THE CENTRAL LIMIT THEOREM
ON COMPACT TOPOLOGICAL SEMIGROUPS

by

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ABSTRACT

The purpose of this paper is to describe and discuss the Central Limit Theorem for compact topological semigroups. In the process of discussing the Central Limit Theorem, we exhibit some of the stable laws on a compact topological semigroup and briefly discuss the "Domain of Attraction" problem.

It turns out that the stable laws on a compact topological semigroup are the limit laws of the n-fold convolution of a probability measure on a compact topological semigroup. These limit laws are in fact the idempotent probability measures on the compact topological semigroup. These idempotent probability measures have as their support a completely simple semigroup and as a result we can identify the idempotent probability measures. Every completely simple semigroup can be written as a disjoint union of groups or as the Rees product of a group with two index sets. (These two structure theorems are actually the same.) As a simple compact semigroup is completely simple, the group components are compact. Thus we can write the idempotent probability measure as the product of the normalized Haar measure on the group with two probability measures defined on the two index sets. Finally, the limit law of the n-fold convolution of a probability measure can be determined by just considering the support of the probability measure and the structure of the simple compact semigroup. Some examples are then discussed using the above results.

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CHAPTER O - THE INTRODUCTION

The problem we want to consider is what happens when we take the limit of the n-fold convolution of a probability measure on a compact semigroup. This problem can be thought of to be a special case of the Central Limit Theorem. So we will give a quick survey of how these two problems are actually the same.

Suppose S is a measurable semigroup having a Borel field, where the binary operation is measurable. Furthermore suppose μ and ν are Borel measures on S. Then we can define an operation on the set of all measures on the Borel field of S called convolution and written $\mu * \nu$. It is defined as follows. Let E be a Borel set, then

$$\mu * \nu\ (E) = \mu \times \nu(\{(a,b) \mid ab \in E\}).$$

Notice that convolution is associative as the semigroup is associative. Also convolution is commutative if the semigroup is commutative.

Let (Ω, F, P) be a probability space. Then X is a random variable on S or a random element of the semigroup (S, B) if

$$X: \quad (\Omega, F, P) \to (S, B)$$

is measurable. As P is a probability measure, so also is PoX^{-1}, where

$$(PoX^{-1})\ (E) = P\ (\{\omega\varepsilon\Omega \mid X(\omega)\varepsilon E\}).$$

PoX^{-1} is said to be the probability distribution of the random variable

X. X_1 and X_2 are said to be identically distributed random variables over a semigroup $(S,\mathcal{B})$ if and only if $P \circ X_1^{-1} = P \circ X_2^{-1}$.

Let X and Y be two identically distributed variables on the semigroup $(S,\mathcal{B})$; that is, suppose they are both defined on the same set Ω. Then we define

$$XY: \Omega \to S$$

as follows: Let $\omega \in \Omega$, then

$$XY\ (\omega) = X(\omega) \cdot Y(\omega) \quad \text{and} \quad X(\omega) \cdot Y(\omega) \in S.$$

We can define another operation with X and Y, namely (X,Y). The Borel field on $S \times S$ is just the product Borel field $\mathcal{B} \times \mathcal{B}$.

Define

$$(X,Y): \ \Omega \to S \times S,$$

where $S \times S$ has the Borel field $\mathcal{B} \times \mathcal{B}$ as follows:

$$(X,Y)\ (\omega) = (X(\omega),Y(\omega)).$$

Two random variables X and Y are <u>independent</u> if and only if

$$P \circ (X,Y)^{-1} = (P \circ X^{-1}) \times (P \circ Y^{-1})$$

where this is the product measure. Let $X_1,\ldots,X_n$ be identically distributed random variables on the semigroup $(S,\mathcal{B})$. For arbitrary n, $(X_1,\ldots,X_n)$ is defined similarly as (X_1,X_2), and independence is also defined similarly, that is $\{X_1,\ldots,X_n\}$ is independent if and only if

$$P \circ (X_1,\ldots,X_n)^{-1} = (P \circ X_1^{-1}) \times\ldots\times(P \circ X_n^{-1}),$$

where we just have the product measures and $P \circ (X_1,\ldots,X_n)^{-1}$ is defined on the Borel field $\underbrace{\mathcal{B} \times \ldots \times \mathcal{B}}_{n \text{ times}}$. If the semigroup operation in S is written + (as with the real line under addition), then we will write X + Y for XY and talk about sums rather than (semigroups) products.

Now that we know what independence is, given random variables X, Y, we can ask what $P \circ (XY)^{-1}$ looks like. So suppose X,Y are independent random variables, then it can be easily shown that

$$P \circ (XY)^{-1} = (P \circ X^{-1}) * (P \circ Y^{-1}),$$

where $*$ is the convolution of $P \circ X^{-1}$ and $P \circ Y^{-1}$. Similarly if $X_1, \ldots, X_n$ are independent random variables then

$$P \circ (X_1 \ldots X_n)^{-1} = P \circ X_1^{-1} * \ldots * P \circ X_n^{-1} .$$

If $X_1, \ldots, X_n$ are independent and identically distributed random variables, then $P \circ X_1^{-1} = \ldots. = P \circ X_n^{-1}$ and hence

$$P \circ (X_1 \ldots X_n)^{-1} = \underbrace{P \circ X_1^{-1} * \ldots * P \circ X^{-1}}_{n \text{ times}}$$

$$= (P \circ X_1^{-1})^n.$$

Consider for the time being, the real line under addition. To start with we will consider random variables on the real line. So suppose that X is a random variable and for every $k > 0$ and $X_1, \ldots, X_k$ independent with the same distribution as X, there are constants $a_k > 0$, b_k such that the probability distribution of $X_1 + \ldots + X_k$ coincides with that of $a_k X + b_k$. Then X is said to have a <u>stable law</u>.

On the real line, if $X_1, X_2, \ldots$ is an independent and identically distributed sequence of random variables, $a_n > 0$, b_n real and suppose that the probability distribution of $\sum_{k=1}^{n} (a_k X_k + b_k)$ converges to some probability law, then the limit law is stable. (Here the convergence is on the weak topology.) This is known as the <u>Central Limit Theorem</u>. What do the stable laws look like on the real line? They are as follows: Let X have a stable law. Then either X has a normal distribution or there is a number α, $0 < \alpha < 2$, called the exponent of the law and constants $m_1 \geq 0$, $m_2 \geq 0$, β such that

$$\log f_X (u) = iu\beta + m_1 \int_0^\infty \left(e^{iux} - 1 - \frac{iux}{1+x^2}\right) \frac{dx}{x^{1+\alpha}}$$

$$+ m_2 \int_{-\infty}^{0} \left(e^{iux} - 1 - \frac{iux}{1+x^2}\right) \frac{dx}{|x|^{1+\alpha}} .$$

This is found in Leo Breiman's book, Probability, [22] (Theorem 9.27, page 200). The original work was done by Levy [24].

Now we want to restate the Central Limit Theorem in more mathematical language. Let Q be a probability function, $a_n > 0$, b_n real. Suppose

$$\lim_{n\to\infty} \underbrace{Q_n * \ldots * Q_n}_{n \text{ times}} \text{ exists,}$$

where $Q_n(E)=Q(a_nE+b_n)$, then the limit probability measure is a stable probability measure. These two statements are the same as we already have observed as for any random variable, $P \circ X^{-1}$ is a probability measure.

Now that we know that any limit of the n-fold convolution of a particular sequence of probability measures (in the reals when it exists) is a stable probability function, we want to turn the question around and ask the "Domain of Attraction" question. Given a stable probability measure R, the set of all possible probability functions Q such that there exists $a_n > 0$, b_n real such that

$$Q_n * \ldots * Q_n \overset{n}{\to} R,$$

where $Q_n(E) \overset{\forall E}{=} Q(a_nE + b_n)$, is called the Domain of Attraction for the probability measure R. The question is, given a stable R, what is the Domain of Attraction of R ? Consider the real case. One answer to the above problem is as follows given by Leo Breiman, [22], (Theorem 9.34, page 207). The original work here was done by Doeblin, [23]. "F(x) is in the domain of attraction of a stable law with exponent $\alpha < 2$ if and only if there are constants M^+, $M^- \geq 0$, $M^+ + M^- > 0$, such that as $y \to \infty$:

(i) $\text{Lim}\ \dfrac{F(-y)}{1 - F(y)} = \dfrac{M^-}{M^+}$

(ii) For every $\xi > 0$,

$$M^{+} > 0 \Rightarrow \lim \frac{1 - F(\xi y)}{1 - F(y)} = \frac{1}{\xi^{\alpha}},$$

$$M^{-} > 0 \Rightarrow \lim \frac{F(-\xi y)}{F(-y)} = \frac{1}{\xi^{\alpha}}."$$

Here is another special type of probability measure. Q is said to be an <u>idempotent probability measure</u> if and only if $Q * Q = Q$. This is a special case of Q being stable. (We just let $a_k = 1$ and $b_k = 0$.)

Now we want to consider the Central Limit Problem for any semigroup S. Suppose we have a probability measure Q, and a sequence $\{b_n\}_{n=1}^{\infty}$ in S. Assume also that the $\lim_{n \to \infty} Q_n * \ldots * Q_n$ exists where

$$Q_n(E) = Q(Eb_n) \qquad \forall E.$$

The question or central limit problem is, what kind of limits can occur?

The first thing we do is restrict ourselves to a compact semigroup S. Then all we need to show a limit is idempotent is that the b_n's commute with every element of S. In the case of commutative compact semigroups, this is immediate as everything commutes with everything. In the case of non-commutative semigroups, this is a little more difficult. So in some cases of non-commutative compact topological semigroups, we ask a simpler question. Let Q be a probability measure. Assume that

$$\lim_{n \to \infty} \underbrace{Q * \ldots * Q}_{n \text{ times}}$$

exists. What kinds of limits exists? The answer in this case is that the limit probability measure is idempotent. Thus we will be discussing idempotent probability measures in more detail. In this context, the "Domain of Attraction" question is, if R is an idempotent probability measure, what are the possible probability measures Q such that for some sequence b_n in S $\lim_{n \to \infty} Q_n * \ldots * Q_n = R$ where $Q_n(E) = Q(Eb_n)$ or in the non-commutative case, what are the possible probability measures Q such that $\lim_{n\to\infty} \underbrace{Q * * * Q}_{n \text{ times}} = R$.

One possible such Q is trivial and that is R itself. Whether or not there exist less trivial probability measures is a more difficult question.

In order to study these problems, we would therefore like to know what the idempotent (non-trivial) probability measures look like on a compact semigroup. However, we need only consider simple compact semigroups, as once the idempotent probability measures are characterized on simple compact semigroup, they can easily be extended to compact semigroups. If one considers a compact group, then the one idempotent (non-trivial) measure on that group is just the normalized Haar measure on the group. (The normalized Haar measure is just the two-sided invariant measure.) So the question arises as to whether this carries over to compact simple semigroups, and the answer is yes, in a sense. In order to make this idea precise, we will study the structure of semigroups, in particular completely simple semigroups, in detail. One fact that we use immediately is that a compact semigroup, which is not a group (algebraically), has at least one non-identity idempotent. J. G. Wendel makes use of this fact to prove the very useful result, that any non-trivial, non-identity idempotent probability measure on a compact group, (and such a thing exists on each compact group) is the normalized Haar measure. It is this result that enables us to get some results on the structure of non-identity idempotent probability measures on compact semigroup.

This paper will be divided into two major sections. The first section is divided into three subsections. In the first subsection we introduce some preliminary definitions and results.

The second subsection studies the structure of semigroups and introduces the notions of a simple semigroup and a completely simple semigroup. We observe that any completely simple semigroup with identity is actually a group. This result is used several times. As simple compact semigroups are completely simple we find that a simple compact group can be written as a union of its component subgroups. These groups in turn are compact. We also find that any compact semigroup has a completely simple subsemigroup, which is compact, and this is called the kernel of the compact semigroup.

In the second subsection, we study the structure of idempotent probability measures on compact semigroups. The first thing we note is that the support of an idempotent probability measure on a compact semigroup is always a completely simple subsemigroup. Conversely, if we have a compact semigroup and and idempotent probability measure defined on a simple compact subsemigroup, then this idempotent probability measure can be extended to the compact semigroup. So we will then consider only simple compact semigroups.

As mentioned above, the compact simple semigroup S can be written as a union of its compact group components. The number of group components is the same as the number of idempotents of S. If S has a countable number of idempotents, then every idempotent measure on S is a convex combination of the extended normalized Haar measures of the various group components. This completely characterized all the idempotent probability measures on any compact simple semigroup with at most a countable number of idempotent elements. However, there do exist completely simple semigroups with an uncountable number of idempotent elements. If we were to approach this problem of identifying all or at least some of the idempotent probability measures as we did in the case where the completely simple semigroup had only at most a countable number of idempotents, we immediately have a problem of addition. This brings us to the final section in part one.

In this section, we study the structure of completely simple semigroups in another way, so that we can eventually describe idempotent probability measures on a simple compact semigroup. This is done through the Rees-Decomposition Theorem. It turns out that every completely simple semigroup is a direct product of a group G with two arbitrary set X and Y and a particular type of multiplication. The multiplication is defined with the aid of a "sandwich matrix". If the completely simple semigroup is a compact semigroup, then G is a compact group and X,Y are compact Hausdorff spaces. Conversely, if G is a compact group and X,Y are compact Hausdorff spaces, then the Rees Product is a compact simple semigroup.

So now we have apparently different structure theorems for completely simple semigroups. We will compare these two approaches.

Now we are in a position to study the structure of possible idempotent probability measures on any compact simple semigroup, particularly those that have uncountable number of idempotent elements. And so some idempotent probability measures on G x X x Y, where G is a compact group and X, Y are compact Hausdorff spaces, are of the form $\mu \times \alpha \times \beta$, where μ is an idempotent probability measure on G and α, β

are probability measures on X, Y respectively. Notice that μ is the normalized Haar measure on G. In this way we can desribe some idempotent probability measures on a compact simple semigroup with an uncountable number of idempotents.

As this is discussed in all generality, we again compare this result, of the product measures, to the convex combination of the normalized Haar measures on the group components of a compact simple semigroup, where the semigroup only has at most a countable number of idempotents.

In the second major section of this paper, we discuss the convergence of probability measures on compact semigroups. As was noted earlier, this is in a sense a part of the Central Limit Problem. We are considering two types of limits involving a fixed probability measure on a compact semigroup. First we consider the sequence $\{\sum_{i=1}^{n} \frac{\mu^{(i)}}{n}\}_{n \in \omega}$ where μ is a probability measure on a compact semigroup S. This sequence, as we shall see, always converges, regardless of what probability measure we pick. The probability measure to which this sequence converges will be shown to be an idempotent probability measure. This idempotent probability measure can be thought of by our earlier discussion as a stable law.

Next and the other type of sequence we want to consider is the sequence $\{\mu^{(n)}\}_{n \in \omega}$, where μ is a probability measure on a compact semigroup. This sequence need not converge. However, if μ is a regular probability measure (and this is what we are assuming in this paper) on a compact semigroup, then we need only to study the properties of the kernel of the compact semigroup and its relationship to the probability measure μ to determine whether or not the sequence $\{\mu^{(n)}\}_{n=1}^{\infty}$ will converge. This result is due to M. Rosenblatt, [12] and we rely on the fact the every completely simple semigroup can be written as a Rees Product. After this we consider a few examples, first of which is that of a compact group itself. We see that M. Rosenblatt's result is an extension of a prior result on compact groups by Ulf Grenander.

The next set of examples considers a special type of completely simple semigroups. Recall that by the Rees decomposition of a completely simple semigroup, the semigroup is a product of a group and two arbitrary Hausdorff spaces. In these examples the Hausdorff spaces are arbitrary, but the groups that we will consider are cyclic groups and simple groups. Then if we are given a probability measure on a simple compact semigroup, such that the group in the Rees Product is either cyclic or simple, we can immediately determine whether or not the limit of the n-fold convolution of a probability measure on that semigroup exists or does not exist.

All through the paper we will have restricted ourselves to compact semigroups. In the final section we want to take a brief look at just topological semigroups and see what problems there exist there and why compactness is necessary. We find that we have to put stronger conditions on the semigroup and these conditions are immediately satisfied by compact semigroups. This way we can get a somewhat of an extension in the results of convergence of the n-fold convolution of a probability measure on a topological semigroup. So what we know for compact groups can somewhat be extended to compact semigroups and that can be extended somewhat to topological semigroups.

CHAPTER I - PRELIMINARIES

It will be necessary first to establish a few conventions and to give a few definitions. Let S be a set with an operation o: S × S → S so that for all s_1, s_2, s_3, ε S, $(s_1 o s_2)$ o s_3 = s_1 o $(s_2 o s_3)$. Then S will be called a semigroup. Henceforth o will just be denoted by juxtaposition, that is $s_1 o s_2 = s_1 s_2$, where s_1, s_2 ε S. Furthermore, if S has a topology defined on it, which is Hausdorff and such that o is jointly continuous from S × S to S, then S will be a topological Hausdorff semigroup, or simply a topological semigroup. (In this paper we will only consider those topological semigroups whose topologies are Hausdorff.) A topological semigroup S is a compact topological Hausdorff semigroup or a compact topological semigroup if the topology on S is both Hausdorff and compact and multiplication is continuous.

Now we need a few definitions from measure theory so that we can define an operation called convolution on a particular set of functions. Let (S,$\mathcal{T}$) be a topological semigroup with topology $\mathcal{T}$. Let $\mathcal{B}$ denote the Borel Field over (S,$\mathcal{T}$). (See any book on Measure Theory for the definition of a Borel Field, as an example, Halmos.) The Borel Field over (S,$\mathcal{T}$) is also represented by $\mathcal{B}$(S).

Let S be a compact topological Hausdorff semigroup, $\mathcal{B}$(S) the Borel Field on S. Denote the set of all regular probability measures on S by M(S), that is

$$M(S) = \{\ \nu: \mathcal{B}(S) \to [0,1]\ |\ \nu \text{ is a regular measure and } \nu(S) = 1\}.$$

We define a multiplication on (S) called convolution, which will be denoted by $*$, as follows: Let $\nu, \Gamma \in \mathcal{M}(S)$, then $\nu * \Gamma$: $\mathcal{B}(S) \to [0,1]$ with $\nu * \Gamma(\mathcal{B}) = \nu \times \Gamma \{ (x,y) \mid xy \in (\mathcal{B})\}$, where $\nu \times \Gamma$ is the product measure on $\mathcal{B}(S) \times \mathcal{B}(S)$. $\nu * \Gamma(S) = \nu \times \Gamma\ (S \times S) = \nu(S) \times \Gamma(S) = 1$. Hence $\nu * \Gamma \in \mathcal{M}(S)$. As S is a semigroup and hence associative, it follows immediately that for $\nu, \Gamma, \rho \in \mathcal{M}(S)$, $(\nu * \Gamma) * \rho = \nu * (\Gamma * \rho)$ and hence $\mathcal{M}(S)$ is a semigroup. We would further like to know whether or not $\mathcal{M}(S)$ is compact when S is compact. We need to know this to derive some properties of the probability measures on a compact semigroup and in particular on a compact group. If $(S,\mathcal{T})$ is a compact topological Hausdorff semigroup, then $\mathcal{M}(S)$ is also a compact topological Hausdorff semigroup with the weak star topology where the operation on $\mathcal{M}(S)$ is convolution (See Hille and Phillips [25] for details).

Now we are going to leave $\mathcal{M}(S)$ and consider some other properties elements of semigroups could have. Let S be a semigroup, $s \in S$, then s is said to be idempotent if $s \cdot s = s$. Numakura [2], has proven that if S is a compact topological semigroup, then S must possess idempotent elements. Furthermore if S contains an identity; that is, if there exists a $1 \in S$ such that $1 \cdot s = s \cdot 1 = s$ for all $s \in S$ and if S is not a group, then S has at least one other idempotent element besides the identity element. This claim will be proved later in the paper. Consider, however, a few examples. Define $\mathcal{R} = \{ x \mid -\infty < x < \infty \}$.

i) Let $S = [0,1]$ with the induced topology of $\mathcal{R}$ and ordinary multiplication for the semigoup operation. S is not a group since 1/2, for example, does not have an inverse in S. The inverse is 2 and $2 \notin [0,1]$. So S must have at least one idempotent element. One idempotent is 1 which is also the identity element and there is also a second idempotent element, namely 0.

ii) Let $S = \{ A \in (R)_2 \mid ||A|| \leq 1 \}$, where $(R)_2$ is the ring of all 2×2 matrices over the reals. The norm is the usual operator norm [22]. Give S the induced topology from $(R)_2$, which has the usual matrix topology. Then S with matrix multiplication is a compact topological semigroup. Since for $A, B \in S$, $||AB|| \leq ||A|| \cdot ||B|| \leq 1$, S is closed under multiplication. Therefore S must have idempotent elements. It has four idempotent elements, one of which is the identity element, and they are as follows:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

iii) Let $S^- = [1,\infty)$ with the induced topology from R and with ordinary multiplication. S^- has only one idempotent and that is 1, which is also the identity element of S^-. S^-, however, is not a group, but only a semigroup and the semigroup is not a compact topological semigroup. This shows us that in the absence of compactness there may not exist idempotents other than the identity. Now define

$S = S^- \cup \{ \infty \}$ with the one point compactification and multiplication as follows:

$$a \circ b = \begin{cases} ab \text{ if } a, b \in [1.\infty) \\ \infty \text{ if } a = \infty \text{ or } b = \infty \end{cases}$$

Now S is a compact semigroup which has two idempotents namely 1 and ∞. Notice that $[0,1]$ and $[1,\infty]$ are homeomorphic and semigroup isomorphic (if we define $1/\infty = 0$) by the transformation $f\colon [1,\infty] \to [0,1]$, $f(x) = 1/x$.

iv) Let $S^{-} = [a,\infty)$, where $a > 1$, with the induced topology from $\mathbb{R}$ and with ordinary multiplication. In this case S^{-} has no idempotent. Now define $S = S^{-} \cup \{\infty\}$ with the one point compactification and multiplication as follows:

$$d \circ b = \begin{cases} db \text{ if } d, b \in [a,\infty) \\ \infty \text{ if } d = \infty \text{ or } b = \infty \end{cases} .$$

Now S is a compact semigroup which has one idempotent namely ∞. Notice that $[0,1/a]$ and $[a,\infty]$ (where $a > 1$) are homeomorphic and semigroup isomorphic by the same transformation as in Example (iii). Observe that ∞ is not an identity as $\infty = b \cdot \infty = \infty \cdot b \neq b$ for any b, $1 < b < \infty$.

v) Let X be any non-empty compact space. Then X becomes a compact topological semigroup with the multiplication $(x,y) \to x$. Every element is an idempotent element. (Note that if X has two or more elements, no element is an identity element.

vi) Consider the following semigroups Zn:

$$Zn = \begin{cases} \{0,1,\ldots,n-1\} \text{ if n is not a prime number.} \\ \{1,\ldots,n-1\} \text{ if n is a prime number.} \end{cases}$$

Give Zn the discrete topology and the following multiplication. Let $a, b \in Zn$ then

$$a \circ b = \begin{cases} ab \text{ if } ab \in Zn \\ c \text{ if } ab \notin Zn \text{ and } c \in Zn \\ \quad \text{where } ab \equiv c (\text{mod } n). \end{cases}$$

From group and ring theory, we know that if n is a prime number, then Zn is a group and Zn in that case has only one idempotent element and that element is the identity 1. If n is not a prime number, then Zn is not a group and clearly

0 and 1 are idempotent elements of Zn and 1 is still the identity of Zn.

