AN EXPOSITION OF A THEOREM OF GOLOD AND ŠAFAREVIČ WITH APPLICATIONS TO NIL ALGEBRAS AND PERIODIC GROUPS

by

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TO THE MEMORY OF MY FATHER

MICHALAKIS SAPARILLAS
(1891 - 1959)

ABSTRACT

A theorem by Golod and Šafarevič with application to nil algebras and periodic groups is clearly proved in this thesis. The applications settle negatively Kuroš's question: Is a finitely generated algebraic algebra, finite-dimensional? and Burnside's question: Is a finitely generated periodic group finite?

Remarks and theorems on subjects related to the main theorem are in Chapter 1, the proof of the theorem is in Chapter 2, and the applications of it are in Chapter 3.

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INTRODUCTION

The purpose of this thesis is to give a clear exposition of a theorem of Golod and Šafarevič [6] and some of its consequences. The theorem, published in 1964, is a remarkable result. Its proof is rather short, but it provides the answer to many questions. Two of the questions which this paper will discuss are more closely related than originally appeared. The problems referred to are the Kuroš Problem [15], and the general Burnside Problem [1] and the construction of examples which solve these problems is fairly straight-forward (given the main theorem of Golod-Šafarevič).

This thesis has an example of an infinite dimensional nil algebra with a finite number of generators over a countable field. This is a negative answer to the Kuroš question which was asked in 1941: Let A be a finitely generated, algebraic algebra. Is A finite-dimensional (as a vector space)? The history of the question is very interesting. Kuroš discussed several special cases [15], all with affirmative answers. Jacobson and Levitzki [13],[16],[17] settled the question affirmatively for algebras of bounded degree. In the meantime, many special cases had been studied. Then, in 1964, Golod announced that the answer to the Kuroš question was negative. At the same time, he gave a negative answer to the Burnside problem: Let G be a finitely generated periodic group. Is G finite?

Burnside [1] considered the following three cases with affirmative answers.

(1) G of exponent 2,

(2) G of exponent 3,

(3) G of exponent 4, and G with two generators.

In 1940, Sanov [22] obtained an affirmative answer for exponent 4 and an arbitrary (but finite) number of generators. Marshall Hall Jr. [9] gave an affirmative answer for exponent 6. The answer is still unknown for G of exponent 5.

Then Novikov, in 1959, announced [20] that the answer is no, if the exponent of G $n \geq 72$ and the number of generators is at least 2. (The proof of [20] appeared in 1968 by P.S. Novikov and S.I. Adyan [21], where $n \geq 72$ has been replaced by odd $n \geq 4381$.)

In 1964, Golod constructed a finitely generated group which is periodic and infinite, which settled negatively the original Burnside problem.

CHAPTER 1.

This chapter is to make the reader familiar with a few terms and some symbols which are closely related to the main part of this thesis. In addition, some definitions will be given, while it will be assumed that the reader is acquainted with the most basic ones.

1.1 Free Semigroups and Generators

Let $X = \{x_1, x_2, \ldots, x_d\}$ be a set of d noncommuting indeterminates, and let S_X consist of all finite sequences of elements of X,

$$S_X = \{x_{i_1} x_{i_2} \ldots x_{i_n} | x_{i_k} \varepsilon X\}.$$

Define a binary operation, that is, a multiplication on S_X, as follows: For any two elements of S_X, say $s_1 = x_{i_1} x_{i_2} \ldots x_{i_n}$ and $s_2 = x_{j_1} x_{j_2} \ldots x_{j_m}$, their product $s_1 \cdot s_2$ is the product obtained by juxtaposition of s_1 and s_2:

$$s_1 \cdot s_2 = x_{i_1} x_{i_2} \ldots x_{i_n} x_{j_1} \ldots x_{j_m}.$$

For example, if $s_1 = x_1 x_1 x_3$ and $s_2 = x_3 x_2$, then $s_1 \cdot s_2 = x_1 x_1 x_3 \cdot x_3 x_2 = x_1 x_1 x_3 x_3 x_2$.

With this definition of multiplication S_X becomes a semigroup; we call it the <u>free semigroup on X</u>. Note that the binary operation which we just defined is associative. For example:

$$(x_1 x_1 x_3 \cdot x_3 x_2) \cdot x_1 x_4 x_3 = (x_1 x_1 x_3 x_3 x_2) \cdot x_1 x_4 x_3$$

$$= x_1 x_1 x_3 x_3 x_2 x_1 x_4 x_3$$

$$= x_1 x_1 x_3 \cdot (x_3 x_2 x_1 x_4 x_3)$$

$$= x_1 x_1 x_3 \cdot (x_3 x_2 x_1 \cdot x_4 x_3)$$

The elements of S_X are often called "words" but in this thesis, they will be called monomials. We may say that the element x_i of X has length 1 if we consider x_i as a word. However, talking in terms of monomials x_i has degree 1. Now we are ready to define the degree of a monomial which is simply the number of occuring x_i's. For example, the monomial $x_2x_2x_5x_4$ is of degree 4.

If in a word we have a succession of indeterminates all the same, say $x_ix_i \dots x_i$, (m times), then we write x_i^m.

We let 1 be a symbol not in X(we call 1 the "empty word" or the "monomial of degree 0"), and define $1\cdot s = s\cdot 1 = s$ for all $s\varepsilon S_X$. Thus we have a semigroup $\{1\}\cup S_X$, the free semigroup with identity on X.

Remark: 1.1.1 The number of distinct monomials of a given degree n is the number of ways of choosing (in order) n indeterminates from the set X. This number in this case is d^n.

Example: Assume that $X = \{x_1,x_2,x_3,x_4\}$ is the set of four non-commuting indeterminates. Then the number of monomials of degree 3 is $4^3 = 64$.

The monomials of degree 2 are 16 in number and they are the following:

x_1x_1	x_1x_2	x_1x_3	x_1x_4	x_2x_1	x_2x_2	x_2x_3	x_2x_4
x_3x_1	x_3x_2	x_3x_3	x_3x_4	x_4x_1	x_4x_2	x_4x_3	x_4x_4

The elements of S_X, that is the monomials, are of the form

$$\prod_{k=1}^{n} x_{i_k} = x_{i_1} x_{i_2} \cdots x_{i_n} \qquad x_{i_k} \varepsilon X.$$

We say that X is a set of generators of S_X. It is often convenient to work with $S_X^1 = \{1\} \cup S_X$ rather than S_X. We index S_X^1 by the index set Ω: $S_X^1 = \{s_w | w \varepsilon \Omega\}$.

1.2 Vector Spaces Over a Field F and Algebras Over a Field F.

Let T be the vector space over a field F with a basis S_X^1. Denote T by $F[x_1, x_2, \ldots, x_d]$. Then $T = F[x_1, x_2, \ldots, x_d] = \{\sum_\Omega a_w s_w | a_w \varepsilon F$ and $a_w \neq 0$ for only finitely many $w \varepsilon \Omega\}$. Each element of T is uniquely expressed as a linear combination of elements of S_X^1 over the field F. (Note that $s_i \neq s_j$ if $i \neq j$).

Define addition in T by

$$\sum_\Omega a_w s_w + \sum_\Omega b_w s_w = \sum_\Omega (a_w + b_w) s_w \qquad a_w, b_w \varepsilon F.$$

Addition is obviously well defined since $a_w + b_w \varepsilon F$.

Define scalar multiplication by

$$a(\sum_\Omega a_w s_w) = \sum_\Omega (a a_w) s_w \qquad a, a_w \varepsilon F.$$

Note that $\sum_\Omega a_w s_w = \sum_\Omega b_w s_w$ if and only if $a_w = b_w$ for all $w \varepsilon \Omega$. Then $0 = \sum_\Omega 0 s_w$ and $\sum_\Omega a_w s_w = 0$ implies $a_w = 0$ for all $w \varepsilon \Omega$.

Example: Let S_X^e be the semigroup $\{e, a, a^2\}$, where $ea = ae = a$, $ea^2 = a^2 e = a^2$, $aa^2 = a^2 a = e$. Then $T = \{xe + ya + za^2 | x, y, z \ \varepsilon F\}$ is a vector space over the field F. Let $xe + ya + za^2$ and $x'e + y'a + z'a^2$ be any two elements of T. Then it is natural to write:

$$(xe + ya + za^2)(x'e + y'a + z'a^2) = xx'ee + xy'ea + xz'ea^2$$

$$+ yx'ae + yy'aa + yz'aa^2 + zx'a^2e + zy'a^2a + zz'a^2a^2$$

$$= xx'e + xy'a + xz'a^2 + yz'e + yz'a + yy'a^2 + zy'e + zz'a + zx'a^2$$

$$= (xx' + yz' + zy')e + (xy' + yx' + zz')a + (xz' + yy' + zx')a^2$$

Definition 1.2.1 An Algebra A is a ring which is a vector space over a field F. In addition, the following holds:

$$a(uv) = (au)v = u(av) \quad \text{for all } a \in F,\ u,v \in A.$$

Now let us define multiplication on T over F. Let $u,v \in T$, where

$$u = \sum_{i \in \Omega} a_i s_j \text{ and } v = \sum_{j \in \Omega} b_j s_j .$$

Then

$$uv = \sum_{i \in \Omega} a_i s_i \cdot \sum_{j \in \Omega} b_j s_j = \sum_{i,j \in \Omega} (a_i b_j)(s_i s_j) \qquad (*)$$

The above multiplication is clearly well defined since s_i, s_j are elements in S_X^1 where multiplication is already defined.

Theorem 1.2.2 With the multiplication defined in (*), T is a ring with identity.

Proof: Let $u = \Sigma a_i s_i$, $v = \Sigma b_j s_j$, $w = \Sigma c_k s_k$ be elements of T. Then the multiplication (*) is associative, since

$$\begin{aligned}(uv)w &= (\sum_\Omega a_i s_i \cdot \sum_\Omega b_j s_j)\sum_\Omega c_k s_k = \sum_\Omega a_i b_j (s_i s_j)\sum_\Omega c_k s_k \\ &= \sum_\Omega (a_i b_j) c_k (s_i s_j) s_k = \sum_\Omega a_i (b_j c_k) s_i (s_j s_k) \qquad (**) \\ &= \sum_\Omega a_i s_i \cdot \sum_\Omega b_j c_k (s_j s_k) = \sum_\Omega a_i s_i \cdot (\sum_\Omega b_j s_j \cdot \sum_\Omega c_k s_k) = u(vw).\end{aligned}$$

(**) since $s_i, s_j, s_k \in S_X^1$.

The distributive law holds also, since

$$u(v + w) = \Sigma a_i s_i(\Sigma b_j s_j + \Sigma c_j s_j) = \Sigma a_i s_i(\Sigma(b_j + c_j)s_j)$$

$$= \Sigma a_i(b_j + c_j)s_i s_j = \Sigma(a_i b_j + a_i c_j)s_i s_j$$

$$= \Sigma(a_i b_j s_i s_j + a_i c_j s_i s_j) = \Sigma a_i b_j s_i s_j + \Sigma a_i c_j s_i s_j$$

$$= \Sigma a_i s_i \Sigma b_j s_j + \Sigma a_i s_i \Sigma c_j s_j = uv + uw$$

Similarly $(u + v)w = uw + vw$.

The identity of T is the monomial of degree 0 denoted by 1. Hence T is a ring with identity.

Theorem 1.2.3 T is an algebra over F, called the free semigroup algebra on S_X^1 over F.

Proof: Let $a, a_i, b_j \in F$, $u = \Sigma a_i s_i$ and $v = \Sigma b_j s_j \in T$. Then:

$$a(uv) = a\Sigma a_i s_i \Sigma b_j s_j = a\Sigma a_i b_j s_i s_j = \Sigma(aa_i)b_j s_i s_j$$

$$= \Sigma aa_i s_i \Sigma b_j s_j = (a\Sigma a_i s_i)\Sigma b_j s_j = (au)v$$

$$= \Sigma aa_i s_i \Sigma b_j s_j = \Sigma(aa_i)b_j s_i s_j = \Sigma a_i(ab_j)s_i s_j$$

$$= \Sigma a_i s_i(\Sigma ab_j s_j) = \Sigma a_i s_i(a\Sigma b_j s_j) = u(av)$$

Hence T is an algebra over F.

It is worthwhile to observe that the elements of X do not commute with each other, but they do commute with elements of F.

1.3 Homogeneous Polynomials and Subvector-Spaces of T Over F.

In this section, we will call the elements of T polynomials. This is why the x_i are called non-commuting indeterminates.

Definition 1.3.1 A homogeneous polynomial of degree n is a linear combination of distinct monomials each of degree n .

If u is a homogeneous polynomial of degree i we denote the degree i by $\partial(u) = i$. Here is an example:
Let $x_1 x_2 + x_2 x_2 = u$. Then u is a homogeneous polynomial and $\partial(u) = 2$.
Let $x_1 + x_1 x_2 x_1 = v$. Clearly v is not a homogeneous polynomial.

Theorem 1.3.2 Let T_n be the set of all homogeneous polynomials of T of degree n. Then T_n is a subvector space of T over F.

Proof: At first note that T_n is a subset of T . We need to show that T_n is itself a vector space over F . Also observe that $T_n \neq \emptyset$ since $x_i^n \varepsilon T_n$.

Now let $u = \Sigma a_i s_i$ and $v = \Sigma b_i s_i \in T_n$ where a_i, b_i, c are in F. It is clear that $\partial(s_i) = n$ for each s_i that appears in u or in v. Then

$$cu + v = c\Sigma a_i s_i + \Sigma b_i s_i = \Sigma c a_i s_i + \Sigma b_i s_i = \Sigma(ca_i + b_i)s_i$$

is in T_n .

