AN EXPOSITION OF GARRISON'S NON-TRANSFORMATION METHOD

by

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ABSTRACT.

This thesis is, basically, a clear explanation of B. Kvarda Garrison's non-transformation proof of Mann's Density Theorem.

Chapter 1 consists of preliminary work which includes lemmas, observations, and a basic construction. The theorem is proved in Chapter 2 and in Chapter 3 examples of Garrison's method are presented. Chapter 4 is the conclusion of the paper. Some possible applications of the method are stated in this chapter.

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INTRODUCTION.

This thesis is based on a new, non-transformation proof of Mann's Density Theorem. The proof was first given by B. Kvarda Garrison [3]. In this paper we generalize some of the arguments in her proof, creating out of them several lemmas which may be of independent interest. Before expanding on this, we must make some definitions and provide some notation.

Let P be the set of positive integers. If A and B are subsets of P, then

$$A + B = A \cup B \cup \{a + b : a \in A \text{ and } b \in B\}.$$

If m and n are positive integers such that $m \leq n$, then $[m,n] = \{x \in P : m \leq x \leq n\}$. If $m = 1$, then $[m,n]$ is called a fundamental set.

If A and F are subsets of P with F finite, then A(F) is the number of integers in $A \cap F$. When $F = [m,n]$, we shall omit the parentheses and write $A[m,n]$ for $A([m,n])$. If $m = 1$, we shall write $A(n)$ for $A[m,n]$.

Definition. If A is a subset of P, then the Schnirelmann Density of A, denoted d(A), is the greatest lower bound of $\{A(n)/n : n \in P\}$. That is

$$d(A) = \text{glb}\{A(n)/n : n \in P\}.$$

Using this definition of density, L. Schnirelmann [8] was able to prove the following lemma: if $C = A + B$, then

$$d(C) \geq d(A) + d(B) - d(A)d(B).$$

This is known as the Landau-Schnirelmann Inequality. It first appeared in 1930. The next year, Edmund Landau and Schnirelmann noted that for every example they had found, the expression $d(A)d(B)$ could be dropped from this inequality, provided that $d(A) + d(B) \leq 1$. They conjectured the following theorem: if $d(A) + d(B) \leq 1$, then

$$d(C) \geq d(A) + d(B).$$

The simplicity of this conjecture captured the attention of many mathematicians in the 1930's with the result that a considerable amount of research time was spent on it. A. Khinchin [5] writes that in 1935 half the English mathematicians were devoting their full attention to this problem. Nevertheless, the proof eluded everyone until 1942 when H. Mann [6] published his famous theorem which may be stated in the following form:

<u>Mann's Density Theorem</u>. Let A and B be subsets of P, let $C = A + B$, and let R be any fundamental set. Then either $C(R) = P(R)$ or there exists a fundamental set $W = [1,m]$ such that $m \notin C$, $W \subset R$, and

$$\frac{C(R)}{P(R)} \geq \frac{A(W) + B(W)}{P(W)}.$$

The Landau-Schnirelmann conjecture follows immediately from this theorem.

Mann attacked the theorem in a completely different manner from his contemporaries. His proof relies on transforming the sets A and B whereas other mathematicians had been searching for a "straightforward" proof involving the counting of elements of the various sets in certain intervals.

Another proof (F. Dyson [1]) has appeared since Mann's proof. However, although the methods differ, Dyson's proof also depends on the transformation of A and B. Neither of the proofs is particularly illuminating as to what happens when two sets are added together. Also any attempt to generalize either proof to two or more dimensions (in a certain way - see conclusion) has failed.

Until recent months, no new proofs of the theorem had appeared. Then, in 1968, B. Kvarda Garrison [3] proved a slightly weaker form of Mann's Theorem without transforming A or B in any manner. Garrison's Theorem the same as that of Dyson. The Landau-Schnirelmann conjecture also follows immediately from this theorem. The theorem is stated in the following form:

<u>Garrison's Theorem</u>. Let A and B be subsets of P, let $C = A + B$, and let R be any fundamental set for which $P(R - C) = k \geq 1$. Then there exists a fundamental set W such that

$$\frac{C(R)}{P(R)} \geq \frac{A(W) + B(W)}{P(W)}.$$

Although complicated, Garrison's proof involves only the counting of elements of A, B, and C in certain intervals and does not involve the transformation of the sets A and B. Thus, Garrison has finally discovered the proof that eluded so many great mathematicians in the 1930's.

Originally, Garrison looked for a new proof of Mann's Theorem in the hope that it might generalize, in a certain way, to the n-dimensional case. However, after she found the new proof, Garrison constructed an example [4] which clearly demonstrates that the above theorem is not true in the certain n-dimensional case. Despite this discovery, her proof may yet be useful for other applications. There are other more general, systems than the natural numbers for which it is not known whether the Landau-Schnirelmann conjecture holds or not (see conclusion). Modifications of Garrison's method may provide a proof in some of these systems.

Garrison's proof is divided into cases. An induction argument is applied to the cases and the number of cases is eventually shown to be finite. Much of the work in her proof is repetious. In this paper this material is gathered together into more general lemmas and constructions and some of the proofs have been simplified. These lemmas comprise Chapter 1 of this paper. Although everything in Chapter 1 is applied in the proof of the theorem or in other lemmas, a good portion of the material seems to be interesting in itself.

The proof of Garrison's Theorem appears in Chapter 2. I have kept Garrison's general format of dividing the proof into cases. However, more emphasis has been placed on the induction argument and more detail has been included.

In Chapter 3, two examples are presented which clearly illustrate the method of proof. These examples were not easily found but it is clear now that they are very general in nature and many more can be found fairly quickly. No other examples of Garrison's method have, to my knowledge, been found and thus I do not know if there exist examples which are different in nature from the ones presented here.

Chapter 4 is the conclusion of this paper. In it, I attempt to show in which directions we can go from this point and to illustrate the research potential of the thesis.

Although simple counting arguments are used throughout the paper, the proofs are very complicated. In order to increase the understanding of the proof it is suggested that the paper be read in the following order: (1) Chapter 1 (thoroughly), (2) Chapter 2 (lightly), (3) Chapter 3 (thoroughly), (4) Chapter 2 (thoroughly), (5) Chapter 4.

CHAPTER 1.

PRELIMINARY LEMMAS AND OBSERVATIONS.

We shall first prove two basic results that do not require any further definitions.

Lemma 1. Let R be a non-empty , finite subset of P and let k , m , and n be positive integers such that m + n = k. If S and T are two non-empty sets that partition R such that

$$\frac{P(S)}{m} \geq \frac{P(T)}{n}$$

then

$$\frac{P(R) - k}{P(R)} \geq \frac{P(T) - n}{P(T)}.$$

Proof. Let us assume that $\frac{P(R) - k}{P(R)} < \frac{P(T) - n}{P(T)}$.

Then $P(R)P(T) - kP(T) < P(R)P(T) - nP(R)$ implies $kP(T) > nP(R)$ implies $mP(T) + nP(T) > nP(S) + nP(T)$ implies $\frac{P(T)}{n} > \frac{P(S)}{m}$. This is a contradiction. Therefore,

$$\frac{P(R) - k}{P(R)} \geq \frac{P(T) - n}{P(T)} .$$

<u>Lemma 2</u>. Let A , B , and X be subsets of P with X finite. Let $g \in P - (A + B)$ such that if $x \in X$, then $x < g$. Let $Y = \{g - x : x \in X\}$. If $P(X) = A(X) + B(X) - u$, then

$$P(Y) \geq A(Y) + B(Y) + u.$$

<u>Proof</u>. If $a \in A \cap X$, then $g - a \in Y - B$. Thus $A(X) \leq P(Y - B) = P(Y) - B(Y)$. Hence

$$B(Y) \leq P(Y) - A(X) = P(X) - A(X) = B(X) - u$$

which implies $B(X) \geq B(Y) + u$. Similarly, $A(X) \geq A(Y) + u$. Therefore, $A(X) + B(X) \geq A(Y) + B(Y) + 2u$ which implies that

$$P(Y) = P(X) = A(X) + B(X) - u \geq A(Y) + B(Y) + u.$$

The next observation is a very old result first proved by Schnirelmann using a counting argument. Our proof, however is based on Lemma 2.

<u>Observation 1</u>. If $g \in P - (A + B)$, then

$$P(g) \geq A(g) + B(g) + 1.$$

<u>Proof</u>. Suppose that $P(g - 1) \geq A(g - 1) + B(g - 1)$. Then

$$\begin{aligned} P(g) &= P(g - 1) + 1 \\ &\geq A(g - 1) + B(g - 1) + 1 \end{aligned}$$

$$= A(g) + B(g) + 1 .$$

Now suppose that $P(g - 1) < A(g - 1) + B(g - 1)$. Then $P(g - 1) = A(g - 1) + B(g - 1) - u$, for some $u > 0$. Note that $[1 , g - 1] = \{g - x : x \in [1 , g - 1]\}$. By Lemma 2 ,

$$\begin{aligned} P(g - 1) &\geq A(g - 1) + B(g - 1) + u \\ &> A(g - 1) + B(g - 1). \end{aligned}$$

But this contradicts our assumption that $P(g - 1) < A(g - 1) + B(g - 1)$. Therefore ,

$$P(g) \geq A(g) + B(g) + 1 .$$

Before proving any more lemmas we shall make some definitions.

<u>Definition 1.</u> For $k \in P$, let

$$H_k = \{(y_1 , y_2 , \ldots , y_k) \in P^k : y_1 < y_2 < \ldots < y_k\}.$$

<u>Definition 2</u>. Let $k \in P$ and let $\Gamma = (g_1 , g_2 , \ldots , g_k) \in H_k$. Then

$$i(x , \Gamma) = \begin{cases} 0 & \text{if } 1 \leq x < g_1 \\ s & \text{if } g_s \leq x < g_{s+1} \text{ and } 0 < s < k \\ k & \text{if } g_k \leq x . \end{cases}$$

If it is clear which $\Gamma \varepsilon H_k$ we are referring to, we write i_x for $i(x, \Gamma)$.

<u>Definition 3</u>. Let Γ be as in Definition 2. Let $a \varepsilon P$ be such that $a \leq g_1$. Then

$$M(a,\Gamma) = \{x\varepsilon[a,g_k] : P[a,x] \geq A[a,x] + B[a,x]+i_x-\delta_a\}$$

where $\delta_a = \begin{cases} 0 & \text{if } a = 1 \\ 1 & \text{if } a > 1 \end{cases}$.

