COUNTABLE THEORIES CATEGORICAL IN UNCOUNTABLE POWER

by

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§0 Introduction and Terminology

Most of this thesis is concerned with countable complete theories having only infinite models. A theory T is categorical in power λ if all models of T which have power λ are isomorphic. Łos focused attention on $\aleph_1$-categorical theories with his conjecture [5] that a countable theory was $\aleph_1$-categorical if and only if it was categorical in every uncountable power. Vaught suggested another property of $\aleph_1$-categorical theories. In [21], he conjectured that an $\aleph_1$-categorical theory was either $\aleph_0$-categorical or had exactly $\aleph_0$ isomorphism types of countable models. Morley [7] proved the Łos conjecture. In the course of the proof he attached an ordinal α_T to each countable complete theory T and conjectured that if T were $\aleph_1$-categorical then α_T would be finite. In his thesis [6] Marsh introduced the notions of "algebraic closure" and "strongly minimal set."

This thesis investigates the properties of $\aleph_1$-categorical theories. The principal tools of this investigation are the notion of strongly minimal set, Vaught's two cardinal theorem, and two of Morley's theorems. The theorems of Morley state that a theory categorical in any uncountable power is totally transcendental [7; 3.8] and that if X is a subset of the universe of a model of a totally transcendental theory there is a model of T prime over X.

Section 1 of this thesis exhibits the basic properties of algebraic closure and strongly minimal sets. The simplest sorts of $\aleph_1$-categorical theories, strongly minimal theories, are studied in Section 2. In Section 3 the notion of strongly minimal is combined with the idea of Vaught's two cardinal theorem [9] to reprove the Łos conjecture. A slightly more complicated sort of $\aleph_1$-categorical theories, those which are almost strongly minimal, is considered in Section 4. Vaught's conjecture is proved in Section 5. Morley's conjecture is proved in section 6. Section 7 deals with a problem in definability. Section 8 summarizes some related results.

Sections 1, 3 and 5 are from [1]. To the best of the author's knowledge those results in Sections 2, 4, 6 and 7 not explicitly credited elsewhere are new.

The notation used here combines that of Morley [7] and Shoenfield [20]. We deal with countable first order languages. For convenience, we will assume that each such language L contains only relation symbols and constants. L has variables $v_0, v_1, \ldots$. Following [20; p. 17] we let $A_{w_0, \ldots w_n}(a_0, \ldots, a_n)$ denote the formula obtained by replacing each occurrence of the variable w_i by the term a_i for $0 \le i \le n$. Whenever such an expression occurs we assume no variable has become bound by the substitution. We omit the subscripted variables $w_0, \ldots w_n$ if they are clear from context. For each natural number k we admit quantifiers $\exists!^k v$ and $\exists^{\le k} v$ which mean intuitively "there exist exactly k elements v" and "there exist at most k elements v" respectively. If c is a constant in L and $\mathfrak{A}$ is an L-structure then the value of c in $\mathfrak{A}$ is denoted as in [20] by $(c)_{\mathfrak{A}}$.

We may extend the language L in several ways. If $\mathfrak{A}$ is an L-structure there is a natural extension $L(\mathfrak{A})$ of L obtained by adjoining to L a constant a for each $a \varepsilon |\mathfrak{A}|$ (the universe of $\mathfrak{A}$). For each sentence $A(a_1, \ldots, a_n) \varepsilon L(A)$ we say $\mathfrak{A}$ satisfies $A(a_1, \ldots, a_n)$ and write $\mathfrak{A} \models A(a_1, \ldots, a_n)$ if in Shoenfield's notation $\mathfrak{A}(A(a_1, \ldots, a_n)) = T$ [20; p. 19]. If $\mathfrak{A}$ is an L-structure and X is a subset of $|\mathfrak{A}|$ then L(X) is the language obtained by adjoining to L a name x for each $x \varepsilon X$. $(\mathfrak{A}, X)$ is the natural expansion of $\mathfrak{A}$ to an L(X)-structure. A structure $\mathfrak{B}$ is an <u>inessential expansion</u> [20; p. 141] of an L-structure $\mathfrak{A}$ if $\mathfrak{B} = (\mathfrak{A}, X)$ for some $X \subseteq |\mathfrak{A}|$. We also extend a language L by adding additional relation symbols. If L' extends L and $\mathfrak{A}'$ is an L'-structure the <u>reduct</u>, denoted $\mathfrak{A}'|L$, <u>of</u> $\mathfrak{A}'$ <u>to</u> L is the L-structure obtained from $\mathfrak{A}'$ by omitting those relations and constants which occur in L' but not L. Shoenfield calls this concept "restriction" [20; p. 43].

$S_n(L)$ denotes the set of formulas of L with free variables among $v_0, \ldots, v_{n-1}$. If A is a formula such that $u_1, \ldots u_n$ in the natural order are the free variables in A, then $A(\mathfrak{A})$ is the set of n-tuples $b_1, \ldots, b_n$ such that $A_{u_1, \ldots, u_n}(b_1, \ldots b_n)$. If p is a unary predicate symbol we abbreviate $pv_0(\mathfrak{A})$ by $p(\mathfrak{A})$. If $\mathfrak{A}$ is an L-structure $Y \subseteq |\mathfrak{A}|$ and $X \subseteq |\mathfrak{A}|^k$ then X is said to be <u>definable in</u>

$(\mathfrak{A}, Y)$ if there is a formula A in $S_k(L(Y))$ such that $X = A(\mathfrak{A})$. X is said to be <u>definable</u> <u>in</u> $\mathfrak{A}$ if X is definable in $(\mathfrak{A}, |\mathfrak{A}|)$.

A consistent set of L-sentences is a <u>theory</u> in L. If T and T' are theories in L then T' extends T if $T \subseteq T'$. If T is a theory in a language L then T' is an <u>inessential extension</u> of T if there is a model $\mathfrak{A}$ of T and a subset X of $|\mathfrak{A}|$ such that $T' = Th(\mathfrak{A}, X)$ (i.e. the set of all sentences in L(X) true of $(\mathfrak{A}, X)$). T' is a <u>principal</u> <u>extension</u> of T if T' is an inessential extension of T by a finite number of constants and a set of nonlogical axioms for T' can be obtained by adjoining a finite set of sentences to a set of nonlogical axioms for T.

$\mathfrak{A} \leq \mathfrak{B}$ and $\mathfrak{A} \equiv \mathfrak{B}$ abbreviate "$\mathfrak{A}$ is an elementary substructure of $\mathfrak{B}$", "$\mathfrak{A}$ is elementarily equivalent to $\mathfrak{B}$" respectively [20; pp. 72-74]. Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are L-structures, $X \subseteq |\mathfrak{A}|$ and f is a 1 - 1 map of X into $|\mathfrak{B}|$. Let $\mathfrak{B}'$ be the L(X)-structure obtained by setting $(x)_{\mathfrak{B}'} = f(x)$ for each $x \in X$. Then f is an <u>elementary</u> <u>monomorphism</u> <u>(elementary embedding)</u> of X into $|\mathfrak{B}|$ if $(\mathfrak{A}, X) \equiv \mathfrak{B}'$. The cardinality of a set X is denoted by $\kappa(X)$; we abbreviate $\kappa(|\mathfrak{A}|)$ by $\kappa(\mathfrak{A})$.

Let Γ be a subset of $S_k(L)$. Then Γ is a <u>k-type</u> in T if there is some model $\mathfrak{A}$ of T and elements $a_1, \ldots a_k \in |\mathfrak{A}|$ such that $\mathfrak{A} \models A(a_1, \ldots a_k)$ if and only if $A \in \Gamma$. If $\mathfrak{A}$ is a model of T and $X \subseteq |\mathfrak{A}|$ then a k-type <u>Γ is</u> <u>realized</u> <u>in</u> X if there exist $x_1, \ldots x_k \in X$ such that $\mathfrak{A} \models A(x_1, \ldots x_k)$ for each $A \varepsilon \Gamma$. A k-type Γ is a principal k-type in T if there is a formula $A \varepsilon S_k(L(\mathfrak{A}))$ such that for each formula B in Γ $\mathfrak{A} \models \forall v_o, \ldots \forall v_{k-1} (A \rightarrow B)$. Since T is complete there is one 0-types, truth ~~and falsehood~~.

Following Morley [7] we assume that each $T = \Sigma^*$ for some Σ and thus that each n-ary formula Φ is equivalent in T to an n-ary relation A. $N(T)$ is set of all substructures of models of T. The following summarizes with slight changes in notation the second paragraph of §2 in [7]. If $\mathfrak{A}$ is an L-structure $\mathcal{D}(\mathfrak{A})$ is the set of all open sentences in $L(\mathfrak{A})$ which are true in $(\mathfrak{A}, |A|)$. If $\mathfrak{A} \varepsilon N(T)$, $T(\mathfrak{A}) = \mathcal{D}(\mathfrak{A}) \cup T$ is a complete theory in $L(\mathfrak{A})$. Let $S_k(\mathfrak{A})$ denote the Boolean algebra whose elements are the equivalence classes into which $S_k(L(\mathfrak{A}))$ is partitioned by the relation of equivalence in $T(\mathfrak{A})$, and whose operations of intersection, union, and complementation are those induced by conjunction, disjunction and negation respectively. The Stone space of $S_1(\mathfrak{A})$, the set of dual prime ideals of $S_1(\mathfrak{A})$, is a topological space denoted $S(\mathfrak{A})$. A dual prime ideal of $S_k(\mathfrak{A})$ is a k-type of $T(\mathfrak{A})$. This is a special case of the definition of k-type in the preceding paragraph. Note that if $p \varepsilon S(\mathfrak{A})$ and $\mathfrak{A}'$ is an inessential expansion of $\mathfrak{A}$ p is naturally a member of $S(\mathfrak{A}')$.

§1. Algebraic Closure and Strongly Minimal Sets

This section contains the definitions used, and the basic theorems proved, by Marsh [6]. For completeness, the results are reproved in a manner similar to that used by Marsh.

Let $\mathcal{A}$ be an L-structure and X a subset of $|\mathcal{A}|$. The *algebraic closure* of X, denoted by cl(X), is the union of all finite subsets of $|\mathcal{A}|$ definable in $(\mathcal{A}, X)$. This notion was explored by Park as "obligation" in [11]. X *spans* Y if $Y \subseteq cl(X)$. X is *independent* if for each $x \in X$, $x \in cl(X-\{x\})$. X is a *basis* for Y if X is an independent subset of Y which spans Y. If every basis for Y has the same cardinality μ, we define the *dimension* of Y to be μ and write $\dim(Y) = \mu$.

Let $\mathcal{A}$ be an L-structure. A subset X of $|\mathcal{A}|$ is *minimal in* $\mathcal{A}$ if X is infinite, definable in $\mathcal{A}$, and for any subset Y of $|\mathcal{A}|$ which is definable in $\mathcal{A}$ either $Y \cap X$ or $X - Y$ is finite.

If $D \in S_1(L(\mathcal{A}))$ and $X = D(\mathcal{A})$ then X is *strongly minimal in* $\mathcal{A}$ if for any elementary extension $\mathcal{B}$ of $\mathcal{A}$ $D(\mathcal{B})$ is minimal in $\mathcal{B}$. Let $\mathcal{A}_0$ and $\mathcal{A}_1$ be models of a complete theory T. Since up to isomorphism any two models of T have a common elementary extension, $D(\mathcal{A}_0)$ is strongly minimal in $\mathcal{A}_0$ if and only if $D(\mathcal{A}_1)$ is strongly minimal in $\mathcal{A}_1$. Thus, without ambiguity we define a formula $D \in S_1(L)$ to be *strongly minimal in T* if there is a model $\mathcal{A}$ of T such that $D(\mathcal{A})$ is strongly minimal in $\mathfrak{A}$.