Note that a basis for T_n consists of all distinct monomials of degree n , hence $\dim T_n = d^n$.

Example: Let $T = F[x_1, x_2, x_3]$. Then

$$\dim T_0 = 3^0 = 1,\ \dim T_1 = 3^1 = 3,\ \dim T_2 = 3^2 = 9$$
$$\dim T_3 = 3^3 = 27,\ \dim T_4 = 3^4 = 81 \text{ and so on.}$$

T_0 has basis $\{1\}$ (the identity of S_X^1), and may be identified with the field F .

T_1 has basis $\{x_1, x_2, x_3\}$

T_2 has basis $\{x_1x_1, x_1x_2, x_1x_3, x_2x_1, x_2x_2, x_2x_3, x_3x_1, x_3x_2, x_3x_3\}$

and T_3 has basis

$$\{x_1^3, x_1^2x_2, x_1^2x_3, x_1x_2x_3, x_1x_3x_2, x_1x_2x_1, x_1x_3x_1, x_1x_2^2, x_1x_3^2$$
$$x_2^3, x_2^2x_1, x_2^2x_3, x_2x_1x_3, x_2x_3x_1, x_2x_1x_2, x_2x_3x_2, x_2x_1^2, x_2x_3^2$$
$$x_3^3, x_3^2x_1, x_3^2x_2, x_3x_1x_2, x_3x_2x_1, x_3x_1x_3, x_3x_2x_3, x_3x_1^2, x_3x_2^2\}$$

The polynomial $x_1^2x_2x_1 + x_3^3x_2 + x_2^2x_1^2 + x_2x_1x_3^2$ is a homogeneous polynomial of degree 4 and hence is in T_4.

<u>Definition 1.3.3</u> Let $W_1, W_2, \ldots, W_k, \ldots$ be subspaces of the vector space W. We shall say that W is <u>the direct sum</u> of $W_1, W_2, \ldots, W_k, \ldots$ and we write $W = W_1 \oplus W_2 \oplus \ldots \oplus W_k \oplus \ldots$ if any of the following equivalent conditions hold:

(i) $W = W_1 + W_2 + \ldots + W_k + \ldots$ and $W_1, \ldots, W_k, \ldots$ are independent. (That is if $\alpha_1 + \alpha_2 + \ldots + \alpha_k = 0$, $\alpha_i \varepsilon W_i$ implies that each $\alpha_i = 0$ (for any k).)

(ii) Each vector $\alpha \neq 0$ in W can be uniquely expressed in the form $\alpha = \alpha_{i_1} + \ldots + \alpha_{i_k}$ with $\alpha_{i_j} \varepsilon W_{i_j}$ (for some k) where the indices are distinct and $\alpha_{i_j} \neq 0$, $1 \leq j \leq k$.

(iii) $W = W_1 + W_2 + \ldots + W_k + \ldots$ and, for each $j \geq 1$, the subspace W_j is disjoint from (has intersection $\{0\}$ with) the sum

$$(W_1 + \ldots + W_{j-1} + W_{j+1} + \ldots).$$

Theorem 1.3.4 Let $T = F[x_1, x_2, \ldots, x_d]$ be the vector space over a field F and let T_n be the subspace of T of all homogeneous polynomials of degree n, for $n = 0,1,2, \ldots$. Then T is the direct sum of $T_0, T_1, \ldots, T_n, \ldots$ and we write $T = T_0 \oplus T_1 \oplus \ldots \oplus T_n \oplus \ldots$

Proof: Let T_j be the subspace of T of all homogeneous polynomials of degree j. Then

$$T_j \cap (T_0 + T_1 + \ldots + T_{j-1} + T_{j+1} + \ldots) = \{0\}$$

because the subspaces $T_0, T_1, \ldots, T_{j-1}, T_{j+1}, \ldots$ have only homogeneous polynomials of degrees $0, 1, \ldots, j-1, j+1, \ldots$ respectively. Also, clearly $T = T_0 + T_1 + \ldots + T_n + \ldots$. Hence

$$T = T_0 \oplus T_1 \oplus \ldots \oplus T_n \oplus \ldots .$$

Corollary 1.3.5 Each element $u \in T$ can be uniquely expressed as a sum of homogeneous polynomials.

Proposition 1.3.6 $T_n = T_{n-1}x_1 \oplus T_{n-1}x_2 \oplus \ldots \oplus T_{n-1}x_d$.

Proof: The elements in $T_{n-1}x_i$, $(i = 1,2, \ldots, d)$ are of degree n, and clearly

$$T_n = T_{n-1}x_1 + T_{n-1}x_2 + \ldots + T_{n-1}x_d.$$

Moreover

$$T_{n-1}x_i \cap (T_{n-1}x_1 + \ldots + T_{n-1}x_{i-1} + T_{n-1}x_{i+1} + \ldots + T_{n-1}x_d) = \{0\}$$

because the x_i's do not commute. Hence

$$T_n = T_{n-1}x_1 \oplus T_{n-1}x_2 \oplus \ldots \oplus T_{n-1}x_d.$$

<u>Example 1.3.7</u> Let $T = F[x_1,x_2,x_3]$. The basis elements for T_2 are grouped into the following sets:

$$\{x_1x_1,x_2x_1,x_3x_1\},\{x_1x_2,x_2x_2,x_3x_2\},\{x_1x_3,x_2x_3,x_3x_3\}$$

where $\quad T_1x_1$ = subspace spanned by $\{x_1x_1,x_2x_1,x_3x_1\}$

$\quad T_1x_2$ = subspace spanned by $\{x_1x_2,x_2x_2,x_3x_2\}$

$\quad T_1x_3$ = subspace spanned by $\{x_1x_3,x_2x_3,x_3x_3\}$

Hence

$$T_2 = T_1x_1 \oplus T_1x_2 \oplus T_1x_3 .$$

1.4 Ideals and Quotient Algebras

The following section consists of some definitions and results which are actually part of (used for) the proof of the main result in Chapter 2, but are put here to get the reader even more familiar with the basic structure we shall be working with.

Let H be a subset of T which consists of nonzero homogeneous polynomials $f_1,f_2, \ldots$ such that $2\leq\partial(f_1)\leq\partial(f_2)\leq \ldots$ and let $\partial(f_i) = n_i$. Now rewriting, we have $2\leq n_1\leq n_2\leq \ldots$.

Let the number of all those f_j's which have degree i be denoted by r_i. This number is assumed to be finite (for each i) and is possibly zero.

Let $\mathfrak{A}$ be the intersection of all ideals of T which contain H. $\mathfrak{A}$ is then an ideal, which is in fact the smallest ideal of T containing the set H. This ideal is called the <u>ideal generated by H.</u>

In what follows, the subset H which generates the ideal $\mathfrak{A}$ will always be as described above. In particular, H contains only homogeneous polynomials of degree ≥ 2.

Theorem 1.4.1 Let T be an algebra, $H = \{ f_1, f_2, \dots \}$ and $\mathfrak{A}$ the ideal generated by H in T. Then the elements of $\mathfrak{A}$ are all elements of T which may be represented in the form $\sum_{i\in I} a_i f_i b_i$, where I is a finite set, a_i and b_i are elements of T and the f_i are elements of H.

Proof: Let $B = \{ \sum_{i=1}^{n} a_i f_i b_i \mid a_i, b_i \in T,\ f_i \in H,\ n = 1,2,3, \dots \}$.

Then $H \subset B \subset \mathfrak{A}$.

Now since $\mathfrak{A}$ is the intersection of all the ideals of T which contain H, to get $\mathfrak{A} \subset B$ it is sufficient to show that B is an ideal of T. But this is obvious. Hence $B \subset \mathfrak{A}$ and $\mathfrak{A} \subset B$, so $\mathfrak{A} = B$.

Remark 1.4.2 Consider the element $\sum_{i\in I} a_i f_i b_i$ of $\mathfrak{A}$. By expressing each a_i and each b_i as a sum of homogeneous polynomials, and then multiplying out, we see that in fact, $\mathfrak{A}$ is the set of all elements of this form $\sum_{j\in J} a'_j g_j b'_j$, where J is a finite set, a'_j and b'_j are homogeneous elements of T, and the g_j are elements of H.

Corollary 1.4.3 Let $r = u_0 + u_1 + u_2 + \dots + u_s \in \mathfrak{A}$ where $u_i \in T_i$; then each $u_i \in \mathfrak{A}$.

Proof: By preceding remark,

$$r = \sum_{j\in J} a_j g_j b_j, \qquad \text{where}$$

each $g_j \in H$ and the a_j, b_j are homogeneous polynomials. By collecting summands of equal degree, we get

$$r = v_0 + v_1 + \dots + v_t, \quad \text{where} \quad v_i \in T_i \cap \mathfrak{A}.$$

But by corollary 1.3.5, r can be expressed uniquely as a sum of homogeneous polynomials. Hence (since we assume $u_s \neq 0 \neq v_t$) we must have $s=t$ and $u_0=v_0$, $u_1=v_1$, ..., $u_s=v_s$. But $v_0, \ldots, v_s \in \mathfrak{A}$, hence $u_0, \ldots, u_s \in \mathfrak{A}$. (Note that since H contains only polynomials of degree ≥ 2, this gives us $u_0 = u_1 = 0$.)

Remark 1.4.4 Let A_i be the quotient vector space $(T_i + \mathfrak{A})/\mathfrak{A}$ over F ; then A_i is a vector subspace of the quotient vector space $T/\mathfrak{A}$ over F.

Theorem 1.4.5 Let A be the quotient vector space $T/\mathfrak{A}$ over F. Then as a vector space, $A = A_0 \oplus A_1 \oplus A_2 \oplus \ldots \oplus A_n \oplus \ldots$, where $A_i = (T_i + \mathfrak{A})/\mathfrak{A}$.

Proof: Let $a \in A$. Then $a = u + \mathfrak{A}$ where $u \in T$. But then u can be written uniquely as the sum of u_i's, i.e. $u = u_0 + u_1 + \ldots + u_n$, where $u_i \in T_i$ and

$$a = (u_0 + u_1 + \ldots + u_n) + \mathfrak{A} = (u_0 + \mathfrak{A}) + (u_1 + \mathfrak{A}) + \ldots + (u_n + \mathfrak{A})$$

where each $u_k + \mathfrak{A} \in (T_k + \mathfrak{A})/\mathfrak{A} = A_k$. Hence $A = A_0 + A_1 + \ldots A_n + \ldots$.

Now we need to show that a can be written in only one way as a sum of elements of the different A_i. Hence, suppose that

(1) $a = (u_0 + \mathfrak{A}) + \ldots + (u_n + \mathfrak{A}) = (v_0 + \mathfrak{A}) + \ldots + (v_m + \mathfrak{A})$

$$u_i, v_i \in T_i$$

We want to show that $u_i + \mathfrak{A} = v_i + \mathfrak{A}$. From (1) we have that

$$(u_0 + \ldots + u_n) + \mathfrak{A} = (v_0 + \ldots + v_m) + \mathfrak{A}$$

hence $\qquad u_0 + \dots + u_n \equiv (v_0 + \dots + v_m) \quad (\text{mod}\ \mathfrak{A})$

Now if $m \geq n$

$$(u_0 - v_0) + (u_1 - v_1) + \dots + (u_n - v_n) + (0 - v_{n+1}) + \dots + (0 - v_m) \equiv 0\ (\text{mod}\ \mathfrak{A})$$

Hence $\qquad (u_0 - v_0) + \dots + (u_n - v_n) + (0 - v_{n+1}) + \dots + (0 - v_m) \in \mathfrak{A}$

where $u_i - v_i \in T_i$ and therefore $u_i - v_i \in \mathfrak{A}$, $\forall i$, by corollary 1.4.3.

So $u_i + \mathfrak{A} = v_i + \mathfrak{A}$ which shows the uniqueness i.e. $u_i \equiv v_i$ $(\text{mod}\ \mathfrak{A})$. Consequently, $A = A_0 \oplus A_1 \oplus \dots \oplus A_n \oplus \dots$.

<u>Theorem 1.4.6</u> If $\mathfrak{A}_n = T_n \cap \mathfrak{A}$, then $A_n \cong T_n/\mathfrak{A}_n$, where $\mathfrak{A}_n$ and A_n are regarded as vector spaces over F.

<u>Proof</u>: Consider the vector space T over F as an additive group. Then the ideal $\mathfrak{A}$ is a normal subgroup of T, and by the Second Isomorphism Theorem of group theory, we have the following diagram and isomorphisms.

$$A_n = \frac{T_n + \mathfrak{A}}{\mathfrak{A}} \simeq \frac{T_n}{T_n \cap \mathfrak{A}} = \frac{T_n}{\mathfrak{A}_n} \quad . \text{(i)}$$

$$\begin{tikzcd} & T_n + \mathfrak{A} \arrow[dl, no head] \arrow[dr, no head] & \\ T_n \arrow[dr, no head] & & \mathfrak{A} \arrow[dl, no head] \\ & T_n \cap \mathfrak{A} & \end{tikzcd}$$

Because $\mathfrak{A} \cap T_0 = \{0\}$, $\mathfrak{A} \cap T_1 = \{0\}$ we have the following particular cases:

$$A_0 \simeq T_0 \simeq F \qquad \text{(ii)}$$

$$A_1 \simeq T_1 \qquad \text{(iii)}$$

<u>Remark 1.4.6</u> Note that $\mathfrak{A} = \mathfrak{A}_0 \oplus \mathfrak{A}_1 \oplus \dots \oplus \mathfrak{A}_n \oplus \dots$ where $\mathfrak{A}_0 = \{0\} = \mathfrak{A}_1$

CHAPTER 2.