The previous three definitions are made in order to clarify the following definition which will prove to be very useful for the rest of the paper.

<u>Definition 4</u>. If $M(a,\Gamma) \neq \phi$, let

$$\alpha = \alpha(a,\Gamma) = 1 + \max\{x : x \varepsilon M(a,\Gamma)\}.$$

Thus $\alpha-1 = \max\{x \varepsilon [a,g_k] : P[a,x] \geq A[a,x]+B[a,x]+i_x-\delta_a\}$. Also let $s = i_{\alpha-1}$ and $t = k - s$. Note that $0 \leq s \leq k$, $0 \leq t \leq k$, and $\alpha-1 \leq g_k$.

Now we shall prove some lemmas using the notation we have developed in Definitions 1 through 4, in each case assuming α exists.

<u>Lemma 3</u>. If α exists and $\alpha \le x \le g_k$, then

$$P[\alpha,x] \le A[\alpha,x] + B[\alpha,x] + i_x - s - 1 .$$

<u>Proof</u>. From Definition 4 , we have

$$\begin{aligned} P[\alpha,x] &= P[a,x] - P[a,\alpha-1] \\ &\le A[a,x] + B[a,x] + i_x - \delta_a - 1 - A[a,\alpha-1] \\ &\qquad - B[a,\alpha-1] - s + \delta_a \\ &= A[\alpha,x] + B[\alpha,x] + i_x - s - 1 . \end{aligned}$$

Therefore , $P[\alpha,x] \le A[\alpha,x] + B[\alpha,x] + i_x - s - 1$.

<u>Observation 2</u>. If α exists , $\alpha \le x \le g_k$, and $P[\alpha,x] \ge A[\alpha,x] + B[\alpha,x] + u$, then $u \le i_x - s - 1$ which implies $s + u + 1 \le i_x$.

<u>Lemma 4</u>. Assume α exists and $t > 0$. If $g_{s+1} \notin A \cup B$, then

(i) $\alpha < g_{s+1}$ and

(ii) $\alpha \in A \cap B$.

<u>Proof of (i)</u>. Since $t > 0$, $s < k$. Clearly $\alpha \le g_{s+1}$ since $\alpha-1 < g_{s+1}$. If $\alpha = g_{s+1}$, then $i_\alpha = s+1$ and

$$P[a,\alpha] = P[a,\alpha-1] + 1$$

$$\geq A[a,\alpha-1] + B[a,\alpha-1] + s - \delta_a + 1$$
$$= A[a,\alpha] + B[a,\alpha] + i_\alpha - \delta_a$$

which contradicts the maximality of $\alpha-1$.
Hence $\alpha < g_{s+1}$. Note, therefore, that $i_\alpha = s$.

<u>Proof of (ii)</u>. From the definition of α, we get

$$\begin{aligned} A[a,\alpha-1] &+ B[a,\alpha-1] + s - \delta_a + 1 \\ &\leq P[a,\alpha-1] + 1 \\ &= P[a,\alpha] \\ &< A[a,\alpha] + B[a,\alpha] + s - \delta_a \\ &= A[a,\alpha-1] + A[\alpha,\alpha] + B[a,\alpha-1] + B[\alpha,\alpha] + s - \delta_a. \end{aligned}$$

Cancelling on both sides of the inequality, we obtain: $1 < A[\alpha,\alpha] + B[\alpha,\alpha]$. Therefore, $A[\alpha,\alpha] = B[\alpha,\alpha] = 1$ and thus $\alpha \in A \cap B$.

<u>Lemma 5</u>. Assume α exists and $t > 0$. If $g_{s+j} \notin A + B$ for all j, $1 \leq j \leq t$, then

(i) $P(g_{s+1} - \alpha) \geq A(g_{s+1} - \alpha) + B(g_{s+1} - \alpha) + 1$

and

(ii) $g_{s+j} - \alpha \notin A \cup B$ for all j, $1 \leq j \leq t$.

Proof of (i). Notice that $[1,g_{s+1} - \alpha] = \{g_{s+1} - x : x \in [\alpha, g_{s+1} - 1]\}$. By Lemma 3 ,

$$P[\alpha,g_{s+1} - 1] \leq A[\alpha,g_{s+1} - 1] + B[\alpha,g_{s+1} - 1] - 1.$$

Therefore ,

$$P[\alpha,g_{s+1}-1] = A[\alpha,g_{s+1}-1] + B[\alpha,g_{s+1}-1] - 1 - u$$

for some $u \geq 0$. From Lemma 2 , we get

$$P(g_{s+1}-\alpha) \geq A(g_{s+1}-\alpha) + B(g_{s+1}-\alpha) + 1 + u$$
$$\geq A(g_{s+1}-\alpha) + B(g_{s+1}-\alpha) + 1.$$

Proof of (ii). Suppose there exists a $j_0 \in [1,t]$ such that $g_{s+j_0} - \alpha \in A \cup B$. Then since , by Lemma 4 , $\alpha \in A \cap B$, we have $g_{s+j_0} - \alpha + \alpha \in A + B$. That is , $g_{s+j_0} \in A + B$ for some $j_0 \in [1,t]$. This contradicts our assumption that $g_{s+j} \notin A + B$ for all j , $1 \leq j \leq t$. Therefore ,

$$g_{s+j} - \alpha \notin A \cup B \text{ for all } j \text{ , } 1 \leq j \leq t.$$

Before proving any more lemmas , we shall make a very important definition. The definition will involve the construction of a sequence of positive integers , $x_1, x_2, \ldots$, around which the proof of our theorem will be built.

Definition 5. Let $\Gamma = (g_1, g_2, \ldots, g_k) \in H_k$. If $\alpha(1,\Gamma)$ exists, let $x_1 = \alpha(1,\Gamma)$, $s_1 = i(x_1-1,\Gamma)$, and $t_1 = k - s_1$.

Assume x_1, s_1, and t_1 exist. If $t_1 > 0$, if $\Gamma_1 = (g_{s_1+1}-x_1, \ldots, g_k-x_1) \in H_{t_1}$, and if $\alpha(1,\Gamma_1)$ exists, then let $x_2 = \alpha(1,\Gamma_1)$, $s_2 = i(x_2-1,\Gamma_1)$, and $t_2 = t_1 - s_2$.

Let $n \geq 3$ and assume x_i, s_i, and t_i exist, $1 \leq i < n$. If $t_{n-1} > 0$, if

$$\Gamma_{n-1} = (g_{s_1+\ldots+s_{n-1}+1}-x_{n-1}, \ldots, g_k-x_{n-1}) \in H_{t_{n-1}},$$

and if $\alpha(x_{n-2},\Gamma_{n-1})$ exists, then let $x_n = \alpha(x_{n-2},\Gamma_{n-1})$, $s_n = i(x_n-1,\Gamma_{n-1})$, and $t_n = t_{n-1} - s_n$.

Observation 3. If x_i is defined, $i = 1,2,\ldots,n$, then

$$\sum_{i=1}^{n} s_i + t_n = k.$$

Proof. If $n = 1$, $t_1 = k - s_1$ implies $s_1 + t_1 = k$. Now assume the observation is true for all i, $1 \leq i < n$. Thus,

$$\sum_{i=1}^{n-1} s_i + t_{n-1} = k.$$

But $t_n = t_{n-1} - s_n$. Therefore,

$$\sum_{i=1}^{n} s_i + t_n = \sum_{i=1}^{n-1} s_i + s_n + t_n$$

$$= \sum_{i=1}^{n-1} s_i + t_{n-1}$$

$$= k.$$

We now prove some very important lemmas concerning the construction in Definition 5.

<u>Lemma 6</u>. Let $\Gamma = (g_1, g_2, \ldots, g_k) \in H_k$ and assume $g_i \notin A + B$, $1 \le i \le k$. Let $n \ge 3$ and assume x_i exists , $1 \le i \le n-1$. If $t_{n-1} > 0$, then

$$P[x_{n-2}, g_{s_1+\ldots+s_{n-1}+1} - x_{n-1}] \ge$$

$$A[x_{n-2}, g_{s_1+\ldots+s_{n-1}+1} - x_{n-1}] + B[x_{n-2}, g_{s_1+\ldots+s_{n-1}+1} - x_{n-1}].$$

<u>Proof</u>. Recall from the definitions of x_{n-1} and s_{n-1} that $g_{s_1+\ldots+s_{n-1}} - x_{n-2} \le x_{n-1} - 1 < g_{s_1+\ldots+s_{n-1}+1} - x_{n-2}$. Also from Definition 5 , we have

$$\Gamma_{n-2} = (g_{s_1+\ldots+s_{n-2}+1} - x_{n-2}, \ \ldots \ , g_k - x_{n-2}) \in H_{t_{n-2}}.$$

Thus $i(g_{s_1+\ldots+s_{n-1}+1} - x_{n-2}, \Gamma_{n-2}) = s_{n-1} + 1$. From Lemma 3 , we obtain

$$P[x_{n-1}, g_{s_1+\ldots+s_{n-1}+1}-x_{n-2}] \le A[x_{n-1}, g_{s_1+\ldots+s_{n-1}+1}-x_{n-2}]$$
$$+ B[x_{n-1}, g_{s_1+\ldots+s_{n-1}+1}-x_{n-2}] + s_{n-1} + 1 - s_{n-1} - 1$$
$$= A[x_{n-1}, g_{s_1+\ldots+s_{n-1}+1}-x_{n-2}]$$
$$+ B[x_{n-1}, g_{s_1+\ldots+s_{n-1}+1}-x_{n-2}].$$

Thus , $P[x_{n-1}, g_{s_1+\ldots+s_{n-1}+1}-x_{n-2}] =$
$A[x_{n-1}, g_{s_1+\ldots+s_{n-1}+1}-x_{n-2}] + B[x_{n-1}, g_{s_1+\ldots+s_{n-1}+1}-x_{n-2}] - u$
for some $u \ge 0$. Notice that , $[x_{n-2}, g_{s_1+\ldots+s_{n-1}+1}-x_{n-1}] =$
$\{g_{s_1+\ldots+s_{n-1}+1}-x : x \in [x_{n-1}, g_{s_1+\ldots+s_{n-1}+1}-x_{n-2}]\}$.
Therefore , by Lemma 2 , we have

$$P[x_{n-2}, g_{s_1+\ldots+s_{n-1}+1}-x_{n-1}] \ge A[x_{n-2}, g_{s_1+\ldots+s_{n-1}+1}-x_{n-1}]$$
$$+ B[x_{n-2}, g_{s_1+\ldots+s_{n-1}+1}-x_{n-1}].$$

The next lemma tells us exactly when the x_i's exist in the special case when $g_i \notin A + B$, $i = 1,2,\ldots,k$.