We now exhibit some of Marsh's results. It is trivial that $X \subseteq cl(X)$ and that $cl(X) \subseteq cl(Y)$ if $X \subseteq Y$.

Lemma 1. $cl(cl(X)) = cl(X)$

Proof. Let $\mathfrak{A}$ be an L-structure and $X \subseteq |\mathfrak{A}|$. Suppose $X \varepsilon cl(cl(X))$. Then there exist elements $a_1, \ldots, a_n$ in $cl(X)$ and a formula $A \varepsilon S_{n+1}(L)$ such that for some k $\mathfrak{A} \models A(x, a_1, \ldots, a_n) \wedge \exists !^k v_0\ A(v_0, a_1, \ldots, a_n)$. But then for each i, $1 \leq i \leq n$, there exists a formula B_i in $S_1(L(X))$ and an integer k_i such that $\mathfrak{A} \models B_i(a_i) \wedge \exists !^{k_i} v_0\ B_i(v_0)$. Let A_1 be the formula $\exists v_1, \ldots, \exists v_n (\wedge^n_{i=1} B_i (v_i). \wedge. A. \wedge. \exists !^k v_0\ A)$. Then $A_1 \varepsilon S_1(L(X))$, $A_1(\mathfrak{A})$ is finite, and $\mathfrak{A}$ $A_1(x)$ so $x \varepsilon cl(X)$.

The next lemma asserts that within strongly minimal sets the "exchange principle" holds.

Lemma 2. Let D be in $S_1(L)$ and $\mathfrak{A}$ be an L-structure. Suppose $D(\mathfrak{A}) = X$ is strongly minimal in $\mathfrak{A}$. If $Y \subseteq X$, a and b are elements of X, $a \varepsilon cl(Y)$, $b \varepsilon cl(Y)$, and $b \varepsilon cl((Y \cup \{a\}) - cl(Y))$ then $a \varepsilon cl(Y \cup \{b\})$.

Proof. We suppose that Y is empty since if that case is proved the general case may be deduced by adjoining names for the members of Y to L. Since $b \varepsilon cl(\{a\})$ there exists an $A \varepsilon S_2(L)$ such that $\mathfrak{A} \models A(a, b) \wedge \exists !^k v_1\ A(a, v_1)$

for some positive integer k. If $(A(v_0,b) \wedge D)(\mathfrak{A})$ is finite the lemma is proved. If not, since $D(\mathfrak{A})$ is strongly minimal, for some positive integer m $(\sim A(v_0,b) \wedge D)(\mathfrak{A})$ has cardinality m. Write $C(v_1)$ for $D(v_1) \wedge \exists !^m v_0(\sim A(v_0,v_1) \wedge D(v_0))$. Then $\mathfrak{A} \models C(b)$ and $b \notin cl(\emptyset)$, so $C(\mathfrak{A})$ is infinite. Write $B(v_0)$ for $D(v_0) \wedge \exists !^k v_1 A(v_0,v_1)$. Since $\mathfrak{A}$ $B(a)$ and $a \in cl(\emptyset)$, $B(\mathfrak{A})$ is infinite. Let $b_0, \ldots, b_k$ be chosen from $C(\mathfrak{A})$. Since $(D(v_0) \wedge \sim A(v_0,b_i))$ $(\mathfrak{A})$ has cardinality m for each $i \leq k$, and $B(\mathfrak{A})$ is infinite and contained in $D(\mathfrak{A})$, for some $a' \in B(\mathfrak{A})$ $\mathfrak{A} \models A(a',b_i)$ for $i = 0, \ldots, k$. But this contradicts the definition of B so the lemma is proved.

The proof we have just given differs from Marsh's in that we did not invoke an elementary extension of $\mathfrak{A}$. The following is proved from the exchange principle just as the corresponding result is proved in the theory of vector spaces.

Lemma 3. Let X be strongly minimal in $\mathfrak{A}$ and let $Y \subseteq X$. If Z is an independent subset of Y then Z can be extended to a basis for Y. Moreover, any two bases for Y have the same cardinality.

The relationship between strongly minimal sets and elementary monomorphisms is expressed in the following lemma.

Lemma 4. Let D be strongly minimal in a complete theory T and let $\mathfrak{A}$ and $\mathfrak{B}$ be models of T. Let f be a 1 - 1 map from $X \subseteq D(\mathfrak{A})$ into $D(\mathfrak{B})$ such that X, f(X) are independent in $\mathfrak{A}$, $\mathfrak{B}$ respectively. Then f is an elementary monomorphism.

Proof. By using the device of adjoining names it suffices to treat the case in which X is a singleton, say $\{x\}$. Let A be any formula in $S_1(L)$. Since D is strongly minimal, just one of $(D \wedge A)(\mathfrak{A})$, $(D \wedge \sim A)(\mathfrak{A})$ is infinite. Without loss of generality we may suppose $(D \wedge A)(\mathfrak{A})$ is infinite. Then $(D \wedge A)(\mathfrak{B})$ is infinite since T is complete. Since $\{x\}$ is independent in $\mathfrak{A}$ and $x \in D(\mathfrak{A})$ we have $x \in A(\mathfrak{A})$. Similarly $f(X) \in A(\mathfrak{B})$; since A is arbitrary f is an elementary monomorphism.

One may easily obtain the following slight variant on Proposition 4 of [6].

Lemma 5. Let $\mathfrak{A}$ and $\mathfrak{B}$ be L-structures, $X \subseteq |\mathfrak{A}|$ and f mapping X into $\mathfrak{B}$ an elementary monomorphism. Then f can be extended to an elementary monomorphism of cl(X) into $\mathfrak{B}$. The image of this extension is cl(f(X)).

§2 Strongly Minimal Theories

A theory T is said to be a <u>strongly minimal theory</u> if the formula $v_0 = v_0$ is strongly minimal in T, that is if the universe of each model of T is strongly minimal. The following result is due to Marsh [6].

Theorem 1. If T is a countable strongly minimal theory then T is categorical in every uncountable power.

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be models of T each with power $\lambda > \aleph_0$. Let X be a basis for $|\mathcal{A}|$ and Y a basis for $|\mathcal{B}|$. Then $\kappa(X) = \kappa(Y) = \lambda$ since for each X a subset of a model of T $\kappa(\text{cl}(X)) \leq \kappa(X) + \aleph_0$. Then there exists a 1 - 1 map f from X onto Y. Then by Lemma 4 f is an elementary monomorphism. By Lemma 5 f extends to an isomorphism of $\mathcal{A}$ and $\mathcal{B}$.

If we drop for the moment the assumption that L is countable we could still prove in the same way Lemmas 1 through 5. Then, similarly to the proof of Theorem 1, we obtain

Theorem 1'. If T is a strongly minimal first order theory in a language L for each $\lambda > \kappa(L)$, T is categorical in power λ.

We return to countable languages. $\mathcal{A}$ is a <u>prime model</u> of T if for each model $\mathcal{B}$ of T there is an elementary embedding of $\mathcal{A}$ into $\mathcal{B}$.

Theorem 2. If T is a strongly minimal theory and $\mathfrak{A}$ is a prime model of T then T is $\aleph_0$-categorical if and only if for every finite $X \subseteq |\mathfrak{A}|$ cl(X) is finite.

Proof. Suppose for some finite $X \subseteq |\mathfrak{A}|$ cl(X) is infinite. Let $T_1 = Th(\mathfrak{A}, X)$. Then T_1 is $\aleph_0$-categorical if and only if T is $\aleph_0$-categorical. Let Γ be the set of formulas of the form $\sim (A \wedge \exists !^k v_0\, A)$ as k ranges through the positive integers and A through $S_1(L(X))$. Γ is consistent because cl(ϕ) in $(\mathfrak{A}, X)$ is infinite. If $B \in S_1(L(X))$ generated Γ then both $B(\mathfrak{A})$ and $\sim B(\mathfrak{A})$ would be infinite which is impossible since $|\mathfrak{A}|$ is strongly minimal. Γ is therefore not principal; thus by Ryll-Nardjewski's theorem [20, p. 91] T_1 and hence T is not $\aleph_0$-categorical.

Assuming that for each finite $X \subseteq |\mathfrak{A}|$, cl(X) is finite, we prove by contradiction that for each n T has only finitely many n-types. The theorem then follows by Ryll-Nardjewski's theorem. There are only two 0-types since T is complete. Let n + 1 be the least natural number such that there are infinitely many n + 1 types. Then some n-type Γ has infinitely many extensions. Since T has only finitely many n-types Γ must be principal and hence be realized in $\mathfrak{A}$ by $\langle a_1, \ldots, a_n \rangle$. Since $cl(\{a_1, \ldots, a_n\})$ is finite there is a formula $B \in S_1(L(\{a_1, \ldots a_n\}))$ such that $B(\mathfrak{A}) = cl(\{a_1, \ldots a_n\})$. Then if $T' = Th(\mathfrak{A}, \{a_1, \ldots, a_n\})$, $\sim B$ generates a principal 1-type in T'. For, supposing $C \in S_1(L(\{a_1, \ldots, a_n\}))$ and both $\exists v_0 (C \wedge \sim B)$ and

$\exists v_0(C \wedge \sim B)$ are consistent with T', then either $(C \wedge \sim B)(\mathcal{A})$ or $(\sim C \wedge \sim B)(\mathcal{A})$ is a finite nonempty set since $|\mathcal{A}|$ is strongly minimal. But then $B(\mathcal{A}) = cl(a_1, \ldots, a_n)$. Hence there are only $\kappa(B(\mathcal{A})) + 1$ 1-types in T' so Γ has only finitely many extensions. So for each n T has only finitely many n-types and T is $\aleph_0$-categorical.

Although to the author's knowledge this result has never been published, it has probably been known for some years. A related result, also unpublished and due to Vaught, as far as we know, states that there are no finitely axiomatizable, $\aleph_0$-categorical strongly minimal theories.

In [7] Morley makes the following definition. For each ordinal α and each $\mathcal{A} \in N(T)$, subspaces $S^\alpha(\mathcal{A})$ and $Tr^\alpha(\mathcal{A})$ of $S(\mathcal{A})$ are defined inductively by

(1) $S^\alpha(\mathcal{A}) = S(\mathcal{A}) - \cup_{\beta<\alpha} Tr^\beta(\mathcal{A})$

(2) $p \in Tr^\alpha(\mathcal{A})$ if (i) $p \in S^\alpha(\mathcal{A})$ and (ii) for every map $(f^*\colon\ S(\mathcal{B}) \to S(\mathcal{A}))$ where $\mathcal{B} \in N(T)$ and f is a monomorphism from $\mathcal{A}$ into $\mathcal{B}$, $f^{*-1}(p) \cap S^\alpha(\mathcal{B})$ is a set of isolated points in $S^\alpha(\mathcal{B})$. (See [7; p. 519] for the definition of f^*.) If $i_{\mathcal{A}\mathcal{B}}$ is an elementary embedding of $\mathcal{A}$ into $\mathcal{B}$ then $i^*_{\mathcal{A}\mathcal{B}}$ maps $S(\mathcal{B})$ onto $S(\mathcal{A})$. Note that $q \in i^{*-1}_{\mathcal{A}\mathcal{B}}(p)$ is equivalent to $q \cap S_1(L(\mathcal{A})) = p$.