This chapter is devoted to the proof of the main Golod-Šafarevič theorem. In the first part of the chapter, we will derive some results which will be used to give a short proof of the first theorem. The second part of the chapter deals with the proof of the first theorem. We give two different proofs; one is a re-worked and expanded version of the proof in the paper by Fisher and Struik [5]. The other proof, which has to do with homology, is a re-worked and expanded version of a proof from Herstein's book "Noncommutative Rings" [11]. In the same part, two more theorems follow which are re-worked from the original paper by Golod and Šafarevič [6]. Finally, the last part of this chapter deals with some corollaries and special cases [5], [19].

2.1 Some Subspaces and Their Dimensions

In this section before the derivation of the results necessary for the proof of the main theorem, let us recall the various notations we have introduced up to this point.

We have a field F and d noncommutative indeterminates over F, which are $x_1, x_2, \ldots, x_d$. Also,

$$T = F[x_1, x_2, \ldots, x_d]$$

is the free associative algebra over F in the x_i and, moreover:

$$T = T_0 \oplus T_1 \oplus \ldots \oplus T_n \oplus \ldots$$

where each T_n is the subspace of T consisting of all the homogeneous polynomials of degree n. Recall that:

$$H = \{f_1, f_2, \ldots, f_n, \ldots\},$$

where each f_i is a homogeneous polynomial of T and

$$2 \leq \partial(f_1) \leq \partial(f_2) \leq \cdots$$

($\partial(f_i) = n_i$ is the degree of f_i). $\mathfrak{A}$ is the ideal of T generated by H, and r_i is the number of polynomials f_j in H which have degree i. The quotient algebra $A = T/\mathfrak{A}$ is also of the form:

$$A = A_0 \oplus A_1 \oplus \cdots \oplus A_n \oplus \cdots$$

where $A_i = (T_i + \mathfrak{A})/\mathfrak{A}$. We mentioned that $\dim T_n = d^n$. Now let $\dim A_n = b_n$ and observe:

2.1.1 $\qquad 1 = d^0 = \dim T_0 = \dim A_0 = b_0$ $\qquad$ and

2.1.2 $\qquad d = \dim T_1 = \dim A_1 = b_1$.

Proposition 2.1.3 Recall that $\mathfrak{A}_n = T_n \cap \mathfrak{A}$ (by definition), and that $A_n \simeq T_n/\mathfrak{A}_n$. Let S_n be a complementary subspace of $\mathfrak{A}_n$ in T_n; that is, $T_n = \mathfrak{A}_n \oplus S_n$. Then $\dim S_n = \dim A_n = b_n$.

Proof: $T_n = \mathfrak{A}_n \oplus S_n$ gives $\dim T_n = \dim \mathfrak{A}_n + \dim S_n$.

Also $\quad A_n \simeq T_n/\mathfrak{A}_n$ gives $\dim T_n = \dim \mathfrak{A}_n + \dim A_n$.

These two equalities give the desired result .

Proposition 2.1.4 Dim $\mathfrak{A}_2 \leq r_2$, where $\mathfrak{A}_2 = T_2 \cap \mathfrak{A}$ and r_2 is the number of f_j of degree 2.

Proof: Look at $\mathfrak{A}$ as a vector space. Recall that

$$\mathfrak{A} = \mathfrak{A}_2 \oplus \mathfrak{A}_3 \oplus \cdots \oplus \mathfrak{A}_n \oplus \cdots$$

where each $\mathfrak{A}_n$ is a subvector space of $\mathfrak{A}$.

Now $\mathfrak{A}_2$ has a basis of elements of the form mf_jn, where m, n are monomials and $\partial(mf_jn) = 2$. If the degree of mf_jn is 2, then $\partial(f_j) \leq 2$. But $\partial(f_j) > 1$ always. Hence $\partial(f_j) = 2$ which implies that m, n are constants. In other words, a basis for $\mathfrak{A}_2$ is a set of linearly independent

f_j of degree 2. If the f_j of degree 2 were linearly independent, then $\dim \mathfrak{A}_2$ would equal r_2.

Since the number r_2 does not necessarily denote linearly independent f_i of degree 2, we have

$$\dim \mathfrak{A}_2 \leq r_2.$$

Definition 2.1.5 Let $J = \mathfrak{A}_{n-1}x_1 \oplus \cdots \oplus \mathfrak{A}_{n-1}x_d$.

Note 2.1.6 To prove that the sum J is direct, we need to show that if $g_1, g_2, \ldots, g_d \in T_{n-1}$, then $g_1x_1 + \ldots + g_dx_d = 0$ implies $g_1 = \ldots = g_d = 0$.

Proof: Each g_i is the sum of distinct monomials of degree n-1. Therefore, g_ix_i is the sum of distinct monomials of degree n. If $i \neq k$, then the monomials in g_ix_i are distinct from the monomials in g_kx_k. Therefore, the set of all monomials involved in $g_1x_1 + \ldots + g_dx_d$ is a set of distinct monomials, hence is a set of linearly independent monomials. Therefore, every coefficient in $g_1x_1 + \ldots + g_dx_d = 0$ must be zero. Therefore:

$$g_1 = g_2 = \ldots = g_d = 0$$

Proposition 2.1.7 $\dim J = d \dim \mathfrak{A}_{n-1}$

Proof: Since $J = \mathfrak{A}_{n-1}x_1 \oplus \cdots \oplus \mathfrak{A}_{n-1}x_d$ is a direct sum and

$$\dim \mathfrak{A}_{n-1}x_i = \dim \mathfrak{A}_{n-1},$$

it follows that:

$$\dim J = d \dim \mathfrak{A}_{n-1}.$$

<u>Example 2.1.8</u> Let $g_1, g_2, \ldots, g_m$ be a basis for $\mathfrak{A}_{n-1}$. Then we have the following bases for each $\mathfrak{A}_{n-1}x_i$ $(i = 1, 2, \ldots, d)$.

$B_1 = \{g_1x_1, g_2x_1, \ldots, g_mx_1\}$ forms a basis for $\mathfrak{A}_{n-1}x_1$.

$B_2 = \{g_1x_2, g_2x_2, \ldots, g_mx_2\}$ forms a basis for $\mathfrak{A}_{n-1}x_2$.

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$B_d = \{g_1x_d, g_2x_d, \ldots, g_mx_d\}$ forms a basis for $\mathfrak{A}_{n-1}x_d$

The elements of the above sets are linearly independent, for suppose that:

$$\sum_{j=1}^{m} a_jg_j = g \qquad \text{where } g\varepsilon\mathfrak{A}_{n-1} \subset T_{n-1},\ a_j\varepsilon F$$

write

$$g = \sum_{i=1}^{n} b_iu_i \qquad\qquad b_i\varepsilon F$$

where the u_i are distinct monomials of degree n-1. (We have d^{n-1} possible u_i's.) Suppose

$$0 = gx_k = \sum_{i=1}^{n}(b_iu_i)x_k = \sum_{i=1}^{n} b_i(u_ix_k)$$

Then if the u_ix_k are distinct (monomials of degree n), then they are linearly independent, therefore b_i's = 0, therefore g = 0. Let u_1, u_2 be distinct monomials of degree n-1. Then u_1x_k and u_2x_k are distinct since T is the free associative algebra over F. Thus the set $\{g_1x_i, g_2x_i, \ldots, g_mx_i\}$ consists of linearly independent elements. Hence, B_i is a basis for $\mathfrak{A}_{n-1}x_i$.

<u>Proposition 2.1.9</u> Prove that $\dim T_n = db_{n-1} + \dim J$

<u>Proof</u>: We know that $T_{n-1} = \mathfrak{A}_{n-1} \oplus S_{n-1}$, where S_{n-1} is the complement of $\mathfrak{A}_{n-1}$ in T_{n-1}. Let $s_1, s_2, \ldots, s_{b_{n-1}}$ be a basis for S_{n-1} and let $g_1, g_2, \ldots, g_m$ be a basis for $\mathfrak{A}_{n-1}$. (S_{n-1} and $\mathfrak{A}_{n-1}$ are considered as vector spaces over F.) The elements $s_i x_j$ and $g_k x_j$, where $i = 1, 2, \ldots, b_{n-1}$, $j = 1, 2, \ldots, d$, and $k = 1, 2, \ldots, m$ form a basis for T_n for the following reasons: The $s_i x_j$ and $g_k x_j$ are all of degree n and:

$$\dim T_n = b_{n-1}d + md = (b_{n-1} + m)d = d^{n-1}d = d^n \qquad (1)$$

where $b_{n-1} + m = \dim T_{n-1} = d^{n-1}$. Finally the set $\{s_i x_j\} \cup \{g_k x_j\}$, (i,j,k as before) consists of linearly independent vectors, for suppose not, then:

$$\sum_{i,j}^{b_{n-1},d} a_{ij} s_i x_j = \sum_{kl}^{m,d} b_{kl} g_k x_l$$

implies

$$\sum_j (\sum_i a_{ij} s_i) x_j = \sum_k (\sum_l b_{kl} g_k) x_l$$

Hence for all e = 1, 2, ..., d

$$\sum_i (a_{ie} s_i) x_e = \sum_k (b_{ke} g_k) x_e$$

implies

$$\Sigma(a_{ie} s_i) x_e + \Sigma(-b_{ke} g_k) x_e = 0$$

By example 2.1.8, $a_{ie} = 0 = -b_{ke}$

Thus the set $\{s_i x_j\} \cup \{g_k x_j\}$ $(i = 1,2, \ldots, b_{n-1};\ j = 1, 2, \ldots, d;$ and $k = 1, 2, \ldots, m)$ consists of linearly independent elements, moreover, it has d^n elements all of degree n, therefore:

$$\{s_i x_j\} \cup \{g_k x_j\}$$

is a basis for T_n. So

$$T_n = S_{n-1}x_1 \oplus \cdots \oplus S_{n-1}x_d \oplus \mathfrak{A}_{n-1}x_1 \oplus \cdots \oplus \mathfrak{A}_{n-1}x_d$$

Hence:

$$\dim T_n = db_{n-1} + d\dim \mathfrak{A}_{n-1}$$

or

$$\dim T_n = db_{n-1} + \dim J$$

Definition 2.1.10 Let L be the vector space spanned by all elements of degree n of the form $v_i f_j$, where f_j is in the set H and $\{v_i\}$ is a set of homogeneous polynomials of degrees up to n-2, which forms a basis for $S_0 \oplus S_1 \oplus \cdots \oplus S_{n-2}$

Proposition 2.1.11 $\dim L \le \sum_{i=2}^{n} b_{n-i} r_i$ where L is defined above and where $b_{n-i} = \dim A_{n-i}$ and r_i is the number of those polynomials f_i in H of degree i.

Proof: Let $\partial(v_{i_j}) = i$. Then:

v_{0_1} forms a basis for S_0

$v_{1_1}, v_{1_2}, \ldots, v_{1_{b_1}}$ form a basis for S_1

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$v_{n-2_1}, v_{n-2_2}, \ldots, v_{n-2_{b_i}}$ form a basis for S_i .

The elements v_{0_1}, v_{1_1}, ..., $v_{1_{b_1}}$, v_{2_1}, ..., $v_{2_{b_2}}$, ...,

v_{n-2_1}, ..., $v_{n-2_{b_{n-2}}}$ form a basis for $S_0 \oplus S_1 \oplus \ldots \oplus S_{n-2}$.

Let

$$H = \{f_{2_1}, f_{2_2}, \ldots, f_{2_{r_2}}, f_{3_1}, f_{3_2}, \ldots, f_{3_{r_3}}, f_{4_1}, \ldots, f_{4_{r_n}}, \ldots\}$$

where f_{i_j} has degree i (and for some i's , perhaps there are no f_{i_j}'s.) Let $\partial(G_i)$ denote the degree of the elements of G_i .

$$\{f_{2_1}, f_{2_2}, \ldots, f_{2_{r_2}}\} = G_2 \qquad \text{where } \partial(G_2) = 2$$

$$\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array} \qquad \begin{array}{ccc} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array}$$

$$f_{n_1}, f_{n_2}, \ldots, f_{n_{r_n}} = G_n \qquad \text{where } \partial(G_n) = n$$

Now we observe that the following elements are homogeneous polynomials of degree n.

$$\begin{array}{cccc} v_{(n-i)_1} f_{i_1}, & v_{(n-i)_1} f_{i_2}, & \ldots, & v_{(n-i)_1} f_{i_{r_i}} \\ v_{(n-i)_2} f_{i_1}, & v_{(n-i)_2} f_{i_2}, & \ldots, & v_{(n-i)_2} f_{i_{r_i}} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ v_{(n-i)_{b_{n-i}}} f_{i_1}, & v_{(n-i)_{b_{n-i}}} f_{i_2}, & \ldots, & v_{(n-i)_{b_{n-i}}} f_{i_{r_i}} \end{array}$$

Also they span

$$S_{n-i}f_{i_1} + S_{n-i}f_{i_2} + \dots + S_{n-i}f_{i_{r_i}} = S_{n-i}G_i$$

and their number is $b_{n-i}r_i$. By summing $S_{n-i}G_i$ over $i = 2, \dots, n$ we get

$$L = S_{n-2}G_2 + S_{n-3}G_3 + \dots + S_1G_{n-1} + S_0G_n$$

and hence

$$\dim L \le \dim (S_{n-2}G_2) + \dim (S_{n-3}G_3) + \dots + \dim (S_0G_n)$$

$$\le b_{n-2}r_2 + b_{n-3}r_3 + \dots + b_1r_{n-1} + b_0r_n$$

$$= \sum_{j=2}^{n} b_{n-j}r_j$$

Hence $\dim L \le \sum_{j=2}^{n} b_{n-j}r_j$

<u>Proposition 2.1.12</u> $\dim \mathfrak{A}_n \le \dim J + \dim L$.