These examples, then illustrate that a compact semigroup always has an idempotent element and if one idempotent element is an identity, then the semigroup has at least one idempotent element. These examples are just examples of the above statement. A proof will not be given of the above claim.

We will see that while $M(S)$ is a semigroup, it is almost never a group (ie. there exists an element $\nu \in M(S)$ such that ν^{-1} does not exist). First of all, if S has just one element (then S is a group), we can immediately conclude that $M(S)$ also has just one element, namely unit mass on the one element of S and hence $M(S)$ is a group. On the other hand if S has more than one element, then $M(S)$ has more than one element. (eg. unit mass at each element of S).

Before we can go on, we must define the notion of an inverse element. Let S be a semigroup and a, b $\in$ S; then a, b are <u>inverse</u> of each other if and only if aba = a and bab = b. Now let G be a group and s, t $\in$ G; then s, t are <u>inverse</u>* of each other if and only if st = ts = e, where e is the identity of G. The question that arises is, as G is also a semigroup, are inverses and inverses* the same? First of all suppose s, t are inverse*, then that means st = ts = e. Therefore sts = se = s and tst = et = t. Therefore, inverse* are inverses. Now suppose s, t are inverses. That is, sts = s and tst = t. As G is a Group, s and t have inverse* say s' and t' respectively. Therefore stss' = ss' and hence we get ste = e and therfore st = e. Also tstt' = tt' and hence ts = e. We can thus conclude that st = e = ts and s and t are inverse*. Also from group theory we know that inverse are unique and hence s = t' and t = s'. Thus we can conclude that inverses and inverse* are the same in a group and the definition of inverse has been properly extended from groups to semigroups. It should be mentioned that in a semigroup, in= verses need not exist. An example of this is the semigroup $[a,\infty)$, $a > 1$ and using ordinary real number multiplication. Also if inverses do exist then not every element need have an inverse as in the example $[a,\infty]$ as defined in a previous example.

Now we want to go back and show that for any compact semigroup S, $\mathcal{M}(S)$ is almost never a compact group, but just a compact semigroup. First, however, we will consider a compact topological group G. J. G. Wendel [1] (Section 2) proved that a necessary and sufficient condition that $\nu \in \mathcal{M}(G)$ have an inverse is that ν be unit mass on an element $g \in G$. Now if a compact group G has two or more elements, then any probability measure ν on G which has positive mass on a finite number of points (more than one) does not have an inverse probability measure, as any probability measure μ that could be an inverse to ν , would be nothing more than unit mass on any element of G. This however, is impossible as ν then would be the inverse of μ and and hence ν is the unit mass of an element of G, but we assumed that ν had positive mass on a finite number of elements (more that one). Such measures as exist because G has two or more elements. Let g_1, g_2 be two elements of G. Define μ as follows: $\mu: G \to G$, $\mu(g_1) = \mu(g_2) = 1/2$, $\mu(g) = 0$ for all $g \in G \sim \{g_1, g_2\}$. Hence $\mu \in M(G)$, μ does not have an inverse in $M(G)$ and hence $M(G)$ is not a group. So in the case where G is a compact group, $M(G)$ is not a group but just a semigroup. Now consider a compact topological semigroup S. If S has an identity e, then unit mass on e is the identity in $M(S)$. On the other hand, if S does not have an identity, then $M(S)$ does not have an identity. J. G. Wendel proved that a necessary and sufficient condition that $\mu \in M(G)$ have an inverse is that μ be unit mass on an element $g \in G$. This result is also true for semigroups. Now we want to show that if S does not have an identity, then $M(S)$ does not have an identity. For suppose ν, $\mu \in M(S)$ are inverses of each other, then by Wendel's result, ν, μ are unit masses of some element s, t of S such that s, t are inverses of each other. If, furthermore η is an identity of $M(S)$, then $\nu * \mu = \mu * \nu = \eta$. As η is an identity of $M(S)$ and as η is itself an inverse of η, there exists an element $e \in S$ such that η is unit mass on e. But as $\nu * \mu = \eta$ we can only conclude that st = e and similarly ts = e. Furthermore e is an identity in S. (It should be obvious that if a semigroup S has an identity, then it is unique: For suppose e and e' are identities of S, then e = ee' = e'e = e' and therefore e = e'.) But we assumed that S had no identity and therefore $\mathcal{M}(S)$ has no identity. Therefore $\mathcal{M}(S)$ is only a semigroup and not a group. Observe, incidentally, that S can be imbedded in $\mathcal{M}(S)$ by that mapping that takes each element s to the probability measure that has unit mass on s in $\mathcal{M}(S)$. This mapping is a one-to-one mapping.

Let G be a group with a topology $\mathcal{T}$ and the topology is Hausdorff. Then G is a <u>topological group</u> if the multiplication is jointly continuous and inversion is continuous. G is a compact <u>topological group</u> if G is a topological group and the topology is compact. From now on, instead of saying compact topological semigroup (group), we will just say compact semigroup (group) meaning compact topological semigroup (group). The topologies are always Hausdorff.

Now consider a compact group G. $\mathcal{M}$ (G) is therefore a compact semigroup, under convolution, with identity (the identity being unit mass on the identity of G). Hence there exists at least one other idempotent measure e in $\mathcal{M}(G)$. What is e? In this paper, J. G. Wendel [1] proved that there is exactly one non-zero, non-identity probability measure on G and this measure is the normalized Haar measure on G. Hence all the idempotent measures on any compact group G are known. For a detailed study of Haar measure, see Hewitt and Ross [19]. Here are brief description and some properties of Haar measures: To start with one considers $C_{oo}(G)$. where G is a group and $C_{oo}(G)$ is the set of all complex-valued continuous functions f on G such that there exists a compact subset F of G (depending on f) such that f(x) = 0 for all $x \varepsilon G-F$. In our case, as G is a compact group, $C_{oo}(G)$ is just the set of all complex-valued continuous functions f on G. On this set $C_{oo}(G)$, the Haar Integral is defined and from this integral we get a set function λ, which has the following properties:

(i) $0 < \lambda(U)$ for all non-empty open sets U in G.

(ii) $\lambda(U) < \infty$ for all open set U in G.

(iii) $\lambda(aB) = \lambda(B) = \lambda(Bc)$ for all $B \varepsilon G$ and for all $a, c \varepsilon G$.

If G is a compact group, then $\lambda(G) < \infty$. Let ν be the measure generated by λ (in the sense of Caratheodory). If $\nu(G) = b \neq 1$, then ν will be normalized by defining $\mu = 1/b \cdot \nu$ and μ is called the normalized Haar measure on a compact group G and its properties are:

(i) $0 \leq \mu(B) \leq 1$ for all Borel sets B.

(ii) $\mu(G) = 1$.

(iii) $\mu(aB) = \mu(Bv) = \mu(B)$ for all Borel sets B and for all $a, v \varepsilon G$.

(iv) μ is the unique measure on the compact group G having the above three properties.

There is one other question which will be given consideration here and its importance will become apparent later. Also this result will be able to be transferred to locally compact groups. Suppose that S is a group with a topology such that multiplication is jointly continuous. Then does it follow that S is a topological group; that is, that the inversion map $x \to x^{-1}$, $x \in S$, is also continuous? If S is a compact topological semigroup (that is, S is a topological semigroup and the topology is compact) and S is a group, then the inversion map is automatically continuous and hence S is a compact topological group. More generally, if the semigroup of a group S with a topology is a locally compact topological semigroup, then S is a locally compact topological group. To prove this it is sufficient to show that if U is a neighborhood of the identity e, then there exists a neighborhood V of e such that $V^{-1} = \{v^{-1} \mid v \in V\} \subset U$. Since S is a locally compact semigroup we may let $\{V_\alpha\}_\alpha$ be the collection of all compact neighborhoods of e (this collection is not empty since S is locally compact). If for one α, $V_\alpha^{-1} \subset U$, then inversion is continuous. So suppose that for every V_α, $V_\alpha^{-1} \not\subset U$, then $V_\alpha^{-1} \sim U \neq \phi$ and $\bigcap_\alpha V_\alpha^{-1} \sim U \neq \phi$ since V_α^{-1} is compact. But $\bigcap_\alpha V_\alpha^{-1} \sim U \subset {}_\alpha V_\alpha^{-1} = \{e\}$ implies that $e \notin U$ which is contradiction. Hence for every neighborhood U of e, there exists a neighborhood V of e such that $V^{-1} \subset U$ and therefore the inversion map is continuous. Therefore, S is a topological group and hence a compact (locally compact) topological group. However, if G is a topological group and the topology is not locally compact, then inversion need not be continuous. For example, consider the following example:

Let $G = R$, the real line as an additive abelian group. Let T be the topology of G whose base is $\{[a,b) \mid -\infty < a \leq x < b < \infty\}$. Define g_1: $R \times R \to R$ as follows:

$$g_1\ (x,y) = x + y.$$

Define g_2: $R \to R$ as follows:

$$g_2\ (x) = -x.$$

Then g_1 is jointly continuous everywhere on $R \times R$(since if [a,b) is a neighborhood of a, then [a,b/2) is also a neighborhood of a), but g_2 is not continuos: Consider the point $0 \in \mathcal{R}$. [0,b) with b > 0 is a neighborhood of 0 and there is no neighborhood V of 0 such that $-V \subset [0,b)$. Therefore G is a topological semigroup and G is a group, but G is not a topological group. Observe that the topology on G is not locally compact.

Following is a slightly different example, whose importance is realized when we discuss how a compact topological semigroup can be written as a union of compact groups:

If H is a locally compact, non-compact, topological group, then H^o, the one-point compactification of H with the point at infinity as a zero, is a compact semitopological semigroup, but not a topological semigroup [10] (1.1 (a) page 146). (If S is a semigroup with topology $\mathcal{T}$ and the multiplication is continuous in each variable, then the semigroup is called a <u>semitopological semigroup</u>. Hence every topological semigroup is a semitopological semigroup.) So consider the set $S = \{x \mid 0 < x < \infty\}$ with the induced topology of the real numbers and having normal multiplication as the operation. Then S is a locally compact, non-compact, topological group. However by the last result, if we one-point compactify it, S^o is a semitopological semigroup but not a topological semigroup. Thus multiplication is not even jointly continuous, just continuous in each variable.

Consider the Semitopological group $\mathcal{R}$ of additive numbers with the topology $\mathcal{T}$ defined as follows: $U \in \mathcal{T}$ if and only if there is a countable set C (which may be empty) such that $U \cup C$ is open in the usual topology of R. R has the following properties [22] (Theorem 9.27 page 200):

a) Although a semitopological group, R is not a topological semigroup.
b) R is not completely regular, whereas any topological group is completely regular.
c) Inversion in R is continuous.
d) The only compact sets in R are finite sets.
e) R , the one-point compactification of R with the operation extended by
$\alpha + \infty = \infty + \alpha = \infty$
is a quasi-compact semitopological semigroup.

(A topological space is quasi-compact if and only if each given open covering of it contains a finite open subcover.) Notice that inversion is continuous, whereas multiplication is only continuous in each variable separately. Now we go back to compact groups and idempotent probability measures on them.

Consider a compact group G, J. G. Wendel [1] showed using integrals that a compact group G has a unique non-zero idempotent probability measure, other than unit mass at the identiy. What about non-compact groups? Do they have at least one one other idempotent probability measure, other than the above identity measure? For a complete separable metric group G, K. R. Parthasarathy [18] has proved (without reserting to integrals) if the group has one or more non-identity idempotent probability measure, then the idempotent measures are actually the normalized Haar measure of some compact subgroup (that is assigning zero mass to sets disjoint from this subgroup). There is another result on another class of topological groups. It is a special class of locally compact groups and it was proven by Ulf Grenander [17]. The result is as follows: An idempotent probability measure on a commutative, locally compact group has all of its mass concentrated on a compact subgroup.

Notice that for the conclusion of the last two results, we had to assume that idempotent probability measures existed and the conclusion was, that of the support of the measure was a certain compact subgroup of the original group. In considering the problem of what idempotent measures look like on non-compact topological groups, we immediately lose the existence of a finite Haar measure. However if an idempotent measure does exist, we know that it is the normalized Haar measure on some compact subgroup of the group and hence by the uniqueness of the Haar measure we know that that particular subgroup cannot be the support of another idempotent probability measure.

We will be using the foregoing results on various groups, to determine, if possible, the idempotent probability measures on semigroups. In order to see how this is done, we now will discuss some of the properties of semigroups and how they can be written.

CHAPTER II - SOME RESULTS ON THE STRUCTURE OF COMPACT SEMIGROUPS

We have briefly considered compact topological groups. We have observed that there is one and only one non-zero, non-identity idempotent probability measure on a compact topological group and that this measure is the normalized Haar measure. Can we make use of this in studying what idempotent probability measures on compact topological semigroups look like? We also know that if G is a group and a compact topological semigroup, then G is a compact topological group. So now we want to consider the structure of compact topological semigroups. This turns out to be the union of compact groups. Hence for some compact semigroups we will be able to use the normalized Haar measures on the compact group, which are the components of this semigroup, to determine what the idempotent measures on that particular compact semigroup look like. In this section there will also be a few examples to illustrate some of the results discussed. For all of this we have to introduce some concepts on the structure of semigroups.

Let S be a semigroup. R is a right ideal of S if $RS \subset R$; L is a left ideal of S if $SL \subset L$; M is a two-sided ideal of S if $MS \cup SM \subset M$. F is a subsemigroup of S if $FF \subset F$. L is a minimal left ideal of S if $\phi \subsetneqq L \subset S$ and for any left ideal K, $\phi \neq K \subset L$ implies $K = L$. Similar definitions are understood for minimal right ideals and minimal two-sided ideals. Notice that a minimal (right, left, two-sided)ideal can be a set containing a single element or the whole set. An element $s \in S$ is called a zero of S if st = ts = s for all t ε S. An element s ε S is called a right zero of S if st = s for all t ε S and s ε S is called a left zero of S if ts = s for all t ε S. Let 0 be a zero of S. From now on a zero of S will be denoted by 0 and it is always unique. For suppose 0 and 0' are zeros of S. Then 0' = 00' = 0'0 = 0. That is 0' = 0. Note that {0} is a minimal two-sided ideal of S. Furthermore, if S has a zero, then {0} is the unique minimal two-sided ideal of S as any two-sided ideal M must contain 0, since MS $\cup$ SM $\subset$ M and M $\cdot$ 0 = {0} (as 0 ε S). Similarly if a ε S is a left (right zero) of S, then {a} is minimal left (right) ideal of S. Again let R be a left ideal of S, that is SR $\subset$ R. In particular a ε S and hence aR = {a}. Therefore, as above, {a} is the unique minimal left (right) ideal of S as a belongs to every left (right) ideal of S. If G is a group, then G itself is its only (left, right, two-sided) ideal as the identity e of G is in every ideal and hence {e}G = G = G{e}.

L(S) will denote the set of all minimal left ideals of S and R(S) will denote the set of all minimal right ideals of S. Let $\mathcal{B}$ and A be index sets such that

$$L(S) = \{L_\beta \mid \beta \in \mathcal{B},\ L_\beta \text{ is a minimal left ideal of } S\} \text{ and}$$

$$R(S) = \{R_\alpha \mid \alpha \in A,\ R_\alpha \text{ is a minimal right ideal of } S\}.$$

Let t ε S. Then t is an idempotent element of S if t $\cdot$ t = t. If a semigroup S contains any minimal left ideal, then S possesses a minimal two-sided ideal which is equal to the union of all the minimal left ideals of S [3] (Theorem page 184, Chapter V.) The minimality of the minimal left ideal insures the existence of a minimal two-sided ideal (as two-sided ideals exist, namely S itself). Similarly, if S contains any minimal right ideal, S again possesses a minimal two-sided ideal which is equal to the union of all the minimal right ideals of S. Ljapin has also shown that every minimal left ideal and also every minimal right ideal contains an idempotent [3] (Section 3.17 Chapter IV, page 156). Moreover, if S contains a minimal left ideal L and a minimal right ideal R, then G = RL is a group where RL = R ∩ L [3] (Theorem under Section 3.2, page 189, Chapter V). Since L is a left ideal, RL ⊂ SL ⊂ L and since R is a right ideal RL ⊂ RS ⊂ R. Hence RL ⊂ L and RL ⊂ R, hence RL ⊂ R ∩ L. Clearly R ∩ L ⊂ RL and hence RL = R ∩ L. Hence if L(S) ≠ ϕ and R(S) ≠ ϕ, then there exists a minimal two-sided ideal, and it will be called K, such that

$$K = \bigcup_{\beta \in B} L_\beta = \bigcup_{\alpha \in A} R_\alpha = \bigcup_{\alpha \in A} \bigcup_{\beta \in B} G_{\alpha\beta}$$

where $G_{\alpha\beta} = R_\alpha \cap L_\beta$ and the components of each union are non-empty and pointwise nonintersecting [3] (Theorem under Section 2.2, page 183, Chapter V). In the case of compact semigroups, Numakura [2] (Lemma 7, page 103 and Corollary, page 107) has shown that L(S) ≠ ϕ and R(S) ≠ ϕ.

A semigroup S is said to be <u>simple</u> if it has no proper two-sided ideals. So any group is simple with the above definition of simiplicity. If S is a compact topological semigroup, then Numakura [2] (Corollary, page 107) has shown that every ideal (left, right, two-sided) is closed and therefore compact (remember that all topologies are Hausdorff). By the comment above, if S is a compact topological semigroup, it contains minimal left and minimal right ideals and hence a minimal two-sided ideal. Hofman and Nostert showed further that in fact only one minimal two-sided ideal exists in a compact topological semigroup. It will be called the kernel of S (denoted by K) [4](Chapter A, Section 1). Hence if S is a compact semigroup, K is also a compact topological semigroup, since every ideal is closed and a two-sided ideal is a semigroup (as are left and right ideals). For every pair (α,β) ($\alpha \in A$ and $\beta \in B$). $R_\alpha \times L_\beta$ is compact and multiplication is jointly continuous by the Tychonoff Theorem and the fact that multiplication is jointly continuous in S, respectively. Therefore, $G_{\alpha\beta} = R_\alpha L_\beta = R_\alpha \cap L_\beta$ is a topological semigroup whose topology is compact. We then have that $G_{\alpha\beta}$ is a group whose topology is compact and multiplication is jointly continuous, and hence by a previous remark, the inversion map is also continuous. Thus $G_{\alpha\beta}$ is a compact topological group. Hence, if S is a compact semigroup, its kernel K is the union of compact topological groups.

We know that every minimal left ideal and every minimal right ideal of a semigroup has an idempotent element. As a compact topological semigroup S has minimal left ideals and minimal right ideals and hence each minimal left (right) ideal has at least one idempotent, we then know that S also has idempotents. So let T denote the set of all idempotents of the compact semigroup S. Is it possible to write the kernel K of a compact topological semigroup in terms of its idempotents? Yes, and Numakura [2] (Theorem 2, page 104) has given us the following formula for the kernel K of a compact semigroup S:

$$K = \bigcap_{e \in T} SeS$$

From this formula it is easy to see that K is compact, for {e} is compact and hence SeS is compact. Also multiplication is continuous. If S has a zero, then it is immediate by the above formula that $K = \{0\}$.

Recall that

$$K = \bigcup_{\alpha \in A} R_\alpha$$

$$= \bigcup_{\beta \in B} L_\beta$$

$$= \bigcup_{\alpha \in A} \bigcup_{\beta \in B} G_{\alpha\beta} .$$

$G_{\alpha\beta}$ is called a group component of K. Denote the identity of $G_{\alpha\beta}$ by $e_{\alpha\beta}$. It will be shown later that the $G_{\alpha\beta}$ are isomorphic. It is immediate that $e_{\alpha\beta}$, for all $\alpha \in A$, $\beta \in B$, is an idempotent belonging to S. Recall that if $L_{\beta 1}$ and $L_{\beta 2}$ are two minimal left ideals of S, than either $L_{\beta 1} \cap L_{\beta 2} = \phi$ or $L_{\beta 1} = L_{\beta 2}$. This is also true for minimal right ideals. If S is a simple semigroup, S still can have minimal left ideals or minimal right ideals including itself. For example, when S is a group. Observe, that if S is a commutative semigroup and S is simple, then S contains no proper minimal left and no proper minimal right ideals as all ideals (left, right, two-sided) are two-sided ideals.

Lemma 1 Let S be a simple compact topological (Hausdorff) semigroup and x ($\neq 1$, if $1 \in S$) a non-zero idempotent of S. Then there exists a group component X of S, for which x is the identity.

Proof

$S = \bigcup_{\beta \in B} L_\beta$, where $L_{\beta 1} \cap L_{\beta 2} = \phi$ whenever $\beta_1 \neq \beta_2$. (S has minimal left ideals as S is a compact semigroup. This was noted earlier.) Hence there exists a unique $\gamma \in B$ such that $x \in L_\gamma$. Similarly $S = \bigcup_{\alpha \in A} R_\alpha$ where $R_{\alpha 1} \cap R_{\alpha 2} = \phi$ whenever $\alpha_1 \neq \alpha_2$ and hence there exists a unique $\rho \in A$ such that $x \in R_\rho$ Set $X = R \cap L_\gamma$, $x \in X$ as $x \in R_\rho$ and $x \in L_\gamma$ and $x \cdot x = x$ as x was an idempotent element in S. As noted before X is a group and this group contains an idempotent. As a group only has one idempotent, x is the identity element for the group X. Hence for every idempotent element in a compact simple group, there is one group component whose identity element is this idempotent element.

Suppose $0 \in S$, then $\{0\} \subset S$ is a minimal two-sided ideal and as was pointed out before, it is the unique minimal two-sided ideal. S is simple implies that $S = \{0\}$ as S cannot have any proper two-sided ideals. Hence any simple semigroup containing two or more elements cannot have a zero element. Suppose S is simple and contains an identity element. The question that arises is what does S look like if S is simple and has an identity? Before that question can be partially answered, we need another concept.

