BOOLEAN TOPOI AND MODELS OF ZFC

by

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ABSTRACT

A topos is a cartesian-closed category with a subobject classifier. A topos is Boolean if its subobject classifier has a Boolean algebra structure. Boolean-valued models of set theory are examples of Boolean topoi. The main result characterizes those Boolean topoi which are Boolean-valued models of ZFC. In order that the work be fairly self contained, introductory chapters on category theory and the model theory of ZFC are included. Also included is an introductory account of the elementary theory of topoi up to the proof that in any topos the subobject classifier has a Heyting algebra structure.

DEDICATION

To the memory of my father

Joseph Martin Cunningham

PREFACE

Much work has been done on the use of categorical algebra in the foundations of foundations of mathematics (e.g. Lawvere [11]-[14] and MacLane [16] and [17]). The first attempt at a category theoretic characterization of sets was Lawvere [10]. Mitchell in [19] showed that categories satisfying Lawvere's axioms were models for a finitely axiomatizable set theory Z_1 which is strictly weaker than ZFC (Zermelo--Fraenkel set theory with the Axiom of Choice) in that the full axiom scheme of Replacement does not hold.

Following the suggestion of my supervisor, Dr. Harvey Gerber, I have tried to make this thesis as self contained as possible. It is assumed that the reader is familiar with first-order theories and their models (e.g. Shoenfield [25, Chapters 1-5]). Also some acquaintance with basic terminology concerning lattices and Boolean algebras is desirable for §I.5, the examples of Chapter II, and §III.6.

Chapter I presents the usual axiomatization of ZFC and demonstrates the equivalence of another axiomatization which is technically useful in Chapter IV. Terminology concerning models of ZFC is introduced in §I.4. In §1.5 the concept of a Boolean-valued model is defined. The treatment of Boolean-valued models of ZFC is taken from Jech [7], Rosser [24], and Solovay and Tennenbaum [26].

Chapter II presents an introduction to category theory. The main references used in its writing were MacLane [18] and Stone [27]; references which were used to a lesser extent were Freyd [4] and Pareigis [21].

Chapter III is primarily taken from Freyd [5], although some use is made of Benabou and Celeyrette [1] and Kock and Wraith [8].

Chapter IV is taken from Mitchell [19].

Many results have been obtained beyond these. Lawvere and Tierney (see Lawvere and Tierney [15] and Tierney [29]) have shown that Cohen's method of forcing (see Cohen [2], Felgner [3], Jech [7], Mostowski [20], and Takeuti and Zaring [28]) can be done category theoretically in Lawvere's Elementary Theory of the Category of Sets [10], by using a category of sheaves construction. The last part of this construction shows that, as might be suspected by analogy with Boolean-valued models (see Solovay and Tennenbaum [26] for instance), one can collapse the appropriate Boolean topos to a two-valued topos via a category of fractions constuction (see Gabriel and Zisman [6]).

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Dr. Harvey Gerber for not only putting up with him but also for paying the cost of thesis typing.

Last but not least, any readability which this thesis may possess is not a fault of the author, but a virtue of the typist, Linda Cowan, without whom the author never could have met his deadlines.

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CHAPTER I

AN INTRODUCTION TO THE MODEL THEORY OF ZFC

I.1 The axioms of ZFC

The formal language $\mathcal{L}$ of ZFC is the first-order language with equality whose only nonlogical symbol is the binary relation symbol $\in$, which we shall always write between its arguments, e.g. $x \in y$ (read x *is an element of* y). We suppose $\mathcal{L}$ to be formulated so that its primitive logical symbols are $\sim$(*not*), $\vee$(*or*), $\exists$(*there exists*), and $=$(*equals*). We use both subscripted and unsubscripted lower case Latin letters $x, y, z, u, v, x_1, x_2, x_3, \ldots$ to denote variables of $\mathcal{L}$. Terms, atomic formulas, and formulas of $\mathcal{L}$ are defined in the usual way. Lower case Greek letters with or without subscripts $\phi, \psi, \theta, \phi_1, \phi_2, \phi_3, \ldots$ are used as metavariables ranging over formulas of $\mathcal{L}$. The notation $\phi(x_1, \ldots, x_n)$ implicitly denotes the fact that the variables $x_1, \ldots, x_n$ occur free in ϕ. $\phi_{x_1,\ldots,x_n}[t_1, \ldots, t_n]$ is used to denote the formula obtained by simultaneously substituting the terms $t_1, \ldots, t_n$ for the free occurrences of the variables $x_1, \ldots, x_n$ in $\phi(x_1, \ldots, x_n)$. When no possibility for confusion arises we may write $\phi(t_1, \ldots, t_n)$ for $\phi_{x_1,\ldots,x_n}[t_1, \ldots, t_n]$. We use the symbols $=_{df}$ (*is defined to be equal to*) and $\equiv_{df}$ (*is defined to be equivalent to*) as metalogical connectives introducing abbreviations for terms and formulas, e.g. see (1.1)-(1.7) below.

$\phi \wedge \psi \equiv_{df} \sim(\sim\phi \vee \sim\psi)$ (*ϕ and ψ*) (1.1)

$\phi \rightarrow \psi \equiv_{df} \sim\phi \vee \psi$ (*ϕ implies ψ*) (1.2)

$\phi \leftrightarrow \psi \equiv_{df} (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$ (*ϕ is equivalent to ψ*) (1.3)

$\forall x\phi \equiv_{df} \sim\exists x \sim \phi$ (*for all* x *ϕ holds*) (1.4)

$x \notin y \equiv_{df} \sim(x \in y)$ (x *is not an element of* y) (1.5)

$x \neq y \equiv_{df} \sim(x=y)$ (x *does not equal* y) (1.6)

$\exists_{\leq 1} x\phi(x) \equiv_{df} \exists x\forall y(\phi(y) \rightarrow x=y)$ (*there exists at most one* x *such that* $\phi(x)$) (1.7)

(where y does not occur in $\phi(x)$)

The logical axioms and rules of inference of the first-order theory of $\mathcal{L}$ we shall take to be those of Shoenfield [25, pp. 20-21] or any equivalent formulation. The nonlogical axioms of ZFC are listed and explained below.

AXIOM 1 - The Axiom of Extensionality (abbreviated *AxExt*)

$$\forall x\forall y(x=y \leftrightarrow \forall z(z \in x \leftrightarrow z \in y))$$

AxExt specifies the relationship between the symbols = and $\in$ by asserting that the equality relation is completely determined by the element relation.

AXIOM 2 - The Axiom of the Null Set (abbreviated *AxNull*)

$$\exists x\forall y(y \notin x)$$

AxNull asserts the existence of an elementless set. By *AxExt* there can be only one such set, hence we may introduce a constant symbol $\emptyset$ to stand for it.

$x=\emptyset \equiv_{df} \forall y(y \notin x)$ (x *is the null set*) (1.8)

AXIOM 3 - The Axiom of Unordered Pairs (abbreviated *AxPair*)

$$\forall x \forall y \exists z \forall t (t \in z \leftrightarrow (t=x \vee t=y))$$

AxPair says that given any two sets x and y there is a third set z whose only elements are x and y. By *AxExt* for given x and y such a z is unique, hence the following definitions make sense:

$z=\{x,y\} \equiv_{df} \forall t(t \in z \leftrightarrow (t=x \vee t=y))$	(z *is the unordered pair of* x *and* y)	(1.9)
$\{x\} \equiv_{df} \{x,x\}$	(*singleton* x)	(1.10)
$\langle x,y \rangle \equiv_{df} \{x,\{x,y\}\}$	(*the ordered pair of* x *and* y)	(1.11)
$\langle x_1,\ldots,x_n \rangle \equiv_{df} \langle\langle x_1,\ldots,x_{n-1}\rangle,x_n\rangle$ (for $n \geq 3$)	(*the ordered n-tuple of* $x_1,\ldots,x_n$)	(1.12)

AXIOM 4 - The Axiom of Union (abbreviated *AxUnion*)

$$\forall x \exists y \forall z (z \in y \leftrightarrow \exists t (t \in x \wedge z \in t))$$

AxUnion says that given a set x the collection of elements of elements of x forms a set y. By *AxExt* for a given x such a y is unique, hence the following definitions make sense:

$y=\bigcup x \equiv_{df} \forall z(z \in y \leftrightarrow \exists t(t \in x \wedge z \in t))$	(y *is the union of* x)	(1.13)
$x \cup y =_{df} \bigcup\{x,y\}$	(*the union of* x *and* y)	(1.14)
$\{x_1,\ldots,x_n\} =_{df} \{x_1,\ldots,x_{n-1}\} \cup \{x_n\}$ (for $n \geq 2$)	(*the unordered n-tuple of* $x_1,\ldots,x_n$)	(1.15)

AXIOM 5 - The Axiom of Infinity (abbreviated *AxInf*)

$$\exists x (\emptyset \in x \wedge \forall y (y \in x \rightarrow y \cup \{y\} \in x))$$

AxInf asserts the existence of an infinite set.

We define subset relations as follows:

$y \subseteq x \equiv_{df} \forall z(z \in y \rightarrow z \in x)$ (*y is a subset of* x) (1.16)

$y \subset x \equiv_{df} y \subseteq x \wedge y \neq x$ (*y is a proper subset of* x) (1.17)

AXIOM 6 - The Axiom of the Power Set (abbreviated *AxPower*)

$$\forall x \exists y \forall z(z \in y \leftrightarrow z \subseteq x)$$

AxPower says that given a set x, the collection of all subsets of x forms a set y. By *AxExt* for given x this y is unique. Hence we define

$y = \underline{PS}(x) \equiv_{df} \forall z(z \in y \leftrightarrow z \subseteq x)$ (*y is the power-set of* x) (1.18)

AXIOM 7 - The Axiom Scheme of Replacement (abbreviated *AxRepl*)

For each formula $\phi(x,y,t_1,\ldots,t_k)$ of $\mathcal{L}$ having exactly k+2 free variables the following formula, which will be referred to as $AxRepl^{\phi}$, is a nonlogical axiom.

$$\forall t_1 \ldots \forall t_k [\forall x \exists_{\leq 1} y \phi(x,y,t_1,\ldots,t_k) \rightarrow \forall u \exists v \forall r(r \in v \leftrightarrow \exists s(s \in u \wedge \phi(s,r,t_1,\ldots,t_k)))]$$

$AxRepl^{\phi}$ says that if for fixed $t_1,\ldots,t_k$ the formula ϕ determines a partial functional relation then the image of any set u under the partial function determined by ϕ is also a set v.

AXIOM 8 - The Axiom of Foundation (abbreviated *AxFound*)

$$\forall x(x \neq \emptyset \rightarrow \exists y(y \in x \wedge \forall z(z \in y \rightarrow z \notin x)))$$

The purpose of *AxFound* is to prevent the existence of sets containing $\in$-cycles or infinite descending $\in$-chains. This then allows us to describe the universe of all sets as a hierarchy of sets built up from the null set $\emptyset$ by the operations of power-set and union (see Theorem II, p. 16).

AXIOM 9 - The Axiom of Choice (abbreviated *AC*)

$$\forall x \exists f \forall z (z \in x \wedge z \neq \emptyset \rightarrow \exists u \exists y (u \in f \wedge u = \langle z,y \rangle \wedge y \in z \wedge \forall u_1 \forall y_1 (u_1 \in f \wedge u_1 = \langle z,y_1 \rangle \rightarrow y = y_1)))$$

This formulation of *AC* says that for every set x there is a choice function f which picks out one member y from each nonempty $z \in x$.

This completes the list of nonlogical axioms of ZFC. It should be noted that the above is not a minimal set of axioms; *AxNull*, *AxPair* and *AxUnion* are redundant, being easily proved from *AxRepl*.

We shall sometimes need to consider systems weaker than ZFC; two such systems are ZF and ZF^0 which are obtained by deleting respectively from the above list (i.) *AC* and (ii.) both *AxFound* and *AC*. A theorem scheme which is often added as an axiom scheme in place of *AxRepl* to yield still weaker systems is given by the following.

THEOREM I (*Aussonderung*): Let $\psi(x,t_1,\ldots,t_k)$ be any formula of $\mathcal{L}$ having exactly k+1 free variables (where $k \geq 0$). Then

$$ZF^0 \vdash \forall t_1 \ldots \forall t_k \forall x \exists y \forall z (z \in y \leftrightarrow z \in x \wedge \psi(z,t_1,\ldots,t_k)).$$

Proof: This is immediate from $AxRepl^{\phi}$ where $\phi(x,y,t_1,\ldots,t_k)$ is the formula $x=y \wedge \psi(x,t_1,\ldots,t_k)$.□

We next define the notion of a bounded quantifier as a sequence of symbols of the from $\exists z \in x$ or $\forall z \in x$ where

$$\exists z \in x \phi(z) \equiv_{df} \exists z (z \in x \wedge \phi(z)) \text{ and} \tag{1.19}$$

$$\forall z \in x \phi(z) \equiv_{df} \forall z (z \in x \rightarrow \phi(z)). \tag{1.20}$$

A formula of $\mathcal{L}$ is said to be <u>limited</u> if it can be written so that all its quantifiers are bounded.

Two systems which are weaker than ZF are Z and Z^{lim} which are obtained from ZF by deleting *AxRepl* and adding respectively

(i.) *Aussonderung* for every formula $\psi(x,t_1,...,t_k)$ having k+1 free variables and

(ii.) *Aussonderung* for every limited formula $\psi(x,t_1,...,t_k)$ having k+1 free variables.

I.2 Basic definitions

In Cantor's original conception of set theory it was an accepted principle that the collection of all objects having a certain property was a set. However Russell's paradox demonstrated that this viewpoint was rather too naive. It has since been recognized that an adequate set theory must provide a means of talking about two different kinds of collections: sets and classes, a class being a collection of objects satisfying a certain property. Hence we would like to define in our language $\mathcal{L}$ an abstraction operator $\{x|\quad\}$ operating on formulas, where $\{x|\ \phi\}$ is to be read "the class of x such that ϕ".

Occurrences of the variable x in $\{x|\ \phi\}$ are treated as being bound. In particular if x does not occur free in ϕ the notation is regarded as denoting $\emptyset$. We make the following definitions:

$$a\in\{x|\ \phi(x)\} \equiv_{df} \phi(a) \qquad (2.1)$$

$$\{x|\ \phi(x)\}\in b \equiv_{df} \exists y\in b\forall z(z\in y\leftrightarrow\phi(z)) \tag{2.2}$$

$$\{x|\ \phi(x)\}\in\{x|\ \psi(x)\} \equiv_{df} \exists y(y\in\{x|\ \psi(x)\} \wedge \forall z(z\in y\leftrightarrow\phi(z))) \tag{2.3}$$

$$\{x|\ \phi(x)\}=a \equiv_{df} a=\{x|\ \phi(x)\} \equiv_{df} \forall z(z\in a\leftrightarrow\phi(z)) \tag{2.4}$$

$$\{x|\ \phi(x)\}=\{x|\ \psi(x)\} \equiv_{df} \forall z(z\in\{x|\ \phi(x)\}\leftrightarrow z\in\{x|\ \psi(x)\}) \tag{2.5}$$

In the above, and in general in the following,we usually assume that variables not mentioned in subformulas such as $\phi,\psi,\ldots,$ and which occur in other parts of a definition involving these subformulas, do not occur in these subformulas. For example in (2.2) it is understood that y and z do not occur in $\phi(x)$.

Formulas involving occurrences of the abstraction operator are really just abbreviations for formulas of $\mathcal{L}$. An explicit procedure for reducing a formula ϕ of the language having the operator $\{\ |\ \}$ to a formula ϕ^* of $\mathcal{L}$ is given below.

$$(x\in y)^* \equiv_{df} x\in y \tag{2.6}$$

$$(a\in\{x|\ \phi(x)\})^* \equiv_{df}(\phi(a))^* \equiv_{df} \phi^*(a) \tag{2.7}$$

$$(\{x|\ \phi(x)\}\in b)^* \equiv_{df} \exists y\in b\forall z(z\in y \leftrightarrow \phi^*(z)) \tag{2.8}$$

$$(\{x|\ \phi(x)\}\in\{x|\ \psi(x)\})^* \equiv_{df} \exists y(\psi^*(y)\ \wedge \forall z(z\in y\leftrightarrow\phi^*(z))) \tag{2.9}$$

$$(x=y)^* \equiv_{df} x=y \tag{2.10}$$

$$(a=\{x|\ \phi(x)\})^* \equiv_{df} (\{x|\ \phi(x)\}=a)^* \equiv_{df} \forall y(y\in a\leftrightarrow\phi^*(y)) \tag{2.11}$$

$$(\{x|\ \phi(x)\}=\{x|\ \psi(x)\})^* \equiv_{df} \forall z((z\in\{x|\ \phi(x)\})^* \longleftrightarrow (z\in\{x|\ \psi(x)\})^*) \quad (2.12)$$

$$(\sim\phi)^* \equiv_{df} \sim\phi^* \quad (2.13)$$

$$(\phi \vee \psi)^* \equiv_{df} \phi^* \vee \psi^* \quad (2.14)$$

$$(\exists x\phi)^* \equiv_{df} \exists x\phi^* \quad (2.15)$$

A detailed proof that this reduction procedure actually determines a unique ϕ^* may be found in Takeuti and Zaring [28, p.11].

When it is reasonably clear that a certain class described by an abstraction operator is actually a set, we may make use of this fact without explicit mention.

By a <u>class term</u> we mean either an individual variable symbol or a class symbol of the form $\{x|\ \phi\}$ where ϕ is a formula of $\mathcal{L}$. Metavariables ranging over class terms will be denoted by upper case Latin letters A,B,C,....

Below we list a number of formal definition schemes for some convenient abbreviations of expressions in $\mathcal{L}$. For the most part the notation is either fairly standard or mnemonic. The list is intended mainly as a reference; for the most part our definitions in the rest of this work will tend to be more informal, although we will always try to indicate enough so that it will be clear that our discussion is explicitly formalizable.

$$A\subseteq B \equiv_{df} \forall x(x\in A \longleftrightarrow x\in B) \qquad \text{(A \textit{is a subclass of} B)} \qquad (2.16)$$

$A \subset B \equiv_{df} A \subseteq B \wedge A \neq B$	(A *is a proper subclass of* B)	(2.17)
$\mathcal{U} =_{df} \{x \mid x=x\}$	(*the class of all sets*)	(2.18)
$A \times B =_{df} \{x \mid \exists a \exists b(a \in A \wedge b \in B \wedge x=\langle a,b\rangle)\}$	(*the cartesian product of* A *and* B)	(2.19)
$A^{-1} =_{df} \{u \mid \exists x \exists y(u=\langle x,y\rangle \wedge \langle y,x\rangle \in A)\}$	(*the inverse of* A)	(2.20)
$\bigcup A =_{df} \{x \mid \exists y(y \in A \wedge x \in y)\}$	(*the union of* A)	(2.21)
$A \cup B =_{df} \{x \mid x \in A \vee x \in B\}$	(*the union of* A *and* B)	(2.22)
$\bigcap A =_{df} \{x \mid \forall y(y \in A \rightarrow x \in y)\}$	(*the intersection of* A)	(2.23)
$A \cap B =_{df} \{x \mid x \in A \wedge x \in B\}$	(*the intersection of* A *and* B)	(2.24)
$A \restriction B =_{df} A \cap (B \times \mathcal{U})$	(A *restricted to* B)	(2.25)
$\underline{\mathit{Rel}}(R) \equiv_{df} R \subseteq \mathcal{U} \times \mathcal{U}$	(R *is a relation*)	(2.26)
$\underline{\mathit{Rel}}(R,A) \equiv_{df} R \subseteq A \times A$	(R *is a relation on* A)	(2.27)
$xRy \equiv_{df} \langle x,y\rangle \in R$	(x *is* R *to* y)	(2.28)
$\underline{\mathit{Refl}}(R) \equiv_{df} \forall x(xRx)$	(R *is reflexive*)	(2.29)
$\underline{\mathit{Refl}}(R,A) \equiv_{df} \forall x(x \in A \rightarrow xRx)$	(R *is reflexive on* A)	(2.30)
$x \not R y \equiv_{df} \langle x,y\rangle \notin R$	(x *is not* R *to* y)	(2.31)

$\underline{Irrefl}(R) \equiv_{df} \forall x(x \not R x)$ (R *is irreflexive*) (2.32)

$\underline{Irrefl}(R,A) \equiv_{df} \forall x(x \in A \rightarrow x \not R x)$ (R *is irreflexive on* A) (2.33)

$\underline{Symm}(R) \equiv_{df} \forall x \forall y(xRy \rightarrow yRx)$ (R *is symmetric*) (2.34)

$\underline{Symm}(R,A) \equiv_{df} \forall x \forall y(x \in A \wedge y \in A \wedge xRy \rightarrow yRx)$ (2.35)
(R *is symmetric on* A)

$\underline{Antisymm}(R) \equiv_{df} \forall x \forall y(xRy \wedge yRx \rightarrow x=y)$ (2.36)
(R *is antisymmetric*)

$\underline{Antisymm}(R,A) \equiv_{df} \forall x \forall y(x \in A \wedge y \in A \wedge xRy \wedge yRx \rightarrow x=y)$ (2.37)
(R *is antisymmetric on* A)

$\underline{RTrans}(R) \equiv_{df} \forall x \forall y \forall z(xRy \wedge yRz \rightarrow xRz)$ (2.38)
(R *is relationally transitive*)

$\underline{RTrans}(R,A) \equiv_{df} \forall x \forall y \forall z(x \in A \wedge y \in A \wedge z \in A \wedge xRy \wedge yRz \rightarrow xRz)$ (2.39)
(R *is relationally transitive on* A)

$\underline{Trich}(R) \equiv_{df} \forall x \forall y(xRy \vee x=y \vee yRx)$ (2.40)
(R *satisfies the law of trichotomy*)

$\underline{Trich}(R,A) \equiv_{df} \forall x \forall y(x \in A \wedge y \in A \rightarrow xRy \vee x=y \vee yRx)$ (2.41)
(R *satisfies the law of trichotomy on* A)

$\underline{PreOrd}(R) \equiv_{df} \underline{Rel}(R) \wedge \underline{Refl}(R) \wedge \underline{RTrans}(R)$ (2.42)
(R *is a preordering*)

$\underline{\mathit{PreOrd}}(R,A) \equiv_{df} \underline{\mathit{Rel}}(R,A) \land \underline{\mathit{Refl}}(R,A) \land \underline{\mathit{RTrans}}(R,A)$ (2.43)

(R *is a preordering on* A)

$\underline{\mathit{Equiv}}(R) \equiv_{df} \underline{\mathit{PreOrd}}(R) \land \underline{\mathit{Symm}}(R)$ (2.44)

(R *is an equivalence relation*)

$\underline{\mathit{Equiv}}(R,A) \equiv_{df} \underline{\mathit{PreOrd}}(R,A) \land \underline{\mathit{Symm}}(R,A)$ (2.45)

(R *is an equivalence relation on* A)

$\underline{\mathit{ParOrd}}(R) \equiv_{df} \underline{\mathit{PreOrd}}(R) \land \underline{\mathit{Antisymm}}(R)$ (2.46)

(R *is a partial ordering*)

$\underline{\mathit{ParOrd}}(R,A) \equiv_{df} \underline{\mathit{PreOrd}}(R,A) \land \underline{\mathit{Antisymm}}(R,A)$ (2.47)

(R *is a partial ordering on* A)

$\underline{\mathit{LinOrd}}(R) \equiv_{df} \underline{\mathit{ParOrd}}(R) \land \underline{\mathit{Trich}}(R)$ (2.48)

(R *is a linear ordering*)

$\underline{\mathit{LinOrd}}(R,A) \equiv_{df} \underline{\mathit{ParOrd}}(R,A) \land \underline{\mathit{Trich}}(R,A)$ (2.49)

(R *is a linear ordering on* A)

$\underline{\mathit{Minimal}}(x,R) \equiv_{df} \sim\exists y(y \neq x \land yRx)$ (x *is* R-*minimal*) (2.50)

$\underline{\mathit{Minimal}}(x,R,A) \equiv_{df} \sim\exists y(y \in A \land y \neq x \land yRx)$ (2.51)

(x *is* R-*minimal on* A)

$\underline{\mathit{Maximal}}(x,R) \equiv_{df} \sim\exists y(y \neq x \land xRy)$ (x *is* R-*maximal*) (2.52)

$\underline{\mathit{Maximal}}(x,R,A) \equiv_{df} \sim\exists y(y \in A \land y \neq x \land xRy)$ (2.53)

(x *is* R-*maximal on* A)

$\underline{\mathit{Least}}(x,R) \equiv_{df} \forall y(x \neq y \to xRy)$ (x *is* R-*least*) (2.54)

$\underline{\mathit{Least}}(x,R,A) \equiv_{df} x \in A \wedge \forall y(y \in A \wedge x \neq y \to xRy)$ (2.55)
(x *is* R-*least in* A)

$\underline{\mathit{Greatest}}(x,R) \equiv_{df} \forall y(x \neq y \to yRx)$ (x *is* R-*greatest*) (2.56)

$\underline{\mathit{Greatest}}(x,R,A) \equiv_{df} x \in A \wedge \forall y(y \in A \wedge x \neq y \to yRx)$ (2.57)
(x *is* R-*greatest in* A)

$\underline{\mathit{WellOrd}}(R) \equiv_{df} \underline{\mathit{LinOrd}}(R) \wedge \forall x((x \times U) \cap R \neq \emptyset \to \exists y(\underline{\mathit{Least}}(y, R \upharpoonright x, x)))$ (2.58)
(R *is a well-ordering*)

$\underline{\mathit{WellOrd}}(R,A) \equiv_{df} \underline{\mathit{LinOrd}}(R,A) \wedge \forall x(x \neq \emptyset \wedge x \subseteq A \to \exists y(\underline{\mathit{Least}}(y, R \upharpoonright x, x)))$ (2.59)
(R *is a well-ordering of* A)

$\underline{\mathit{Unary}}(A) \equiv_{df} \forall x \forall y \forall z(\langle x,y \rangle \in A \wedge \langle x,z \rangle \in A \to y=z)$ (2.60)
(A *is unary*)

$\underline{\mathit{Biunary}}(A) \equiv_{df} \underline{\mathit{Unary}}(A) \wedge \underline{\mathit{Unary}}(A^{-1})$ (2.61)
(A *is biunary*)

$\underline{\mathit{Fnc}}(A) \equiv_{df} \underline{\mathit{Rel}}(A) \wedge \underline{\mathit{Unary}}(A)$ (A *is a function*) (2.62)

$\underline{\mathit{One\text{-}one}}(A) \equiv_{df} \underline{\mathit{Rel}}(A) \wedge \underline{\mathit{Biunary}}(A)$ (2.63)
(A *is a one-one function*)

$\underline{\mathit{Dom}}(A) =_{df} \{x \mid \exists y(\langle x,y \rangle \in A)\}$ (*the domain of* A) (2.64)

$\underline{\mathit{Rg}}(A) =_{df} \{y \mid \exists x(\langle x,y \rangle \in A)\}$ (*the range of* A) (2.65)

$A"B =_{df} \underline{Rg}(A \upharpoonright B)$ (*the image of* B *under* A) (2.66)

$A \circ B =_{df} \{u \mid \exists x \exists y \exists z(u=\langle x,z\rangle \wedge \langle x,y\rangle \in A \wedge \langle y,z\rangle \in B)\}$ (2.67)

(A *composed with* B)

$F : A \longrightarrow B \equiv_{df} \underline{Fnc}(F) \wedge \underline{Dom}(F)=A \wedge \underline{Rg}(F) \subseteq B$ (2.68)

(F *maps* A *into* B)

$F : A \xrightarrow[\text{onto}]{} B \equiv_{df} \underline{Fnc}(F) \wedge \underline{Dom}(F)=A \wedge \underline{Rg}(F)=B$ (2.69)

(F *maps* A *onto* B)

$F : A \xrightarrow{1-1} B \equiv_{df} F : A \longrightarrow B \wedge \underline{One\text{-}one}(F)$ (2.70)

(F *maps* A *one-one into* B)

$F : A \xrightarrow[\text{onto}]{1-1} B \equiv_{df} F : A \xrightarrow{1-1} B \wedge F : A \xrightarrow[\text{onto}]{} B$ (2.71)

(F *maps* A *one-one onto* B)

$y=F(x) \equiv_{df} \langle x,y\rangle \in F$ (y *is* F *of* x) (2.72)

${}^{x}y =_{df} \{f \mid f : x \longrightarrow y\}$ (*the set of all functions from* x *into* y) (2.73)

$\underline{E} =_{df} \{u \mid \exists x \exists y(u=\langle x,y\rangle \wedge x \in y)\}$ (*the element relation*) (2.74)

$\underline{Trans}(A) \equiv_{df} \forall x \forall y(x \in A \wedge y \in x \rightarrow y \in A)$ (2.75)

(A *is transitive*)

$\underline{On}(x) \equiv_{df} \underline{Trans}(x) \wedge \underline{Wellord}(\underline{E} \upharpoonright x, x)$ (2.76)

(x *is an ordinal*)

$x+1 =_{df} x \cup \{x\}$ (x *plus one*) (2.77)

$$\underline{SuccOn}(x) \equiv_{df} \underline{On}(x) \wedge \exists y(\underline{On}(y) \wedge x=y+1) \quad (2.78)$$

(x *is a successor ordinal*)

$$\underline{LimOn}(x) \equiv_{df} \underline{On}(x) \wedge x \neq \emptyset \wedge \sim\underline{SuccOn}(x) \quad (2.79)$$

(x *is a limit ordinal*)

$$\bar{\bar{x}}=\bar{\bar{y}} \equiv_{df} \exists f(f : x \xrightarrow[\text{onto}]{1-1} y) \quad (2.80)$$

(x *and* y *are equipollent*)

$$\underline{Card}(x) \equiv_{df} \underline{On}(x) \wedge \forall y(\underline{On}(y) \wedge \bar{\bar{x}}=\bar{\bar{y}} \wedge x \neq y \rightarrow x \in y) \quad (2.81)$$

(x *is a cardinal*)

$$x=\omega \equiv_{df} \underline{LimOn}(x) \wedge \forall y(\underline{LimOn}(y) \wedge x \neq y \rightarrow x \in y) \quad (2.82)$$

(x *is* ω)

From now on we will try to present most of our definitions more informally, leaving it to the reader to satisfy himself that our definitions are actually explicitly formalizable in $\mathcal{L}$.

Having defined ordinal and cardinal numbers, we will assume, as needed, that the reader is familiar with some of their elementary properties: e.g. simple ordinal and cardinal arithmetic, transfinite induction, transfinite recursion, the Schröder-Bernstein Theorem, Cantor's Theorem, etc. We shall usually denote ordinals by lower case Greek letters $\alpha, \beta, \gamma, \ldots$, relying on the context to prevent confusion between these and metavariables for formulas of $\mathcal{L}$.

Below we simultaneously define the sequences of $\aleph$ and ω numbers by transfinite recursion over the ordinals.

i.) $\aleph_0 =_{df} \omega_0 =_{df} \omega$ (2.83)

ii.) $\aleph_{\alpha+1} =_{df} \omega_{\alpha+1} =_{df}$ [the least ordinal γ such that $\sim\exists f(f : \gamma \xrightarrow[\text{onto}]{1-1} \omega_\alpha)$]

iii.) $\aleph_\lambda =_{df} \omega_\lambda =_{df} \bigcup_{\beta<\lambda} \omega_\beta$ for λ a limit.

In the above we write $\beta<\lambda$ for $\beta\in\lambda$ and assume that the reader can figure out an indexed union. If λ is a limit ordinal then the cofinality of λ, $\underline{cf}(\lambda)$, is defined by

$$\underline{cf}(\lambda) =_{df} [\text{the least ordinal } \beta \text{ such that } \exists f(f : \beta \longrightarrow \lambda \wedge \bigcup \underline{Rg}(f)=\lambda)]. \tag{2.84}$$

A cardinal $\aleph_\alpha$ is said to be regular if $\underline{cf}(\omega_\alpha)=\omega_\alpha$. $\aleph_\alpha$ is said to be singular if $\underline{cf}(\omega_\alpha)<\omega_\alpha$.

Next we wish to show that *AxFound* actually allows us to describe the universe of all sets $\mathcal{U}$ as a hierarchy built up from the null set $\emptyset$ by power-set and union operations.

For each ordinal α we define a set V_α as follows:

i.) $V_0 =_{df} \emptyset$ (2.85)

ii.) $V_{\beta+1} =_{df} \underline{PS}(V_\beta)$

iii.) $V_\lambda =_{df} \bigcup_{\beta<\lambda} V_\beta$ for λ a limit ordinal.

Finally, let

$$V =_{df} \{x \mid \exists\alpha(\underline{On}(\alpha) \wedge x\in V_\alpha)\}. \tag{2.86}$$

The following propositions about V are easy to prove.

PROPOSITION 1: For each ordinal α, V_α is transitive.□

PROPOSITION 2: If $\alpha<\beta$ then $V_\alpha \in V_\beta$.□

PROPOSITION 3: If $\alpha\leq\beta$ then $V_\alpha \subseteq V_\beta$.□

Next we define a rank function, *rank* : V → *On* by

$$\mathit{rank}(x) =_{df} [\text{least ordinal } \alpha \text{ such that } x \in V_{\alpha+1}]. \tag{2.87}$$

The following propositions about *rank*(x) are easy to prove.

PROPOSITION 4: If $x\in y$ then $\mathit{rank}(x)<\mathit{rank}(y)$.□

PROPOSITION 5: If $x\subseteq y$ then $\mathit{rank}(x)\leq\mathit{rank}(y)$.□

PROPOSITION 6: If α is an ordinal then $\mathit{rank}(\alpha)=\alpha$.□

THEOREM II: $ZF\vdash V=\mathcal{U}$.

Proof: The proof is by *reductio ad absurdum.* Suppose there exists a set $x\in\mathcal{U}$ such that $x\notin V$. The first claim is that by *AxFound* we may assume without loss of generality that every element of x is in V. To see this define the transitive closure of x, *TransCl*(x), as follows:

$$\text{i.)}\quad T_0(x) =_{df} \{x\} \tag{2.88}$$

$$\text{ii.)}\quad T_{n+1}(x) =_{df} \bigcup T_n(x)$$

$$\text{iii.)}\quad \mathit{TransCl}(x) =_{df} \bigcup_{n\in\omega} T_n(x).$$

By *Aussonderung* $\{y \mid y\in \mathit{TransCl}(x) \wedge y\notin V\}$ is a set and by *AxFound* it has an $\in$-minimal element which has the property that all of its elements are in V while it is not. Hence we could have taken x to be this element to begin with.