<u>Proof</u>: Here J and L are as they have been previously defined by definitions 2.1.5 and 2.1.10 respectively.

Now we take $u \in \mathfrak{A}_n$, and we wish to show that:

$$u = w + v$$

where $w \in J$ and $v \in L$. By proposition 1.4.1, u is the <u>sum</u> of polynomials of the form

$$a_if_jb_k \qquad a_i, b_k \in T$$

where a_i, b_k, are homogeneous polynomials and $\partial(a_if_jb_k) = n$. (We know nothing about $\partial(a_i)$, $\partial(b_k)$, and $\partial(f_j)$.)

Now if we get simply $a_if_jb_k \in J$, we have <u>case I,</u> while if we get

$$a_if_jb_k = w'f_j + v'f_j$$

where $w'f_j \in J$ and $v'f_j \in L$, we have the <u>case II</u>.

Thus each polynomial $a_i f_j b_k \varepsilon$ J + L, and hence, the sum of a lot of them, namely u, also ε J + L.

Case I: Assume $\partial(b_k)\geq 1$. We can write $b_k = b_k' x_n$ ($b_k' \varepsilon T_{j-1}$, if $b_k \varepsilon T_j$).

Now
$$a_i f_j b_k = a_i f_j b_k' x_n = (a_i f_j b_k') x_n \varepsilon \mathfrak{A}_{n-1} x_n \subset J$$

since
$$a_i f_j b_k' \varepsilon \mathfrak{A}_{n-1} \quad (n = 1, 2, \ldots, d).$$

Case II: Assume that $\partial(b_k) = 0$, that is $a_i f_j b_k$ is a homogeneous polynomial of the form $a_i f_j$, where $\partial(a_i f_j) = n$. Now a_i is homogeneous and say has degree k, so:

$$a_i \varepsilon T_k = \mathfrak{A}_k \oplus S_k ;$$

let
$$a_i = w' + v', \qquad w' \varepsilon \mathfrak{A}_k , v' \varepsilon S_k.$$

therefore
$$a_i f_j = w' f_j + v' f_j$$

First look at $w' f_j$. Now $\partial(w') = k = \partial(a_i)$, and $\partial(a_i f_j) = n$. Say $\partial(f_j) = k'$, so $k + k' = n$. Note $k' \geq 2$. Now $w' \varepsilon \mathfrak{A}_k \subset \mathfrak{A}$. Since $\partial(f_j)\geq 2$, we can write $f_j = hx_m$, where $x_m \varepsilon \{x_1, x_2, \ldots, x_d\}$.

Then $w' \varepsilon \mathfrak{A} \Rightarrow w'h \varepsilon \mathfrak{A}$, but $w'h$ is homogeneous and

$$\begin{aligned}\partial(w'h) &= \partial(w'f_j) - 1 \\ &= k + k' - 1 \\ &= n - 1\end{aligned}$$

Therefore $w'h \varepsilon \mathfrak{A} \cap T_{n-1} = \mathfrak{A}_{n-1}$. Therefore:

$$w' f_j = (w'h) x_m \varepsilon \mathfrak{A}_{n-1} x_m \subset J$$

Next, look at $v'f_j$. Now $v' \in S_k \subset T_k$, so v' is a homogeneous polynomial of degree k. We still have $k + k' = n$ and $k' \geq 2$. So

$$k \leq n - 2$$

i.e. $v' \in S_k \subset S_0 \oplus S_1 \oplus \ldots \oplus S_{n-2}$. Since $\{v_1, v_2, \ldots\}$ is a basis of homogeneous polynomials for $S_0 \oplus S_1 \oplus \ldots \oplus S_{n-2}$, we can write v' as a linear combination of some of these basis elements, say v' is a linear combination of

$$v_{i_1}, v_{i_2}, \ldots, v_{i_p} .$$

Since $v', v_{i_1}, \ldots, v_{i_p}$ are all homogeneous, and the $v_{i_1}, \ldots, v_{i_p}$ are linearly independent, it must be the case that

$$k = \partial(v') = \partial(v_{i_1}) = \ldots = \partial(v_{i_p})$$

Thus

$$v'f_j = c_{i_1}(v_{i_1}f_j) + c_{i_2}(v_{i_2}f_j) + \ldots + c_{i_p}(v_{i_p}f_j);$$

But

$$v_{i_1}f_j, \ldots, v_{i_p}f_j$$

are among the polynomials which span L. (Each has degree n, and is of the proper form.) Thus:

$$v'f_j \in L.$$

2.2 Golod-Šafarevič Theorem and its Proofs

Theorem 2.2.1 (Golod and Šafarevič)

(i) $$b_n \geq db_{n-1} - \sum_{i=2}^{n} r_i b_{n-i} \qquad n \geq 2$$

(ii) $$b_n \geq db_{n-1} - \sum_{n_i \leq n} b_{n-n_i} \qquad n \geq 2$$

<u>Note</u>: The same notation used previously holds here. Also (i) and (ii) are the same. Because some of the n_i's may be equal, (ii) may be written as follows:

$$b_n \geq db_{n-1} - \sum_{\{i|n_i \leq n\}} b_{n-n_i}$$

<u>Proof of (i)</u> : (Fisher and Struick) This proof is using the dimensionality of the various subspaces.

Clearly we know that

$$T_n = \mathfrak{U}_n \oplus S_n$$

And therefore

$$\dim T_n = \dim \mathfrak{U}_n + \dim S_n.$$

(1) Let $n = 2$. Then we have that

$$\dim T_2 = \dim \mathfrak{U}_2 + \dim S_2$$

$$d^2 \leq r_2 + b_2 \qquad \text{(by 2.1.4 and 2.1.3)}$$

$$b_2 \geq d^2 - r_2$$

$$b_2 \geq dd - r_2 \cdot 1$$

$$b_2 \geq db_1 - r_2 b_0 \qquad \text{(by 2.1.2 and 2.1.1)}$$

which is statement (i) for $n = 2$.

(2) Let $n \geq 2$. Then

$$\dim T_n = \dim \mathfrak{U}_n + \dim S_n$$

$$\leq \dim J + \dim L + b_n \qquad (*)$$

$$\leq \dim J + \sum_{i=2}^{n} r_i b_{n-i} + b_n \qquad (**)$$

(*)by 2.1.12 and 2.1.3, (**) by 2.1.11.

But since, by 2.1.9, we have that

$$\dim T_n = db_{n-1} + \dim J$$

we have

$$db_{n-1} \le \sum_{i=2}^{n} r_i b_{n-i} + b_n$$

and therefore

$$b_n \ge db_{n-1} - \sum_{i=2}^{n} r_i b_{n-i} \qquad n \ge 2.$$

<u>Proof of (ii)</u> (Herstein) This proof is using homology which was used in the original proof by Golod and Šafarevič.

Suppose that we can exhibit linear mappings, ϕ, ψ so that the following sequence is exact

$$\underbrace{A_{n-n_1} \oplus \ldots \oplus A_{n-n_k} \oplus \ldots}_{n_i \le n} \xrightarrow{\phi} \underbrace{A_{n-1} \oplus A_{n-1} \oplus \ldots \oplus A_{n-1}}_{d \text{ times}} \xrightarrow{\psi} A_n \to 0 \quad (1)$$

Then

$$\dim(A_{n-1} \oplus \ldots \oplus A_{n-1}) = \operatorname{rank} \psi + \operatorname{nullity} \psi,$$

or

$$\begin{aligned} db_{n-1} &= \dim \operatorname{Im} \psi + \dim \ker \psi \\ &= b_n + \dim \operatorname{Im} \phi \end{aligned}$$

by exactness, so

$$db_{n-1} \le b_n + \sum_{n_i \le n} b_{n-n_i} \qquad n \ge 2.$$

Hence:

$$b_n \ge db_{n-1} - \sum_{n_i \le n} b_{n-n_i} \qquad n \ge 2.$$

Now our objective is that of defining the ϕ and ψ. First we shall define mappings Φ and Ψ for the following sequence

$$\underbrace{T_{n-n_1} \oplus \ldots \oplus T_{n-n_k} \oplus \ldots}_{n_i \leqq n} \xrightarrow{\Phi} \underbrace{T_{n-1} \oplus T_{n-1} \oplus \ldots \oplus T_{n-1}}_{d \text{ times}} \xrightarrow{\Psi} T_n \to 0 \quad (2)$$

where Φ and Ψ are linear. We are not interested if the sequence is exact or not at the T - level. However, we want to induce the proper ϕ and ψ from the Φ and Ψ so the sequence will be exact at the A-level.

Define Ψ by:

$$\Psi : t_1 \oplus \ldots \oplus t_d \to t_1 x_1 + t_2 x_2 + \ldots + t_d x_d \text{ (for } t_i \varepsilon T_{n-1})$$

If $u \varepsilon T_{n-1} \oplus \ldots \oplus T_{n-1}$, then u may be written uniquely as

$$u = t_1 \oplus \ldots \oplus t_d$$

where t_i is in the ith T_{n-1} in the above direct sum. Hence if we define

$$\Psi(u) = t_1 x_1 + \ldots + t_d x_d$$

Ψ is well define and obviously, if $a,b \varepsilon F$ and $u,v \varepsilon T_{n-1} \oplus \ldots \oplus T_{n-1}$, we have

$$\Psi(au + bv) = a\Psi(u) + b\Psi(v)$$

and hence Ψ is linear.

Define Φ by:

$$\Phi : s_{n-n_1} \oplus \ldots \oplus s_{n-n_k} \oplus \ldots \to u_1 \oplus u_2 \oplus \ldots \oplus u_d$$

where

$$s_{n-n_1} \oplus \ldots \oplus s_{n-n_k} \oplus \ldots \varepsilon T_{n-n_1} \oplus \ldots \oplus T_{n-n_k} \oplus \ldots$$

and $u_1 \oplus \ldots \oplus u_d \varepsilon T_{n-1} \oplus \ldots \oplus T_{n-1}$.

The way we follow to get Φ is the following: If

$$s_{n-n_1} \oplus \cdots \oplus s_{n-n_k} \oplus \cdots \in T_{n-n_1} \oplus \cdots \oplus T_{n-n_k} \oplus \cdots$$

then, recalling that $\partial(f_i) = n_i$, we see that

$$\sum_{n_i \leq n} s_{n-n_i} f_i \in T_n.$$

$(\partial(s_{n-n_i}) + \partial(f_i) = n - n_i + n_i = n)$. As an element in T_n, we can write

$$\sum_{n_i \leq n} s_{n-n_i} f_i = \sum_{i=1}^{d} u_i x_i$$

where the u_i are uniquely determined elements in T_{n-1}. Hence Φ is well defined and like Ψ, Φ is linear.

Proposition 2.2.2 Let Ψ be defined as above. Then sequence (2) is exact at T_n.

Proof: To show exactness at T_n, we need to show that Ψ is an onto homomorphism. Now if $w \in T_n$, then w can be written uniquely as follows:

$$w = t_1 x_1 + \cdots + t_d x_d$$

where $t_1, t_2, \ldots, t_d$ are in T_{n-1} and such that

$$\Psi(t_1 \oplus \cdots \oplus t_d) = t_1 x_1 + \cdots t_d x_d = w$$

Hence Ψ is onto and since Ψ is linear, the sequence

$$\underbrace{T_{n-1} \oplus \cdots \oplus T_{n-1}}_{d \text{ times}} \to T_n \to 0$$

is exact.

Recall that $\mathfrak{A}_{n-1} = \mathfrak{A} \cap T_{n-1}$. Since $\mathfrak{A}_{n-1} \subseteq T_{n-1}$, obviously we have:

$$\underbrace{\mathfrak{A}_{n-1} \oplus \mathfrak{A}_{n-1} \oplus \cdots \oplus \mathfrak{A}_{n-1}}_{d \text{ times}} \subseteq \underbrace{T_{n-1} \oplus \cdots \oplus T_{n-1}}_{d \text{ times}}$$

If we can show that

$$\Psi(\mathfrak{A}_{n-1} \oplus \dots \oplus \mathfrak{A}_{n-1}) = \{\Psi(t_1 \oplus \dots \oplus t_d) | t_i \in \mathfrak{A}_{n-1}\} \subseteq \mathfrak{A}_n$$

then we can induce a new homomorphism

$$\psi : \frac{T_{n-1} \oplus \dots \oplus T_{n-1}}{\mathfrak{A}_{n-1} \oplus \dots \oplus \mathfrak{A}_{n-1}} \to \frac{T_n}{\mathfrak{A}_n}$$

given by

$$\begin{aligned}\psi(t_1 \oplus \dots \oplus t_d + \mathfrak{A}_{n-1} \oplus \dots \oplus \mathfrak{A}_{n-1}) &= \Psi(t_1 \oplus \dots \oplus t_d) + \mathfrak{A}_n \\ &= t_1x_1 + \dots + t_dx_d + \mathfrak{A}_n\end{aligned}$$

<u>Proposition 2.2.3</u> $\Psi(\mathfrak{A}_{n-1} \oplus \dots \oplus \mathfrak{A}_{n-1}) \subseteq \mathfrak{A}_n$.