An element p of $S(\mathcal{A})$ is algebraic if $p \in Tr^0(\mathcal{A})$; p is transcendental in rank α if $p \in Tr^\alpha(\mathcal{A})$.

Morley defines T to be totally transcendental if there is an ordinal β such that if $\alpha \geq \beta$ and $\mathcal{A} \in N(T)$ $S^\alpha(\mathcal{A}) = \phi$.

The least such β is called α_T. The notion of total transcendence will be discussed in detail in the next section. However, we wish to note here the value of α_T for strongly minimal theories.

Theorem 3. If T is a strongly minimal theory, $\alpha_T = 2$.

Proof. For each model $\mathcal{A}$ of T there is a unique point in $S^1(\mathcal{A})$, the only member of $S(\mathcal{A})$ which is not realized in $\mathcal{A}$. The uniqueness is guaranteed by the strong minimality of $|\mathcal{A}|$.

An example to indicate that the converse of this theorem is false and a partial converse will both be exhibited in §4.

The prototype strongly minimal theory is the theory of algebraically closed fields of characteristic 0. Other examples include the theory of infinite, divisible, torsion-free, abelian groups and the theory of the integers with the successor relation. Each of these theories has $\aleph_0$ isomorphism types of countable models.

An example of an $\aleph_0$-categorical strongly minimal theory is obtained by adjoining nonlogical axioms to the pure theory of equality asserting that the universe is infinite. A less trivial example was noticed by Łos [5]. Let T_p be theory of infinite abelian groups such that each element has order p. Then for each prime p T_p is an $\aleph_0$ categorical strongly minimal theory.

§3 $\aleph_1$ Categorical Theories

A theory T is <u>totally transcendental</u> if for each model $\mathcal{A}$ of T and each countable substructure $\mathcal{B}$ of $\mathcal{A}$ $S(\mathcal{B})$ is countable Morley proved in [7] that this definition is equivalent to his original definition mentioned in §2.

If $\mathcal{A}$ is a model of T, $\mathcal{B} \in N(T)$, and $\mathcal{B} \subseteq |\mathcal{A}|$ then $\mathcal{A}$ is <u>prime over</u> $\mathcal{B}$ if every elementary embedding of $\mathcal{B}$ into a model $\mathcal{C}$ of T can be extended to an embedding of $\mathcal{A}$ into $\mathcal{C}$. $\mathcal{A}$ is a prime model of T if $\mathcal{A}$ is prime over ϕ.

We wish to show that if T is totally transcendental then for each $X \in N(T)$ there is a model $\mathcal{A}$ of T which is prime over X. Morley's original proof in [7] depended on his notion of transcendence rank. To avoid this notion we must prove from our definition of totally transcendental that for each model $\mathcal{A}$ of T and each $X \subset |\mathcal{A}|$ the principal 1-types are dense in S(X). That is we must show for each $A \in S_1'(L(X))$ such that $(\mathcal{A}, X) \models \exists v_0 A$, there is a principal 1-type $\Gamma \in S(X)$ with $A \in \Gamma$. Suppose for contradiction T, $\mathcal{A}$, X and A constitute a counter-example. Construct a sequence of formulas in $S_1(L(X))$ as follows. Let A_1 be A. If A_n has been chosen with $n \geq 1$ choose $B \in S_1(L(X))$ such that $\mathcal{A} \models \exists v_0(A \wedge B)$ and $\mathcal{A} \models \exists v_0(A \wedge \sim B)$. Let $A_{2n} = A \wedge B$ and $A_{2n+1} = A \wedge \sim B$. For any real number α, $1 \leq \alpha \leq 2$, there is a 1-type Γ in S(X) containing $A_{[\alpha]}$, $A_{[2\alpha]}$, $A_{[4\alpha]}$, ... Let X* consist of all x in X such that x, the name of x, occurs in some A_n. Then X* is countable and $\kappa(S(X^*)) = 2^{\aleph_0}$ which contradicts T being totally transcendental. Now we may apply 4.3 of [7] to obtain:

Lemma 6. If T is totally transcendental and X ε $N(T)$ then there is a model $\mathcal{A}$ of T prime over X.

In his proof of the two cardinal theorem Vaught [21] showed that if T is a complete theory, D a formula in $S_1(T)$ and $\mathcal{A}$, $\mathcal{B}$ models of T such that $\mathcal{A} \prec \mathcal{B}$, $D(\mathcal{A}) = D(\mathcal{B})$ but $|\mathcal{A}| \neq |\mathcal{B}|$ then there exists a model C of T with $\kappa(C) = \aleph_1$ and $\kappa(D(C)) = \aleph_0$. If a theory T has such models $\mathcal{A}$ and $\mathcal{B}$ we say <u>T satisfies the hypothesis of the two cardinal theorem</u>. No theory T which satisfies the hypothesis of the two cardinal theorem can be $\aleph_1$ categorical. For in addition to C by the Löwenheim Skolem theorem T has a model $\mathcal{A}$ with $\kappa(\mathcal{A}) = \kappa(D(\mathcal{A})) = \aleph_1$. An obvious consequence is

Lemma 7. If D is strongly minimal in a complete $\aleph_1$-categorical theory T and if $\mathcal{A}$ is a proper elementary substructure of a model $\mathcal{B}$ of T then $D(\mathcal{A})$ is a proper subset of $D(\mathcal{B})$.

We wish to obtain strongly minimal formulas in totally transcendental theories. To this end we prove

Lemma 8. Let T be a complete totally transcendental theory then either some principal extension of T has a strongly minimal formula or some inessential extension of T satisfies the hypothesis of the two cardinal theorem.

Proof. Let T be a complete totally transcendental theory and $\mathcal{A}$ a prime model of T. There is a formula D ε $S_1(L(\mathcal{A}))$ such that $D(\mathcal{A})$ is minimal in $\mathcal{A}$. Consider all sets $X \subseteq |\mathcal{A}|$ definable in $\mathcal{A}$. The sets X form a subalgebra $\mathcal{B}$ of the Boolean algebra formed by all subsets of $|\mathcal{A}|$. It is clear that the Stone space of $\mathcal{B}$ is homeomorphic to the space $S(\mathcal{A})$ described in §1. But then there exists X in $\mathcal{B}$ such that X is infinite and for no Y in $\mathcal{B}$ are $X \cap Y$ and $X - Y$ both infinite. Otherwise there would be $2^{\aleph_0}$ dual prime ideals of $\mathcal{B}$ whence $\kappa(S(\mathcal{A})) = 2^{\aleph_0}$. But $\kappa(\mathcal{A}) = \aleph_0$ so $\kappa(S(\mathcal{A})) = \aleph_0$ since T is totally transcendental. Choose $D \in S_1(L(\mathcal{A}))$ such that $X = D(\mathcal{A})$ then $D(\mathcal{A})$ is minimal in $\mathcal{A}$.

Form L' by adjoining to L the constants which occur in D. Let $\mathcal{A}'$ be an inessential expansion of $\mathcal{A}$ to an L'-structure. If $X = D(\mathcal{A}) = D(\mathcal{A}')$ is strongly minimal in $\mathcal{A}'$ then D is strongly minimal in T'. The theorem is then proved. For, T' is a principal extension of T since any n-tuple from $|\mathcal{A}|$ realizes a principal type.

If X is not strongly minimal in $\mathcal{A}'$ then there exists an inessential expansion $\mathcal{B}''$ of an elementary extension $\mathcal{B}'$ of $\mathcal{A}'$ such that for some $C \in S_1(\mathcal{B}'')$ both $D(\mathcal{B}'') \cap C(\mathcal{B}'')$ and $D(\mathcal{B}'') - C(\mathcal{B}'')$ are infinite. Now C can be written as $C'(v_0, c_1, \ldots c_m)$ where $C' \in S_{m+1}(L')$ and $c_1, \ldots c_m$ are constants adjoined in expanding $\mathcal{B}'$ to $\mathcal{B}''$. Let $L'' = L \cup \{c_1 \ldots c_m\}$. Consider all possible expansions $\mathcal{A}''$ of $\mathcal{A}'$ to an L''-structure. Since X is minimal for each such $\mathcal{A}''$ one of $D(\mathcal{A}'') \cap C(\mathcal{A}'')$ or $D(\mathcal{A}'') - C(\mathcal{A}'')$ is finite. If as $\mathcal{A}''$ varies neither $D(\mathcal{A}'') \cap C(\mathcal{A}'')$ nor $D(\mathcal{A}'') - C(\mathcal{A}'')$ can have arbitrarily large finite cardinality then for some positive integer N the formula $\exists^{\leq N} v_0(D \wedge C') \vee \exists^{\leq N} v_0(D \wedge \sim C')$ is valid in $\mathcal{A}'$ and hence in $\mathcal{B}'$. This contradicts the fact that $(D \wedge C)(\mathcal{B}')$ and $(D \wedge \sim C)(\mathcal{B}')$ are both infinite. Without loss suppose that as $\mathcal{A}''$

varies $D(\mathcal{A}'') \cap C(\mathcal{A}'')$ can have arbitrarily large finite cardinality

Let $L^* = L' \cup \{p\}$ where p is a unary relation symbol. Let Γ be the set of nonlogical axioms of T. Let Δ be a set of sentences which are true in an L^*-structure C^* just if there is an elementary substructure C_1 of $C^*|L''$ such that $|C_1| = p(C^*)$ and such that $(D \wedge C)(C^*) = (D \wedge C)(C_1)$. Let A^n be a sentence which is true in $\mathcal{A}''$ if and only if $(D \wedge C)(\mathcal{A}'')$ has cardinality greater than n. Let T^* be a theory with nonlogical axioms $\Gamma \cup \Delta \cup \{A^n | \ n < w\}$. To show that T'' is consistent it suffices to show for arbitrary k that the set of nonlogical axioms $\Gamma \cup \Delta \cup \{A^n : n \leq k\}$ yields a consistent theory $T_k{}^*$. To obtain a model of $T_k{}^*$ choose $\mathcal{A}''$ such that $(D \wedge C)(\mathcal{A}'')$ has finite cardinality greater than k. Let $\mathcal{B}''$ be a proper elementary extension of $\mathcal{A}''$. Expand $\mathcal{B}''$ to $\mathcal{B}^*$ by letting $p(\mathcal{B}^*) = |\mathcal{A}''|$. Then $\mathcal{B}^*$ is a model of $T_k{}^*$. Hence T^* is consistent. Let $\mathcal{B}^*$ now be a model of T^* then $Th(\mathcal{B}^*| L'')$ is an inessential extension of T which satisfies the hypothesis of the two cardinal theorem.

Our original proof of Lemma 8 used Keisler's result [3 ; 4.26] that an $\aleph_1$ categorical theory cannot have the finite cover property.

By Theorem 3.8 of [7] if T is $\aleph_1$ categorical T is totally transcendental. Thus by Lemma 8 if T is $\aleph_1$ categorical a principal extension of T has a strongly minimal formula. We now derive some properties of strongly minimal formulas in $\aleph_1$-categorical theories to make use of this fact.

Lemma 9. Let D be strongly minimal in a complete theory T. If μ is an infinite cardinal there is a model $\mathcal{A}$ of T with $\kappa(\mathcal{A}) = \dim(D(\mathcal{A})) = \mu$. If $\mathcal{A}$ is a model of T and n is finite with $\dim D(\mathcal{A}) = n$ then there is a model $\mathcal{B}$ of T with $\dim(D(\mathcal{B})) = n + 1$.

