{r}' - \underline{r}_o) \times \underline{1} \quad , \end{aligned}$$

where $\underline{1}$ is a unit vector representing the point forces at $\underline{r}_o$ and at $\underline{r}'$. Then equation (A.1) becomes:

$$\begin{aligned} C_{ijkl}\, U_{lm,jk}(\underline{r},\underline{r}') = {} & \delta_{im}\left[-\delta(\underline{r}-\underline{r}') + \delta(\underline{r}-\underline{r}_o)\right] \\ & - F_i\left[\delta(\underline{r}-\underline{r}_o-\underline{\epsilon}) - \delta(\underline{r}-\underline{r}_o+\underline{\epsilon})\right] \\ & - F_i'\left[\delta(\underline{r}-\underline{r}_o-\underline{\epsilon}') - \delta(\underline{r}-\underline{r}_o+\underline{\epsilon}')\right] . \end{aligned} \qquad (A.2)$$

Expanding to first order terms gives,

$$\begin{aligned} \delta(\underline{r}-\underline{r}_o-\underline{\epsilon}) &= \delta(\underline{r}-\underline{r}_o) - \epsilon_p \frac{\partial}{\partial x_p}\delta(r-r_o) - \cdots\cdots \\ \delta(\underline{r}-\underline{r}_o+\underline{\epsilon}) &= \delta(\underline{r}-\underline{r}_o) + \epsilon_p \frac{\partial}{\partial x_p}\delta(\underline{r}-\underline{r}_o) + \cdots\cdots . \end{aligned}$$

So to first order, we have

$$\delta(\underline{r}-\underline{r}_0-\underline{\epsilon}) - \delta(\underline{r}-\underline{r}_0+\underline{\epsilon}) = -2\epsilon_p \frac{\partial}{\partial x_p}\delta(\underline{r}-\underline{r}_0) \; .$$

Then equation (A.2) becomes

$$\begin{aligned} C_{ijkl}\, U_{lm,kj}(\underline{r},\underline{r}') &= \delta_{im}\left[\delta(\underline{r}-\underline{r}_0) - \delta(\underline{r}-\underline{r}')\right] \\ &\quad + 2\left(F_i \epsilon_p + F_i' \epsilon_p'\right)\delta(\underline{r}-\underline{r}_0)_{,p} \end{aligned}$$

(A.3)

Since $\underline{F} \parallel \underline{\epsilon}'$ and $\underline{F}' \parallel \underline{\epsilon}$, then $F_i = C_1 \epsilon_i'$ and $F_i' = C_2 \epsilon_i$. Also, since $\underline{\epsilon} \times \underline{F} = \underline{\epsilon}' \times \underline{F}'$, $\varepsilon_{ijk}\epsilon_j F_k = \varepsilon_{ijk}\epsilon_j' F_k'$. Combining these two conditions gives:

$$\varepsilon_{ijk}\,\epsilon_j\, C_1 \epsilon_k' = \varepsilon_{ijk}\,\epsilon_j'\, C_2\, \epsilon_k$$

$$\text{or } -\varepsilon_{ijk}\,\epsilon_k\, C_1 C_j' = \varepsilon_{ijk}\,\epsilon_j'\, C_2\, \epsilon_k$$

$$\Rightarrow C_1 = -C_2$$

So

$$F_i = C_1 \epsilon_i' \quad \text{and} \quad F_i' = -C_1 \epsilon_i \; ,$$

(A.4)

Now

$$\varepsilon_{ijk} F_j \epsilon_k = \varepsilon_{ijk} F_j' \epsilon_k'$$

$$= -\tfrac{1}{4}\, \varepsilon_{ijk} \left(r_j' - r_{oj}\right) 1_k \,.$$

So

$$\varepsilon_{ijk} F_j \epsilon_k + \varepsilon_{ijk} F_j' \epsilon_k'$$

$$= \tfrac{1}{4}\, \varepsilon_{ijk} \left(r_k' - r_{ok}\right) 1_j - \tfrac{1}{4}\, \varepsilon_{ijk} \left(r_j' - r_{oj}\right) 1_k \,.$$

or

$$0 = \varepsilon_{ijk} \Big[F_j \epsilon_k + F_j' \epsilon_k'$$

$$- \tfrac{1}{4} \left(r_k' - r_{ok}\right) 1_j + \tfrac{1}{4} \left(r_j - r_{oj}\right) 1_k \Big]$$

Taking the antisymmetrical part of the above term in brackets gives

$$0 = \frac{F_j \epsilon_k - F_k \epsilon_j}{2} + \frac{F_j' \epsilon_k' - F_k' \epsilon_j'}{2}$$

$$- \tfrac{1}{4} \left(r_k' - r_{ok}\right) \delta_{jm} + \tfrac{1}{4} \left(r_j' - r_{oj}\right) \delta_{km}$$

Using equation (A.4), we have

$$0 = \tfrac{1}{2}\left(C_1 \epsilon_j' \epsilon_k - C_1 \epsilon_k' \epsilon_j - C_1 \epsilon_j \epsilon_k' + C_1 \epsilon_k \epsilon_j'\right) - \tfrac{1}{4}(r_k' - r_{ok})\delta_{jm} + \tfrac{1}{4}(r_j' - r_{oj})\delta_{km} \quad ,$$

$$0 = C_1 \epsilon_j' \epsilon_k - C_1 \epsilon_k \epsilon_j' - \tfrac{1}{4}(r_k' - r_{ok})\delta_{jm} + \tfrac{1}{4}(r_j' - r_{oj})\delta_{km} \quad ,$$

or again from (A.4)

$$F_j \epsilon_k + F_j' \epsilon_k' = \tfrac{1}{4}(r_j' - r_{oj})\delta_{km} - \tfrac{1}{4}(r_k' - r_{ok})\delta_{jm} \quad . \tag{A.5}$$