<u>Proof</u>: Take $t_1, \dots, t_d$ such that t_i is in the ith $\mathfrak{A}_{n-1}$ then

$$\Psi(t_1 \oplus \dots \oplus t_d) = t_1x_1 + \dots + t_dx_d$$

$t_1, t_2, \dots t_d$ are in $\mathfrak{A}_{n-1}$ and hence in $\mathfrak{A}$; since $\mathfrak{A}$ is an ideal then $t_1x_1, t_2x_2, \dots, t_dx_d$ are in $\mathfrak{A}$ and therefore their sum is in $\mathfrak{A}$. But it is also in T_n. Hence

$$t_1x_1 + \dots + t_dx_d \in \mathfrak{A}_n,$$

and so

$$\Psi(\mathfrak{A}_{n-1} \oplus \dots \oplus \mathfrak{A}_{n-1}) \subseteq \mathfrak{A}_n.$$

Now if we show that

$$\frac{T_{n-1} \oplus \dots \oplus T_{n-1}}{\mathfrak{A}_{n-1} \oplus \dots \oplus \mathfrak{A}_{n-1}} \simeq \frac{T_{n-1}}{\mathfrak{A}_{n-1}} \oplus \dots \oplus \frac{T_{n-1}}{\mathfrak{A}_{n-1}}$$

then we can induce the required mapping

$$\psi : A_{n-1} \oplus \dots \oplus A_{n-1} \to A_n$$

Proposition 2.2.4

$$\frac{T_{n-1} \oplus \ldots \oplus T_{n-1}}{\mathfrak{A}_{n-1} \oplus \ldots \oplus \mathfrak{A}_{n-1}} \simeq \frac{T_{n-1}}{\mathfrak{A}_{n-1}} \oplus \ldots \oplus \frac{T_{n-1}}{\mathfrak{A}_{n-1}}$$

Proof: We need to find an onto map γ such that

$$\gamma : T_{n-1} \oplus \ldots \oplus T_{n-1} \twoheadrightarrow \frac{T_{n-1}}{\mathfrak{A}_{n-1}} \oplus \ldots \oplus \frac{T_{n-1}}{\mathfrak{A}_{n-1}}$$

and such that $\ker \gamma = \mathfrak{A}_{n-1} \oplus \ldots \oplus \mathfrak{A}_{n-1}$. So define

$$\gamma(t_1 \oplus \ldots \oplus t_d) = \bar{t}_1 \oplus \ldots \oplus \bar{t}_d$$

$$= (t_1 + \mathfrak{A}_{n-1}) \oplus \ldots \oplus (t_d + \mathfrak{A}_{n-1}).$$

Let $t_1 \oplus \ldots \oplus t_d \in \ker \gamma$, then

$$\gamma(t_1 \oplus \ldots \oplus t_d) = (t_1 + \mathfrak{A}_{n-1}) \oplus \ldots \oplus (t_d + \mathfrak{A}_{n-1})$$

$$= (0 + \mathfrak{A}_{n-1}) \oplus \ldots \oplus (0 + \mathfrak{A}_{n-1}).$$

Since we are working with direct sum, this holds if and only if:

$$t_i + \mathfrak{A}_{n-1} = 0 + \mathfrak{A}_{n-1},$$

that is $t_i \in \mathfrak{A}_{n-1}$

so $(t_1 \oplus \ldots \oplus t_d) \in (\mathfrak{A}_{n-1} \oplus \ldots \oplus \mathfrak{A}_{n-1}),$

therefore $\ker \gamma = \mathfrak{A}_{n-1} \oplus \ldots \oplus \mathfrak{A}_{n-1}.$

Now if θ is the natural map such that

$$\theta : T_{n-1} \oplus \ldots \oplus T_{n-1} \to \frac{T_{n-1} \oplus \ldots \oplus T_{n-1}}{\mathfrak{A}_{n-1} \oplus \ldots \oplus \mathfrak{A}_{n-1}}$$

then there exists an isomorphism σ such that

$$\sigma : \frac{T_{n-1} \oplus \ldots \oplus T_{n-1}}{\mathfrak{A}_{n-1} \oplus \ldots \oplus \mathfrak{A}_{n-1}} \twoheadrightarrow \frac{T_{n-1}}{\mathfrak{A}_{n-1}} \oplus \ldots \oplus \frac{T_{n-1}}{\mathfrak{A}_{n-1}},$$

Thus the mapping Ψ induces

$$\psi : A_{n-1} \oplus \dots \oplus A_{n-1} \to A_n \qquad \text{given by}$$

$$\psi\left((t_1 + \mathfrak{A}_{n-1}) \oplus \dots \oplus (t_d + \mathfrak{A}_{n-1})\right) = (t_1x_1 + \dots + t_dx_d) + \mathfrak{A}_n$$

where $\qquad \Psi(t_1 \oplus \dots \oplus t_d) = t_1x_1 + \dots + t_dx_d.$

We can now consider Φ. Suppose that $s_{n-n_1}, s_{n-n_2}, \dots, s_{n-n_k}, \dots$ are in $\mathfrak{A}_{n-n_1}, \mathfrak{A}_{n-n_2}, \dots, \mathfrak{A}_{n-n_k}, \dots$ respectively. We must show that $u_1, u_2, \dots, u_d$ defined by $\Sigma s_{n-n_i}f_i = \Sigma u_ix_i$ are in $\mathfrak{A}_{n-1}$. Since Φ is linear it suffices to do so for each s_{n-n_i} in $\mathfrak{A}_{n-n_i}$. Note that $\partial(s_{n-n_i}f_i) = n-n_i + n_i = n$. Since $\partial(f_i) = n_i$ implies that

$$f_i = \sum_{j=1}^{d} g_{ij}x_j$$

where $g_{ij} \in T_{n_i-1}$. Therefore:

$$s_{n-n_i}f_i = s_{n-n_i}\sum_{j=1}^{d} g_{ij}x_j = \sum_{j=1}^{d}(s_{n-n_i}g_{ij})x_j = \sum_{j=1}^{d} u_jx_j$$

where $u_j = s_{n-n_i}g_{ij}$ and $\partial(u_j) = n - n_i + n_i - 1 = n - 1$. Thus $u_j \in T_{n-1}$. But $u_j = s_{n-n_i}g_{ij} \in \mathfrak{A}$, as s_{n-n_i} is in the ideal $\mathfrak{A}$.

Therefore, $u_j \in \mathfrak{A} \cap T_{n-1} = \mathfrak{A}_{n-1}$. Therefore Φ induces a map:

$$\phi : A_{n-n_1} \oplus \dots \oplus A_{n-n_k} \oplus \dots \to A_{n-1} \oplus \dots \oplus A_{n-1}$$

given by

$$\phi\left((s_{n-n_1} + \mathfrak{A}_{n-n_1}) \oplus \dots \oplus (s_{n-n_k} + \mathfrak{A}_{n-n_k}) \oplus \dots\right) = (u_1 \oplus \dots \oplus u_d) + (\mathfrak{A}_{n-1} \oplus \dots \oplus \mathfrak{A}_{n-1}),$$

$$= (u_1 + \mathfrak{A}_{n-1}) \oplus \dots \oplus (u_d + \mathfrak{A}_{n-1}),$$

where $\qquad \Phi(s_{n-n_1} \oplus \dots \oplus s_{n-n_k} \oplus \dots) = u_1 \oplus \dots \oplus u_d.$

<u>Proposition 2.2.5</u> The sequence

$$\underbrace{A_{n-n_1} \oplus \ldots \oplus A_{n-n_k} \oplus \ldots}_{n_i \leq n} \xrightarrow{\phi} \underbrace{A_{n-1} \oplus \ldots \oplus A_{n-1}}_{d \text{ times}} \xrightarrow{\psi} A_n \to 0 \qquad (1)$$

is exact.

<u>Proof</u>: To show the exactness of (1), we must prove exactness at A_n and exactness at $A_{n-1} \oplus \ldots \oplus A_{n-1}$. To show exactness at A_n, we need to show that ψ is a homomorphism onto. So let $t + \mathfrak{A}_n$ be in A_n, where $t \in T_n$. We want to find some $(t_1 + \mathfrak{A}_{n-1}) \oplus \ldots \oplus (t_d + \mathfrak{A}_{n-1}) \in A_{n-1} \oplus \ldots \oplus A_{n-1}$, where $t_1 \oplus \ldots \oplus t_d \in T_{n-1} \oplus \ldots \oplus T_{n-1}$, such that,

$$t + \mathfrak{A}_n = \psi((t_1 + \mathfrak{A}_{n-1}) \oplus \ldots \oplus (t_d + \mathfrak{A}_{n-1}))$$

$$= t_1x_1 \oplus \ldots \oplus t_dx_d + \mathfrak{A}_{n-1}$$

where $\Psi(t_1 \oplus \ldots \oplus t_d) = t_1x_1 \oplus \ldots \oplus t_dx_d$. But Ψ is onto by 2.2.2. Hence, ψ is onto and the sequence (1) is exact at A_n.

Now we need to show that the sequence (1) is exact at $A_{n-1} \oplus \ldots \oplus A_{n-1}$. That is, we need to show that $\text{Im } \phi = \ker \psi$.

(i) $\text{Im } \phi \subseteq \ker \psi$, that is $\phi\psi = 0$. So, if s_{n-n_1}, s_{n-n_2}, ..., s_{n-n_k}, ... are elements of T_{n-n_1}, T_{n-n_2}, ..., T_{n-n_k}, ... respectively, so

$$(s_{n-n_1} \oplus s_{n-n_2} \oplus \ldots \oplus s_{n-n_k} \oplus \ldots)\Phi\Psi = u_1x_1 + u_2x_2 + \ldots + u_dx_d.$$

where $\sum_{i=1}^{d} u_ix_i = \sum_{n_i \leq n} s_{n-n_i} f_i$; but the f_i's generate $\mathfrak{A}$, thus $\sum_{n_i \leq n} s_{n-n_i} f_i \in \mathfrak{A}$ and so $\sum_{i=1}^{d} u_ix_i \in \mathfrak{A}$. But $\sum_{i=1}^{d} u_ix_i \in T_n$ too.

So $\Phi\Psi$ maps $T_{n-n_1} \oplus T_{n-n_2} \oplus \ldots \oplus T_{n-n_k} \oplus \ldots$ into $\mathfrak{A}_n = T_n \cap \mathfrak{A}$ and so $A_{n-n_1} \oplus A_{n-n_2} \oplus \ldots \oplus A_{n-n_k} \oplus \ldots$ is mapped into 0 by $\phi\psi$, as follows:

Let $\bar{s}_{n-n_1} \oplus \ldots \oplus \bar{s}_{n-n_k} \oplus \ldots$ be an element of $A_{n-n_1} \oplus \ldots \oplus A_{n-n_k} \oplus \ldots$. Then:

$$\begin{aligned}
(\bar{s}_{n-n_1}\oplus\cdots\oplus\bar{s}_{n-n_k}\oplus\cdots)\phi\psi &= (s_{n-n_1}+\mathfrak{A}_{n-n_1}\oplus\cdots\oplus s_{n-n_k}+\mathfrak{A}_{n-n_k}\oplus\cdots)\phi\psi \\
&= \left(\Phi(s_{n-n_1}\oplus\cdots\oplus s_{n-n_k}\oplus\cdots)+\mathfrak{A}_{n-1}\oplus\cdots\oplus\mathfrak{A}_{n-1}\right)\psi \\
&= (u_1\oplus\cdots\oplus u_d+\mathfrak{A}_{n-1}\oplus\cdots\oplus\mathfrak{A}_{n-1})\psi \\
&= \Psi(u_1\oplus\cdots\oplus u_d) + \mathfrak{A}_n \\
&= (u_1x_1 + \ldots + u_dx_d) + \mathfrak{A}_n \\
&= 0 + \mathfrak{A}_n,
\end{aligned}$$

since $u_1x_1 + \ldots + u_dx_d \in \mathfrak{A}_n$. Hence $\operatorname{Im}\phi \subseteq \ker\psi$.

(ii) $\ker\psi \subseteq \operatorname{Im}\phi$. Here we want to show that if $\bar{t}_1\oplus\ldots\oplus\bar{t}_d \in \ker\psi$, then $\bar{t}_1\oplus\ldots\oplus\bar{t}_d \in \operatorname{Im}\phi$. That is we want to find some $\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_d$ in A_{n-1}, where $\bar{u}_1 \oplus \ldots \oplus \bar{u}_d \in \operatorname{Im}\phi$ and such that:

$$(\bar{t}_1 \oplus \ldots \oplus \bar{t}_d) - (\bar{u}_1 \oplus \ldots \oplus \bar{u}_d) = 0.$$

That is
$$\bar{t}_1 - \bar{u}_1 \oplus \ldots \oplus \bar{t}_d - \bar{u}_d = 0,$$

or
$$\bar{t}_i - \bar{u}_i = 0$$

$(i = 1, 2, \ldots, d)$ (by the direct sum), or

$t_i - u_i + \mathfrak{A}_{n-1} = 0$, or $t_i - u_i \in \mathfrak{A}_{n-1} = \mathfrak{A} \cap T_{n-1}$, or $t_i - u_i \in \mathfrak{A}$.

Also $\bar{u}_1 \oplus \dots \oplus \bar{u}_d \in \operatorname{Im} \phi$ implies that there exist

$\bar{s}_{n-n_1}, \bar{s}_{n-n_2}, \dots, \bar{s}_{n-n_k}, \dots$ in $A_{n-n_1}, A_{n-n_2}, \dots, A_{n-n_k}, \dots$

such that:

$$\begin{aligned}
\bar{u}_1 \oplus \dots \oplus \bar{u}_d &= \phi(\bar{s}_{n-n_1} \oplus \dots \oplus \bar{s}_{n-n_k} \oplus \dots) \\
&= \phi((s_{n-n_1} + \mathfrak{A}_{n-n_1}) \oplus \dots \oplus (s_{n-n_k} + \mathfrak{A}_{n-n_k}) \oplus \dots) \\
&= u_1 \oplus \dots \oplus u_d + \mathfrak{A}_{n-1} \oplus \dots \oplus \mathfrak{A}_{n-1}, \\
&= (u_1 + \mathfrak{A}_{n-1}) \oplus \dots \oplus (u_d + \mathfrak{A}_{n-1}),
\end{aligned}$$

where $\sum_{i=1}^{d} u_i x_i = \sum_{n_i \leq n} s_{n-n_i} f_i$, for some $s_{n-n_1}, \dots, s_{n-n_k}, \dots$ in $T_{n-n_1}, \dots, T_{n-n_k}, \dots$, and $u_i \in$ ith T_{n-1}.