Now *rank*$\upharpoonright$x : x $\longrightarrow$ *On* and by *AxRepl*, *rank*"x is a set. Hence ρ=$\bigcup$*rank*"x is an ordinal greater than or equal to the rank of any element of x. As V_ρ is transitive this means that for all y$\in$x, y$\in V_\rho$. Therefore x$\subseteq V_\rho$ and x$\in V_{\rho+1}$, which is a contradiction.□

I.3 Other axiomatizations of ZFC

The purpose of this section is to introduce some other axiomatizations of ZFC which will prove to be technically useful in Chapter IV.

First we wish to define what we mean by the relativization of a formula. Let $\theta(x)$ be a formula of $\mathcal{L}$ with exactly one free variable and let A $=_{df}$ $\{x|\ \theta(x)\}$. In (3.1)-(3.5) we define ϕ^A, the relativization of the formula ϕ to the class A, by induction over formulas ϕ of $\mathcal{L}$.

$$(y\in z)^A \equiv_{df} y\in z \tag{3.1}$$

$$(y=z)^A \equiv_{df} y=z \tag{3.2}$$

$$(\sim\psi)^A \equiv_{df} \sim\psi^A \tag{3.3}$$

$$(\psi_1 \vee \psi_2)^A \equiv_{df} \psi_1^A \vee \psi_2^A \tag{3.4}$$

$$(\exists y\psi)^A \equiv_{df} \exists y(y\in A \wedge \psi^A) \tag{3.5}$$

Next, let $\phi(x_1,\ldots,x_n)$ be a formula of $\mathcal{L}$ with exactly n free variables. Let W be a class and let $w_0\in W$. We say that w_0 mirrors $\phi(x_1,\ldots,x_n)$ in W if (3.6) is provable.

$$\forall x_1 \in w_0 \ldots \forall x_n \in w_0 (\phi^{w_0}(x_1,\ldots,x_n) \longleftrightarrow \phi^W(x_1,\ldots x_n)) \tag{3.6}$$

Finally, if ϕ_1 and ϕ_2 are any two formulas of $\mathcal{L}$ in prenex normal form, we say that ϕ_1 is a truncation of ϕ_2 if ϕ_1 can be obtained from ϕ_2 by deleting some initial segment of the prefix of ϕ_2.

THEOREM III (The Generalized Reflection Principle):

Suppose that for every ordinal α we have defined a set W_α such that

i.) if $\beta \leq \gamma$ then $W_\beta \subseteq W_\gamma$ and

ii.) if λ is a limit ordinal then $W_\lambda = \bigcup_{\beta<\lambda} W_\beta$.

Let $W =_{df} \{x \mid \exists\alpha(\mathit{On}(\alpha) \wedge x \in W_\alpha)\}$ and let $\phi(x_1,\ldots,x_n)$ be any formula of $\mathcal{L}$ with exactly n free variables, which is in prenex normal form. Then it is provable in ZF that given any ordinal α, there exists a limit ordinal $\lambda>\alpha$ such that W_λ mirrors $\phi(x_1,\ldots,x_n)$ and all of its truncations in W.

Proof: The proof is by induction over formulas ϕ of $\mathcal{L}$.

Case 1: ϕ is quantifier free. In this case ϕ^A is ϕ for any class term A; hence we take λ to be the first limit ordinal above α.

Case 2: ϕ is $\sim\psi$. By the induction hypothesis we can find a limit ordinal $\lambda>\alpha$ such that W_λ mirrors ψ and all its truncations in W. But for all $x_1,\ldots,x_n \in W_\lambda$, $\psi^{W_\lambda}(x_1,\ldots,x_n) \longleftrightarrow \psi^W(x_1,\ldots,x_n)$ is provable if and only if $\sim\psi^{W_\lambda}(x_1,\ldots,x_n) \longleftrightarrow \sim\psi^W(x_1,\ldots,x_n)$ is provable.

Case 3: $\phi(x_1,\ldots,x_n)$ is $\exists x\psi(x,x_1,\ldots,x_n)$

Define the functional relation *wrank* : $W \longrightarrow \mathit{On}$ by

$$\underline{wrank}(w) = \alpha \equiv_{df} w \in W_{\alpha+1} \wedge \forall\beta\in\alpha(w \notin W_{\beta+1}) \tag{3.7}$$

for all $w \in W$.

Let F_ϕ be the n-place function defined by

$$F_\phi(x_1,\ldots,x_n) =_{df} \tag{3.8}$$
$$=_{df} \{x \mid x\in W \wedge \psi^W(x,x_1,\ldots,x_n) \wedge \forall\beta\in\underline{wrank}(x)\forall z\in W_\beta(\sim\psi^W(z,x_1,\ldots,x_n))\}$$

for all $x_1,\ldots,x_n \in W$.

From (3.8) it is clear that

$$\exists x\in W\psi^W(x,x_1,\ldots,x_n) \longleftrightarrow \exists x\in F_\phi(x_1,\ldots,x_n)\psi^W(x_1,\ldots,x_n) \tag{3.9}$$

We next define a sequence of ordinal $\{\lambda_k\}_{k\in\omega}$ as follows:

$$\lambda_0 =_{df} [\text{the least limit ordinal above } \alpha \text{ such that } W_{\lambda_0} \text{ mirrors } \psi(x,x_1,\ldots,x_n) \text{ and all its truncations in } W] \tag{3.10}$$

Let $\theta(x_1,\ldots,x_m)$ be any truncation of ϕ beginning with an existential quantifier and suppose $x_1,\ldots,x_m$ to be an exhaustive list of the free variables of θ.

$$\lambda_{2k+1,\theta} =_{df} [\text{the least ordinal above } \lambda_{2k} \text{ such that for all } a_1,\ldots,a_m\in W_{\lambda_{2k}},\ F_\theta(a_1,\ldots,a_m) \subseteq W_{\lambda_{2k+1,\theta}}] \tag{3.11}$$

and

$$\lambda_{2k+1} =_{df} \cup\{\lambda_{2k+1,\theta} \mid \theta \text{ is a truncation of } \phi \text{ beginning with an existential quantifier}\} \quad (3.12)$$

$$\lambda_{2k+2} =_{df} [\text{the least ordinal above } \lambda_{2k+1} \text{ such that } W_{\lambda_{2k+2}} \text{ mirrors } \psi(x,x_1,\ldots,x_n) \text{ and all its truncations}] \quad (3.13)$$

λ_0 and λ_{2k+2} are well defined by our induction hypothesis. $\lambda_{2k+1,\theta}$ is well defined for each θ a truncation of ϕ beginning with an existential quantifier, for if we let

$$\overset{m}{\underset{i=1}{\times}} y =_{df} \{z \mid \exists y_1 \epsilon y \ldots \exists y_m \epsilon y (z = \langle y_1,\ldots,y_m\rangle)\} \quad (3.14)$$

then $\underline{wrank}\ F_\theta{}'' \overset{m}{\underset{i=1}{\times}} W_{\lambda_{2k}}$ is a set by $\underline{AxRepl}$. Since there are only a finite number of θ's satisfying the conditions posited, λ_{2k+1} is obviously well defined.

Let $\lambda =_{df} \bigcup_{k\epsilon\omega} \lambda_k = \bigcup_{k\epsilon\omega} \lambda_{2k}$. λ is obviously a limit ordinal, so by the continuity hypothesis about W, $W_\lambda = \bigcup_{k\epsilon\omega} W_{\lambda_k} = \bigcup_{k\epsilon\omega} W_{\lambda_{2k}}$. We now claim that W_λ mirrors ϕ and all its truncations in W. We prove this claim by induction over all truncations θ of ϕ.

<u>Subcase 1</u>: <u>θ is quantifier free</u>. Same as <u>Case 1</u>.

<u>Subcase 2</u>: <u>θ is $\sim\xi$</u>. Same as <u>Case 2</u>.

<u>Subcase 3</u>: <u>$\theta(x_1,\ldots,x_m)$ is $\exists x\xi(x,x_1,\ldots,x_m)$</u>. Suppose that $a_1,\ldots,a_m \epsilon W_\lambda$. Choose $k<\omega$ large enough so that $a_1,\ldots,a_m \epsilon W_{\lambda_{2k}}$.

Then

$$
\begin{aligned}
\theta^{W}(a_1,\ldots,a_m) &\Leftrightarrow \exists x(x\in W \wedge \xi^{W}(x,a_1,\ldots,a_m)) \\
&\Leftrightarrow \exists x\in F_\theta(a_1,\ldots,a_m)\,\xi^{W}(x,a_1,\ldots,a_m) \\
&\Leftrightarrow \exists x\in W_{2k+1,\theta}\,\xi^{W}(x,a_1,\ldots,a_m) \\
&\Rightarrow \exists x\in W_{\lambda_{2k+1}}\,\xi^{W}(x,a_1,\ldots,a_m) \\
&\Rightarrow \exists x\in W_\lambda\,\xi^{W}(x,a_1,\ldots,a_m) \\
&\Leftrightarrow \exists x\in W_\lambda\,\xi^{W_\lambda}(x,a_1,\ldots,a_m) \\
&\Leftrightarrow \theta^{W_\lambda}(a_1,\ldots,a_m)
\end{aligned}
\tag{3.15}
$$

Conversely, if $\exists x\in W_\lambda \xi^{W_\lambda}(x,a_1,\ldots,a_m)$ holds, there is an $a\in W_\lambda$ such that $\xi^{W_\lambda}(a,a_1,\ldots,a_m)$ holds. By the induction hypothesis this implies $\xi^{W}(a,a_1,\ldots,a_m)$ holds. Therefore $\exists x(x\in W \wedge \xi^{W}(x,a_1,\ldots,a_m))$ holds.

This then completes both inductions.□

<u>COROLLARY III.1</u> (The Reflection Principle, abbreviated $\mathcal{RP}$)

If $\phi(x_1,\ldots,x_n)$ is any formula in $\mathcal{L}$ in prenex normal form with exactly n free variables then given any ordinal α there exists a limit ordinal $\lambda>\alpha$ such that V_λ mirrors $\phi(x_1,\ldots,x_n)$ and all of its truncations in V.□

By the <u>Bounding Principle</u> (abbreviated $\mathcal{BP}$) we mean the formula scheme (3.16)

$$\forall t_1\ldots\forall t_k\forall x(\forall u\in x\exists v\phi(u,v,t_1,\ldots,t_k) \longleftrightarrow \exists y\forall u\in x\exists v\in y\phi(u,v,t_1,\ldots,t_k) \tag{3.16}$$

where $\phi(u,v,t_1,\ldots,t_k)$ is any formula in $\mathcal{L}$ with exactly k+2 free variables.

<u>THEOREM IV</u>: For all formulas ϕ in $\mathcal{L}$, $Z^{\lim}+\mathcal{BP}\vdash\phi$ if and only if $ZF\vdash\phi$.

<u>Proof</u>: ($\Rightarrow$) It is enough to show that $ZF\vdash\mathcal{BP}$. We show in fact something which is seemingly stronger.

<u>LEMMA 1</u>: $Z^{\lim}+\mathcal{RP}\vdash\mathcal{BP}$.

<u>Proof</u>: Fix $t_1,\ldots t_k$ and x. Let $\alpha =_{df} \bigcup\{\underline{rank}(x),\underline{rank}(t_1),\ldots,\underline{rank}(t_k)\}$. By $\mathcal{RP}$ there is a limit ordinal $\lambda>\alpha$ such that V_λ mirrors

$$\forall u\in x\exists v\phi(u,v,t_1,\ldots,t_k) \tag{3.17}$$

and all its truncations.

In particular V_λ mirrors (3.18).

$$u\in x \rightarrow \phi(u,v,t_1,\ldots,t_k) \tag{3.18}$$

So we have (3.19)

$$(u\in x \rightarrow \phi(u,v,t_1,\ldots,t_k)) \longleftrightarrow (u\in x \rightarrow \phi(u,v,t_1,\ldots,t_k))^{V_\lambda} \longleftrightarrow$$
$$\longleftrightarrow (u\in x \rightarrow \phi^{V}(u,v,t_1,\ldots,t_k)) \tag{3.19}$$

Hence (3.20) holds.

$$(u\in x \rightarrow (\phi^{V_\lambda}(u,v,t_1,\ldots,t_k)\longleftrightarrow\phi(u,v,t_1,\ldots,t_k))) \tag{3.20}$$

From the fact that V_λ mirrors (3.17) we have (3.21)

$$\forall u\in x\exists v\phi(u,v,t_1,\ldots,t_k)\longleftrightarrow\forall u\in x\exists v\in V_\lambda\phi^{V_\lambda}(u,v,t_1,\ldots,t_k) \tag{3.21}$$

which by (3.20) is equivalent to (3.22).

$$\forall u\in x\exists v\phi(u,v,t_1,\ldots,t_k)\longleftrightarrow\forall u\in x\exists v\in V_\lambda\phi(u,v,t_1,\ldots,t_k) \tag{3.22}$$

This completes the proof of Lemma 1.□

(⇐) To show the converse we have to prove that $Z^{lim}+\mathcal{BP} \vdash \mathit{AxRepl}$. Informally we can see this by noting that any instance of *AxRepl* may be replaced by use of $\mathcal{BP}$, to get a bound on the image we want, followed by a use of *Aussonderung*, to carve out the exact set we want. The content of $\mathcal{RP}$ is that our use of *Aussonderung* may be replaced by a limited instance of *Aussonderung*. To be strictly more formal requires showing that we can actually prove $\mathcal{RP}$ in $Z^{lim}+\mathcal{BP}$, since *AxRepl* was used in proving $\mathcal{RP}$.

<u>LEMMA 2</u>: $Z^{lim}+\mathcal{BP}$ $\mathcal{RP}$.

<u>Proof</u>: The proof simply requires a careful look at the proof of Theorem III in the case W = V. Our key use of *AxRepl* there was in (3.11), in which we needed to get a bound on $\lambda_{2k+1,\theta}$. But $\mathcal{BP}$ is all we really need there. This then proves Lemma 2.□

Our proof of Theorem IV is now complete.□

We may make use of *AC* and its equivalents without comment in the following. The most frequently employed equivalent of *AC* of which we make use is <u>Cantor's law of trichotomy</u>, which is expressed by (3.23)

$$\forall x \exists y (y = \bar{\bar{x}}) \tag{3.23}$$

For other equivalents of *AC* the reader is referred to Cohen [2], Felgner [3], Jech [7], Krivine [9], Mostowski [20], Shoenfield [25], and Takeuti and Zaring [28].

I.4 Classical models of ZFC

A (classical) model of ZFC is a structure $\mathfrak{A} = \langle A, e \rangle$, where A is a set and $e \subseteq A \times A$, which is a model of the first-order theory ZFC in the usual sense. For the definition of a model of a first-order theory the reader is referred to Shoenfield [25].

We do not allow the universe A of the structure $\mathfrak{A}$ to be a proper class because if we did we would not be able to express the fact that $\mathfrak{A} \models ZFC$ in a single sentence of $\mathcal{L}$. This is a consequence of the fact that ZFC is not finitely axiomatizable. However, when we insist that A be a set we can say that $\mathfrak{A}$ satisfies the infinite axiom scheme *AxRepl* by saying that A is closed under a finite number of operations, e.g. Gödel's $\mathcal{F}_1$-$\mathcal{F}_8$. See Cohen [2], Felgner [3], Jech [7], Mostowski [20], Shoenfield [25], or Takeuti and Zaring [28].

A model $\mathfrak{A} = \langle A, e \rangle$ of ZFC is said to be a standard model if $e = \underline{E} \upharpoonright A$; otherwise it is said to be nonstandard. A standard model is called transitive if its universe is a transitive set. Since this work is concerned with models of ZFC, the following axioms concerning the existence of models are of considerable interest to us.

The Model Axiom (abbreviated *M*)

$\exists x$(x is a model of ZFC)

The Standard Model Axiom (abbreviated *SM*)

$\exists x$(x is a standard model of ZFC)

The Standard Transitive Model Axiom (abbreviated STM)

$$\exists x(x \text{ is a standard transitive model of ZFC})$$

We have already seen that these axioms are formalizable in $\mathcal{L}$. The following relationship between the axioms is evident

$$\text{ZFC} \vdash STM \to SM \to M \to [\text{ZFC is consistent}]. \tag{4.1}$$

Hence by Gödel's completeness theorem, if ZFC is consistent then the addition of any of the axioms M, SM, or STM yields a set theory strictly stronger than ZFC. Also by Gödel's completeness theorem we know that

$$\text{ZFC} \vdash [\text{ZFC is consistent}] \longleftrightarrow M \tag{4.2}$$

Also

$$\text{ZFC} \vdash SM \longleftrightarrow STM \tag{4.3}$$

See Corollary V.2. However

$$\text{ZFC} \vdash \sim(M \to SM). \tag{4.4}$$

A proof of this may be found in Takeuti and Zaring [28, p.243] and Cohen [2, p.104].

A structure $\langle x,B\rangle$ is said to be B-extensional if (4.5) holds.

$$\underline{\mathit{Rel}}(B,x) \wedge \forall p \forall q(p\in x \wedge q\in x \wedge p\neq q \to \exists r(r\in x \wedge ((rBp \wedge \sim rBq) \vee (rBq \wedge \sim rBp)))) \tag{4.5}$$

The structure $\langle x,B\rangle$ is said to be B-well-founded if (4.6) holds.

$$\underline{\mathit{Rel}}(B,x) \wedge \forall z(z \subseteq x \wedge z\neq\emptyset \rightarrow \exists p(p\in z \wedge \forall q(q\in x \wedge qBp \rightarrow q\notin z))) \tag{4.6}$$

THEOREM V (Mostowski's Transitive Collapse Theorem):

Suppose $\langle x,B\rangle$ is B-extensional and B-well-founded. Then there exists a unique transitive set t and a unique function $f : x \xrightarrow[\text{onto}]{1-1} t$ such that for all y, z$\in$x, yBz if and only if f(y)$\in$f(z).

Proof: Let $\emptyset_{\langle x,B\rangle}$ denote the B-minimal element of x, which exists by B-well-foundedness and is unique by B-extensionality. Define a hierarchy of pseudo-ranks in x as follows by induction on the ordinals:

$$P_0 = \emptyset_{\langle x,B\rangle} \tag{4.7}$$

$$P_{\alpha+1} = \{z\in x|\ \forall q(q\in x \wedge qBz \rightarrow q\in P_\alpha)\} \tag{4.8}$$

$$P_\lambda = \bigcup_{\alpha<\lambda} P_\alpha \text{ for } \lambda \text{ a limit ordinal.} \tag{4.9}$$

Define a pseudo-rank function $\underline{\mathit{prank}} : x \longrightarrow \underline{\mathit{On}}$ by

$$\underline{\mathit{prank}}(z) = [\text{the least ordinal } \alpha \text{ such that } z\in P_{\alpha+1}] \tag{4.10}$$

LEMMA 1: There exists an ordinal α such that for all ordinals $\beta>\alpha$, $P_\beta \setminus P_\alpha = \emptyset$.

Proof: Suppose not, i.e. suppose that $\forall\alpha(\underline{\mathit{On}}(\alpha) \rightarrow \sim x \subseteq P_\alpha)$. By B-well-foundedness we may assume without loss of generality that x is B-minimal with this property. But by *AxRepl*, $\underline{\mathit{prank}}$"x is a set, hence $\rho = \bigcup \underline{\mathit{prank}}$"x is an ordinal which is greater than or equal to the pseudo-rank of every element of x. Thus $x \subseteq P_{\rho+1}$ which contradicts our hypothesis. This proves Lemma 1.□

Define the function $f : x \longrightarrow V$ by

$$f(\emptyset_{\langle x,B\rangle}) = \emptyset \tag{4.11}$$

$$f(y) = \{f(z) \mid z \in x \wedge \underline{prank}(z) < \underline{prank}(y) \wedge zBy\} \tag{4.12}$$

LEMMA 2: f is injective.

Proof: Suppose $z_1 \neq z_2$. We want to show that this implies that $f(z_1) \neq f(z_2)$. The proof is by induction on max $(\underline{prank}(z_1), \underline{prank}(z_2))$. By B-extensionality $z_1 \neq z_2$ implies $\exists z_0 \in x((z_0Bz_1 \wedge \sim z_0Bz_2) \vee \vee (z_0Bz_2 \wedge \sim z_0Bz_1))$. Suppose we have that $z_0Bz_1 \wedge \sim z_0Bz_2$. As $\underline{prank}(z_0) < \underline{prank}(z_1)$ we have by our induction hypothesis that

$$\begin{aligned} f(z_1) &= \{f(z) \mid z \in x \wedge \underline{prank}(z) < \underline{prank}(z_1) \wedge zBz_1\} \neq \\ &\neq \{f(z) \mid z \in x \wedge \underline{prank}(z) < \underline{prank}(z_2) \wedge zBz_2\} = f(z_2) \end{aligned} \tag{4.13}$$

since $f(z_0) \in f(z_1)$, whereas $f(z_0) \neq f(z_2)$. This proves Lemma 2.□

Let $t =_{df} f''x$.

LEMMA 3: t is transitive.

Proof: If $u \in v$ and $v \in f(z)$ for some $z \in x$, then $v = f(y)$ for some $y \in x$. But then $u = f(q)$ for some $q \in x$. This proves Lemma 3.□

LEMMA 4: z_1Bz_2 if and only if $f(z_1) \in f(z_2)$.

Proof: By the definition of f, z_1Bz_2 implies that $f(z_1) \in f(z_2)$. Conversely, $f(z_1) \in f(z_2)$ implies that $f(z_1) = f(y)$ for some yBz_2. By Lemma 2 this implies $z_1 = y$. Hence z_1Bz_2. This proves Lemma 4.□

LEMMA 5: f is unique.

Proof: The obvious induction on *prank* suffices to prove Lemma 5.□

This completes the proof of Theorem V.□

COROLLARY V.1: If $B = \underline{E}\restriction x$ and $z \subseteq x$ is transitive then $f\restriction z$ is the identity function of z.

Proof: Follows from the uniqueness part of Theorem V.□

We will refer to $f : x \longrightarrow t$ as the collapsing function of $\langle x,B\rangle$ and to t as the transitive collapse of $\langle x,B\rangle$.

COROLLARY V.2: $ZFC \vdash SM \longleftrightarrow STM$.□

Remark: Theorem V cannot be used to show that $M \to STM$ because the assertion that $\langle x,B\rangle$ is B-well-founded is strictly stronger than the assertion that $\langle x,B\rangle$ satisfies *AxFound*.

A regular cardinal $\aleph_\alpha$ is said to be (strongly) inaccessible if α is a limit ordinal and $\forall x(\overline{\overline{x}} < \aleph_\alpha \to \overline{\overline{\underline{PS}(x)}} < \aleph_\alpha)$.

The following axiom will often be useful:

The Axiom of Inaccessible Cardinals (abbreviated I)

$$\exists\alpha(\underline{On}(\alpha) \wedge [\aleph_\alpha \text{ is strongly inaccessible}]).$$

We will usually use ι to represent an inaccessible cardinal.

PROPOSITION 7: It is provable in ZFC that if ι is a strongly inaccessible cardinal then V_ι forms a standard transitive model of ZFC. Thus $ZFC \vdash I \to STM$.

Proof: See Takeuti and Zaring [28, p.131].□

We will denote the system ZFC + I by ZFCI.

I.5 Boolean valued models of ZFC

Let $\mathcal{B}$ be a complete Boolean algebra which will remain fixed throughout this section.

A $\mathcal{B}$-valued interpretation of $\mathcal{L}$ consists of the following:

1.) a set u, called the universe for the interpretation and

2.) two functions $R_0 : u \times u \longrightarrow \mathcal{B}$ and $R_1 : u \times u \longrightarrow \mathcal{B}$ which satisfy Condition (*) below.

For every closed formula σ of $\mathcal{L}(u)$ (the language $\mathcal{L}$ with constant symbols for elements of u adjoined) we define a truth value in $\mathcal{B}$, $[\![\sigma]\!]$, by recursion as follows:

$$[\![a_1 = a_2]\!] =_{df} R_0(a_1,a_2) \text{ for all } a_1,a_2 \in u \tag{5.1}$$

$$[\![a_1 \in a_2]\!] =_{df} R_1(a_1,a_2) \text{ for all } a_1,a_2 \in u \tag{5.2}$$

$$[\![\sim\phi]\!] =_{df} \sim[\![\phi]\!] \tag{5.3}$$

$$[\![\phi \vee \psi]\!] =_{df} [\![\phi]\!] \vee [\![\psi]\!] \tag{5.4}$$

$$[\![\exists x\phi(x)]\!] =_{df} \sup_{\mathcal{B}}\{[\![\phi(a)]\!] \mid a \in u\} \tag{5.5}$$

We say that a sentence σ of $\mathcal{L}(u)$ is $\mathcal{B}$-valid if $[\![\phi]\!] = 1^{(\mathcal{B})}$, the greatest element of $\mathcal{B}$.

For R_0 and R_1 to be part of a $\mathcal{B}$-valued interpretation of $\mathcal{L}$ we also require that they satisfy

Condition (*): the sentences of $\mathcal{L}$ asserting that = is an equivalence relation and $\in$ is substitutive with respect to = are $\mathcal{B}$-valid.

Let $\mathcal{M}$ be a model of ZFC. In the rest of this section all of our considerations will be carried out in $\mathcal{M}$ unless indicated otherwise.

We define the $\mathcal{B}$-valued universe for $\mathcal{M}$, $M^{(\mathcal{B})}$, by induction as follows:

$$\text{i.)} \quad M_{\circ}^{(\mathcal{B})} =_{df} \emptyset$$

$$\text{ii.)} \quad M_{\beta+1}^{(\mathcal{B})} =_{df} \{f \mid \underline{Fnc}(f) \wedge \underline{Dom}(f) \subseteq M_{\beta}^{(\mathcal{B})} \wedge \underline{Rg}(f) \subseteq \mathcal{B}\} \tag{5.6}$$

$$\text{iii.)} \quad M_{\lambda}^{(\mathcal{B})} =_{df} \bigcup_{\beta<\lambda} M_{\beta}^{(\mathcal{B})} \text{ for } \lambda \text{ a limit ordinal.}$$

Finally let

$$M^{(\mathcal{B})} =_{df} \{x \mid \exists\alpha(\underline{On}(\alpha) \wedge x \in M_{\alpha}^{(\mathcal{B})}\}. \tag{5.7}$$

There is a natural embedding of the universe M of $\mathcal{M}$ into the $\mathcal{B}$-valued universe $M^{(\mathcal{B})}$. We denote this embedding by $\check{}: M \longrightarrow M^{(\mathcal{B})}$ which we define by $\in$-recursion as follows:

$$\check{\emptyset} =_{df} M_0^{(\mathcal{B})}$$
$$\check{x} =_{df} [\text{the unique constant function } \{\check{y} \mid y \in x\} \longrightarrow \{1^{(\mathcal{B})}\}] \tag{5.8}$$

We now construct a $\mathcal{B}$-valued interpretation of $\mathcal{L}$ with universe $M^{(\mathcal{B})}$. Let $\underline{Brank} : M^{(\mathcal{B})} \longrightarrow \underline{On}$ be the function defined by

$$\underline{Brank}(x) =_{df} [\text{least ordinal } \alpha \text{ such } x \in M_{\alpha+1}^{(\mathcal{B})}]. \tag{5.9}$$

We now define $[\![x=y]\!]$ and $[\![x \in y]\!]$ by recursion on $\langle\underline{Brank}(x), \underline{Brank}(y)\rangle$, in the canonical well ordering of $\underline{On} \times \underline{On}$.

$$[\![x=y]\!] =_{df} \inf_{\mathcal{B}}\{x(z)\Rightarrow[\![z\in y]\!] \mid z\in \underline{\mathcal{Dom}}(x)\} \wedge \wedge \inf_{\mathcal{B}}\{y(z)\Rightarrow[\![z\in x]\!] \mid z\in \underline{\mathcal{Dom}}(y)\} \tag{5.10}$$

where

$$u\Rightarrow v =_{df} \sim u \vee v \tag{5.11}$$

for all $u,v\in\mathcal{B}$ and

$$[\![x\in y]\!] =_{df} \sup_{\mathcal{B}}\{y(z) \wedge [\![z=x]\!] \mid z\in \underline{\mathcal{Dom}}(y)\} \tag{5.12}$$

Proofs that these interpretations satisfy Condition (*) are straightforward and may be found in Rosser [24].

We will denote by $\mathcal{M}^{(\mathcal{B})}$ the $\mathcal{B}$-valued structure with universe $M^{(\mathcal{B})}$ and interpretations of = and $\in$ given by (5.10) and (5.12) above.

THEOREM VI: If ϕ is provable in the first-order theory of $\mathcal{L}$ then ϕ is $\mathcal{B}$-valid in $\mathcal{M}^{(\mathcal{B})}$.

Proof: This is proved in Rasiowa and Sikorski [22] and in Rosser [24], using the formulation of the first order predicate calculus given in Rosser [23].□

In $\mathcal{M}$ we cannot actually prove that $\mathcal{M}^{(\mathcal{B})}$ satisfies all the axioms of ZFC without contradicting Gödel's incompleteness theorem. However, in $\mathcal{M}$ we can check that $\mathcal{M}^{(\mathcal{B})}$ satisfies each axiom of ZFC. If we work in ZFCI and assume that $M \subseteq V_{\iota}$, where ι is an inaccessible cardinal, we are able to look at $\mathcal{M}^{(\mathcal{B})}$ "from the outside" and see that the following theorem is true.

THEOREM VII: All the axioms of ZFC are $\mathcal{B}$-valid in $M^{(\mathcal{B})}$.

Proof: See Jech [7] or Rosser [24].□

We say that $M^{(\mathcal{B})}$ is separated if and only if for all $x,y \in M^{(\mathcal{B})}$, $[\![x=y]\!] = 1^{(\mathcal{B})}$ implies x=y. The reader should note that $M^{(\mathcal{B})}$ as we have defined it is not necessarily separated. We would like to construct a separated version of $M^{(\mathcal{B})}$, $M_s^{(\mathcal{B})}$, in M. However there is a difficulty, namely that the equivalence classes in the equivalence relation $[\![\_=\_]\!] = 1^{(\mathcal{B})}$ are proper classes in M. To get around this we use the following trick of Scott's.

Let

$$[x]_s =_{df} \{y \in M^{(\mathcal{B})} \mid [\![x=y]\!] = 1^{(\mathcal{B})} \wedge \wedge \forall z \in M^{(\mathcal{B})} ([\![x=z]\!] = 1^{(\mathcal{B})} \rightarrow \mathit{rank}(y) \leq \mathit{rank}(z))\} \tag{5.13}$$

$[x]_s$ is then a set in M, called the Scott equivalence class of x. Further $[x]_s = [y]_s$ if and only if $[\![x=y]\!] = 1^{(\mathcal{B})}$. Hence there is a quotient map $\pi : M^{(\mathcal{B})} \longrightarrow M_s^{(\mathcal{B})}$ defined by

$$\pi(x) =_{df} [x]_s \tag{5.14}$$

which satisfies

$$[\![\phi(x_1,\ldots,x_n)]\!] = [\![\phi(\pi(x_1),\ldots,\pi(x_n))]\!] \tag{5.15}$$

for all closed formulas ϕ with parameters $x_1,\ldots,x_n \in M^{(\mathcal{B})}$.

CHAPTER II

AN INTRODUCTION TO CATEGORY THEORY

II.1 Categories and metacategories

In this section we shall describe the notion of a category informally by means of axioms, without recourse to any set theory. Objects of our intuition which obey the axioms we shall call "metacategories". The term "category" we shall reserve for realizations of metacategories within set theory. We shall always be working with categories in order to make our discussions more concrete to those readers who favor set theory as a foundation of mathematics, however, it is important to realize that our set theoretic discussions using categories are logically unneccessary and that perfectly abstract discussions using metacategories are possible, and perhaps even preferable to those who favor category theory as a foundation of mathematics.

It is assumed that the following concepts are intuitively meaningful:

i.) the notion of an object and

ii.) the notion of an arrow from an object to an object.

Regarding the notion of an arrow, it is assumed that we are able to distinguish which particular object lies at the head and which particular object lies at the tail of a given arrow.

We shall usually use upper case Latin letters A, B, C, ... to label objects and lower case Latin letters preceded by a dot .a, .b, .c, ... to label arrows, though we reserve the right to explicitly deviate from this notation whenever it is convenient.

We say that two object labels A and B are equal, which we denote by

A = B, if A and B are both labels for the same object. Similarly, we say that two arrow labels .a and .b are equal, written .a = .b, if they are both labels for the same arrow.

A metagraph consists of objects A, B, C, ...; arrows between these objects .a, .b, .c, ...; and two operators, *Domain* and *Codomain*, assigning objects to arrows as follows:

i.) the operator *Domain* assigns to each arrow .a as in Figure 2.1 the object *Domain* (.a) = A lying at its tail; and

ii.) the operator *Codomain* assigns to each arrow .a as in Figure 2.1 the object *Codomain* (.a) = B lying at its head.

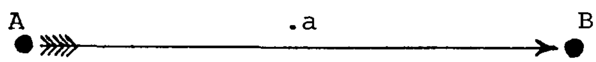


Figure 2.1

We often abbreviate the assertion ".a is an arrow such that *Domain* (.a) = = A and *Codomain* (.a) = B" by either ".a : A $\longrightarrow$ B" or "A $\xrightarrow{.a}$ B"

A metacategory is a metagraph with two additional operators, *Identity* and *Composition*, which are described below and which satisfy Axioms I and II:

i.) *Identity* is an operator which assigns to each object A an arrow

$$\mathit{Identity}\ (A) =_{df} .1_A : A \longrightarrow A; \text{ and} \tag{1.1}$$

ii.) *Composition* is a partial operator from pairs of arrows to arrows which assigns to every pair of arrows (.a,.b) such that *Codomain* (.a) = = *Domain* (.b) an arrow

$$\mathit{Composition}\ (.a,.b) =_{df} .a\circ.b =_{df} .a\circ b =_{df} .a.b =_{df} \\ =_{df} .ab : \mathit{Domain}\ (.a) \longrightarrow \mathit{Codomain}\ (.b). \tag{1.2}$$

Axiom I (Associativity) For any collection of objects and arrows in the configuration of Figure 2.2 it is the case that

$$\mathit{Composition}(\mathit{Composition}(.a,.b),.c) = \mathit{Composition}(.a,\mathit{Composition}(.b,.c))$$

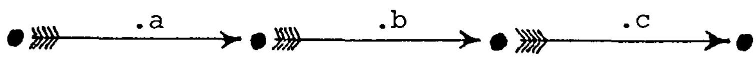


Figure 2.2

Axiom II (Unit Law) For any $.a : A \longrightarrow B$ it is the case that $.1_A .a = .a$ and $.a.1_B = .a.$

The above definitions and axioms can be expressed in the language of set theory. By a category we shall mean an interpretation of the category axioms in ZFC or ZFCI. We indicate how such an interpretation is to be carried out in the following.