Proof. Suppose that D, T satisfy the hypothesis. If μ is infinite by the completeness theorem there is certainly a model C of T with $\kappa(C) = \dim(D(C)) = \mu$. Let $\mathcal{A}$ be a model of T with $\dim(D(\mathcal{A}))$ finite. Let c be a new constant symbol and T' the theory obtained by adjoining to $Th(\mathcal{A}, D(\mathcal{A}))$ the nonlogical axioms D(c) and $\{c \neq b \mid b \in D(\mathcal{A})\}$. Let $\mathcal{A}'$ be a model of T'; by renaming the elements of $|\mathcal{A}'|$, if necessary we may assume that the interpretation of b in $\mathcal{A}'$ is b for $b \in D(\mathcal{A})$. Applying Lemma 4 to $Th(\mathcal{A}, D(\mathcal{A}))$ we see that T' is complete. Again applying Lemma 4 the 1-type Γ in $L' = L \cup \{c\}$ of a point $d \in D(\mathcal{A}') - cl(D(\mathcal{A}) \cup \{(c)_{\mathcal{A}'}\})$ is independent of the choice of d and $\mathcal{A}'$. Suppose Γ were principal and generated by $A \in S_1(L')$ then $A(\mathcal{A}')$ would be contained in $D(\mathcal{A}')$ infinite and disjoint from $D(\mathcal{A})$. This would contradict the strong minimality of D in T. Thus Γ is not principal and by Ehrenfreuht's theorem there is a model $\mathcal{B}'$ of T' omitting Γ. Let $\mathcal{B} = \mathcal{B}'|L$ then $\dim D(\mathcal{B}) = 1 + \dim D(\mathcal{A})$ since if X is a basis for $D(\mathcal{A})$, $X \cup \{(c)_{\mathcal{B}}\}$ is clearly a basis for $D(\mathcal{B})$.

Let $\mathcal{A} \prec \mathcal{B}$; then $\mathcal{B}$ is a <u>prime extension</u> of $\mathcal{A}$ if $|\mathcal{A}| \subseteq |\mathcal{B}|$ and every proper elementary embedding f of $\mathcal{A}$ into a model

C extends to an elementary embedding of $\mathcal{B}$ into C. Let $\mathcal{A} \prec \mathcal{B}$; then $\mathcal{B}$ is a <u>minimal extension</u> of $\mathcal{A}$ if $|\mathcal{A}| \neq |\mathcal{B}|$ and $\mathcal{A} \prec C \prec \mathcal{B}$ implies $C = \mathcal{A}$ or $C = \mathcal{B}$.

Lemma 10. Let D be strongly minimal in a theory T which does not satisfy the hypothesis of the two cardinal theorem. Let $\mathcal{A}$ be a model of T and X a basis for $D(\mathcal{A})$ then $\mathcal{A}$ is prime over X. The isomorphism type of $\mathcal{A}$ is uniquely determined by the cardinality of X. Moreover $\mathcal{A}$ has a minimal prime extension $\mathcal{B}$ and there exists $y \in D(\mathcal{B}) - D(\mathcal{A})$ such that $X \cup \{y\}$ is a basis for $D(\mathcal{B})$.

Proof. Assuming the hypothesis let $\mathcal{A}'$ be an elementary submodel of $\mathcal{A}$ prime over X then $D(\mathcal{A}') = D(\mathcal{A}) = cl(X)$. Hence $\mathcal{A}' = \mathcal{A}$ or T would satisfy the hypothesis of the two cardinal theorem. Suppose C is a model of T, that Z is a basis for D(Z) and that $\kappa(Z) = \kappa(X)$. Let f be a bijection from X to Z then f is an elementary monomorphism by Lemma 4. Since $\mathcal{A} = \mathcal{A}'$ is prime over X, f can be extended to an elementary monomorphism g of $\mathcal{A}$ into C. By Lemma 5 $g(D(\mathcal{A})) = D(C)$ so g is an isomorphism of $\mathcal{A}$ and C; otherwise T would satisfy the hypothesis of the two cardinal theorem. To show that $\mathcal{A}$ has a minimal prime extension let C now be a proper elementary extension of $\mathcal{A}$. Since T does not satisfy the hypothesis of the two cardinal theorem there exists $y \in D(C) - D(\mathcal{A})$. Let C_1 be an elementary submodel of C prime over $X \cup \{y\}$. Then C_1 has an elementary submodel C_2

prime over X and hence isomorphic to $\mathcal{A}$. Since $y \notin cl(X)$, C_1 is a proper extension of $\mathcal{A}$. Further if $C_2 \prec \mathcal{B} \prec C_1$ and $|C_2| \neq |\mathcal{B}|$ then $D(C_1) = D(\mathcal{B})$ by the exchange principle (Lemma 2) and so $C_1 = \mathcal{B}$ since T does not satisfy the hypothesis of the two cardinal theorem. Thus C_1 is a minimal extension of C_2. Let f be an elementary embedding of C_2 properly into a model $\mathcal{D}$ of T. By Lemma 7 there exists $y' \in (D(\mathcal{D}) - D(f(C_2)))$. Let g extend f with domain $g = |C_2| \cup \{y\}$ and $g(y) = y'$. Since D is strongly minimal in $Th(C_2, |C_2|)$ by Lemma 4 g is an elementary monomorphism. Since C_1 contains a model prime over $|C_2| \cup \{y\}$ and C_1 is a minimal extension of C_2, C_1 must be prime over $|C_2| \cup \{y\}$. Therefore g extends to an elementary embedding of C_1 in $\mathcal{D}$. Hence C_1 is a prime extension of C_2. Now if h is the isomorphism from C_2 onto $\mathcal{A}$ then h can be viewed as an embedding of C_2 properly into C. Let h_1 be the extension of h which embeds C_1 into C. Then $h_1(C_1)$ is the required minimal prime extension of $\mathcal{A}$.

We now combine Lemmas 8 and 10 to obtain

Theorem 4. Let T be an $\aleph_1$-categorical complete theory; then

(i) T is μ categorical for every uncountable cardinal μ.

(ii) there are at most $\aleph_0$ isomorphism types of countable models of T

(iii) each model of T has a minimal prime extension

(iv) every uncountable model of T is saturated

(v) if a countable model of T contains a strictly increasing elementary chain then it is saturated.

(vi) an elementary extension of a saturated model of T is saturated.

Theorem 4 (i) and (iv) are due to Morley [7]. Morley proved Theorem 4 (ii), (iii), and (v) in [8]. Marsh proved Theorem 4 (vi) in [6].

Proof. If T is an $\aleph_1$-categorical complete theory then no inessential extension of T satisfies the hypothesis of the two cardinal theorem. By 3.8 of [7] T is totally transcendental. Hence by Lemma 8 there is a principal extension T' of T which has a strongly minimal formula D. For any model $\mathcal{A}'$ of T' the isomorphism type of $\mathcal{A}'$ is determined by dim(D($\mathcal{A}'$)). Thus T' is categorical in every uncountable power. A fortiori, so is T. Parts (ii) and (iii) are proved similarly.

To prove (iv) consider a model $\mathcal{A}$ of T with $\kappa(\mathcal{A}) = \mu > \aleph_0$. Let X be any subset of $|\mathcal{A}|$ with $\kappa(X) < \mu$. We must show for any $p \in S(X)$ that p is realized in $\mathcal{A}$. Let $\mathcal{B}$ be an elementary submodel of $\mathcal{A}$ prime over X. Then $\kappa(\mathcal{B}) = \kappa(X) + \aleph_0 < \mu$. By the Lowenheim Skolem theorem any point $p \in S(\mathcal{B})$ and hence any point $p \in S(X)$ is realized in an elementary extension $\mathcal{B}'$ of $\mathcal{B}$ with $\kappa(\mathcal{B}') = \kappa(\mathcal{B})$. By (iii) there exists a strictly increasing elementary chain $\langle \mathcal{B}_\gamma \mid \gamma \le \delta \rangle$ such that $\mathcal{B}_0 = \mathcal{B}$, $\mathcal{B}_{\gamma+1}$ is a prime extension of $\mathcal{B}_\gamma$, $\mathcal{B}_\alpha = \cup \{\mathcal{B}_\beta \mid \beta < \alpha\}$ if α is a limit ordinal, and $\mathcal{B}_\delta = \mathcal{B}'$.

Again by (iii) using induction on γ, for each $\gamma \leq \delta$, since $\kappa(\mathcal{B}_\gamma) < \kappa(\mathcal{A})$ there is an elementary embedding of $\mathcal{B}_\gamma$ into $\mathcal{A}$ which preserves inclusions and is the identity on $\mathcal{B}$. Thus p is realized in $\mathcal{A}$ proving (iv).

Since $\kappa(S(\mathcal{A})) = \kappa(\mathcal{A})$ for any model $\mathcal{A}$ of T, there exists an elementary chain $\mathcal{A}_0 \prec \mathcal{A}_1 \preccurlyeq \ldots$ of countable models of T such that $\mathcal{A}_{n+1}$ realizes every point of $S(\mathcal{A}_n)$ for each $n < \omega$. Then $\mathcal{A} = \cup \{\mathcal{A}_n | n < \omega\}$ is clearly saturated. To prove (v) let $\mathcal{B}$, C be two countable models of T each containing a strictly increasing elementary chain; it suffices to show $\mathcal{B}$ and C are isomorphic. Expand $\mathcal{B}$ and C to models $\mathcal{B}'$ and C' of T' by interpreting the new constants in the first member of the chain in $\mathcal{B}$, C respectively. The strictly increasing chains force $\dim(D(\mathcal{B}')) = \dim(D(C')) = \aleph_0$. Hence $\mathcal{B}' \cong C'$ and therefore $\mathcal{B} \cong C$. Note that (vi) is now immediate.

We wish to produce a proof of Morley's theorem that a countabl first order theory is categorical in power $\aleph_1$ if and only it is categorical in every uncountable power which does not depend on the notion of transcendence rank. In this endeavor we use two results from Morley. In Lemma 6 we showed that 4.3 of [7] whose proof does not depend on the notion of transcendence rank could be applied without reference to the notion of transcendence rank. We also use 3.8 of [7] but Morley's original proof did not employ the notion of transcendence rank.

Lemma 11. Let $\mathcal{A}$ be a model of a totally transcendental theory T. Let $X \subseteq |\mathfrak{A}|$ have cardinality $\geq \aleph_1$. There exists an elementary extension $\mathfrak{B}$ of $\mathfrak{A}$ and a subset Y of $|\mathfrak{B}|$ which properly contains X such that each n-type in T which is realized in Y is realized in X.

Proof: By essentially the same argument that was used in the first paragraph of the proof of Lemma 8 there exists $X' \subseteq X$ with $\kappa(X') = \aleph_1$ such that for each $Z \subseteq |\mathfrak{A}|$ definable in some inessential expansion of $\mathfrak{A}$ just one of $X' \cap Z$ and $X' - Z$ has power $\aleph_1$. (Instead of considering all subsets of $|\mathfrak{A}|$ as in Lemma 8 consider subsets of X and replace "infinite" by "of cardinality $\aleph_1$".)