<u>Lemma 7</u>. Let $\Gamma = (g_1, g_2, \ldots, g_k) \in H_k$ be given. Suppose that $g_i \notin A + B$, $1 \le i \le k$. If $k > 0$, then $x_1 = \alpha(1,\Gamma)$, $s_1 = i(x_1-1,\Gamma)$, and $t_1 = k - s_1$ exist. Let $n > 1$. If x_i

exists , $1 \le i \le n-1$, and $t_{n-1} > 0$, then x_n , s_n , and t_n exist.

<u>Proof</u>. By Observation 1 , $P(g) \ge A(g) + B(g) + 1$, and thus $g_1 \in M(1,\Gamma)$. Hence $\alpha(1,\Gamma)$ exists. Therefore , $x_1 = \alpha(1,\Gamma)$, $s_1 = i(x_1-1,\Gamma)$, and $t_1 = k - s_1$ exist.

Suppose $t_1 > 0$. By Lemma 4(i) , $x_1 < g_{s_1+1}$. Thus $g_{s_1+1}-x_1 > 0$ and $\Gamma_1 \in H_{t_1}$. By Lemma 5(i) ,

$$P(g_{s_1+1}-x_1) \ge A(g_{s_1+1}-x_1) + B(g_{s_1+1}-x_1) + 1.$$

Thus $g_{s_1+1}-x_1 \in M(1,\Gamma_1)$ which implies $\alpha(1,\Gamma_1)$ exists. Therefore $x_2 = \alpha(1,\Gamma_1)$, $s_2 = i(x_2-1,\Gamma_1)$, and $t_2 = t_1 - s_2$ exist.

Let $n \ge 3$ and suppose x_{n-1} exists and $t_{n-1} > 0$. Clearly $x_{n-1} < g_{s_1+\ldots+s_{n-2}+s_{n-1}+1}-x_{n-2}$. Thus $0 < x_{n-2} < g_{s_1+\ldots+s_{n-1}+1}-x_{n-1}$. Hence $\Gamma_{n-1} \in H_{t_{n-1}}$. By Lemma 6 , $g_{s_1+\ldots+s_{n-1}+1}-x_{n-1} \in M(x_{n-2},\Gamma_{n-1})$. Therefore , $\alpha(x_{n-2},\Gamma_{n-1})$ exists and thus $x_n = \alpha(x_{n-2},\Gamma_{n-1})$, $s_n = i(x_n-1,\Gamma_{n-1})$, and $t_n = t_{n-1} - s_n$ exist.

Observation 4. If $g_i \notin A + B$, $1 \leq i \leq k$, then there exists an $m \in P$ such that $t_m = 0$. Thus the sequence of x_i's is finite.

Proof. If x_1 exists , then , since $1 \leq s_1 \leq k$, we have $0 \leq t_1 < k$.

Suppose x_i exists , $1 \leq i \leq n-1$, and $t_{n-1} > 0$. Then x_n , s_n , and t_n exist. Since $1 \leq s_n \leq t_{n-1}$, we have

$$0 \leq t_n < t_{n-1} < \ldots < t_2 < t_1 < k.$$

Since k is finite , there must exist an $m \in P$ such that $t_m = 0$. Therefore x_{m+1} does not exist and the sequence of x_i's is finite.

In the next two lemmas we assume $g_i \notin A + B$, for all i , $1 \leq i \leq k$.

Lemma 8. Suppose $x_1 = \alpha(1,\Gamma)$ exists and $t_1 > 0$. Let $\hat{k}$ be any integer such that $s_1 < \hat{k} \leq k$ and let $\hat{\Gamma}_1 = (g_{s_1+1}-x_1,\ldots,g_{\hat{k}}-x_1)$. Assume

$$\frac{(A + B)(g_k)}{P(g_k)} \geq \frac{P(g_{\hat{k}}-x_1) - \hat{t}_1}{P(g_{\hat{k}}-x_1)}$$

where $\hat{t}_1 = \hat{k} - s_1$. Then $\hat{x}_2 := \alpha(1,\hat{r}_1)$ exists and

(i) If $\hat{t}_2 = 0$, then $\dfrac{(A + B)(g_k)}{P(g_k)} \geq \dfrac{A(\hat{x}_2-1) + B(\hat{x}_2-1)}{P(\hat{x}_2-1)}$,

(ii) If $\hat{t}_2 > 0$, and $\dfrac{P(\hat{x}_2-1)}{\hat{s}_2} \leq \dfrac{P[\hat{x}_2, g_{\hat{k}}-x_1]}{\hat{t}_2}$, then

$$\frac{(A + B)(g_k)}{P(g_k)} \geq \frac{A(\hat{x}_2-1) + B(\hat{x}_2-1)}{P(\hat{x}_2-1)},$$

(iii) If $\hat{t}_2 > 0$ and $\dfrac{P(\hat{x}_2-1)}{\hat{s}_2} > \dfrac{P[\hat{x}_2, g_{\hat{k}}-x_1]}{\hat{t}_2}$, then

$$\frac{(A + B)(g_k)}{P(g_k)} \geq \frac{P[x_1, g_{\hat{k}}-\hat{x}_2] - \hat{t}_2}{P[x_1, g_{\hat{k}}-\hat{x}_2]}.$$

<u>Proof</u>. Clearly $\hat{x}_2 = \alpha(1,\hat{r}_1)$ exists.

<u>Proof of (i)</u>. If $\hat{t}_2 = 0$, then $\hat{s}_2 = \hat{t}_1$ and $\hat{x}_2-1 = g_{\hat{k}}-x_1$. Therefore, by the definition of $\hat{x}_2$,

$$\frac{(A + B)(g_k)}{P(g_k)} \geq \frac{P(g_{\hat{k}}-x_1) - \hat{t}_1}{P(g_{\hat{k}}-x_1)}$$

$$= \frac{P(\hat{x}_2-1) - \hat{s}_2}{P(\hat{x}_2-1)}$$

$$\geq \frac{A(\hat{x}_2-1) + B(\hat{x}_2-1)}{P(\hat{x}_2-1)}.$$

Proof of (ii). We shall use Lemma 1 for the first time. Let $R = [1, g_{\hat{k}} - x_1]$, $T = [1, \hat{x}_2 - 1]$, $S = [\hat{x}_2, g_{\hat{k}} - x_1]$, $n = \hat{s}_2$, $m = \hat{t}_2$, and $k = \hat{t}_1$. By Lemma 1,

$$\frac{(A+B)(g_k)}{P(g_k)} \geq \frac{P(g_{\hat{k}} - x_1) - \hat{t}_1}{P(g_{\hat{k}} - x_1)}$$

$$\geq \frac{P(\hat{x}_2 - 1) - \hat{s}_2}{P(\hat{x}_2 - 1)}$$

$$\geq \frac{A(\hat{x}_2 - 1) + B(\hat{x}_2 - 1)}{P(\hat{x}_2 - 1)}.$$

Proof of (iii). Once again, we shall use Lemma 1, with $R = [1, g_{\hat{k}} - x_1]$, $S = [1, \hat{x}_2 - 1]$, $T = [\hat{x}_2, g_{\hat{k}} - x_1]$, $n = \hat{t}_2$, $m = \hat{s}_2$, and $k = \hat{t}_1$. By Lemma 1,

$$\frac{(A+B)(g_k)}{P(g_k)} \geq \frac{P(g_{\hat{k}} - x_1) - \hat{t}_1}{P(g_{\hat{k}} - x_1)}$$

$$\geq \frac{P[\hat{x}_2, g_{\hat{k}} - x_1] - \hat{t}_2}{P[\hat{x}_2, g_{\hat{k}} - x_1]}$$

$$= \frac{P[x_1, g_{\hat{k}} - \hat{x}_2] - \hat{t}_2}{P[x_1, g_{\hat{k}} - \hat{x}_2]}.$$

<u>Lemma 9</u>. Let $n \geq 4$ and suppose x_i , s_i , and t_i exist , $1 \leq i \leq n-1$. Also suppose that $t_{n-1} > 0$. Let $\hat{k}$ be any integer such that $\sum_{i=1}^{n-1} s_i < \hat{k} \leq k$ and let

$$\hat{\Gamma}_{n-1} = (g_{s_1+\ldots+s_{n-1}+1} - x_{n-1}, \ldots, g_{\hat{k}} - x_{n-1}).$$

Assume

$$\frac{(A + B)(g_k)}{P(g_k)} \geq \frac{P[x_{n-2}, g_{\hat{k}} - x_{n-1}] - \hat{t}_{n-1}}{P[x_{n-2}, g_{\hat{k}} - x_{n-1}]}$$

where $\hat{t}_{n-1} = \hat{k} - \sum_{i=1}^{n-1} s_i$. Then $\hat{x}_n = \alpha(x_{n-2}, \hat{\Gamma}_{n-1})$, $\hat{s}_n$, and $\hat{t}_n$ exist and if either

(i) $\hat{t}_n = 0$ or

(ii) $\hat{t}_n > 0$ and $\dfrac{P[x_{n-2}, \hat{x}_n - 1]}{\hat{s}_n} \leq \dfrac{P[\hat{x}_n, g_{\hat{k}} - x_{n-1}]}{\hat{t}_n}$,

then

$$\frac{(A + B)(g_k)}{P(g_k)} \geq \frac{P[x_{n-3}, g_{s_1+\ldots+s_{n-2}+\hat{s}_n} - x_{n-2}] - \hat{s}_n}{P[x_{n-3}, g_{s_1+\ldots+s_{n-2}+\hat{s}_n} - x_{n-2}]}.$$

<u>Proof</u>. Clearly $\hat{x}_n = \alpha(x_{n-2}, \hat{\Gamma}_{n-1})$, $\hat{s}_n$, and $\hat{t}_n$ exist. By the definition of $\hat{x}_n$, we have

$$P[x_{n-2}, \hat{x}_n - 1] \geq A[x_{n-2}, \hat{x}_n - 1] + B[x_{n-2}, \hat{x}_n - 1] + \hat{s}_n - 1.$$