By a graph we mean an ordered four-tuple $\langle \underline{Obj},\underline{Arr},\underline{dom},\underline{cod}\rangle$ such that $\underline{Obj}$ and $\underline{Arr}$ are sets and $\underline{dom}$ and $\underline{cod}$ are functions such that $\underline{dom} : \underline{Arr} \longrightarrow \underline{Obj}$ and $\underline{cod} : \underline{Arr} \longrightarrow \underline{Obj}$. For any graph $\mathcal{G} = \langle \underline{Obj},\underline{Arr},\underline{dom},\underline{cod}\rangle$ we define the set of composable pairs of arrows of $\mathcal{G}$, denoted by $\underline{Arr}\,{}_{\mathcal{G}}\underline{Arr}$ or $\underline{Arr}\,\underline{Arr}$, by the following:

$$\underline{Arr} \circ_{\mathcal{G}} \underline{Arr} =_{df} \{\langle .a,.b\rangle \in \underline{Arr}\times\underline{Arr} \mid \underline{dom}(b) = \underline{cod}(a)\} \qquad (1.3)$$

Finally a category $\mathcal{a}$ (over a graph $\mathcal{G}$) is an ordered six-tuple $\mathcal{a} = \langle \underline{Obj},\underline{Arr},\underline{dom},\underline{cod},.\underline{id},.\underline{comp}\rangle$ such that

i.) $\mathcal{G} = \langle \underline{Obj},\underline{Arr},\underline{dom},\underline{cod}\rangle$ is a graph;

ii.) $.\underline{id} : \underline{Obj} \longrightarrow \underline{Arr}$ is a function such that for all $A \in \underline{Obj}$, $\underline{dom}(.\underline{id}(A)) = \underline{cod}(.\underline{id}(A)) = A$;

iii.) $.\underline{comp} : \underline{Arr} \circ \underline{Arr} \longrightarrow \underline{Arr}$ is a function such that for all $\langle .a, .b \rangle \in \underline{Arr} \circ \underline{Arr}$ we have that $\underline{dom}(.\underline{comp}(\langle .a, .b \rangle)) = \underline{dom}(.a)$ and $\underline{cod}(.\underline{comp}(\langle .a, .b \rangle)) = \underline{cod}(.b)$;

iv.) Axiom I holds in $\mathcal{A}$; i.e. for all $\langle .a, .b \rangle, \langle .b, .c \rangle \in \underline{Arr} \circ \underline{Arr}$ we have that $.\underline{comp}(\langle .\underline{comp}(\langle .a, .b \rangle), .c \rangle) = .\underline{comp}(\langle .a, .\underline{comp}(\langle .b, .c \rangle) \rangle)$; and

v.) Axiom II holds in $\mathcal{A}$, i.e. for all $.a \in \underline{Arr}$ we have that $.\underline{comp}(\langle .a, .\underline{id}(\underline{cod}(.a)) \rangle) = .a$ and $.\underline{comp}(\langle .\underline{id}(\underline{dom}(.a)), .a \rangle) = .a$.

Categories will usually be denoted by upper case English script letters $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{F}, \ldots$. If $\mathcal{A} = \langle \underline{Obj}_{\mathcal{A}}, \underline{Arr}_{\mathcal{A}}, \underline{dom}_{\mathcal{A}}, \underline{cod}_{\mathcal{A}}, .\underline{id}_{\mathcal{A}}, .\underline{comp}_{\mathcal{A}} \rangle$ is a category then

i.) the elements of $\underline{Obj}_{\mathcal{A}}$ will be called <u>objects</u> (in $\mathcal{A}$) and denoted by upper case Latin letters A, B, C, ...;

ii.) the elements of $\underline{Arr}_{\mathcal{A}}$ will be called <u>arrows</u>, <u>morphisms</u>, or <u>maps</u> (in $\mathcal{A}$) and denoted by lower case Latin letters preceded by a dot .a, .b, .c, ...;

iii.) if $.a \in \underline{Arr}_{\mathcal{A}}$ then the object $\underline{dom}_{\mathcal{A}}(.a)$ will be called the <u>domain of .a</u> and will also be denoted by $\underline{dom}_{\mathcal{A}}(a)$;

iv.) if $.a \in \underline{Arr}_{\mathcal{A}}$ then the object $\underline{cod}_{\mathcal{A}}(a)$ will be called the <u>codomain of .a</u> and will also be denoted by $\underline{cod}_{\mathcal{A}}(a)$;

v.) if $A \in \underline{Obj}_{\mathcal{A}}$ then the morphism $.\underline{id}_{\mathcal{A}}(A)$ will be called the <u>identity morphism</u> (<u>arrow</u> or <u>map</u>) on A and will also be denoted by $.1_A$;

vi.) if $\langle .a, .b \rangle \in \underline{Arr}_{\mathcal{A}} \circ \underline{Arr}_{\mathcal{A}}$ we will say that $\langle .a, .b \rangle$ is a <u>composable pair of morphisms</u> (<u>arrows</u> or <u>maps</u>) in $\mathcal{A}$ and $.\underline{comp}_{\mathcal{A}}(\langle .a, .b \rangle)$ will be called the <u>composition of .a and .b</u> and will also be denoted by $.a \circ .b$, $.a \cdot b$, $.a.b$, or $.ab$; and vii.) if $A,B \in \underline{Obj}_{\mathcal{A}}$ then the set

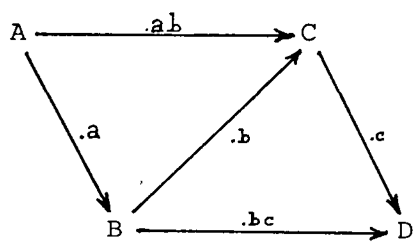
$$\underline{Hom}_{\mathcal{A}}(A,B) =_{df} \{.a \in \underline{Arr}_{\mathcal{A}} \mid \underline{dom}_{\mathcal{A}}(a) = A \wedge \underline{cod}_{\mathcal{A}}(a) = B\} \qquad (1.4)$$

Of course we may deviate somewhat from the above notations when we find it convenient. We shall usually be quite explicit when we introduce new notations, however we will not always call attention to obvious simplifications such as the dropping of dots or parentheses in complicated expressions.

In the following we shall frequently employ diagrams whose vertices consist of labels for objects and whose directed edges consist of labels for arrows and pictures of arrows. We say that such a diagram _is commutative_ (or _commutes_) if for each pair of vertices c_1 and c_2, any two paths formed by following directed edges from c_1 to c_2 yield, via composition of arrow labels, "equal arrows" (i.e. equal arrow labels).

For example we may rewrite Axioms I and II as follows:

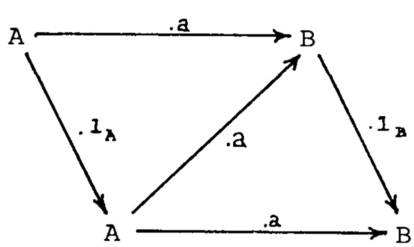
Axiom I: For all $A \xrightarrow{.a} B \xrightarrow{.b} C \xrightarrow{.c} D$, the diagram



(1.5)

commutes.

Axiom II: For all $A \xrightarrow{.a} B$, the diagram



(1.6)

commutes.

Examples of categories:

1.) In ZFCI we will denote the inaccessible cardinal by ι (iota). V_ι is a standard transitive model of ZFC. The category whose objects are the elements of V_ι, whose morphisms are ordered triples $\langle x,f,y\rangle \in V_\iota$ such that $f : x \longrightarrow y$, and whose notions of domain, codomain, identity, and composition are defined in the obvious manner, is called the category of sets and is denoted by $\mathcal{S}$.

Note that we now have two different notations for functions in set theory. First there is the ordinary set theoretic notation in which functions are written on the left and compositions are written from the right to the left. Second, there is the category theoretic notation in $\mathcal{S}$ in which functions are denoted .a, .b, .c, ... and written and composed on the right. The presence or absence of dots on the function symbols should indicate which notation is being used for any given expression.

2.) It will be noted that our definition of category may be carried out in ZFCI or relativized to V_ι. This then gives us two notions of category. A category in the sense of V_ι will be called a small category. Categories in ZFCI which are not small will be called large categories.

3.) Any partially ordered set (or poset) $\langle x,\leq\rangle$ may be made into a category $\mathcal{O}(x,\leq)$ by letting

$$\underline{Obj}_{\mathcal{O}(x,\leq)} = x \text{ and } \overline{\overline{\underline{Hom}_{\mathcal{O}(x,\leq)}(z_1,z_2)}} = \begin{cases} 1 & \text{if } z_1 \leq z_2 \\ 0 & \text{otherwise} \end{cases} \qquad (1.7)$$

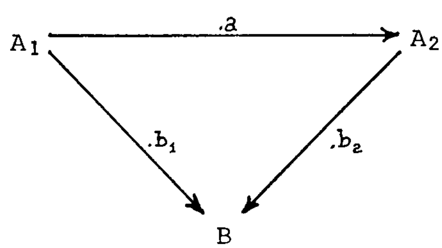
for all $z_1, z_2 \in x$. Define domain, codomain, identity and composition the obvious manner. Such a category will be called a partially ordered category or pocategory.

4.) Let $\mathcal{A} = \langle \underline{Obj}_{\mathcal{A}}, \underline{Arr}_{\mathcal{A}}, \underline{dom}_{\mathcal{A}}, \underline{cod}_{\mathcal{A}}, .\underline{id}_{\mathcal{A}}, .\underline{comp}_{\mathcal{A}} \rangle$ be any category. The *dual category of* $\mathcal{A}$, $\mathcal{A}^{op}$, is the category obtained from $\mathcal{A}$ by reversing all the arrows, i.e. formally $\mathcal{A}^{op} = \langle \underline{Obj}_{\mathcal{A}}, \underline{Arr}_{\mathcal{A}}, \underline{cod}_{\mathcal{A}}, \underline{dom}_{\mathcal{A}}, .\underline{id}_{\mathcal{A}}, .\underline{comp}_{\mathcal{A}} \rangle$.

5.) Suppose $\mathcal{A}$ is a category and $B \in \underline{Obj}_{\mathcal{A}}$. Then the *category of arrows in* $\mathcal{A}$ *over* B (or the *comma category of* $\mathcal{A}$ *over* B) is the category denoted by $\mathcal{A}{\downarrow}B$ whose objects are arrows in $\mathcal{A}$ of the form $A \xrightarrow{.b} B$ and whose arrows are described by the requirement that the elements of

$$\underline{Hom}_{\mathcal{A}\downarrow B}(A_1 \xrightarrow{.b_1} B,\ A_2 \xrightarrow{.b_2} B)$$

are all commutative triangles in $\mathcal{A}$ of the form



(1.8)

Types of morphisms:

1.) A morphism $.m : A \longrightarrow B$ is said to be a *monomorphism* (or *monic*) if for every pair of morphisms $L \underset{.\ell_2}{\overset{.\ell_1}{\rightrightarrows}} A$, $.\ell_1 m = .\ell_2 m$ implies $.\ell_1 = .\ell_2$. We will also write "$.m : A \rightarrowtail B$" or "$A \overset{.m}{\rightarrowtail} B$" for "$.m : A \longrightarrow B$ is a monomorphism".

2.) A morphism $.e : A \longrightarrow B$ is said to be an *epimorphism* (or *epic*) if for every pair of morphisms $B \underset{.r_2}{\overset{.r_1}{\rightrightarrows}} R$, $.er_1 = .er_2$ implies $.r_1 = .r_2$. We will also write "$.e : A \twoheadrightarrow B$" or "$A \overset{.e}{\twoheadrightarrow} B$" for "$.e : A \longrightarrow B$ is an epimorphism".

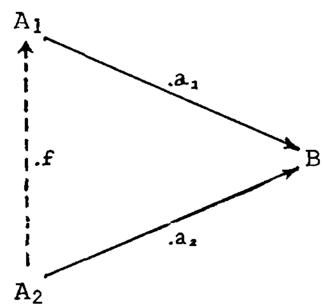
3.) A morphism $.i : A \longrightarrow B$ is said to be an *isomorphism* (or *iso* or *invertible*) if there exists a morphism $.j : B \longrightarrow A$ such that $.ij = .1_A$

and $.ji = .1_B$. We will also write "$.i : A \Longrightarrow B$" or "$A \overset{.i}{\Longrightarrow} B$" for "$.i : A \longrightarrow B$ is an isomorphism". We say that two objects A and B are <u>isomorphic</u>, denoted by $A \cong B$, if there exists an $.i : A \Longrightarrow B$.

4.) A morphism $.f : A \longrightarrow A$, whose domain and codomain are the same object, is said to be an <u>endomorphism</u> (or <u>endo</u>).

5.) An endomorphism which is also an isomorphism is said to be an <u>automorphism</u> (or <u>auto</u>).

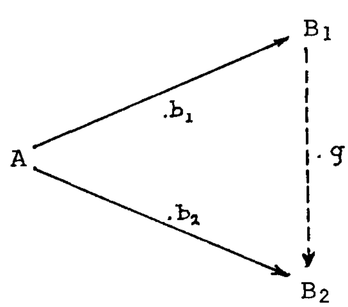
6.) Let $.a_1 : A_1 \longrightarrow B$ and $.a_2 : A_2 \longrightarrow B$ be two morphisms with common codomain B. We say <u>$.a_2$ factors through $.a_1$</u> if there exists a morphism $.f : A_2 \longrightarrow A_1$ such that



(1.9)

commutes.

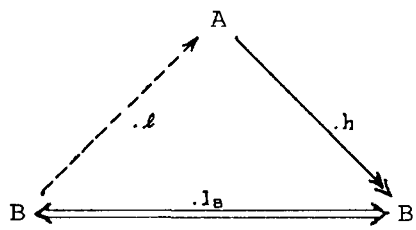
Dually, let $.b_1 : A \longrightarrow B_1$ and $.b_2 : A \longrightarrow B_2$ be two morphisms with common domain A. We say that <u>$.b_2$ factors through $.b_1$</u> if there exists a morphism $.g : B_1 \longrightarrow B_2$ such that



(1.10)

commutes.

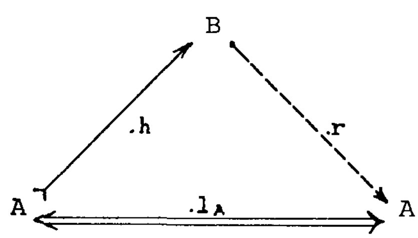
7.) An epimorphism .h : $A \twoheadrightarrow B$ is said to be a split epimorphism (or a split epic or simply to split) if $.1_B$ factors through .h, i.e. if there exists a map $.\ell : B \longrightarrow A$, called a section of .h, such that



(1.11)

commutes.

8.) A monomorphism .h : $A \rightarrowtail B$ is said to be a split monomorphism (or a split monic or simply to split) if $.1_A$ factors through .h, i.e. if there exists a map $.r : B \longrightarrow A$, called a retraction of .h, such that

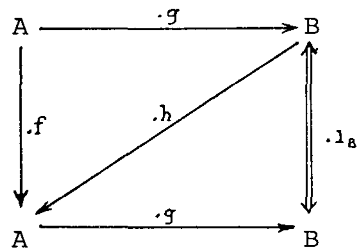


(1.12)

commutes.

9.) An endomorphism .f : $A \longrightarrow A$ is said to be an idempotent if .ff = .f.

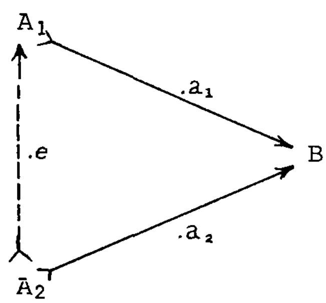
10.) An idempotent .f : $A \longrightarrow A$ splits if there exists an object B and morphisms $A \underset{.h}{\overset{.g}{\rightleftarrows}} B$ such that the diagram



(1.13)

commutes.

11.) Let $.a_1 : A_1 \rightarrowtail B$ and $.a \; : A \rightarrowtail B$ be two monics with common codomain B. We write $.a_2 \leq .a_1$ if $.a_2$ factors through $.a_1$, i.e. if there exists a map $.e : A_2 \rightarrowtail A_1$ such that



(1.14)

commutes.

We write $.a_1 \simeq .a_2$ if $.a_1 \leq .a_2$ and $.a_2 \leq .a_1$. "$\simeq$" is then an equivalence relation on monics with codomain B. The corresponding equivalence classes of monics are called the <u>subobjects of B</u>. The collection of subobjects of B also has a natural ordering, the one induced on it by $\leq$ above, which we shall also denote by $\leq$. It is often convenient to abuse our language by calling a monic with codomain B a subobject of B and writing $A \overset{.a}{\rightarrowtail} B$ with the intention that it be read ".a is a subobject of B" (or even "A is a subobject of B").

II.2 Functors and natural transformations

Let $\mathcal{A}_i = \langle \underline{Obj}_i, \underline{Arr}_i, \underline{dom}_i, \underline{cod}_i, .\underline{id}_i, .\underline{comp}_i \rangle$ be a category for i = 1,2.

A (covariant) functor $:F : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$ is an ordered pair of functions $:F = \langle F_{\underline{Obj}}, F_{\underline{Arr}} \rangle$ such that

i.) $F_{\underline{Obj}} : \underline{Obj}_1 \longrightarrow \underline{Obj}_2$;

ii.) $F_{\underline{Arr}} : \underline{Arr}_1 \longrightarrow \underline{Arr}_2$;

iii.) for all $.a \in \underline{Arr}_1$, $\underline{dom}_2(F_{\underline{Arr}}(a)) = F_{\underline{Obj}}(\underline{dom}_1(a))$ and $\underline{cod}_2(F_{\underline{Arr}}(a)) = F_{\underline{Obj}}(\underline{cod}_1(a))$; and

iv.) for all $\langle .a, .b \rangle \in \underline{Arr}_1 \circ \underline{Arr}_1$, $F_{\underline{Arr}}(.\underline{comp}_1(a,b)) =$
$= .\underline{comp}_2(F_{\underline{Arr}}(a), F_{\underline{Arr}}(b))$.

A contravariant functor $:G : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$ is a covariant functor $:G^{op} : \mathcal{A}_1^{op} \longrightarrow \mathcal{A}_2$.

Functors will usually be denoted by upper case script Latin letters preceded by two dots $:\mathcal{A}$, $:\mathcal{B}$, $:\mathcal{C}$, ..., although we shall occasionally explicitly deviate from this convention. If $:F = \langle F_{\underline{Obj}}, F_{\underline{Arr}} \rangle$ is a functor as above, we shall normally write $A{:}F$ for $F_{\underline{Obj}}(A)$ where $A \in \underline{Obj}_1$ and $.a{:}F$ for $F_{\underline{Arr}}(a)$ where $.a \in \underline{Arr}_1$.

The identity functor $:1_{\mathcal{A}}$ on any category $\mathcal{A}$ is the ordered pair $:1_{\mathcal{A}} =_{df} \langle I_1, I_2 \rangle$ where I_1 is the identity function on $\underline{Obj}_{\mathcal{A}}$ and I_2 is the identity function on $\underline{Arr}_{\mathcal{A}}$.

If $:F : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$ is a functor then $\mathcal{A}_1$ is called the domain of $:F$ and $\mathcal{A}_2$ is called the codomain of $:F$. However we will usually write simply $:F : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$ or $\mathcal{A}_1 \xrightarrow{:F} \mathcal{A}_2$.

Finally if $:F = \langle F_{\underline{Obj}}, F_{\underline{Arr}} \rangle : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$ and $:G = \langle G_{\underline{Obj}}, G_{\underline{Arr}} \rangle : \mathcal{A}_2 \longrightarrow \mathcal{A}_3$, we define the <u>composition of :F and :G</u> to be the functor

$$\langle G_{\underline{Obj}} \circ F_{\underline{Obj}},\ G_{\underline{Arr}} \circ F_{\underline{Arr}} \rangle =_{df} :F \circ :G =_{df} :F:G =_{df} :F \circ G =_{df} :FG. \qquad (2.1)$$

Note that the functions on the far left hand side of the above equation are written in the set theoretic notation with the functions on the left.

A functor $:F : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$ is called a <u>constant functor</u> if there exists an $A_2 \in \underline{Obj}_{\mathcal{A}_2}$ such that for all $A_1 \in \underline{Obj}_{\mathcal{A}_1}$, $A_1{:}F = A_2$ and for all $.a \in \underline{Arr}_{\mathcal{A}_1}$, $.a{:}F = .1_{A_2}$. $:F$ is then called the <u>constant functor on A_2</u> from $\mathcal{A}_1$ and denoted by $:A_2$.

A functor $:F : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$ is said to be <u>full</u> if for all $A, B \in \underline{Obj}_{\mathcal{A}_1}$ and all $.a_2 \in \underline{Hom}_{\mathcal{A}_2}(A{:}F, B{:}F)$, there exists an $.a_1 \in \underline{Hom}_{\mathcal{A}_1}(A,B)$ such that $.a_1 \in \underline{Hom}_{\mathcal{A}_1}(A,B)$ such that $.a_1{:}F = .a_2$.

A functor $:F : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$ is said to be <u>faithful</u> if for all $.a_1, .a_2 \in \underline{Arr}_{\mathcal{A}_1}$ $.a_1 \neq .a_2$ implies $.a_1{:}F \neq .a_2{:}F$.

A functor $:F = \langle F_{\underline{Obj}}, F_{\underline{Arr}} \rangle : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$ is said to be an <u>isomorphism of categories</u> if $F_{\underline{Obj}}$ and $F_{\underline{Arr}}$ are both bijections.

We say that a category <u>$\mathcal{A}_1$ is a subcategory of the category $\mathcal{A}_2$</u>, denoted $\mathcal{A}_1 \subseteq \mathcal{A}_2$ if

i.) $\underline{Obj}_{\mathcal{A}_1} \subseteq \underline{Obj}_{\mathcal{A}_2}$ (we denote the inclusion function by $\underline{in}^{\mathcal{A}_1,\mathcal{A}_2}_{\underline{Obj}} : \underline{Obj}_{\mathcal{A}_1} \longrightarrow \underline{Obj}_{\mathcal{A}_2}$)

ii.) $\underline{Arr}_{\mathcal{A}_1} \subseteq \underline{Arr}_{\mathcal{A}_2}$ (we denote the inclusion function by $\underline{in}^{\mathcal{A}_1,\mathcal{A}_2}_{\underline{Arr}} : \underline{Arr}_{\mathcal{A}_1} \longrightarrow \underline{Arr}_{\mathcal{A}_2}$)

iii.) $\underline{dom}_{\mathcal{A}_1} = \underline{dom}_{\mathcal{A}_2} \restriction \underline{Arr}_{\mathcal{A}_1}$;

iv.) $\underline{cod}_{\mathcal{A}_1} = \underline{cod}_{\mathcal{A}_2} \restriction \underline{Arr}_{\mathcal{A}_1}$;

v.) $\underline{.id}_{\mathcal{A}_1} = \underline{.id}_{\mathcal{A}_2} \restriction \underline{Obj}_{\mathcal{A}_1}$; and

vi.) $\underline{.comp}_{\mathcal{A}_1} = \underline{.comp}_{\mathcal{A}_2} \restriction \underline{Arr}_{\mathcal{A}_1} \circ \underline{Arr}_{\mathcal{A}_1}$.

The functor $:I^{\mathcal{A}_1,\mathcal{A}_2} = \langle in^{\mathcal{A}_1,\mathcal{A}_2}_{\underline{Obj}}, in^{\mathcal{A}_1,\mathcal{A}_2}_{\underline{Arr}} \rangle$ is called <u>the inclusion functor</u> and it is obviously faithful. If $:I^{\mathcal{A}_1,\mathcal{A}_2}$ is also full then we say that $\mathcal{A}_1$ is a <u>full subcategory</u> of $\mathcal{A}_2$.

Note that there are other definitions of subcategory in the literature. The definition above coincides with that of MacLane [18, p.15].

<u>Examples</u>: 1.) Let $\langle x, \leq_x \rangle$ and $\langle y, \leq_y \rangle$ be any two posets and let $f : x \longrightarrow y$ be any order preserving function. Then f induces in the obvious way a functor $:P_f : \mathcal{O}(x, \leq_x) \longrightarrow \mathcal{O}(y, \leq_y)$ between the corresponding pocategories.

2.) Let $\mathcal{O}_{\mathcal{S}}$ denote the subcategory of $\mathcal{S}$ whose objects are partially ordered sets and whose morphisms are order preserving maps. Let $\mathcal{O}_{\mathcal{C}}$ denote the category whose objects are small pocategories and whose morphisms are the functors between them. Note that both $\mathcal{O}_{\mathcal{S}}$ and $\mathcal{O}_{\mathcal{C}}$ are large categories. From the definition of pocategory it is obvious that there is a functor $:J : \mathcal{O}_{\mathcal{S}} \longrightarrow \mathcal{O}_{\mathcal{C}}$ that is an isomorphism of categories.

Let $:F$ and $:G$ be functors from $\mathcal{A}_1$ to $\mathcal{A}_2$, i.e. $\mathcal{A}_1 \underset{:G}{\overset{:F}{\rightrightarrows}} \mathcal{A}_2$. A <u>natural transformation</u> from $:F$ to $:G$ $:\eta: :F \longrightarrow :G$ is a set of arrows in $\mathcal{A}_2$ indexed by objects of $\mathcal{A}_1$, $:\eta = \{.\eta_A \in \underline{Arr}_{\mathcal{A}_2} | A \in \underline{Obj}_{\mathcal{A}_1}\}$ such that

i.) for all $A \in \underline{Obj}_{\mathcal{A}_1}$, $\underline{dom}_{\mathcal{A}_2}(\eta_A) = A{:}F$ and $\underline{cod}_{\mathcal{A}_2}(\eta_A) = A{:}G$ and

ii.) for all $A, B \in \underline{Obj}_{\mathcal{A}_1}$ and all $.a \in \underline{Hom}_{\mathcal{A}_1}(A,B)$, the diagram

$$\begin{array}{ccc} A : F & \xrightarrow{\;.\eta_A\;} & A : G \\ \Big\downarrow {\scriptstyle .a:F} & & \Big\downarrow {\scriptstyle .a:G} \\ B : F & \xrightarrow{\;.\eta_B\;} & B : G \end{array} \qquad (2.2)$$

commutes (in $\mathcal{A}_2$).

The elements of $:\eta = \{.\eta_A \in \underline{\mathit{Arr}}_{\mathcal{A}_2} \mid A \in \underline{\mathit{Obj}}_{\mathcal{A}_1}\}$ are called the <u>components</u> of the natural transformation $:\eta$. The element $.\eta_A$ is called the <u>A-component of $:\eta$</u> and may also be denoted by $.A:\eta$.

If $:F$ and $:G$ are contravariant functors, a <u>natural transformation</u> from $:F$ to $:G$, $:\eta : :F \longrightarrow :G$ is a natural transformation $:\eta^{op} : :F^{op} \longrightarrow :G^{op}$.

In general we will denote natural transformations by lower case Greek letters preceded by two dots $:\alpha$, $:\beta$, $:\gamma$, $\cdots$.

A <u>natural isomorphism</u> (or <u>natural equivalence</u>) is a natural transformation whose components are all isomorphisms.

Let $:F : \mathcal{A} \longrightarrow \mathcal{B}$ be a functor and let $B \in \underline{\mathit{Obj}}_{\mathcal{B}}$. Then $:B$ denotes the constant functor on B from $\mathcal{A}$. A natural transformation $:\eta : :B \longrightarrow :F$ is called a <u>cone on $:F$ from B</u>. B is called the <u>vertex</u> of the cone and $:F$ is called the <u>base</u>. We also use the notation $:\eta : B \longrightarrow\!\!\!\!\lhd\!\!\!\!\rightarrow :F$ to abbreviate "$:\eta : :B \longrightarrow :F$ is a cone on $:F$ from B". Dually, a natural transformation $:\varepsilon : :F \longrightarrow :B$ is called a <u>cocone on $:F$ to B</u>. B is called the <u>vertex</u> of the cocone and $:F$ is called the <u>base</u>. We also use the notation

$:\varepsilon : :\mathcal{F} \longrightarrow\!\!\triangleright\!\!\rightarrow B$ to abbreviate "$:\varepsilon : :\mathcal{F} \longrightarrow :B$ is a cocone on $:\mathcal{F}$ to B".

If $\mathcal{A}_1$ and $\mathcal{A}_2$ are categories we define the product category $\mathcal{A}_1 \times \mathcal{A}_2$ in the obvious manner:

i.) $\underline{Obj}_{\mathcal{A}_1\times\mathcal{A}_2} = \underline{Obj}_{\mathcal{A}_1} \times \underline{Obj}_{\mathcal{A}_2}$;

ii.) $\underline{Arr}_{\mathcal{A}_1\times\mathcal{A}_2} = \bigcup\{\underline{Hom}_{\mathcal{A}_1}(A_1,B_1) \times \underline{Hom}_{\mathcal{A}_2}(A_2,B_2) \mid$
$|\langle A_1,A_2\rangle,\langle B_1,B_2\rangle \in \underline{Obj}_{\mathcal{A}_1\times\mathcal{A}_2}\}$;

iii.) $\underline{dom}_{\mathcal{A}_1\times\mathcal{A}_2} = (\underline{dom}_{\mathcal{A}_1} \times \underline{dom}_{\mathcal{A}_2}) \restriction \underline{Arr}_{\mathcal{A}_1\times\mathcal{A}_2}$;

iv.) $\underline{cod}_{\mathcal{A}_1\times\mathcal{A}_2} = (\underline{cod}_{\mathcal{A}_1} \times \underline{cod}_{\mathcal{A}_2}) \restriction \underline{Arr}_{\mathcal{A}_1\times\mathcal{A}_2}$;

v.) $\underline{.id}_{\mathcal{A}_1\times\mathcal{A}_2} = \underline{.id}_{\mathcal{A}_1} \times \underline{.id}_{\mathcal{A}_2}$; and

vi.) $\underline{.comp}_{\mathcal{A}_1\times\mathcal{A}_2} = (\underline{.comp}_{\mathcal{A}_1} \times \underline{.comp}_{\mathcal{A}_2}) \restriction \underline{Arr}_{\mathcal{A}_1\times\mathcal{A}_2} \circ \underline{Arr}_{\mathcal{A}_1\times\mathcal{A}_2}$

It is a simple matter to verify that $\mathcal{A}_1 \times \mathcal{A}_2$ satisfies the definition of a category. By the simple iteration of the above construction we may define the product of the n categories $\mathcal{A}_1, \ldots, \mathcal{A}_n$, which we will denote by $\mathcal{A}_1 \times \ldots \times \mathcal{A}_n$ or $\overset{n}{\underset{i=1}{\times}} \mathcal{A}_i$.

A functor from a product of n categories is called a multifunctor (with n-arguments). By looking at dual categories for certain arguments it is possible to speak of multifunctors being contravariant in certain arguments and covariant in others.

If $\mathcal{A}_1 \times \ldots \times \mathcal{A}_n$ is a product of n categories, there exist n special multifunctors $:\mathcal{P}r_i : \mathcal{A}_1 \times \ldots \times \mathcal{A}_n \longrightarrow \mathcal{A}_i$ for i=1, ..., n. These multifunctors $:\mathcal{P}r_n$ will be called projection functors and they are defined in the obvious manner. We may consider any multifunctor of n-argument to give rise to functors and multifunctors of less than arguments by holding certain arguments fixed.

Example: Let $\mathcal{A}$ be any small category. Then $:\underline{Hom}_{\mathcal{A}}(\_\_,\_\_) : \mathcal{A}^{op} \times \mathcal{A} \longrightarrow \mathcal{S}$ is a multifunctor with two arguments or a <u>bifunctor</u>.

Note that $:\underline{Hom}_{\mathcal{A}}(\_\_,\_\_)$ is contravariant in the first argument and covariant in the second. To see this let us suppose that $A_1, A_2, B \in \underline{Obj}_{\mathcal{A}}$ and $.f : A_1 \longrightarrow A_2$. Then $.\underline{Hom}_{\mathcal{A}}(f,B) : \underline{Hom}_{\mathcal{A}}(A_2,B) \longrightarrow \underline{Hom}_{\mathcal{A}}(A_1,B)$ is the set theoretic function which sends $A_2 \xrightarrow{.a} B$ to $A_1 \xrightarrow{.f} A_2 \xrightarrow{.a} B$. Similarly, if $A, B_1, B_2 \in \underline{Obj}_{\mathcal{A}}$ and $.g : B_1 \longrightarrow B_2$ then $.\underline{Hom}_{\mathcal{A}}(A,g) : \underline{Hom}_{\mathcal{A}}(A,B_1) \longrightarrow \underline{Hom}_{\mathcal{A}}(A,B_2)$ is the set theoretic function which sends $A \xrightarrow{.b} B_1$ to $A \xrightarrow{.b} B_1 \xrightarrow{.g} B_2$.

For each $A \in \underline{Obj}_{\mathcal{A}}$ we obtain the two functors $:\underline{Hom}_{\mathcal{A}}(A,\_\_)$ and $:\underline{Hom}_{\mathcal{A}}(\_\_,A)$ by holding A fixed. The functor $:\underline{Hom}_{\mathcal{A}}(A,\_\_)$ is called the <u>covariant Hom-functor</u> <u>(with A fixed)</u> and $:\underline{Hom}_{\mathcal{A}}(\_\_,A)$ is called the <u>contravariant Hom-functor</u> <u>(with A fixed)</u>. A covariant (resp. contravariant) functor $:F : \mathcal{A} \longrightarrow \mathcal{S}$ is said to be <u>representable</u> with A as its <u>representing object</u> if $:F$ is naturally isomorphic to the covariant (resp. contravariant) Hom-functor with A fixed. It is easy to see that naturally isomorphic representable functors must have isomorphic representing objects.