We now construct a theory T' such that the required $\mathfrak{B}$ and Y can be obtained from a model $\mathfrak{B}'$ of T'. Let L' be the language obtained by adjoining a new constant c to $L(\mathfrak{A})$. Let T' have nonlogical axioms Λ where:

$$\Lambda = Th(\mathfrak{A}, |\mathfrak{A}|) \cup \{A(c) \mid \exists Z[Z \subset |\mathfrak{A}| \wedge \kappa(Z) < \aleph_0 \wedge A \in S_1(L(Z)) \wedge \kappa\{a|a \in X' \wedge (\mathfrak{A}, Z) \models A(a)\} = \aleph_1\}.$$

To show Λ is consistent consider fixed finite $Z_1 \subseteq |\mathfrak{A}|$ and $A \in S_1(L(Z_1))$ such that $A(c) \in \Lambda$. By choice of X' there are at most $\aleph_0$ members a of X' such that $(\mathfrak{A}, Z) \models \sim A(a)$. Since there are only $\aleph_0$ formulas $A \in S_1(L(Z))$ we have for all but $\aleph_0$ members a of X' and for each $A \in S_1(L(Z))$ such that $A(c) \in \Lambda$ $(\mathfrak{A}, Z) \models A(a)$. This shows Λ is consistent. Let $\mathfrak{B}'$ be a model of T' and $\mathfrak{B} = \mathfrak{B}'|L$. Without loss, since $\mathfrak{B}' \models Th(\mathfrak{A}, |\mathfrak{A}|)$, we may suppose $\mathfrak{B}$ is an elementary extension of $\mathfrak{A}$.

Let Y be $(c)_{\mathcal{B}}$, and $Y = X \cup \{y\}$. Then $\mathcal{B}$ and y satisfy the conditions of the theorem. For, let Γ be an n-type realized in Y by $x_1, \ldots x_{n-1}, y$. Then for each $A' \in S_1(L(x_1, \ldots \ _{n-1}))$ where $A' = A(v_0, X_1, \ldots, X_{n-1})$ either $A'(c)$ or $\sim A'(c)$ is in Λ. Moreover $A(v_1 \ldots v_n) \in \Gamma$ is equivalent to $A''(c) \in \Lambda$ which is in turn, equivalent to $\kappa(\{a \in X' \mid (\mathcal{A}, \{x_1, \ldots x_{n-1}\}) \models A'(a)\}) = \aleph_1$. But then since there are only countably many $A' \in S_1(L(X_1, \ldots X_n))$ some $x \in X'$ must be such that $\langle x_1, \ldots x_{n-1}, x \rangle$ realizes Γ.

Careful examination of the proof of 4.3 of [7] yields the following: let $\mathcal{A}$ be an L-structure with subset X such that $\mathcal{A}$ is prime over X and constructed as in 4.3 [7] then for each n, every n-type in $Th(\mathcal{A}, X)$ realized in $(\mathcal{A}, X)$ is principal. With this observation we can reformulate the last lemma to read: if $Th(\mathcal{A})$ is totally transcendental and $\kappa(\mathcal{A}) \geq \aleph_1$ then $\mathcal{A}$ has a proper elementary extension C such that for each n each n-type in $Th(\mathcal{A})$ realized in C is already realized in $\mathcal{A}$. To see this first apply Lemma 10 with $X = |\mathcal{A}|$ to obtain $\mathcal{B}$ and Y. Then let C be prime over $|\mathcal{A}| \cup \{y\}$.

Theorem 5. Let T be μ-categorical for some $\mu > \aleph_0$ then T is $\aleph_1$-categorical.

Proof. Assume the hypothesis and for contradiction that T is not $\aleph_1$-categorical. By 3.8 of [7] T is totally transcendental. Then by Lemma 8 there is an inessential extension T' of T in a language L' extending L which satisfies the hypothesis of the two cardinal theorem. Hence there exists a model $\mathcal{C}$ of T' and a formula D ε $S_1(L')$ such that $\kappa(\mathcal{C}) = \aleph_1$ and $\kappa(D(\mathcal{C})) = \aleph_0$. Let $\mathcal{C}' = (\mathcal{C}, D(\mathcal{C}))$. Applying Lemma 11 and Tarski's lemma we obtain an elementary extension $\mathcal{B}'$ of $\mathcal{C}'$ and an X with $\kappa(X) = \mu$, and $|\mathcal{C}'| \subseteq X \subseteq |\mathcal{B}'|$ such that every n-type from $Th(\mathcal{C}')$ realized in X is realized in $\mathcal{C}'$. Lemma 11 can be applied because $D(\mathcal{C}')$ is countable. Now we use the idea in the reformulation of Lemma 11. Let $\mathcal{A}' \prec \mathcal{B}'$ be prime over X. Suppose for contradiction there exists d in $D(\mathcal{A}') - D(\mathcal{C})$ then there exists A ε $S_1(L'(X))$ generating the 1-type in $L'(D(\mathcal{C}))$ realized by d. Now A has the form $A(v_0, a_1, \ldots a_m)$ where $a_1, \ldots a_m \in X$ and B ε $S_1(L'(D(\mathcal{C})))$. Choose $(c_1, \ldots c_m)$ in $|\mathcal{C}|^m$ realizing the same $Th(\mathcal{C}')$ type in $\mathcal{C}'$ as $(a_1, \ldots a_m)$ does in $\mathcal{B}'$. Then $B(v_0, c_1, \ldots c_m)(\mathcal{C}') \cap (D(\mathcal{C}') - D(\mathcal{C})) \neq \emptyset$ which is absurd. Hence $D(\mathcal{A}') = D(\mathcal{C})$. But then $\mathcal{A}'|L$ is a model T of power μ in some inessential expansion of which is definable a set of cardinality $\aleph_0$. But using the completeness theorem and Tarski's lemma it is easy to construct a model of T of power μ such that any set definable in an inessential expansion of it has power μ. Hence T is not μ-categorical contrary to hypothesis.

This completes our development of the main results known concerning theorems categorical in power. The spirit of our approach is captured in the following observation which is obvious from the proof of Theorem 4: a theory is categorical in uncountable power if and only if it is totally transcendental and no inessential extension of it satisfies the hypothesis of the two cardinal theorem.

§4 Almost Strongly Minimal Theories

A theory T is <u>almost strongly minimal</u> if there is a principal extension T' of T with strongly minimal formula D' such that for each model $\mathcal{A}'$ of T' $|\mathcal{A}'| = \mathrm{cl}(D(\mathcal{A}'))$. In this section we prove a theorem characterizing such theories. We also make use of this concept to prove the promised partial converse to Theorem 3.

If T is almost strongly minimal then T is $\aleph_1$ categorical. For, let $\mathcal{A}$ and $\mathcal{B}$ be models of T with $\kappa(\mathcal{A}) = \kappa(\mathcal{B}) = \aleph_1$. Let $\mathcal{A}'$ and $\mathcal{B}'$ be inessential expansions of $\mathcal{A}$ and $\mathcal{B}$ which are models of T'. Then by Theorem 1 and Lemma 5 $\mathcal{A}'$ is isomorphic to $\mathcal{B}'$. Hence $\mathcal{A}$ is isomorphic to $\mathcal{B}$.

Let T be $\aleph_1$-categorical and $\mathcal{A}$ a model of T. We prove some technical lemmas about the structure of $S(\mathcal{A})$. $S^n(\mathcal{A})$ and $Tr^n(\mathcal{A})$ are defined in Section 2, just before the statement of Theorem 3.

If T is an $\aleph_1$-categorical theory we can associate with any model $\mathcal{A}$ of T a strictly increasing elementary chain $\mathcal{A} = \mathcal{A}_0 \prec \mathcal{A}_1 \prec \cdots$ by letting $\mathcal{A}_{n+1}$ be a minimal prime extension of $\mathcal{A}_n$ for each n. Note that by Lemma 10 this chain is unique up to isomorphism. If p is an element of $S(\mathcal{A})$ we say p is <u>first realized</u> in $\mathcal{A}_n$ if p is realized in $\mathcal{A}_n$ but p is not realized in $\mathcal{A}_m$ for each $m < n$. For each n we take $\{d_1, \ldots d_n\}$ to be a subset of $D(\mathcal{A}_n) - D(\mathcal{A})$ which is algebraically independent in $L(\mathcal{A})$. Thus by Lemma 10 $\mathcal{A}_n$ is prime over $|\mathcal{A}| \cup \{d_1, \ldots d_n\}$.

Lemma 12. Let T be an $\aleph_1$-categorical theory, with strongly minimal formula D, $\mathcal{A}$ a model of T and $p \in S(\mathcal{A})$. There exists a natural number n such that

(i) p is first realized in $\mathcal{A}_n$.

(ii) There exists a formula $A \in S_{n+1}(L(\mathcal{A}))$ such that $\exists v_1, \ldots \exists v_n A$ is in p and if $\mathcal{B}$ is an elementary extension of $\mathcal{A}$ and $y_1, \ldots, y_n$, b are elements of $|\mathcal{B}|$ such that $\mathcal{B} \models A(b, y_1, \ldots, y_n)$ and b realizes p then $y_1, \ldots, y_n$ are algebraically independent in $L(\mathcal{A})$ but $\{y_1, \ldots, y_n\} \subseteq cl(|\mathcal{A}| \cup \{b\})$.

Proof. (i) Let $\mathcal{B}$ be an elementary extension of $\mathcal{A}$ such that p is realized in $\mathcal{B}$ by b. By Lemma 10 $\mathcal{B}$ is prime over $|\mathcal{A}| \cup Y$ where Y is a subset of $D(\mathcal{B}) - D(\mathcal{A})$ which is algebraically independent in $L(\mathcal{A})$. Then for some n there is a formula $B \in S_{m+1}(L(\mathcal{A}))$ and elements $y_1, \ldots y_n$ in Y such that $B(v_0, y_1, \ldots y_n)$ generates the principal type in $Th(\mathcal{B}, |\mathcal{A}| \cup Y)$ realized by b. Let n be chosen as small as possible. Then $\mathcal{A}_n$ is isomorphic to the model prime over $|\mathcal{A}| \cup \{b\}$. Hence p is realized in $\mathcal{A}_n$ but not in $\mathcal{A}_m$ for $m < n$.

(ii) Let p be first realized in $\mathcal{A}_n$ by c and suppose $d_1, \ldots d_n$ are a basis for $D(\mathcal{A}_n)$ in $L(\mathcal{A})$. Let $A(c, v_1, \ldots v_n)$ generate the principal n-type in $Th(\mathcal{A}_n, |\mathcal{A}_n| \cup \{c\})$ realized by $d_1, \ldots, d_n$. Clearly

$\exists v_1 \ldots \exists v_n A$ is in p.

For $i = 1, \ldots n$ there is a formula $A_i \in S_2(L(\mathcal{A}))$ and an integer N such that

$$\mathcal{A}_n \models A_i(c,d_i) \wedge \exists^{\leq N} v_1 A_i(c,v_1)$$

For let $A_i(c,v_1)$ generate the principal type in $Th(\mathcal{A}_n, |\mathcal{A}| \cup \{c\})$, realized by d_i and suppose $A_i(c,v_1)(\mathcal{A}_n)$ is infinite. Then since D is strongly minimal and $A_i(c,v_1)(\mathcal{A}_n) \subseteq D(\mathcal{A}_n)$ there is an element $c' \in D(\mathcal{A})$ such that $\mathcal{A}_n \models \exists v_1(A_i(c,v_1) \wedge v_1 = c')$. But then for each element e of $A_i(c,v_1)(\mathcal{A}_n)$ $\mathcal{A}_n \models e = c'$. This is absurd so each $A_i(c,v_1)(\mathcal{A}_n)$ is finite and we may assume each of these sets has less than N elements. Note that in $Th(\mathcal{A}_n, |\mathcal{A}| \cup \{c\})$ $A(c,v_1, \ldots v_n) \rightarrow \bigwedge_1^n A_i(c,v_i)$.