Let S be a simple semigroup. S is a <u>completely simple semigroup</u> if it contains at least one minimal left ideal and at least one minimal right ideal and has no proper two-sided ideal [3] (Lemma 9, page 105 and Theorem 4L and 4R, page 107). Let us consider, for example, groups. Let G be a group, so that G has no proper two-sided ideals. The only minimal left ideal and the only minimal right ideal of the group G is the group itself. Therefore all groups are completely simple. One would think that any simple semigroup ought to be completely simple. This however is not the case. Stefan Schwarz gives an example [5] (page 229), and analogous examples due to O. Anderson, can be found in this book [6]. These examples hold basically for the reason that the semigroup, though it is simple, it does not have an idempotent element. However, a simple compact topological semigroup is completely simple [2] (Lemma 7, page 103). There is another definition of completely simple semigroups. It has been shown that the two definitions are the same [3] (Theorem 5, page 107). An idempotent f is said to be <u>under</u> another one if $ef = f = fe$. An idempotent e is <u>primitive</u> if there are no non-zero idempotents under e. A simple semigroup S is said to be <u>completely simple</u> if every idempotent element of S is primitive, and for each $a \in S$ there exists idempotents e and f such that $ea = a = af$. There is also another characterization of simplicity. S is a simple semigroup if and only if for every x of S, S   S = S. We then get the result, that a completely simple semigroup S with the identity 1 is a group. For consider any element x. Since S is simple we know that there exists elements a and b such that $a x b = 1$. This comes from the above characterization of simplicity. It is immediate that xba and bax are idempotents as for example

$$xba \cdot xba = xb(a\ x\ b)a = xb \cdot 1a = xba.$$

But as S is completely simple, 1 is a primitive idempotent. As xba and bax are under 1, we then get that $xba = bax = 1$. Therefore the inverse of x is ab and therefore S is a group. Thus we have a partial answer to the question: If S is a simple semigroup and has an identity, what can we say about the semigroup? If the simple semigroup is also a completely simple semigroup, then it is a group. If the simple semigroup is not completely simple, we cannot say anything. In the case of simple compact semigroups, we know that they are then immediately completely simple (as they have at least one minimal left ideal and at least one minimal right ideal) and hence if a simple compact semigroup has an identity, we know then that it is a compact topological group, by a previous remark.

<u>Lemma 2</u> Let S be a completely simple semigroup and $x(\neq 1, 1 \in S)$ a non-zero idempotent of S. Then there exists a group component X of S for which x is the identity.

The proof of the above lemma folows identically as for Lemma 1. So we have the result that for each idempotent element there is one and only one group component of a completely simple semigroup. Hence, if a completely simple semigroup S has more than one idempotent, we know that S can be represented as the union of groups whose identities form the set of all idempotents of S. Furthermore, given a completely simple semigroup S and writing T for the set of all idempotents in S we have

$$R(S) = \{Se \mid e \varepsilon T\}$$
$$L(S) = \{eS \mid e \varepsilon T\} ,$$

and the groups of S are of the form $e_\mu S e_\gamma$ with e_μ, e_γ εT [2] (Lemma 9, page 105 and Theorem 4L and 4R, page 107).

Suppose S is a semigroup and L is a minimal left ideal. Is SL this minimal two-sided ideal that exists? Clearly, by a simple calculation, SL is a minimal two-sided ideal. Just note that if S also has a minimal right ideal, then this is the minimal two-sided ideal as it is unique. For compact semigroups we have a more definite answer. Recall that Numakura [2] (Lemma 7, page 103 and Corollary, page 107) has shown that any compact semigroup has a minimal left and a minimal right ideal. He has a further result [2] (Theorem 5, page 107): "Let R and L be a right and a left minimal ideal of a compact semigroup S, respectively, and K be the kernel of S. Then LR = K and RL is a group." Just recall that if a semigroup S has a minimal left ideal and a minimal right ideal, then S has a kernel. In another section we will be discussing in further detail the structure of completely simple semigroups.

CHAPTER III - EXAMPLES

In this section we will be dealing mainly with examples demonstrating some of the results in the previous section. Before we give these examples, another major concept will be introduced. This is the concept of the support (or carrier) of a measure function on a semigroup S.

Let S be a topological semigroup and $x \in S$. A neighborhood of x, denoted by V_x is a set such that there exists an open set O_x such that $x \in O_x \subset V_x$. A neighborhood of a point need not be a Borel set. Let μ be a measure function in $\mathcal{B}(S)$, the set of all Borel sets of S. The support of μ on a topological semigroup S is defined as follows:

$$C(\mu) = \{x \in S \mid \text{for all } V_x,\ \mu(V_x) > 0 \text{ such that } V_x \in \mathcal{B}(S)\}.$$

Recall that $M(S)$ is the set of all probability measure μ on a topological semigroup S. If S is a compact topological semigroup then $M(S)$ (with the weak-star topology) is also a compact topological semigroup. If $\mu \in M(S)$, then it is easy to show that $C(\mu)$ is closed and hence compact. J. S. Pym [7] has shown that if S is a locally compact topological semigroup and $\mu, \gamma \in M(S)$, then

$\overline{C(\mu * \gamma)} = \overline{C(\mu) \cdot C(\gamma)}$ (the bar denotes closure). Hence if S is a compact topological semigroup and $\mu, \gamma \in M(S)$, then

$C(\mu * \gamma) = C(\mu) \cdot C(\gamma)$. This follows immediately from the fact that the Cartesian product of compact sets is compact and multiplication is jointly continuous. Now we will consider a few examples.

The next few examples just show or verify a few properties using the support of a measure when we consider an algebraic system having defined on it a compact topology and the appropriate operations being continuous.

1) Let G be a compact topological group with $g \varepsilon G$ and $H \varepsilon \mathcal{B}(G)$. μ_g is unit mass on g if

$$\mu_g(H) = \begin{cases} 1 \text{ if } g \varepsilon H \\ 0 \text{ if } g \notin H. \end{cases}$$

Clearly μ_g is a probability measure on G. Recall that $\mu \varepsilon \mathcal{M}(G)$ is invertible if and only if μ is a unit mass measure μ_h for some $h \varepsilon G$. It will be shown that $\mu_g * \mu_{g^{-1}} = \mu_1$. $C(\mu_1) = \{1\}$ and in general $C(\mu_g) = \{g\}$. Since G is a compact group,

$$C(\mu_g * \mu_{g^{-1}}) = C(\mu_g) \cdot C(\mu_{g^{-1}}) ,$$

$$C(\mu_g) \cdot C(\mu_{g^{-1}}) = \{gg^{-1}\}.$$

So $C(\mu_g * \mu_{g^{-1}}) = C(\mu_g) \cdot C(\mu_{g^{-1}}) = \{gg^{-1}\} = \{1\} = C(\mu_1)$. Hence $\mu_g * \mu_{g^{-1}} = \mu_1$ as each unit mass measure is uniquely defined by its support.

2) This example will generalize the above to compact topological semigroup. Let S be a compact topological semigroup with identity. Let $g \varepsilon S$ have an inverse, say g^{-1}. (We say $h \varepsilon S$ has an inverse if $h \cdot k = k \cdot h = 1$). Then as before we have $C(\mu_g * \mu_{g^{-1}}) = C(\mu_g) \cdot C(\mu_{g^{-1}}) = \{gg^{-1}\} = \{1\} = C(\mu_1)$.

Hence $\mu_g * \mu_{g^{-1}} = \mu_1$.

3) Now we want to generalize to multiplication. That is we will show that for $s, t \varepsilon S$, $\mu_s * \mu_t = \mu_{st}$.

$$\{st\} = C(\mu_s) \cdot C(\mu_t) = C(\mu_s * \mu_t)$$

$$\{st\} = C(\mu_{st}).$$

By the uniqueness of the support of a unit mass measure, we then deduce that $\mu_{st} = \mu_s * \mu_t$.

4) This example will illustrate some of the results on compact semigroups. We will determine its kernel, show that it has minimal left and right ideals of which we are assured.

Let R be the set of all real numbers. $(R)_n$ will denote the set of all $n \times n$ matrices whose entries are from the field R. Let $A \varepsilon (R)_n$, then $\| \ \|$ is a funtion from $(R)_n$ to R. It is called the operator norm. For a complete definition of this norm see the book "Real Analysis" by Royden [22]. I will be using some properties of this norm which will also be described in the above reference. The main property, that I will use is the triangular inequality which says that for A, B ε $(R)_n$, $||A|| \cdot ||B|| \geq ||AB||$.

Let $S = \{A \varepsilon (R)_2 \mid ||A|| \leq 1\}$ where $(R)_2$ is the set of all the 2×2 matrices over the reals. Clearly S is a semigroup under matrix multiplication. Let A , B ε S then $||AB|| \leq ||A|| \; ||B|| \leq 1 \cdot 1 = 1$. Also, S is compact. Notice that S has at least one, in fact it has an infinite number of proper left ideals L and at least one proper right ideal: namely

$$L = \{A \varepsilon S \mid A = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}\}$$

and $R = \{B \varepsilon S \mid B = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\}$.

S has a minimal left ideal and a minimal right ideal and in this case it is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ as S has a zero. Hence S has a minimal two-sided ideal, which is the kernel of S, and that is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Hence S is not simple. S has other proper, again an infinite number of, two-sided ideals. Let S^n $\{A \varepsilon S \mid ||A|| \leq 1/n\}$ where n is a strictly positive integer. Then $SS^n \subset S^n$ and $S^nS \subset S^n$ since

$||AB|| \leq ||A|| \; ||B||$ for all A, B ε $(R)_2$ Observe that the minimal left ideal, minimal right ideal, minimal two-sided ideal or kernel in this semigroup are identical namely $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. L and R are not minimal left and minimal right ideals, respectively, as we can define L^n and R^n for all strictly positive integers, similarly as S^n. S, of course, is not completely simple. In the first case S is not even simple, but also there exists an idempotent that is under $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. In matter of fact, there are two idempotents under $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and these are $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, as for example $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.
The question arises, does S have a subsemigroup which is simple? (For we are interested in completely simple semigroups.) The reason for this question is to determine whether or not we can define an idempotent probability measure on S and this question will be answered later. Consider the following subset of S.

$$G = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \mid 0 < \theta < 2\pi \right\}.$$

Then $G \subset S$ since

$\Theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ rotates all points in the plane around the origin by the angle θ. The distance of the point from the origin is kept constant and hence $||\Theta|| = 1$, hence $\Theta \; \varepsilon \; S$. By a simple calculation, it can be shown that $G \cdot G = G$. If $\theta = 0$, then

$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and therefore, G has an identity. We will show that G is a group by showing that it is completely simple. As $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the only idempotent in G, all idempotents are primitive. Further for any a ε G, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} a = a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. All we have to show is that G is simple. Notice that G is homeomorphic and isomorphic to [0, 2π) which has addition of the real line modulo 2π. That is for a, b ε [0, 2π)

$$a + b = \begin{cases} a + b & \text{if } 0 \leq a + b < 2\pi \\ a + b - 2\pi & \text{if } 2\pi \leq a + b < 4\pi. \end{cases}$$

It is immediate that [0, 2π) is simple (has no two-sided ideals) and therefore G is simple. As [0, 2π) is commutative, so is G. Therefore,G is a completely simple semigroup (compact in fact) with identity (see [2] (Lemma 8, page 105)) and therefore G is a group. In this we could have shown that G was a group simply by showing what the inverse of $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ was and it simply is $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. This group is generally know as the Rotation Group.

S has another subsemigroup which is also a group Consider $H = \{A \in (R)_2 \mid \|A\| \leq 1 \text{ and } \det = \pm 1\}$. Clearly H ⊂ S and H•H ⊂ H. Let A,B ε H, then $\|AB\| \leq \|A\|\cdot\|B\| \leq 1$. If the condition had been that $\|A\| = 1$, we would not be assured that H is closed under multiplication. Furthermore det AB = detA •detB = ±1 as detA = ±1 and detB = ±1 . Det $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$ and hence $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ε H and therefore,HH = H. H is still compact and so H is a proper subsemigroup of S as $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \not\in H$. Hence it is neither a left, right or two-sided ideal of S. In this case it is difficult to determine whether or not H is simple. However it is immediate that H is a group as H • H = H and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ε H and the determinent of all matrices in H are nonzero and hence invertible. Now consider the subset F of H where: $F = G \cup \{e_1\}$, where G is the rotation group as mentioned above and $e_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Clearly $e_1 \neq G$, so $F \supsetneq G$. F is a compact semigroup as G and e_1G are compact sets and hence a finite union of compact sets is also compact. Observe that $G \cap e_1G = \phi$ and G is a simple semigroup. Therefore F is a simple semigroup and hence a completely simple semigroup as F is compact. [2] (Lemma 7, page 103) F also has an identity, namely $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Therefore F is a group, as F is a completely simple semigroup with identity. Now we will show that F is a subset of H. First observe that if $\Phi \in G$, then $\det \Phi = 1$. Let $A \in F$, then

$$A = \begin{cases} \Phi & \text{for some } \Phi \in G \text{ if } \det A = 1 \\ e_1\Phi & \text{for some } \Phi \in G \text{ if } \det A = -1. \end{cases}$$

Also $||A|| \leq ||\Phi||\ ||e_1|| = ||\Phi|| = 1$ if $\det A = -1$ when $||e_1|| = 1$. or $||A|| = ||\Phi|| = 1$ if $\det A = 1$.

Therefore $A \in H$ and hence $F \subset H$.

So we have at least three subsemigroups of S that are groups G, F, H and they are related as follows, $G \subsetneq F \subset H$. Their importance will become apparent later when we illustrate the theorem on the convergence of sequences of probability measures. Note that, the groups G and F are both abelian.

6) This example is just an extension of the above example. The importance lies in the fact that the groups derived are no longer abelian.

Define $S = \{A \in (R)_n \mid ||A|| \leq 1\}$,

where $(R)_n$ is the set of all $n \times n$ matrices over the reals. Since for $A \in S$, we have $||A|| \leq 1$, we know that S is compact. Again notice that S has at least one left ideal L and at least one right ideal N, namely:

$$L = \left\{ A \in S \mid A = \underbrace{\begin{pmatrix} a_1 & 0 & \dots & 0 \\ a_2 & 0 & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ a_n & 0 & \dots & 0 \end{pmatrix}}_{n \text{ columns}} \right\}$$

and

$$N = \left\{ B \in S \mid B = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ 0 & 0 & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & 0 \end{pmatrix} \right\} n \text{ rows} \right\}$$

as with n = 2, the minimal left ideal, the minimal right ideal and the minimal two-sided ideal, which also is the kernel of S, is the zero matrix so that S is not simple. Does S have subsemigroups which are simple? The answer is yes and the first subsemigroup is as follows: Let G be the set of all orthogenal real matrices such that the determinant of each matrix is 1. P is orthogenal if $P^* = P^{-1}$ (P^* means the transpose of P - see any book on matrix theory for definitions). Let P, Q ε G, then $(PQ)^* = Q^*P^* = Q^{-1}P^{-1} = (PQ)^{-1}$. Also G is a closed subset of S and hence G is compact with identity [21] . Now we will show that G is simple. Let P ε G. Consider G P G. Since P ε G, P^* ε G and hence $G = G P P^* \subset G P G$. Therefore G P G = G for all P ε G and hence G is simple. But because G is compact, G is completely simple and it follows that G is a group.

Now define H (as in example 4) as follows:

$H = \{A \in (R)_n \mid ||A|| \leq 1, \det A \pm 1\}$. Then H is a closed and hence compact subsemigroup of S is neither a left, right, or two-sided ideal of H. $0 \notin H$ and hence $H \subsetneq S$. Again observe that $H \cdot H = H$ and hence H is a compact group. Define F = G $\cup$ e, G where

$$e_1 = \begin{pmatrix} -1 & 0 & \dots & \dots & . & 0 \\ 0 & 1 & 0 & \dots & . & 0 \\ 0 & 0 & 1 & 0 \dots & . & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot & \\ \cdot & \cdot & \cdot & \dots & \cdot & \\ \cdot & \cdot & \cdot & \dots & . & 0 \\ 0 & \cdot & \cdot & \dots & 0 & 1 \end{pmatrix}$$

Then det $e_1 = -1$. Clearly again as before $F \subset H$ and F is also compact. Furthermore F is simple and hence a group. Again $G \subsetneq F \subset H$

and neither G nor F in this case (actually for $n \leq 3$) are commutative.

6) The next example just shows us some of the consequence of certain properties. Let S be a commutative completely simple semigroup without identity. We have considered the general case when S had an identity. Then any right or left ideal is a two-sided ideal. As S is completely simple, there exists minimal left and right ideals, that is there exists at least one minimal ideal in S. But by the definition of completely simple, S is automatically simple and hence it has no proper ideals, thus S is the only ideal. Now S must contain idempotents as S is completely simple (see [3] (Section 3.17, Chapter IV, page 156). Recall that every idempotent is an identity for one of the group components of S.

$$\text{Therefore } S = \bigcup_{\beta \in B} L_\beta = \bigcup_{\alpha \in A} R_\alpha = \bigcup_{\alpha \in A} \bigcup_{\beta \in B} R_\alpha L_\beta = \bigcup_{\alpha \in A} \bigcup_{\beta \in B} G_{\alpha\beta}.$$

But $L_\beta = R_\alpha$ for some α and some β as S is abelian and furthermore $S = L_\beta = R_\alpha = S$ for all $\alpha \in A$ and $\beta \in B$ as S is simple and thus $L(S) = \{S\} = R(S)$ and hence $G_{\alpha\beta} = S$. Therefore,S is a group and hence has an identity. Therefore,every commutative, completely simple semigroup is a group. Just two more observations, if S is a simple commutative semigroup with a zero, then as before $S = \{0\}$. Also,simple commutative semigroups are completely simple as all ideals, left, right, two-sided are all two-sided. As the semigroup is simple its only non-empty ideal is itself and hence it has a minimal left ideal and a minimal right ideal. Hence every simple commutative semigroup is a completely simple commutative semigroup and hence it is a group.

7) This example will illustrate all the results on compact simple semigroups. The minimal left ideals and the minimal right ideals and the group components will be clearly defined. The reason for this example will become apparent later when we will further study the structure of completely simple semigroups.

Let G be any compact topological group with identity e. Define G^o to be the semigroup $G \cup \{0\}$ where $g \cdot 0 = 0 \cdot g = 0$ for all $g \in G^o$ and $g_1 g_2$ is the same as before for all $g_1, g_2 \in G$. Let $\mathcal{G}^o$ be the set of all 3×3 matrices whose entries are from G^o such that at most one entry is non-zero. Define $P = \begin{pmatrix} e & e & e \\ e & e & e \\ e & e & e \end{pmatrix}$. Let $A \in \mathcal{G}^o$, then either every entry is zero and we denote A as (0), or exactly one entry is not zero, say the (i, j) position for some i = 1, 2, 3 and j = 1, 2, 3. This non-zero element is some g in G. Denote A as $(g)_{ij}$. That is,every matrix of $\mathcal{G}^o$ will be written as $(g)_{ij}$ where $g \in G$ and i = 1, 2, 3 and j = 1, 2, 3 or (0).

So $\mathcal{G}^o = \{(g)_{ij} \mid g \in G, i = 1,2,3 \text{ and } j = 1, 2, 3\} \cup \{(0)\}$.

Define a new multiplication $\circ$ in $\mathcal{G}^o$ as follows:

Let $A, B \in \mathcal{G}^o$ then $A \circ B = A\,P\,B$

where on the right hand side we have ordinary matrix multiplication. As there generally is no additive operation in the group G, we define a formal sum as follows:

Let $\{a_\lambda\}_\lambda$ be a sequence in G^o. Then

$$\sum_\lambda a_\lambda = \begin{cases} \{a_\mu\} \text{if } a_\gamma = 0 \text{ for all } \gamma \neq \mu \\ \text{undefined otherwise} \end{cases}$$

It should be immediate that any formal sum in the matrix A P B has at most one non-zero term and hence it is defined. Therefore A P B is always defined, and it is also immediate that $A P B \varepsilon \mathcal{G}^o$ Therefore $\mathcal{G}^o$ is a semigroup as multiplication is associate as ordinary matrix multiplication is associative. Define $\mathcal{G} = \mathcal{G}^o - \{(0)\}$, where (0) is zero matrix. $\mathcal{G}$ has no (right, left, two-sided) zero divisors (a is a <u>two-sided zero divisor in a semigroup</u> G with zero if there exists a, b ε S, a $\neq$ 0, b $\neq$ 0 such that ab = ba = 0. Similarly definitions hold for <u>left and right zero divisors</u>.) Give $\mathcal{G}$ the induced matrix topology derived from G. Then it is immediate that multiplication is continuous. Therefore G is a topological semigroup. As $\mathcal{G}$ is a compact semigroup, it then follows that $\mathcal{G}$ is also a compact semigroup. Recall that

$$(a)_{ij} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \quad \text{such that}$$

$$\alpha_{1k} = \begin{cases} a \text{ if } i = 1 \text{ and } j = k \\ 0 \text{ otherwise.} \end{cases}$$

Denote the non-zero element in the (i,j) position as a_{ij}. Now observe that $(a)_{ij} \circ (b)_{lm} = (ab)_{im}$ for all $(a)_{ih}$, $(b)_{lm} \varepsilon \mathcal{G}$. $\mathcal{G}$ has no proper two-sided ideals as G is a compact group with no zeros and hence $\mathcal{G}$ is simple. As $\mathcal{G}$ is compact, $\mathcal{G}$ must have at least one minimal left ideal and at least one minimal right ideal, as $\mathcal{G}$ is completely simple (a compact simple semigroup is a completely simple semigroup). Furthermore $\mathcal{G}$ must have idempotents. As a matter of fact, $\mathcal{G}$ has 3 minimal left ideals, 3 minimal right ideals and hence 9 group components and therefore 9 idempotents. These are as follows: First we identify the idempotent elements. These are

$\{(e)_{ij} \mid 1 \leq i, j \leq 3\}$ where e is the multiplicative identity of G. Let us consider $(e)_{ij}$; $(e)_{ij} \circ (e)_{ij} = (e)_{ij}$ by the rule given above. Suppose $(g)_{ij}$ is an idempotent. Then

$(g)_{ij} \circ (g)_{ij}) = (gg)_{ij} = (g)_{ij}$,

that is $g^2 = g$ and hence $g = e$ as G is a group. Therefore, these are the only idempotents of G. The minimal left ideals are as follows and will be labelled L_1, L_2, L_3:

$L_j = \{(g)_{ij} \mid g \in G, i = 1, 2, 3\}$

for $j = 1, 2, 3$.

The minimal right ideals are as follows and will be labelled R_1, R_2, R_3:

$R_i = \{(g)_{ij} \mid g \in G, j = 1, 2, 3\}$

for $i = 1, 2, 3$.