A <u>natural transformation of multifunctors</u> is in a sense already defined since a multifunctor is a special kind of functor. Notice that a natural transformation of multifunctors is a natural transformation in each argument.

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be categories. We define the <u>category of functors from</u> <u>$\mathcal{A}_1$ to $\mathcal{A}_2$</u>, $\underline{Func}(\mathcal{A}_1,\mathcal{A}_2)$, to be the category whose objects are functors $:F : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$ and whose morphisms are natural transformations between such functors. The domain, codomain, identity and composition functions are defined in the obvious manner. It is a simple matter to check that $\underline{Func}(\mathcal{A}_1, \mathcal{A}_2)$ satisfies the category axioms.

<u>Examples of functor categories</u>: Let $\mathbb{1}$ denote the category with one object and one arrow. Let $\mathbb{2}$ denote the category with two objects having just one arrow between them and whose only endomorphisms are identity maps. If $\mathcal{A}$ is any category then $\underline{Func}(\mathbb{1}, \mathcal{A})$ is isomorphic to $\mathcal{A}$. $\underline{Func}(\mathbb{2}, \mathcal{A})$ is called the <u>category of arrows in $\mathcal{A}$</u> and it is isomorphic to the category whose objects are arrows $.a : A_1 \longrightarrow A_2$ in $\mathcal{A}$ and whose morphisms $.f : a_1 \longrightarrow a_2$, where $.a_1 : A_{11} \longrightarrow A_{12}$ and $.a_2 : A_{21} \longrightarrow A_{22}$ are arrows in $\mathcal{A}$, are ordered pairs of arrows in $\mathcal{A}$, $.f = \langle h,k \rangle$, such that

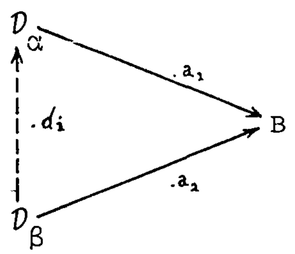
$$\begin{array}{ccc} A_{11} & \xrightarrow{.a_1} & A_{12} \\ \downarrow{.h} & & \downarrow{.k} \\ A_{21} & \xrightarrow{.a_2} & A_{22} \end{array} \qquad (2.3)$$

commutes.

II.3 Limits and colimits

We now formalize our notion of a diagram more precisely; a diagram in a category $\mathcal{A}$ is a functor $:\mathcal{D} : \mathcal{I} \longrightarrow \mathcal{A}$, where $\mathcal{I}$ is a small category called the shape (or index category) of the diagram $:\mathcal{D}$. We think of each of the elements of $Obj_{\mathcal{I}}$ and $Arr_{\mathcal{I}}$ as being a label for its image under $:\mathcal{D}$. Generally we will denote objects in an index category $\mathcal{I}$ by $\alpha, \beta, \gamma, \ldots$ and arrows in $\mathcal{I}$ by .i, .j, .k,.... We will often write $\mathcal{D}_\alpha$ for $\alpha{:}\mathcal{D}$ and $.d_i$ for $.i{:}\mathcal{D}$.

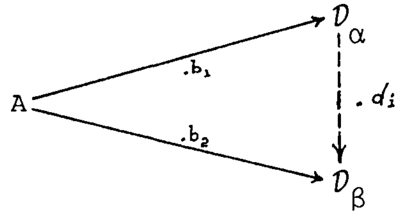
Let $.a_1 : \mathcal{D}_\alpha \longrightarrow B$ and $.a_2 : \mathcal{D}_\beta \longrightarrow B$ be arrows in $\mathcal{A}$ with common codomain B. We say that $.a_2$ factors uniquely through $.a_1$ in the diagram $:\mathcal{D}$ if there exists a unique arrow $.i : \beta \longrightarrow \alpha$ in $\mathcal{I}$ such that



(3.1)

commutes.

Dually, if $.b_1 : A \longrightarrow \mathcal{D}_\alpha$ and $.b_2 : A \longrightarrow \mathcal{D}_\beta$ are arrows in $\mathcal{A}$ with common domain A then we say that $.b_2$ factors uniquely through $.b_1$ in the diagram $:\mathcal{D}$ if there exists a unique arrow $.i : \alpha \longrightarrow \beta$ in $\mathcal{I}$ such that



(3.2)

commutes.

If $:\mathcal{D} : \mathcal{J} \longrightarrow \mathcal{A}$ is a diagram in $\mathcal{A}$ and $A \in \underline{Obj}_{\mathcal{A}}$ then a <u>universal arrow from $:\mathcal{D}$ to A</u> is an ordered pair $\langle \alpha, .a \rangle$ where $\alpha \in \underline{Obj}_{\mathcal{J}}$ and $.a : \mathcal{D}_\alpha \longrightarrow A$ is an arrow in $\mathcal{A}$ such thatfor all $\beta \in \underline{Obj}_{\mathcal{J}}$ and all arrows $.b : \mathcal{D}_\beta \longrightarrow$ in $\mathcal{A}$, .b factors uniquely through.a in $:\mathcal{D}$. Dually, <u>a universal arrow from A to $:\mathcal{D}$</u> is an ordered pair $\langle \alpha, .a \rangle$ where $\alpha \in \underline{Obj}_{\mathcal{J}}$ and $.a : A \longrightarrow \mathcal{D}_\alpha$ is an arrow in $\mathcal{A}$ such that for all $\beta \in \underline{Obj}_{\mathcal{J}}$ and all arrows $.b : A \longrightarrow \mathcal{D}_\beta$ in $\mathcal{A}$, .b factors uniquely through .a in $:\mathcal{D}$.

Let $\underline{Diag}(\mathcal{J},\mathcal{A}) =_{df} \underline{Func}(\mathcal{J},\mathcal{A})$ be the category of diagrams in $\mathcal{A}$ of shape $\mathcal{J}$. Define that <u>generalized diagonal functor</u> $:\nabla : \mathcal{A} \longrightarrow \underline{Diag}(\mathcal{J},\mathcal{A})$ to be the functor which

i.) sends each $A \in \underline{Obj}_{\mathcal{A}}$ to the constant functor from $\mathcal{J}$ to A, $:A : \mathcal{J} \longrightarrow \mathcal{A}$ and

ii.) sends each arrow $.a : A \longrightarrow B$ in $\mathcal{A}$ to the natural transformation $:\eta : :A \longrightarrow :B$ whose components are all equal to .a.

If $\mathcal{D} \in \underline{Obj}_{\underline{Diag}(\mathcal{J},\mathcal{A})}$ then a <u>limit of the diagram $:\mathcal{D}$ in $\mathcal{A}$</u> is a universal arrow from $:\nabla$ to $\mathcal{D}$ in $\underline{Diag}(\mathcal{J},\mathcal{A})$ and a <u>colimit of the diagram $:\mathcal{D}$ in $\mathcal{A}$</u> is a universal arrow from $\mathcal{D}$ to $:\nabla$ in $\underline{Diag}(\mathcal{J},\mathcal{A})$.

Below we give another characterization of limits and colimits in order to unravel the above definitions slightly.

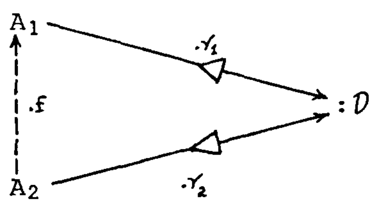
Let $$\underline{Cone}_{\mathcal{A}}(A,:\mathcal{D}) =_{df} \underline{Hom}_{\underline{\mathcal{D}iag}(\mathcal{I},\mathcal{A})}(A:\nabla,\mathcal{D}) \qquad (3.3)$$

and

$$\underline{Cocone}_{\mathcal{A}}(:\mathcal{D},A) =_{df} \underline{Hom}_{\underline{\mathcal{D}iag}(\mathcal{I},\mathcal{A})}(\mathcal{D},A:\nabla) \qquad (3.4)$$

where $A \in \underline{Obj}_{\mathcal{A}}$ and $:\mathcal{D} : \mathcal{I} \longrightarrow \mathcal{A}$.

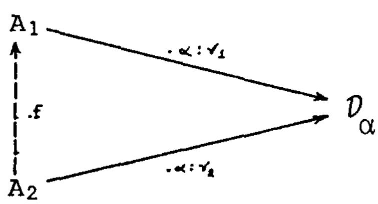
If $A_1, A_2 \in \underline{Obj}_{\mathcal{A}}$, we say that $:\gamma_2 \in \underline{Cone}_{\mathcal{A}}(A_2,:\mathcal{D})$ <u>factors uniquely through</u> $:\gamma_1 \in \underline{Cone}_{\mathcal{A}}(A_1,:\mathcal{D})$ if there exists an unique arrow $.f : A_2 \longrightarrow A_1$ in $\mathcal{A}$ such that



(3.5)

commutes.

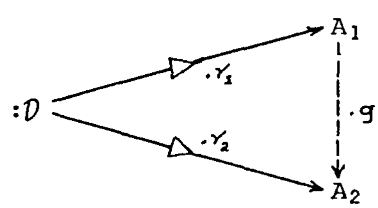
i.e.



(3.6)

commutes for all $\alpha \in \underline{Obj}_{\mathcal{I}}$

Dually, we say that $:\gamma_2 \in \underline{Cocone}_{\mathcal{A}}(:\mathcal{D},A_2)$ <u>factors uniquely through</u> $:\gamma_1 \in \underline{Cocone}_{\mathcal{A}}(:\mathcal{D},A_1)$ if there exists an unique $.g : A_1 \longrightarrow A_2$ in $\mathcal{A}$ such that



(3.7)

commutes.

A <u>universal cone on $:\mathcal{D} : \mathcal{I} \longrightarrow \mathcal{A}$</u> is an ordered pair $\langle L, :\pi \rangle$ where $L \in \underline{Obj}_{\mathcal{A}}$ and $:\pi \in \underline{Cone}_{\mathcal{A}}(L, :\mathcal{D})$ such that for every $A \in \underline{Obj}_{\mathcal{A}}$ and every $:\alpha \in \underline{Cone}_{\mathcal{A}}(A, :\mathcal{D})$, $:\alpha$ factors uniquely through $:\pi$.

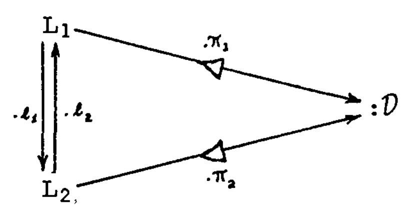
Dually, a <u>universal cocone on $:\mathcal{D} : \mathcal{I} \longrightarrow \mathcal{A}$</u> is an ordered pair $\langle C, :\iota \rangle$ where $C \in \underline{Obj}_{\mathcal{A}}$ and $:\iota \in \underline{Cocone}_{\mathcal{A}}(:\mathcal{D}, C)$ such that for every $B \in \underline{Obj}_{\mathcal{A}}$ and every $:\beta \in \underline{Cocone}_{\mathcal{A}}(:\mathcal{D}, B)$, $:\beta$ factors uniquely through $:\iota$.

<u>PROPOSITION 1</u>: $\langle L, :\pi \rangle$ is a limit of $:\mathcal{D}$ in $\mathcal{A}$ if and only if $\langle L, :\pi \rangle$ is a universal cone on $:\mathcal{D}$.□

<u>PROPOSITION 2</u>: $\langle C, :\iota \rangle$ is a colimit of $:\mathcal{D}$ in $\mathcal{A}$ if and only if $\langle C, :\iota \rangle$ is a universal cocone on $:\mathcal{D}$.□

<u>PROPOSITION 3</u>: Limits are unique up to isomorphism.

Proof: Let $\langle L_1, :\pi_1 \rangle$ and $\langle L_2, :\pi_2 \rangle$ be two limits of $:\mathcal{D}$ in $\mathcal{A}$. Then $:\pi_1$ factors uniquely through $:\pi_2$ and vice versa, so there exist maps $.\ell_1 : L_1 \longrightarrow L_2$ and $.\ell_2 : L_2 \longrightarrow L_1$ such that $:\pi_1 = :\ell_1\pi_2$ and $:\pi_2 = :\ell_1\pi_1$ in the diagram below



(3.8)

The claim is that $.\ell_1\ell_2 = .1_{L_1}$ and $.\ell_2\ell_1 = .1_{L_2}$. To see this observe that $:\ell_1\ell_2\pi_1 = :\ell_1\pi_2 = :\pi_1$, but $:\pi_1$ factors uniquely through itself as $:\pi_1 = :1_{L_1}\pi_1$, whence $.\ell_1\ell_2 = .1_{L_2}$. Similarly, $.\ell_2\ell_1 = .1_{L_2}$.□

<u>PROPOSITION 4</u>: Colimits are unique up to isomorphism.□

Notation and terminology:

Let $\langle L, :\pi \rangle$ be a limit of the diagram $:\mathcal{D}$ in $\mathcal{A}$. We call L the limit object and denote it by $\underline{Lim}:\mathcal{D}$; we call $:\pi$ the limit transformation and refer to its components as projections. If $\langle A, :\alpha \rangle$ is any pair such that $A \in \underline{Obj}_{\mathcal{A}}$ and $:\alpha \in \underline{Cone}_{\mathcal{A}}(A, :\mathcal{D})$ then the unique $.\ell : A \longrightarrow L$ such that $:\alpha = :\ell\pi$ is called the limit morphism for $:\alpha$ and is denoted by $.\underline{lim}^{\mathcal{D}}_{\mathcal{A}}(\alpha)$ or simply $.\underline{lim}(\alpha)$.

Similarly, should $\langle C, :\iota \rangle$ be a colimit of the diagram $:\mathcal{D}$ in $\mathcal{A}$, we call C the colimit object and denote it by $\underline{CoLim}:\mathcal{D}$, $:\iota$ the colimit transformation and refer to its components as injections. If $\langle B, :\beta \rangle$ is any pair such that $B \in \underline{Obj}_{\mathcal{A}}$ and $:\beta \in \underline{Cocone}_{\mathcal{A}}(:\mathcal{D}, B)$ then the unique $.c : C \longrightarrow B$ such that $:\iota c = :\beta$ is called the colimit morphism and is denoted by $.\underline{colim}^{\mathcal{D}}_{\mathcal{A}}(\beta)$ or simply $.\underline{colim}(\beta)$.

Types of limits and colimits:

1.) By the empty diagram in a category $\mathcal{A}$ we mean the diagram in $\mathcal{A}$ whose shape is the empty category.

If a limit of the empty diagram exists in $\mathcal{A}$, we call its limit object a terminal object in $\mathcal{A}$. We use the symbol $\underline{1}$ to denote a terminal object.

If a colimit of the empty diagram exists in $\mathcal{A}$, we call its colimit object an initial object in $\mathcal{A}$. We use the symbol $\underline{0}$ to denote an initial object.

The following two propositions are immediate from the above definitions:

PROPOSITION 5: $\underline{1}$ is a terminal object of if and only if $\overline{\underline{Hom}_{\mathcal{A}}(A, \underline{1})} = 1$ for all $A \in \underline{Obj}_{\mathcal{A}}$. □

PROPOSITION 6: $\underline{0}$ is an initial object of $\mathcal{A}$ if and only if $\overline{\overline{Hom_{\mathcal{A}}(\underline{0},B)}} = 1$ for all $B \in Obj_{\mathcal{A}}$. □

Examples of initial and terminal objects:

(1.1) In $\mathcal{S}$, $\emptyset$ is an initial object and any singleton is a terminal object.

(1.2) In any lattice $\langle L,\leq\rangle$ regarded as a pocategory $\mathcal{O}(L,\leq)$, the least element 0 and the greatest element 1 are initial and terminal objects respectively.

2.) A category $\mathcal{I}$ is said to be discrete if all its arrows are identity arrows. A diagram $:\mathcal{D} : \mathcal{I} \longrightarrow \mathcal{A}$ is said to be discrete if its shape $\mathcal{I}$ is discrete.

Limits and colimits of discrete diagrams are called products and coproducts respectively. We shall denote the product of $:\mathcal{D}$ by $\underset{\alpha\in\mathcal{I}}{\otimes}\mathcal{D}_\alpha$ or by $\mathcal{D}_1\otimes\ldots\otimes\mathcal{D}_n$ if $\mathcal{I}$ is a discrete category with n elements. Similarly we shall denote the coproduct of $:\mathcal{D}$ by $\underset{\alpha\in\mathcal{I}}{\oplus}\mathcal{D}_\alpha$ or by $\mathcal{D}_1\oplus\ldots\oplus\mathcal{D}_n$ if is a discrete category with n elements. The limit and colimit natural transformations for $\underset{\alpha\in\mathcal{I}}{\otimes}\mathcal{D}_\alpha$ and $\underset{\alpha\in\mathcal{I}}{\oplus}\mathcal{D}_\alpha$ are denoted by $:\underline{pr}$ and $:\underline{in}$ and their components are called projections and injections respectively.

Examples of products and coproducts:

(2.1) In $\mathcal{S}$ the cartesian product is a product and the disjoint union is a coproduct.

(2.2) In any lattice $\langle L,\leq\rangle$ regarded as a pocategory $\mathcal{O}(L,\leq)$, the meet $\wedge$ and the join $\vee$ correspond to the notions of product and coproduct respectively.

3.) A limit of a diagram of the shape $\alpha \overset{.i}{\underset{.j}{\rightrightarrows}} \beta$ is called an *equalizer* and is denoted by $\langle \underline{Eqz}(d_i, d_j),\ :\underline{eqz}(d_i, d_j) \rangle$. The object $\underline{Eqz}(d_i, d_j)$ is called the *equalizer object*, while the morphism $.\alpha{:}\underline{eqz}(d_i, d_j)$ is called the *equalizer morphism* and is denoted by $.\underline{eqz}(d_i, d_j)$.

A colimit of a diagram of the shape $\alpha \overset{.i}{\underset{.j}{\rightrightarrows}} \beta$ is called a *coequalizer* and is denoted by $\langle \underline{Coeqz}(d_i, d_j),\ :\underline{coeqz}(d_i, d_j) \rangle$. The object $\underline{Coeqz}(d_i, d_j)$ is called the *coequalizer object*, while the morphism $.\beta{:}\underline{coeqz}(d_i, d_j)$ is called the *coequalizer morphism* and is denoted by $.\underline{coeqz}(d_i, d_j)$.

The following two propositions are easily established:

PROPOSITION 7: Equalizers are monic.□

PROPOSITION 8: Coequalizers are epic.□

This may all be summarized by the following commutative diagram:

$$\underline{Eqz}(d_i, d_j) \xrightarrow{.\underline{eqz}(d_i, d_j)} \mathcal{D}_\alpha \overset{.d_i}{\underset{.d_j}{\rightrightarrows}} \mathcal{D}_\beta \xrightarrow{.\underline{coeqz}(d_i, d_j)} \underline{Coeqz}(d_i, d_j) \tag{3.9}$$

4.) A limit of a diagram of the shape

$$\begin{array}{ccc} & & \beta \\ & & \downarrow{\scriptstyle .j} \\ \alpha & \xrightarrow{.i} & \gamma \end{array} \tag{3.10}$$

is called a *pullback*. Often we shall simply write:

$$\begin{array}{ccc} \overline{P}_{f,g} & \xrightarrow{.f_g} & B \\ \downarrow{\scriptstyle .g_f} & & \downarrow{\scriptstyle .g} \\ A & \xrightarrow{.f} & C \end{array} \tag{3.11}$$

is a pullback.

$\overline{P}_{f,g}$ is called *the pullback object of* $.f$ *and* $.g$ or just the *pullback of* $.f$ *and* $.g$. $.\overline{f}_g$ is called the *pullback of* $.f$ *along* $.g$ and $.\overline{g}_f$ is called the *pullback of* $.g$ *along* $.f$.

Dually, a colimit of a diagram of the shape

$$\begin{array}{ccc} \alpha & \xrightarrow{.i} & \beta \\ \downarrow{\scriptstyle .j} & & \\ \gamma & & \end{array} \qquad (3.12)$$

is called a *pushout*. Often we shall simply write:

$$\begin{array}{ccc} A & \xrightarrow{.f} & B \\ \downarrow{\scriptstyle .g} & & \downarrow{\scriptstyle .\underline{g}^f} \\ C & \xrightarrow{.\underline{f}^g} & \underline{P}^{f,g} \end{array} \qquad (3.13)$$

is a pushout.

$\underline{P}^{f,g}$ is called the *pushout object of* $.f$ *and* $.g$ or just the *pushout of* $.f$ *and* $.g$. $.\underline{f}^g$ is called the *pushout of* $.f$ *along* $.g$ and $.\underline{g}^f$ is called the *pushout of* $.g$ *along* $.f$.

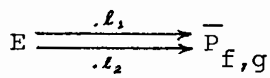
PROPOSITION 9: Pullbacks of monics are monic.

Proof: Let

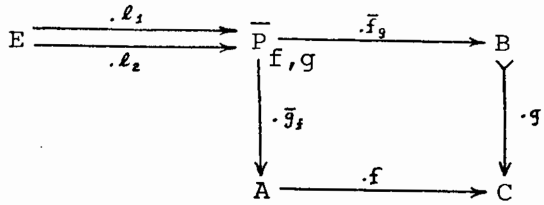
$$\begin{array}{ccc} \overline{P}_{f,g} & \xrightarrow{.\overline{f}_g} & B \\ \downarrow{\scriptstyle .\overline{g}_f} & & \downarrow{\scriptstyle .g} \\ A & \xrightarrow{.f} & C \end{array} \qquad (3.14)$$

be a pullback where $.g$ is a monic.

We want to show that $.\overline{g}_f$ is monic, i.e. we want to show that given any two maps



such that $.\ell_1\overline{g}_f = .\ell_2\overline{g}_f$, $.\ell_1 = .\ell_2$. Consider the diagram

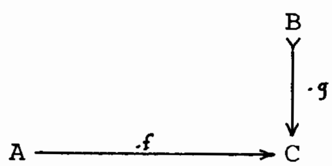


(3.15)

If $.\ell_1\overline{g}_f = .\ell_2\overline{g}_f$ then

$$.\ell_1\overline{f}_g g = .\ell_1\overline{g}_f f = .\ell_2\overline{g}_f f = .\ell_2\overline{f}_g g \tag{3.16}$$

and, since $.g$ is monic $.\ell_1\overline{f}_g = .\ell_2\overline{f}_g$. So $\{.\ell_1\overline{g}_f = .\ell_2\overline{g}_f, .\ell_1\overline{f}_g = .\ell_2\overline{f}_g\}$ determines a cone $:\pi$ from E to the diagram



(3.17)

using (3.16) and $.\ell_1$ and $.\ell_2$ both play the role of $.\underline{\lim}(\pi)$. But $.\underline{\lim}(\pi)$ is unique. So $.\ell_1 = .\ell_2$.□

<u>PROPOSITION 10:</u> Pushouts of epics are epic.

<u>Proof:</u> Dual to that of Proposition 9 above.□

5.) A pullback of the form

$$\begin{array}{ccc} K & \xrightarrow{.k_1} & A \\ {\scriptstyle .k_2}\downarrow & & \downarrow{\scriptstyle .f} \\ A & \xrightarrow{.f} & B \end{array} \qquad (3.18)$$

is called a <u>kernel pair</u>.

We may also refer to the pair $\langle .k_1, .k_2 \rangle$ as a <u>kernel pair for .f</u>.

Dually, a pushout of the form

$$\begin{array}{ccc} A & \xrightarrow{.f} & B \\ {\scriptstyle .f}\downarrow & & \downarrow{\scriptstyle .c_2} \\ B & \xrightarrow{.c_1} & C \end{array} \qquad (3.19)$$

is called a <u>cokernel pair</u> and $\langle .c_1, .c_2 \rangle$ is called a <u>cokernel pair for .f</u>.

In different categories limits and colimits of various types may or may not exist. A category $\mathcal{A}$ is said to be (<u>finitely</u>) <u>complete</u> if every (finite) diagram $:\mathcal{D} : \mathcal{I} \longrightarrow \mathcal{A}$ has a limit in $\mathcal{A}$. $\mathcal{A}$ is said to be (<u>finitely</u>) <u>cocomplete</u> if every (finite) diagram $:\mathcal{D} : \mathcal{I} \longrightarrow \mathcal{A}$ has a colimit in $\mathcal{A}$. $\mathcal{A}$ is (<u>finitely</u>) <u>bicomplete</u> if it is both (finitely) complete and (finitely) cocomplete.

Proofs of the following two propositions may be found in MacLane [18, pp.108-109], Pareigis [21, p.85], and Stone [27, pp.11-12b].

<u>PROPOSITION 11</u>: A category $\mathcal{A}$ is (finitely) complete if and only if it has (finite) products and equalizers.□

<u>PROPOSITION 12</u>: A category $\mathcal{A}$ is (finitely) cocomplete if and only if it has (finite) coproducts and coequalizers.□

II.4 Adjoint pairs and continuous functors

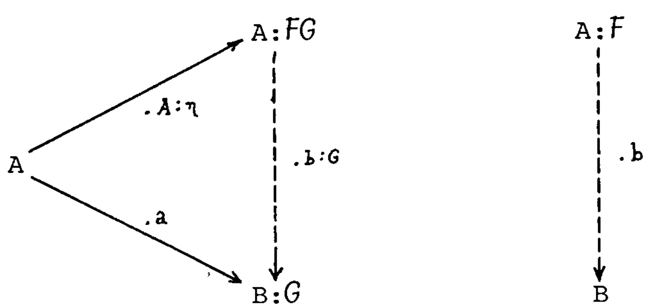
Let $:F : \mathcal{A} \longrightarrow \mathcal{B}$ and $:G : \mathcal{B} \longrightarrow \mathcal{A}$ be a pair of covariant functors. Such a pair is said to be an <u>adjoint pair</u>, $:F$ being the <u>left adjoint</u> and $:G$ being the <u>right adjoint</u> of the pair, denoted $:F \dashv :G$, if any of the four following equivalent conditions are satisfied:

i.) there exist a pair of natural transformations

$$:\eta : :1_{\mathcal{A}} \longrightarrow :FG \quad \text{and} \quad :\varepsilon : :GF \longrightarrow :1_{\mathcal{B}}, \tag{4.1}$$

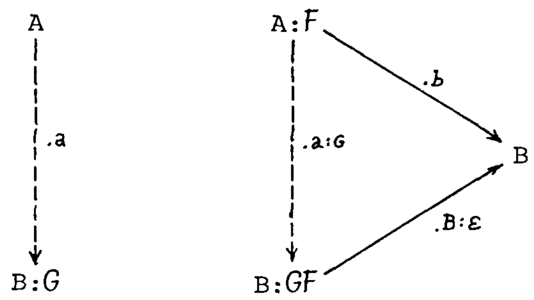
called respectively the <u>unit</u> and <u>counit</u> of the pair, such that $:\eta F \circ :F\varepsilon = :1_{:F}$ and $:G\eta \circ :\varepsilon G = :1_{:G}$;

ii.) there exists a natural transformation $:\eta : :1_{\mathcal{A}} \longrightarrow :FG$, called the <u>unit</u> of the pair, such that for all $A \in \underline{Obj}_{\mathcal{A}}$, $B \in \underline{Obj}_{\mathcal{B}}$, and $.a \in \underline{Hom}_{\mathcal{A}}(A, B{:}G)$ there exists an unique $.b \in \underline{Hom}_{\mathcal{B}}(A{:}F, B)$ such that $.A{:}\eta \circ .b{:}G = .a$;



(4.2)

iii.) there exists a natural transformation $:\varepsilon : :GF \longrightarrow :1_{\mathcal{B}}$, called the <u>counit</u> of the pair, such that for all $A \in \underline{Obj}_{\mathcal{A}}$, $B \in \underline{Obj}_{\mathcal{B}}$, and $.b \in \underline{Hom}_{\mathcal{B}}(A{:}F, B)$ there exists an unique $.a \in \underline{Hom}_{\mathcal{A}}(A, B{:}G)$ such that $.a{:}G \circ .B{:}\varepsilon = .b$;



(4.3)

and iv.) there exists a natural isomorphism $:\theta$, called the <u>adjunction isomorphism</u>, such that

$$.\theta_{A,B} : \underline{Hom}_{\mathcal{B}}(A{:}F,B) \Longrightarrow \underline{Hom}_{\mathcal{A}}(A,B{:}G). \tag{4.4}$$

for all $A \in \underline{Obj}_{\mathcal{A}}$ and $B \in \underline{Obj}_{\mathcal{B}}$.

By an <u>adjunction</u> we mean an adjoint pair together with a specified adjunction isomorphism. Proofs that the four definitions of adjoint pair are equivalent and that an adjunction may be specified by specifying either the unit or counit may be found in MacLane [18, Chapter IV], Pareigis [21, Chapter 2], and Stone [27].

Suppose $:F : \mathcal{A} \longrightarrow \mathcal{B}$ and $:G : \mathcal{B} \longrightarrow \mathcal{A}$ are a pair of contravariant functors. We say that $:F$ and $:G$ are <u>adjoint on the left</u> if there exists a natural isomorphism $:\theta$ such that

$$.\theta_{A,B}: \underline{Hom}_{\mathcal{B}}(A{:}F,B) \Longrightarrow \underline{Hom}_{\mathcal{A}}(B{:}G,A) \tag{4.5}$$

for all $A \in \underline{Obj}_{\mathcal{A}}$ and $B \in \underline{Obj}_{\mathcal{B}}$. We say that $:F$ and $:G$ are <u>adjoint on the right</u> if there exists a natural isomorphism $:\theta$ such that

$$.\theta_{A,B} : \underline{Hom}_{\mathcal{B}}(B,A{:}F) \Longrightarrow \underline{Hom}_{\mathcal{A}}(A,B{:}G) \tag{4.6}$$

for all $A \in \underline{Obj}_{\mathcal{A}}$ and $B \in \underline{Obj}_{\mathcal{B}}$.

Proofs of the following two propositions may be found in Freyd [4], MacLane [18], Pareigis [21], and Stone [27].

PROPOSITION 13: Left adjoints and right adjoints (adjoints on the left and adjoints on the right) to a given covariant (contravariant) functor are unique to within natural isomorphism when they exist.□

PROPOSITION 14: Suppose $\mathcal{A}_1 \underset{:G_2}{\overset{:F_1}{\rightleftarrows}} \mathcal{A}_2 \underset{:G_1}{\overset{:F_2}{\rightleftarrows}} \mathcal{A}_3$ and $:F_1 \longrightarrow\!\!\dashv :G_2$ and $:F_2 \longrightarrow\!\!\dashv :G_1$. Then $:F_1F_2 \longrightarrow\!\!\dashv :G_1G_2$.□

The following theorem specifies four special cases of Freyd's Adjoint Functor Theorem which we shall use frequently.

THEOREM I: Let $:F : \mathcal{A} \longrightarrow \mathcal{B}$ be a covariant functor and $:G : \mathcal{A} \longrightarrow \mathcal{B}$ be a contravariant functor. Then

i.) if $:F$ has a left adjoint then $:F$ is continuous;

ii.) if $:F$ has a right adjoint then $:F$ is cocontinuous;

iii.) if $:G$ has an adjoint on the left then $:G$ is contracocontinuous; and

iv.) if $:G$ has an adjoint on the right then $:G$ is contracontinuous.

Proof: See Freyd [4], MacLane [18], Pareigis [21], or Stone [27].□

A subcategory $\mathcal{A}$ of $\mathcal{B}$ is called a reflective subcategory if the inclusion functor has a left adjoint $:R$. Such an $:R$ is called a reflector. Dually, $\mathcal{A}$ is a coreflective subcategory if the inclusion functor has a right adjoint $:C$. Such a $:C$ is called a coreflector.

CHAPTER III

CARTESIAN-CLOSED CATEGORIES AND TOPOI

III.1 Cartesian closed categories

A category $\mathcal{C}$ is said to be <u>cartesian-closed</u> if

i.) it is finitely bicomplete and

ii.) for every $A \in \underline{Obj}_{\mathcal{C}}$ the functor $:(\_\otimes A)$ has a right adjoint, which we will denote by $:(A \uparrow \_)$.

The counit of the adjunction is called the <u>evaluation natural transformation</u> and it is denoted by $:\underline{ev} : :(A\uparrow\_)\circ:(\_\otimes A) \longrightarrow :1_{\mathcal{C}}$. There exists a natural isomorphism $:\phi : :\underline{Hom}_{\mathcal{C}}(\_\otimes A,B) \longrightarrow :\underline{Hom}_{\mathcal{C}}(\_,A\uparrow B)$ for each $A,B \in \underline{Obj}_{\mathcal{C}}$, hence the set-valued functor $:\underline{Hom}_{\mathcal{C}}(\_\otimes A,B) : \mathcal{C} \to \mathcal{S}$ is representable with $A\uparrow B$ as its representing object. If $C \in \underline{Obj}_{\mathcal{C}}$ and $f \in \underline{Hom}_{\mathcal{C}}(C\otimes A,B)$ then $f.\phi_C \in \underline{Hom}_{\mathcal{C}}(C,A\uparrow B)$ is called the <u>cartesian adjoint of f</u>. Similarly if $g \in \underline{Hom}_{\mathcal{C}}(C,A\uparrow B)$ then $g.\phi_C^{-1} \in \underline{Hom}_{\mathcal{C}}(C\otimes A,B)$ is called the <u>cartesian adjoint of g</u>. We denote passage either way along the adjunction isomorphism by a superscript "*", e.g. $f.\phi_C = f^*$ and $g.\phi_C^{-1} = g^*$.