Moreover, if $\bar{t}_1 \oplus \bar{t}_2 \oplus \dots \oplus \bar{t}_d \in \ker\psi$, means that

$$\psi(\bar{t}_1 \oplus \bar{t}_2 \oplus \dots \oplus \bar{t}_d) = 0$$

which implies that

$$\Psi(t_1 \oplus t_2 \oplus \dots \oplus t_d) \in \mathfrak{A}_n = T_n \cap \mathfrak{A}$$

Hence $$\Psi(t_1 \oplus t_2 \oplus \dots \oplus t_d) \in \mathfrak{A}.$$

<u>Conclusion</u>: So we need to show that if $\Psi(t_1 \oplus \dots \oplus t_d) \in \mathfrak{A}$, then we can find elements $u_1, u_2, \dots, u_d$ in T_{n-1} such that

$$t_i - u_i \in \mathfrak{A} \qquad \text{for } i = 1, 2, \dots, d$$

and such that $\Sigma u_i x_i = \sum_{n_i \leq n} s_{n-n_i} f_i$ for some s_{n-n_i} in the appropriate T_{n-n_i}.

Suppose then, that $\Psi(t_1 \oplus \dots \oplus t_d) = \sum_{i=1}^{d} t_i x_i \in \mathfrak{A}$. Since $\mathfrak{A}$ is a two-sided ideal generated by the f_j, we have that the elements in $\mathfrak{A}$ can be written in the following form and hence:

$$\sum_{i=1}^{d} t_i x_i = \Sigma a_{kq} f_q b_{kq} + \Sigma c_q f_q$$

where the a_{kq}, b_{kq}, c_q are homogeneous and where the degree of b_{kq} is at least 1. On comparing degree on both sides, we may even assume that the $a_{kq} f_q b_{kq}$, $c_q f_q$ are all in T_n. Since the b_{kq} are of degree at least 1,

$$b_{kq} = \sum_{m=1}^{d} d_{kqm} x_m$$

where d_{kq} is any homogeneous polynomial or constant. Then

$$\Sigma a_{kq} f_q b_{kq} = \sum_{k,q,m=1}^{d} a_{kq} f_q d_{kqm} x_m = \sum_{m=1}^{d} d_m x_m$$

where

$$d_m = \sum_{k,q} a_{kq} f_q d_{kqm}.$$

But since $f_q \varepsilon \mathfrak{A}$ we have that $d_m \varepsilon \mathfrak{A}$. If we write

$$\Sigma c_q f_q = \sum_{i=1}^{d} u_i x_i$$

we then have that

$$\sum_{i=1}^{d} t_i x_i = \sum_{i=1}^{d} d_i x_i + \sum_{i=1}^{d} u_i x_i$$

implies

$$t_i = d_i + u_i \qquad i = 1, 2, \ldots, d$$

hence

$$t_i - u_i = d_i \varepsilon \mathfrak{A} .$$

But $\Phi(c_1 \oplus \ldots \oplus c_k \oplus \ldots) = u_1 \oplus \ldots \oplus u_d$ by the definition of Φ; hence we have proved (ii).

The two inclusions (i) and (ii) give us the desired result, and hence we have proved exactness of (1) at $A_{n-1} \oplus \cdots \oplus A_{n-1}$. This proves proposition 2.2.5, and hence also Theorem 2.2.1(ii).

<u>Definition 2.2.6</u> The power series

$$P_A(t) = \sum_{n=0}^{\infty} b_n t^n$$

is called the Poincare function of the algebra A.

The following two theorems and corollary 2.3.1 are reworked from the original paper by Golod and Šafarevič.

<u>Theorem 2.2.7</u>

$$P_A(t)(1 - dt + \sum_{i=2}^{\infty} r_i t^i) \geq 1,$$

where inequality between power series is understood coefficient-wise.

<u>Proof</u>: Recall that

$$A = A_0 \oplus A_1 \oplus \cdots \oplus A_n \oplus \cdots \qquad (1)$$

and that the numbers $b_n = \dim A_n$, $n \geq 0$ are all finite. For the dimensions of the subspaces of A we obtained the inequality: (Theorem 2,2.1 (ii))

$$b_n \geq db_{n-1} - \sum_{n_i \leq n} b_{n-n_i} \qquad (n \geq 1) \qquad (2)$$

Multiplying this inequality by t^n and adding up for all $n \geq 1$, we obtain an inequality for the series:

$$\sum_{n=1}^{\infty} b_n t^n \geq \sum_{n=1}^{\infty} db_{n-1} t^n - \sum_{n=1}^{\infty} \sum_{n_i \leq n} t^n b_{n-n_i} \qquad (3)$$

If we set in the last sum $n - n_i = m$, and from the definition of r_i, we see that:

$$\sum_{n=1}^{\infty}\Big(\sum_{n_i\le n} t^n b_{n-n_i}\Big) = \sum_{n_i}\Big(\sum_{n=n_i}^{\infty} t^n b_{n-n_i}\Big) = \sum_{n_i} t^{n_i}\Big(\sum_{n=n_i}^{\infty} t^{n-n_i} b_{n-n_i}\Big)$$

$$= \sum_{n_i} t^{n_i}\Big(\sum_{m=0}^{\infty} t^m b_m\Big) = \sum_{n_i} t^{n_i} P_A(t)$$

$$= \Big(\sum_{n_i} t^{n_i}\Big) P_A(t) = \Big(\sum_{i=2}^{\infty} r_i t^i\Big) P_A(t) \tag{4}$$

On the other hand

$$\sum_{n=1}^{\infty} b_n t^n = \sum_{n=0}^{\infty} b_n t^n - 1 = P_A(t) - 1 \tag{5}$$

since $b_0 = 1$, and

$$\sum_{n=1}^{\infty} d b_{n-1} t^n = \sum_{n=1}^{\infty} d b_{n-1} t t^{n-1} = dt\, P_A(t) \tag{6}$$

Therefore, the inequality (3) yields:

$$P_A(t) - 1 \geq dt\, P_A(t) - \Big(\sum_{i=2}^{\infty} r_i t^i\Big) P_A(t), \tag{7}$$

hence

$$P_A(t)\Big(1 - dt + \sum_{i=2}^{\infty} r_i t^i\Big) \geq 1 \tag{8}$$

This proves theorem 2.2.7.

Theorem 2.2.8 (Golod and Šafarevič) *If the coefficients of the power series*

$$\Big(1 - dt + \sum_{i=2}^{\infty} r_i t^i\Big)^{-1}$$

are non-negative, then

$$P_A(t) \geq \Big(1 - dt + \sum_{i=2}^{\infty} r_i t^i\Big)^{-1} \tag{9}$$

and the algebra A *is infinite-dimensional.*

Proof: The inequality (9) is obtained from (8) by multiplying both sides by the power series

$$F(t) = \Big(1 - dt + \sum_{i=2}^{\infty} r_i t^i\Big)^{-1}, \tag{10}$$

which by assumption has non-negative coefficients. It remains to show that the algebra A is infinite-dimensional. For this purpose, it is sufficient to show that $b_n > 0$ for an infinite number of values of n, and this follows from (10) if we can show that the power series F(t) is not a polynomial in t. We set

$$1 + \sum_{i=2}^{\infty} r_i t^i = U(t) \qquad (11)$$

Then

$$F(t)(U(t) - dt) = 1$$

i.e.

$$F(t)U(t) = 1 + dtF(t) \qquad (12)$$

Since both F(t) and U(t) have non-negative coefficients, and U(t) is not a polynomial, then clearly F(t)U(t) is not a polynomial. Hence the left hand side of (12) is not a polynomial. Hence the right hand side of (12) is not a polynomial. Hence F(t) is not a polynomial.

2.3 Conditions on r_i

Corollary 2.3.1 If the numbers r_i satisfy the inequalities $r_i \le s_i$, and all the coefficients of the power series:

$$(1 - dt + \sum_{i=2}^{\infty} s_i t^i)^{-1}$$

are non-negative, then A is infinite dimensional.

Proof: Let

$$F = 1 - dt + \sum_{n=2}^{\infty} r_n t^n,$$

$$G = 1 - dt + \sum_{n=2}^{\infty} s_n t^n,$$

$$U = G - F = \sum_{n=2}^{\infty} (s_n - r_n) t^n.$$

We have then:

$$F = G - U = G\,(1 - UG^{-1}),\ \text{and}\ G^{-1} \geq 0,\ U \geq 0,\ \text{from which we find:}$$

$$F^{-1} = G^{-1}(1 - UG^{-1})^{-1}.$$

Now since $U \geq 0$ and $G^{-1} \geq 0$, we have $UG^{-1} \geq 0$, which implies $-UG^{-1} \leq 0$, which implies $1 - UG^{-1} \leq 1$, which implies

$$(1 - UG^{-1})^{-1} \geq 1,$$

(for if $$1 - UG^{-1} = 1 - a_1 t - a_2 t^2 - \dots$$

and $$(1 - UG^{-1})^{-1} = 1 + b_1 t + b_2 t^2 + \dots$$

then $$(1 - a_1 t - a_2 t^2 \dots)(1 + b_1 t + b_2 t^2 + \dots) = 1;$$

computing, we get

$$1 = 1$$

$$-a_1 + b_1 = 0 \Rightarrow b_1 = a_1 \geq 0$$

$$b_2 - a_1 b_1 - a_2 = 0 \Rightarrow b_2 = a_1 b_1 + a_2 \geq 0$$

$$\vdots$$

$$b_n - a_1 b_{n-1} - a_2 b_{n-2} - \dots - a_n = 0 \Rightarrow b_n = a_1 b_{n-1} + a_2 b_{n-2} + \dots + a_n \geq 0\ ).$$

Hence:

$$F^{-1} = G^{-1}(1 - UG^{-1})^{-1} \geq 0$$

But $F^{-1} = (1 - dt + \sum_{i=2}^{\infty} r_i t^i)^{-1}$. Hence, by Theorem 2.2.8, A is infinite-dimensional.

<u>Corollary 2.3.2</u> If for each $i = 2, 3, \dots, r_i \leq \left(\frac{d-1}{2}\right)^2$, then the algebra A is infinite-dimensional.

<u>Proof</u>: Since $r_i \leq \left(\frac{d-1}{2}\right)^2$, we need to examine the coefficients of

$$\left(1 - dt + \sum_{i=2}^{\infty}\left(\frac{d-1}{2}\right)^2 t^i\right)^{-1}$$

and apply Corollary 2.3.1. So we have

$$1 - dt + \sum_{i=2}^{\infty}\left(\frac{d-1}{2}\right)^2 t^i = 1 - dt + \left(\frac{d-1}{2}\right)^2(-1-t+1+t+t^2+t^3+\ldots)$$

But

$$-(1+t)+1+t+t^2+t^3+\ldots = -(1+t) + \frac{1}{1-t} = \frac{-(1+t)(1-t)+1}{1-t}$$

$$= \frac{-1+t^2+1}{1-t} = \frac{t^2}{1-t}$$

To continue the above we have

$$1 - dt + \sum_{i=2}^{\infty}\left(\frac{d-1}{2}\right)^2 t^i = 1 - dt + \left(\frac{d^2-2d+1}{4}\right)\left(\frac{t^2}{1-t}\right)$$

$$= \frac{(1-dt)(4-4t) + (d^2-2d+1)t^2}{4(1-t)}$$

$$= \frac{4-4dt-4t+4dt^2+d^2t^2-2dt^2+t^2}{4(1-t)}$$

$$= \frac{4-4(d+1)t + (d+1)^2t^2}{4(1-t)}$$

$$= \frac{(2-(d+1)t)^2}{4(1-t)}$$

Taking the inverse of the above, we have

$$\left(1 - dt + \Sigma\left(\frac{d+1}{2}\right)^2 t^i\right)^{-1} = \frac{4(1-t)}{(2-(d+1)t)^2} = \frac{(1-t)}{\left(1 - \frac{d+1}{2}t\right)^2}$$

$$= (1-t)\left(\sum_{n=1}^{\infty} n\left(\frac{d+1}{2}t\right)^{n-1}\right)$$

$$= (1-t)\left(1 + \sum_{n=1}^{\infty} (n+1)\left(\frac{d+1}{2}t\right)^n\right)$$

$$= 1 + \sum_{n=1}^{\infty}(n+1)\left(\frac{d+1}{2}\right)^{n} t^{n} - t - \sum_{n=1}^{\infty}(n+1)\left(\frac{d+1}{2}\right)^{n} t^{n+1}$$

$$= 1 + \sum_{n=1}^{\infty}(n+1)\left(\frac{d+1}{2}\right)^{n} t^{n} - \sum_{n=1}^{\infty} n\left(\frac{d+1}{2}\right)^{n-1} t^{n}$$

$$= 1 + \sum_{n=1}^{\infty}\left((n+1)\left(\frac{d+1}{2}\right)^{n} - n\left(\frac{d+1}{2}\right)^{n-1}\right)t^{n}$$

$$= 1 + \sum_{n=1}^{\infty}\left(\left(\frac{d+1}{2}\right)^{n-1}\left(\frac{(n+1)(d+1)-2n}{2}\right)\right)t^{n}$$

$$= 1 + \sum_{n=1}^{\infty}\left(\left(\frac{d+1}{2}\right)^{n-1}\left(\frac{(n+1)d-(n-1)}{2}\right)\right)t^{n} \qquad (*)$$

Now since $d \geq 1$, we have that $\frac{d+1}{2} \geq 1$, and also $(n+1)d-(n-1) \geq 2$. So (*) has non-negative coefficients. Hence, by Corollary 2.3.1, A is infinite dimensional.