Observe that $G = \bigcup_{i=1}^{3} R_i = \bigcup_{i=1}^{3} L_i$.

Also $R_i \cap L_j = R_iL_j$. Now R_iL_j is a group and we will denote it by G_{ij}. $G_{ij} = \{(g)_{ij} \mid g \in G\}$. It is immediate that $G_{ij} \cong G_{lm} \cong G$ for all $1 \leq i, j, l, m \leq 3$. Also notice that

$$G = \bigcup_{i=1}^{3} \bigcup_{j=1}^{3} G_{ij}.$$

Furthermore $(e)_{ij}$ is the identity of G_{ij}. Hence the set of idempotents are the identities of the group components.

This example seems to have come out of nowhere. It will become clear later on why this example arises naturally in the study of completely simple semigroups. It does demonstrate how a completely simple semigroup breaks down into a union of groups. All of these groups are isomorphic to each other and to the original group. We needed compactness to get this semigroup completely simple. The above example can be created or extended to all positive integers n. If $n = 1$, it is not necessary to adjoin a zero to G as the set of 1×1 matrices with entries from G is just G itself.

Now that we have considered these various examples, we want to discuss idempotent probability measures on compact topological semigroups. Before we discuss the idempotent probability measures on compact semigroups, we want to discuss idempotent probability measures first on compact simple semigroups. Recall that on semigroups, idempotent probability measures are the stable laws on that semigroup.

CHAPTER IV - IDEMPOTENT MEASURES

We are interested in the convergence of a sequence of probability measures and the stable laws of a given compact semigroup S. We find that some sequences of probability measures converge to idempotent probability measures and we find that these particular sequences are in fact stable laws. So with this in mind, we want to discuss idempotent probability measures on a compact topological semigroup. In particular we want to be able to determine what these idempotent probability measures look like on a topological semigroup. In this section we will consider just compact topological semigroups, and mainly just simple ones. For a compact topological group we already know the answer. J. G. Wendel [1] has proved that every compact topological group G has a non-zero, non-identity idempotent measure, namely the normalized Haar measure. As some compact semigroups are just unions of compact groups, one would think that at least some of the idempotent probability measures would be some linear combination of the normalized Haar measures on the various compact groups.

From now on for any two measure functions on an algebraic system, we will denote convolution simply by juxtaposition, that is if μ,γ are two measures, then $\mu * \gamma = \mu\gamma$. Let S be a topological semigroup, then e is an <u>idempotent probability measure</u> if $e \neq 0$, $e \neq 1$, if 0 or $1 \in S$ respectively and $e * e = e$. Now assume that S is a compact semigroup and $e \in \mathcal{M}(S)$ is an idempotent probability measure, then B. M. Kloss [8] has shown that the support of ε (denoted by $C(\varepsilon)$) is a closed simple subsemigroup of S and hence compact. Therefore, the support of any idempotent probability measure on a compact semigroup is a completely simple compact semigroup (as a simple compact has minimal left and minimal right ideals). P will always denote the support of an idempotent measure μ on S, so that it is a completely simple compact semigroup. So the problem of determining what idempotent probability measures look like on a compact semigroup has been reduced to the study of completely simple compact semigroups or simple compact semigroups as both concepts are the same for compact semigroups.

Suppose μ is an idempotent measure on a compact semigroup S and $C(\mu) = P$ contains a finite number of idempotents. In the succeeding pages, unless otherwise mentioned, all simple compact semigroups will contain only a finite number of idempotents. If P contains only a finite number of idempotents, then we know that $P = \bigcup_{i=1}^{s} \bigcup_{j=1}^{r} G_{ij}$, s and r some positive integers, is the group decomption of P, then μ restricted to the Borel subsets of G_{ik} is an invariant positive finite measure on the group G_{ik}. We know that μ is finite as $G_{ik} \subset P$ and $\mu(P) = 1$. $(G_{ik}) > 0$ as G_{ik} is contained in the support of μ and G_{ik} is an open set. μ restricted to G_{ik} is defined as follows: Let B be a Borel set of P, then $B \cap G_{ik}$ is a Borel set as G_{ik} is a Borel set and the intersection of Borel sets is another Borel set. Similarly any Borel set in G_{ik} is derived in this manner. So consider any Borel set B in G_{ik}. There exists a Borel set B' in P such that $B = B' \cap G_{ik}$. Then the restriction of to G_{ik} will be denoted by μ_{ik} where $\mu_{ik}(B) = \mu(B' n G_{ik})$. Now we will show that μ_{ik} is invariant. Let $a \varepsilon G_{ik}$. Then $a \varepsilon P$ and so

$\mu_{ik}(aB) = \mu(a(B' \cap G_{ik})) = \mu(B' \cap G_{ik}) = \mu_{ik}(B)$ as μ is invariant, being an idempotent measure on P. Similarly μ_{ik} is left invariant and hence μ_{ik} is invariant on G_{ik}. Therefore μ_{ik} is a positive invariant finite measure on G_{ik}. Furthermore, if $P = \bigcup_{i=1}^{s} R_i = \bigcup_{j=1}^{r} L_j$ with $G_{ij} = R_i L_j$ where the R_i's are the minimal right ideals and the L_j's are the minimal left ideals and if μ is an idempotent probability measure on P, one has

$\mu(R_i)\ \mu(L_j) = \mu(R_i L_j) = \mu(G_{ij})$ [9] (Theorem 1.1, page 67 and Corollary, page 99)

Let $g_{\alpha\beta}$ be an element of $G_{\alpha\beta}$ and let $e_{\alpha\rho}$ be the identity in $G_{\alpha\beta}$. We will recall that for a compact simple semigroup P we have

$$P = \bigcup_{\alpha\varepsilon\Lambda_1} R_\alpha = \bigcup_{\beta\varepsilon\Lambda_2} L_\beta = \bigcup_{\alpha\varepsilon\Lambda_1} \bigcup_{\beta\varepsilon\Lambda_2} G_{\alpha\beta}$$

(and this holds for all completely simple semigroups S). Since P is simple and compact, we have the following results [9] (Page 99):

a) $L_\beta g_{\alpha\delta} = L_\delta$; $g_{\alpha\delta} R_\gamma = R_\alpha$

b) $\{e_{\alpha\beta},\ \alpha\ \varepsilon\ \Lambda_1\}$ is the set of all idempotent elements belonging to L_β. Each of them is a right unit of L_β. The set $\{e_{\alpha\beta},\ \beta\ \varepsilon\ \Lambda_2\}$ is the set of all idempotents belonging to R_α. Each of them is a left unit of R_α.

c) Any two minimal left ideals L_α, L_β are isomorpnic under the mapping $x \to xe_{\gamma\beta}$. The inverse mapping is $y \to ye_{\beta\gamma}\ \varepsilon\ L_\gamma$.

d) $g_{\alpha\beta}\ L_\delta = G_{\alpha\delta}$, $R_\gamma g_{\alpha\beta} = G_{\gamma\beta}$.

e) $G_{\alpha\beta}\ g_{\gamma\delta} = G_{\alpha\delta}$, $g_{\alpha\beta} G_{\gamma\delta} = G_{\alpha\delta}$.

f) $G_{\alpha\beta} G_{\gamma\delta} = G_{\alpha\delta}$.

g) Any two groups $G_{\alpha\beta}$ and $G_{\gamma\delta}$ are topologically isomorphic by, for example, the mapping

$$a_{\gamma\delta}\ \varepsilon\ G_{\gamma\delta} \to e_{\alpha\beta}\ a_{\gamma\delta}\ e_{\gamma\beta}\ \varepsilon\ G_{\alpha\beta}.$$

The inverse to this map is given by

$$b_{\alpha\beta}\ \varepsilon\ G_{\alpha\beta} \to e_{\gamma\beta}\ b_{\alpha\beta}\ e_{\alpha\delta}\ \varepsilon\ G_{\gamma\delta}.$$

For (a), $g_{\gamma\delta} \varepsilon G_{\gamma\delta} = R_\gamma L_\delta = R_\gamma \cap L_\delta$ and therefore $g_{\gamma\delta} \varepsilon L_\delta$. Hence $L_\beta g_{\gamma\delta} \subset L_\delta$ as L_δ is a left ideal. But L_δ is a minimal left ideal and $L_\beta g_{\gamma\delta} \neq \phi$ and hence $L_\beta g_{\gamma\delta} = L_\delta$. Similarly $g_{\gamma\delta} \varepsilon R_\gamma$ and hence $g_{\gamma\delta} R_\alpha \subset R_\gamma$ as R_γ is a right ideal and since R_γ is minimal $g_{\gamma\delta} R_\alpha = R_\gamma$. It is easy to check that $L_\beta g_{\gamma\delta}$ is a left ideal and $g_{\gamma\delta} R_\alpha$ is a right ideal. Then (b) is immediate as are (c) and (g) since the mappings are given. For (e) observe that $G_{\alpha\beta} = R_\alpha L_\beta$ and using (a) we get the desired result as for example $G_{\alpha\beta} g_{\gamma\delta} = R_\alpha L_\beta g_{\gamma\delta} = R_\alpha L_\delta = G_{\alpha\delta}$ ($L_\beta g_{\gamma\delta} = L_\delta$ by (a)). Now consider (d). $g_{\alpha\beta} \varepsilon G_{\alpha\beta} = R_\alpha L_\beta$. Hence $g_{\alpha\beta} L_\gamma \subset R_\alpha L_\beta L_\gamma = R_\alpha L_\gamma = G_{\alpha\gamma}$ as L_γ is a left ideal. By the minimality of all ideals, $g_{\alpha\beta} L_\gamma = R_\alpha L_\beta L_\gamma$ and hence the desired result. Finally for (f) observe again that $G_{\alpha\beta} = R_\alpha L_\beta$. Therefore $G_{\alpha\beta} G_{\gamma\delta} = R_\alpha L_\beta R_\gamma L_\delta = R_\alpha (L_\beta R_\gamma) L_\delta = R_\alpha L_\gamma L_\delta = R_\alpha (L_\gamma L_\delta) = R_\alpha L_\delta = G_{\alpha\delta}$. The first bracket is a result of (a) and the second is that L_δ is a left ideal. This proves all the above claims. Now observe that $e_{\alpha\beta} \cdot e_{\gamma\delta} \varepsilon G_{\alpha\delta}$ but in general $e_{\alpha\beta} \cdot e_{\gamma\delta} = e_{\alpha\delta}$ need not hold. We have however that $e_{\alpha\beta} e_{\alpha\gamma} = e_{\alpha\gamma}$ and $e_{\alpha\beta} e_{\gamma\beta} = e_{\gamma\beta}$.

Let us consider example (7) in Chapter III. Recall that $\mathcal{G}$ is the set of all 3 × 3 matrices over the compact topological group G which have one and only one nonzero entry, and an element of G is of the form $(a)_{ij}$ where $a \neq 0$ and is at the (i, j) entry of the 3 × 3 matrix. Multiplication was defined so that $(a)_{ij} \circ (b)_{lk} = (ab)_{ik}$. With this definition of multiplicative conditions (a) - (g), as given above, are immediately satisfied. Notice however that in this particular case $(e)_{\alpha\beta} \circ (e)_{\gamma\delta} = (e)_{\alpha\delta}$. Also in this case $(e)_{\alpha\beta} \circ (a)_{\gamma\delta} \circ (e)_{\gamma\beta} = (eae)_{\alpha\beta} = (a)_{\alpha\beta}$ because of the definition of multiplication and hence the mapping to show that two group components are isomorphic is trivial.

Thus we have considered some of the properties of an idempotent probability measure μ on a simple compact topological semigroup S. We have shown that if μ is an idempotent probability measure on a compact semigroup S, then its support is a simple compact semigroup and if we restrict μ to each group component of $C(\mu)$, it still is a positive, finite, invariant measure function. Furthermore, we have looked at some particular cases of multiplication of minimal left and minimal right ideals and group components by elements of the group components. The result as given above [9] (page 99) also holds in the general case. We have only considered the case when the simple compact topological group S contains a finite number of idempotents. It is also true for any simple compact topological semigroup S.

Now that we have some properties of an idempotent probability measure on a compact semigroup when restricted to its group component, we would like to go the other way. What happens when we consider idempotent probability measures on the group components? Can we extend each measure and then add them? What do we get? This is what we consider now.

Let S be a simple compact semigroup with a finite number of idempotents. Then $S = \bigcup_{i=1}^{s} \bigcup_{j=1}^{r} G_{ij}$. Let μ_{ik} denote the normalized Haar measure on the group G_{ik}. G_{ik} is compact as S is a compact semigroup. Let $g_{ik} \in G_{ik}$, then g_{ik} will also denote the probability measure of unit mass on g_{ik}. The meaning of g_{ik} will generally be clear from the context of its use. Convolution is just juxtaposition as was mentioned earlier. Now we extend the normalized Haar measure μ_{ik} on G_{ik} to all Borel sets E of S by putting $\mu_{ik}(E) = \mu_{ik}(E \cap G_{ik})$. Stefan Schwarz [9] (Lemma 1, 2, page 100) has proved the following results on the measure functions μ_{ik} and g_{jl}:

a) $g_{ik}\,\mu_{jl} = \mu_{ik}\,g_{jl} = \mu_{il}$.

b) $\mu_{ik}\,\mu_{jl} = \mu_{il}$.

c) If S is a compact semigroup and P is a simple (closed) subsemigroup and $\gamma \in M(S)$ with $C(\gamma) \subset P$ then $\mu_{ik}\,\gamma\,\mu_{jl} = \mu_{il}$.

Notice the similarities between these results and the results just mentioned dealing with multiplication of minimal left and minimal right ideals and group components by elements of the group components. The result (a) here is similar to result (e) before and the result (b) here is similar to result (f) before. The proof of condition (a) is rather lengthy and can be found in [9] (Lemma 1, 2, page 100). For (b) we use (a) by observing that $\mu_{ik}\,\mu_{jl} = (\mu_{ik}\,e_{ik})\,(e_{jl}\mu_{jl}) = \mu_{ik}\,(e_{ik}\,e_{jl})\,\mu_{jl}$. Denote $e_{ik}\,e_{jl} = g_{il}$ and we get $\mu_{ik}\,\mu_{jl} = \mu_{ik}\,(g_{il}\,\mu_{jl}) = \mu_{ik}\,\mu_{il}$. Again by (a) and observing that μ_{il} is an idempotent probability measure belonging to $M(S)$, we finally have

$$\mu_{ik}\,\mu_{il} = \mu_{ik}\,(e_{il}\,\mu_{il}) = (\mu_{ik}\,e_{il})\,(\mu_{il}) = \mu_{il}\,\mu_{il} = \mu_{il}.$$

For (c), see [9] (Lemma 1, 2, page 100).

We are now in a position to state the result which tells us what the idempotent probability measures on a simple compact semigroup P with a finite number of idempotents look like. This then will give us a partial answer to what the stable laws look like on a simple compact topological semigroup. The result was proven by Stefan Schwarz [9] (Theorem 1,2, page 102) and is as follows: Let S be a compact topological semigroup and P a closed simple subsemigroup of S that contains a finite number of idempotents. Let $P = \bigcup_{i=1}^{s} \bigcup_{k=1}^{r} G_{ik}$ be its decomposition into its group components. Then every idempotent probability measure ε belonging to $M(S)$ with $C(\varepsilon) = P$ is of the form

$$\varepsilon = \sum_{i=1}^{s} \sum_{k=1}^{r} \delta_i \eta_k \mu_{ik},$$

where δ_i, η_k are positive numbers satisfying $\sum_{i=1}^{s} \delta_i = 1 = \sum_{k=1}^{r} \eta_k$ and μ_{ik} is the normalized Haar measure extended to P and S. Conversely, if δ_i, η_k are positive numbers satisfying $\sum_{i=1}^{s} \delta_i = 1 = \sum_{k=1}^{r} \eta_k$, then $\varepsilon = \sum_{i=1}^{s} \sum_{k=1}^{r} \delta_i \eta_k \mu_{ik}$ is an idempotent probability measure belonging to $M(S)$ whose support is exactly P.

So for any compact topological semigroup whose closed simple subsemigroup contains only a finite number of idempotents, we can construct all idempotent probability measures on this compact semigroup by simply considering each simple (closed) subsemigroup. This result will give us a partial answer to what some of the stable laws are on a compact semigroup.

We will give a quick sketch of the proof for the above result. First we need the fact that the μ_{ik}'s are completely determined by means of a fixed μ_{ij} say μ_{11} and the idempotent elements belonging to P, since we have $\mu_{ik}(E) = \mu_{11}(e_{1k}\, Ee_{11})$ for any Borel subset E of G_{ik} or alternatively $\mu_{ik} = e_{ik}\, \mu_{11} e_{1k}$ where the e_{ij} are the idempotent probability measures (by conditions (a) - (g) of the first set and conditions (e) of the second set.)

Write $\mu = \mu^2 \varepsilon M(S)$ with $C(\mu) = P$ in the form

$$\mu = \sum_{i=1}^{s} \sum_{k=1}^{r} t_{ik} \mu_{ik} \quad (\text{as } P = \bigcup_{i=1}^{s} \bigcup_{k=1}^{r} G_{ik}) \text{ with}$$

$$\sum_{i=1}^{s} \sum_{k=1}^{r} t_{ik} = 1, \; t_{ik} > 0.$$

We then get

$$\Big(\sum_{i=1}^{s} \sum_{k=1}^{r} t_{ik} \mu_{ik}\Big) \cdot \Big(\sum_{j=1}^{s} \sum_{l=1}^{r} t_{jl} \mu_{jl}\Big) = \sum_{i=1}^{s} \sum_{l=1}^{r} t_{il} \mu_{il},$$

and hence by condition (a) - (c) above we get

$$\sum_{i=1}^{s} \sum_{k=1}^{r} \sum_{j=1}^{s} \sum_{l=1}^{r} t_{ik} t_{jl} \mu_{il} = \sum_{i=1}^{s} \sum_{l=1}^{r} t_{il} \mu_{il}$$

and thus

$$\sum_{k=1}^{r} \sum_{j=1}^{s} t_{ik} t_{jl} = t_{il} \qquad (*) .$$

Set $\sum_{k=1}^{r} t_{ik} = \delta_i$ and $\sum_{j=1}^{s} t_{jl} = \eta_l$. Then (*) implies $t_{il} = \delta_i \eta_l$.

Clearly $\sum_{i=1}^{s} \delta_i = 1 = \sum_{l=1}^{r} \eta_l$ and $\mu = \sum_{i=1}^{s} \sum_{l=1}^{r} \delta_i \eta_l \mu_{il}$.

Conversely, let $\mu_1 = \sum_{i=1}^{s} \sum_{l=1}^{r} \delta_i \eta_l \mu_{il}$ be an element of $M(S)$ where δ_i, η_l are positive numbers satisfying $\sum_{i=1}^{s} \delta_i = 1 = \sum_{l=1}^{r} \eta_l$.

$$\mu_1^{\,2} = \Big(\sum_{i=1}^{s} \sum_{l=1}^{r} \delta_i \eta_l \mu_{il}\Big)\cdot\Big(\sum_{j=1}^{s} \sum_{k=1}^{r} \delta_j \eta_k \mu_{jk}\Big) = \sum_{i=1}^{s} \sum_{l=1}^{r} \sum_{j=1}^{s} \sum_{k=1}^{r} \delta_i \eta_l \delta_j \eta_k \mu_{ik}$$

$$= \Big(\sum_{l=1}^{r} \eta_l\Big)\cdot\Big(\sum_{j=1}^{s} \delta_j\Big)\cdot \sum_{i=1}^{s} \sum_{k=1}^{r} \delta_i \eta_k \mu_{ik} = \mu_1 .$$

Thus $\mu_1^{\,2} = \mu_1$ and μ_1 is an idempotent probability measure.

Consider now the following examples on the above result.

(i) Taking s = t = 3 gives us example (7) in Chapter III . Suppose we are also looking for idempotent probability measures whose support is not all of G. Consider $G_1 = \bigcup_{i=1}^{3} \bigcup_{j=1}^{2} G_{ij}$ where G is still the same as before. Then any idempotent measure on G_1 whose support is exactly G_1 would be of the form $\varepsilon = \sum_{i=1}^{3} \sum_{j=1}^{2} \delta_i \eta_j \mu_{ij}$ where μ_{ij} is the normalized Haar measure on G_{ij} and $\sum_{i=1}^{3} \delta_i = \eta_1 + \eta_2 = 1$. Observe that G_1 is a simple closed subsemigroup of G. Therefore, any idempotent measure on G_1 is also an idempotent measure on G, but the support of this idempotent measure is strictly contained in G. There of course exist other simple subsemigroups (in this case all subsemigroups are closed as G_{ij} is open and closed for all i=1,2,3 and j=1,2,3). Some of these are $G_2 = \bigcup_{i=1}^{2} \bigcup_{j=1}^{2} G_{ij}$, $G_3 = \bigcup_{i=1}^{3} G_{ij}$ for some fixed j and so on. In all of these cases we must redefine the matrix P so that muliplication is well defined. Clearly the subset $H = G_{11} \cup G_{12} \cup G_{21}$ is not a subsemigroup as $G_{21} G_{12} = G_{22}$ and $G_{22} \not\subset H$ (for any matrix P). Hence H is not closed under multiplication, but intuitively speaking, any submatrix of the 3 × 3 matrix is a simple subsemigroup of G. In general if a compact semigroup S has a closed subgroup, then one idempotent probability measure on S is just the extension of the normalized Haar measure on the group to all of S.

One condition that was imposed was that $\delta_i \neq 0$ and $\eta_k \neq 0$. If however, $\delta_i = 0$ for some i or $\eta_k = 0$ for some k, then the resulting measure is still an idempotent measure on S, if the resulting subset of P by eliminating those group components for which $\delta_i = 0$ or $\eta_k = 0$, is still a subsemigroup. This was demonstrated in example (i) above. The support of the resulting probability measure will no longer be P, but the corresponding subsemigroup of P. We have to have a subsemigroup, otherwise nothing will make sense except for the formal sum $\sum_{i=1}^{s} \sum_{j=1}^{r} \delta_i \eta_j \mu_{ij}$.

We will make a general intuitive remark here which will make sense (or more sense) later when we further discuss completely simple semigroups. Suppose S is a compact semigroup. Then S has a Kernel K. Suppose that this kernel has a finite number of idempotents. Then

$$K = \bigcup_{i=1}^{s} \bigcup_{j=1}^{r} G_{ij}.$$

K has as many simple closed subsemigroups as there are submatrices of an s × r matrix. Hence, if we want all the idempotent probability measures on S, we must consider all of these simple subsemigroups, but there are just a finite number of these.