<u>PROPOSITION 1</u>: Let $\mathcal{C}$ be a cartesian-closed category; $A,B,C \in \underline{Obj}_{\mathcal{C}}$; and $\underline{0}$ and $\underline{1}$, initial and terminal objects respectively. Then the following are naturally isomorphic:

i.) $\underline{0} \cong \underline{0}\otimes A$

ii.) $(A\oplus B)\otimes C \cong (A\otimes C)\oplus(B\otimes C)$

iii.) $A\uparrow\underline{1} \cong \underline{1}$

iv.) $C\uparrow(A\otimes B) \cong (C\uparrow A)\otimes(C\uparrow B)$

v.) $\underline{0}\uparrow A \cong \underline{1}$

vi.) $(B\oplus C)\uparrow A \cong (B\uparrow A)\otimes(C\uparrow A)$

vii.) $A \cong \underline{1}\uparrow A$

viii.) $(C\otimes B)\uparrow A \cong C\uparrow(B\uparrow A)$

<u>Proof:</u> i.) Since $\underline{0}$ is an initial object and for all $X\in \underline{Obj}_{\mathcal{C}}$, $\underline{Hom}_{\mathcal{C}}(\underline{0}\otimes A,X) \cong \underline{Hom}_{\mathcal{C}}(\underline{0},A\uparrow X)$, we have that

$$\overline{\overline{\underline{Hom}_{\mathcal{C}}(\underline{0}\otimes A,X)}} = \overline{\overline{\underline{Hom}_{\mathcal{C}}(\underline{0},A\uparrow X)}} = 1 \quad \text{for all } X\in \underline{Obj}_{\mathcal{C}} \tag{1.1}$$

Therefore $\underline{0}\otimes A$ is an initial object.

ii.) Since $:(\_\otimes C)$ has a right adjoint, it must be cocontinuous. In particular $:(\_\otimes C)$ preserves coproducts so that

$$(A\oplus B)\otimes C = (A\oplus B):(\_\otimes C) \cong A:(\_\otimes C)\oplus B:(\_\otimes C) = (A\otimes C)\oplus(B\otimes C). \tag{1.2}$$

iii.) Since $\underline{1}$ is a terminal object,

$$\overline{\overline{\underline{Hom}_{\mathcal{C}}(X,A\uparrow \underline{1})}} = \overline{\overline{\underline{Hom}_{\mathcal{C}}(X\otimes A,\underline{1})}} = 1 \tag{1.3}$$

for all $X\in \underline{Obj}_{\mathcal{C}}$, hence $A\uparrow\underline{1}$ is a terminal object.

iv.) Since $:(C\uparrow\_)$ has a left adjoint, it must be continuous. In particular $:(C\uparrow\_)$ preserves products so that

$$C\uparrow(A\otimes B) = (A\otimes B):(C\uparrow\_) \cong A:(C\uparrow\_)\otimes B:(C\uparrow\_) = (C\uparrow A)\otimes(C\uparrow B) \tag{1.4}$$

v.) For all $X\in \underline{Obj}_{\mathcal{C}}$

$$\underline{Hom}_{\mathcal{C}}(X,\underline{0}\uparrow A) \cong \underline{Hom}_{\mathcal{C}}(X\otimes\underline{0},A) \cong \underline{Hom}_{\mathcal{C}}(\underline{0}\otimes X,A) \cong \underline{Hom}_{\mathcal{C}}(\underline{0},A), \tag{1.5}$$

and the latter is a singleton, hence the former is also. Thus $\underline{0} \Uparrow A$ is a terminal object.

vi.) Observe that the contravariant functor $:(\_ \Uparrow A)$ is its own adjoint on the right, for if $X_1, X_2 \in \underline{Obj}_{\mathcal{C}}$ then

$$\underline{Hom}_{\mathcal{C}}(X_1, X_2 \Uparrow A) \cong \underline{Hom}_{\mathcal{C}}(X_1 \otimes X_2, A) \cong \underline{Hom}_{\mathcal{C}}(X_2 \otimes X_1, A) \cong \underline{Hom}_{\mathcal{C}}(X_2, X_1 \Uparrow A) \quad (1.6)$$

Hence $:(\_ \Uparrow A)$ is contracontinuous. In particular $:(\_ \Uparrow A)$ carries coproducts to products so that

$$(B \oplus C) \Uparrow A = (B \oplus C):(\_ \Uparrow A) \cong B:(\_ \Uparrow A) \otimes C:(\_ \Uparrow A) = (B \Uparrow A) \otimes (C \Uparrow A). \quad (1.7)$$

vii.) It is easy to see that there is an isomorphism $.i_A : A \otimes \underline{1} \Longrightarrow A$ natural in A; its cartesian adjoint $.i_A^* : A \Longrightarrow \underline{1} \Uparrow A$ is also an isomorphism natural in A.

viii.) The functor $:\underline{Hom}_{\mathcal{C}}(\_ \otimes C \otimes B, A) : \mathcal{C} \longrightarrow \mathcal{S}$ is representable in two ways

$$:\underline{Hom}_{\mathcal{C}}(\_ \otimes C \otimes B, A) \cong :\underline{Hom}_{\mathcal{C}}(\_, (C \otimes B) \Uparrow A) \quad (1.8)$$

$$:\underline{Hom}_{\mathcal{C}}(\_ \otimes C \otimes B, A) \cong :\underline{Hom}_{\mathcal{C}}(\_ \otimes C, B \Uparrow A) \cong :\underline{Hom}_{\mathcal{C}}(\_, C \Uparrow (B \Uparrow A)). \quad (1.9)$$

Since $(C \otimes B) \Uparrow A$ and $C \Uparrow (B \Uparrow A)$ both function as representing objects for the same functor, it follows that they must be isomorphic. □

<u>PROPOSITION 2</u>: If there exists a map $.f : A \longrightarrow \underline{0}$ in a cartesian-closed category $\mathcal{C}$ then $A \cong \underline{0}$.

<u>Proof</u>:

$$A \xrightarrow{.f} \underline{0} \Longrightarrow A \otimes \underline{0} \xrightarrow{.pr_1} A = .1_A \quad (1.10)$$

and

$$\underline{0} \longrightarrow A \xrightarrow{.f} \underline{0} = .1_{\underline{0}}. \square \tag{1.11}$$

PROPOSITION 3: A cartesian-closed category $\mathcal{C}$ is isomorphic to the category $\mathbb{1}$ if and only if there exist a map $\underline{1} \longrightarrow \underline{0}$ in $\mathcal{C}$.□

PROPOSITION 4: Let $\mathcal{A}$ be any small category and $\mathcal{C}$ be any cartesian-closed category. Then $\underline{Func}(\mathcal{A},\mathcal{C})$ is a cartesian-closed category.

Proof: First we want to show that $\underline{Func}(\mathcal{A},\mathcal{C})$ is finitely bicomplete. Let $:\mathcal{D} : \mathcal{I} \longrightarrow \underline{Func}(\mathcal{A},\mathcal{C})$ be a finite diagram in $\underline{Func}(\mathcal{A},\mathcal{C})$. For each $A \in \underline{Obj}_{\mathcal{A}}$ let $:\mathcal{D}^A : \mathcal{I} \longrightarrow \mathcal{C}$ be the diagram in $\mathcal{C}$ defined by

$$\mathcal{D}^A_\alpha =_{df} A{:}\mathcal{D}_\alpha \text{ for each } \alpha \in \underline{Obj}_{\mathcal{I}} \text{ and} \tag{1.12}$$

$$.d^A_i =_{df} .A{:}d_i \text{ for each } .i \in \underline{Arr}_{\mathcal{I}}. \tag{1.13}$$

For each $.a \in \underline{Hom}_{\mathcal{A}}(A_1,A_2)$ let $:\mathcal{D}^a : :\mathcal{D}^{A_1} \longrightarrow :\mathcal{D}^{A_2}$ be the natural transformation defined by $.\alpha{:}\mathcal{D}^a =_{df} .a{:}\mathcal{D}_\alpha$ for all $\alpha \in \underline{Obj}_{\mathcal{I}}$. We construct the limit $\langle L, :\pi \rangle$ of $:\mathcal{D}$ as follows:

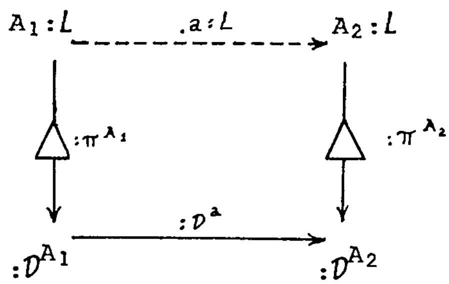
i.) $:L : \mathcal{A} \longrightarrow \mathcal{C}$ is the functor defined by

$$A{:}L =_{df} \underline{Lim}_{\mathcal{C}}{:}\mathcal{D}^A \text{ for each } A \in \underline{Obj}_{\mathcal{A}} \tag{1.14}$$

and if $.a \in \underline{Hom}_{\mathcal{A}}(A_1,A_2)$ and if $\langle A_1{:}L, :\pi^{A_1} \rangle$ and $\langle A_2{:}L, :\pi^{A_2} \rangle$ are limits of $:\mathcal{D}^{A_1}$ and $:\mathcal{D}^{A_2}$ respectively in $\mathcal{C}$ then

$$.a{:}L =_{df} \underline{\ell im}^{\pi^{A_2}} (\pi^{A_1} \circ \mathcal{D}^a), \tag{1.15}$$

i.e. $.a{:}\mathcal{L}$ is the unique map making



(1.16)

commute; and

ii.) $:\pi : \mathcal{L} \multimap {:}\mathcal{D}$ is the natural transformation whose components $.\pi_\alpha$ are natural transformations $:\pi_\alpha : {:}\mathcal{L} \longrightarrow {:}\mathcal{D}_\alpha$ defined for each $\alpha \in \underline{Obj}$ by $.A{:}\pi_\alpha =_{df} .\alpha{:}\pi^A$ for each $A \in \underline{Obj}_{\mathcal{A}}$ where $:\pi^A : A{:}\mathcal{L} \multimap {:}\mathcal{D}^A$ is the limit transformation.

It is easy to see that $\langle \mathcal{L}, {:}\pi \rangle$ is a limit of $:\mathcal{D}$ in $\underline{Func}(\mathcal{A},\mathcal{C})$. We construct the colimit of $:\mathcal{D}$ by the obvious dual construction.

Product and hom relations are obtained on $Func(\mathcal{A},\mathcal{C})$ by letting $:F \otimes G$ be the functor from $\mathcal{A}$ to $\mathcal{C}$ defined by

$$A{:}F \otimes G =_{df} A{:}F \otimes A{:}G \quad \text{and} \tag{1.17}$$

$$.a{:}F \otimes G =_{df} .a{:}F \otimes .a{:}G. \tag{1.18}$$

and letting $:F \pitchfork G$ be the functor defined by

$$A{:}F \pitchfork G =_{df} A{:}F \pitchfork A{:}G \tag{1.19}$$

$$.a{:}F \pitchfork G =_{df} .a{:}F \pitchfork .a{:}G \tag{1.20}$$

where $:F$ and $:G$ are functors from $\mathcal{A}$ to $\mathcal{C}$, $A \in \underline{Obj}_{\mathcal{A}}$, and $.a \in \underline{Arr}_{\mathcal{A}}$. It is easy to see that these definitions yield the required pairs of adjoints needed to make $\underline{Func}(\mathcal{A},\mathcal{C})$ cartesian-closed.□

Remark: (The "Kelly view" of full reflective subcategories.)

Let $\mathcal{A}$ be a full reflective subcategory of $\mathcal{B}$ and let $:R : \mathcal{B} \longrightarrow \mathcal{A}$ be the reflector. Since $:R$ is the left adjoint of the inclusion functor for all $A \in \underline{Obj}_{\mathcal{A}}$ and $B \in \underline{Obj}_{\mathcal{B}}$ there is an isomorphism $\underline{Hom}_{\mathcal{A}}(B{:}R,A) \cong \underline{Hom}_{\mathcal{B}}(B,A)$ (we omit writing applications of the inclusion functor). But since $\mathcal{A}$ is a full subcategory of $\mathcal{B}$ we have that $\underline{Hom}_{\mathcal{A}}(B{:}R,A) \cong \underline{Hom}_{\mathcal{B}}(B{:}R,A)$. This says that up to natural isomorphism we may identify $\mathcal{A}$ with the full subcategory of $\mathcal{B}$ whose objects are the elements of $\{A \in \underline{Obj}_{\mathcal{B}} \mid \forall B \in \underline{Obj}_{\mathcal{B}}(\underline{Hom}_{\mathcal{B}}(B{:}R,A) \cong \underline{Hom}_{\mathcal{B}}(B,A))\}$. In the following we shall make such identifications without further comment.

PROPOSITION 5: Let $\mathcal{A}$ be a full refective subcategory of the cartesian-closed category $\mathcal{C}$ with $:R : \mathcal{C} \longrightarrow \mathcal{A}$ the reflector. Then $:R$ preserves products if and only if for all $A \in \underline{Obj}_{\mathcal{A}}$ and all $C \in \underline{Obj}_{\mathcal{C}}$, $C \Uparrow A \in \underline{Obj}_{\mathcal{A}}$.

Proof: (⇒) Suppose $:R$ preserves products, i.e. for all $D \in \underline{Obj}_{\mathcal{C}}$, $C \otimes D{:}R \cong C{:}R \otimes D{:}R$. By the above remark it is enough to show that

$$\underline{Hom}_{\mathcal{C}}(D{:}R, C \Uparrow A) \cong \underline{Hom}_{\mathcal{C}}(D, C \Uparrow A) \tag{1.21}$$

But we have the following chain of isomorphisms:

$$\begin{aligned} \underline{Hom}_{\mathcal{C}}(D{:}R, C \Uparrow A) &\cong \underline{Hom}_{\mathcal{C}}(D{:}R \otimes C, A) \cong \underline{Hom}_{\mathcal{C}}((D{:}R \otimes C){:}R, A) \cong \\ &\cong \underline{Hom}_{\mathcal{C}}(D{:}RR \otimes C{:}R, A) \cong \underline{Hom}_{\mathcal{C}}(D{:}R \otimes C{:}R, A) \cong \\ &\cong \underline{Hom}_{\mathcal{C}}((D \otimes C){:}R, A) \cong \underline{Hom}_{\mathcal{C}}(D \otimes C, A) \cong \\ &\cong \underline{Hom}_{\mathcal{C}}(D, C \Uparrow A). \end{aligned} \tag{1.22}$$

($\Leftarrow$) Conversly, suppose we have that for all $A \in \underline{Obj}_{\mathcal{A}}$ and $C \in \underline{Obj}_{\mathcal{C}}$ we have that $C \pitchfork A \in \underline{Obj}_{\mathcal{A}}$. We want to show that for all $D \in \underline{Obj}_{\mathcal{C}}$, $C \otimes D:R \cong C:R \otimes D:R$. It is enough to show that

$$:\underline{Hom}_{\mathcal{C}}(C \otimes D:R,\_) \cong :\underline{Hom}_{\mathcal{C}}(C:R \otimes D:R,\_) \tag{1.23}$$

for then the representing objects must be isomorphic. But we have the following chain of isomorphisms natural in A:

$$\begin{aligned} \underline{Hom}_{\mathcal{C}}(C \otimes D:R,A) &\cong \underline{Hom}_{\mathcal{C}}(C \otimes D,A) \cong \underline{Hom}_{\mathcal{C}}(D \otimes C,A) \cong \\ &\cong \underline{Hom}_{\mathcal{C}}(D,C \pitchfork A) \cong \underline{Hom}_{\mathcal{C}}(D:R,C \pitchfork A) \cong \\ &\cong \underline{Hom}_{\mathcal{C}}(D:R \otimes C,A) \cong \underline{Hom}_{\mathcal{C}}(C \otimes (D:R),A) \cong \\ &\cong \underline{Hom}_{\mathcal{C}}(C,D:R \pitchfork A) \cong \underline{Hom}_{\mathcal{C}}(C:R,D:R \pitchfork A) \cong \\ &\cong \underline{Hom}_{\mathcal{C}}(C:R \otimes D:R,A). \square \end{aligned} \tag{1.24}$$

Let $\mathcal{A}$ be a category and $B \in \underline{Obj}_{\mathcal{A}}$. We are next going to define three functors. (1) $:\Sigma_B : \mathcal{A}\downarrow B \longrightarrow \mathcal{A}$, which is defined for all $\mathcal{A}$; (2) $:\chi_B : \mathcal{A} \longrightarrow \mathcal{A}\downarrow B$, which is defined for all $\mathcal{A}$ having finite products; and (3) $:\Pi_B : \mathcal{A}\downarrow B \longrightarrow \mathcal{A}$, which is defined for all cartesian-closed $\mathcal{A}$.

(1) $:\Sigma_B : \mathcal{A}\downarrow B \longrightarrow \mathcal{A}$ is the forgetful functor defined in the obvious manner by

$$\begin{pmatrix} A \\ \big\downarrow .b \\ B \end{pmatrix} :\Sigma_B =_{df} A \quad \text{for all} \begin{pmatrix} A \\ \big\downarrow .b \\ B \end{pmatrix} \in \underline{Obj}_{\mathcal{A}\downarrow B} \tag{1.25}$$

and

$$\left(\begin{array}{ccc} A_1 & \xrightarrow{.a} & A_2 \\ & \searrow \quad \swarrow & \\ & B & \end{array}\right) : \Sigma_B =_{df} .a \text{ for all } \left(\begin{array}{ccc} A_1 & \xrightarrow{.a} & A_2 \\ & \searrow \quad \swarrow & \\ & B & \end{array}\right) \in \underline{Arr}_{\mathcal{A}\downarrow B} . \tag{1.26}$$

(2) $:\chi_B : \mathcal{A} \longrightarrow \mathcal{A}\downarrow B$ is defined by

$$A:\chi_B =_{df} \left(\begin{array}{c} A\otimes B \\ \downarrow .pr_2 \\ B \end{array}\right) \text{ for all } A \in \underline{Obj}_{\mathcal{A}} \tag{1.2 }$$

$$(A_1 \xrightarrow{.a} A_2):\chi_B =_{df} \left(\begin{array}{ccc} A_1\otimes B & \xrightarrow{.a\otimes 1_B} & A_2\otimes B \\ & {\scriptstyle .pr_2}\searrow \quad \swarrow{\scriptstyle .pr_2} & \\ & B & \end{array}\right) \tag{1.28}$$

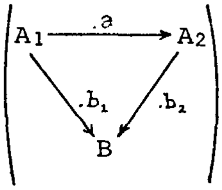
for all $.a \in \underline{Hom}_{\mathcal{A}}(A_1, A_2)$

(3) $:\Pi_B : \mathcal{A}\downarrow B \longrightarrow \mathcal{A}$ is defined by

$$\left(\begin{array}{c} A \\ \downarrow .b \\ B \end{array}\right) :\Pi_B =_{df} \overline{P}_{1_B\uparrow b, 1_B^*} , \tag{1.29}$$

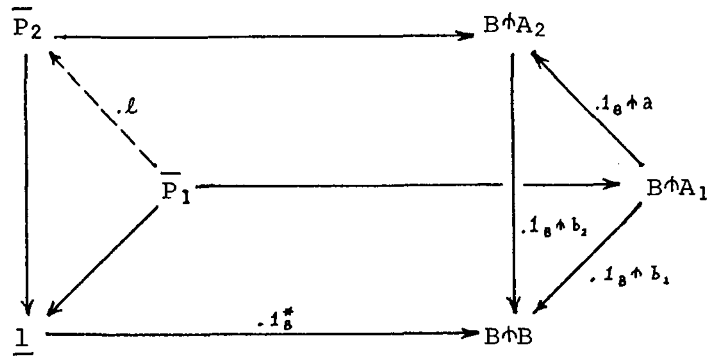
where $.1_B^* : \underline{1} \longrightarrow B\uparrow B$ is the cartesian adjoint of $\underline{1}\otimes B \Longrightarrow B$ and $\overline{P}_{1_B\uparrow b, 1_B^*}$ denotes the pullback of $.1_B\uparrow b : B\uparrow A \longrightarrow B\uparrow B$ and $.1_B^*$

$$\begin{array}{ccc} \overline{P}_{1_B\uparrow b, 1_B^*} & \longrightarrow & B\uparrow A \\ \downarrow & & \downarrow .1_B\uparrow b \\ \underline{1} & \xrightarrow{.1_B^*} & B\uparrow B \end{array} \tag{1.30}$$



$:\Pi_B = .\ell$ (1.31)

where $.\ell$ is the limit morphism in (1.32), in which the front and bottom faces are pullbacks



(1.32)

PROPOSITION 6: Let $\mathcal{A}$ be any category and $B \in \underline{Obj}_{\mathcal{A}}$. Then

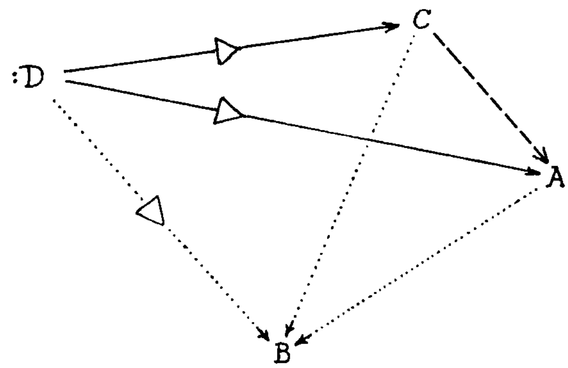
i.) $:\Sigma_B : \mathcal{A}\downarrow B \longrightarrow \mathcal{A}$ preserves and reflects colimits, equalizers, pullbacks and monomorphisms when they exist;

ii.) if $\mathcal{A}$ is finitely bicomplete then $:\Sigma_B \dashv :\chi_B$ and furthermore if there exists a functor $:\tilde{\Pi}_B : \mathcal{A}\downarrow B \longrightarrow \mathcal{A}$ such that $:\chi_B \dashv :\tilde{\Pi}_B$ for all $B \in \underline{Obj}_{\mathcal{A}}$ then $\mathcal{A}$ is cartesian-closed; and

iii.) if $\mathcal{A}$ is cartesian-closed then $:\Sigma_B \dashv :\chi_B \dashv :\Pi_B$ for all $B \in \underline{Obj}_{\mathcal{A}}$

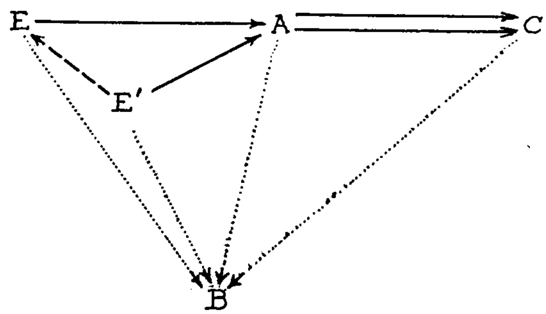
Proof: i.) This part of the Proposition is evident from (1.33)-(1.36) below, where the whole diagrams are in $\mathcal{A}\downarrow B$ and removing the portion with the dotted arrows (········➛) is intended to illustrate the action of $:\Sigma_B$.

colimits:



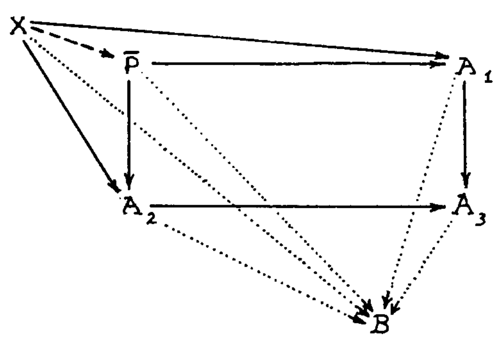
(1.33)

equalizers:



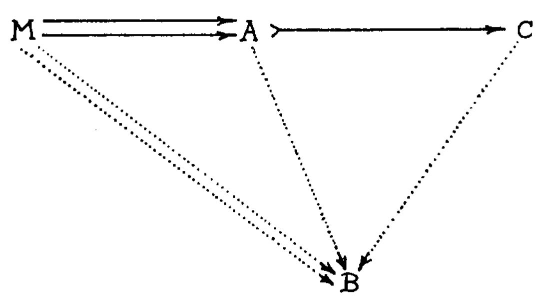
(1.34)

pullbacks:



(1.35)

monomorphisms:

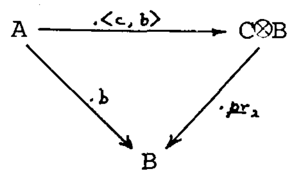


(1.36)

ii.) First we need to show that

$$\underline{Hom}_{\mathcal{A}}\left(\left(\begin{matrix} A \\ \downarrow .b \\ B \end{matrix}\right):\Sigma_B, C\right) \cong \underline{Hom}_{\mathcal{A}\downarrow B}\left(\left(\begin{matrix} A \\ \downarrow .b \\ B \end{matrix}\right), C:\chi_B\right) \tag{1.37}$$

The required natural bijection is that which associates the map $.c : A \longrightarrow C$ in $\underline{Hom}_{\mathcal{A}}\left(\left(\begin{matrix} A \\ \downarrow .b \\ B \end{matrix}\right):\Sigma_B, C\right)$ with the map



$$\text{in } \underline{Hom}_{\mathcal{A}\downarrow B}\left(\left(\begin{matrix} A \\ \downarrow .b \\ B \end{matrix}\right), C:\chi\right). \tag{1.38}$$

Secondly we note that if $:\chi_B$ has a right adjoint $:\tilde{\Pi}_B$ then

$$\begin{aligned} :\underline{Hom}_{\mathcal{A}}(\_\otimes B, A) &\cong :\underline{Hom}_{\mathcal{A}}(\_:\chi_B \circ \Sigma_B, A) \cong \\ &\cong :\underline{Hom}_{\mathcal{A}\downarrow B}(\_:\chi_B, A:\chi_B) \cong \\ &\cong :\underline{Hom}_{\mathcal{A}}(\_, A:\chi_B\ \tilde{\Pi}_B). \end{aligned} \tag{1.39}$$

Hence we can take B A to be $A:\chi_B \circ \tilde{\Pi}_B$, so that $\mathcal{A}$ must be cartesian-closed.

iii.) Observe that $:\underline{Hom}_{\mathcal{A}}(C,\_)$ is continuous since it preserves products and equalizers. Hence the pullback diagram (1.30) used to define $:\Pi_B$ gives rise to the pullback diagram (1.40) in $\mathcal{S}$

$$
\begin{array}{ccc}
\underline{Hom}_{\mathcal{A}}\left(C, \begin{pmatrix} A \\ \downarrow .b \\ B \end{pmatrix} :\Pi_B\right) & \longrightarrow & \underline{Hom}_{\mathcal{A}}(C, B\pitchfork A) \\
\downarrow & & \downarrow .\underline{Hom}_{\mathcal{A}}(C, 1_B \pitchfork b) \\
\underline{Hom}_{\mathcal{A}}(C, \underline{1}) & \longrightarrow & \underline{Hom}_{\mathcal{A}}(C, B\pitchfork B)
\end{array}
\tag{1.40}
$$

which is isomorphic to the diagram

$$
\begin{array}{ccc}
\underline{Hom}_{\mathcal{A}}\left(C, \begin{pmatrix} A \\ \downarrow .b \\ B \end{pmatrix} :\Pi_B\right) & \longrightarrow & \underline{Hom}_{\mathcal{A}}(C\otimes B, A) \\
\downarrow & & \downarrow .\underline{Hom}_{\mathcal{A}}(C\otimes B, b) \\
\underline{1} & \xrightarrow{\quad .x \quad} & \underline{Hom}_{\mathcal{A}}(C\otimes B, B)
\end{array}
\tag{1.41}
$$

where $.x$ is the injection taking $\underline{1}$ to $.\underline{pr}_2 : C\otimes B \longrightarrow B$.

If we view $\underline{Hom}_{\mathcal{A}}\left(C, \begin{pmatrix} A \\ \downarrow .b \\ B \end{pmatrix} :\Pi_B\right)$ as a subset of $\underline{Hom}_{\mathcal{A}}(C\otimes B, A)$ we have that

$$
\underline{Hom}_{\mathcal{A}}\left(C, \begin{pmatrix} A \\ \downarrow .b \\ B \end{pmatrix} :\Pi_B\right) =
$$

$$
= \left\{ .g \in \underline{Hom}_{\mathcal{A}}(C\otimes B, A) \;\middle|\; \begin{array}{ccc} C\otimes B & \xrightarrow{.g} & A \\ & {}_{.pr_2}\searrow \quad \swarrow_{.b} & \\ & B & \end{array} \text{ commutes} \right\} = \tag{1.42}
$$

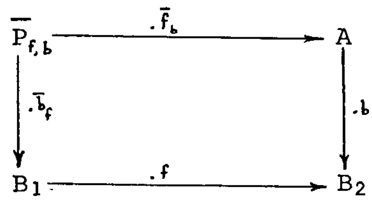
$$
= \underline{Hom}_{\mathcal{A}\downarrow B}\left(\begin{pmatrix} C\otimes B \\ \downarrow .\underline{pr}_2 \\ B \end{pmatrix}, \begin{pmatrix} A \\ \downarrow .b \\ B \end{pmatrix} \right) =
$$

$$
= \underline{Hom}_{\mathcal{A}\downarrow B}\left(C:\chi_B, \begin{pmatrix} A \\ \downarrow .b \\ B \end{pmatrix} \right) .
$$

Therefore $:\chi_B \longmapsto :\Pi_B$. □

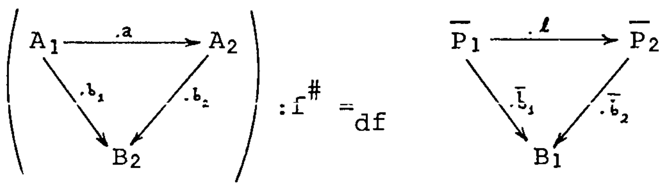
Now let $\mathcal{A}$ be any finitely complete category and let $.f : B_1 \longrightarrow B_2$ be an arrow in $\mathcal{A}$. Define the functor $:f^{\#} : \mathcal{A}{\downarrow}B_2 \longrightarrow \mathcal{A}{\downarrow}B_1$ by

$$\left(\begin{array}{c} A \\ \downarrow .b \\ B_2 \end{array}\right) :f^{\#} =_{df} \begin{array}{c} \overline{P}_{f,b} \\ \downarrow .\overline{b}_f \\ B_1 \end{array} \quad \text{in the pullback (1.44)} \tag{1.43}$$



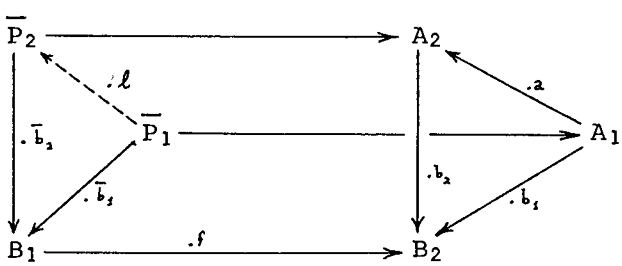
(1.44)

and



in (1.46) (1.45)

where the front and bottom faces are pullbacks and $.\ell$ is the indicated limit morphism.



(1.46)

Also define the functor $:\Sigma_f : \mathcal{A}\downarrow B_1 \longrightarrow \mathcal{A}\downarrow B_2$ by

$$\left(\begin{array}{c} A \\ \downarrow .b \\ B_1 \end{array}\right):\Sigma_f =_{df} \begin{array}{c} A \\ \downarrow .b \\ B_1 \\ \downarrow .f \\ B_2 \end{array} \tag{1.47}$$

and

$$\left(\begin{array}{ccc} A_1 & \xrightarrow{.a} & A_2 \\ & {\scriptstyle .b_1}\searrow \quad \swarrow{\scriptstyle .b_2} & \\ & B_1 & \end{array}\right):\Sigma_f =_{df} \begin{array}{ccc} A_1 & \xrightarrow{.a} & A_2 \\ & {\scriptstyle .b_1 f}\searrow \quad \swarrow{\scriptstyle .b_2 f} & \\ & B_2 & \end{array} \tag{1.48}$$

<u>PROPOSITION 7</u>: Let $\mathcal{A}$ be any finitely bicomplete category. Then

i.) $:\Sigma_f \longrightarrow\!\!\!| :f^{\#}$ for all $.f \in \underline{Arr}_{\mathcal{A}}$ and

ii.) $:f^{\#}$ has a right adjoint $:\Pi_f$ for all $.f \in \underline{Arr}_{\mathcal{A}}$ if and only if $\mathcal{A}\downarrow B$ is cartesian-closed for all $B \in \underline{Obj}_{\mathcal{A}}$.

<u>Proof</u>: Consider $.f:B_1 \longrightarrow B_2$ as an object in $\mathcal{A}\downarrow B_2$. We may define the functors

$$:\Sigma_{(B_1 \xrightarrow{.f} B_2)} : (\mathcal{A}\downarrow B_2)\downarrow(B_1 \xrightarrow{.f} B_2) \longrightarrow \mathcal{A}\downarrow B_2 \tag{1.49}$$

$$:\chi_{(B_1 \xrightarrow{.f} B_2)} : \mathcal{A}\downarrow B_2 \longrightarrow (\mathcal{A}\downarrow B_2)\downarrow(B_1 \xrightarrow{.f} B_2) \tag{1.50}$$

$$:\Pi_{(B_1 \xrightarrow{.f} B_2)} : (\mathcal{A}\downarrow B_2)\downarrow(B_1 \xrightarrow{.f} B_2) \longrightarrow \mathcal{A}\downarrow B_2 \tag{1.51}$$

But $(\mathcal{A}\downarrow B_2)\downarrow(B_1 \xrightarrow{.f} B_2) \cong \mathcal{A}\downarrow B_1$. Thus $:\Sigma_f \cong :\Sigma_{(B_1 \xrightarrow{.f} B_2)}$, $:f^{\#} \cong :\chi_{(B_1 \xrightarrow{.f} B_2)}$ and we may take $:\Pi_f =_{df} :\Pi_{(B_1 \xrightarrow{.f} B_2)}$. In this way Proposition 7 reduces to Proposition 6.□

III.2 Elementary topoi

An <u>(elementary) topos</u> $\mathcal{E}$ is a cartesian-closed category with a <u>subobject classifier</u>, i.e. with an object $\Omega \in \underline{Obj}_{\mathcal{E}}$ and a monomorphism $.\underline{true} : \underline{1} \rightarrowtail \Omega$ such that for every subobject $.m : B \rightarrowtail A$ in $\mathcal{E}$ there exists a unique morphism $.\underline{ch}(m) : A \longrightarrow \Omega$, called the <u>characteristic function of .m</u>, making (2.1) into a pullback.

$$\begin{array}{ccc} B & \overset{.m}{\rightarrowtail} & A \\ \big| & & \big\downarrow {\scriptstyle .\underline{ch}(m)} \\ \underline{1} & \underset{.\underline{true}}{\rightarrowtail} & \Omega \end{array} \tag{2.1}$$

If $.a : A \longrightarrow \Omega$ is an arrow in $\mathcal{E}$ we let $.[\ | a]$ denote the pullback of $.\underline{true}$ along $.a$, i.e.

$$.[\ | a] =_{df} .\overline{\underline{true}}_a \tag{2.2}$$

Suppose $\mathcal{A}$ is any category with pullbacks. Define the set valued contravariant functor $:\underline{Sub}_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{S}$ by

i.) $A:\underline{Sub}_{\mathcal{A}} =_{df} \{x |\ x \text{ is a subobject of } A\}$ for all $A \in \underline{Obj}_{\mathcal{A}}$.

ii.) if $.a \in \underline{Hom}_{\mathcal{A}}(B,A)$ we define $.a:\underline{Sub}_{\mathcal{A}} : A:\underline{Sub}_{\mathcal{A}} \longrightarrow B:\underline{Sub}_{\mathcal{A}}$ to be the function which sends each subobject $.m : A' \rightarrowtail A$ to its pullback along $.a$, i.e.

$$m.a:\underline{Sub}_{\mathcal{A}} =_{df} .\overline{m}_a \tag{2.3}$$

$$\begin{array}{ccc} \overline{P}_{m,a} & \xrightarrow{.\overline{m}_a} & B \\ {\scriptstyle .\overline{a}_m}\downarrow & & \downarrow{\scriptstyle .a} \\ A' & \xrightarrow{.m} & A \end{array} \qquad (2.4)$$

PROPOSITION 8: Let $\mathcal{C}$ be a cartesian-closed category. Then $\mathcal{C}$ is a topos if and only if $:\underline{Sub}_{\mathcal{C}}$ is representable.