An even stronger condition on r_i is the following due to Golod.

Corollary 2.3.3 Let r_i and A be as previously defined. If

$$r_i \leq \varepsilon^2 (d-2\varepsilon)^{i-2}$$

where ε is any positive number such that $d-2\varepsilon > 0$, then A is infinite dimensional.

Proof: It is sufficient to examine the coefficients of

$$\left(1 - dt + \sum_{i=2}^{\infty}\varepsilon^2 (d-2\varepsilon)^{i-2} t^{i}\right)^{-1} \qquad (1)$$

We have that

$$1 - dt + \sum_{i=2}^{\infty} \varepsilon^2 (d-2\varepsilon)^{i-2} t^{i} = 1 - dt + \varepsilon^2 t^2 \left(1 + (d-2\varepsilon)t + (d-2\varepsilon)^2 t^2 + \ldots\right)$$

$$= 1 - dt + \varepsilon^2 t^2 \left(\frac{1}{1-(d-2\varepsilon)t}\right)$$

$$= \frac{(1-dt)(1 - dt + 2\varepsilon t) + \varepsilon^2 t^2}{1 - (d-2\varepsilon)t}$$

$$= \frac{1 - 2dt + d^2t^2 + 2\varepsilon t - 2d\varepsilon t^2 + \varepsilon^2 t^2}{1 - (d-2\varepsilon)t}$$

$$= \frac{1 - 2(d-\varepsilon)t + (d-\varepsilon)^2 t^2}{1 - (d-2\varepsilon)t}$$

$$= \frac{(1 - (d-\varepsilon)t)^2}{1 - (d-2\varepsilon)t} \qquad (2)$$

Taking the inverse of (2), we have (1) which is equal to

$$\frac{1-(d-2\varepsilon)t}{(1-(d-\varepsilon)t)^2} = \left(1 - (d-2\varepsilon)t\right)\frac{1}{(1-(d-\varepsilon)t)^2} = \left(1 - (d-2\varepsilon)t\right)\left(\sum_{n=1}^{\infty} n(d-\varepsilon)^{n-1}t^{n-1}\right)$$

$$= \left(1 - (d-2\varepsilon)t\right)\left(1 + \sum_{n=1}^{\infty}(n+1)(d-\varepsilon)^n t^n\right)$$

$$= 1 + \sum_{n=1}^{\infty}(n+1)(d-\varepsilon)^n t^n - (d-2\varepsilon)t - (d-2\varepsilon)t\sum_{n=1}^{\infty}(n+1)(d-\varepsilon)^n t^n$$

$$= 1 + \sum_{n=1}^{\infty}(n+1)(d-\varepsilon)^n t^n - (d-2\varepsilon)\left(t + \sum_{n=1}^{\infty}(n+1)(d-\varepsilon)^n t^{n+1}\right)$$

$$= 1 + \sum_{n=1}^{\infty}(n+1)(d-\varepsilon)^n t^n - (d-2\varepsilon)\sum_{n=1}^{\infty} n(d-\varepsilon)^{n-1}t^n$$

$$= 1 + \sum_{n=1}^{\infty}(d-\varepsilon)^{n-1}\left((n+1)(d-\varepsilon) - (d-2\varepsilon)n\right)t^n$$

$$= 1 + \sum_{n=1}^{\infty}(d-\varepsilon)^{n-1}(nd + d - n\varepsilon - \varepsilon - nd + 2n\varepsilon)t^n$$

$$= 1 + \sum_{n=1}^{\infty}(d-\varepsilon)^{n-1}\left(d + (n-1)\varepsilon\right)t^n$$

Since $d-2\varepsilon>0 \Rightarrow \frac{d}{2} > \varepsilon \Rightarrow d-\varepsilon > \varepsilon > 0$.

Hence all the coefficients of (1) are nonnegative and, by corollary 2.3.1, A is infinite dimensional.

<u>Corollary 2.3.4</u> Let $d = 2$ and $r_i = 0$ for $i = 2, 3, \ldots, 9$ and $r_i = 0$ or 1 for $i \geq 10$. Then A is infinite dimensional.

Proof: Here corollary 2.3.2. does not apply for

(1) $r_i = 0 \le \left(\frac{2-1}{2}\right)^2 = \frac{1}{4}$ but (2) $r_i = 1 > \left(\frac{2-1}{2}\right)^2 = \frac{1}{4}$

So we use corollary 2.3.3 and we choose $\varepsilon = \frac{1}{4}$. Then

$$d - 2\varepsilon > 0 \qquad \text{i.e.} \quad 2 - \frac{1}{2} > 0$$

Clearly for $i = 2, 3, \ldots, 9$ $r_i \le \frac{1}{16}\left(2 - \frac{2}{4}\right)^i$. Now suppose that $i \ge 10$. Then

$$r_i \le 1 \le \frac{\left(2 - \frac{1}{2}\right)^8}{16} \le \frac{\left(2 - \frac{1}{2}\right)^{i-2}}{16}$$

Expanding $\left(2 - \frac{1}{2}\right)^8$ using the binomial theorem, we find that the first four terms add to 19, so $\left(2 - \frac{1}{2}\right)^8 > 16$. Hence $\varepsilon^2(d-2\varepsilon)^8 > 1$. Since

$$(d - 2\varepsilon)^i < (d - 2\varepsilon)^{i+1}$$

if $(d - 2\varepsilon) > 1$, this is sufficient to prove corollary 2.3.4.

Corollary 2.3.6. below is re-worked from a paper due to Newman.[19]

Lemma 2.3.5 The following two conditions are equivalent

(i) There exists $0<\varepsilon<d/2$ such that

$$r_i \le \varepsilon^2(d - 2\varepsilon)^{i-2} \qquad \text{for } i = 2, 3, \ldots .$$

(ii) There exists $0 < k < d$ such that

$$r_i \le \left(\frac{d-k}{2}\right)^2 k^{i-2} \qquad \text{for } i = 2, 3, \ldots .$$

Proof: Set $d - k = 2\varepsilon$. Then $0<\varepsilon<d/2$ if and only if $0<k<d$, and

$$\varepsilon^2(d-2\varepsilon)^{i-2} = \left(\frac{d-k}{2}\right)^2 k^{i-2}$$

Corollary 2.3.6 There is a positive integer N such that, if $r_i=0$ for $i<N$ and $r_i \leq (d-1)^i$ for $i \geq N$, then A is infinite dimensional.

Proof: Let N be an integer satisfying $N \geq 4d$ and $\left(1+\frac{1}{2d}\right)^{N-2} \geq N^2$. Put $k = \frac{(N-2)d}{N}$, then for $i \geq 2$, by Lemma 2.3.5 (ii), we have:

$$\left(\frac{d-k}{2}\right)^2 k^{i-2} = \left(\frac{d}{2} - \frac{(N-2)d}{2N}\right)^2 \left(\frac{(N-2)d}{N}\right)^{i-2}$$

$$= \left(\frac{dN-Nd+2d}{2N}\right)^2 \left(\frac{(N-2)d}{N}\right)^{i-2}$$

$$= \frac{d^2}{N^2}\left(\frac{N-2}{N}\right)^{i-2} d^{i-2} \tag{1}$$

Since $\left(1+\frac{1}{2d}\right)^{N-2} \geq N^2$, it follows that $\frac{1}{N^2} \geq \frac{1}{\left(1+\frac{1}{2d}\right)^{N-2}}$ (2)

Note that since $N \geq 4d$ implies $\frac{2}{N} \leq \frac{2}{4d}$. Hence

$$\left(\frac{N-2}{N}\right)^{i-2} = \left(1-\frac{2}{N}\right)^{i-2} \geq \left(1-\frac{2}{4d}\right)^{i-2} \tag{3}$$

Substituting (2) and (3) in (1), we have that (1)

$$\geq \frac{1}{\left(1+\frac{1}{2d}\right)^{N-2}}\left(1-\frac{2}{4d}\right)^{i-2} d^i \tag{4}$$

$$\geq \left(1-\frac{1}{d}\right)^{i-2} d^i \tag{5}$$

We have (5) because when $i \geq N$, then

$$\frac{\left(1-\frac{1}{2d}\right)^{i-2}}{\left(1+\frac{1}{2d}\right)^{N-2}} \geq \frac{\left(1-\frac{1}{2d}\right)^{i-2}}{\left(1+\frac{1}{2d}\right)^{i-2}} \geq \left(1-\frac{1}{d}\right)^{i-2}$$

provided that

$$\frac{1-\frac{1}{2d}}{1+\frac{1}{2d}} \geq 1-\frac{1}{d}$$

So

$$1 - \frac{1}{2d} \geq (1 - \frac{1}{d})(1 + \frac{1}{2d}) = 1 - \frac{1}{d} + \frac{1}{2d} - \frac{1}{2d^2} = 1 - \frac{1}{2d} - \frac{1}{2d^2}$$

That is

$$0 \geq -\frac{1}{2d^2}$$

Now since $d^2 \geq (d-1)^2$ we have that

$$(1 - \frac{1}{d})^{i-2} d^i = \left(\frac{d-1}{d}\right)^{i-2} d^i = (d-1)^{i-2} d^2$$

$$\geq (d-1)^{i-2}(d-1)^2 = (d-1)^i \geq r_i$$

Hence A is infinite dimensional.

CHAPTER 3.

In this chapter we will construct some examples of nil algebras and periodic groups. Before this, however, we will state clearly the Kuroš problem and the Burnside question adding all the definitions necessary to understand them.

3.1 Algebraic and Nil Algebras

Definition 3.1.1 An algebra, A, is finitely-generated if there is a finite subset $a_1, \ldots, a_r$ (called its generators) such that every element of A can be obtained from the generators by a finite number of additions, multiplications, and/or scalar multiplications.

Definition 3.1.2 Let A be an algebra over a field F; $a \varepsilon A$ is said to be algebraic over F if there is a non-zero polynomial $p(x) \varepsilon F[x]$ such that $p(a) = 0$. That is

$$(1) \qquad p(a) = k_n a^n + k_{n-1} a^{n-1} + \ldots + k_1 a + k_0 = 0$$

where $k_i \varepsilon F$. The equation (1) may differ for different $a \varepsilon A$.

Definition 3.1.3 An algebra A over F is said to be algebraic over F if every $a \varepsilon A$ is algebraic over F.

The following theorem is a very interesting one and we will see soon that it gives us the converse of the Kuroš problem.

Theorem 3.1.4 If A is a finite-dimensional (as a vector space) algebra over F, then it is algebraic over F.

Proof: Let $a \in A$, and let $n = \dim A$. Then the $n+1$ elements $a, a^2, a^3, \ldots, a^n, a^{n+1}$, are linearly dependent over F. Thus there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_{n+1}$ in F such that they are not all zero and such that

$$\text{(i)} \qquad \alpha_1 a + \alpha_2 a^2 + \ldots + \alpha_n a^n + \alpha_{n+1} a^{n+1} = 0$$

Thus $p(a) = 0$, where $p(x)$ is the non-zero polynomial

$$\text{(ii)} \qquad p(x) = \alpha_1 x + \alpha_2 x^2 + \ldots + \alpha_{n+1} x^{n+1} \qquad \text{in } F[x]$$

Hence a is algebraic over F. But since a was any element of A, we can conclude that every element of A is algebraic and therefore A is algebraic over F.

Definition 3.1.5 Let A be an algebra over F; $a \in A$ is said to be nilpotent if there exists a positive integer n such that $a^n = 0$.

Definition 3.1.6 Let A be an algebra over F such that $A^n = (0)$ for some positive integer n; then A is said to be a nilpotent algebra over F.

Definition 3.1.7 An algebra A over F is nil if every element of A is nilpotent.

Theorem 3.1.8 If A is a nil algebra over F, then A is algebraic over F.

Proof: Since A is nil, this implies that for $a \in A$, there exists a positive integer n such that $a^n = 0$. Clearly a is algebraic over F since it satisfies the following polynomial.

$$1x^n + 0x^{n-1} + \dots + 0x + 0$$

i.e. $$1a^n + 0a^{n-1} + \dots + 0a + 0 = 0$$

Hence A is algebraic over F.

3.2 Kuroš's Problem

First we will define the locally finite algebras. Then we will discuss the Kuroš Problem.

Definition 3.2.1. An algebra A over a field F is locally finite if every finite subset of A generates a finite dimensional subalgebra.

We have seen that any finite dimensional algebra is algebraic (Theorem 3.1.4), hence any locally finite algebra is algebraic. Now the following question (an analog to the Burnside's Problem on groups), was raised by Kuroš in 1941.

Problem 3.2.2 Is every algebraic algebra locally finite?

In other words, if A is an algebraic algebra over F, does a finite number of elements of A generate a finite dimensional subalgebra of A? Or, is a finitely generated algebraic algebra finite dimensional?

As Jacobson says, "A number of interesting open questions on algebraic algebras seem to hinge on the answer to this problem." Some of these are the following:

Question 3.2.3 If A and B are algebraic, then is $A \otimes B$ algebraic?

It is easy to see that if A and B are locally finite, then $A \otimes B$ is locally finite. Hence an affirmative answer to Kuroš's problem would provide an affirmative answer to 3.2.3.