Suppose $\mathcal{B}$ is an elementary extension of $\mathcal{A}$ and $y_1, \ldots y_n$, b are elements of $|\mathcal{B}|$ such that $\mathcal{B} \models A(b, y_1, \ldots, y_n)$ and b realizes p. Since b realizes p $Th(\mathcal{B}, |\mathcal{A}| \cup \{b\}) = Th(\mathcal{A}_n, |\mathcal{A}| \cup \{c\})$. Then since $A(c, v_1, \ldots v_n)$ generates the principal type realized by $d_1, \ldots, d_n$ which are algebraically independent in $L(\mathcal{A})$, $y_1, \ldots y_n$ are algebraically independent in $L(\mathcal{A})$. Similarly, since each $d_i \in cl(|\mathcal{A}| \cup \{c\})$ each $y_i \in cl(|\mathcal{A}| \cup \{c\})$.

Lemma 13. Let T be an $\aleph_1$-categorical theory with strongly minimal formula D and $\mathcal{A}$ a model of T. If p is an element of $S(\mathcal{A})$ which is not realized in $\mathcal{A}$ for $k < m$ then

$p \in S^m(\mathcal{A})$.

Proof. The proof is by induction on m. If $m = 0$, the result is evident. Suppose the lemma holds for each model of T and each $m < s$. Let $\mathcal{A}$ be a model of T and $p \in S(\mathcal{A})$ such that p is not realized in $\mathcal{A}_m$ for $m < s$. Then by Lemma 12 (i) p is first realized in $\mathcal{A}_n$ for some $n \geq s$. Choose $A \in S_{n+1}(L(\mathcal{A}))$ satisfying Lemma 12 (ii). Let $\mathcal{B}$ be an elementary extension of $\mathcal{A}$ such that $D(\mathcal{B}) - D(\mathcal{A})$ is infinite. Let $\mathcal{B}_{n-1} \succ \mathcal{B}$ and suppose $y_1, \ldots, y_{n-1}$ are a basis in $L(\mathcal{B})$ for $D(\mathcal{B}_{n-1})$. Let $b_1, b_2 \ldots$ enumerate $D(\mathcal{B}) - D(\mathcal{A})$.

Let f_j be an isomorphism of $\mathcal{A}_n$ into $\mathcal{B}_{m-1}$ determined by $f_j(d_i) = y_i$ for $1 \leq i \leq n - 1$ and $f_j(d_n) = b_j$. Let $a_j = f_j(c)$. Let q_j be the element of $S(\mathcal{B})$ realized by a_j. Then q_j is not realized in $\mathcal{B}_k$ for any $k < n-1$. For suppose $\mathcal{B}' \succ \mathcal{B}$ and $a_j' \in \mathcal{B}'$ realizes q_j. Then if $\mathcal{B}^j$ is prime over $|\mathcal{B}| \cup \{a_j\}$ there is an isomorphism of $\mathcal{B}^j$ into $\mathcal{B}'$. But by Lemma 12 (ii) $\{y_1, \ldots y_{n-1}\} \subseteq cl(|\mathcal{A}| \cup \{a_j\})$ so $\{y_1, \ldots y_{n-1}\} \subseteq D(\mathcal{B}^j)$. Thus $D(\mathcal{B}^j)$ and hence $D(\mathcal{B}')$ contain at least $n - 1$ elements algebraically independent in $L(\mathcal{B})$. So q_j is not realized in any $\mathcal{B}_k$ with $k < n - 1$ and in particular with $k < s - 1$ so $q_j \in S^{s-1}(\mathcal{B})$ by induction. Since each a_j realizes p each $q_j \in i^{*}_{\mathcal{A}\mathcal{B}}{}^{-1}(p)$. Moreover there are infinitely many distinct q_j. For if not there exsits an elementary extension $\mathcal{C}$ of $\mathcal{B}$ and $c \in |\mathcal{C}|$ such that c realizes infinitely many of the q_j. Then $\mathcal{C} \models B_n(c, b_j)$ for infinitely many b_j

in $D(\mathcal{B}) - D(\mathcal{A})$. But since c realizes p $B_n(c, v_1)(C)$ is finite.

Thus $i^{*-1}_{\mathcal{A}\mathcal{B}}(p) \cap S^{s-1}(\mathcal{B})$ is infinite so $p \in S^s(\mathcal{B})$ which was to be proved.

Lemma 14. Let T be an $\aleph_1$-categorical theory with strongly minimal formula D and prime model $\mathcal{A}'$ such that for each model $\mathcal{A}$ of T $|\mathcal{A}| = cl(D(\mathcal{A}) \cup |\mathcal{A}'|)$. For each model $\mathcal{A}$ of T if $p \in S(\mathcal{A})$ is first realized in $\mathcal{A}_m$ then $p \in Tr^m(\mathcal{A})$.

Proof. The proof is by induction on m. If m = 0 the lemma is clearly true. Suppose the lemma holds for m < n and let $\mathcal{A}$ be a model of T and $p \in S(\mathcal{A})$ such that p is first realized in $\mathcal{A}_n$ by c.

Let $B \in S_{n+1}(L(\mathcal{A}))$ and suppose $B(v_0, d_1, \ldots, d_n)$ generates the principal type in $Th(\mathcal{A}_n, |\mathcal{A}| \cup \{d_1, \ldots d_n\})$ realized by c. Then by hypothesis $B(v_0, d_1, \ldots d_n)(\mathcal{A}_n)$ is finite, say with cardinality k. Let A be chosen as in 12 (ii). Let $C = A \wedge B$. Clearly $\exists v_1, \ldots, \exists v_n$ C is in p. Let $\mathcal{B}$ be an elementary extension of $\mathcal{A}$ and suppose $y_1, \ldots, y_n$ is a basis in $L(\mathcal{B})$ for $D(\mathcal{B}_n)$. Then $C(v_0, y_1, \ldots, y_n)(\mathcal{B}_n)$ has cardinality k so there at most k types in $S(\mathcal{B})$ which are realized in $C(v_0, y_1, \ldots, y_n)(\mathcal{B}_n)$. Let $q \in i^{*-1}_{\mathcal{A}\mathcal{B}}(p)$. Let $\mathcal{B}' \geq \mathcal{B}$ and suppose q is realized in $\mathcal{B}'$ by b. Since b realizes p there exist $y_1', \ldots y_n'$ in $D(\mathcal{B}')$ such that $\mathcal{B}' \models C(b, y_1', \ldots, y')$ and by Lemma 12 (ii) $y_1', \ldots, y_n'$ are algebraically independent in $L(\mathcal{A})$. Moreover $b \in cl(|\mathcal{A}| \cup \{y_1', \ldots, y_n'\})$. Thus if $y_1', \ldots, y_n'$ are not algebraically independent in $L(\mathcal{B})$ q is realized in $\mathcal{B}_k$ for some

$k < n$ and $q \notin S^n(\mathcal{A})$. If $y_1', \ldots y_n'$ are algebraically independent in $L(\mathcal{B})$, let f map y_i to y_i'. Then f extends to an isomorphism taking a model prime over $y_1', \ldots y_n'$ into $\mathcal{B}_n$. Then $f(b)$ is in $C(v_0, y_1, \ldots, y_n)(\mathcal{B}_n)$ so q is one of the at most k types in $S(\mathcal{B})$ which are realized by a member of $C(v_0, y_1, \ldots, y_n)(\mathcal{B}_n)$. Thus for every $\mathcal{B} \succ \mathcal{A}$ $i^{*\,-1}_{\mathcal{A}\mathcal{B}}(p) \cap S^n(\mathcal{B})$ is finite so $p \in Tr^n(\mathcal{A})$.

Lemma 15. Let T be $\aleph_1$-categorical with strongly minimal formula D. Let $\mathcal{A}$ be a model of T and $p \in S(\mathcal{A})$ such that p is first realized in $\mathcal{A}_m$. Let C be the formula in $S_{m+1}(L(\mathcal{A}))$ associated with p in the proof of Lemma 14. Suppose $C(v_0, d_1, \ldots d_m)(\mathcal{A}_m)$ is infinite. Then $p \in S^{m+1}(\mathcal{A})$.

Proof. The proof is by induction on m. Let $m = 1$. Let $\mathcal{B} \succ \mathcal{A}$ and suppose $D(\mathcal{B}) - D(\mathcal{A})$ is infinite. Let $b_1, b_2, \ldots$ enumerate $D(\mathcal{B}) - D(\mathcal{A})$. Since $C(v_0, d_1)(\mathcal{A}_1)$ is infinite, for each $j < \omega$ $C(v_0, b_j)(\mathcal{B})$ is infinite. Thus for each j there is a type $q_j \in S^1(\mathcal{B})$ with $C(v_0, b_j) \in q_j$. As in Lemma 13 infinitely many q_j are distinct. Each $q_j \in i^{*\,-1}_{\mathcal{A}\mathcal{B}}(p)$ so $p \in S^2(\mathcal{A})$.

Suppose the lemma holds for each $m < n$ and $\mathcal{A}$ is a model of T with $p \in S(\mathcal{A})$ such that p is first realized in $\mathcal{A}_n$ and $C(v_0, d_1, \ldots d_n)(\mathcal{A}_n)$ is infinite. Let $\mathcal{B} \succ \mathcal{A}$ such that $D(\mathcal{B}) - D(\mathcal{A})$ is infinite. Let $y_1, \ldots y_{n-1}$ be a basis in $L(\mathcal{B})$ for $\mathcal{B}_{n-1}$ and let $b_1, b_2 \ldots$ enumerate $D(\mathcal{B}) - D(\mathcal{A})$. For each j consider $C(v_0, y_1, \ldots, y_{n-1}, b_j)(\mathcal{B}_{n-1})$ which is infinite since $C(v_0, d_1, \ldots d_n)(\mathcal{A}_n)$ is. Consider the types in $Th(\mathcal{B}_n, |\mathcal{B}| \cup \{y_1, \ldots y_{n-1}\})$ of elements of $C(v_0, y_1, \ldots y_{n-1}, b_j)$

$(\mathcal{B}_{n-1})$. If some such type q is realized infinitely often in $\mathcal{B}_{n-1}$ let $q_j = i^*_{\mathcal{B}\mathcal{B}'_{n-1}}(q)$ where $\mathcal{B}'_n = \mathcal{B} \cup \{y_1, \ldots, y_n\}$. Then by induction $q_j \epsilon\ S^n(\mathcal{B})$. If each type in $S(\mathcal{B}'_n)$ is realized at most finitely often in $C(v_0, y_1, \ldots, y_n)$ $(\mathcal{B}_n)$ then there are infinitely many such types in $S(\mathcal{B}'_n)$ so there exists $q'_j \ \epsilon\ S^1(\mathcal{B}'_n)$ with $C(v_0, y_1, \ldots v_{n-1}, b_j)$ in q'_j. Let $\mathcal{C} \succcurlyeq \mathcal{B}_n$ and $c_j\ \epsilon\ |\mathcal{C}|$ such that c_j realizes q'. Then if C' is prime over $|\mathcal{B}| \cup \{c_j\}$ the dimension in $L(\mathcal{B})$ of $D(C')$ is s for some $s \geq n$.