(ii) The following example considers a common semigroup of 2 × 2 matrices. A direct approach to finding some idempotent probability measures seems almost impossible. But by using the above results, we can quickly find a nontrivial idempotent probability measure on this set. We have considered this example before under the structure of semigroups.

Let $S = \{A \in (R)_2 \mid ||A|| \leq 1\}$ with the induced topology from $(R)_2$. Then S is a compact semigroup. Define H as follows:

$$H = \{A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \mid 0 \leq \theta < 2\} .$$

H is a subsemigroup of S and furthermore H is simple and closed and hence compact. Therefore H is a simple compact topological semigroup with identity (Also $H \cong G$ (topologically and isomorphically) where G is the circle group.) By the above result we get the following. Let μ be the normalized Haar measure on H (as H is a compact group). Extend μ to S as before, that is for all Borel subsets E of S, $\mu(E) = \mu(E \cap H)$. Then $\mu \in M(S)$ and $\mu * \mu = \mu$, by the previous result. (Of course, this could be verified directly, but it is much more difficult.) The result that we used above was that the support of an idempotent probability measure on a compact semigroup S is a simple (closed) subsemigroup of S. As H is a closed simple subsemigroup of S and since H is a group, we found a non-trivial idempotent probability measure on S. The trivial one is of course unit mass on the identity.

Now we know what all the idempotent probability measures look like on a simple compact topological semigroup having only a finite number of idempotents. They are namely the convex combination of the normalized Haar measures of the various group components. We also know that if S is a compact semigroup and ε is an idempotent probability measure on the kernel of S, the ε can be extended to the whole compact semigroups S.

However, there are many other simple compact semigroups other than those which contain just a finite number of idempotents. These are those simple compact semigroups which contain an infinite number of idempotents and we will look at those now.

Let S be a compact simple semigroup and P a simple closed subsemigroup of S. Then P is compact. One question that we want to consider now is, what can be said about the structure of idempotent probability measures, if P contains an infinite number of idempotents such that the support of these idempotent measure is P? First we will consider the case when P contains a countable number of idempotents. Therefore:

$$P = \bigcup_{j \varepsilon J} L_j = \bigcup_{i \varepsilon I} R_i$$

$$= \bigcup_{i \varepsilon I} \bigcup_{j \varepsilon J} G_{iJ} \qquad (G_{ij} = R_i L_j)$$

Here $\phi \neq I \subset \omega$ and $\phi \neq J \subset \omega$ and at least one of I or J has to be equal to ω where $\omega = \{0, 1, 2, \ldots\}$. In other words, one of the index set can be finite or infinite countable. We will consider the case when both I and J are countabley infinite. The approach will follow along the same lines as that of the finite case.

Suppose that μ is an idempotent probability measure on P, then μ_{ik} (μ restricted to G_{ik}) is an invariant positive measure on the group G_{ik} and furthermore $\mu(R_i)\ \mu\ (L_k) = \mu(G_{ik})$ [9].

Now let μ_{ik} denote the normalized Haar measure on the group G_{ik}. G_{ik} is a group, in fact it is a topological group as P is a compact semigroup. Recall that Numakura has proven that every minimal ideal (left, right, two-sided) of a compact semigroup is closed and compact, and hence every group, being the product of a compact minimal right ideal and a compact minimal left ideal is also compact [2] (Lemma 7, page 103 and Corollary page 107). Hence by Wendel's result [1], a normalized Haar measure exists on G_{ik}, even though P contains an infinite number of idempotents. At this point we should backtrack a little to the finite case. In the finite case, all the G_{ik} were open, but in the infinite case this need not hold. In fact if all the G_{ik} are open, then P could only have a finite number of group components. This should become clearer in the section under the Rees Theorem. The reason why P would only have a finite number of G_{ik} if all the G_{ik} are open is that then the G_{ik}'s are open and closed and pairwise disjoint. If we had an infinite number of open G_{ik} we would immediately have an open covering of P which does not have a finite subcover and hence violating compactness. Hence P would only have a finite number of G_{ik}'s.

Coming back to the case when P has an infinite number of idempotents (with the restrictions imposed above), consider the normalized Haar measures μ_{ik} defined on G_{ik}. Extend these to all Borel subsets E of S by setting $\mu_{ik}(E) = \mu_{ik}(G_{ik} \cap E)$. Recall that S is a compact semigroup and $P \subset S$ is such that it is simple with a countably infinite number of idempotents. By the above remark and since

$$P = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} G_{ik},$$

if μ is an idempotent probability measure $\varepsilon\ \mathcal{M}(S)$ and $C(\mu) = P$, then we necessarily have that $\mu = \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} t_{ik}\mu_{ik}$ with positive numbers t_{ik} satisfying $\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} t_{ik} = 1$.

Again as in the finite case we know $\mu_{ik} = e_{ik}\ \mu_{11}\ e_{ik}$, where e_{ij} are the unit mass probability measures on $e_{ij}\ \varepsilon\ G_{ij}$. Now let $\mu = \mu^2\ \varepsilon\ \mathcal{M}(S)$, with $C(\mu) = P$, be in the form of $\mu = \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} t_{ik}\mu_{ik}$

with $\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} t_{ik} = 1$, $t_{ik} > 0$. Then we have the following:

$$\mu^2 = \left(\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} t_{ik}\, \mu_{ik}\right)\cdot\left(\sum_{j=1}^{\infty} \sum_{l=1}^{\infty} t_{jl}\, \mu_{jl}\right) = \sum_{i=1}^{\infty} \sum_{l=1}^{\infty} t_{ik}\, \mu_{il} = \mu.$$

Define $\mu_i = \sum_{k=1}^{\infty} t_{ik}\, \mu_{ik}$

and $\overline{\mu}_i = \sum_{l=1}^{\infty} t_{il}\, \mu_{il}$.

Then

$$\left(\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} t_{ik}\, \mu_{ik}\right)\cdot\left(\sum_{j=1}^{\infty} \sum_{l=1}^{\infty} t_{jl}\, \mu_{jl}\right) = \left(\sum_{l=1}^{\infty} \mu_i\right)\cdot\left(\sum_{j=1}^{\infty} \overline{\mu}_j\right)$$

$$= \sum_{j=1}^{\infty} \mu_1\, \overline{\mu}_j + \sum_{j=1}^{\infty} \mu_2\, \overline{\mu}_j + \dots$$

$$= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \mu_{ij}.$$

Consider $\mu_i\, \overline{\mu}_j = \left(\sum_{k=1}^{\infty} t_{ik}\, \mu_{ik}\right)\cdot\left(\sum_{l=1}^{\infty} t_{jl}\, \mu_{jl}\right)$

$$= \sum_{l=1}^{\infty} t_{i1} t_{jl} \mu_{i1} \mu_{jl} + \sum_{l=1}^{\infty} t_{i2} t_{jl} \mu_{i2} \mu_{jl} + \dots$$

$$= \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} t_{ik}\, t_{jl}\, \mu_{ik}\, \mu_{jl}.$$

That is

$$\left(\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} t_{ik}\mu_{ik}\right)\cdot\left(\sum_{j=1}^{\infty} \sum_{l=1}^{\infty} t_{jl}\mu_{jl}\right) = \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} t_{ik} t_{jl} \mu_{ik} \mu_{jl}.$$

But by a previous remark, $\mu_{ik}\, \mu_{jl} = \mu_{il}$. Hence

$$\left(\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} t_{ik}\, \mu_{ik}\right)\left(\sum_{j=1}^{\infty} \sum_{l=1}^{\infty} t_{jl}\, \mu_{jl}\right) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} t_{ik}\, t_{ik}\, \mu_{il}.$$

This equation is true for every Borel set in S. Therefore the coefficients of μ_{il} are equal; that is because

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} t_{ik}\, t_{jl}\, \mu_{il} = \sum_{i=1}^{\infty} \sum_{l=1}^{\infty} t_{il}\, \mu_{il},$$

and hence we have $\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} t_{ik}\, t_{jl} = t_{il}$. (A)

Set $\sum_{k=1}^{\infty} t_{ik} = \delta_i$ and $\sum_{j=1}^{\infty} t_{jl} = \eta_l$ then by (A) we have that

$t_{il} = \delta_i \eta_l$. Thus

$\mu = \sum_{i=1}^{\infty} \sum_{l=1}^{\infty} \delta_i \eta_l \mu_{il}$ where $\delta_i > 0$ and $\eta_l > 0$ and

$$\sum_{i=1}^{\infty} \delta_i = \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} t_{ik} = 1 = \sum_{j=1}^{\infty} \sum_{j=1}^{\infty} t_{jl} = \sum_{l=1}^{\infty} \eta_l .$$

Conversely let $\mu_1 = \sum_{i=1}^{\infty} \sum_{l=1}^{\infty} \delta_i \eta_l \mu_{il}$ be an element belonging to $M(S)$

where δ_l, η_l are positive numbers satisfying $\sum_{i=1}^{\infty} \delta_i = \sum_{l=1}^{\infty} \eta_l = 1$.

We then have

$$\mu_1{}^2 = \left(\sum_{i=1}^{\infty} \sum_{l=1}^{\infty} \delta_i \eta_l \mu_{il}\right)\cdot\left(\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \delta_j \eta_k \mu_{jk}\right)$$

$$= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} \delta_i \eta_l \delta_j \eta_k \mu_{ik}$$

$$= \left(\sum_{l=1}^{\infty} \eta_l\right)\cdot\left(\sum_{j=1}^{\infty} \delta_i\right)\cdot\left(\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \eta_i \mu_k \; {}_{ik}\right) = \mu_1 .$$

The question of convergence of the infinite sums in the infinite case is no problem since all of the above infinite series converge uniformly and absolutely (as all terms are positive), so that rearrangement in all the series is possible and the sums remain the same. Hence all the above operations are well defined.

We have only considered one case when P contains an infinite number of idempotents and that is when both the index sets I and J are countably infinite. The other two cases which actually is just one case, and that is when either I or J is non-empty and finite, follows in a similar manner and thus the case when P has countably infinite number of idempotents, we get the following result. We summarize this result as follows:

Theorem

Let S be a compact topological semigroup and P such a closed simple subsemigroup of S that contains a countable number of idempotents. Let $P = \bigcup_{i \varepsilon I} \bigcup_{j \varepsilon J} G_{ij}$ be its decomposition into its group components, where I and J are non-empty index sets, both subsets of $\omega = \{0, 1, 2, \dots \}$ and at least one equal to ω. Let μ_{ik} denote the normalized Haar measure on G_{ik}. Then every idempotent $E \varepsilon M(S)$ with $C(E) = P$ is of the form

$E = \sum_{i \varepsilon I} \sum_{K \varepsilon J} \delta_i \eta_k \mu_{ik}$ where δ_i, η_k are positive members satisfying $\sum_{i \varepsilon I} \delta_i = \sum_{k \varepsilon J} \eta_k = 1$.

Conversely, if δ_i, η_k are positive numbers satisfying $\sum_{i \varepsilon I} \delta_i = 1 = \sum_{j \varepsilon J} \eta_j$,

then $\sum_{i \varepsilon I} \sum_{j \varepsilon J} \delta_i \eta_j \mu_{ij}$ is an idempotent probability measure belonging to $M(S)$ whose support is exactly P.

As in the finite case, we will consider what will happen if $\delta_i = 0$ for some $i \varepsilon I$. The support of the resulting idempotent probability measure will be a subsemigroup of P. Intuitively, we can think (this will be made precise in the section "The Rees Theorem") of P as a set of I x J matrices. If $\delta_i = 0$ for some $i \varepsilon I$, then effectively we have a set of $I - \{i\} \times J$ matrices. Hence no δ_i can be zero as the resulting probability measure will not have as its support all of P .

If S is a compact semigroup, whose closed simple subsemigroups contain at most a countable many idempotents, then we can find all the idempotent probability measures on S by simply using the above result over and over again on each simple closed subsemigroup.

The final case that has to be considered is if P contains an uncountably number of idempotents. Suppose we consider the following definition of summation of an uncountable number of nonnegative numbers. Define $\sum_{\alpha \epsilon I} a_\alpha = \sup \{ \sum_{i=1}^{K} a_{\alpha i} \mid \forall \{\alpha_1, \ldots, \alpha_k\} \subset I\}$. This definition then gives us that fact, that if $\sum_{\alpha \epsilon I} a_\alpha < \infty$, then at most a countable number of the a_α's are non-zero. So consider $\mu = \sum_{\alpha \epsilon I} \sum_{\beta \epsilon J} a_{\alpha\beta} \mu_{\alpha\beta}$. Then $\mu^2 = \mu$ by the above calculation, where $\sum_{\alpha \epsilon I} \sum_{\beta \epsilon J} a_{\alpha\beta} = 1$, but $C(\mu) \subsetneq P$ since most of the $a_{\alpha\beta}$'s are zero. (A similar remark was made with regard to the finite and countably infinite cases if some of the $t_{ik} \neq 0$.) This means that we can only consider compact simple semigroups P with at most a countably infinite number of idempotents, if we want to apply the above result and method of obtaining idempotent probability measures. If we do consider compact simple semigroups P with an uncountable number of idempotents and we want convex combinations of the extended normalized Haar measures of the group components (which still exist), we immediately return to the case of a compact simple semigroup P having at most a countable number of idempotents. It is possible to have a compact simple semigroup P with an uncountable of idempotents as will be shown later. Furthermore it is possible that such compact semigroup, can have idempotent probability measures on them.

Consider the following example:

Example 8. Let G be any compact group. We will take example 7 and expand the index sets. The index sets will be the set of all positive integers. With the multiplication and the particular matrix P extended to cover the larger index set, we get a semigroup whose multiplication is associative and continuous. Hence G is a topological semigroup. Observe however that each G_{ij} is an open and closed and compact subsets of G.

Therefore observe that G is not compact. But $E = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \delta_i \eta_j \mu_{ij}$ where $\sum_{i=1}^{\infty} \delta_i = 1 = \sum_{j=1}^{\infty} \eta_j$, δ_i, η_j all positive and μ_{ij} is the normalized Haar measure on $G_{ij} \cong G$ for all i and j. Furthermore, $C(E) = G$ and all the idempotent probability measures belonging to $M(G)$ are of this form. So for some particular locally compact semigroups, the result proved earlier also holds. In this case it holds since each group component is compact. However, for an arbitrary completely simple locally compact group, the group component need not be compact, in fact they are generally just locally compact. Furthermore, the only results on locally compact groups are on commutative locally compact groups. Notice that we had to have a completely simple locally compact semigroup. A simple locally compact semigroup need not be a completely simple semigroup, but simplicity was all that was needed in compact semigroups to get a completely simple semigroup. As a result, locally compact semigroups are more difficult to work with and to find idempotent probability measures on them, whereas for compact simple semigroups with at most a countably number of idempotents, idempotent probability measures are relatively easy to characterize. The above example does give us a class of locally compact semigroups for which we can quite easily find idempotent probability measures.

So now we have considered the case of a simple compact topological semigroup containing at most a countable number of idempotents. We have been able to characterize all idempotent probability measures on such semigroups whose support is the semigroup. Now we want to take one step further and consider simple compact topological semigroups that contain an uncountable number of idempotents. In order to do this, we shall first discuss the Rees-Theorem and how this relates to completely simple semigroups of which simple compact semigroups are just a part.

CHAPTER V - THE REES THEOREM

In the last section we considered a compact semigroup whose simple closed subsemigroup contained only a countable number of idempotents. We were able to characterize all idempotent measures on such semigroups. But when a compact simple semigroup contained an uncountable number of idempotents, the above method failed to characterize idempotent probability measures on such semigroups. Hence we must reconsider such semigroups in order to be able to describe idempotent probability measures on simple compact semigroups containing an uncountable number of idempotents. In order to do this we will first consider the Rees decomposition theorem of completely simple semigroups. Then we will try to relate our previous decomposition theorem on completely simple semigroups to the Rees decomposition theorem. With the Rees decomposition theorem we then can give a partial characterization of idempotents probability measures on a simple compact semigroup, and again this result will be compared to the previous result in the last section.

Let G be a group and $0 \notin G$. Define G^o as follows:

$$G^o = G \cup \{0\} \text{ and}$$

$$g_1 \cdot g_2 = \begin{cases} g_1 \cdot g_2 & \text{if } g_1, g_2 \quad G \\ 0 & \text{if } g_1 = 0 \text{ or } g_2 = 0. \end{cases}$$

Then G^o is called a <u>group with zero</u>. Let S be a semigroup. Similarly define S^o if S has no zero. If S does have a zero then $S^o = S$. Note, if a semigroup S has a zero, then that zero is always unique. Suppose $0, 0' \in S$ are two zeros, then

$$0 = 0 \cdot 0' = 0' \cdot 0 = 0.$$

That is $0 = 0'$ and hence there exists only one zero is S. Let A be a (left, right, two-sided) ideal of S. A is an 0-minimal (left, right, two-sided) ideal of S^o if $\{0\} \subset A$ is the only (left, right, two-sided) proper ideal, respectively, contained in A. A semigroup S with a zero element 0 is called 0-simple if $S^2 \neq 0$ and $\{0\}$ is the only proper two-sided ideal of S. Clearly $S^2 \neq 0$ implies that $S^2 = S$ as $\{0\}$ is the only proper two-sided ideal of S. Observe that if S is a semigroup without a zero, then if A is a 0-minimal (left, right, two-sided) ideal of S^o, then $A - \{0\}$ is a minimal (left, right, two-sided) ideal of S and conversely. A semigroup S with a zero element 0 is called completely 0-simple if S is 0-simple and S has at least one 0-minimal left ideal and at least one 0-minimal right ideal. Clifford and Preston [11] (pages 76 to 83) have shown that if S is a completely 0-simple semigroup, then S is the union of its 0-minimal left (right) ideals. Furthermore, they have shown that S is 0-simple if and only if $S a S = S$ for all $a \in S$, $a \neq 0$. Also if $\{R_i \mid i \in I\}$ and $\{L_\lambda \mid \lambda \in \Lambda\}$ are its 0-minimal right ideals and its 0-minimal left ideals, respectively, then for every $i \in I$ and $\lambda \in I$ and $\lambda \in \Lambda$, $H_{i\lambda} = R_i \cap L_\lambda$ is a (maximal) 0-subgroup of S; furthermore $H_{i\mu} = H_{i\lambda} H_{j\mu}$ for every $i, j \in I$ and $\lambda, \mu \in \Lambda$. They have also shown that every completely 0-simple semigroup with identity is a group with zero. In other words, a completely 0-simple semigroup has " basically " the same properties as a completely simple semigroup except that a completely 0-simple semigroup has a 0 as one of its elements. Furthermore, let $H_{i\mu}$ and $H_{j\lambda}$ be two subgroups derived from R_i , L_μ and R_j , L_λ , respectively, then

$H_{i\mu} \cap H_{j\lambda} = \{0\}$.

Let G^o be a group with zero. Let X be any set and for $i \varepsilon X$, $i \to a_i$ (for $a \varepsilon G^o$) be a mapping of X into G^o. We define Σ on G^o as follows:

$$\sum_{i \varepsilon X} a_i = \begin{cases} a_i \text{ if } a_i \neq 0 \text{ and } a_j = 0 \quad j \neq i \\ 0 \text{ if } a_i = 0 \text{ for all } i \quad X \\ \text{undefined if } a_i \neq 0 \text{ and } a_k \neq 0,\ i \neq k. \end{cases}$$

Now let X and Y be any sets, by an X × Y matrix over G^o we mean a mapping A of X × Y into G^o. If $(i, j) \varepsilon X \times Y$ and $a_{ij} = A((i,j))$, then we may write $A = (a_{ij})$ and speak of a_{ij} as the entry of A lying in the i^{th} row and the j^{th} column of A. Let X, Y, Z be sets. Let $A = (a_{ij})$ be an X × Y matrix over G^o and let $B = (b_{jk})$ be a Y × Z matrix over G^o. If for every pair $(i,k) \varepsilon X \times Z$, the sum $C_{ik} = \sum_{j \varepsilon Y} a_{ij} b_{ik}$ is defined, where $a_{ij} \varepsilon G^o$ and $b_{jk} \varepsilon G^o$ for all $i \varepsilon X$, $j \varepsilon Y$, $k \varepsilon Z$, then we define the matrix product C = AB of A and B to be the X × Z matrix $C = (c_{ik})$ over G^o. A matrix A over G^o is called row-monomial if each row of A contains at most one non-zero element of G^o. The set of all row- monomial X × Y matrices over G^o is a semigroup.

Now we will consider a slightly different type of semigroup of matrices over G^o. Let I and Λ be arbitrary sets. The elements of I will be denoted by k, j, k, ... , and those of Λ by $\lambda, \mu, \gamma, \ldots$. By a Rees I × Λ matrix over G^o we mean an I × Λ matrix over G^o having at most one non-zero entry. Let A be a Rees I × Λ matrix over G^o, then

$$A = \begin{cases} (a)_{i\gamma} & \text{where } a \neq 0 \text{ in the } (i, \gamma) \text{ entry} \\ 0 & \text{if } b_{i\mu} = 0 \text{ for all } (i, \mu) \varepsilon I \times \Lambda. \end{cases}$$

Now let $P = (p_{\lambda i})$ be an arbitrary but fixed $\Lambda \times I$ matrix over G^o. P is called a sandwich matrix. Define the binary operation (o), using P, on the set of Rees I × Λ matrices over G^o as follows:

$$A \circ B = A\ P\ B.$$

It follows immediately that if A, B are Rees $I \times \Lambda$ matrices, then so is $A \circ B$. Also $(\circ)$ is associative as

$$A \circ (B \circ C) = A\ P\ (B \circ C) = A\ P\ (B\ P\ C)$$
$$= (A\ P\ B)\ (P\ C)$$
$$= (A \circ B)\ P\ C$$
$$= (A \circ B) \circ C.$$

Therefore,the set of Rees $I \times \Lambda$ matrices over G^o is a semigroup with respect to the operation $(\circ)$. This semigroup is called the Rees $I \times \Lambda$ matrix semigroup over the group with zero G^o with sandwich matrix P and it will be denoted by $M^o\ (G;\ I,\ \Lambda;\ P)$. G is called the structure group of M^o. Notice that for every different sandwich matrix P', we get a different binary operation on the set of the Rees $I \times \Lambda$ matrices. Note that in example (7) and (8) we considered an example where $I = \Lambda$ which were subsets of the integers and the sandwich matrix had as its entries the identity element of the group for all $(i, \lambda) \in I \times \Lambda$.

Another approach to the Rees matrix semigroup (as used by M.Rosenblatt, Heble) is to begin with the set $G^o \times I \times \Lambda$ consisting of all ordered triples, where G^o is a group with a zero and $I \times \Lambda$ are arbitrary sets. Multiplication is defined by $(\circ)$ as follows:

$$(a;\ i,\ \lambda) \circ (b;\ j,\ \mu) = (ap_{\lambda j}\ b;\ i,\ \mu)$$

where P is a mapping from $\Lambda \times I \to G^o$, (from henceforth $\circ$ is just denoted by juxtaposition).