Using (A.5) in (A.3) gives

$$C_{ijkl}\, u_{lm,jk}(\underline{r},\underline{r}') = \delta_{im}\left[\delta(\underline{r}-\underline{r}_o) - \delta(\underline{r}-\underline{r}')\right] + \tfrac{1}{2}\left[(r_i' - r_{oi})\delta_{pm} - (r_p' - r_{op})\delta_{pm}\right]\delta(\underline{r}-\underline{r}_o)_{,p} \quad , \tag{A.6}$$

$$\text{for } \underline{r} \in B .$$

Now we are ready to apply the Gauss divergence theorem on $u_i(\underline{r})$ and $u_{lm}(\underline{r},\underline{r}')$ as before. In addition to equation (A.6) Green's function, $u_{lm}(\underline{r},\underline{r}')$ satisfies the equation

$$C_{ijkl}\, u_{lm,k}(\underline{r},\underline{r}')\, n_j = 0, \qquad \text{for } \underline{r} \text{ on } \partial B = S_1 . \tag{A.7}$$

We follow the same procedure as in the mixed case except now we use all of B as the integration region :

$$\int_{S_1} \left[u_i(\underline{r})\, u_{lm,k}(\underline{r},\underline{r}') - u_{lm}(\underline{r},\underline{r}')\, u_{i,k}(\underline{r}) \right] n_j C_{ijkl}\, dS$$
$$= \int_B \left[u_i(\underline{r})\, u_{lm,k}(\underline{r},\underline{r}') - u_{lm}(\underline{r},\underline{r}')\, u_{i,k}(\underline{r}) \right]_{,j} C_{ijkl}\, dV. \tag{A.8}$$

The right hand side of equation (A.8) is

$$\int_B \Big[u_i(\underline{r})\, u_{lm,kj}(\underline{r},\underline{r}') + u_{i,j}(\underline{r})\, u_{lm,k}(\underline{r},\underline{r}')$$
$$- u_{lm}(\underline{r},\underline{r}')\, u_{i,kj}(\underline{r})$$
$$- u_{lm,j}(\underline{r},\underline{r}')\, u_{i,k}(\underline{r}) \Big] C_{ijkl}\, dV .$$

Because of the symmetry of C_{ijkl}, and by using the field equations and boundary values for $u_i(\underline{r})$, the right hand side of equation (A.8) becomes:

$$\int_B \left[u_i(\underline{r})\, u_{\ell m,kj}(\underline{r},\underline{r}') + u_{\ell m}(\underline{r})\, f_\ell(\underline{r}) \right] C_{ijk\ell}\, dV \; .$$

Using (A.7), the left hand side of (A.8) becomes

$$- \int_{S_1} u_{\ell m}(\underline{r},\underline{r}')\, u_{i,k}(\underline{r})\, n_j\, C_{ijk\ell}\, dS$$

$$= - \int_{S_1} u_{\ell m}(\underline{r},\underline{r}')\, t_\ell(\underline{r})\, dS \; .$$

Combining the left and right hand side gives

$$- \int_{S_1} u_{\ell m}(\underline{r},\underline{r}')\, t_\ell(\underline{r})\, dS$$

$$= \int_B \left[u_i(\underline{r})\, u_{\ell m,kj}(\underline{r},\underline{r}') + u_{\ell m}(\underline{r},\underline{r}')\, f_\ell(\underline{r}) \right] C_{ijk\ell}\, dV .$$

Then using (A.6) we have

$$\int_{S_1} u_{\ell m}(\underline{r},\underline{r}')\, t_\ell(\underline{r})\, dS + \int_B u_{\ell m}(\underline{r},\underline{r}')\, f_\ell(\underline{r})\, dV$$

$$= - \int_B u_i(\underline{r}) \Big\{ \delta_{im} \left[\delta(\underline{r}-\underline{r}_0) - \delta(\underline{r}-\underline{r}') \right]$$

$$+ \tfrac{1}{2} \left[(r_i' - r_{0i})\delta_{pm} - (r_p' - r_{0p})\delta_{pm} \right] \delta(\underline{r}-\underline{r}_0)_{,j} \Big\} \cdot dV$$

$$= u_m(\underline{r}_0) + u_m(\underline{r}') - \tfrac{1}{2}(r_p' - r_{0p}) \left[u_{m,p}(\underline{r}_0) - u_{pm}(\underline{r}_0) \right] \qquad \text{(A.9)}$$

Notice that the terms involving $\underline{r}_0$ represent a rigid translation and rotation about $\underline{r}_o$. That is, the traction (Neumann) boundary value problem is unique up to an arbitrary rigid displacement. So regarding the terms involving $\underline{r}_o$ as arbitrary constants that can be specified independently from the rest of the problem, equation (A.9) becomes

$$u_m(\underline{r}') = \int_{\partial B = S_1} u_{\ell m}(\underline{r},\underline{r}')\, t_\ell \, dS + \int_{B} u_{\ell m}(\underline{r},\underline{r}')\, f_\ell \, dV. \tag{A.10}$$

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