Let q_j be the type in $S(\mathcal{B})$ realized by c_j. Construct $C_j\ \epsilon\ S_{s+1}(L(\mathcal{B}))$ from q_j just as C is constructed from p. Then as in Lemma 12 (ii) any $\mathcal{B}' \succ \mathcal{B}$ which realizes q, must contain a set of s elements in $D(\mathcal{B}')$ which is algebraically independent in $L(\mathcal{B})$. So q_j is first realized in $\mathcal{B}_s$. Since $s \geq n$ by Lemma 13 $q_j\ \epsilon\ S^n(\mathcal{B})$. As in Lemma 13 infinitely many of the q_j are distinct. Hence $p\ \epsilon\ S^{n+1}(\mathcal{A})$.

Lemma 16. If T is $\aleph_1$ categorical and for some model of T, $p\ \epsilon\ S(\mathcal{A})$ is first realized in $\mathcal{A}_m$ implies $p\ \epsilon\ Tr^m(\mathcal{A})$ for every m, then T is almost strongly minimal.

Proof. Since T is $\aleph_1$ categorical, by Lemma 8 there is a principal extension T' of T in a language L' with a formula $D\ \epsilon\ S_1(L')$ which is strongly minimal in T'. For each model of T let $\mathcal{A}'$ be the natural expansion of $\mathcal{A}$ to an L'-structure. Let $\mathcal{A}$ be as in the hypothesis. We may assume for contradiction that there is an elementary extension $\mathcal{B}'$ of $\mathcal{A}'$ and an element $b\ \epsilon\ cl(D(\mathcal{B}') \cup |\mathcal{A}|)$. Let p be the element of

$S(\mathfrak{A})$ realized by b. By Lemma 12 (i) p is first realized in $\mathfrak{A}_n$ for some n. Choose C as in the proof of Lemma 14. If $C(v_0, d_1, \ldots d_n)(\mathfrak{A}'_n)$ is finite, mapping $\mathfrak{A}'_n$ into $\mathfrak{B}'$ yields a contradiction. Hence p satisfies the hypothesis of Lemma 15. Thus $p \in S^{n+1}(\mathfrak{A}')$. But p is realized in $\mathfrak{A}'_n$ so by hypothesis $p \in Tr^n(\mathfrak{A}')$. From this contradiction we conclude that for each $\mathfrak{B}' \succ \mathfrak{A}'$ and each $b \in |\mathfrak{B}'|$ there are natural numbers n and k, a formula $A_b \in S_{n+1}(L(\mathfrak{A}'))$ and elements $d_1, \ldots, d_n \in D(\mathfrak{B}')$ such that $\mathfrak{B}' \models A_b(d_1, \ldots, d_n, k) \wedge \exists !^k v_0 A_b(d_1, \ldots, d_n, v_0)$. But then by the compactness theorem $Th(\mathfrak{A}', |\mathfrak{A}'|)$ models

$$\forall v_0\ \exists v_1, \ldots, \exists v_N \Big(\bigvee_{i=1}^{p} (A_i(v_1, \ldots, v_N, v_0) \wedge \exists^{\le k} v_0 A_i(v_1, \ldots, v_N, v_0)) \wedge \bigwedge_{1}^{N} D(v_i)\Big)$$

for some natural numbers N and k. But then if $\mathfrak{B}$ is a prime model of T, $\mathfrak{B}$ must model that sentence with the constants from $|\mathfrak{A}|$ replaced by names of members of $|\mathfrak{B}|$. If the set of these constants is X, $Th(\mathfrak{B}, X)$ is the required principal extension of T.

Let T be an $\aleph_1$-categorical theory, $\mathfrak{A}$ a model of T and p an element of $S(\mathfrak{A})$. Let T' be a principal extension of T with strongly minimal formula D. For each model $\mathfrak{B}$ of T let $\mathfrak{B}'$ be the natural expansion of $\mathfrak{B}$ to a model of T. Then p is naturally a member of $S(\mathfrak{A}')$. Moreover p is first realized in $\mathfrak{A}_m$ if and only if as a member of $S(\mathfrak{A}')$ it is first realized in $\mathfrak{A}'_m$. Furthermore for each α $S^{\alpha}(\mathfrak{A}) = S^{\alpha}(\mathfrak{A}')$.

With this observation we can collect the preceding results as

Theorem 6. Let T be an $\aleph_1$-categorical theory. The following are equivalent:

(i) T is almost strongly minimal

(ii) For some model $\mathfrak{A}$ of T and every natural number m, if p is first realized in $\mathfrak{A}_m$ then $p \in Tr^m(\mathfrak{A})$.

(iii) For every model $\mathfrak{A}$ of T and every natural number m $p \in Tr^m(\mathfrak{A})$ if and only if p is first realized in $\mathfrak{A}_m$.

By Lemma 12 and Theorem 6 (iii) if T is almost strongly minimal α_T is finite. From Theorem 6 we can further deduce the following partial converse to Theorem 3.

Corollary. If T is $\aleph_1$-categorical and $\alpha_T = 2$ then T is almost strongly minimal.

Proof. $\alpha_T = 2$ implies that for each model $\mathfrak{A}$ of T each $p \in S(\mathfrak{A})$ is in $Tr^0(\mathfrak{A})$ or $Tr^1(\mathfrak{A})$. Thus for each natural number n, if $p \in Tr^n(\mathfrak{A})$ then p is first realized in $\mathfrak{A}_n$. Hence by Theorem 6, T is almost strongly minimal.

The following construction shows that for each positive integer n there is an almost strongly minimal theory T with $\alpha_T = n$.

Let T be a totally transcendental theory in a first order language L which has only relation symbols and constants. For each positive integer k we construct a theory $T^{(k)}$. The language $L^{(k)}$ is formed by adjoining to L a unary relation symbol p and binary relation symbols $f_1, \ldots, f_k$. Let $\mathcal{A}$ be a prime model of T. We will construct a prime model $\mathcal{B}$ for $T^{(k)}$. Let $|\mathcal{B}| = |\mathcal{A}| \cup |\mathcal{A}|^k$. Without loss we may assume $|\mathcal{A}| \cap |\mathcal{A}|^k = \emptyset$. Following Shoenfield [20; p. 18[if q is a symbol in L and $\mathcal{A}$ is an L-structure $(q)_{\mathcal{A}}$ denotes the interpretation of q in $\mathcal{A}$. Define $(p)_{\mathcal{B}}$ to be $|\mathcal{A}|$. Each $(f_i)_{\mathcal{B}}$ is the graph of the ith coordinate function mapping $|\mathcal{A}|^k$ onto $|\mathcal{A}|$. If q is an n-ary relation symbol in L then $(q)_{\mathcal{B}} = (q)_{\mathcal{A}}$. If c is a constant in L $(c)_{\mathcal{B}} = (c)_{\mathcal{A}}$. Let $T^{(k)}$ be Th($\mathcal{B}$). Then $\mathcal{B}$ is a prime model of $T^{(k)}$ and it is easy to verify

Lemma 17. If k is a positive integer and T is totally transcendental

(i) $T^{(k)}$ is a totally transcendental theory.

(ii) If T is $\aleph_1$ categorical so is $T^{(k)}$

(iii) If T is a strongly minimal theory then for each k $T^{(k)}$ is an almost strongly minimal but not a strongly minimal theory. Moreover $\alpha_{T^{(k)}} = k + 1$.

Applying Lemma 17 (iii) with k = 1 shows that the corollary to Theorem 6 could not be strengthened to: if T is $\aleph_1$ categorical and $\alpha_T = 2$ then T is strongly minimal.

In each of the particular $\aleph_1$-categorical theories T that we have considered so far there has been a strongly minimal formula. This is not always the case. There is no strongly minimal formula in the theory of algebraically closed projective planes of characteristic zero formulated in a language containing unary relations picking out points and lines and the incidence relation. But if a principal extension is formed naming a line and two points not on the line, not only is this set strongly minimal but its closure is the entire model.

We now exhibit an $\aleph_1$-but not $\aleph_0$-categorical theory $T^{\#}$ which is not almost strongly minimal. For convenience in presenting the next set of examples we will consider languages which have function symbols. We first define a structure for the language L whose only nonlogical symbol is a ternary function symbol f: let Q denote the set of rationals and let $|\mathfrak{A}|$ consist of all pairs $(q,1-q)$, $q \in Q$. Define $F = (f)_{\mathfrak{A}}$ by:

$$F(\phi,\Psi,x) = -\phi + \Psi + x;\ \phi,\ \Psi,\ \chi \in |\mathfrak{A}|$$

where addition in $|\mathfrak{A}|$ is point-wise. We show that <u>every n-ary relation definable in $\mathfrak{A}$ is a Boolean combination of relations R of the form</u>:

(1) $R(\xi_1, \ldots, \xi_n) \leftrightarrow q_1 \xi_1 + \ldots + q_n \xi_n = 0$,

where $q_1, \ldots, q_n$ are all in Q and $\Sigma_{1 \le j \le n}\, q_j = 0$. We first show that if $A \in S_n(L)$ is an arbitrary atomic formula $s = t$ then the n-ary relation defined by A in $\mathfrak{A}$ has the form (1). This is easily accomplished by showing that for any term s containing at most the variables $z_1, \ldots, z_n$ there exist $s_1, \ldots s_n \in Q$ with $\Sigma_{1 \le j \le n}\, s_j = 1$, such that if i_j is the name of a_j for $1 \le j \le n$ then

$$(s[i_1, \ldots, i_n]) = \Sigma_{1 \le j \le n}\, s_j \alpha_j .$$

For the rest suppose that the relation defined in $\mathfrak{A}$ by B is a

Boolean combination of relations of the form (1) for all B in L with length less than that of A. It is sufficient to show that A defines the same kind of relation. The only case which is not obvious is that in which A has the form $\exists z B$. Without loss assume that $B \in s_{n+1}(L)$ and that z is z_{n+1}. From the induction hypothesis, using disjunctive normal form, we may suppose that B defines an (n+1) -any relation S of the form:

$$(2) \quad S(\xi_1, \ldots, \xi_{n+1}) \leftrightarrow \bigvee_{1 \le r \le m} [\,\&_{1 \le s \le m_r} \{q_{r,s,1}\xi_1 + \ldots + q_{r,s,n+1}\xi_{n+1} = 0\}$$

$$\&_{1 \le s \le m'_r} \{q'_{r,s,1}\xi_1 + \ldots + q'_{r,s,n+1} \neq 0\}],$$

where for each pair (r,s) we have $\Sigma_{1 \le j \le n+1} q_{r,s,j} =$ $\Sigma_{1 \le j \le n+1} q'_{r,s,j} = 0$. We have to show that the n-ary relation T on $|\mathfrak{a}|$ defined by :

$$T(\xi_1, \ldots, \xi_n) \leftrightarrow \exists \xi_{n+1}\, S(\xi_1, \ldots, \xi_n, \xi_{n+1})$$

is a Boolean combination of relations of the form (1). Since $\exists$ and $\vee$ commute we need only examine (2) when m = 1, i.e. we may suppose that