Associativity is easily verified. Let $0 = \{(0;\ i,\ \mu)\ |\ 0 \in G^o$ $(i, \mu) \in I \times \Lambda\}$. It is immediate the 0 is the zero of $G^o \times I \times \Lambda$. Notice now that the $p_{\lambda j}$ represents the sandwich matrix and $(a;\ i,\ \lambda)$ is just a Rees $I \times \Lambda$ matrix. Hence $G^o \times I \times \Lambda \cong M^o\ (G;\ I,\ \Lambda;\ P)$.

Now suppose that P contains no zeros in any of its entries, then M^o (G; I, Λ; P) has no proper zero divisors and hence M^o - {0} is called the Rees I × Λ matrix semigroup without zero over the group G with sandwich matrix P and it will be denoted by M(G; I, Λ; P). The question we want to look at is for what P will M^o(G; I, Λ; P) have no proper zero divisors? A partial answer, as was noted above, is if P has no zero entries. It is possible, however, to get a complete answer.

A matrix P over a group with zero is regular if and only if each row and each column of P contains at least one non-zero entry. The Rees I × Λ matrix semigroup M^o(G; I, Λ; P) over a group with zero and with sandwich matrix P is regular if and only if P is a regular Λ × I matrix [11] (Lemma 3.1, page 89). Recall that a semigroup S is regular if for every a ε S, there exists a b ε S such that a b a = a. A further result is that a Rees matrix semigroup is 0-simple if and only if it is regular and if so, it is completely 0-simple [11] (Theorem 3.3, page 90). Notice that a Rees matrix semigroup has built in it minimal left ideals and minimal right ideals. A minimal left ideal is just the subsets of the form of Rees I × {μ} matrices and a minimal right ideal is just the subsets of the form of Rees {i} × Λ matrices. The notation is not technically correct, but we use the subsets of the Rees I × Λ matrices whose isomorphic copies look like the above.

Let M^o be regular. Then

$$M^o = \bigcup_{\lambda \varepsilon \Lambda} L_\lambda \cup \{0\}$$

where $L_\lambda = \{(a)_{i\lambda} \mid a \varepsilon G,\ i \varepsilon I\}$ and also

$$M^o = \bigcup_{i \varepsilon I} R_i \cup \{0\}$$

where $R_i = \{(a)_{i\lambda} \mid a \in G \text{ and } \lambda \in \Lambda\}$. Notice that the 0-minimal left ideals are $L_\lambda \cup \{0\}$ and the 0-minimal right ideals are $R_i \cup \{0\}$ and $L_\lambda \cap L_{\lambda 1} = \{0\}$ $(\lambda \neq \lambda_1)$and similarly $R_i \cap R_{i_1} = \{0\}$ $(i \neq i_1)$. Therefore,M^o is completely 0-simple if M^o is regular. Furthermore $G_{i\lambda} = R_i \cap L_\lambda$ is a group and is isomorphic to G. Intuitively

$$G_{i\lambda} = \{(g)_{i\lambda} \mid g \in G\}.$$

The important theorem that was proven by Rees is as follows. A semigroup S is completely 0-simple if and only if it is isomorphic with a regular Rees-matrix semigroup over a group with zero [11] (Theorem 3.5, page 94).

Suppose now that we have a completely simple semigroup S, then S^o is a completely 0-simple semigroup. It is therefore isomorphic with a regular Rees matrix semigroup over a group with zero. S is then isomorphic to a Rees matrix semigroup over a group as the zeros of each "line up". In some discussions,the writer states that a completely simple semigroup S is isomorphic to the Rees Product $G \times X \times Y$, where the multiplication is defined in the same manner as was described in the alternative way of constructing Rees matrix semigroups (as mentioned earlier; that is, the Rees products and Rees matrix semigroups over a group are the same), where G is a group and X, Y are arbitrary sets; sometimes $X \times G \times Y$ is written instead of $G \times X \times Y$. Everything is still the same, just the first and the second coordinates have been interchanged. Both ways of writing will be used throughout the remaining part of this paper. Furthermore, if G is a compact topological group, I, Λ compact Hausdorff spaces and P a regular $\Lambda \times I$ sandwich matrix, then $M[G; I, \Lambda; P]$ is a compact topological semigroup which is simple [12] (page 430 to 432). As M is just $G \times I \times \Lambda$, M is immediately compact. M has the corresponding matrix topology and as multiplication is matrix multiplication with a sandwich matrix, it then becomes reasonable that multiplication is continuous. Hence is a compact topological semigroup.

Now we will show that our previous decomposition into groups of completely simple semigroups is the same up to isomorphism to the Rees matrix semigroup. Let us first consider a completely simple semigroup S containing a finite number of idempotents. Then

$$S = \bigcup_{i=1}^{S} \bigcup_{j=1}^{r} G_{ij}.$$

In this case, the set $I = \{1, 2, \ldots, S\}$ and the set $\Lambda = \{1, 2, \ldots, r\}$ and the group is G_{11}. The sandwich matrix is then easily determined by observing that $G_{\alpha\beta}\, G_{\gamma\delta} = G_{\alpha\delta}$. So consider $p_{\alpha\beta}$. Take $G_{1\beta}$ and $G_{\alpha 1}$, let $g_{1\beta} \in G_{1\beta}$ and $g_{\alpha 1} \in G_{\alpha 1}$ such that $g_{1\beta} \circ g_{\alpha 1} = g_{11}$. But $g_{1\beta}, g_{11}, g_{\alpha 1} \in G$ and so $g_{1\beta}\, p_{\beta\alpha}\, g_{\alpha 1} = g_{11}$ for some $p_{\beta\alpha}$. Hence we can solve for $p_{\beta\alpha}$ as a value in G as G is a group. Thus we have determined the sandwich matrix P. We can, of course, heavily rely on the fact that for all i, λ, j, μ, $G_{i\lambda} \cong G_{j\mu}$. Then $G \times I \times \Lambda$ with the multiplication $(a; i, \lambda)$ $(b; j, \mu) = (a\, p_{\lambda j}\, b; i, \mu)$ is a completely simple semigroup. Now $(a; i, \lambda) \in G_{i\lambda}$ and $(b; j, \mu) \in G_{j\mu}$ and $(a\, p_{\lambda j}\, g; i, \mu) \in G_{i\mu}$. Then clearly $S \cong G \times I \times \Lambda$. Hence a completely simple semigroup containing a finite number of idempotents is a Rees product. Similarly a completely simple semigroup containing a countable number of idempotent can also be written as a Rees product or a Rees $I \times \Lambda$ matrix semigroup without zero over the group G with sandwich matrix P.

If S is a simple compact semigroup and hence a completely simple semigroup, then because S is a completely simple semigroup, S can be written as a Rees Product $G \times X \times Y$. But since S is compact we know that G is compact and furthermore that X and Y are also compact and Hausdorff, as S is Hausdorff. This then gives us a complete characterization of the structure of completely simple semigroups and simple compact topological semigroups.

One of the questions we are looking at is what are the idempotent probability measures on compact topological semigroups. We have an answer if the simple compact topological semigroup contains only a countable number of idempotents. We now know that a compact simple semigroup S is isomorphic to the Rees product $G \times X \times Y$ (or the Rees $X \times Y$ matrix semigroup without zero over the group G with sandwich matrix P) and S is compact if and only if G, X, Y are compact (Hausdorff) and G is a topological group. Then $E = \mu \times \gamma \times \beta$ is an idempotent probability measure where μ is the normalized Haar measure on G, γ and β are probability measures on X and Y respectively and E is the product measure of μ, γ, β[12] (pages 430 to 432).

Now, conversely, suppose S is a simple compact semigroup, so that $S = G \times X \times Y$, where G is a compact topological group and X, Y are compact Hausdorff spaces. Let μ be a regular idempotent probability measures. What can we say about μ? Since μ is regular we know that $\mu = E \times \alpha \times \beta$, where E is the normalized Haar measure on G and α, β are regular probability measures on X, Y respectively [12] (pages 430 to 432). So for any regular idempotent probability measures on a compact semigroup we know its decomposition with regard to the Rees product. Note that if the simple compact semigroup contains only a countable number of idempotents, that is, the cardinality of X and Y is at most w, then all idempotent probability measures on the simple compact topological semigroups are regular measures. Now how does the above result compare to our earlier result?

Consider a compact simple semigroup S with a finite number of idempotents. Then as shown before, any idempotent probability measure on S can be written as $E = \sum_{\alpha=1}^{s} \sum_{\beta=1}^{r} \delta_\alpha \eta_\beta \mu_{\alpha\beta}$ where $S = \bigcup_{\alpha=1}^{s} \bigcup_{\beta=1}^{r} G_{\alpha\beta}$ and $\sum_{\alpha=1}^{s} \delta_\alpha = 1 = \sum_{\beta=1}^{r} \eta_\beta$ and δ_σ, η_β are positive numbers. Recall that $\mu_{\alpha\beta}$ is the normalized Haar measure on $G_{\alpha\beta}$ and can alternatively be written as

$$\mu_{\alpha\beta} = e_{\alpha 1} \mu_{11} e_{1\beta}$$

(where juxtaposition is convolution and $e_{\alpha j}$ is the unit mass measure). Then

$$E = \sum_{\alpha=1}^{s} \sum_{\beta=1}^{r} \delta_\alpha \eta_\beta \mu_{\alpha\beta} = \sum_{\alpha=1}^{s} \sum_{\beta=1}^{r} \delta_\alpha \eta_\beta e_{\alpha 1} \mu_{11} e_{1\beta}$$

$$= \left(\sum_{\alpha=1}^{s} \delta_\alpha p_{\alpha .}\right) \mu_{11} \left(\sum_{\beta=1}^{r} \eta_\beta e_{1\beta}\right).$$

Denote $\gamma_1 = \sum_{\alpha=1}^{s} \delta_\alpha e_{\alpha 1}$ and $\gamma_r = \sum_{\beta=1}^{r} \eta_\beta e_{1\beta}$ we have

$$E = \gamma_1 \mu_{11} \gamma_r$$

with $C(\gamma_1)$ = {set of all idempotents in L_1}
and $C(\gamma_r)$ = {set of all idempotents in R_1} and μ_{11} is the normalized Haar measure on the compact group G_{11}. Observe that the cardinality of $C(\gamma_1)$ is s and that of $C(\gamma_r)$ is r and so if S has a finite number of idempotents, the idempotent probability measures that we get on S are the same whether we write S as a union of its group components or write S as a Rees matrix semigroup over a group without zero and sandwich matrix P. We can find all idempotent probability measures easier by considering the group decompostion of S, but it should be clear what the probability measures on the sets I = {1, ... , s} and Λ = {1, ... , r} have to be.

There is one comment that is appropriate here. Consider a completely simple semigroup $S = \bigcup_{\alpha=1}^{s} \bigcup_{\beta=1}^{r} G_{\alpha\beta}$. Recall that $e_{\alpha\beta}\, G_{\beta\gamma}\, e_{\gamma\delta} = G_{\alpha\delta}$. S. Schwarz makes a statement that $e_{\alpha\beta}\, e_{\beta\gamma}\, e_{\gamma\delta} = e_{\alpha\delta}$ need not hold. When we first discussed this, this was not explained, but with the Rees matrix semigroup over a group G without zero and sandwich matrix P, $M =$ $= M(G; I, \Lambda; P)$, this statement can now be explained. M is a completely simple semigroup if and only if P is regular (that is P has at least one nonzero element in each row and column of P). This sandwich matrix P will determine the multiplication of the elements of M. This sandwich matrix need not take two idempotent elements into another idempotent element as the entries of P are arbitrary so long as each row and column has one non-zero entry. For example, if P just has the identity element of the group on the diagonal, that is $P_{\lambda\lambda} = p_{ii} = e$, then $e_{ij} \quad e_{ik} = e_{ik}$. Otherwise anything could happen.

Now we know that every compact simple topological semigroup S can be written as the direct product of a compact group G and two compact Hausdorff spaces X and Y [12] (page 430 to page 432) and thus some of the idempotent probability measures of such a semigroup S can then be easily described. Let μ be the normalized Haar measure of G and α,β regular probability measures on X and Y. Then the product measure $\mu \times \alpha \times \beta$ is an idempotent probability measure on S [11]. As μ is unique and fixed, the idempotent probability measures on S of the above form depend only on the probability measures α and β on X and Y, respectively. And of course we saw this when we considered simple compact topological semigroups S containing an finite number of idempotents as the only condition on the coefficients of μ_{ij} was the δ_i and η_j be positive for all i and j and $\sum_{i=1}^{s} \delta_i = 1 = \sum_{j=1}^{r} \eta_j$. Notice that all this says is that α is a probability (discrete) measure on the set $\{1, 2, \ldots, s\}$ and β is a probability (discrete) measure in the set $\{1, 2, \ldots, r\}$. Hence $E = \alpha \times \mu \times \beta$ is an idempotent probability measure on

$\{1, 2, \ldots, s\} \times G \times \{1, 2, \ldots, r\}$ [15].

For arbitrary compact Hausdorff spaces X and Y and the compact group G, there is no condition to tell us that all the idempotent probability measures on S are of the form $\mu \times \alpha \times \beta$. Stefan Schwarz makes the following statement: [15] (pages 121 - 122)

> "If T is a closed simple subsemigroup of S, there need not exist in general an idempotent $E \in M(S)$ with $C(E) = T$. But such an idempotent always exists if T contains only a finite number of idempotent elements."

Thus we have been able to determine idempotent probability measures on some simple compact topological semigroups. Now that we can determine some idempotent probability measures on a simple compact semigroup, we are in a position to determine some idempotent probability measures on compact semigroups. We first determine the kernel of a compact semigroup. This always exists as every compact semigroup has at least one minimal left ideal and at least one minimal right ideal and thus we are assumed of a minimal two-sided ideal which is unique. Then we determine the idempotent probability measures, if this is possible, on the kernel. Then we can extend these measures to the whole semigroup. They will still be idempotent probability measures and the support will always be the kernel of the semigroup. Furthermore, if we can find any other closed simple subsemigroups of the compact semigroup, then we can also define idempotent probability measures on these semigroups and extend them to the whole semigroup and these as well will be idempotent probability measures on that semigroup.

Now that we know what some of the idempotent probability measures look like, we again turn our attention to the question of the convergence of the following sequences $\{\mu^{(n)}\}_{n \in w}$ and $\{1/n \sum_{i=1}^{n} \mu^{(i)}\}_{n \in w}$ where μ is any probability measure on a compact group. By studying the convergences of these type of sequences we will get a partial answer to the question of stable laws on a compact semigroup.

CHAPTER VI - CONVERGENCE OF PROBABILITY MEASURES

Finally we are ready to describe the Central Limit Theorem for compact semigroups. As was mentioned in the introduction, in one case we consider a probability measure μ on a compact topological semigroup S and we asked whether or not the sequence $\{\mu^{(n)}\}_{n \varepsilon w}$ converges. We now want to describe the conditions under which this sequence will converge. If it converges, then its limit is a stable law. In this section as well we will describe some of the stable laws of an abstract compact semigroup.

Let S be a compact semigroup and μ a probability measure on S. Then $\mu^{(n)}$ will be defined inductively as follows:

$\mu^{(1)} = \mu$, $\mu^{(2)} = \mu * \mu$, $\mu^{(3)} = \mu * (\mu * \mu)$, ... , $\mu^{(n+1)} =$

$= \mu * \mu^{(n)}$. Note that convolution is an associative operation.

Now consider the following sequence $\gamma_n = 1/n \sum_{i=1}^{n} \mu^{(i)}$, n = 1, 2, 3, ...

(This can be thought of as taking the average of the probabilities of various samples, or if we compare this to the convergences of a sequence of real numbers, this would be a Cesaro Sum.) Rosenblatt [13] (Theorem 7, page 193) has proven the following amazing fact. If μ is any probability measure on a compact topological semigroup, then the averages γ_n converge in the weak star topology to an idempotent measure γ ($\gamma^{(2)} = \gamma$) which also satisfies the relation

$\mu * \gamma = \gamma * \mu = \gamma$.

Furthermore (Rosenblatt [13] Lemma 13 and Theorem 8, page 194) the mass of the sequence of the probability measures γ_n concentrates on the kernel

K (every compact semigroup S has a kernel as every compact semigroup S has at least one minimal left ideal and at least one minimal right ideal and hence a minimal two-sided ideal which is completely simple). As $n \to \infty$; that is given any open set G with $K \subset G$ and any fixed $\varepsilon > 0$, there is an m sufficiently large so that for $n > m$, $\gamma_n(G) > 1 - \varepsilon$ Therefore, the support of the idempotent probability measure $\gamma = \lim_{n\to\infty} 1/n \sum_{i=1}^{n} \mu^{(i)}$ is the kernel of S. If the kernel K of S contains a finite or countable number of idempotent elements, we then know exactly what the idempotent probability measures look like as was shown in Chapter IV. We also know exactly what K looks like.

Before we continue we would like to return to compact groups. Ulf Grenander [2] (Corollary, page 107) proved that for a given non-trivial probability distribution μ, the limit of $\mu^{(n)}$ as $n \to \infty$ exists if and only if $C(\mu)$ is not contained in any coset of any closed proper, normal subgroup of G. The limit of $\mu^{(n)}$ is the normalized Haar measure on G. Let G be a group and H a subgroup, then gH is a left coset of G for any $g \in G \sim H$. Notice that if μ and γ are probability measures on G satisfying the above condition, then $\lim_{n\to\infty} \mu^{(n)} = \lim_{n\to\infty} \gamma^{(n)}$ is the normalized Haar measure on G as G is a compact group and the normalized Haar measure is the only non-trivial idempotent probability measure on G. So any number of probability measures satisfying the above condition will have the same limit. This then gives us an easy condition to check for probability measures that will converge on a compact group. So what are the non-trivial stable laws on a compact group? They are, the normalized Haar measure on the compact group, and the trivial idempotent probability measure on the compact group. (More will be said later about the Domain of Attraction question.)

Up to now, we have been successful in extending results from compact groups to simple compact semigroups, as simple compact semigroups are just the disjoint union of isomorphic compact groups. Can we extend the above result to compact topological semigroups? Rosenblatt [13] (Lemma 14, Theorem 9 and Corollary, pages 194 to 196) has given us a necessary and sufficient condition for $\mu^{(n)}$ to converge, where μ is a probability measure on a compact semigroup. First, however, we must clarify a few notational conventions that he uses. Recall that a compact semigroup S has a unique kernel K and $K = G \times X \times Y$, where G is a compact group and X and Y are compact Hausdorff spaces. Multiplication is defined as follows for $s = (g; x, y)$ and $t = (g'; x', y')$:

$$s \bullet t = (g; x, y) \bullet (g'; x', y') = (g\ (x'y)\ g'; x, y')$$

where (xy) is a continuous mapping of $X \times Y$ into G. This continuous map is just the $(xy)^{th}$ entry in the sandwich matrix, which was mentioned in the construction of the Rees matrix groups. Now we can state the result of Rosenblatt [13] (Lemma 14, Theorem 9 and Corollary, pages 194 to 196) as follows: "Let γ be a regular probability measure on S whose support generates S. The sequence of measures $\gamma^{(n)}$ will not converge in the weak star topology if and only if there is a proper closed normal subgroup G' of G such that $XY \subset G'$ and the support of γ is contained in $(G' \times X \times Y)^{-1} (gG' \times X \times Y)$ where $g \in G$ and $g \notin G'$." Above we mentioned that (xy) is a continuous mapping into G. Actually the mapping is $\phi : \{X \times Y \to G$ and we just write (xy) instead of $\phi(x, y)$. So XY = $= \{xy \mid x \in X \text{ and } y \in Y\}$, $XY \subset G'$ just means that $\phi(X \times Y) = \{\phi((x, y)) \mid (x, y) \in X \times Y\}$ is contained in G'. Later on we will show that this result by Rosenblatt is an extension from compact groups, as shown by Grenander, to compact semigroups.

Now we want to derive several particular results from the main result mentioned above. Consider the case where $G' = \{e\}$ and $\phi(X \times Y) = \{e\}$ (that is $\phi((x,y))=e$ for all $(x, y) \in X \times Y$). In order to get another result, we must explain what $(G' \times X \times Y)^{-1}$ is and we must recall a few definitions. Two elements a and b of a semigroup S are said to be inverses of each other if

$$a\ b\ a = a \text{ and } b\ a\ b = b.$$

An element a in a semigroup S is regular if $a \in a\ S\ a$, in particular, a regular element b of S also has an inverse namely $c = x\ b\ x$ where $b = b\ x\ b$ since

$$\begin{aligned} b\ c\ b &= b(x\ b\ x)b \\ &= (b\ x\ b)\ (x\ b) \\ &= b\ x\ b \\ &= b \end{aligned}$$

and also
$$\begin{aligned} c\ b\ c &= (x\ b\ x)\ b\ (x\ b\ x) \\ &= x(b\ x\ b)\ (x\ b\ x) \\ &= x\ b(x\ b\ x) \\ &= x(b\ x\ b)x \\ &= x\ b\ x \\ &= c. \end{aligned}$$

Since by definition any completely simple semigroup S is a simple semigroup, we observe that for any $a \in S$, $S = a\ S\ a$ and hence we immediately deduce that S is regular. (A semigroup S is regular if every element in S is regular.) Therefore, every element in S has an inverse (if S is simple). In general, an inverse is not unique. But a completely simple semigroup, that is not a group, does not have an identity, as a completely simple semigroup with identity is a group. Let T be a subsemigroup of S, then T^{-1} is the set of all inverses of all elements in T. Coming back now to the above, we have the following. By the Rees Theorem (and where $G' < G$ and is closed), $G' \times X \times Y$ is a completely simple semigroup, hence a regular semigroup and hence $(G' \times X \times Y)^{-1}$ exists.

Now suppose $\phi((x,y)) \equiv e$ for all $(x, y) \in X \times Y$. Now G' just has to be a proper normal closed subgroup of G. Consider $(G' \times X \times Y)^{-1}$ $(gG' \times X \times Y)$ for some $g \in G \sim G'$. If $\phi(X, Y) = \{e\}$, then for $s =$ $(g; x, y)$ and $t = (g'; x', y')$

$$\begin{aligned} s \cdot t &= (g; x, y) \cdot (g'; x', y') \\ &= (g \ (x', y)g'; x, y') \\ &= (gg'; x, y'). \end{aligned}$$

What however is $(G' \times X \times Y)^{-1}$? The claim is that $((x'y)^{-1}g^{-1}(xy')^{-1} ; x',y')$ is an inverse of $(g;x,y)$ where $\phi((x,y))=xy$. Recall that if $a,b \in S$ and S is a semigroup then a is an inverse of b if and only if $aba = a$ and $bab = b$.