Proof: (⇒) If $\mathcal{C}$ is a topos, we want to show that there exists a natural isomorphism $:\eta : :\underline{Sub}_{\mathcal{C}} \longrightarrow :\underline{Hom}_{\mathcal{C}}(\_,\Omega)$. Define $:\eta$ component-wise by

$$m.A:\eta =_{df} .\underline{ch}(m) \qquad m\in A:\underline{Sub}_{\mathcal{C}},\ A\in\underline{Obj}_{\mathcal{C}} \qquad (2.5)$$

$.A:\eta$ is easily seen to be a bijection. To show that it is natural we need to show that given any $.a : B \longrightarrow A$ in $\mathcal{C}$

$$\begin{array}{ccc} A:\underline{Sub}_{\mathcal{C}} & \xrightarrow{.A:\eta} & A:\underline{Hom}_{\mathcal{C}}(\_,\Omega) \\ {\scriptstyle .a:\underline{Sub}_{\mathcal{C}}}\downarrow & & \downarrow{\scriptstyle .a:\underline{Hom}_{\mathcal{C}}(\_,\Omega)} \\ B:\underline{Sub}_{\mathcal{C}} & \xrightarrow{.B:\eta} & B:\underline{Hom}_{\mathcal{C}}(\_,\Omega) \end{array} \qquad (2.6)$$

commutes.

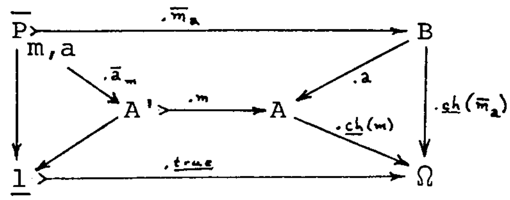
Let $m\in A:\underline{Sub}_{\mathcal{C}}$. Then

$$m.A:\eta \circ .a:\underline{Hom}_{\mathcal{C}}(\_,\Omega) = .a\circ\underline{ch}(m)$$

and $m.a:\underline{Sub}_{\mathcal{C}}.B:\eta = .\underline{ch}(\overline{m}_a)$

so that we must show that $.a\circ\underline{ch}(m) = .\underline{ch}(\overline{m}_a)$.

Consider the diagram



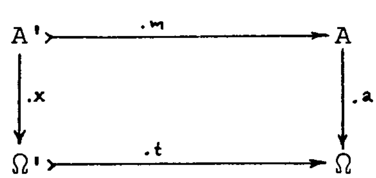
(2.7)

We know that the two inner trapezoids are pullbacks, as is the outside rectangle, and the left hand triangle commutes. But then $.\bar{m}_a$ is the pullback of .<u>true</u> along .a.<u>ch</u>(m) as pullbacks of pullbacks are pullbacks. But $.\underline{ch}(\bar{m}_a)$ is unique so $.a.\underline{ch}(m) = .\underline{ch}(\bar{m}_a)$.

(⇐) Conversely, suppose there exists a natural isomorphism $:\eta : :\underline{Sub}_{\mathcal{C}} \longrightarrow :\underline{Hom}_{\mathcal{C}}(\_,\Omega)$. As for each $.a \in \underline{Hom}_{\mathcal{C}}(A,\Omega)$, $.a : A \longrightarrow \Omega$ is the unique map such that $1_\Omega.\underline{Hom}_{\mathcal{C}}(a,\Omega) = a$ and since $:\underline{Sub}_{\mathcal{C}} \cong :\underline{Hom}_{\mathcal{C}}(\_,\Omega)$ it follows that for each $.m \in A:\underline{Sub}_{\mathcal{C}}$ there exists a unique $.a : A \longrightarrow \Omega$ such that

$$(1_\Omega.\Omega:\eta^{-1}).a:\underline{Sub}_{\mathcal{C}} = .m \qquad (2.8)$$

Thus $1_\Omega.\Omega:\eta^{-1}$ plays the role of .<u>true</u> and the unique $.a : A \longrightarrow \Omega$ specified by (2.8) above plays the role of the characteristic function of .m. By (2.8) $.t =_{df} 1_\Omega:\Omega:\eta : \Omega' \rightarrowtail \Omega$ has the property that for all $.m : A' \rightarrowtail A$ there exists a unique $.a : A \longrightarrow \Omega$ such that there exists an $.x : A' \longrightarrow \Omega'$ making (2.9) into a pullback



(2.9)

In particular if .a factors through .t then (2.10) is a pullback and the uniqueness condition on .a implies that there can be only one map from $A \longrightarrow \Omega'$. Therefore $\Omega' \cong \underline{1}$ and we are done.

$$\begin{array}{ccc} A & \overset{.1_A}{\Longleftrightarrow} & A \\ {\scriptstyle .x}\downarrow & & \downarrow{\scriptstyle .a} \\ \Omega' & \overset{.t}{\rightarrowtail} & \Omega \end{array} \qquad (2.10)$$

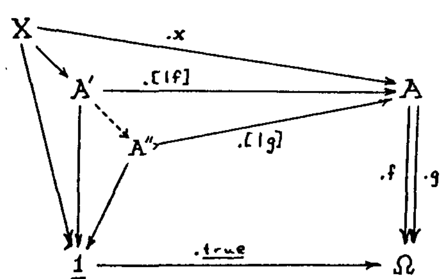
.□

<u>Remark</u>: Note that in any topos $\mathcal{E}$ the usual ordering of subobjects of $A \in \underline{Obj}_{\mathcal{E}}$ induces an ordering of $\underline{Hom}_{\mathcal{E}}(A,\Omega)$. Further note that if $.f,.g \in \underline{Hom}_{\mathcal{E}}(A,\Omega)$ then $.f \leq .g$ if and only if for all $X \in \underline{Obj}_{\mathcal{E}}$ and $.x \in \underline{Hom}_{\mathcal{E}}(X,A)$ (2.11) commutes implies that (2.12) commutes.

$$\begin{array}{ccc} X & \overset{.x}{\longrightarrow} & A \\ \downarrow & & \downarrow{\scriptstyle .f} \\ \underline{1} & \overset{.true}{\longrightarrow} & \Omega \end{array} \qquad (2.11)$$

$$\begin{array}{ccc} X & \overset{.x}{\longrightarrow} & A \\ \downarrow & & \downarrow{\scriptstyle .g} \\ \underline{1} & \overset{.true}{\longrightarrow} & \Omega \end{array} \qquad (2.12)$$

to see this consider the diagram (2.13)



(2.13)

III.3 The representability of relations and partial maps

Throughout this section let $\mathcal{E}$ be a topos, $A,B,C,\ldots \in \underline{Obj}_{\mathcal{E}}$ and $.a,.b,.c,\ldots \in \underline{Arr}_{\mathcal{E}}$.

A relation between A and B is a subobject of $A \otimes B$.

We define the set valued contravariant functor $:\underline{Rel}\ (\_,B)$:

by

$$:\underline{Rel}_{\mathcal{E}}(\_,B) =_{df} :(\_\otimes B)\circ :\underline{Sub}_{\mathcal{E}} \tag{3.1}$$

PROPOSITION 9: $:\underline{Rel}_{\mathcal{E}}(\_,B)$ is representable.

Proof:
$$\begin{aligned} :\underline{Rel}_{\mathcal{E}}(\_B) &= :(\_\otimes B)\circ :\underline{Sub}_{\mathcal{E}} \\ &\cong :(\_\otimes B)\circ :\underline{Hom}_{\mathcal{E}}(\_,\Omega) \cong \\ &\cong :\underline{Hom}_{\mathcal{E}}(\_\otimes B,\Omega) \cong :\underline{Hom}_{\mathcal{E}}(\_,B\pitchfork\Omega) \quad \square \end{aligned} \tag{3.2}$$

A partial map $.f$ from A to B is a map from a subobject of A to B. More formally it is a pair of arrows of the form

$$\begin{array}{ccc} A' & \xrightarrow{\ .f''\ } & B \\ {\scriptstyle .f'}\big\downarrow & & \\ A & & \end{array} \tag{3.3}$$

We will denote partial maps by lower case script Latin letters preceded by a dot $.f,.g,.h,\ldots$. The subobject component of $.f$ will be denoted by $.f'$ and the other component by $.f''$. Composition of partial maps is defined by pulling back i.e. if

$$\cdot f = \begin{array}{ccc} A' & \xrightarrow{\cdot f''} & B \\ \big\downarrow{\scriptstyle \cdot f'} & & \\ A & & \end{array} \tag{3.4}$$

and

$$\cdot g = \begin{array}{ccc} B' & \xrightarrow{\cdot g''} & C \\ \big\downarrow{\scriptstyle \cdot g'} & & \\ B & & \end{array} \tag{3.5}$$

then

$$\cdot fg = \begin{array}{ccccc} A'' & \longrightarrow & B' & \xrightarrow{\cdot g''} & C \\ \big\downarrow & & \big\downarrow{\scriptstyle \cdot g'} & & \\ A' & \xrightarrow{\cdot f''} & B & & \\ \big\downarrow{\scriptstyle \cdot f'} & & & & \\ A & & & & \end{array} \tag{3.6}$$

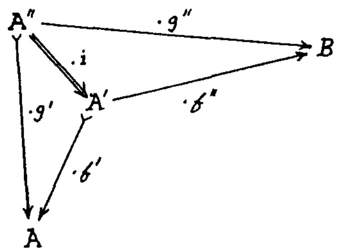
where the top left hand rectangle in (3.6) is a pull back.

Two partial maps $\cdot f$ and $\cdot g$ as in (3.7) and (3.8)

$$\begin{array}{ccc} A' & \xrightarrow{\cdot f''} & B \\ \big\downarrow{\scriptstyle \cdot f'} & & \\ A & & \end{array} \tag{3.7}$$

$$\begin{array}{ccc} A'' & \xrightarrow{\cdot g''} & B \\ \big\downarrow{\scriptstyle \cdot g'} & & \\ A & & \end{array} \tag{3.8}$$

are said to be <u>equivalent</u>, written $.f \simeq .g$, if there exists an isomorphism $.i : A'' \Longrightarrow A'$ such that

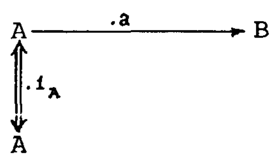


(3.9)

commutes.

Let $\underline{Par}_{\mathcal{E}}(A,B)$ denote the set of equivalence classes of partial maps from A to B under the equivalence relation "$\simeq$" above.

By identifying the morphism $.a \in \underline{Hom}_{\mathcal{E}}(A,B)$ with the partial map



(3.10)

we may view $\underline{Hom}_{\mathcal{E}}(A,B)$ as a subset of $\underline{Par}_{\mathcal{E}}(A,B)$. This identification determines a natural transformation

$$:\alpha \ : \ :\underline{Hom}_{\mathcal{E}}(\_,B) \longrightarrow :\underline{Par}_{\mathcal{E}}(\_,B) \tag{3.11}$$

Define the set-valued contravariant functor $:\underline{Par}_{\mathcal{E}}(\_,B) : \mathcal{E} \longrightarrow \mathcal{S}$ by

$$A:\underline{Par}_{\mathcal{E}}(\_,B) =_{df} \underline{Par}_{\mathcal{E}}(A,B) \tag{3.12}$$

and if $.a \in \underline{Hom}_{\mathcal{E}}(A_1,A_2)$ then

$$.a:\underline{Par}_{\mathcal{E}}(\_,B) \ : \ \underline{Par}_{\mathcal{E}}(A_2,B) \longrightarrow \underline{Par}_{\mathcal{E}}(A_1,B)$$

is the set theoretic functions which sends $.f \in \underline{Par}_{\mathcal{E}}(A_2,B)$ to $.af \in \underline{Par}_{\mathcal{E}}(A_1,B)$.

For every partial map $.f \in \underline{Par}_{\mathcal{E}}(A,B)$ there is an associated relation between A and B, namely

$$.\Gamma_f =_{df} .\langle f', f'' \rangle : A' \rightarrowtail A \otimes B. \tag{3.13}$$

$.\Gamma_f$ is a subobject of $A \otimes B$ because $.f'$ is a subobject of A.

Thus there is a natural transformation

$$:\gamma \ : \ :\underline{Par}_{\mathcal{E}}(\_,B) \longrightarrow :\underline{Rel}_{\mathcal{E}}(\_,B) \cong \underline{Hom}_{\mathcal{E}}(\_,B\uparrow\Omega) \tag{3.14}$$

By composing $:\alpha$ with $:\gamma$ we get a natural transformation

$$:\alpha\circ\gamma \ : \ :\underline{Hom}_{\mathcal{E}}(\_,B) \longrightarrow :\underline{Hom}_{\mathcal{E}}(\_,B\uparrow\Omega). \tag{3.15}$$

The map inducing $:\alpha\circ\gamma$ is called the <u>singleton map</u> and it is denoted by $.\{\_\} : B \rightarrowtail B\uparrow\Omega$. It is easily checked that it is a monomorphism. $.\{\_\}$ may also be described as follows.

The <u>diagonal subobject</u> $.\Delta$ of $B \otimes B$ is the subobject

$$.\Delta =_{df} .\langle 1_B, 1_B \rangle : B \rightarrowtail B \otimes B. \tag{3.16}$$

The <u>Kronnecker-delta</u> $.\delta$ is the characteristic function of $.\Delta$, i.e.

$$.\delta =_{df} .\underline{ch}(\Delta) : B \otimes B \longrightarrow \Omega. \tag{3.17}$$

$.\{\_\} : B \rightarrowtail B\uparrow\Omega$ is the cartesian-adjoint of $.\delta$, i.e. $.\{\_\} = .\delta^*$.

PROPOSITION 10: (Unique existentiation)

Let $.a : C \longrightarrow A$ be given. Then there exists a $.q : Q \rightarrowtail A$ which factors through $.a$ and such that

$$\begin{array}{ccc} Q & \overset{.1_Q}{\Longleftrightarrow} & Q \\ {\scriptstyle .c}\downarrow & & \downarrow{\scriptstyle .q} \\ C & \underset{.a}{\longrightarrow} & A \end{array} \qquad (3.18)$$

is a pullback

and such that given any $.f : X \longrightarrow A$ which factors through $.a$ and such that

$$\begin{array}{ccc} X & \overset{.1_X}{\Longleftrightarrow} & X \\ {\scriptstyle .p}\downarrow & & \downarrow{\scriptstyle .f} \\ C & \underset{.a}{\longrightarrow} & A \end{array} \qquad (3.19)$$

is a pullback,

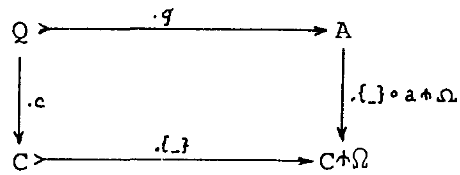
it is the case $.f$ factors uniquely through $.q$.

Proof: Define the natural transformation $:\sigma : :\underline{Hom}_{\mathcal{E}}(\_,A) \longrightarrow :\underline{Rel}_{\mathcal{E}}(\_,C)$ by letting $.X:\sigma$ be the set theoretic function which takes each $.y \in \underline{Hom}_{\mathcal{E}}(X,A)$ to the relation

$$.r =_{df} .\underline{eqz}(.\underline{pr}_1 \circ y, .\underline{pr}_2 \circ a) : R \longrightarrow X \otimes C \overset{.pr_1 \circ y}{\underset{.pr_2 \circ a}{\rightrightarrows}} A \qquad (3.20)$$

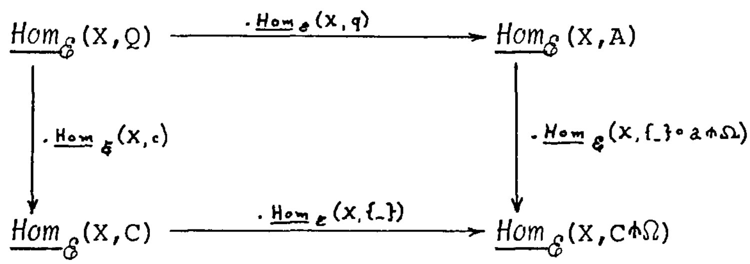
$:\sigma$ may also be thought of as a natural transformation from $:\underline{Hom}_{\mathcal{E}}(\_,A)$ to $:\underline{Hom}_{\mathcal{E}}(\_,C \pitchfork \Omega)$, and as such it is induced by the map $A \overset{.\{\_\}}{\rightarrowtail} A \pitchfork \Omega \xrightarrow{.a \pitchfork \Omega} C \pitchfork \Omega$.

Let $.q : Q \rightarrowtail A$ be the pullback of $.\{\_\} : C \rightarrowtail C \pitchfork \Omega$ along $.\{\_\} \circ .a \pitchfork \Omega : A \longrightarrow C \pitchfork \Omega$ in (3.21)



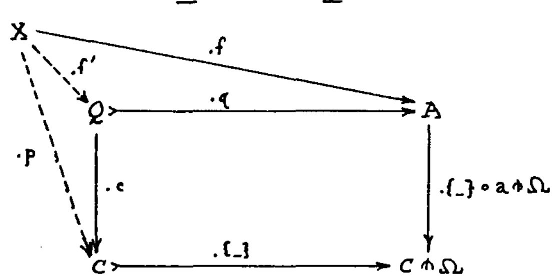
(3.21)

Since :$\underline{Hom}_{\mathcal{E}}(X,\_)$ is continuous (3.22) is also a pullback.



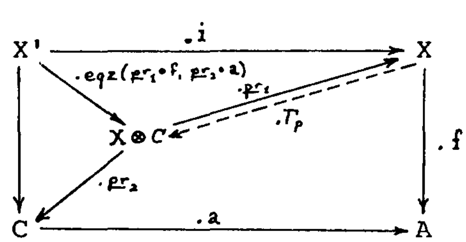
(3.22)

Since $.\underline{Hom}_{\mathcal{E}}(X,q)$ ia an injection we may take $\underline{Hom}_{\mathcal{E}}(X,Q) \subseteq \underline{Hom}_{\mathcal{E}}(X,A)$. Then $\underline{Hom}_{\mathcal{E}}(X,Q)$ is described by the requirement that $.f \in \underline{Hom}_{\mathcal{E}}(X,A)$ may be regarded as an element $.f'$ of $\underline{Hom}_{\mathcal{E}}(X,Q)$ if and only if there exists a map $.p : X \longrightarrow C$ such that $.p \circ \{\_\} = .f \circ \{\_\} \circ a \pitchfork \Omega$.



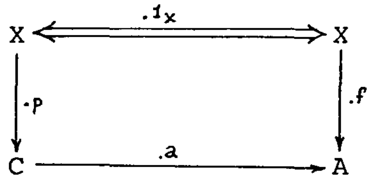
(3.23)

Now consider (3.24) in which the outside rectangle is a pullback.



(3.24)

The existence of .p, which implies the existence of $.\Gamma_p = .\langle 1_X, p\rangle$. implies

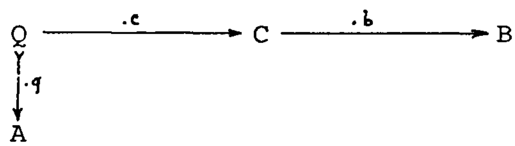


(3.25)

is a pullback. The unique factorization property of .q is apparent from (3.23).□

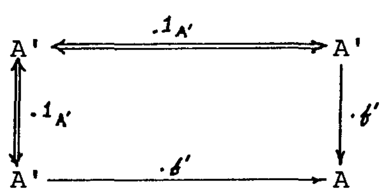
<u>PROPOSITION 11</u>: In any topos $\mathcal{E}$, $:\underline{Par}_{\mathcal{E}}(\_,B)$ is representable.

<u>Proof</u>: Let $.r : C \rightarrowtail A \otimes B$ be a relation. Let $.a = .r.\underline{pr}_1$ and $.b = .r.\underline{pr}_2$. Construct $.q : Q \rightarrowtail A$ from .a as in Proposition 10 above. Then



(3.26)

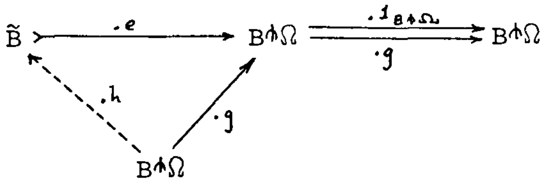
is a partial map from A to B. This operation of associating a partial map with a relation gives rise to a natural transformation $:\mu : :\underline{Rel}_{\mathcal{E}}(\_,B) \longrightarrow :\underline{Par}_{\mathcal{E}}(\_,B)$. The natural transformation $:\gamma\mu : :\underline{Par}_{\mathcal{E}}(\_,B) \longrightarrow :\underline{Par}_{\mathcal{E}}(\_,B)$, is the identity natural transformation since if $.f \in \underline{Par}_{\mathcal{E}}(A,B)$, (3.27) is a pullback with the factorization property required in Proposition 10.



(3.27)

Thus $:\mu\gamma : \underline{Rel}_{\mathcal{E}}(\_,B) \longrightarrow :\underline{Rel}_{\mathcal{E}}(\_,B)$ is an idempotent natural transformation. Since $:\underline{Rel}_{\mathcal{E}}(\_,B) \cong :\underline{Hom}_{\mathcal{E}}(\_,B{\uparrow}\Omega)$ it follows that $:\mu\gamma$ is induced by an idempotent endomorphism $.g : B{\uparrow}\Omega \longrightarrow B{\uparrow}\Omega$. This map $.g$ may also be described as the cartesian-adjoint of the characteristic function of $.\langle\{\_\},1_B\rangle : B \longrightarrow (B{\uparrow}\Omega)\otimes B$, i.e. $.g = .\underline{ch}(\langle\{\_\},1_B\rangle)^*$.

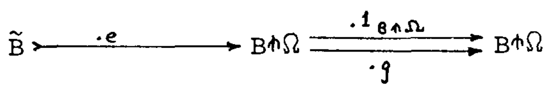
Let $\tilde{B} =_{df} \underline{Eqz}(1_{B\uparrow\Omega},g)$ and $.e =_{df} .\underline{eqz}(1_{B\uparrow\Omega},g)$. Consider the diagram (3.28).



(3.28)

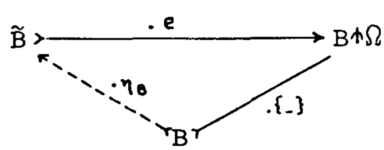
The existence of the limit morphism $.h$ in (3.28) says that the idempotent $.g$ splits. This splitting of $.g$ induces a splitting of $:\mu\gamma$ which shows that $:\underline{Par}_{\mathcal{E}}(\_,B) \cong :\underline{Hom}_{\mathcal{E}}(\_,\tilde{B})$. □

<u>PROPOSITION 12:</u> Let



(3.29)

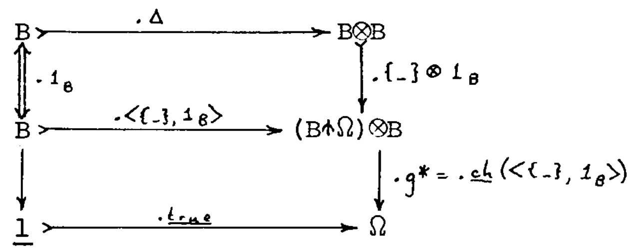
be as in the proof above of Proposition 11. Then $.\{\_\} : B \rightarrowtail B{\uparrow}\Omega$ factors uniquely through $.e$, i.e. there exists a unique map $.\eta_B : B \rightarrowtail \tilde{B}$ such that



(3.30)

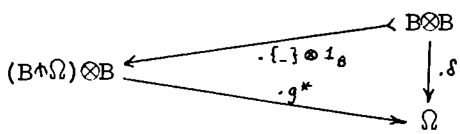
commutes.

<u>Proof:</u> Consider the diagram (3.31).



(3.31)

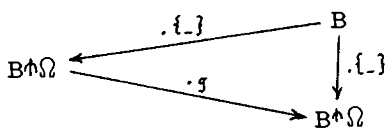
The bottom rectangle is a pullback by the definition of .g. The top rectangle is a pullback because $.\{\_\}\otimes 1_B$ is a monic. Thus $.\{\_\}\otimes 1_B \circ g^{*} = .\underline{ch}(\Delta) = .\delta$. Therefore



(3.32)

commutes

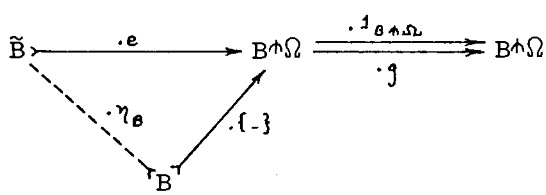
and by cartesian-adjointness



(3.33)

commutes.

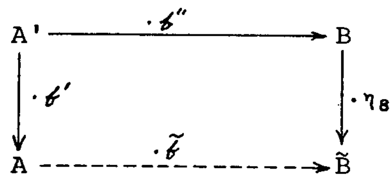
Now let $.\eta_B$ be the indicated limit morphism in the equalizer diagram (3.34).



(3.34)

□

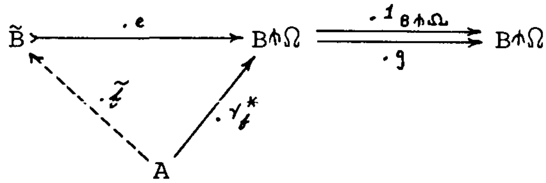
PROPOSITION 13: Given any partial map $.f \in \underline{Par}_{\mathcal{S}}(A,B)$ there exists a unique $.\tilde{f} : A \longrightarrow \tilde{B}$ such that



(3.35)

is a pullback.

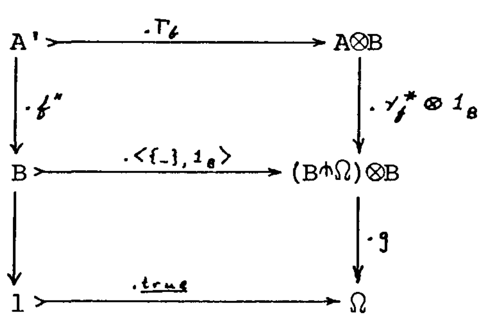
Proof: Let $.g$ be as in Proposition 12 above. Let $.\Gamma_f = .\langle f', f''\rangle : A' \rightarrowtail A\otimes B$ be the graph of $.f$. Let $.\gamma_f =_{df} .\underline{ch}(\Gamma_f) : A\otimes B \longrightarrow \Omega$ and let $.\gamma_f^* : A \longrightarrow B \pitchfork \Omega$ be the cartesian-adjoint of $.\gamma_f$. We want to define $.\tilde{f}$ as the indicated limit morphism in the equalizer diagram (3.36).



(3.36)

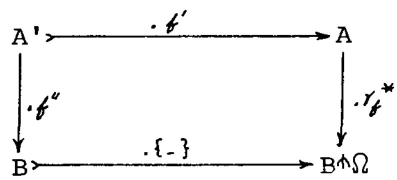
Lemma 1: $.\gamma_f^* = .\gamma_f^* \, g.$

Proof: By cartesian-adjointness it is enough to show that $.\gamma_f = .\gamma_f^* \otimes 1_B \circ g : A\otimes B \longrightarrow \Omega$. To do this it is sufficient to show that the outside of (3.37) is a pullback.



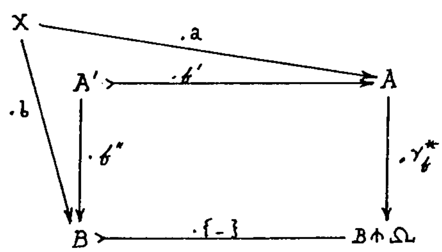
(3.37)

But the lower rectangle is a pullback by the definition of .g. Hence it is enough to show that the top rectangle is a pullback. To do this it is sufficient to show that (3.38) is a pullback.



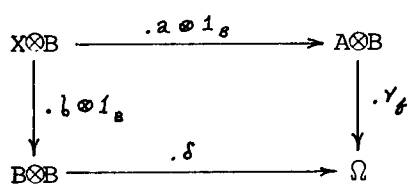
(3.38)

So suppose that we have two morphisms .a : X ⟶ A and .b : X ⟶ B such that (3.39) commutes.



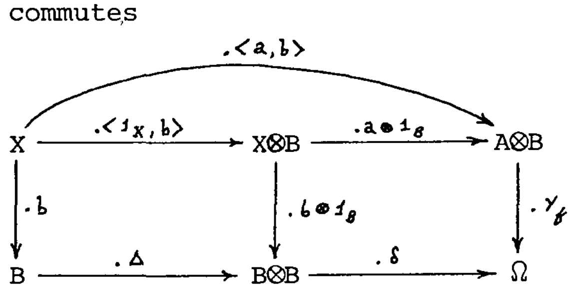
(3.39)

By cartesian-adjointness (3.40) commutes.



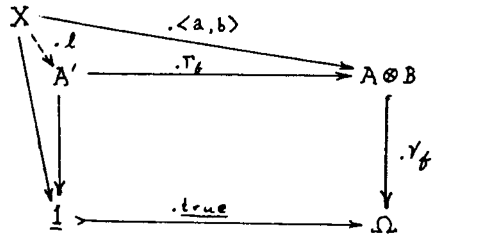
(3.40)

Therefore (3.41) commutes



(3.41)

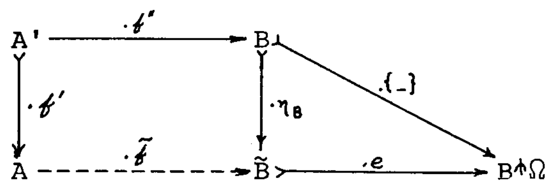
But $.\delta = .\underline{ch}(\Delta)$, so $.\Delta\delta$ factors through $.\underline{true}$. Hence $.\langle a,b\rangle\circ\gamma_{\delta}$ factors through $.\underline{true}$. We then take the limit morphism $.\ell : X \longrightarrow A'$ in (3.42) to be our limit morphism for (3.39).



(3.42)

Therefore Lemma 1 is proved.□

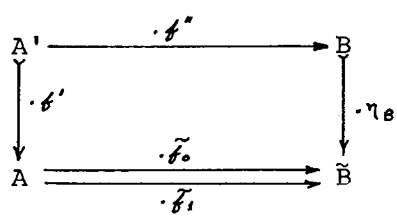
Now look at (3.43).



(3.43)

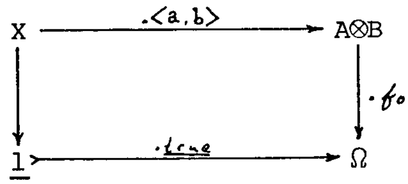
The outside of (3.43) is a pullback by the proof of Lemma 1 [see (3.38)] and the fact that $.\tilde{\delta}e = .\gamma^{*}_{\delta}$. But since .e is monic, this implies that the rectangle is a pullback. This proves the existence of $.\tilde{\delta}$.

To show that $.\tilde{\delta}$ is unique, suppose there exist two maps $.\tilde{\delta}_0$ and $\tilde{\delta}_1$ each making (3.44) into a pullback.



(3.44)

Let $.\delta_i = .(\tilde{\delta}_i \circ e)^{*} : A\otimes B \longrightarrow \Omega$ for i = 1,2. By symmetry and the fact that .e is monic it is enough to show that $.\delta_0 \leq .\delta_1$ in order to prove the uniqueness of $.\tilde{f}$. We apply the Remark at the end of III.2. Suppose (3.45) commutes

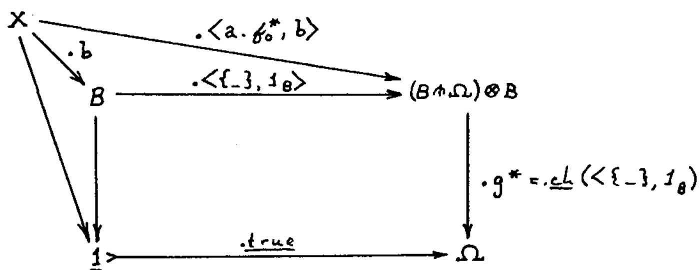


(3.45)

Now

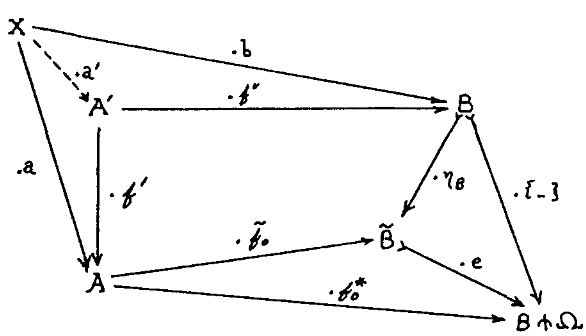
$$
\begin{aligned}
.\langle a,b\rangle f_0 &= .\langle 1_X,b\rangle.(a\otimes 1_B.f_0) = .\langle 1_X,b\rangle.(af_0^*)^* = \\
&= .\langle 1_X,b\rangle.(a\tilde{f}_0 e)^* = .\langle 1_X,b\rangle.(a\tilde{f}_0 eg)^* = \\
&= .\langle 1_X,b\rangle.(af_0^* g)^* = .\langle 1_X,b\rangle.a\otimes 1_B.f_0^*\otimes 1_B.g^* = \\
&= .\langle a,b\rangle.f_0^*\otimes 1_B.g^* = .\langle a.f_0^*,b\rangle.g^* = \\
&= .\langle a.f_0^*,b\rangle.\underline{ch}(\langle\{\_\},1_B\rangle).
\end{aligned} \tag{3.46}
$$

So we have the situation pictured in (3.47) which commutes.