By Observation 2 , $s_{n-2} + \hat{s}_n \le i(\hat{x}_n - 1, \Gamma_{n-3}) = i$, say , and $g_{s_1+\dots+s_{n-2}+\hat{s}_n} - x_{n-3} \le g_{s_1+\dots+s_{n-3}+i} - x_{n-3} \le \hat{x}_n - 1$ which implies

$$P[x_{n-2}, \hat{x}_n - 1] \ge P[x_{n-2}, g_{s_1+\dots+s_{n-2}+\hat{s}_n} - x_{n-3}]$$

$$= P[x_{n-3}, g_{s_1+\dots+s_{n-2}+\hat{s}_n} - x_{n-2}].$$

Now suppose (i) holds. Then $0 = \hat{t}_{n-1} - \hat{s}_n$ which implies $\hat{s}_n = \hat{t}_{n-1}$. Thus $\hat{k} = \sum_{i=1}^{n-1} s_i + \hat{t}_{n-1} = \sum_{i=1}^{n-1} s_i + \hat{s}_n$ which implies $\hat{x}_n - 1 = g_{\hat{k}} - x_{n-1}$. Therefore , we have ,

$$\frac{(A + B)(g_k)}{P(g_k)} \ge \frac{P[x_{n-2}, g_{\hat{k}} - x_{n-1}] - \hat{t}_{n-1}}{P[x_{n-2}, g_{\hat{k}} - x_{n-1}]}$$

$$= \frac{P[x_{n-2}, \hat{x}_n - 1] - \hat{s}_n}{P[x_{n-2}, \hat{x}_n - 1]}$$

$$\ge \frac{P[x_{n-3}, g_{s_1+\dots+s_{n-2}+\hat{s}_n} - x_{n-2}] - \hat{s}_n}{P[x_{n-3}, g_{s_1+\dots+s_{n-2}+\hat{s}_n} - x_{n-2}]}.$$

Now suppose (ii) holds. Then , by Lemma 1 ,

$$\frac{(A + B)(g_k)}{P(g_k)} \ge \frac{P[x_{n-2}, g_{\hat{k}} - x_{n-1}] - \hat{t}_{n-1}}{P[x_{n-2}, g_k - x_{n-1}]}$$

$$\ge \frac{P[x_{n-2}, \hat{x}_n - 1] - \hat{s}_n}{P[x_{n-2}, \hat{x}_n - 1]}$$

$$\geq \frac{P[x_{n-3}, g_{s_1+\ldots+s_{n-2}+\hat{s}_n-x_{n-2}}] - \hat{s}_n}{P[x_{n-3}, g_{s_1+\ldots+s_{n-2}+\hat{s}_n-x_{n-2}}]}.$$

Corollary. Let $n = 3$ and let all other hypotheses be the same as in the previous lemma. Then

$$\frac{(A + B)(g_k)}{P(g_k)} > \frac{P(g_{s_1+\hat{s}_3-x_1}) - \hat{s}_3}{P(g_{s_1+\hat{s}_3-x_1})}.$$

Proof. Use the same proof as in Lemma 9 , letting $x_0 = 0$ and $P[0,b] = P(b)$, for any $b \in P$.

CHAPTER 2.

PROOF OF THE THEOREM.

Before going on with the proof we shall state the theorem to be proved once more.

Theorem. Let A and B be subsets of P, let C = A + B, and let R be any fundamental set for which $P(R - C) = k \geq 1$. Then there exists a fundamental set $W \subset R$ such that

$$\frac{C(R)}{P(R)} \geq \frac{A(W) + B(W)}{P(W)}.$$

Thus in order to prove the theorem we must find a fundamental set W which satisfies the conditions of the theorem. The proof will consist of a systematic search for such a W.

Proof of the theorem.

Let $\{g_1, g_2, \ldots, g_k\}$ be the k elements in R - C listed in the usual order so that $\Gamma = (g_1, g_2, \ldots, g_k) \in H_k$. Clearly $R = [1,N]$ for some $N \geq g_k$ and thus, if $N > g_k$, we have

$$\frac{C(R)}{P(R)} = \frac{P(R) - k}{P(R)} = \frac{P(g_k) + P[g_k+1,N] - k}{P(g_k) + P[g_k+1,N]}$$

$$\geq \frac{P(g_k) - k}{P(g_k)}$$

$$= \frac{C(g_k)}{P(g_k)}.$$

Therefore, it is sufficient to assume $R = [1,g_k]$.

By Lemma 7, $x_1 = \alpha(1,\Gamma)$, $s_1 = i(x_1-1,\Gamma)$, and $t_1 = k - s_1$ exist. Clearly $1 \leq s_1 \leq k$ and $0 \leq t_1 < k$.

<u>Case 1.1</u>. Suppose $t_1 = 0$. Then $s_1 = k$ and $x_1-1 = g_k$ which implies $P(R) \geq A(R) + B(R) + k$ or $C(R) \geq A(R) + B(R)$. Thus we have

$$\frac{C(R)}{P(R)} \geq \frac{A(R) + B(R)}{P(R)}.$$

Therefore, let $W = R$ and we are done.

<u>Case 1.2</u>. Suppose $t_1 > 0$ and $\dfrac{P(x_1-1)}{s_1} \leq \dfrac{P[x_1,g_k]}{t_1}$.

Then, by Lemma 1 and the choice of x_1, we have

$$\frac{C(R)}{P(R)} \geq \frac{P(x_1-1) - s_1}{P(x_1-1)} \geq \frac{A(x_1-1) + B(x_1-1)}{P(x_1-1)}.$$

Therefore, let $W = [1, x_1 - 1]$ and we are done.

Case 1.3. Suppose $t_1 > 0$ and $\frac{P(x_1-1)}{s_1} > \frac{P[x_1, g_k]}{t_1}$.

Then, by Lemma 1, we get

$$\frac{C(R)}{P(R)} \geq \frac{P[x_1, g_k] - t_1}{P[x_1, g_k]}$$

$$= \frac{P(g_k - x_1) + 1 - t_1}{P(g_k - x_1) + 1}$$

$$> \frac{P(g_k - x_1) - t_1}{P(g_k - x_1)}.$$

We cannot immediately find a W that satisfies the theorem. Therefore, we must divide case 1.3 into three subcases: 2.1, 2.2, and 2.3. From this point on in the proof we are under the assumptions in case 1.3, although they will not be stated with each case.

By Lemma 7, $x_2 = \alpha(1, \Gamma_1)$, $s_2 = i(x_2 - 1, \Gamma_1)$, and $t_2 = t_1 - s_2$ exist. Clearly $1 \leq s_2 \leq t_1$ and $0 \leq t_2 < t_1 < k$.

Case 2.1. Suppose $t_2 = 0$. Then, by Lemma 8,

$$\frac{C(R)}{P(R)} \geq \frac{A(g_k - x_1) + B(g_k - x_1)}{P(g_k - x_1)}.$$

Therefore, let $W = [1, g_k - x_1]$ and we are done.

Case 2.2. Suppose $t_2 > 0$ and $\frac{P(x_2-1)}{s_2} \leq \frac{P[x_2, g_k - x_1]}{t_2}$.

Then, by Lemma 8,

$$\frac{C(R)}{P(R)} \geq \frac{A(x_2-1) + B(x_2-1)}{P(x_2-1)}.$$

Therefore, let $W = [1, x_2-1]$ and we are done.

Case 2.3. Suppose $t_2 > 0$ and $\frac{P(x_2-1)}{s_2} > \frac{P[x_2, g_k - x_1]}{t_2}$.

Then, by Lemma 8,

$$\frac{C(R)}{P(R)} \geq \frac{P[x_1, g_k - x_2] - t_2}{P[x_1, g_k - x_2]}.$$

Just as in case 1.3, we cannot immediately find a W that satisfies the theorem. Therefore, we subdivide case 2.3 into three subcases: 3.1, 3.2, and 3.3.

Since $t_2 > 0$, we may apply Lemma 7 to show that $x_3 = \alpha(x_1, \Gamma_2)$, $s_3 = i(x_3-1, \Gamma_2)$, and $t_3 = t_2 - s_3$ exist. Clearly $1 \leq s_3 \leq t_2$ and $0 \leq t_3 < t_2 < t_1 < k$.

Using the lemmas from Chapter 1, we will be able to handle case 3.1 and 3.2 together.

Case 3.1. Suppose $t_3 = 0$.

or

Case 3.2. Suppose $t_3 > 0$ and

$$\frac{P[x_1,x_3-1]}{s_3} \le \frac{P[x_3,g_k-x_2]}{t_3}.$$

Using the corollary to Lemma 9, we obtain

$$\frac{C(R)}{P(R)} > \frac{P(g_{s_1+s_3}-x_1) - s_3}{P(g_{s_1+s_3}-x_1)}.$$

Clearly $s_1 < s_1 + s_3 < k$. Let $k' = s_1 + s_3$ and we have

$$\frac{C(R)}{P(R)} > \frac{P(g_{k'}-x_1) - s_3}{P(g_{k'}-x_1)}.$$

Before proceeding with this case, we shall look at our sequence of x_i's under k' instead of k. Let $\Gamma' = (g_1, g_2, \ldots, g_{k'})$. Clearly $\Gamma' \varepsilon H_{k'}$. Since $g_1 \varepsilon M(1,\Gamma')$, $x_1' = \alpha(1,\Gamma')$, $s_1' = i(x_1'-1,\Gamma')$, and, $t_1' = k' - s_1'$ exist.

Note that $x_1-1 \varepsilon [1,g_{k'}]$ and thus $x_1-1 \varepsilon M(1,\Gamma')$ which implies $x_1' \ge x_1$. However $k' < k$. Thus $x_1' \le x_1$. Therefore, $x_1' = x_1$, $s_1' = s_1$, and $t_1' = k' - s_1 = s_3$.

Clearly $\Gamma_1' = (g_{s_1+1}-x_1, \ldots, g_{k'}-x_1) \in H_{t_1'}$. Since $t_1' = s_3 > 0$, $x_2' = \alpha(1,\Gamma_1')$, $s_2' = i(x_2'-1,\Gamma_1')$, and $t_2' = t_1' - s_2'$ exist. Clearly $1 \le s_2' \le t_1'$ and $0 \le t_2' < t_1' \le t_2 < t_1 < k$.

We now subdivide the combined cases, 3.1 and 3.2, into three subcases: 2.1', 2.2', and 2.3'.

Case 2.1'. Suppose $t_2' = 0$. By Lemma 8

$$\frac{C(R)}{P(R)} \geq \frac{A(g_{s_1+s_3}-x_1) + B(g_{s_1+s_3}-x_1)}{P(g_{s_1+s_3}-x_1)}.$$

Therefore, let $W = [1,g_{s_1+s_3}-x_1]$ and we are done.

Case 2.2'. Suppose $t_2' > 0$ and $\frac{P(x_2'-1)}{s_2'} \leq \frac{P[x_2',g_{k'}-x_1]}{t_2'}$.