In the coming sections, we shall give some examples of infinite dimensional algebras.

Also, we like to mention that Kuroš's question has an affirmative answer for algebras with a polynomial identity (PI - algebras) and hence for algebras of bounded degree. The results are due to Kaplansky which generalize earlier results by Jacobson and by Malcev.

3.3 PI - Algebras and Bounded Algebras

Definition 3.3.1 An algebra A over a field F is said to satisfy a polynomial identity if there is an $f \neq 0$ in $F[x_1, \ldots, x_d]$, the free algebra over F in the noncommuting variables $x_1, x_2, \ldots, x_d$ for some d, such that $f(a_1, \ldots, a_d) = 0$ for all $a_1, \ldots, a_d$ in A. An algebra A which satisfies a polynomial identity is called a PI - algebra.

Example 3.3.2 Let A be a nil algebra of bounded index of nilpotency. That is, $x^k = 0$ holds for every x for some fixed k. Then A is a PI - algebra.

Example 3.3.3 Any commutative algebra A over F is a PI - algebra, for it satisfies the polynomial identity $f(x_1,x_2) = 0$, where $f(x_1,x_2) = x_1x_2 - x_2x_1$.

We mention the following results to give an idea of what was known regarding the Kuroš problem prior to the work of Golod and Šafarevič. If A is finite dimensional over F, of dimension n, then every element in A satisfies a polynomial of degree n+1 over F. This defines the notion of an algebraic algebra of bounded degree over F.

<u>Definition 3.3.4</u> A is said to be an <u>algebraic algebra of bounded degree</u> over F if there exists an integer n such that given $a \in A$, there exists a polynomial $x^n + \alpha_1 x^{n-1} + \ldots + \alpha_n \in F[x]$ satisfied by a.

i.e. $a^n + \alpha_1 a^{n-1} + \ldots + \alpha_n = 0$.

<u>Lemma 3.3.5</u> If A is algebraic of bounded degree over F, then A is a PI algebra. [11]

<u>Theorem 3.3.6</u> If A is an algebraic algebra over F satisfying a polynomial identity, then A is locally finite. [11]

<u>Theorem 6.4.4</u> If A is an algebraic algebra of bounded degree over F, then it is locally finite. [11]

3.4 Periodic Groups and Locally Finite Groups

<u>Definition 3.4.1</u> A group G is said to be a <u>periodic</u> or <u>torsion</u> group if every element in G is of finite order.

<u>Definition 3.4.2</u> The <u>order of an element</u> b is the smallest positive integer n such that $b^n = 1$, if it exists. If there is such an n, we say that b has finite order.

Definition 3.4.3 If $b^n = 1$, with n fixed, for all $b \in G$, and n is the smallest positive integer for which this is true, then n is called the exponent of G.

Definition 3.4.4 A group G is said to be locally finite if every finitely generated subgroup of G is finite.

Definition 3.4.5 G is a finitely generated group if G contains a finite set of elements $g_1, g_2, \ldots, g_r$ (called its generators) such that every element can be expressed as a finite product of the generators and their inverses.

Theorem 3.4.5 Every locally finite group is a torsion group.

Proof: Let G be a locally finite group. We want to show that every element of G has finite order. That is, the subgroup generated by that element is finite. But the subgroup generated by a given element is certainly finitely generated, hence is finite, which implies that the given element has finite order.

Hence G is a torsion group.

Example 3.4.6 The group Z+ of integers is not a torsion group since a single element does not have finite order. Hence Z+ is not locally finite.

Example 3.4.7 This is an example of an infinite group which is locally finite. Take an infinite dimensional vector space V over the field of integers module p, Zp. Then V is an abelian group. Now take any finite subset of V, $a_1, a_2, \ldots, a_n$, then the subgroup generated by this subset is just the set of all $\sum_{i=1}^{n} \xi_i a_i$, where $\xi_i \in Zp$. There are only finitely many choices of each ξ_i. Hence, only a finite number of elements of the subgroup generated by the a_i.

Hence that finitely generated subgroup is finite. Hence the group is locally finite.

3.5 Burnside Problem

The converse to Theorem 3.4.5 is the Burnside Problem which originally was asked in 1904. We state two versions of the Burnside Problem.

1. Original Burnside Problem. Is every torsion group locally finite? An equivalent version of this question is: Is a finitely generated periodic group finite?

2. Burnside Problem for Exponent N. Let G be a torsion group in which $x^N = 1$ for all $x \in G$, N a fixed positive integer. Is G then locally finite?

These problems have answers now and they are as follows:

1. As a result of the work of Golod and Šafarevič, the original Burnside problem is answered in the negative. In the following section, we will exhibit a finitely generated periodic group which is infinite.

However, for matrix groups, Burnside himself settled the original Burnside Problem in the affirmative, by the following:

Theorem 3.5.1 (Burnside) A torsion group of matrices over a field is locally finite.

2. Novikov in 1959, announced the existance of an infinite group G_N generated by two elements in which $x^N = 1$ holds for all $x \in G$. This is true for any odd $N \geq 4381$. The proof done by induction appeared in 1968 in paper nearly 300 pages long, which gives an actual construction.

Regardless of the answer to the Burnside Problem, for exponent N, the following problem is still an interesting one.

Restricted Burnside Problem for Exponent N: Among all the finite groups on K generators with exponent N, is there a largest one?

The answer is "Yes", if N is prime, done by Kostrikin.

If N is prime and ≥ 4381, we have two results:

(a) There is a largest finite group of exponent N in two generators (Kostrikin).

(b) There is an infinite group of exponent N in two generators (Novikov and Adyan).

3.6 Settling the Kuroš Problem and the Original Burnside Problem in Negative

In this Section, we are ready to apply the Golod-Šafarevič theorem to construct a finitely generated nil algebra which is infinite-dimensional and a finitely generated infinite periodic group.

This settles the Kuroš and Burnside problems negatively.

<u>Theorem 3.6.1</u> If F is any countable field, there exists an infinite dimensional nil algebra over F generated by two elements.

<u>Proof:</u> Let $T = F[x_1, x_2]$. Then

$$T = F \oplus T_1 \oplus \dots \oplus T_n \oplus \dots$$

where the elements of T_i are homogeneous of degree i. Let

$$T' = T_1 \oplus T_2 \oplus \dots \oplus T_n \oplus \dots$$

T' is an ideal, since if $u \in T'$ and $r \in T$, then $ru \in T'$ and $ur \in T'$, because $\partial(ru)$ and $\partial(ur)$ are always ≥ 1 since $\partial(u) \geq 1$. Also, T' is a vector space with a countable basis since the basis of each T_i is finite. Hence, by Lemma 3.6.2, T' is countable. Now let

$$s_1, s_2, \dots, s_n, \dots$$

be the elements of T'. Pick $m_1 = 10$ and raise s_1 to the m_1 power so

$$s_1^{m_1} = s_{11} \oplus s_{12} \oplus \dots \oplus s_{1,k_1}$$

$s_{1j} \in T_{9+j}$, $1 \leq j \leq k_1 - 9$ and $s_1^{m_1} \in T_{10} \oplus T_{11} \oplus \dots \oplus T_{k_1}$.

Choose $m_2 > 0$ so that

$$s_2^{m_2} = s_{2,k_1+1} \oplus s_{2,k_1+2} \oplus \dots \oplus s_{2,k_2}$$

$s_{2,k_1+j} \in T_{k_1+j}$, $1 \leq j \leq k_2 - k_1$ and $s_2^{m_2} \in T_{k_1+1} \oplus T_{k_1+2} \oplus \dots \oplus T_{k_2}$.

Having chosen $m_1, \dots, m_{n-1}$, with corresponding $k_1 < k_2 < \dots < k_{n-1}$, choose $m_n > 0$ so that

$$s_n^{m_n} = s_{n,k_{n-1}+1} \oplus s_{n,k_{n-1}+2} \oplus \dots \oplus s_{n,k_n}$$

$s_{n,k_{n-1}+j} \in T_{k_{n-1}+j}$, $1 \leq j \leq k_n - k_{n-1}$ and $s_n^{m_n} \in T_{k_{n-1}+1} \oplus T_{k_{n-1}+2} \oplus \dots \oplus T_{k_n}$.

Clearly $k_1 < k_2 < \ldots < k_n < \ldots$.

Now let $\mathfrak{A}$ be the ideal of T generated by all the s_{ij}. Notice that for that choice of the s_{ij}'s, we have $r_k = 0$, $2 \leq k \leq 9$. and $r_k = 0$ or 1 for $k \geq 10$, by construction. Hence, by corollary 2.3.4, we have that $T/\mathfrak{A}$ is infinite dimensional. Now since $\mathfrak{A} \subseteq T'$, we form the quotient algebra $T'/\mathfrak{A}$ which is obviously infinite dimensional. But $T'/\mathfrak{A}$ is a nil algebra by construction, for if $\bar{s}_i \varepsilon T'/\mathfrak{A}$ then $\bar{s}_i = s_i + \mathfrak{A}$, and $\bar{s}_i^{m_i} = (s_i + \mathfrak{A})^{m_i} = s_i^{m_i} + \mathfrak{A} = \mathfrak{A}$, hence, $\bar{s}_i^{m_i} = \bar{0}$. Hence the algebra $T'/\mathfrak{A}$ is the required finitely generated algebraic algebra (in fact, a nil algebra) which is infinite dimensional.

Lemma 3.6.2 Let V be a vector space with a countable basis over a countable field F. Then V is countable.

Proof: Let $B = \{v_1, v_2, \ldots, v_n, \ldots\}$ be a countable basis for V, and let $B_n = \{v_1, v_2, \ldots, v_n\}$ be a subset of B. Now let $\bar{B}_n$ be the subspace of V spanned by B_n. Then $\bar{B}_n$ is countable since there is a natural one to one correspondence between $\bar{B}_n$ and $\underbrace{F \times F \times \ldots \times F}_{n \text{ times}}$

But then
$$V = \bigcup_{n=1}^{\infty} \bar{B}_n$$
is the countable union of countable sets and hence countable.

Let F be a finite field with p elements and let $\mathfrak{A}$ be the ideal in $T = F[x_1, x_2]$ as in Theorem 3.6.1 and let $T' = T_1 \oplus T_2 \oplus \ldots \oplus T_n \oplus \ldots$ If $A = T/\mathfrak{A}$ then $a_1 = x_1 + \mathfrak{A}$ and $a_2 = x_2 + \mathfrak{A}$ is the generating set for $T'/\mathfrak{A}$.

<u>Definition 3.6.3</u> A group $G \neq \{1\}$ is a p-group if every element of G except the identity has order a power of the prime p.

<u>Lemma 3.6.4</u> Let G be the multiplicative semigroup in A generated by $1 + a_1$, $1 + a_2$. Then G is a group, and is in fact, a p-group.

<u>Proof</u>: Obviously G is the subset of A consisting of all finite power products of the elements $1 + a_1$, $1 + a_2$, (with non-negative exponents). Hence:

$$G \subset \{1 + a \mid \text{for some } a \in T'/\mathfrak{A}\}.$$

But the algebra $T'/\mathfrak{A}$ is a nil algebra (Theorem 3.6.1) and therefore, each $a \in T'/\mathfrak{A}$ is nilpotent, i.e. for some n we have $a^n = \bar{0}$. Now take n large enough that $p^n > n$. Then

$$a^{p^n} = a^n a^{p^n - n} = \bar{0} \qquad (a^n = \bar{0})$$

and

$$(1 + a)^{p^n} = 1 + p^n a + \tfrac{1}{2}(p^n - 1)p^n a + \dots + p^n a^{p^n - 1} + a^{p^n} = 1 + a^{p^n} = 1 .$$

This is because all the coefficients are 0, since they are divisible by p and F is the finite field with p elements. Hence G contains a multiplicative identity 1. Hence G is a semigroup with identity. Also, since powers of the same element commute, we have

$$1 = (1 + a)^{p^n} = (1 + a)(1 + a)^{p^n - 1} = (1 + a)^{p^n - 1}(1 + a);$$

that is $1 + a$ has a multiplicative inverse $(1 + a)^{p^n - 1}$, which is clearly in G.

Therefore, G is a group. Moreover, G is a p-group.

<u>Lemma 3.6.5</u> Let A be an algebra over a field F and let G be a finite subset of A which is a group under multiplication. Then the linear combinations of the elements of G form a finite dimensional subalgebra B over F.

<u>Proof:</u> Let $G = \{a_1, a_2, \ldots, a_n\}$ be a finite subset of A and moreover, let G be a multiplicative group. Then the elements of the subalgebra generated by G are of the form

$$\sum_{i=1}^{n} \xi_i a_i \qquad (\xi_i \varepsilon F)$$

The subalgebra looked at as a vector space is spanned by $a_1, \ldots, a_n$. Therefore, it has a finite basis and hence is finite-dimensional.

<u>Theorem 3.6.6</u> If p is any prime, there is an infinite group G generated by two elements in which every element has finite order a power of p.

<u>Proof</u>: Let G be the group in Lemma 3.6.4. Then G is a p-group, and it remains to show that G is infinite. Assume that G is finite. Since G is finite, the linear combinations of the elements of G form a finite dimensional algebra B over F, as in Lemma 3.6.5. Since $1, 1 + a_1, 1 + a_2$, are in G, then the elements

$$a_1 = (1 + a_1) - 1$$

$$a_2 = (1 + a_2) - 1$$

$$1 = (1 + a_1) - a_1 = (1 + a_2) - a_2$$

are in B. Observing that $1, a_1, a_2$ generate the algebra A, we get $A = B$, contradicting that A is infinite-dimensional over F. Therefore, B is infinite dimensional and hence G is infinite.

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