(3.47)

Therefore $.a.f_0^* = .b.\{\_\}$. Now consider the diagram (3.48).



(3.48)

Since the inner quadrilateral with lower right hand corner $\tilde{B}$ is a pullback by (3.44) and since .e is monic, the inner quadrilateral with $B\uparrow\Omega$ in the lower right hand corner is a pullback also. Therefore there exists a map $.a' : X \longrightarrow A'$ such that $.a'\delta' = .a$ and $.a'\delta'' = .b$. From (3.44) we conclude that $.a\tilde{\delta}_1 = .b\eta_B$. Multiplying on the right by .e we get

$$.a\delta_1^* = .a\tilde{\delta}_1 e = .b\eta_B e = .b.\{\_\} \qquad (3.49)$$

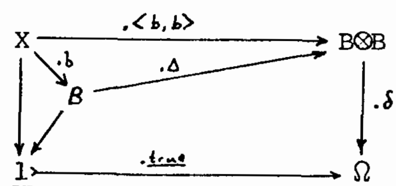
Looking at cartesian adjoints yields

$$.a\otimes 1_B.\delta_1 = .b\otimes 1_B.\delta \qquad (3.50)$$

which implies

$$.\langle a,b\rangle.\delta_1 = .\langle 1_X,b\rangle.a\otimes 1_B.\delta_1 = \langle 1_X,b\rangle.b\otimes 1_B.\delta - .\langle b,b\rangle.\delta \qquad (3.51)$$

But the outside of (3.52) obviously commutes.



(3.52)

Thus $.\delta_0 \leq .\delta_1$. By symmetry $.\delta_1 \leq .\delta_0$. Therefore $.\delta_0 = .\delta_1$ and by cartesian-adjointness $.\delta_0^* = .\delta_1^*$. Since .e is monic we have $.\tilde{\delta}_0 = .\tilde{\delta}_1$. This then completes the proof of Proposition 13.□

III.4 The fundamental theorem of topoi

Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be topoi. A functor $:L : \mathcal{E}_1 \longrightarrow \mathcal{E}_2$ is called a <u>logical-morphism</u> if

(i.) $:L$ is finitely bicontinuous,

(ii.) $:(\_\pitchfork\_)\circ :L \cong :(\_:L \pitchfork \_:L)$, and

(iii.) $\Omega^{\mathcal{E}}:L \cong \Omega^{\mathcal{E}_2}$.

<u>THEOREM I</u> (The fundamental theorem of topoi):

Let $\mathcal{E}$ be a topos and $B \in \underline{Obj}_{\mathcal{E}}$; then $\mathcal{E}\downarrow B$ is a topos. Furthermore, if $.f : B_1 \longrightarrow B_2$ is an arrow in $\mathcal{E}$ then the functor $:f^{\#} : \mathcal{E}\downarrow B_2 \longrightarrow \mathcal{E}\downarrow B_1$ (see III.1) is a logical morphism.

<u>Proof</u>: First we wish to show that $\mathcal{E}\downarrow B$ is cartesian-closed. So given objects $\begin{pmatrix} A \\ \downarrow .f \\ B \end{pmatrix}$ and $\begin{pmatrix} C \\ \downarrow .g \\ B \end{pmatrix}$ in $\mathcal{E}\downarrow B$, we wish to construct $\begin{pmatrix} A \\ \downarrow .f \\ B \end{pmatrix} \pitchfork \begin{pmatrix} C \\ \downarrow .g \\ B \end{pmatrix}$.

Let $.k : B \otimes A \longrightarrow \tilde{B}$ be the unique map making

$$\begin{array}{ccc} A & \xrightarrow{\quad .f \quad} & B \\ \Big\downarrow{\scriptstyle \langle f, 1_A \rangle} & & \Big\downarrow{\scriptstyle .\eta_B} \\ B \otimes A & \xrightarrow{\quad .k \quad} & \tilde{B} \end{array} \tag{4.1}$$

into a pullback.

Let $.k^* : B \longrightarrow A \pitchfork \tilde{B}$ be the cartesian-adjoint of $.k$. Let $.\overline{g}$ be the pushout of $.g$ along $.\eta_C$.

$$\begin{array}{ccc} C & \xrightarrow{\quad .g \quad} & B \\ \Big\downarrow{\scriptstyle .\eta_C} & & \Big\downarrow{\scriptstyle .\eta_B} \\ \tilde{C} & \xrightarrow{\quad .\overline{g} \quad} & \tilde{B} \end{array} \tag{4.2}$$

Then define $\left(\begin{array}{c}\bar{P}\\ \downarrow .b\\ B\end{array}\right)$ to be the pullback of $.A\pitchfork\bar{g}$ along $.k^*$.

$$\begin{array}{ccc} \bar{P} & \xrightarrow{\quad .a \quad} & A\pitchfork\tilde{C} \\ \Big\downarrow .b & & \Big\downarrow .A\pitchfork\bar{g} \\ B & \xrightarrow{\quad .k^* \quad} & A\pitchfork\tilde{B} \end{array} \tag{4.3}$$

Then define

$$\left(\begin{array}{c}A\\ \downarrow .f\\ B\end{array}\right) \pitchfork \left(\begin{array}{c}C\\ \downarrow .g\\ B\end{array}\right) =_{df} \left(\begin{array}{c}\bar{P}\\ \downarrow .b\\ B\end{array}\right) \tag{4.4}$$

Now suppose that we are given $\left(\begin{array}{c}X\\ \downarrow .h\\ B\end{array}\right)$. Consider the diagram (4.5) below.

$$\begin{array}{ccccc} \underline{Hom}_{\mathcal{E}\downarrow B}\left(\left(\begin{array}{c}X\\ \downarrow .h\\ B\end{array}\right),\left(\begin{array}{c}\bar{P}\\ \downarrow .b\\ B\end{array}\right)\right) & \longrightarrow & \underline{Hom}_{\mathcal{E}}(X,\bar{P}) & \xrightarrow{.\underline{Hom}_{\mathcal{E}}(X,a)} & \underline{Hom}_{\mathcal{E}}(X,A\pitchfork\tilde{C}) \cong \underline{Par}_{\mathcal{E}}(X\otimes A,C) \\ \Big\downarrow & & \Big\downarrow .\underline{Hom}_{\mathcal{E}}(X,b) & & \Big\downarrow .\underline{Hom}_{\mathcal{E}}(X,A\pitchfork\bar{g}) \quad \vdots \\ \underline{1} & \longrightarrow & \underline{Hom}_{\mathcal{E}}(X,B) & \xrightarrow{.\underline{Hom}_{\mathcal{E}}(X,k^*)} & \underline{Hom}_{\mathcal{E}}(X,A\pitchfork\tilde{B}) \cong \underline{Par}_{\mathcal{E}}(X\otimes A,B) \end{array} \tag{4.5}$$

The left-hand square is a pullback by the definition of $\mathcal{E}\downarrow B$ and the middle square is a pullback by the definition of $\bar{P}$ and the continuity of $:\underline{Hom}_{\mathcal{E}}(X,\_)$.

Note that on the bottom line of the diagram, the map $\underline{1} \longrightarrow \underline{Par}_{\mathcal{E}}(X\otimes A,B)$ corresponds to $X\otimes A \xrightarrow{.h\otimes 1_A} B\otimes A \xrightarrow{.k} \tilde{B}$, i.e. it is the element of $\underline{Par}_{\mathcal{E}}(X\otimes A,B)$ obtained from the pullback (4.6)

$$
\begin{array}{ccccc}
Q & \xrightarrow{\;.r\;} & A & \xrightarrow{\;.f\;} & B \\
\big\downarrow{\scriptstyle .\langle s, 1_A\rangle} & & \big\downarrow{\scriptstyle .\langle f, 1_A\rangle} & & \big\downarrow{\scriptstyle .\eta_B} \\
X \otimes A & \xrightarrow{\;.h \otimes 1_A\;} & B \otimes A & \xrightarrow{\;.k\;} & \tilde{B}
\end{array} \qquad (4.6)
$$

But this is determined by the pullback

$$
\begin{array}{ccc}
Q & \xrightarrow{\;.r\;} & A \\
\big\downarrow{\scriptstyle .s} & & \big\downarrow{\scriptstyle .f} \\
X & \xrightarrow{\;.h\;} & B
\end{array} \qquad (4.7)
$$

which is the product of $\begin{pmatrix} X \\ \downarrow .h \\ B \end{pmatrix}$ and $\begin{pmatrix} A \\ \downarrow .f \\ B \end{pmatrix}$ in $\mathcal{E}{\downarrow}B$. Thus

$$
\underline{Hom}_{\mathcal{E}}\left(\left(\begin{pmatrix} X \\ \downarrow .h \\ B \end{pmatrix}, \begin{pmatrix} A \\ \downarrow .f \\ B \end{pmatrix} \pitchfork \begin{pmatrix} C \\ \downarrow .\overline{g} \\ B \end{pmatrix}\right)\right) \cong \underline{Hom}_{\mathcal{E}\downarrow B}\left(\left(\begin{pmatrix} X \\ \downarrow .h \\ B \end{pmatrix} \otimes \begin{pmatrix} A \\ \downarrow .f \\ B \end{pmatrix}, \begin{pmatrix} C \\ \downarrow .\overline{g} \\ B \end{pmatrix}\right)\right) \qquad (4.8)
$$

and $\mathcal{E}{\downarrow}B$ is cartesian-closed.

Next we show that $\mathcal{E}{\downarrow}B$ has a subobject classifier, by Proposition 8 it is enough to show that the functor $:\underline{Sub}_{\mathcal{E}\downarrow B}$ is representable. Simply note that

$$
\begin{aligned}
\begin{pmatrix} A \\ \downarrow .f \\ B \end{pmatrix} :\underline{Sub}_{\mathcal{E}\downarrow B} &\cong A:\underline{Sub}_{\mathcal{E}} \cong A:\underline{Hom}_{\mathcal{E}}(\_,\Omega^{\mathcal{E}}) \cong \\
&\cong \underline{Hom}_{\mathcal{E}}\left(\left(\begin{pmatrix} A \\ \downarrow .f \\ B \end{pmatrix} :\Sigma_B, \Omega^{\mathcal{E}}\right)\right) \cong \underline{Hom}_{\mathcal{E}\downarrow B}\left(\left(\begin{pmatrix} A \\ \downarrow .f \\ B \end{pmatrix}, \Omega^{\mathcal{E}}:\chi_B\right)\right).
\end{aligned} \qquad (4.9)
$$

Hence $\Omega^{\mathcal{S}\downarrow B} = \Omega^{\mathcal{S}}:\chi_B$.

Next we wish to show that $:f^{\#}$ is a logical morphism. First by Proposition 7 we have that $:\Sigma_f \dashv :f^{\#} \dashv :\Pi_f$. Hence $:f^{\#}$ is finitely bicontinuous.

Next by using the technique employed in the proof of Proposition 7 we can see that it is sufficient to show that $:\chi_B$ preserves exponentiation. We will do this by showing that $(A\pitchfork C):\chi_B$ and $A:\chi_B \pitchfork C:\chi_B$ function as representing objects for the same functor. First we note that given any $\left(\begin{array}{c} D \\ \downarrow .d \\ B \end{array}\right)$ we have that

$$D\otimes A \cong \left(\left(\begin{array}{c} D \\ \downarrow .d \\ B \end{array}\right) \otimes A:\chi_B\right):\Sigma_B. \tag{4.10}$$

This is because both (4.11) and (4.12) are pullbacks.

$$\begin{array}{ccc} \left(\left(\begin{array}{c} D \\ \downarrow .d \\ B \end{array}\right) \otimes A:\chi_B\right):\Sigma_B & \xrightarrow{\cdot pr_2\otimes pr_3} & A\otimes B \\ \downarrow \cdot pr_1 & & \downarrow \cdot pr_2 \\ D & \xrightarrow{\cdot d} & B \end{array} \tag{4.11}$$

$$\begin{array}{ccc} D\otimes A & \xrightarrow{\cdot pr_2\otimes pr_1 \circ \cdot 1_A\otimes d} & A\otimes B \\ \downarrow \cdot pr_1 & & \downarrow \cdot pr_2 \\ D & \xrightarrow{\cdot d} & B \end{array} \tag{4.12}$$

Hence we have that

$$\underline{Hom}_{\mathcal{E}\downarrow B}\left(\left(\begin{array}{c} D \\ \big\downarrow .d \\ B \end{array}\right), (A\pitchfork C):\chi_B\right) \cong \underline{Hom}_{\mathcal{E}}\left(\left(\begin{array}{c} D \\ \big\downarrow .d \\ B \end{array}\right):\Sigma_B, A\pitchfork C\right) \cong$$

$$\begin{aligned}
&\cong \underline{Hom}_{\mathcal{E}}(D, A\pitchfork C) \cong \underline{Hom}_{\mathcal{E}}(D\otimes A, C) \cong \\
&\cong \underline{Hom}_{\mathcal{E}}\left(\left(\left(\begin{array}{c} D \\ \big\downarrow .d \\ B \end{array}\right)\otimes A:\chi_B\right):\Sigma_B, C\right) \cong \\
&\cong \underline{Hom}_{\mathcal{E}\downarrow B}\left(\left(\begin{array}{c} D \\ \big\downarrow .d \\ B \end{array}\right)\otimes A:\chi_B, C:\chi_B\right) \cong \\
&\cong \underline{Hom}_{\mathcal{E}\downarrow B}\left(\left(\begin{array}{c} D \\ \big\downarrow .d \\ B \end{array}\right), A:\chi_B\pitchfork C:\chi_B\right).
\end{aligned} \tag{4.13}$$

Finally the Ω condition on $:f^{\#}$ follows easily for

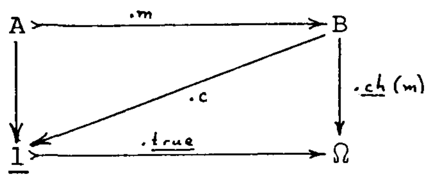
$$\Omega^{\mathcal{E}\downarrow B_2}:f^{\#} \cong \Omega^{\mathcal{E}}:\chi_{B_2}\circ f^{\#} \cong \Omega^{\mathcal{E}}:\chi_{B_1} \cong \Omega^{\mathcal{E}\downarrow B_1}. \square \tag{4.14}$$

III.5 Morphisms in a topos

Throughout this section let $\mathcal{E}$ be an elementary topos, $A,B,C,\ldots \in \underline{Obj}$ and $.a,.b,.c,\ldots \in \underline{Arr}_{\mathcal{E}}$.

PROPOSITION 14: Monomorphisms are equalizers in $\mathcal{E}$.

Proof: Let $A \overset{.m}{\rightarrowtail} B$ be a monomorphism in $\mathcal{E}$. Let $.c : B \longrightarrow \underline{1}$ be the unique such map. Consider the diagram



(5.1)

The outside part is a pullback and the whole thing commutes. Hence $A = \mathcal{E}qz(.c \circ \underline{true}, .\underline{ch}(m))$ and $.m = .\underline{eqz}(.c \circ \underline{true}, .\underline{ch}(m))$. □

<u>PROPOSITION 15</u>: Monomorphisms which are also epimorphisms are isomorphisms in $\mathcal{S}$.

<u>Proof</u>: Suppose that $A \xrightarrow{.a} B$ is both monic and epic. By Proposition 14 it is an equalizer, so suppose

$$A \xrightarrow{.a} B \underset{.r_2}{\overset{.r_1}{\rightrightarrows}} R \qquad (5.2)$$

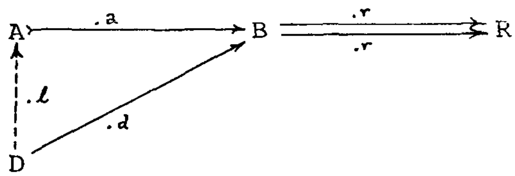
is the relavent equalizer diagram. As .a is epic we have that $.ar_1 = .ar_2$ if and only if $.r_1 = .r_2$. So we may denote the map $.r_1 = .r_2$ by .r and the relavent diagram now becomes

$$A \xrightarrow{.a} B \underset{.r}{\overset{.r}{\rightrightarrows}} R \qquad (5.3)$$

The fact that this is an equalizer diagram says that given any $.d : D \longrightarrow B$ such that

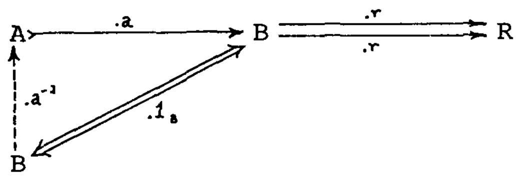
$$D \xrightarrow{.d} B \underset{.r}{\overset{.r}{\rightrightarrows}} R \qquad (5.4)$$

commutes, there is a unique limit morphism $.\ell : D \longrightarrow A$ such that



(5.5)

commutes. In particular, taking D = B and $.d = .1_B$ we define $.a^{-1}$ to be the limit morphism in (5.6)



(5.6)

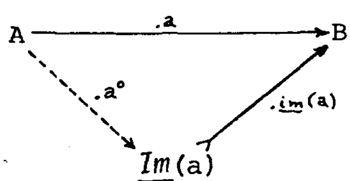
.□

If $A \xrightarrow{.a} B$ is an arrow in $\mathcal{E}$ then the equalizer of the cokernel pair of .a is called the <u>image of .a</u> and is denoted by $\underline{Im}(a) \xrightarrow{.\underline{im}(a)} B$.

<u>PROPOSITION 16</u>: For every arrow $A \xrightarrow{.a} B$, $\underline{Im}(a) \xrightarrow{.\underline{im}(a)} B$ is the smallest subobject of B through which .a factors.

<u>Proof</u>: We must show that

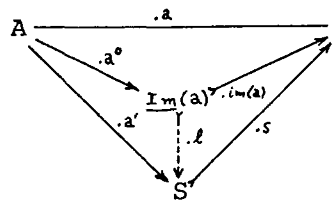
i.) there exists a unique morphism $A \xrightarrow{.a^{\circ}} \underline{Im}(a)$ such that



(5.7)

commutes and

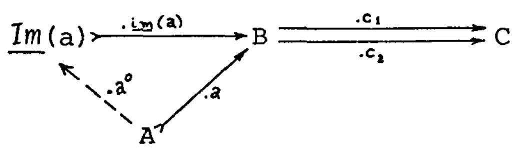
ii.) given any subobject $S \xrightarrow{.s} B$ through which .a factors (into $A \xrightarrow{.a'} S \longrightarrow B$ say) then there exists a unique morphism $\underline{Im}(a) \xrightarrow{.\ell} S$ making



(5.8)

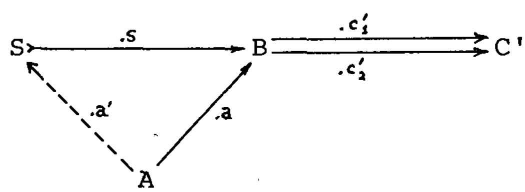
commute.

i.) is obvious, given the cokernal pair of .a is $B \underset{.c_2}{\overset{.c_1}{\rightrightarrows}} C$ we define .a to be the indicated limit morphism in the equalizer diagram (5.9) below:



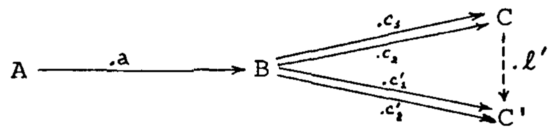
(5.9)

If $S \overset{.s}{\rightarrowtail} B$ is a subobject of B and $B \underset{.c_2'}{\overset{.c_1'}{\rightrightarrows}} C'$ is the cokernel pair of .s then .a factors through .s if and only if $.ac_1' = .ac_2'$. The only if part is easy to see, as $.ac_1' = .a'sc_1' = .a'sc_2' = .ac_2'$. Conversely, if $.ac_1' = .ac_2'$ then as .s is the equalizer of its cokernel pair the required map $A \xrightarrow{.a'} S$ is the limit morphism in the equalizer diagram (5.10) below.



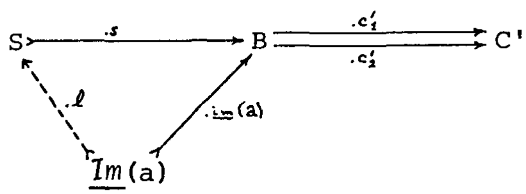
(5.10)

So if we are given an $S \overset{.s}{\rightarrowtail} B$ through which .a factors we have that $.ac_1' = .ac_2'$. Now note that $.\underline{im}(a) \circ c_1' = .\underline{im}(a) \circ c_2'$. To see this look at the diagram (5.11)



(5.11)

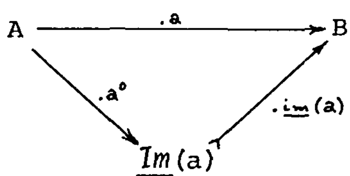
as $.ac'_1 = .ac'_2$ there exists a colimit morphism $.\ell' : C \longrightarrow C'$ making appropriate things commute. Hence $.\underline{im}(a)\circ c_1 = .\underline{im}(a)\circ c_2$ implies that $.\underline{im}(a)\circ c'_1 = .\underline{im}(a)\circ c_1\circ \ell' = .\underline{im}(a)\circ c_2\circ \ell' = .\underline{im}(a)\circ c'_2$. But this now implies the existence of a limit morphism $.\ell : \underline{Im}(a) \rightarrowtail S$ in the equalizer diagram (5.12) below.



(5.12)

□

PROPOSITION 17: Let .a and $.a^o$ be as in Proposition 16 above. Then

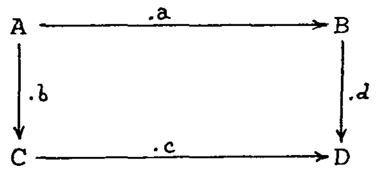


(5.13)

commutes, and

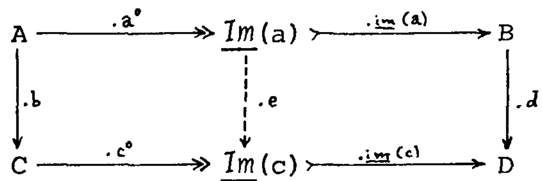
i.) $A \xrightarrow{.a^o} \underline{Im}(a)$ is epic, hence every morphism .a has an epic-monic factorization;

ii.) if $A \xrightarrow{.e} S \xrightarrow{.c} B$ is a epic-monic factorization of .a then there exists a unique isomorphism $.i : \underline{Im}(a) \Longrightarrow S$; and iii.) Given that



(5.14)

commutes, there exists a unique $.e : \underline{Im}(a) \longrightarrow \underline{Im}(c)$ such that

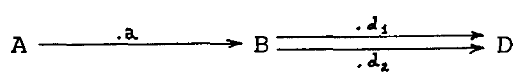


(5.15)

commutes.

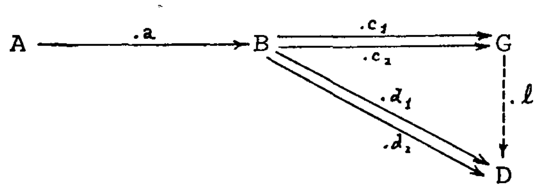
<u>Proof</u>: i.) First note that $\underline{Im}(a) = B$ if and only if .a is a epimorphism.

Let $B \underset{.c_2}{\overset{.c_1}{\rightrightarrows}} G$ be the cokernel pair of .a. If .a is epic then $.ac_1 = .ac_2$ implies that $.c_1 = .c_2$. Hence $.1_B$ equalizes the cokernel pair of .a. Conversely, suppose that $\underline{Im}(a) = B$. Look at



(5.16)

then we want $.ad_1 = .ad_2$ implies $.d_1 = .d_2$. Let $.\ell$ be the colimit morphism in the cokernel pair diagram (5.17) below where $.ad_1 = .ad_2$.



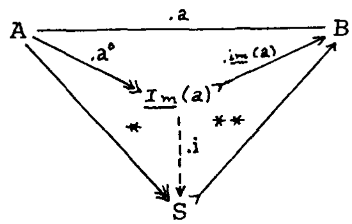
(5.17)

Then $.d_1 = .c_1\ell$ and $.d_2 = .c_2\ell$. So $\underline{Im}(a) = B$ implies $.c_1 = .c_2$

whence $.d_1 = .c_1\ell = .c_2\ell = .d_2$.

Now to show i.) it suffices to show that $\underline{Im}(a^0) = \underline{Im}(a)$. But this follows from Proposition 16.

ii.) Let $A \xrightarrow{.e} S \xrightarrow{.c} B$ be an epic-monic factorization of $A \xrightarrow{.a} B$. By Proposition 16 there is a morphism $\underline{Im}(a) \xrightarrow{.i} S$ such that

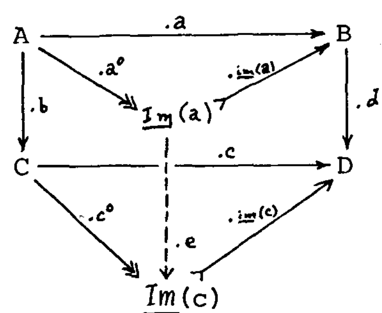


(5.18)

commutes.

As $.a^0$ and $.e$ are epic and * commutes, $.i$ is epic. As $.\underline{im}(a)$ and $.c$ are mono and ** commutes, $.i$ is mono. Hence by Proposition 15 $.i$ is an isomorphism.

iii.) Let $D \underset{.d_2}{\overset{.d_1}{\rightrightarrows}} F$ be the cokernel pair of $.c$. Consider the following diagram



(5.19)

$.\underline{im}(c)$ is the equalizer of $.d_1$ and $.d_2$. Thus $.b.c^0.\underline{im}(c)d_1 =$ $= .bc^0\underline{im}(c)d_2$, which implies

$$.a^0\underline{im}(a)dd_1 = .add_1 = .bcd_1 = .bc^0\underline{im}(c)d_1 = .bc^0\underline{im}(c)d_2 =$$
$$= .bcd_2 = .add_2 = .a^0\underline{im}(a)d_2. \tag{5.20}$$

As $.a^0$ is epic, this implies $.\underline{im}(a)dd_1 = .\underline{im}(a)dd_2$. Hence there is a limit morphism .e from $\underline{Im}(a)$ into the equalizer of $.d_1$ and $.d_2$, namely $\underline{Im}(c)$, and this limit morphism .e makes the diagram (5.19) commute.□

III.6 Heyting algebra valued functors and Boolean topoi

A Heyting algebra is a partially ordered set $\langle H,\leq\rangle$ such that for any two elements $a,b\in H$

i.) the greatest lower bound of a and b, denoted by $(a\wedge b)\in H$, exists;

ii.) the least upper bound of a and b, denoted by $(a\vee b)\in H$, exists;

iii.) the pseudo-complement of a relative to b, defined to be the greatest $x\in H$ such that $a\wedge x \leq b$ and denoted by $(a \Rightarrow b)\in H$, exists; and

iv.) a least element 0 exists.

Recalling Example 3 of II.1 where we established a 1-1 correspondence between partially-ordered sets and partially-ordered categories, we have the following propositions.

PROPOSITION 18: A partially-ordered set $\langle H,\leq\rangle$ is a Heyting algebra if and only if the corresponding partially-ordered category $\mathcal{O}(H,\leq)$ is cartesian-closed.

Proof: Take $a\wedge b = a\otimes b$, $a\vee b = a\oplus b$, $a\Rightarrow b = a^{\pitchfork}b$, and $0 = \underline{0}$. That $\wedge,\vee,\Rightarrow$, and 0 have the right properties is easily seen.□

Now suppose we have a topos $\mathcal{E}$ and $\underline{1}\in\underline{Obj}_{\mathcal{E}}$ is a terminal object. Let $A \overset{.a}{\rightarrowtail} \underline{1}$ and $B \overset{.b}{\rightarrowtail} \underline{1}$ be subobjects of $\underline{1}$. Using the fact that we can talk about images in $\mathcal{E}$ (Propositions 16 and 17) there is a natural way to define greatest lower bounds, least upper bounds, and pseudo-complements on the partially-ordered set $\underline{1}:\underline{Sub}$, namely take $.a \wedge b$ to be $.\underline{im}(a \otimes b)$, $.a \vee b$ to be $.\underline{im}(a \oplus b)$, and $.a \Rightarrow b$ to be $.\underline{im}(a \uparrow b)$. Under this interpretation $\underline{1}:\underline{Sub}_{\mathcal{E}}$ becomes a Heyting algebra with the map $\underline{0} \longrightarrow \underline{1}$ as least element. Thus we have established

PROPOSITION 19: If $\mathcal{E}$ is an elementary topos and $\underline{1}$ is a terminal object in $\mathcal{E}$ then $\underline{1}:\underline{Sub}_{\mathcal{E}}$ has a natural Heyting algebra structure.□

Next we note that if $B\in\underline{Obj}_{\mathcal{E}}$, then since $\begin{array}{c} B \\ \big\downarrow .1_B \\ B \end{array}$ is the terminal object in $\mathcal{E}\downarrow B$, $\mathcal{E}\downarrow B$ is a topos, and $B:\underline{Sub}_{\mathcal{E}} = (.1_B):\underline{Sub}_{\mathcal{E}\downarrow B}$ we have

PROPOSITION 20: If $\mathcal{E}$ is an elementary topos and $B\in\underline{Obj}_{\mathcal{E}}$ then $B:\underline{Sub}_{\mathcal{E}}$ has a natural Heyting algebra structure.□

As the operations of $B:\underline{Sub}_{\mathcal{E}}$ are preserved by pulling back, we may naturally view $:\underline{Sub}_{\mathcal{E}}$ as a Heyting algebra valued functor.

THEOREM II: If $\mathcal{E}$ is an elementary topos and Ω is the subobject classifier in $\mathcal{E}$ then Ω has a natural Heyting algebra structure in $\mathcal{E}$.

Proof: We must show that there exist mappings $\Omega\otimes\Omega \xrightarrow{.\wedge} \Omega$, $\Omega\otimes\Omega \xrightarrow{.\vee} \Omega$, $\Omega\otimes\Omega \xrightarrow{.\Rightarrow} \Omega$, and $\underline{1} \xrightarrow{.0} \Omega$, which satisfy the properties of the Heyting algebra operations. Let $.\underline{false} : \underline{1} \longrightarrow \Omega$ be the characteristic function of $\underline{0} \rightarrowtail \underline{1}$. Take $.0$ to be $.\underline{false}$. Let

(i.) $.\wedge$ be the characteristic function of $\underline{1} \overset{.true \circ \Delta}{\rightarrowtail} \Omega\otimes\Omega$

(ii.) $.\vee$ be the characteristic function of

$.\underline{im}(.\langle 1_\Omega, \underline{true}\rangle \oplus \langle \underline{true}, 1_\Omega\rangle)$

(iii.) $.\Rightarrow$ be the map $.\langle \underline{pr}_1, \wedge\rangle \circ \delta$ where $.\delta$ is the Kronnecker-delta.

It is now easy to directly verify that fact that these maps satisfy the required conditions.□

In a Heyting algebra $\langle H, \leq\rangle$ for any $a \in H$ we define $\sim a$ to be the element $a \Rightarrow 0$. We say that a Heyting algebra is a <u>Boolean algebra</u> if and only if for all $a \in H$, $\sim\sim a = a$.

An elementary topos $\mathcal{E}$ is said to be a <u>Boolean topos</u> if and only if its subobject classifier Ω is a Boolean algebra. We define the map $.\sim : \Omega \longrightarrow \Omega$ to be the characteristic function of $.\underline{false}$.

A Boolean topos is said to be a <u>two-valued topos</u> if $\{.\underline{true}, .\underline{false}\} = \underline{Hom}_{\mathcal{E}}(\underline{1}, \Omega)$.

The following proposition is fairly easy to prove.

<u>PROPOSITION 21</u>: Let $\mathcal{E}$ be a topos and let Ω be the subobject classifier in $\mathcal{E}$. Then the following are equivalent:

i.) $\mathcal{E}$ is a Boolean topos

ii.) $.\underline{true} \oplus \underline{false} : \underline{1} \oplus \underline{1} \longrightarrow \Omega$ is an isomorphism

iii.) $.\sim\sim = .1_\Omega : \Omega \longrightarrow \Omega$.□

CHAPTER IV

TOPOI AND SETS

Throughout this chapter $\mathcal{E} \subseteq V_1$ is a Boolean topos and all objects and morphisms are in $\mathcal{E}$ unless otherwise specified.

IV.1 The language $\mathcal{L}'(\mathcal{E})$ and its interpretation

We now describe a language $\mathcal{L}'(\mathcal{E})$ which is an extension by definitions of the language $\mathcal{L}(V_1)$ (i.e. $\mathcal{L}$ with constant symbols for elements of V_1 adjoined) and indicate how $\mathcal{L}'(\mathcal{E})$ is to be syntactically interpreted in $\mathcal{L}(V_1)$.

Variables: $\mathcal{L}'(\mathcal{E})$ has three sorts of variables:

1.) Object variables, denoted A,B,C,..., which are to be interpreted as ranging over $\mathit{Obj}_{\mathcal{E}}$;

2.) Arrow variables, denoted .a,.b,.c,..., which are to be interpreted as ranging over $\mathit{Arr}_{\mathcal{E}}$; and

3.) Typed variables of type A (for each $A \in \mathit{Obj}_{\mathcal{E}}$), denoted $x_1^A, x_2^A, \ldots$, which are to be interpreted as ranging over $\mathit{Par}_{\mathcal{E}}(\underline{1},A)$.

Terms: Object constants and object variables are object terms; arrow constants and arrow variables are arrow terms. Typed terms are defined inductively as follows:

a.) constants in $\mathit{Par}_{\mathcal{E}}(\underline{1},A)$ and typed variables of type A are terms of type A;

b.) if t^A is a term of type A and $.f \in \mathit{Hom}_{\mathcal{E}}(A,B)$ then $t^A.f$ is a term of type B; and c.) if t_1^A is a term of type A and t_2^B is a term of type B then (t_1^A, t_2^B) is a term of type $A \otimes B$. Typed terms are then interpreted in the obvious manner. (Note: we may occassionally abuse our language by referring to a term of type A as an "element of A".)