By Lemma 8

$$\frac{C(R)}{P(R)} \geq \frac{A(x_2'-1) + B(x_2'-1)}{P(x_2'-1)}.$$

Therefore, let $W = [1,x_2'-1]$ and we are done.

Case 2.3'. Suppose $t_2' > 0$ and

$$\frac{P(x_2'-1)}{s_2'} > \frac{P[x_2', g_{k'}-x_1]}{t_2'}.$$

By Lemma 8

$$\frac{C(R)}{P(R)} \geq \frac{P[x_1, g_{k'}-x_2'] - t_2'}{P[x_1, g_{k'}-x_2']}.$$

Just as in case 2.3, we cannot immediately find a W that satisfies the theorem. Therefore, we subdivide case 2.3' into three subcases: 3.1', 3.2', and 3.3'.

Let $\Gamma_2' = (g_{s_1+s_2'+1}-x_2', \ldots, g_{k'}-x_2')$. Since $t_2' > 0$, we may apply Lemma 7 to show that $x_3' = \alpha(x_1, \Gamma_2')$, $s_3' = i(x_3'-1, \Gamma_2')$, and $t_3' = t_2' - s_3'$ exist. Clearly $1 \leq s_3' \leq t_2'$ and $0 \leq t_3' < t_2' < t_2 < t_1 < k$.

Just as we did with cases 3.1 and 3.2, we state cases 3.1' and 3.2' together and handle them simultaneously.

Case 3.1'. Suppose $t_3' = 0$.

or

Case 3.2'. Suppose $t_3' > 0$ and

$$\frac{P[x_1', x_3'-1]}{s_3'} \leq \frac{P[x_3', g_{s_1+s_3}-x_2']}{t_3'}.$$

Using the corollary to Lemma 9, we obtain

$$\frac{C(R)}{P(R)} > \frac{P(g_{s_1+s_3'}-x_1) - s_3'}{P(g_{s_1+s_3'}-x_1)}.$$

Clearly $s_1 < s_1 + s_3' < s_1 + s_3 < k$. Let $k'' = s_1+s_3'$. Then

$$\frac{C(R)}{P(R)} > \frac{P(g_{k''}-x_1) - s_3'}{P(g_{k''}-x_1)}.$$

Once again, before proceeding with this case, we shall look at our sequence of x_i's under k'' instead of k'. Let $\Gamma'' = (g_1, g_2, \ldots, g_{k''})$. Clearly $\Gamma'' \in H_{k''}$. Since $g_1 \in M(1,\Gamma'')$, $x_1'' = \alpha(1,\Gamma'')$, $s_1'' = i(x_1''-1,\Gamma'')$, and $t_1'' = k'' - s_1''$ exist.

Note that $x_1-1 \in [1,g_{k''}]$ and thus $x_1-1 \in M(1,\Gamma'')$ which implies $x_1'' \geq x_1$. However, $k'' < k'$. Thus $x_1'' \leq x_1$. Therefore, $x_1'' = x_1$, $s_1'' = s_1$, and $t_1'' = k'' - s_1 = s_3'$.

Clearly $\Gamma_1'' = (g_{s_1+1}-x_1, \ldots, g_{k''}-x_1) \in H_{t_1''}$. Since $t_1'' = s_3' > 0$, $x_2'' = \alpha(1,\Gamma_1'')$, $s_2'' = i(x_2''-1,\Gamma_1'')$, and $t_2'' = t_1'' - s_2''$ exist. Clearly $1 \leq s_2'' \leq t_1''$ and $0 \leq t_2'' < t_1'' \leq t_2' < t_1' \leq t_2 < t_1 < k$.

If we continue on in this manner, assuming we never arrive at any form of case 3.3, it is easy to see that either we finish in some form of cases 2.1 or 2.2 or there exists a positive integer m such that $t_2^{(m)} = 0$. Using Lemma 8, we obtain

$$\frac{C(R)}{P(R)} \geq \frac{A(g_{s_1+s_{3,m}}-x_1) + B(g_{s_1+s_{3,m}}-x_1)}{P(g_{s_1+s_{3,m}}-x_1)},$$

where $s_{3,m} = s_3^{(m-1)}$. Therefore, let $W = [1, g_{s_1+s_{3,m}}-x_1]$ and we are done.

We have been tacitly assuming that we never arrive at case $3.3^{(i)}$, $0 \leq i < m$. Thus either we finish or we arrive at case $3.3^{(i)}$, for some i, $0 \leq i < m$.

<u>Case $3.3^{(i)}$</u>. Suppose $t_3^{(i)} > 0$ and

$$\frac{P[x_1, x_3^{(i)}-1]}{s_3^{(i)}} > \frac{P[x_3^{(i)}, g_{s_1+s_{3,i}}-x_2^{(i)}]}{t_3^{(i)}}.$$

Since case $3.3^{(i)}$ is in exactly the same form as case 3.3, it is sufficient to assume we are in case 3.3.

<u>Case 3.3</u>. Suppose $t_3 > 0$ and

$$\frac{P[x_1, x_3-1]}{s_3} > \frac{P[x_3, g_k-x_2]}{t_3}.$$

Then, from case 2.3 and from Lemma 1, we obtain

$$\frac{C(R)}{P(R)} > \frac{P[x_1, g_k - x_2] - t_2}{P[x_1, g_k - x_2]}$$

$$\geq \frac{P[x_3, g_k - x_2] - t_3}{P[x_3, g_k - x_2]}$$

$$= \frac{P[x_2, g_k - x_3] - t_3}{P[x_2, g_k - x_3]}.$$

Once again, we must divide this case into three sub-cases in order to find the proper fundamental set W.

By Lemma 7, since $t_3 > 0$, $x_4 = \alpha(x_2, \Gamma_3)$, $s_4 = i(x_4 - 1, \Gamma_3)$, and $t_4 = t_3 - s_4$ exist. Clearly $1 \leq s_4 \leq t_3$ and $0 \leq t_4 < t_3 < t_2 < t_1 < k$.

Instead of going through cases 4.1, 4.2, 4.3, we shall make our induction assumption.

Let $n \geq 4$ and assume that for every i, $3 \leq i \leq n-1$, either we have finished or we are in some form of case i.3. Thus either we have finished or we are in some form of case (n-1).3. It is sufficient to assume we are in case (n-1).3, since we are dealing with decreasing sequences of $t_i^{(j)}$'s.

Case (n-1).3. Suppose $t_{n-1} > 0$ and

$$\frac{P[x_{n-3},x_{n-1}-1]}{s_{n-1}} > \frac{P[x_{n-1},g_k-x_{n-2}]}{t_{n-1}}.$$

Then, from case (n-2).3 and Lemma 1, we have

$$\begin{aligned}\frac{C(R)}{P(R)} &\geq \frac{P[x_{n-3},g_k-x_{n-2}] - t_{n-2}}{P[x_{n-3},g_k-x_{n-2}]}\\ &\geq \frac{P[x_{n-1},g_k-x_{n-2}] - t_{n-1}}{P[x_{n-1},g_k-x_{n-2}]}\\ &= \frac{P[x_{n-2},g_k-x_{n-1}] - t_{n-1}}{P[x_{n-2},g_k-x_{n-1}]}.\end{aligned}$$

In order to continue our search for a fundamental set W that satisfies the theorem, we subdivide case (n-1).3 into three subcases: n.1, n.2, and n.3.

Since $t_{n-1} > 0$, we apply Lemma 7 to show that $x_n = \alpha(x_{n-2},\Gamma_{n-1})$, $s_n = i(x_n-1,\Gamma_{n-1})$, and $t_n = t_{n-1} - s_n$ exist. Clearly $1 \leq s_n \leq t_{n-1}$ and

$$0 \leq t_n < t_{n-1} < \ldots < t_2 < t_1 < k.$$

We shall state case n.1 and case n.2 at the same time in order to make use of the material that was developed in the first chapter.

Case n.1. Suppose $t_n = 0$.

or

Case n.2. Suppose $t_n > 0$ and

$$\frac{P[x_{n-2},x_n-1]}{s_n} \leq \frac{P[x_n,g_k-x_{n-1}]}{t_n}.$$

Using Lemma 9, we can deal with these two cases together. We get

$$\frac{C(R)}{P(R)} \geq \frac{P[x_{n-3},g_{s_1+\ldots+s_{n-2}+s_n}-x_{n-2}] - s_n}{P[x_{n-3},g_{s_1+\ldots+s_{n-2}+s_n}-x_{n-2}]}.$$

Let $k' = s_1 + \ldots + s_{n-2} + s_n$. (Note: this k' is just a notational convenience; it is not to be confused with the k' that we defined in the combined case 3.1 and 3.2). Thus we have

$$\frac{C(R)}{P(R)} \geq \frac{P[x_{n-3},g_{k'}-x_{n-2}] - s_n}{P[x_{n-3},g_{k'}-x_{n-2}]}$$

Clearly $s_1 + \ldots + s_{n-2} < k' < k = s_1 + \ldots + s_{n-2} + t_{n-2}$. If we had started the whole process with k' instead of k, then our sequence of x_i's would have been exactly the same up to x_{n-2}. Therefore, let

$$\Gamma'_{n-2} = (g_{s_1+\ldots+s_{n-2}+1}-x_{n-2},\ldots,g_{k'}-x_{n-2}).$$

Clearly $\Gamma'_{n-2} \in H_{s_n}$. Since $s_n > 0$, $x'_{n-1} = \alpha(x_{n-3},\Gamma'_{n-2})$, $s'_{n-1} = i(x'_{n-1}-1,\Gamma'_{n-2})$, and $t'_{n-1} = s_n - s'_{n-1}$ exist. Thus we are now just in some form of case (n-1).1, (n-1).2, or (n-1).3. By our induction hypothesis, either we are finished or we are in some form of case (n-1).3. It is sufficient to assume we are in case (n-1).3'. Clearly $1 \le s'_{n-1} \le s_n$ and $0 \le t'_{n-1} < s_n \le t_{n-1} < \dots < t_1 < k$.