So we have $(g; x, y) \cdot ((x'y)^{-1} g^{-1} (xy')^{-1}; x', y') \cdot (g; x, y) =$

$$\begin{aligned} &= (g(x'y) \ ((x'y)^{-1} g^{-1} (xy')^{-1}) \cdot (xy')g; x, y) \\ &= (g; x, y). \end{aligned}$$

Also

$$\begin{aligned} &((x'y)^{-1}g^{-1}(xy')^{-1} ; x',y') \cdot (g;x,y,) \cdot ((x'y)^{-1}g^{-1}(xy')^{-1} ; x',y') = \\ &= ((x'y)^{-1} g^{-1} (xy')^{-1} (xy') g (x'y) (x y)^{-1} g^{-1} (xy')^{-1}; x', y') \\ &= ((x'y)^{-1} g^{-1} (xy')^{-1}; x', y'). \end{aligned}$$

Therefore $(g; x, y)$ and $((x'y)^{-1} g^{-1} (xy')^{-1}; x', y')$ are inverses of each other. Recall that in a semigroup without identity, an element can have many inverses, if it has any inverses. In this case (g; x, y) has as many inverses as the cardinality of $X \times Y$. Recall that $XY \subset G'$ and hence $(x'y)^{-1} (x'y) \in G'$ and therefore $((x'y)^{-1} g^{-1} (xy'); x', y') \in G' \times X \times Y$. That is all inverse of elements of $G' \times X \times Y$ are elements of $G' \times X \times Y$. That is $(G' \times X \times Y)^{-1} \subset G' \times X \times Y$. But similarly for any $(g; x, y) \in G \times X \times Y$, $((x'y)^{-1} g^{-1} (xy')^{-1}; x', y') \in G' \times X \times Y$. Hence (g; x, y) is therefore an inverse for $((x\ y)^{-1} g^{-1} (xy')^{-1}; x', y)$ and therefore $(g; x, y) \in (G' \times X \times Y)$. Hence $G' \times X \times Y \subset (G' \times X \times Y)^{-1}$ and thus $(G' \times X \times Y) = (G' \times X \times Y)^{-1}$. This means that $(G' \times X \times Y)^{-1} \cdot (gG' \times X \times Y) = (G' \times X \times Y) \cdot (gG' \times X \times Y)$.

Now we are assuming that $\phi(X \times Y) = \{e\}$. This if $s \in G' \times X \times Y$ and $t \in (gG' \times X \times Y)$ we have the following where $s = (g'; x, y)$ and $t = (gh ; x' , y')$:

$$\begin{aligned} s \cdot t &= (g'; x, y) \cdot (gh; x', y') \\ &= (g'(x'y)\ gh; x, y') \\ &= (g'gh; x, y') \quad . \end{aligned}$$

But G' is a normal subgroup of G and hence $gG' = G'g$ and hence

$$g'g = gg''$$

for some $g'' \in G'$.

Thus

$$\begin{aligned} s \cdot t &= (g'gh; x, y') \\ &= (gg''h; x, y'). \end{aligned}$$

Hence we can conclude that $s \cdot t \in gG' \times X \times Y$ and so $(G' \times X \times Y) \cdot (gG' \times X \times Y) \subset gG' \times X \times Y$. But $e \in G'$ and $(e; x, y) \cdot (gh; x, y') = (gh; x, y')$ and hence $(G' \times X \times Y)\ (gG' \times X \times Y) = G' \times X \times Y$.

So Rosenblatt's result [13] (Lemma 7 and Theorem 8 and Corollary, pages 194 to 196) for the case $\phi(x, y) \equiv e$ (for all $(x, y) \varepsilon X \times Y$) is as follows: Let γ be a regular probability measure on a compact topological semigroup S whose support generates S. The sequence of measures of $\gamma^{(n)}$ will not converge in the weak star topology if and only if there is a proper closed normal subgroup G' of G such that the support of γ is contained in $(gG' \times X \times Y)$ for some $g \varepsilon G \sim G'$. If the only proper closed normal subgroup G' of G is $G' = \{e\}$, then the support of γ must be contained in $(\{g\} \times X \times Y)$ for some $g \varepsilon G$, $g \neq e$.

Another question we want to discuss is whether or not Rosenblatt's result is an extension of Ulf Grenander's result [17] (Theorem 3.2.4, page 67). One condition Grenander did not mention was that the probability measure on the compact group did not have to be regular. If we note, however, that any probability measure on a compact group is always a regular measure, we immediately have that condition. So the above result is an extension of Ulf Grenander's result as for compact groups, $\phi(\{x\}, \{y\})$ (if one thinks it this way) is always the identity. The group $G \times \{x\} \times \{y\}$, when G is a compact group, is isomorphic and homeomorphic with the compact group G. Unfortunately $\phi(x, y)$ need not be equal to e all the time; it can assume other values in G. The result however is an extension of Ulf Grenander's result.

In the general case, just so that Rosenblatt's result is stated in positive terms, the result is as follows. Let γ be a regular probability measure on S whose support generates S. Then the sequence of probability measures $\gamma^{(n)}$ will converge in the weak star topology if and only if there is no proper closed normal subgroup G' of G with $XY \subset G'$ such that the support of γ is contained in $(G' \times X \times Y)^{-1} (gG' \times X \times Y)$ where g is some element of G not in G'.

A little while ago we compared the convergence of probability measures to summing sequences of real numbers. One major result is that every summable sequence is Cesaro summable and the two sums are the same. For probability measures we have a similar result. Not all probability measures μ have the property that the sequence $\gamma^{(n)}$ converges, but for all probability measures μ, the sequence $\gamma_n = 1/n \sum_{i=1}^{n} \mu^{(i)}$ does converge. The limit measure for this case is an idempotent probability measure. The sequence $\mu^{(n)}$ need not converge, but if it does converge then $\lim_{n\to\infty} \mu^{(n)} = \lim_{n\to\infty} \mu_n$ [13] (Paragraph #2, page 195). That tells us that if the sequence of probability measures $\mu^{(n)}$ converges, the limit is an idempotent probability measure.

So what are the Stable Laws on compact topological semigroup? They are simply the idempotent probability measures. By the previous work on idempotent probability measures, we know what these idempotent probability measures look like by considering the Rees product of any closed simple subsemigroup of the compact topological semigroup.

We are considering one other question and that is the Domain of Attraction question. This question deals with the problem of determining all probability measure μ on a given compact semigroup S such that the sequence of probability measures $\mu^{(n)}$ will converge to a given stable law or idempotent probability measures. To state which probability measures μ yield $\mu^{(n)}$converging to a given stable law is rather difficult, but we can determine whether or not $\mu^{(n)}$ for a given probability measure μ will converge to some stable law. This is the main part of Rosenblatt's result [13] (Lemma 14, Theorem 9 and Corollary, pages 194 to 196) which gives us a condition on the convergence of a sequence of probability measures $\mu^{(n)}$, given the probability measure .

We will now illustrate this result by considering a few examples. Thus we will get a class of groups (compact groups) such that if we form the Rees product using compact regular spaces X, Y and μ is a probability measure defined on $G \times X \times Y$, then the sequence $\{\mu^{(n)}\}_{n \in w}$ will also converge.

There is one other point that should be made before we discuss these examples and that has to do with the question of Domain of Attraction on Compact Groups. As there exists only one non-trivial idempotent probability measure or stable law, any probability measure μ that satisfies Ulf Grenander's result is an element of the Domain of Attraction of the normalized Haar measure. So for Compact Groups, the Domain of Attraction question is fully answered, but this appears difficult to answer in the case we have compact topological semigroups.

CHAPTER VII - EXAMPLES ON THE CONVERGENCE THEOREM

Now that we have the result by M. Rosenblatt, we want to consider a few examples, to see what it means for special examples. The result was as follows: Let γ be a regular probability measure on S whose support generates S. Then $\gamma^{(n)}$ will converge in the weak star topology if and only if there is no proper closed normal subgroup G' of G with $XY \subset G'$ such that the support of γ is contained in

$$(G' \times X \times Y)^{-1} (gG' \times X \times Y)$$

where g is some element of G not in G'. $(K = G \times X \times Y)$ where K is the kernel of S).

Let T be a subset of a semigroup Q. The semigroup generated by T is the set of all products of a finite number of elements of T.

ie. $T = \{a_1^{\alpha_1} \cdot a_2^{\alpha_2} \ldots a_n^{\alpha_n} \mid a_i \in T, \alpha_i \in \{1, 2, \ldots\}, i=1, \ldots n\}$.

In the first example we start with an arbitrary compact topological Hausdorff group and arbitrary X and Y compact Hausdorff spaces. We define the function $\phi: X \times Y \to G$ by $\phi((x,y)) \equiv e$; we want to consider what happens if we consider the closed normal subgroup $G' = \{e\}$. We will use the convention that for $(x, y) \in X \times Y$, $xy = \phi((x,y))$. Thus as $xy = e$ for all $(x, y) \in X \times Y$, $XY = \{e\}$ and hence $XY \subset G'$. This will lead us to a particular class of compact groups for which, for every regular probability measure γ, the sequence $\gamma^{(n)}$ will converge. It will also give us a class of compact groups for which the opposite is true to a certain degree. Then we will restrict out attention to the simple groups. We will study to see what will happen to a probability measure γ on a simple group. The answer is not quite as simple as one would expect and we will show why. Finally we will consider a concrete example of the above.

Example 1

Let G be a compact Hausdorff group and X and Y compact Hausdorff spaces. Define the multiplication on $G \times X \times Y$ as follows:

$$(g_1; x_1, y_1) \cdot (g_2; x_2, y_2) = (g_1\ x_2\ y_1\ g_2;\ x_1\ y_2)$$

where $\phi((x,y)) = xy = e$ for all $x \in X$ and $y \in Y$ and e is the identity of G; that is, $(g_1; x_1, y_1) \cdot (g_2; x_2, y_2) = (g_1 g_2; x_1, y_2)$. Let μ be a regular probability measure on $G \times X \times Y$ such that the support of μ generates $G \times X \times Y$. Let $G' = \{e\}$, then G' is normal and closed in G. Consider the following:

$$\begin{aligned} G' \times X \times Y &= \{e\} \times X \times Y \\ &= \{(e; x, y) \mid x \in X, y \in Y\} \\ &= \{(e; x, y)^{-1}\} = \{(e; x_1, y) \mid x_1 \in X\} \end{aligned}$$

since $(e;\ x, y_1)\ (e; x_1, y) = (e^2; x, y)$

$$= (e; x, y)$$

$$\begin{aligned} \text{Hence } (G' \times X \times Y)^{-1} &= \{(e; x, y)^{-1} \mid x \in X, y \in Y\} \\ &= \{(e; x, y) \mid x \in X, y \in Y\} \\ &= G' \times X \times Y. \end{aligned}$$

Let $g \in G \sim G'$; that is, $g \neq e$,

$$\begin{aligned} \text{then } gG' \times X \times Y &= \{g\} \times X \times Y \\ &= \{(g; x, y) \mid x \in X, y \in Y\} \end{aligned}$$

$$\begin{aligned} \text{Therefore } (G' \times X \times Y)^{-1} (\{g\} \times X \times Y) &= (G' \times X \times Y)\ (\{g\} \times X \times Y) \\ &= \{g\} \times X \times Y. \end{aligned}$$

Suppose now that the support of μ is $\{g\} \times X \times Y = gG' \times X \times Y$. What set does $\{g\} \times X \times Y$ generate? The set $\{g\} \times X \times Y$ generates the set $\langle g\rangle \times X \times Y = \{(g^{(n)}; x, y) \mid n = 1, 2, \ldots, x \in X, y \in Y\}$. Recall that $\phi((x,y))=e$ for all $x \in X$ and for all $y \in Y$. Recall that the support of μ generates our compact semigroup which is $G \times X \times Y$. Therefore $G \times X \times Y = <g> \times X \times Y$. If two completely simple semigroups are equal having the same X and Y then the groups are also equal, that is, $G = <g>$ [13] (#(64), page 195).

$<g> = \{g^n \mid n = 1, 2, 3, \ldots\}$.

Due to the compactness of G, there exists a positive integer n such that $g^n = e$, so that G is a finite cyclic group and hence it has the discrete topology. What does this mean? Let X, Y be arbitrary compact Hausdorff spaces and μ a probability measure on the Rees Product $G \times X \times Y$ with multiplication as above, such that the support of μ generates $G \times X \times Y$. Then if the support of μ is contained in $\{g\} \times X \times Y$ for some $g \in G$, $g \neq e$, $\mu^{(n)}$ will not converge to an idempotent probability measure on $G \times X \times Y$. This gives us a class of compact semigroups for which the sequence $\{\mu^{(n)}\}_{n \in \omega}$ will not converge. Notice that X and Y could have been any compact Hausdorff space. G however had to be cyclic. Notice though that the support of μ had to be $\{g\} \times X \times Y$ with g generating G. Suppose μ_1 is a probability measure on $G \times X \times Y$ and the support of μ, is $\{g_1\} \times X \times Y$ and g_1 does not generate G; then μ does not satisfy the conditions of M. Rosenblatt's result and we cannot conclude anything about the convergence of the sequence $\{\mu_1{}^{(n)}\}_{n \in w}$. Just one other conclusion can be drawn and that is there do not exist compact groups which are infinite cyclic.

Now suppose that G is a compact topological group with a probability measure μ defined on $G \times X \times Y$ whose support generates $G \times X \times Y$, where X and Y are arbitrary compact Hausdorff spaces. Furthermore suppose that the support of μ is contained in $gG' \times X \times Y$, where $G' = \{e\}$ and $g \in G$, $g \neq e$. If the support of μ generates $G \times X \times Y$, then as noted above, G is finite cyclic. If G is not cyclic and not finite then, the normal closed subgroup $G' = \{e\}$ does not satisfy the conditions of M. Rosenblatt theorem. This means for any compact group which is not finite cyclic, we do not have to check the closed normal subgroup $G' = \{e\}$. We can also conclude that if G is not a finite cyclic group but still a compact topological group and μ is a probability measure on $G \times X \times Y$ whose support generates $G \times X \times Y$, then the normal subgroup $G' = \{e\}$ containing XY does not satisfy the conditions that the support of μ is contained in $(G' \times X \times Y)^{-1}$ $(gG' \times X \times Y)$, for any $g \in G$, $g \neq e$. For otherwise G would be cyclic and if the support of μ is contained in $(G' \times X \times Y)^{-1}$ $(gG' \times X \times Y)$ for some $g \in G \sim \{e\}$, the sequence $\{\mu^{(n)}\}_{n \in w}$ will not converge.

Example 2

Let G be a group. G is simple if and only if G has only the two trivial normal subgroups namely $\{e\}$ and G where e is the identity of G. Consider first the groups that are finite and cyclic. Some of these groups are simple and they are those groups of prime order. The others are not simple since they have subgroups. As finite cyclic groups are Abelian, all subgroups are normal. We discussed this case in the last example. We saw that if the support of a probability measure μ was $\{g\} \times X \times Y$ for some $g \neq e$ then the sequence $\{\mu^{(n)}\}_{n \in w}$ did not converge. Now suppose that the support of μ is $(\{g_1\} \times X \times Y) \cup (\{g_2\} \times X \times Y)$ where $g_1 \neq e$ and $g_2 \neq e$ and $g_1, g_2 \in G$ a finite cyclic group of prime order. Then there does not exist an element $g \in G$ such that the support of μ is contained in $\{g\} \times X \times Y$. Hence the normal closed subgroup $\{e\}$ does not work and the sequence $\{\mu^{(n)}\}_{n \varepsilon \omega}$ converges as $\{e\}$ is the only normal closed subgroup properly contained in G.

Consider now all simple compact groups which are not finite cyclic. Then by example 1 and the conclusion in Example 1, the normal closed subgroup $G' = \{e\}$ does not satisfy the conditions in Rosenblatt's result. But $G' = \{e\}$ is the only normal closed subgroup of G not equal to G since G is simple. Therefore if X and Y are compact Hausdorff spaces and μ is a regular probability measure on the Rees product $G \times X \times Y$ for some continuous map $xy \to G$ and the support of μ generates $G \times X \times Y$ then the sequence $\{\mu^{(n)}\}_{n \varepsilon \omega}$ does converge to a probability measure say E. E is therefore an idempotent probability measure. This then enables us to describe a big class of compact topological semigroups and regular probability measures μ on these semigroups for which the sequences $\{\mu^{(n)}\}_{n \varepsilon \omega}$ converges. The compact topological semigroups which we can have are generated by all simple compact groups which are not finite cyclic and any arbitrary compact Hausdorff spaces X and Y together with any regular sandwich matrix. Just note that all regular probability measures μ on these compact semigroups whose support generates the whole semigroup are in the Domain of Attraction of some stable law (which are the idempotent probability measures).

Example 3

Let G be a compact group and $X = Y = \{x\}$. Therefore the mapping, mapping $X \times X \to G$ defined by ϕ is continuous since $\phi(x, y) = h$ for some $h \varepsilon G$. We then get the following Rees product, $G \times \{x\} \times \{x\}$ with the following multiplication:

$$(g_1; x, x) \cdot (g_2; x, x) = (g_1\ \phi(x, x)\ g_2; x, x) = (g, h\ g_2; x, x).$$

Notice that G is homeomorphic to $G \times \{x\} \times \{x\}$ but is not necessarily homomorphic to $G \times \{x\} \times \{x\}$ unless $\phi((x , x))=e$. To show that G is not homomorphic to $G \times \{x\} \times \{x\}$ we just use the following natural map:

$$\theta(g) = (g; x, x) \quad \text{for all } g \in G.$$

$$\begin{aligned} \theta(eg) &= (eg; x, x) \\ &= (h^{-1} hg; x, x) \\ &= (h^{-1} (x, x)g; x, x) \\ &= (h^{-1}; x, x) \cdot (g; x, x) \\ &= \theta(h^{-1}) \cdot \theta(g) \end{aligned}$$

$$\begin{aligned} \theta(e) \cdot \theta(g) &= (e; x, x) \cdot (g; x, x) \\ &= (e\ \phi(x, x)\ g; x, x) \\ &= (e\ h\ g; x, x) \\ &= (h\ g; x, x) \\ &= \theta(hg) \quad . \end{aligned}$$

Since $\theta(g) \neq \theta(h\ g)$, θ is not homomorphic unless $h = e$.

So let us consider the case when $\phi(x,x)=e$. Then G is homeomorphic and isomorphic to $G\times\{x\}\times\{x\}$. For this example Rosenblatt's result takes the following form: Let μ be a regular probability measure on $G \times\{x\}\times\{x\}$ whose support generates $G \times \{ x \} \times \{ x \}$. The sequence of measures $\mu^{(n)}$ will not converge in the weak star topology if and only if there is a proper closed normal subgroup (proper indusion) of G such that $\{x\}\{x\} = \{xx\}$, $\{\phi(x,x)\} \subset G'$ and the support of μ is contained in $(G' \times \{x\} \times \{x\})^{-1} (gG' \times \{x\} \times \{x\})$ for some $g \in G \sim G'$. However $G \cong G \times \{x\} \times \{x\}$. Consider $(G' \times \{x\} \times \{x\})^{-1} (gG' \times \{x\} \times \{x\}) \cong (G')^{-1} gG'$. As G' is a subgroup $(G')^{-1} = G'$ we see that

$$\begin{aligned} (G)^{-1}\ gG' &= G'gG' \\ &= gG'g^{-1}gG' \qquad \text{(as } G' \text{ is normal in G)} \end{aligned}$$

$= gG'G'$

$= gG'.$

So we finally replace the last statement of the result as follows: "the support of μ is contained in gG' for some $g \in G \sim G'$."

Putting all this together for the special case when $\phi(x, x) = e$ and $X = Y = \{x\}$ we get the following result. If μ is a regular probability measure on a compact topological group G whose support generates G, then the sequences of measures $\mu^{(n)}$ will not converge in the weak star topology if and only if there is a proper closed normal subgroup G' of G such that the support of μ is contained in gG' for some $g \in G \sim G'$.

We noticed that Ulf Grenander [17] has proved the same result just using compact groups. His result is as follows:

"For a given probability distribution P the limit of $P^{(n)}$ $n \to \infty$ exists if and only if the support of P is not contained in any coset of any closed proper, normal subgroup of G. The limit of $P^{(n)}$ is the normalized Haar measure on G." Hence the result by M. Rosenblatt is a generalization of the result by Ulf Grenander.

We now give some examples of this:

Example 4

Let us consider a concrete example of a finite cyclic group G of prime order and also compare the convergence (or divergence) of a probability measure μ of $\mu^{(n)}$ and $\sum_{i=1}^{n} \mu^{(i)}/n$.

Let $G = \sigma(3)$ (the group of order 3 under addition as $\sigma(3)$ is a field).

$= \{0, 1, 2\}$

The addition table is as follows:

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Let G have the discrete topology and define a probability measure on G as follows:

$\mu(0) = 0 = \mu(1)$

$\mu(2) = 1$

Let $X = Y = \{x\}$ with $\phi(x, x) = 0$.

Then $G \cong G \times \{x\} \times \{x\}$ where multiplication in $G \times \{x\} \times \{x\}$ is $(g; x, x) + (g_1; x, x) = (g + g_1; x, x)$. Hence we will only consider G. The support of $\mu = \{2\}$. The support of μ generates G. $G' = \{0\}$. Therefore, $G' = G'^{-1}$ and hence

$$\begin{aligned}(G' \times \{x\} \times \{x\}) + (g + G' \times \{x\} \times \{x\}) &= (G')^{-1} + (g + G') \\ &= G' + (g + G') \\ &= g + G' \\ &= \{g\}.\end{aligned}$$

If we let $\{g\} = \{2\}$, then the support of $\mu = (G' \times X \times Y)^{-1} + (gG' \times X \times Y)$
$= \{2\}$.

Now we will calculate $\mu^{(2)}, \mu^{(3)}, \mu^{(4)}, \ldots$. Recall that for any two probability measures λ, γ on a compact semigroup, the convolution of λ, γ is defined as follows, where $\mathcal{B}$ is a Borel set:

$$\lambda + \gamma(\mathcal{B}) = \lambda \times \gamma \{(x, y) \mid x + y \in \mathcal{B}\}.$$

(a) $\{(x, y) \mid x + y = 0\} = \{(0, 0), (1, 2), (2, 1)\}$

(b) $\{(x, y) \mid x + y = 1\} = \{(0, 1), (1, 0), (2, 2)\}$

(c) $\{(x, y) \mid x + y = 2\} = \{(0, 2), (2, 0), (1, 1)\}$.