<u>Relations and atomic formulas</u>:

1.) There are two arrow relation symbols:

i.) the binary relation symbol $\_ =_{\underline{Arr}} \_$ and

ii.) the 3-ary relation symbol $\_ =_{\underline{Arr}} \_ \circ \_$, which are to be interpreted as the relations of equality and composition of arrows respectively. <u>Atomic formulas</u> are formed from arrow terms as follows:

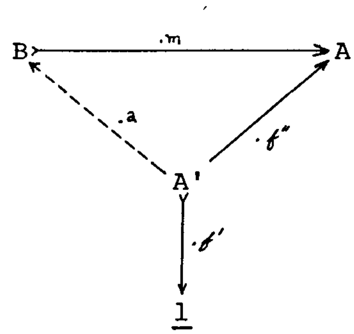
a.) if $.t_1$ and $.t_2$ are arrow terms, $.t_1 =_{\underline{Arr}} .t_1$ is an <u>atomic formula</u>; and

b.) it $.t_1, .t_2$, and $.t_3$ are arrow terms then $.t_1 =_{\underline{Arr}} .t_2 \circ .t_3$ is an <u>atomic formula</u>.

2.) There are relation symbols of type A for each $A \in \underline{Obj}_{\mathcal{E}}$ as follows:

i.) a binary relation symbol $\_ =_A \_$ and

ii.) for each subobject $.m : B \rightarrowtail A$, a unary predicate $\_ \in_m B$, which are to be interpreted as equality restricted to $\underline{Par}_{\mathcal{E}}(\underline{1}, A)$ and factorization of elements of $\underline{Par}_{\mathcal{E}}(\underline{1}, A)$ through $.m$ respectively. For example if $.f \in \underline{Par}_{\mathcal{E}}(\underline{1}, A)$ then $.f \in_m B$ holds if and only if there exists a map $.a : A' \longrightarrow A$ such that (1.1) commutes.



(1.1)

Atomic formulas are formed from typed terms as follows:

a.) if t_1^A and t_2^A are terms of type A then $t_1^A =_A t_2^A$ is an *atomic formula* and

b.) if t^A is a term of type A and $.m : B \rightarrowtail A$ is a subobject of A, then $t^A \in_m B$ is an *atomic formula*.

Formulas:

1.) If ϕ is an atomic formula then ϕ is a *formula*.

2.) If ϕ is a formula then $\sim\phi$ is a *formula*.

3.) If ϕ and ψ are formulas then $\phi \vee \psi$ is a *formula*.

4.) If ϕ is a formula and A is an object variable then $\exists A\phi$ is a *formula*.

5.) If ϕ is a formula and .a is an arrow variable then $\exists .a\phi$ is a *formula*.

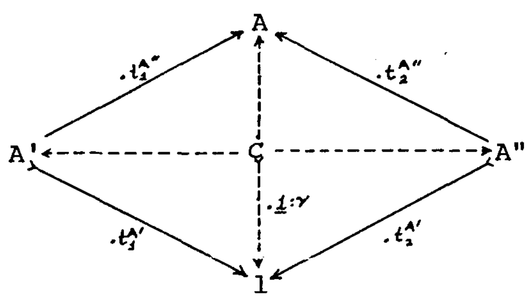
6.) If ϕ is a formula and x^A is a variable of type A then $\exists x^A\phi$ is a *formula*.

Sentences σ of $\mathcal{L}'(\mathcal{C})$ are given a truth value $[\![\sigma]\!]$ in $\mathcal{B}$, the completion of the Boolean algebra $\underline{1}:\mathit{Sub}_{\mathcal{C}}$, according to the following inductive scheme:

1.) $[\![.t_1 =_{\underline{Arr}} .t_2]\!] =_{df} \begin{cases} .1_{\underline{1}} \text{ if } .t_1 \text{ and } .t_2 \text{ are arrow constants denoting the same arrow in } \mathscr{E} \\ .0 : \underline{0} \rightarrowtail \underline{1} \text{ otherwise.} \end{cases}$

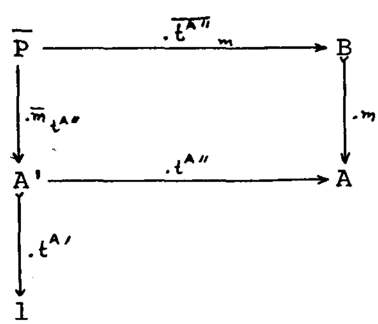
2.) $[\![.t_1 =_{\underline{Arr}} .t_2\ .t_3]\!] =_{df} \begin{cases} .1_{\underline{1}} \text{ if } .t_1 \text{ denotes the composition of the arrows denoted by } .t_2 \text{ and } .t_3 \\ .0 : \underline{0} \rightarrowtail \underline{1} \text{ otherwise.} \end{cases}$

3.) $[\![t_1^A =_A t_2^A]\!] =_{df} .\underline{1}{:}\gamma : C \rightarrowtail \underline{1}$ where $\langle C, {:}\gamma \rangle$ is the colimit of the solid part of (1.2)



(1.2)

4.) $[\![t^A \epsilon_m B]\!] =_{df} .\overline{m}_{t^{A''}} \circ t^{A'}$ i.e. the pullback of $.m : B \rightarrowtail A$ along $.t^{A''}$ composed with $.t^{A'}$ as in the left hand edge of (1.3)



(1.3)

5.) $[\![\sim\phi]\!] =_{df} \sim[\![\phi]\!]$

6.) $[\![\phi \vee \psi]\!] =_{df} [\![\phi]\!] \vee [\![\psi]\!]$

7.) $[\![\exists A\phi(A)]\!] =_{df} \sup_{\mathcal{B}}\{[\![\phi(T)]\!] \mid T\in \underline{Obj}_{\mathcal{S}}\}$

8.) $[\![\exists .a\phi(.a)]\!] =_{df} \sup_{\mathcal{B}}\{[\![\phi(.t)]\!] \mid .t\in \underline{Arr}_{\mathcal{S}}\}$

9.) $[\![\exists x^A\phi(x^A)]\!] =_{df} \sup_{\mathcal{B}}\{[\![\phi(t^A)]\!] \mid t^A\in \underline{Par}_{\mathcal{S}}(\underline{1},A)\}$

We say that a sentence σ is <u>true in the external interpretation</u> if $[\![\sigma]\!] = .1_{\underline{1}}$.

IV.2 <u>The language $\mathcal{L}(\mathcal{S})$: external and internal interpretations</u>.

Let $\mathcal{L}(\mathcal{S})$ be the fragment of $\mathcal{L}'(\mathcal{S})$ obtained by deleting object variables, arrow variables, and the relations $\_ =_{\underline{Arr}} \_$ and $\_ =_{\underline{Arr}} \_\circ\_$.

By the <u>external interpretation of $\mathcal{L}(\mathcal{S})$</u>, we shall mean its $\mathcal{B}$-valued interpretation as a fragment of $\mathcal{L}'(\mathcal{S})$ above.

The <u>internal interpretation of $\mathcal{L}(\mathcal{S})$</u>, which we define below, assigns to each formula $\phi(x_1^{A_1},\ldots,x_n^{A})$ of $\mathcal{L}(\mathcal{S})$, having exactly n free variables of types $A_1,\ldots,A_n$, a truth value $.\|\phi\|$ which is an arrow in $\mathcal{S}$, $.\|\phi\| : A_1\otimes\ldots\otimes A_n \longrightarrow \Omega$. $.\|\phi\|$ is defined inductively as follows:

1.) $.\|x_1^A =_A x_2^A\| =_{df} .\underline{ch}(\Delta) = .\delta : A\otimes A \longrightarrow \Omega$

2.) If $.m : B \rightarrowtail A$ is a subobject of A then

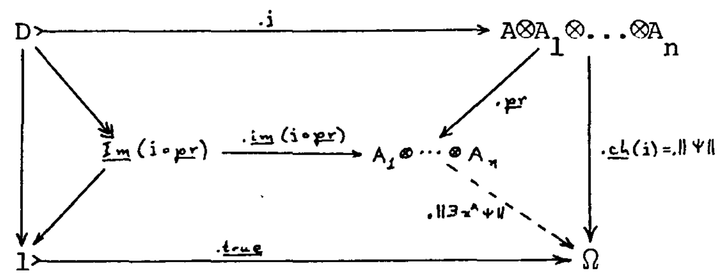
$.\|x^A \in_m B\| =_{df} .\underline{ch}(m) : A \longrightarrow \Omega$

3.) If $f : A \longrightarrow B$ is an arrow in $\mathcal{S}$ and $\psi(x^B)$ is a formula of $\mathcal{L}(\mathcal{S})$ with a free variable of type B then $.\|\psi(x^A.f)\| = .f.\|\psi(x^B)\|$.

4.) $.\|\sim\psi\| =_{df} .\|\psi\|.\sim$

5.) $.\|\psi_1 \vee \psi_2\| =_{df} .\langle \|\psi_1\|,\|\psi_2\|\rangle.\vee$

6.) Let $\psi(x^A, x_1^{A_1}, \ldots, x_n^{A})$ be a formula of $\mathcal{L}(\mathcal{S})$ with exactly n+1 free variables of types $A, A_1, \ldots, A_n$. Let $.i =_{df} .[\mid \|\psi\|]$ be the subobject of $A \otimes A_1 \otimes \ldots \otimes A_n$ whose characteristic function is $.\|\psi\|$. Let D denote the domain of .i. Let $.\underline{pr} : A\otimes A_1 \otimes \ldots \otimes A_n \longrightarrow A_1 \otimes \ldots \otimes A_n$ be the obvious natural projection map. Now define $.\|\exists x^A \; (x^A, x_1^{A_1}, \ldots, x_n^{A})\| =_{df} .\underline{ch}(\underline{im}(i \; \underline{pr}))$ where $.\underline{im}(i \circ \underline{pr})$ is the monic part of $.i \circ \underline{pr}$ as in III.5, Propositions III.16 and III.17. See (2.1)



(2.1)

A formula ϕ of $\mathcal{L}(\mathcal{S})$ is said to be <u>valid in the internal interpretation</u> if $.\|\phi\|$ factors through $.\underline{true} : \underline{1} \rightarrowtail \Omega$. We shall write $\|\phi\| = T$ to abbreviate "ϕ is valid in the internal interpretation".

The following three propositions, which are proved by Mitchell in [19], relate the external and internal interpretations of $\mathcal{L}(\mathcal{S})$.

Define $\exists! x\phi(x)$ by (2.2) in $\mathcal{L}$.

$$\exists! x\phi(x) \equiv_{df} \exists x(\phi(x) \wedge \forall y(\phi(y) \longleftrightarrow (x=y))) \qquad (2.2)$$

<u>PROPOSITION 1</u>: If $\|\forall x^A \exists! y^B \phi(x^A, y^B)\| = T$ then there exists a unique morphism $.g : A \longrightarrow B$ such that $\|\phi(x^A, x^A.g)\| = T$.□

PROPOSITION 2: If $\|\exists! y^B \phi(y^B)\| = \mathcal{T}$ then there exists a morphism $.g : \underline{1} \longrightarrow B$ such that $\|\forall z^{\underline{1}} \phi(z^{\underline{1}}.g)\| = \mathcal{T}$ where $z^{\underline{1}}$ is a variable of type $\underline{1}$ which does not occur free in ϕ.□

Let $\mathcal{WC}$ denote the following axiom:

$\mathcal{WC}$: Let $.a : A \longrightarrow \underline{1}$ and let $A \xrightarrow{.a^{\circ}} \underline{Im}(a) \xrightarrow{.im(a)} \underline{1}$ be the epic-monic factorization of $.a$ (see Proposition III.17). Then $.a^{\circ}$ splits.

We say that a sentence σ of $\mathcal{L}(\mathcal{E})$ is $\underline{\mathcal{E}\text{-absolute}}$ if $\|\sigma\| = \mathcal{T}$ if and only if σ is true in the external interpretation.

PROPOSITION 3: Let $\mathcal{E}$ be a Boolean topos satisfying $\mathcal{WC}$, $\phi(x_1^{A_1},\ldots,x_n^{A_n})$ be a formula of $\mathcal{L}(\mathcal{E})$ with exactly n free variables of types $A_1,\ldots A_n$, and $.p_i \in \underline{Par}_{\mathcal{E}}(\underline{1},A_i)$ for all $i=1,\ldots,n$. Then $\phi(p_1,\ldots,p_n)$ is $\mathcal{E}$-absolute.□

IV.3 Boolean ZFC topoi and two-valued ZFC topoi

A natural numbers object in a topos $\mathcal{E}$ is an object $N \in \underline{Obj}_{\mathcal{E}}$, together with maps

$$\underline{1} \xrightarrow{.o} N \xrightarrow{.s} N$$

such that for any object $X \in \underline{Obj}_{\mathcal{E}}$ together with maps

$$\underline{1} \xrightarrow{.x} X \xrightarrow{.k} X$$

there exists a unique map $.h : N \longrightarrow X$ such that (3.1) commutes.

$$\begin{array}{ccccc} \underline{1} & \xrightarrow{.o} & N & \xrightarrow{.s} & N \\ & \searrow^{.x} & \downarrow{.h} & & \downarrow{.h} \\ & & X & \xrightarrow{.k} & X \end{array} \qquad (3.1)$$

A topos $\mathcal{E}$ is said to satisfy the category form of the Axiom of Choice (abbreviated $\mathcal{CAC}$) if all coequalizers split in $\mathcal{E}$, or equivalently (by the dual of Proposition III.14) if all epimorphisms split in $\mathcal{E}$.

A topos $\mathcal{E}$ is said to satisfy the <u>category form of the Bounding Principle</u> (abbreviated $\mathcal{CBP}$) if for every formula ϕ of $\mathcal{L}'(\mathcal{E})$ with parameters in $\mathcal{E}$ and every $A \in \underline{Obj}_{\mathcal{E}}$, $\mathcal{E}$ satisfies (3.2).

$$\forall x^A \exists B \phi(x^A, B) \longrightarrow \exists C \forall x^A \exists B \exists .b(.b : B \longrightarrow C \wedge \phi(x^A, B)) \qquad (3.2)$$

A topos $\mathcal{E}$ is said to be a <u>Boolean ZFC topos</u> if

i.) $\mathcal{E}$ is Boolean

ii.) $\mathcal{E}$ has a natural numbers object

iii.) $\mathcal{E}$ satisfies $\mathcal{CAC}$, and

iv.) $\mathcal{E}$ satisfies $\mathcal{CBP}$.

A topos $\mathcal{E}$ is said to be a <u>two-valued ZFC topos</u> if it is both a two-valued topos and a Boolean ZFC topos.

In IV.4 and IV.5 we construct two functions:

i.) $\mathfrak{C}$: (Boolean-valued models of ZFC) $\longrightarrow$ (Boolean ZFC topoi)

ii.) $\mathfrak{M}$: (Boolean ZFC topoi) $\longrightarrow$ (Boolean-valued models of ZFC).

One should think of these constructions as taking place in ZFCI and referring to Boolean-valued models of ZFC which, though they are contained in V_ι, may not be sets relative to V_ι.

IV.4 <u>The construction of $\mathfrak{C}[M^{(\mathcal{B})}]$</u>

Let $\mathcal{B}$ be a complete Boolean algebra and let $M^{(\mathcal{B})}$ be a $\mathcal{B}$-valued model of ZFC. Then $\mathfrak{C}[M^{(\mathcal{B})}]$ is the category whose set of objects is the universe of $M^{(\mathcal{B})}$ and whose arrows are ordered triples $\langle x,f,y \rangle$ such that $[\![f : x \longrightarrow y]\!] = 1^{(\mathcal{B})}$, the maximal element of $\mathcal{B}$. Domains, codomains, composition of arrows, and identity arrows are defined in the obvious fashion.

THEOREM I: If $M^{(\mathcal{B})}$ is a $\mathcal{B}$-valued model of ZFC then $\mathfrak{C}[M^{(\mathcal{B})}]$ is a Boolean ZFC topos and $\Omega^{\mathfrak{C}[M^{(\mathcal{B})}]} \cong \mathcal{B}$.

Proof: $\mathfrak{C}[M^{(\mathcal{B})}]$ is easily seen to be finitely bicomplete. Products and coproducts are cartesian products and disjoint unions in $M^{(\mathcal{B})}$. If $f : x \longrightarrow y$ and $g : x \longrightarrow y$, the role of the equalizer of f and g is played by $\{z \mid z \in x \wedge f(z) = g(z)\}$ and its inclusion in x and the role of coequalizer is played by $y/\sim$ and the projection from y to $y/\sim$, where $\sim$ is the smallest equivalence relation on y such that for all $z \in x$, $f(z) \sim g(z)$.

Exponentiation in $\mathfrak{C}[M^{(\mathcal{B})}]$ is given by exponentiation in $M^{(\mathcal{B})}$, i.e. $x \Uparrow y$ is ${}^{x}y$. The counit of the cartesian adjunction is given by ordinary evaluation, i.e. it is the function $e : ({}^{x}y) \times x \longrightarrow y$ defined by $e(f,z) =_{df} f(z)$ for all $f \in {}^{x}y$ and $z \in x$.

$\mathcal{B}$ obviously plays the role of Ω with $.\underline{true} : \underline{1} \rightarrowtail \Omega$ being the function which sends $\emptyset$ to $1^{(\mathcal{B})}$.

ω is the natural numbers object of $\mathfrak{C}[M^{(\mathcal{B})}]$. The map $.0 : 1 \rightarrowtail \omega$ is the canonical inclusion and $.s : \omega \longrightarrow \omega$ is the ordinary successor function.

In order to check that epimorphisms split in $\mathfrak{C}[M^{(\mathcal{B})}]$ let $e : x \longrightarrow y$ be an epimorphism in $\mathfrak{C}[M^{(\mathcal{B})}]$. Then e is a surjection in $M^{(\mathcal{B})}$. By AC there exists a choice function $f : y \longrightarrow x$ picking out a single element of $e^{-1}(z)$ for every $z \in y$. f is the required section of e.

Finally, BP in $M^{(\mathcal{B})}$ translates exactly into CBP in $\mathfrak{C}[M^{(\mathcal{B})}]$.□

COROLLARY I.1: If M is a classical model of ZFC then $\mathfrak{C}[M]$ is a two-valued ZFC topos.□

IV.5 The construction of $\mathfrak{M}[\mathcal{S}]$

A partially ordered set $\langle t,\leq\rangle$ is called a Scott tree if

i.) for all $x \in t$ the set $\hat{x} =_{df} \{z \mid z \in x \wedge z \leq x\}$ is well-ordered by $\leq$;

ii.) t has a greatest element $\ast_t$;

iii.) $\langle t,\leq\rangle$ is $\leq$-well founded and $\langle t,\geq\rangle$ is $\geq$-well founded (see I.4); and

iv.) $\langle t,\leq\rangle$ has no order automorphisms other than the identity.

If $x \in t$ let $\ddagger x \ddagger$ denote the set of immediate predecessors of x in t. In any model of ZFC we can then define the set represented by the Scott tree $\langle t,\leq\rangle$ recursively by insisting that $\langle t,\leq\rangle$ represents a set z if and only if the elements of z are exactly the sets represented by the Scott trees of the form $\langle \hat{a},\leq\restriction\hat{a}\rangle$ where $a \in \ddagger \ast_t \ddagger$.

Now let $\mathcal{S}$ be a Boolean topos. If $.s : B \rightarrowtail A \otimes A$ is a relation from A to A we can interpret $a_1^A \leq_s a_2^A$ to mean $(a_1^A, a_2^A) \in_s B$. In this manner we may express the statement "$\langle A,\leq_s\rangle$ is a Scott tree" as a single sentence $\underline{ST}(A,s)$ of $\mathcal{L}(\mathcal{S})$ with parameters A and .s. We then can say that $\langle A,\leq_s\rangle$ is a Scott tree in $\mathcal{S}$ if $\|\underline{ST}(A,s)\| = T$. If $\mathcal{S}$ also satisfies WC then by Proposition 3, $\underline{ST}(A,s)$ is also externally true in $\mathcal{S}$. By clause ii.) in the definition of Scott tree and Proposition 2 there is a maximal element $.\ast_s : \underline{1} \rightarrowtail A$ in $\mathcal{S}$ and the subobject $.\ddagger\ast_s\ddagger$ of A is well defined. If a^A

is a term of type A, let s_a denote the Scott subtree of $\langle A,\leq_s\rangle$ obtained by restricting $\leq_s$ to $.[b^A|b^A \leq_s a^A]$, the pullback of .<u>true</u> along $.\|b^A \leq_s a^A\|$.

It is easy to see that if $\langle A,\leq_s\rangle$ and $\langle B,\leq_t\rangle$ are two Scott trees in $\mathcal{S}$ and if $f^{A\pitchfork B}$ is a term of type $A\pitchfork B$, we may express the statement "$f^{A\pitchfork B}$ is an order isomorphism from A onto B" in $\mathcal{L}(\mathcal{S})$ by employing the counit of the cartesian adjunction $:\underline{ev}$ to use $f^{A\pitchfork B}$ as a function from A to B.

Let $\mathcal{B}$ denote the completion of the Boolean algebra $\underline{Hom}_{\mathcal{S}}(\underline{1},\Omega)$. Then $\mathfrak{M}[\mathcal{S}]$ is the B-valued structure specified by the following:

i.) the universe of $\mathfrak{M}[\mathcal{S}]$ is the set $\{\langle A,\leq_s\rangle \,|\, \underline{ST}(A,s)\}$

ii.) $[\![\langle A,\leq_s\rangle = \langle B,\leq_t\rangle]\!]^{\mathfrak{M}[\mathcal{S}]} =_{df} \|\exists f^{A\pitchfork B}\ (f^{A\pitchfork B}$ is an order isomorphism from $\langle A,\leq_s\rangle$ onto $\langle B,\leq_t\rangle)\|$.

iii.) $[\![\langle A,\leq_s\rangle \in \langle B,\leq_t\rangle]\!]^{\mathfrak{M}[\mathcal{S}]} =_{df} \|\exists f^{A\pitchfork B}\ (f^{A\pitchfork B}$ is an order isomorphism from $\langle A,\leq_s\rangle$ onto t_b for some term b of type $\ddagger \divideontimes_t \ddagger)\|$.

If $\langle A,\leq_s\rangle$ is a Scott tree in $\mathcal{S}$ and B is an object in $\mathcal{S}$, we say that <u>$\langle A,\leq_s\rangle$ structures B</u> if $B \cong \ddagger \divideontimes_s \ddagger$ in $\mathcal{S}$. An object B is said to be <u>structured</u> if there exists some Scott tree $\langle A,\leq_s\rangle$ in $\mathcal{S}$ which structures B. Note that the category of structured objects in $\mathcal{S}$ and arrows between them is a full subcategory of $\mathcal{S}$.

<u>PROPOSITION 4</u>: If $\mathcal{S}$ satisfies CAC then every object of $\mathcal{S}$ is structured.

<u>Proof</u>: If $\mathcal{S}$ satisfies CAC then every object A can be well ordered in the sense that there exists a subobject $.w : B \rightarrowtail A\otimes A$ such for any two terms of type A, a_1^A and a_2^A, $a_1^A \leq_w a_2^A$ if and only if $\|(a_1^A,a_2^A) \in_w B\| = T$, and .w is a well ordering. The proof of this in $\mathcal{S}$ is just the imitation of the usual proof in sets.

Informally we let $\langle C,\leq_j\rangle$ be the set of strictly descending sequences in A ordered by the requirement that $\langle a_1^A,\ldots,a_n^A\rangle <_j \langle b_1^A,\ldots,b_m^A\rangle$ if and only if m<n and $a_i=b_i$ for all i=1,...,m.

More formally if 0_A denotes the $\leq_w$-minimal element of A we define

$$C =_{df} [(f^{A\uparrow A},a^A)\,|\,\forall a_1^A(a^A \leq_w a_1^A \rightarrow (f^{A\uparrow A},a^A).\underline{ev} = 0_A)\wedge \qquad (5.1)$$
$$\wedge\ \forall a_1^A \forall a_2^A(a_1^A <_w a_2^A <_w a^A \rightarrow (f^{A\uparrow A},a_2^A).\underline{ev} <_w (f^{A\uparrow A},a_1^A).\underline{ev})]$$

and $.j : Y \rightarrowtail C\otimes C$ is the subobject specified by

$$(f_1^{A\uparrow A},a_1^A) <_j (f_2^{A\uparrow A},a_2^A) \equiv_{df} a_2^A <_w a_1^A \wedge$$
$$\wedge\ \ a_3^A(a_3^A <_w a_2^A \rightarrow (f_1^{A\uparrow A},a_3^A).\underline{ev} = (f_1^{A\uparrow A},a_1^A).\underline{ev}) \qquad (5.2)$$

It is now fairly easy to check out that $\langle C,\leq_j\rangle$ is a Scott tree in $\mathcal{S}$ which structures A.□

<u>THEOREM II</u>: Let $\mathcal{S}$ be a Boolean ZFC topos and let $\mathcal{B}$ denote the completion of the Boolean algebra $\underline{Hom}_{\mathcal{S}}(\underline{1},\Omega)$. Then $\mathfrak{M}[\mathcal{S}]$ is a $\mathcal{B}$-valued model of ZFC.

<u>Proof</u>: It is sufficient to show that $\mathfrak{M}[\mathcal{S}]$ satisfies *AxExt*, *AxInf*, *AxPower*, *AxFound*, *Limited Aussonderung*, *BP*, and *AC*.

AxExt: If $[\![\langle A,\leq_s\rangle = \langle B,\leq_t\rangle]\!] = .\underline{true} = 1^{(\mathcal{B})}$ then if we let $x = \langle A,\leq_s\rangle$ and $y = \langle B,\leq_t\rangle$ we have that

$$\|\exists f^{A\uparrow B}(\text{f is an order isomorphism from x to y})\| = T \qquad (5.3)$$

This implies $\|\forall z(z\in x \leftrightarrow z\in y)]\!] = .\underline{true} = 1^{(\mathcal{B})}$. Conversely if $[\![x\neq y]\!] = .\underline{true}$ then since $\langle A,\leq_s\rangle$ is not order isomorphic to $\langle B,\leq_t\rangle$ there must exist either an $a\in\ddagger\ast_s\ddagger$ such that $s_a\notin y$ or a $b\in\ddagger\ast_t\ddagger$ such that $t_b\notin x$. Since *CAC* implies *WC*, Proposition 3 tells us that there exists a Scott tree z such that $\|\sim(z\in x \longleftrightarrow z\in y)\| = T$. For this z, $[\![z\in x \leftrightarrow z\in y]\!] = .\underline{false} = 0^{(B)}$.

AxInf: It is easy to check that in the proof of Proposition 4 the Scott tree $\langle C,\leq_j\rangle$ which structures the natural numbers object N is an ordinal of order type $\geq\omega$.

AxPower: Let $\langle A,\leq_s\rangle$ be a Scott tree. The power set of $\langle A,\leq_s\rangle$ will be the Scott tree $\langle B,\leq_t\rangle$ constructed as follows:

B will be a subobject of $(A\otimes(\ddagger\ast_s\ddagger\pitchfork\Omega))\oplus(\ddagger\ast_s\ddagger\pitchfork\Omega)\oplus\underline{1}$. We will let the single element of $\underline{1}$ be $\ast_t$. Let $\ddagger\ast_t\ddagger =_{df} \ddagger\ast_s\ddagger\pitchfork\Omega$. If a is an element of A and $a\neq\ddagger\ast_s\ddagger$, let $\bar{a}$ be the unique element of $\ddagger\ast_s\ddagger$ such that $a \leq_s \bar{a}$. An element (a,f) of $A\otimes(\ddagger\ast_s\ddagger\pitchfork\Omega)$ is in B if and only if $(f,\bar{a}).\underline{ev}$ factors through $.\underline{true}$. For such an (a,f) we will define $(a,f) \leq_t g$, where g is an element of $\ddagger\ast_s\ddagger\pitchfork\Omega$, if f=g. Finally if $b\in A$, define $(a,f) \leq_t (b,g)$ if f=g and $a \leq_s b$.

The part of the above definition which makes $\langle B,\leq_t\rangle$ the power set of $\langle A,\leq_s\rangle$ is that $\ddagger\ast_t\ddagger = \ddagger\ast_s\ddagger\pitchfork\Omega$. The rest of the construction simply makes sure $\langle B,\leq\rangle$ is a Scott tree with a maximal element $\ast_t$ and appropriate structure below $\ddagger\ast_t\ddagger$.

AxFound: This follows immediately from the well foundedness of Scott trees. We use the fact that *CAC* implies *WC* again in order to use Proposition 3 as we did in the proof of *AxExt*.

Limited Aussonderung: Limited formulas in $\mathcal{L}(\mathfrak{M}[\mathcal{E}])$ may be translated into formulas of $\mathcal{L}(\mathcal{E})$ by identifying limited variables with typed variables.

If $\phi(x)$ is a limited formula with one free variable x in $\mathcal{L}(\mathfrak{M}[\mathcal{S}])$ let $\phi^{\dagger}$ denote its translation into $\mathcal{L}(\mathcal{S})$. Then $\{x \in \langle A, \leq_s \rangle \mid \phi(x)\}$ is the subobject $.[x^A \mid \|\phi^{\dagger}(x^A)\|]$ of A together with the induced ordering. That this is the right set then follows from Proposition 3.

$\mathcal{BP}$: Follows immediately from $\mathcal{CBP}$ and Proposition 3.

$\mathcal{AC}$: As remarked in the proof of Proposition 4, we may use $\mathcal{CAC}$ to well-order any object in $\mathcal{S}$. It then follows from Proposition 3 that we can well order any set in $\mathfrak{M}[\mathcal{S}]$.

This completes the proof of Theorem II.□

COROLLARY II.1: If $\mathcal{S}$ is a two-valued ZFC topos then $\mathfrak{M}[\mathcal{S}]$ is a classical model of ZFC.□

THEOREM III: If $\mathcal{S}$ is a Boolean ZFC topos then $\mathfrak{C}[\mathfrak{M}[\mathcal{S}]]$ is equivalent as a category to $\mathcal{S}$.

Proof: We use the fact that by Proposition 4, every object of $\mathcal{S}$ is structured. Define the functor $:K : \mathfrak{C}[\mathfrak{M}[\mathcal{S}]] \longrightarrow \mathcal{S}$ as follows:

i.) If $\langle A, \leq_s \rangle$ is an object of $\mathfrak{C}[\mathfrak{M}[\mathcal{S}]]$ let $\langle A, \leq_s \rangle :K =_{df} \ddagger \ast_s \ddagger$.

ii.) If $f^{\dagger} = \langle x, f, y \rangle$ is a morphism in $\mathfrak{C}[\mathfrak{M}[\mathcal{S}]]$, where $x = \langle A, \leq_s \rangle$ and $y = \langle B, \leq_t \rangle$, then f is a set of ordered pairs in $\mathfrak{M}[\mathcal{S}]$, whence there is a formula $\psi(a,b)$ of $\mathcal{L}(\mathcal{S})$, with free variables a and b of types $\ddagger \ast_s \ddagger$ and $\ddagger \ast_t \ddagger$ respectively expressing "$\langle s_a, t_b \rangle \in f$" and such that $\|\forall a \exists! b \psi(a,b)\| = T$. Then by Proposition 1 we may define $.f^{\dagger}:K : \ddagger \ast_s \ddagger \longrightarrow \ddagger \ast_t \ddagger$ to be the unique morphism such that $\|\forall a \psi(a, a.f^{\dagger}:K)\| = T$.

$:K$ is obviously surjective on objects and faithful. It is also full for if $\langle A,\leq_s\rangle$ and $\langle B,\leq_t\rangle$ are Scott trees and $.h\in \underline{Hom}_{\mathcal{S}}(\ddagger\ast_s\ddagger,\ddagger\ast_t\ddagger)$ then if we let $f = \{\langle s_a,t_b\rangle | a\in\ddagger\ast_s\ddagger \wedge b = a.h\}$, we have that $.f^{\dagger} = \langle\ddagger\ast_s\ddagger,f,\ddagger\ast_t\ddagger\rangle$ is a morphism in $\mathfrak{C}[\mathfrak{M}[\mathcal{S}]]$ and $.f^{\dagger}:K = .h.$□

Mitchell in [19] proves

PROPOSITION 5: There is no way to define $\mathfrak{M}[\mathcal{S}]$ so that

i.) $\mathfrak{M}[\mathcal{S}]$ is separated and

ii.) there is an equivalence of categories $:K : \mathfrak{C}[\mathfrak{M}[\mathcal{S}]] \longrightarrow \mathcal{S}$ which is definable in $\mathcal{S}$.□

Hence we cannot insist that $\mathfrak{M}[\mathcal{S}]$ be separated. We say that two $\mathcal{B}$-valued models $M_1^{(\mathcal{B})}$ and $M_2^{(\mathcal{B})}$ are weakly isomorphic if $(M_1^{(\mathcal{B})})_s \cong$ $\cong (M_2^{(\mathcal{B})})_s$ (see I.5).

THEOREM IV: If $M^{(\mathcal{B})}$ is a $\mathcal{B}$-valued model of ZFC then $\mathfrak{M}[\mathfrak{C}[M^{(\mathcal{B})}]]$ is weakly isomorphic to $M^{(\mathcal{B})}$.

Proof: We need to define a function $f:M^{(\mathcal{B})} \longrightarrow \mathfrak{M}[\mathfrak{C}[M^{(\mathcal{B})}]]$ which preserves $\|\_=\_\|$ and $\|\_\in\_\|$ and such that for every y in $\mathfrak{M}[\mathfrak{C}[M^{(\mathcal{B})}]]$ there exists an x in $M^{(\mathcal{B})}$ such that $\|y=f(x)\| = 1^{(\mathcal{B})}$ in $\mathfrak{M}[\mathfrak{C}[M^{(\mathcal{B})}]]$. It is provable in ZFC that for every set x there exists a Scott tree $\langle t,\leq\rangle$ which represents x and that every Scott tree represents a set. If x is a set in $M^{(\mathcal{B})}$, let $\langle t,\leq\rangle$ be the Scott tree representing x and let f(x) be the Scott tree $\langle t,\leq_i\rangle$ in $\mathfrak{C}[M^{(\mathcal{B})}]$ where $i = \langle\{\langle a_1,a_2\rangle\in t\times t | a_1\leq a_2\},h,A\times A\rangle$ and h is the inclusion function. This f is the required function.□

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