Case (n-1).3'. Suppose $t'_{n-1} > 0$ and

$$\frac{P[x_{n-3},x'_{n-1}-1]}{s'_{n-1}} > \frac{P[x'_{n-1},g_k,-x_{n-2}]}{t'_{n-1}}.$$

Then, just as in case (n-1).3, we obtain

$$\frac{C(R)}{P(R)} \geq \frac{P[x_{n-2},g_k,-x'_{n-1}] - t'_{n-1}}{P[x_{n-2},g_k,-x'_{n-1}]}.$$

Once again we subdivide into three subcases: n.1', n.2', and n.3'. By Lemma 9, since $t'_{n-1} > 0$, $x'_n = \alpha(x_{n-2},\Gamma'_{n-1})$, $s'_n = i(x'_n-1,\Gamma'_{n-1})$, and $t'_n = t'_{n-1} - s'_n$ exist. Clearly $1 \le s'_n \le t'_{n-1}$ and

$$0 \le t'_n < t'_{n-1} < t_{n-1} < \dots < t_1 < k.$$

We shall state case n.1' and case n.2' together.

<u>Case n.1'</u>. Suppose $t'_n = 0$.

or

<u>Case n.2'</u>. Suppose $t'_n > 0$ and

$$\frac{P[x_{n-2}, x'_n - 1]}{s'_n} \le \frac{P[x'_n, g_{k'} - x'_{n-1}]}{t'_n}.$$

Using Lemma 9, we can deal with case n.1' and n.2' simultaneously. From Lemma 9, we have

$$\frac{C(R)}{P(R)} \ge \frac{P[x_{n-3}, g_{s_1 + \ldots + s_{n-2} + s'_n} - x_{n-2}] - s'_n}{P[x_{n-3}, g_{s_1 + \ldots + s_{n-2} + s'_n} - x_{n-2}]}.$$

Using the same argument as in cases n.1 and n.2, we are finished or we are in some form of case (n-1).3. It is sufficient to assume we are in case (n-1).3". Clearly $1 \le s''_{n-1} \le s'_n$ and $0 \le t''_{n-1} < t'_{n-1} < t_{n-1} < \ldots < t_1 < k$.

Note that

$$1 \le s^{(i)}_n \le t^{(i)}_{n-1} < \ldots < s''_n \le t''_{n-1} < s'_n \le t'_{n-1} < s_n \le t_{n-1} < k,$$

for all i. Therefore if we continue on in the same manner, we will eventually either be finished or there will exist an $m \in P$ such that $s^{(m)}_n = 1$. It is sufficient to assume $s_n = 1$.

Then, since $s_n = 1 > 0$, $x'_{n-1} = \alpha(x_{n-3}, \Gamma'_{n-2})$, $s'_{n-1} = i(x'_{n-1}-1, \Gamma'_{n-2})$, and $t'_{n-1} = s_n - s'_{n-1}$ exist. Clearly $s'_{n-1} = 1$ and $t'_{n-1} = 0$. Therefore we must be in case (n-1).1'.

Since $s'_{n-1} = 1 > 0$, $x'_{n-2} = \alpha(x_{n-4}, \Gamma'_{n-3})$, $s'_{n-2} = i(x'_{n-2}-1, \Gamma'_{n-3})$, and $t'_{n-2} = s'_{n-1} - s'_{n-2}$ exist. Clearly $1 = s'_{n-2}$ and $t'_{n-2} = 0$.

Continuing on in this manner we eventually get $t'_2 = 0$ which, by Lemma 8, implies that we are done.

We have been working under the tacit assumption that we are never in any form of case n.3. It has been shown that, if we work under this assumption, we are done, that is, we are able to find a fundamental set W which satisfies the theorem. Thus, let us assume we have arrived at some form of case n.3. Because of the decreasing nature of the $t_i^{(j)}$'s, it is sufficient to assume we are in case n.3.

By the work which we have just completed, we know how to handle cases (n+1).1 and (n+1).2. Therefore, we are either finished or in case (n+1).3. We have the inequality

$$0 < t_{n+1} < t_n < t_{n-1} < \dots < t_1 < k.$$

Since k is finite, there exists a positive integer m such that $t_m = 1$. If $t_m = 1$, then $t_{m+1} = 0$ and we arrive in case (m+1).1. Since $t_{m+1} = 0$, it is impossible to be in case (m+1).3. We may as well assume that $m+1 = n$ and we know how to handle case n.1. Therefore, we will be able to find a fundamental set $W \subset R$ such that

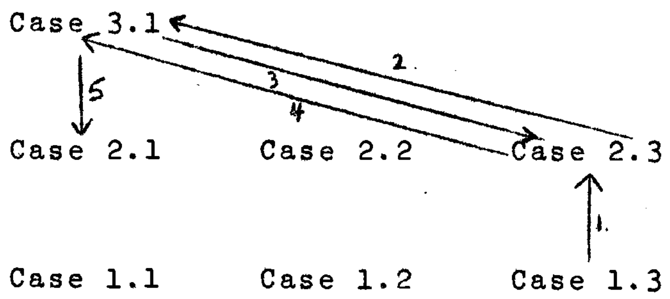
$$\frac{C(R)}{P(R)} \geq \frac{A(W) + B(W)}{P(W)}.$$

and the proof is complete.

CHAPTER 3.

EXAMPLES : ILLUSTRATIONS OF THE METHOD OF PROOF.

We now present two examples which may help to clarify the process that has been described in Chapter 2. Each example will be divided into cases and followed to its termination. The first example can be described by the following flowchart :



where the cases referred to represent either the original cases or some form of them.

Example 1. Let $A = \{1\} \cup [5,11] \cup [15,21] \cup [25,31] \cup \{\overline{35}\}$.

and $B = \{1,10,11,20,\overline{35}\}$. Then

$$C = [1,2] \cup [5,12] \cup [15,22] \cup [25,32] \cup \{\overline{35}\}.$$

Let $R = [1,34]$. Then $k = 8$ and

$$\Gamma = (g_1, g_2, \ldots, g_8)$$
$$= (3,4,13,14,23,24,33,34) \in H_8.$$

Now $M(1,\Gamma) = \{2,3,4,5,6,7,8,9\}$ and thus $x_1 = \alpha(1,\Gamma) = 10$, $s_1 = i(x_1-1,\Gamma) = i(9,\Gamma) = 2$, and $t_1 = k - s_1 = 6$.

Case 1.1. Suppose $t_1 = 0$. We have $t_1 = 6 > 0$. Therefore example 1 does not fall into this case.

Case 1.2. Suppose $t_1 > 0$ and $\frac{P(x_1-1)}{s_1} \leq \frac{P[x_1,g_k]}{t_1}$.

We have $t_1 = 6 > 0$ but

$$\frac{P(x_1-1)}{s_1} = \frac{9}{2} > \frac{25}{6} = \frac{P[x_1,g_k]}{t_1}.$$

Therefore example 1 does not fall into this case.

Case 1.3. Suppose $t_1 > 0$ and $\frac{P(x_1-1)}{s_1} > \frac{P[x_1,g_k]}{t_1}$.

Since we are not in case 1.1 or case 1.2, we must be in case 1.3 which, from the theorem, leads to

$$\frac{C(R)}{P(R)} > \frac{P(g_k-x_1) - t_1}{P(g_k-x_1)} = \frac{P(24) - 6}{P(24)}.$$

$$\text{Now } \Gamma_1 = (g_{s_1+1}-x_1; g_{s_1+2}-x_1, \ldots, g_{s_1+6}-x_1)$$

$$= (3,4,13,14,23,24) \in H_6.$$

We have $M(1,\Gamma_1') = \{2,3,4,5,6,7,8,9\}$ and thus

$x_2 = \alpha(1,\Gamma_1) = 10$, $s_2 = i(9,\Gamma_1) = 2$, and $t_2 = t_1 - s_2 = 4$.

<u>Case 2.1</u>. Suppose $t_2 = 0$. We have $t_2 = 4 > 0$.

Therefore example 1 does not fall into this case.

<u>Case 2.2</u>. Suppose $t_2 > 0$ and

$$\frac{P(x_2-1)}{s_2} \leq \frac{P[x_2, g_k-x_1]}{t_2}.$$

We have $t_2 = 4 > 0$ but

$$\frac{P(x_2-1)}{s_2} = \frac{P(9)}{2} = \frac{9}{2} > \frac{15}{4} = \frac{P[x_2, g_k-x_1]}{t_2}.$$

Therefore example 1 does not fall into this case.

<u>Case 2.3</u>. Suppose $t_2 > 0$ and

$$\frac{P(x_2-1)}{s_2} > \frac{P[x_2, g_k-x_1]}{t_2}.$$

Since we are not in case 2.1 or in case 2.2 we must be in case 2.3. This leads us to

$$\frac{C(R)}{P(R)} > \frac{P[x_1, g_k - x_2] - t_2}{P[x_1, g_k - x_2]} = \frac{P[10,24] - 4}{P[10,24]}.$$

Now $\Gamma_2 = (g_{s_1+s_2+1} - x_2, \ldots, g_{s_1+s_2+4} - x_2)$

$= (13,14,23,24) \ \varepsilon \ H_4.$

We have $24 = g_k - x_2 \ \varepsilon \ M(x_1, \Gamma_2)$ and thus $x_3 = \alpha(10, \Gamma_2) = 25$, $s_3 = i(24, \Gamma_2) = 4$, and $t_3 = t_2 - s_3 = 0$.

Case 3.1. Suppose $t_3 = 0$. We have $t_3 = 0$. Therefore example 1 falls into this case. This leads us to

$$\frac{C(R)}{P(R)} > \frac{P(g_{s_1+s_3} - x_1) - s_3}{P(g_{s_1+s_3} - x_1)}.$$

In order to continue our search for a fundamental set W that satisfies the theorem we go to cases 2.1', 2.2', and 2.3'.

Now $\Gamma_1' = (g_{s_1+1} - x_1, \ldots, g_{s_1+4} - x_1)$

$= (3,4,13,14) \ \varepsilon \ H_4.$

We have $M(1, \Gamma_1') = \{2,3,4,5,6,7,8,9\}$ and thus $x_2' = \alpha(1, \Gamma_1') = 10$, $s_2' = i(9, \Gamma_1') = 2$, and $t_2' = s_3 - s_2' = 2$.

Case 2.1'. Suppose $t_2' = 0$. We have $t_2' = 2 > 0$. Therefore example 1 does not fall into this case.

<u>Case 2.2</u>'. Suppose $t_2' > 0$ and

$$\frac{P(x_2'-1)}{s_2'} \leq \frac{P[x_2', g_{s_1+s_3} - x_1]}{t_2'}.$$

We have $t_2' > 0$ but

$$\frac{P(9)}{2} = \frac{9}{2} > \frac{5}{2} = \frac{P[10,14]}{2}.$$

Therefore example 1 does not fall into this case.