Hence $\mu * \mu\{0\} = \mu \times \mu\{(0, 0), (1, 2), (2, 1)\}$

$= \mu \times \mu\{(0, 0)\} + \mu \times \mu\{(1, 2)\} + \mu \times \mu\{(2, 1)\}$

$= 0 + 0 + 0$

$= 0.$

$\mu * \mu \{1\} = \mu \times \mu \{(0, 1)\} + \mu \times \mu \{(1, 0)\} + \mu \times \mu \{(2, 2)\}$

$= 0 + 0 + 1$

$= 1$

$\mu * \mu \{2\} = \mu \times \mu \{(0, 2)\} + \mu \times \mu \{(2, 0)\} + \mu \times \mu \{(1, 1)\}$

$= 0 + 0 + 0$

$= 0$

Therefore, $\mu^{(2)}\{0\} = \mu^{(2)}\{2\} = 0$ and $\mu^{(2)}\{1\} = 1$.

Now,

$\mu^{(2)} * \mu \{0\} = \mu^{(2)} \times \mu \{(0, 0)\} + \mu^{(2)} \times \mu \{(1, 2)\} + \mu^{(2)} \times \mu\{(2,1)\}$

$= 0 + 1 + 0$

$= 1$

$\mu^{(2)} * \mu \{1\} = \mu^{(2)} \times \mu \{(0, 1)\} + \mu^{(2)} \times \mu \{(1, 0)\} + \mu^{(2)} \times \mu\{(2,2)\}$

$= 0 + 0 + 0$

$= 0$

$\mu^{(2)} * \mu \{2\} = \mu^{(2)} \times \mu \{(0, 2)\} + \mu^{(2)} \times \mu \{(2, 0)\} + \mu^{(2)} \times \mu\{(1,1)\}$

$= 0 + 0 + 0$

$= 0$

Therefore $\mu^{(3)}\{1\} = \mu^{(3)}\{2\} = 0$ and $\mu^{(3)}\{0\} = 1$.

Now, $\mu^{(3)} * \mu \{0\} = \mu^{(3)} \times \mu \{(0, 0)\} + \mu^{(3)} \times \mu \{(1, 2)\} + \mu^{(3)} \times \mu\{(2,1)\}$

$= 0 + 0 + 0$

$= 0$

$$\mu^{(3)} * \mu\{1\} = \mu^{(3)} \times \mu\{(0,1)\} + \mu^{(3)} \times \mu\{(1,0)\} + \mu^{(3)} \times \mu\{(2,2)\}$$
$$= 0 + 0 + 0$$
$$= 0$$

$$\mu^{(3)} * \mu\{2\} = \mu^{(3)} \times \mu\{(0,2)\} + \mu^{(3)} \times \mu\{(2,0)\} + \mu^{(3)} \times \mu\{(1,1)\}$$
$$= 1 + 0 + 0$$
$$= 1$$

Therefore, $\mu^{(4)}\{2\} = \mu\{2\} = 1$ and

$$\mu^{(4)}\{0\} = \mu^{(4)}\{1\} = 0 = \mu\{1\} = \mu\{0\}.$$

Clearly then by induction

$$\mu^{(n)}\{0\} = \begin{cases} 1 \text{ if } n \equiv 0 \pmod 3 \\ 0 \text{ otherwise} \end{cases}$$

$$\mu^{(n)}\{1\} = \begin{cases} 1 \text{ if } n \equiv 2 \pmod 3 \\ 0 \text{ otherwise} \end{cases}$$

$$\mu^{(n)}\{2\} = \begin{cases} 1 \text{ if } n \equiv 1 \pmod 3 \\ 0 \text{ otherwise} \end{cases}$$

Clearly the sequence $\mu^{(n)}$ does not converge. Define $\gamma_n = \sum_{i=1}^{n} \mu^{(i)}/n$.

We need consider only one Borel set, namely $B = \{0\}$.

$$\gamma_n\{0\} = \sum_{i=1}^{n} \mu^{(i)}\{0\}/n.$$

Case 1. $\gamma_{3k}\{0\} = \sum_{i=1}^{3k} k/3k = 1/3$, $k = 1, 2, \ldots$.

Case 2. $\gamma_{3k+1}\{0\} = \sum_{i=1}^{3k} k/(3k+1) \to 1/3$ as $k \to \infty$, $k = 1, 2, \ldots$.

Case 3. $\gamma_{3k+2}\{0\} = \sum_{i=1}^{3k+2} k/(3k+2) \to 1/3$ as $k \to \infty$, $k = 1, 2, \ldots$.

Hence $\lim_{n\to\infty} \gamma_n \{0\} = 1/3$.

Similarly $\lim_{n\to\infty} \gamma_n \{1\} = \lim_{n\to\infty} \gamma_n \{2\} = 1/3$ and therefore $\lim_{n\to\infty} \gamma_n = \gamma$,

where $\gamma\{g\} = 1/3$, $g \in \sigma(3)$.

Notice that γ is an idempotent measure and is the only non-trivial idempotent probability measure on G ($= \sigma(3)$). Observe that $\gamma^{(n)}\{0\}$, $n = 1, 2, 3, \ldots$, does not converge in the usual convergence, but does converge in the Cesaro method.

So in a finite cyclic group we can get measures μ such that $\mu^{(n)}$ does not converge, even in a finite cyclic group of prime order. (that is, a simple cyclic group). However, as mentioned before, for a simple group which is not finite and cyclic, for all measures μ whose support generates the group, $\mu^{(n)}$ will converge and the limit is an idempotent probability measure (that is, a stable law).

Example 5

Let G be the unit circle, with center at the origin, with the induced topology of the plane. Let g be a point on the unit circle. It will be represented by the angle between the line joining g and the origin and the positive x-axis. Addition is then defined as follows:

$$\text{Let } \theta, \theta_1 \in G \quad \theta \oplus \theta_1 = \begin{cases} \theta + \theta_1 & \text{if } \theta + \theta_1 < 2\pi \\ \theta + \theta_1^1 - 2\pi & \text{if } \theta_1^1 + \theta_1 \geq 2\pi \end{cases} .$$

(With the above representation $G = \{ \theta \mid 0 \leq x < 2\pi\}$). The addition is just $\theta \oplus \theta_1 = \theta + \theta_1 \pmod{2\pi}$. With this addition G is a compact topological group. Furthermore it is an abelian group, so that any subgroup of G is immediately normal.

Let us consider all subgroups of G. First we will consider all finite groups. Let G'_n be a finite group, having at least two members, then $G'_n = \{0, 2\pi/n, 2/n \cdot 2\pi, \dots, (n-1)/n \cdot 2\pi\}$ for some positive integer n. These subgroups are all cyclic (that is, each subgroup has at least one element, namely $2\pi/n$ for all n, that will generate the subgroup). Suppose $X = Y = \{x\}$. Then $G'_n \times \{x\} \times \{x\}$ is in 1 - 1 correspondence with G'_n as mentioned earlier. Consider $(G'_n \times \{x\} \times \{x\})^{-1} \cdot (gG'_n \times \{x\} \times \{x\})$ where $g \in G \sim G'n$. Since $X = Y = \{x\}$, then XY is a singleton where $XY \subset G'_n$. Therefore, it is easy to see that

$(G'_n \times \{x\} \times \{x\})^{-1} = G'_n \times \{x\} \times \{x\}$. In any case $(G'_n \times \{x\} \times \{x\})^{-1} \cdot (gG'_n \times \{x\} \times \{x\})$ can have only a finite number of elements as G'_n is finite. Suppose now that μ is a probability measure on $G \times \{x\} \times \{x\}$ such that the support of μ generates $G \times \{x\} \times \{x\}$ and the support is contained in $(G'_n \times \{x\} \times \{x\})^{-1} \cdot (gG'_n \times \{x\} \times \{x\})$ for some $g \in G \sim G'_n$; then we immediately get a contradiction. The support of μ has only a finite number of elements and the set generated by the support of μ still only has a finite number of elements, but G has an infinite number of elements. Therefore,no such μ as described above exists. We therefore must consider infinite subgroups. But there are no infinite ordered subgroups of G that are closed.

Thus we have shown the following. Let μ be a probability measure on $G \times \{x\} \times \{x\}$ whose support generates $G \times \{x\} \times \{x\}$, then there exists no proper closed normal subgroup G' of G with $\{xx\} \subset G'$ such that the support of μ is contained in $(G' \times \{x\} \times \{x\})^{-1} \cdot (gG' \times \{x\} \times \{x\})$ where g is some element of G not in G'. Therefore, for any probability measure μ on $G \times \{x\} \times \{x\}$ whose support generates $G \times \{x\} \times \{x\}$, the sequence $\{\mu^{(n)}\}_{n \in \omega}$ will converge to a probability measure in the weak star topology. This measure will be an idempotent measure whose support is $G \times \{x\} \times \{x\}$ (as the support of an idempotent measure is the kernel of the semigroup $G \times \{x\} \times \{x\}$ which is $G \times \{x\} \times \{x\}$). Also note the idempotent measure will always be the normalized Haar measure as $G \times \{x\} \times \{x\}$ is a group.

Now consider the case when X and Y are arbitary compact Hausdorff spaces. Recall that $G \times X \times Y$ can be thought of as a matrix (Rees matrix). If we consider the finite subgroups of G (as before), then the set

$(G' \times X \times Y)^{-1}\ (gG' \times X \times Y)$

will have only a finite number of elements in each xy coordinate compared with an infinite number of elements in each xy coordinate of $G \times X \times Y$ where $XY \subset G'$. The infinite subgroups of G as noted earlier are not closed. Hence, as before, if μ is a probability measure on $G \times X \times Y$ whose support generates $G \times X \times Y$, then there exists no proper normal closed subgroup G' of G with $XY \subset G'$ such that the support of μ is contained in $(G' \times X \times Y)^{-1} \cdot (gG' \times X \times Y)$ where g is some element of G not in G'. Therefore, for any probability measure μ on $G \times X \times Y$ whose support generates $G \times X \times Y$, the sequence $\{\mu^{(n)}\}_{n \in \omega}$ will converge to a probability measure in the weak star topology.

Let G be the unit circle as above and X and Y two arbitrary compact Hausdorff spaces. The question arises, what are some of the subsets of $G \times X \times Y$ that will generate $G \times X \times Y$? (Let S be a topological semigroup and T a subset of S; then the semigroup is

$$U = \overline{\bigcup_{n=1}^{\infty} T^n} \quad \text{where the bar represents closure and}$$

$$T^n = \{x_1 \cdot x_2 \cdot \ldots \cdot x_n \mid x_i \in T \quad i = 1, \ldots, n\} .)$$

To put it another way, what does the subset of $G \times X \times Y$ generated by $T \times X \times Y$, given some $T \subset G$, look like? First we will show that a set $T \times X' \times Y'$ generates a set $T_1 \times X' \times Y'$ where $X' \subset X$ and $Y' \subset Y$. Consider first $G \times X' \times Y'$ where $X' \subsetneq X$ and $Y' \subsetneq Y$. Then $X' \times Y' \subsetneq X \times Y$ and there is no possibility of $X' \times Y'$ generating $X \times Y$. Let $s, t \in G \times X' \times Y'$ where $s = (g; x, y)$ and $t = (g_1; x_1, y_1)$. Then

$$\begin{aligned} st &= (g; x, y) \cdot (g_1; x_1, y_1) \\ &= (g\ \phi(x_1, y)\ g_1; x, y_1) \quad . \end{aligned}$$

As $s \in G \times X' \times Y'$ we get that $x \in X'$ and similarly $y_1 \in Y'$ as

$t \in G \times X' \times Y'$. Therefore, $(x, y_1) \in X' \times Y'$ and hence $(g\phi(x_1,y)g_1;x,y_1) \in G\times X'\times Y'$. Similarly $ts \in G\times X'\times Y'$. Hence, if $s_1,s_2,\ldots,s_n \in G\times X'\times Y'$, $s_1 s_2 \cdots s_n \in G\times X'\times Y'$. Therefore, as we consider $G\times X'\times Y'$, it is immediate that $G\times X'\times Y'$ will only generate $G\times X'\times Y'$ as $X'\times Y'$ "generates" only $X'\times Y'$. Therefore, the only possible subsets of $G\times X\times Y$ which will generate $G\times X\times Y$ are sets of the form $T\times X\times Y$ where $T \subset G$.

Note, however, that in general if T is a subset of G and $X' \subset X$, $Y' \subset Y$ then $[T \times X' \times Y'] \neq [T] \times X' \times Y'$, where [A] is the set generated by A. Consider the following compact semigroup.

Let $G = [0, 2\pi)$ with ordinary addition modulo 2π. Let $X = Y = \{x\}$ and ϕ $X \times Y \to G$ be as follows

$$\phi(x,x) = 2\pi/5.$$

Let $T = \{2\pi/7\}$. $[T] = \{2\pi/7, 4\pi/7, 6\pi/7, 8\pi/7, 10\pi/7, 12\pi/7, 0\}$.

Consider $[T \times X \times Y]$.

Let $t = 2\pi/7$ and $h = 2\pi/5$.

Then $(t; x, x) + (t; x, x) = t + h + t; x, x)$

$t + h + t = 2\pi/7 + 2\pi/5 + 2\pi/7 = 34\pi/35.$

Hence it is simple to show that

$$[T \times X \times Y] = \{(2n\pi/35; x, x) \mid n = 0, 1, 2, \ldots , 34\}.$$

Let $[T \times X \times Y] = T' \times X \times Y$, then

$$T' = \{2n\pi/35 \mid n = 0, 1, 2, \ldots , 34\}.$$

$$[T] = \{2n\pi/7 \mid n = 0, 1, \ldots , 6\}.$$

Therefore, $T' \neq [T]$ and hence $[T \times X \times Y] \neq [T] \times X \times Y$.

Now what are some of the subsets T of G, where G is defined above, so that for compact Hausdorff spaces X and Y, $T \times X \times Y$ will generate $G \times X \times Y$?. To do that we will first consider $G \times \{x\} \times \{y\}$ for $x \in X$ and $y \in Y$ and suppose $\phi(s, y) = h'$ where $h' \in G$ and $\phi:X \times Y \to G$ is a continuous function. Let $s = (g; x, y)$ and $t = (h; x, y)$ be elements of $G \times \{x\} \times \{y\}$.

$$\begin{aligned} s + t &= (g;\ x,\ y) + (h;\ x,\ y) \\ &= (g + \phi(x,\ y) + h;\ x,\ y) \\ &= (g + h' + h;\ x,\ y). \end{aligned}$$

$g + h' + h \in G$ as G is a group and hence $s + t \in G \times \{x\} \times \{y\}$. Hence we know that $G \times \{x\} \times \{y\}$ is closed under multiplication. Furthermore, $\phi(x, y)$ is just a type of shift operator in the group and is fixed for the choice $\{x\}$ and $\{y\}$. First recall that G is a commutative group so $(g + h' + h; x, y) = (g + h + h'; x, y)$. Hence if a subset T of G generates G, then $T \times \{x\} \times \{y\}$ will generate $G \times \{x\} \times \{y\}$. The reason is that if T generates G and then $\phi(x, y)$ shifts G, but G is a group and so G stays the same. We had to use the commutative property to be able to interchange generating and shifting. Therefore, all we need to consider are subsets of G that will generate G. Any subset T of G that has positive Lebesque measure will generate G. So what does all this mean?

It means that for any regular probability measure μ on $G \times X \times Y$ whose support is $T \times X \times Y$, where T is of the above form, the sequence $\mu^{(n)}$ will converge to an idempotent measure on $G \times X \times Y$. The support of this idempotent measure will be $G \times X \times Y$. Furthermore, if X and Y are at most countable, the idempotent measure will be a convex combination of the normalized Haar measures on each of the groups of the form $G \times \{x\} \times \{y\}$. If X and Y are uncountable, then the idempotent measure will be of the form $E \times \alpha \times \beta$ where E is the normalized Haar measure on

G and α and β are probability measures on X and Y respectively. Hence in this case, any probability measure whose support generates $G \times X \times Y$ is in the Domain of Attraction of some idempotent probability measure on $G \times X \times Y$.

<u>Example 6</u>

Let us consider the following group H. Let G be the group of 2×2 matrices of the form $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, where $0 \leq \theta < 2\pi$, and let H be the group generated by G and $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. As was shown before, this is a group. Furthermore, H is a compact abelian group and G is a closed normal subgroup of H. Let $X = Y = \{x\}$. Let $Q: X \times Y \to H$ be defined as follows:

$$Q(x, y) = e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{for all } x \text{ and } y.$$

Then clearly $H \times X \times Y \cong H$. ($\cong$ means isomorphic). $H \times X \times Y$ and H are also homeomorphic. The mapping showing isomorphism is $f; H \to H \times X \times Y$, defined as follows:

$$f(h) = (h; x, y). \quad \text{Thus}$$

$$\begin{aligned} f(hh') &= (hh'; x, x) \\ &= (h\ e\ h'; x, x) \\ &= (h\ Q(x, x)\ h'; x, x) \\ &= (h; x, x) \cdot (h'; x, x) \\ &= f(h) \cdot f(h'). \end{aligned}$$

Let γ be a regular probability measure on $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} G$, whose support is $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} G$. Then clearly $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} G$ generates H as $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \varepsilon G$. Remember $X = Y = \{x\}$. Since $H \times X \times Y \cong H$, we see that for any subset $S \subset H$, $S \times X \times Y \cong S$. (where $Q(x, x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$). Hence $(G \times X \times Y) \cong G$ and $gG \times X \times Y \cong g\, G$ for some $g \in H \sim G$. As G is a group

$G^{-1} = G$ and therefore $(G \times X \times Y)^{-1} \cong G^{-1}$. Thus

$$(G \times X \times Y)^{-1} \cdot (gG \times X \times Y) \cong G^{-1}\, g\, G$$

$$\begin{aligned} G^{-1}\, g\, G &= G\, g\, G \\ &= g\, G\, G \qquad \text{since } G \text{ is abelian.} \\ &= g\, G \end{aligned}$$

Therefore, $(G \times X \times Y)^{-1} (gG \times X \times Y) = gG \times X \times Y$. So let $g = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, then we have $g \in H \sim G$. Thus by Rosenblatt's result [13] (Lemma 13 and Theorem 8, page 194), the sequence $\gamma^{(n)}$ will not converge in the weak star topology.

Now let us consider arbitrary compact Hausdorff spaces X and Y using the same group H as above. Let μ be a regular probability measure whose support is $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} G \times X \times Y$. Let $\phi: X \times Y \to H$ be a continuous function. Consider the following:

Let $(s; x,y) \in G \times X \times Y$ and $(t; x', y') \in \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} G \times X \times Y$. For convenience let $g = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Then

$$(s; c, y) \cdot (t; x', y') = (s\, \phi(x', y)\, t; x, y').$$

$\phi(x', y) = r \in G$.

So consider $s\, \phi(x', y)\, t = s\, r\, t$. $t = gt'$ for some $t' \in G$.

Then $srt = srgt' = gsrt'$ since H is abelian and since $srt' \in G$, $g(srt') \in G$. Therefore, $(s\phi(x',y)t\ ; x,y') \in gG{\times}X{\times}Y$. We know that $r \in G$ and hence $r^{-1} \in G$. It can easily be shown that $(G{\times}X{\times}Y)(gG{\times}X{\times}Y) = gG{\times}X{\times}Y$. Furthermore, since $(G{\times}X{\times}Y)^{-1} = G{\times}X{\times}Y$,

$$(G \times X \times Y)^{-1} \cdot (gG \times X \times Y) = (G \times X \times Y) \cdot (gG \times X \times Y).$$

For simplicity denote $\phi(x,y) \in G$ as xy. Then $((xy)^{-1}\ ; x,y)$ belongs to $G \times X \times Y$ and $((xy)^{-1}\ ; x,y) \cdot (gh\ ; x,y)$ for arbitrary $h \in G$ is equal to $((xy)^{-1}(xy)gh\ ; x,y) = (gh\ ; x,y)$. This is true for all $(x,y) \in X{\times}Y$ and hence $gG \times X \times Y \subset (G \times X \times Y) \cdot (gG \times X \times Y)$.

This result is true for any arbitrary group. The converse is not always true. In this particular case G is a normal abelian subgroup of H, and therefore we immediately see that

$$gG \times X \times Y = (G \times X \times Y)\ (gG \times X \times Y).$$

Therefore, if $XY \subset G$ and μ is a regular probability measure on $H \times X \times Y$ whose support is $gG \times X \times Y$, then again the sequence $\mu^{(n)}$ will not converge in the weak star topology.

Now we would like to take a look at why, or at least one reason why it is that for a probability measure μ in H of the above type, the sequence $\{\mu^{(n)}\}_{n \in W}$ does not converge. To do this we will look at the support of $\mu^{(n)}$ for arbitrary positive integers n. When n is 1, the support of μ is the set $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} G$. Let $F = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} G$. Also let the set of all Borel subsets of F be denoted by $\mathcal{B}_1$, let the set of all Borel subsets of G be denoted by $\mathcal{B}_2$. Now what is the support of $\mu^{(2)}$?

Let $B \in \mathcal{B}$ the set of all Borel subsets of H.

$$\mu * \mu\ (B) = \mu \times \mu\ \{(x, y)\ |\ xy \in B\}.$$

What does B look like? Claim that $B \subset G$ and hence $B \in \mathcal{B}_2$ For suppose $B \cap F \neq \phi$. Let $B_1 = B \cap F$ and $B_1 \in \mathcal{B}_1$ Then $xy \in B_1$. The only way $xy \in B_1$ is if, without loss of generality, $x \in F$ and $y \in G$. But $\mu(F) = 1$ and $\mu(G) = 0$, as the support of μ is F. So $\mu * \mu\ (B) = 0$ unless $\{(x, y)\ |\ xy \in B\} \subset F \times F$. But $\mu \times \mu\ (F \times F) = 1$ and hence $B \in \mathcal{B}_1 \times \mathcal{B}_1$ and hence $B \in \mathcal{B}_2$ that is the support of $\mu^{(2)} \subset G$.

Now consider the support of $\mu^{(3)}$.

$$\mu^{(3)} = \mu^{(2)} * \mu.$$

The support of $\mu^{(2)}$ is contained in G and the support of μ is contained in F and hence the support of $\mu^{(3)}$ is in F. Thus by induction the support of $\mu^{(n)}$ is contained in F is n is odd and the support of $\mu^{(n)}$ is contained in G if n is even. Thus the support alternates between F and G ($F \cap G = \phi$) and hence the sequence $\mu^{(n)}$ cannot converge. So for this reason for the probability measure μ whose support is $gG \times X \times Y$ the sequence $\mu^{(n)}$ will not converge in the weak star topology.

The above examples illustrate some of the usefulness of the result by M. Rosenblatt. We also noticed that the result of M. Rosenblatt was a generalization of the result by Ulf Grenander on compact groups. In fact, all through this paper we have used results on compact groups and were able to generalize then to simple compact topological groups.

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