<u>Case 2.3</u>'. Suppose $t_2' > 0$ and

$$\frac{P(x_2'-1)}{s_2'} > \frac{P[x_2', g_{s_1+s_3} - x_1]}{t_2'}.$$

Since we are not in case 2.1' or in case 2.2' we must be in case 2.3' which leads us to

$$\frac{C(R)}{P(R)} > \frac{P[x_1, g_{s_1+s_3} - x_2'] - t_2'}{P[x_1, g_{s_1+s_3} - x_2']} = \frac{P[10,14] - 2}{P[10,14]}.$$

Now $\Gamma_2' = (g_{s_1+s_2'+1} - x_2',\ g_{s_1+s_2'+2} - x_2')$

$= (13,14) \in H_2.$

We have $14 = g_{s_1+s_3} - x_2' \in M(10,\Gamma_2')$ and thus

$x_3' = (10,\Gamma_2') = 15$, $s_3' = i(14,\Gamma_2') = 2$, and $t_3' = t_2' - s_3' = 0$.

<u>Case 3.1</u>'. Suppose $t_3' = 0$. We have $t_3' = 0$. Therefore, example 1 falls into case 3.1' which leads us to

$$\frac{C(R)}{P(R)} > \frac{P(g_{s_1+s_3'-x_1}) - s_3'}{P(g_{s_1+s_3'-x_1})}.$$

To continue our search for a fundamental set W we go to case 2.1".

Now $\Gamma_1'' = (g_{s_1+1-x_1}, g_{s_1+s_3'-x_1})$

$$= (3,4) \in H_2.$$

We have $M(1,\Gamma_1'') = \{2,3,4\}$ and thus $x_2'' = \alpha(1,\Gamma_1'') = 5$, $s_2'' = i(4,\Gamma_1'') = 2$, and $t_2'' = s_3' - s_2'' = 0$. Since $t_2'' = 0$, we must be in case 2.1".

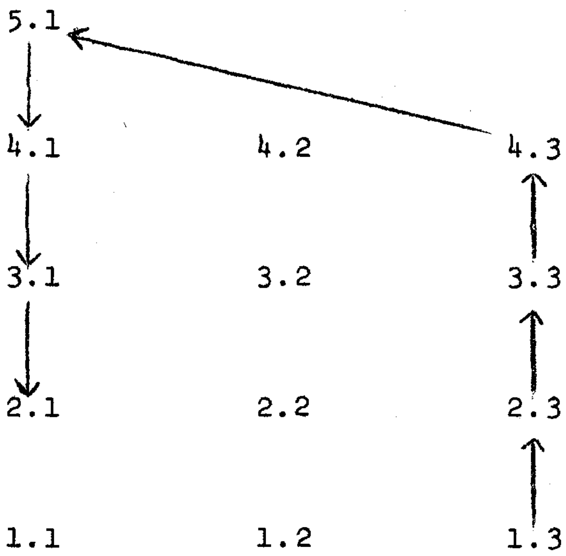
<u>Case 2.1</u>". Suppose $t_2'' = 0$. Then

$$\frac{C(R)}{P(R)} > \frac{P(g_{s_1+s_3'-x_1}) - s_3'}{P(g_{s_1+s_3'-x_1})}$$

$$= \frac{P(4) - 2}{P(4)} \geq \frac{A(4) + B(4)}{P(4)}.$$

Therefore let $W = [1,4]$ and we are done.

Our next example is illustrated by the following flowchart:



where each case represents either itself or some form of it. It is a more complicated example since it goes to case 5.1 before coming back to some form of case 2.1 and ending.

<u>Example 2</u>. Let

$A = \{1\} \cup [10,21] \cup [30,41] \cup [51,61] \cup [70,81] \cup [90,101]$
$\cup [110,121] \cup [130,141] \cup [150,161] \cup \{\overline{170}\}$,

$B = \{1,20,21,40,60,61,80,81,100,120,140,160,\overline{170}\}$. Then

$C = [1,2] \cup [10,22] \cup [30,42] \cup [50,62] \cup [70,82]$
$\cup [90,102] \cup [110,122] \cup [130,142] \cup [150,162]$
$\cup \{\overline{170}\}$.

Let $R = [1,169]$. Then $k = 63$ and

$$g_i = \begin{cases} i + 2 & \text{if } 1 \le i \le 7, \\ i + 15 & \text{if } 8 \le i \le 14, \\ i + 28 & \text{if } 15 \le i \le 21, \\ i + 41 & \text{if } 22 \le i \le 28, \\ i + 54 & \text{if } 29 \le i \le 35, \\ i + 67 & \text{if } 36 \le i \le 42, \\ i + 80 & \text{if } 43 \le i \le 49, \\ i + 93 & \text{if } 50 \le i \le 56, \\ i + 106 & \text{if } 57 \le i \le 63. \end{cases}$$

We have $M(1,\Gamma) = [2,19] \cup [50,59]$. Thus $x_1 = 60$, $s_1 = 21$, and $t_1 = 42$.

<u>Case 1.1</u>. Suppose $t_1 = 0$. We have $t_1 = 42 > 0$. Therefore example 2 does not fall into this case.

<u>Case 1.2</u>. Suppose $t_1 > 0$ and

$$\frac{P(x_1-1)}{s_1} \le \frac{P[x_1,g_k]}{t_1}.$$

We have $t_1 = 42 > 0$ but

$$\frac{P(59)}{21} = \frac{59}{21} > \frac{110}{42} = \frac{P[60,169]}{42}.$$

Therefore example 2 does not fall into this case.

Case 1.3. Suppose $t_1 > 0$ and

$$\frac{P(x_1-1)}{s_1} > \frac{P[x_1,g_k]}{t_1}.$$

Our previous work shows us that example 2 falls into this case. Following the same pattern as the proof of the theorem we obtain

$$\frac{C(R)}{P(R)} > \frac{P(g_k-x_1) - t_1}{P(g_k-x_1)} = \frac{P(109) - 42}{P(109)}.$$

Now $\Gamma_1 = (g_{s_1+1}-x_1,\ldots,g_k-x_1)$ where

$$g_{s_1+i}-x_1 = \begin{cases} i + 2 & \text{if } 1 \le i \le 7, \\ i + 15 & \text{if } 8 \le i \le 14, \\ i + 28 & \text{if } 15 \le i \le 21, \\ i + 41 & \text{if } 22 \le i \le 28, \\ i + 54 & \text{if } 29 \le i \le 35, \\ i + 67 & \text{if } 36 \le i \le 42. \end{cases}$$

We have $M(1,\Gamma_1) = [2,19] \cup [50,59]$. Thus $x_2 = 60$, $s_2 = 21$, and $t_2 = 21$.

Case 2.1. Suppose $t_2 = 0$. We have $t_2 = 21 > 0$. Therefore example 2 does not fall into this case.

Case 2.2. Suppose $t_2 > 0$ and

$$\frac{P(x_2-1)}{s_2} \leq \frac{P[x_2,g_k-x_1]}{t_2}.$$

We have $t_2 = 21 > 0$ but

$$\frac{P(59)}{21} = \frac{59}{21} > \frac{50}{21} = \frac{P[60,109]}{21}.$$

Therefore example 2 does not fall into this case.

Case 2.3. Suppose $t_2 > 0$ and

$$\frac{P(x_2-1)}{s_2} > \frac{P[x_2,g_k-x_1]}{t_2}.$$

Our previous work has shown us that example 2 falls into case 2.3. This leads us to

$$\frac{C(R)}{P(R)} \geq \frac{P[x_1,g_k-x_2] - t_2}{P[x_1,g_k-x_2]}.$$

$$= \frac{P[60,109] - 21}{P[60,109]}.$$

Now $\Gamma_2 = (g_{s_1+s_2+1}-x_2,\ldots,g_k-x_2)$ where

$$g_{s_1+s_2+i}-x_2 = \begin{cases} i + 62 & \text{if } 1 \leq i \leq 7, \\ i + 75 & \text{if } 8 \leq i \leq 14, \\ i + 88 & \text{if } 15 \leq i \leq 21. \end{cases}$$

We have $M(60,\Gamma_2) = \{60\} \cup [62,79]$. Thus $x_3 = 80$, $s_3 = 7$, and $t_3 = 14$.

<u>Case 3.1</u>. Suppose $t_3 = 0$. We have $t_3 = 14 > 0$. Therefore example 2 does not fall into this case.

<u>Case 3.2</u>. Suppose $t_3 > 0$ and

$$\frac{P[x_1, x_3 - 1]}{s_3} \leq \frac{P[x_3, g_k - x_2]}{t_3}.$$

We have $t_3 = 14 > 0$ but

$$\frac{P[60,79]}{7} = \frac{20}{7} > \frac{30}{14} = \frac{P[80,109]}{14}.$$

Therefore example 2 does not fall into this case.

<u>Case 3.3</u>. Suppose $t_3 > 0$ and

$$\frac{P[x_1, x_3 - 1]}{s_3} > \frac{P[x_3, g_k - x_2]}{t_3}.$$

By our previous work, we have seen that example 2 must fall into case 3.3. Just as in the theorem, we obtain

$$\frac{C(R)}{P(R)} \geq \frac{P[x_2, g_k - x_3] - t_3}{P[x_2, g_k - x_3]}$$

$$= \frac{P[60,89] - 14}{P[60,89]}.$$

Now $\Gamma_3 = (g_{s_1+s_2+s_3+1}-x_3, \ldots, g_k-x_3)$ where

$$g_{s_1+s_2+s_3+i}-x_3 = \begin{cases} i + 62 & \text{if } 1 \le i \le 7, \\ i + 75 & \text{if } 8 \le i \le 14. \end{cases}$$

We have $M(60,\Gamma_3) = \{60\} \cup [62,79]$. Thus $x_4 = 80$, $s_4 = 7$, and $t_4 = 7$.

<u>Case 4.1</u>. Suppose $t_4 = 0$. We have $t_4 = 7 > 0$. Therefore example 2 does not fall into this case.

<u>Case 4.2</u>. Suppose $t_4 > 0$ and

$$\frac{P[x_2,x_4-1]}{s_4} \le \frac{P[x_4,g_k-x_3]}{t_4}.$$

We have $t_4 = 7 > 0$ but

$$\frac{P[60,79]}{7} = \frac{20}{7} > \frac{10}{7} = \frac{P[80,89]}{7}.$$

Therefore example 2 does not fall into this case.

<u>Case 4.3</u>. Suppose $t_4 > 0$ and

$$\frac{P[x_2,x_4-1]}{s_4} > \frac{P[x_4,g_k-x_3]}{t_4}.$$

Our previous work has shown us that example 2 falls into this case. Following the pattern of the proof of the theorem, we obtain

$$\frac{C(R)}{P(R)} \geq \frac{P[x_3, g_k - x_4] - t_4}{P[x_3, g_k - x_4]}$$

$$= \frac{P[80,89] - 7}{P[80,89]}.$$

Now $\Gamma_4 = (g_{s_1+s_2+s_3+s_4+1} - x_4, \ldots, g_k - x_4)$ where $g_{s_1+s_2+s_3+s_4+i} - x_4 = i + 82$, $1 \leq i \leq 7$. We have $M(80,\Gamma_4) = \{80\} \cup [82,89]$. Thus $x_5 = 90$, $s_5 = 7$, and $t_5 = 0$.

<u>Case 5.1</u>. Suppose $t_5 = 0$. We have $t_5 = 0$. Therefore example 2 falls into case 5.1 from which we get

$$\frac{C(R)}{P(R)} \geq \frac{P[x_2, g_{s_1+s_2+s_3+s_5} - x_3] - s_5}{P[x_2, g_{s_1+s_2+s_3+s_5} - x_3]}$$

$$= \frac{P[60,69] - 7}{P[60,69]}.$$

Following the pattern of the proof of the theorem, we go to a new form of case 4.1.

Now $\Gamma_3' = (g_{s_1+s_2+s_3+1} - x_3, \ldots, g_{s_1+s_2+s_3+s_5} - x_3)$ where $g_{s_1+s_2+s_3+i} - x_3 = i + 62$, $1 \leq i \leq 7$. We have $M(60,\Gamma_3') = \{60\} \cup [62,69]$. Thus $x_4' = 70$, $s_4' = 7$, and $t_4' = 0$.

<u>Case 4.1'</u>. Suppose $t_4' = 0$. We have $t_4' = 0$. Therefore example 2 falls into case 4.1' from which we obtain

$$\frac{C(R)}{P(R)} \geq \frac{P[x_1, g_{s_1+s_2+s_4'-x_2}] - s_4'}{P[x_1, g_{s_1+s_2+s_4'-x_2}]}$$

$$= \frac{P[60,69] - 7}{P[60,69]}.$$

Once more we follow the pattern of the proof to go to a new form of case 3.1.

Now $\Gamma_2' = (g_{s_1+s_2+1-x_2}, \ldots, g_{s_1+s_2+s_4'-x_2})$ where $g_{s_1+s_2+i-x_2} = i + 62$, $1 \leq i \leq 7$. We have $M(60,\Gamma_2') = \{60\} \cup [62,69]$. Thus $x_3' = 70$, $s_3' = 7$, and $t_3' = 0$.

<u>Case 3.1'</u>. Suppose $t_3' = 0$. We have $t_3' = 0$. Therefore example 2 falls into case 3.1' from which we obtain

$$\frac{C(R)}{P(R)} > \frac{P(g_{s_1+s_3'-x_1}) - s_3'}{P(g_{s_1+s_3'-x_1})}$$

$$= \frac{P(9) - 7}{P(9)}.$$

Following the pattern of the proof, we go to a new form of case 2.1.

Now $\Gamma_1' = (g_{s_1+1-x_1}, \ldots, g_{s_1+s_3'-x_1})$ where

$g_{s_1+i}-x_1 = i + 2$, $1 \le i \le 7$. We have $M(1,\Gamma_1') = [2,9]$.

Thus $x_2' = 10$, $s_2' = 7$, and $t_2' = 0$.

<u>Case 2.1</u>'. Suppose $t_2' = 0$. We have $t_2' = 0$. Therefore example 2 falls into case 2.1'. Hence

$$\frac{C(R)}{P(R)} \ge \frac{A(x_2'-1) + B(x_2'-1)}{P(x_2'-1)}$$

$$= \frac{A(9) + B(9)}{P(9)}.$$

Therefore, since [1,9] is a fundamental set, let W = [1,9] and we are done.

This completes our two illustrations of Garrison's method of proof. It may be interesting to note here that I have not yet found an example that falls into any form of case i.2 for i > 2. However, I believe that some small refinement of the previous two examples or similar examples which will fall into case i.2 for some i > 2 will solve the problem.

CHAPTER 4.

CONCLUSION.

The basic aim of this paper has been to demonstrate clearly Garrison's non-transformation proof of Mann's Density Theorem which, of course, immediately yields the Landau-Schnirelmann conjecture. In order to illustrate the possible consequences of Garrison's new proof, we shall make some definitions which generalize the concepts that were defined in the Introduction. The development of the following ideas is due to A. Freedman [2].

Let S be a non-empty subset of an abelian group $(G,+)$. If x and y are elements of G such that $y - x \in S$, then we write $x < y$. If $x \in S$, then

$$L(x) = \{y \in S : y < x \text{ or } y = x\}.$$

Definition. S is a δ-semigroup if

(i) S is closed under +,

(ii) $0 \notin S$,

(iii) $L(x)$ is finite for each $x \in S$.

Let X be a subset of a δ-semigroup S and let $x \in X$. If, for every $y \in X - \{x\}$, $x \notin L(y)$, then x is a maximal element of X. Let

$$\max(X) = \{x \in X : x \text{ is a maximal element of } X\}.$$

Let $\mathcal{F}$ be a family of non-empty, finite subsets of S and let $x \in F \in \mathcal{F}$. If $F = \{x\}$ or $F - \{x\} \in \mathcal{F}$, then x is called a corner element of F. Let

$$F^* = \{x \in F : x \text{ is a corner element of } F\}.$$

<u>Definition</u>. Let S be a δ-semigroup. Let $\mathcal{F}$ be a non-empty family of non-empty, finite subsets of S. Then $\mathcal{F}$ is a fundamental family of S if

(i) for every $x \in S$, there is an $F \in \mathcal{F}$ with $x \in F$,

(ii) if $F_i \in \mathcal{F}$, $1 \le i \le n$, then $\bigcup_{i=1}^{n} F_i \in \mathcal{F}$,

(iii) if $F_i \in \mathcal{F}$, $1 \le i \le n$, then $\bigcap_{i=1}^{n} F_i \in \mathcal{F}$, if $\bigcap_{i=1}^{n} F_i \ne \phi$,

(iv) if $F \in \mathcal{F}$, then $\max(F) \subset F^*$.

An ordered pair, $(S, \mathcal{F})$, where S is a δ-semigroup and is a fundamental family of S, is called a density space.

If A and B are subsets of S, then

$$A + B = A \cup B \cup \{a + b : a \in A \text{ and } b \in B\}.$$

If A and X are subsets of S with X finite, then $A(X)$ is the number of elements in $A \cap X$.

Definition. If A is a subset of S, then

$$d(A) = \text{glb}\{A(F)/S(F) : F \in \mathcal{F}\},$$

where $\mathcal{F}$ is a fundamental family. d(A) is called the density of A with respect to $\mathcal{F}$.

With these definitions in mind we examine the following statement:

Statement 1. Let $(S,\mathcal{F})$ be a density space. Let A and B be subsets of S, let $C = A + B$, and let $F \in \mathcal{F}$ be such that $S(F - C) = n \geq 1$. Then there exists a $G \in \mathcal{F}$ such that $G \subset F$ and

$$\frac{C(F)}{S(F)} \geq \frac{A(G) + B(G)}{S(G)}.$$

This statement is not true for arbitrary density spaces. However, there are many special cases for which Statement 1 holds and other cases for which its validity is, as yet, unknown. We now examine some of these special cases.

<u>Example 1</u>. Let $S = (P \cup \{0\})^k - \{(0,0,\ldots,0)\}$ and let $\mathcal{F} = \{F \subset S : F$ is finite, $F \neq \phi$, and for every $x \in F$, $L(x) \subset F\}$.

It is easily verified that $(S,\mathcal{F})$ is a density space.

When $k = 1$, $(S,\mathcal{F}) = (P,\mathcal{F})$ is just the positive integers together with the fundamental sets, $[1,n]$, that we defined in the Introduction. Thus, for this example, Statement 1 is identical with Garrison's Theorem which we have proved in Chapter 2.

When $k > 1$, Garrison [3] has shown that Statement 1 is false. However, this does not negate the possibility that the Landau-Schnirelmann Conjecture (generalized in the obvious way) still holds for this case. In order to circumvent Statement 1 and still prove the Landau-Schnirelmann Conjecture, it is necessary to prove the following:

<u>Statement 2</u>. Let the hypothesis be the same as in Statement 1. Then there exists a $G \in \mathcal{F}$ and an $H \in \mathcal{F}$ such that $G \subset F$, $H \subset F$, and

$$\frac{C(F)}{S(F)} \geq \frac{A(G)}{S(G)} + \frac{B(H)}{S(H)}.$$

If this statement is false then the Landau-Schnirelmann

Conjecture is also false for this case. Since Statement 2 is an unsolved problem for $k > 1$, it may be a worthwhile project to attempt to generalize Garrison's method to prove Statement 2 for this case.

Before giving another example, it is necessary to make some more definitions.

<u>Definition</u>. If $\mathcal{F}$ is a fundamental family on S, then

$$[x] = \bigcap_{x \in F \in \mathcal{F}} F .$$

<u>Definition</u>. Let $\mathcal{F}$ be a fundamental family of S. Let x and y be any elements of S such that $x \notin [y]$ and $y \notin [x]$. If $[x] \cap [y] = \phi$, then $\mathcal{F}$ is said to be separated.

<u>Definition</u>. A fundamental family $\mathcal{F}$ on a δ-semigroup S is singularly discrete of order n if

(i) $\mathcal{F}$ is separated,

(ii) for every $x \in S$, $S([x]) \leq n$ with equality holding for some $y \in S$,

(iii) $S([x]) = i$ for at most one $x \in S$, where $i = 2,3,\ldots,n$.

<u>Example 2</u>. Let $(S,\mathcal{F})$ be a density space where $\mathcal{F}$ is singularly discrete of order n.

If $n = 1$ or $n = 2$, Statement 1 is very easily proved. A. E. Olson [7] has proved Statement 1 for the special case $n = 3$ and conjectured it for all n.

I have mentioned these two examples here because they appear to be susceptible to some kind of simple counting argument such as Garrison's method. In the case of example 1 when $k > 1$, both Mann's and Dyson's method of proof have been looked at with no success.

Besides Garrison's method, there may be other research possibilities for this paper. As I mentioned in the Introduction, some of the lemmas and observations in Chapter 1 may be of independent interest.

Referring to Chapter 3, one unsolved problem is to find examples which illustrate Garrison's method beyond case 2.3 and which are different in nature from the ones presented in Chapter 3. Alternatively, if this is impossible, prove it .

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