$$(3) \quad S(\xi_1, \ldots, \xi_{n+1}) \leftrightarrow :$$

$$\&_{1 \le s \le m} (q_{s,1}\xi_1 + \ldots + q_{s,n+1}\xi_{n+1} = 0)\ .\&.$$

$$\&_{1 \le s \le m'} (q'_{s,1}\xi_1 + \ldots + q'_{s,n+1}\xi_{n+1} \neq 0).$$

If $q_{s,n+1} = 0$ for each s there is nothing to prove, because $T(\xi_1, \ldots, \xi_n)$ is clearly equivalent to the r.h.s. of (3) with all the conjuncts in which $q'_{s,n+1} \neq 0$ deleted. Otherwise we may suppose that $q_{1,n+1} \neq 0$ in which case:

$$T(\xi_1, \ldots, \xi_n) \leftrightarrow \exists\xi_{n+1} [\xi_{n+1} = (-q_{1,1}\xi_1 - \ldots - q_{1,n}\xi_n)/q_{1,n+1}$$

$$.\&.\ \underset{2 \le s \le m}{\&} (q_{s,1}\xi_1 + \ldots + q_{s,n+1}\xi_{n+1} = 0)\ .\&.$$

$$\underset{1 \le s \le m'}{\&} (q'_{s,1}\xi_1 + \ldots + q'_{s,n+1}\xi_{n+1} \ne 0)]$$

$$\leftrightarrow :\ \underset{2 \le s \le m}{\&} (\{q_{s,1} - (q_{1,1}q_{s,n+1}/q_{1,n+1})\}\xi_1 + \ldots$$

$$+\{q_{1,n}q_{s,n+1}/q_{1,n+1})\}\xi_n = 0)\ .\&.$$

$$\underset{1 \le s \le m'}{\&} (\{q'_{s,1} - (q_{1,1}q'_{s,n+1}/q_{1,n+1})\}\xi_1 + \ldots$$

$$+\{q'_{s,n} - (q_{1,n}q'_{s,n+1}/q_{1,n+1})\}\xi_n \ne 0).$$

Since $\Sigma_{1 \le j \le n}(q_{1,j}/q_{1,n+1}) = -1$, in each conjunct the sum of the coefficients is zero so the induction is complete.

From (2) we now see that if $B \in S_{n+1}(L)$ then for some positive integer k the formula

$$\exists^{\le k} z_{n+1}\ B\ .v.\ \exists^{\le k} z_{n+1} \sim B$$

is valid in $\mathcal{A}$. In fact we can take $k = \max\{m, m'_1, \ldots, m'_m\}$. This proves that $|\mathcal{A}|$ is strongly minimal in $\mathcal{A}$, and hence by Theorem 1 that Th($\mathcal{A}$) is $\aleph_1$-categorical. Since $|\mathcal{A}|$ has dimension two Th($\mathcal{A}$) is not $\aleph_0$-categorical. It is essential to the construction of $T^{\#}$ below that $\mathcal{A}$ have the following properties: for any $\phi, \Psi \in |\mathcal{A}|$ $\lambda\xi F(\phi, \Psi, \xi)$ is an automorphism of $\mathcal{A}$ mapping ϕ into Ψ, and if $\lambda\xi F(\phi, \Psi, \xi)$, $\lambda\xi F(\phi', \Psi', \xi)$ agree at one point they are identical. These properties may be easily verified from the definition of F given above. Their necessity below is implicit rather than explicit.

Let $L^{\#}$, the language of $T^{\#}$, have as its nonlogical symbols a binary predicate symbol q and a ternary function symbol f. The axioms of $T^{\#}$ are to be such that if $\mathcal{B}$ is any model of $T^{\#}$ then the following conditions are satisfied:

(i) Let $C = \{b \mid b \varepsilon |\mathcal{B}| \;\&\; \mathcal{B}(\exists v_0 q b v_0) = T\}$ and $C_b = \{c \mid c \varepsilon |\mathcal{B}| \;\&\; \mathcal{B}(q\ b\ c) = T\}$ then $C \neq \emptyset$ and $C_b = \emptyset$ just if $b \notin C$.

Also,

$C \cap \bigcup \{C_b \mid b \varepsilon C\} = \emptyset;\ C_b \cap C_{b'} = \emptyset \quad b, b' \varepsilon C.$

(ii) There is a substructure $\mathcal{C}$ of $\mathcal{B}$ such that $|\mathcal{C}| = C$ and $\mathcal{C}|L \equiv \mathcal{A}$, and for each $b \varepsilon C$ there is a substructure $\mathcal{C}_b$ of $\mathcal{B}$ such that $|\mathcal{C}_b| = C_b$ and $\mathcal{C}_b|L \equiv \mathcal{A}$.

(iii) If $b \varepsilon C$ and $c \varepsilon C_b$ then $\lambda x f_{\mathcal{B}}(b, c, x)|C$ is an isomorphism of $\mathcal{C}|L$ and $\mathcal{C}_b|L$ taking b into c. Further if $c' \varepsilon C_b$ then

(4) $f_{\mathcal{B}}(b, c', x) = f_{\mathcal{B}}(c, c', f_{\mathcal{B}}(b, c, x)),\ x \varepsilon C.$

(iv) For all $x, y, z \varepsilon |\mathcal{B}|$ $(x, y, z) \notin$

$$C^3 \cup \bigcup\{C_b^3 \mid b \varepsilon C\} \cup \bigcup\{\{b\} \times C_b \times C \mid b \varepsilon C\}$$

implies $f_{\mathcal{B}}(x, y, z) = z$.

Since (i) - (iv) are all elementary properties, to show that the axioms of $T^{\#}$ can be chosen it's sufficient to exhibit $\mathcal{B}$ satisfying (i) - (iv). Let $|\mathcal{B}| = |\mathcal{A}| \cup (|\mathcal{A}| \times |\mathcal{A}|)$ and define

$$q_{\mathcal{B}}(x, y) \leftrightarrow : x \varepsilon |\mathcal{A}| \;\&\; y \varepsilon \{x\} \times |\mathcal{A}|.$$

Further, $f_{\mathcal{B}}$ is to be the unique function satisfying (iv) and

$$f_{\mathcal{B}} \restriction |\mathcal{A}| = F, \quad f_{\mathcal{B}}(\phi,(\phi,\Psi),\chi) = (\phi,F(\phi,\Psi,\chi))$$

$$f_{\mathcal{B}}((\phi',\phi), (\phi',\Psi), (\phi',\chi)) = (\phi',F(\phi,\Psi,\chi)).$$

for all ϕ, ϕ', Ψ, $\chi \in |\mathcal{A}|$. The reader will easily verify that (i) - (iv) are satisfied.

To show that $T^{\#}$ is $\aleph_1$-categorical consider two models $\mathcal{B}^0$, $\mathcal{B}^1$ of $T^{\#}$ both of power $\aleph_1$ whose universes are disjoint. We carry over all the notation developed for $\mathcal{B}$ adding appropriate superscripts. Since for $j = 0, 1$

$$|\mathcal{B}^j| = C^j \cup \bigcup\{C_b^j \mid b \in C^j\}$$

and since $\kappa(C^j) = \kappa(C_b^j)$ for each $b \in C^j$, we have $\kappa(C^j) = \aleph_1$. Hence $C^0|L$, $C^1|L$ are isomorphic, by G say, since both are models of Th($\mathcal{A}$). For $j = 0, 1$ and each $b \in C^j$ choose $c_b \in C_b^j$. We extend G to all of $|\mathcal{B}^0|$ as follows. Recall from (iii) that for $b \in C^j$, $G_b^j = \lambda x f_{\mathcal{B}^j}(b, c_b, x) \mid C^j$ is an f-isomorphism of C^j and C_b^j. For any $c \in |\mathcal{B}^0| - C^0$ let b be the unique member of C^0 such that $c \in C_b^0$, define

$G(c) = G^1_{G(b)}\, G(G_b^0)^{-1}(c)$. Clearly $G(C_b^0) = C^1_{G(b)}$ for each $b \in C^0$, whence G is a q-isomorphism of $\mathcal{B}^0$ and $\mathcal{B}^1$. To show that G is an f-isomorphism we must show that

(t) $G(f_{\mathcal{B}^0}(x, y, z)) = f_{\mathcal{B}^1}(G(x), G(y), G(z))$.

This is immediate from the choice of G if $x, y, z \in C^0$ or if for some $b \in C_y^0$ we have $x, y, z \in C_b^0$. Suppose that $x = b \in C^0$, $y \in C_b^0$, and $z \in C^0$, then using our last remark and (4) we have:

$$\begin{aligned} G(f_{\mathcal{B}^0}(x, y, z)) &= G(f_{\mathcal{B}^0}(c_b, y, f_{\mathcal{B}^0}(b, c_b, z))) \\ &= f_{\mathcal{B}^1}(G(c_b), G(y), G(f_{\mathcal{B}^0}(b,c_b,z))) \\ &= f_{\mathcal{B}^1}(G^1_{G(b)}G(G^0_b)^{-1}(c_b), G(y), \\ &\qquad G^1_{G(b)}G(G^0_b)^{-1}(G^0_b(z))) \\ &= f_{\mathcal{B}^1}(c_{G(b)}, G(y), f_{\mathcal{B}^1}(G(b), c_{G(b)}, G(z)) \\ &= f_{\mathcal{B}^1}(G(b), G(y), G(z)). \end{aligned}$$

There only remains the case in which (x, y, z) satisfies (iv) with respect to $\mathcal{B}^0$, in which case $f_{\mathcal{B}^0}(x, y, z) = z$. But since G is a q-isomorphism $(G(x), G(y), G(z))$ satisfies (iv) with respect to $\mathcal{B}^1$, whence $f_{\mathcal{B}^1}(G(x), G(y), G(z)) = G(z)$. This completes the proof of (5) and shows that $T^{\#}$ is indeed an $\aleph_1$-categorical theory.

The particular model $\mathcal{B}$ of $T^{\#}$ constructed above is prime, because it has no proper elementary submodels. This follows immediately when one observes that the strongly minimal subset $|\mathcal{A}|$ of $\mathcal{B}$ has dimension two, and that for any a in $|\mathcal{A}|$, $cl(\{a\}) = \{a\}$. By naming a point in $|\mathcal{A}|$ we obtain a structure $(\mathcal{B}, \{a\})$ which has two strongly minimal sets $|\mathcal{A}|$ and C_a of different dimensions.

In [10] Morley conjectured that if T, Σ characterize $\lambda < \aleph_0$ then T has infinitely many algebraic types, (i.e. types realized by only a finite number of points). T, Σ characterize λ if Σ is a set of 1-ary formulas consistent with T and there is a model of T in each cardinal less than λ which omits Σ but every model of T with power λ realizes Σ. Several people, including Shelah [16] have indicated the conjecture is false in general. However we may note the following special case. If T is $\aleph_1$-categorical and T, Σ characterize $\aleph_1$ then T has a principal extension T" with infinitely many algebraic types. For, let T' be a principal extension of T with strongly minimal formula D and let $\mathcal{A}'$ be a prime model of T'. Since T, Σ characterize $\aleph_1$ T and hence T' is not $\aleph_0$ categorical. Hence D($\mathcal{A}'$) has finite dimension. Let T" be Th($\mathcal{A}$, $\mathfrak{X}$) where $\mathfrak{X}$ is a basis for D($\mathcal{A}'$). Lachlan has pointed out that the weakening of the conclusion to allow a principal extension is necessary with the following example.

Let $\mathcal{A}$ be the structure we constructed above whose language contained a single unary function symbol f and whose universe was $\{(q, 1-q) \mid q \in Q\}$. Define a structure $\mathcal{B}$ by letting $|\mathcal{B}| = |\mathcal{A}| \times |\mathcal{A}|$, and interpret two nonlogical symbols over this set: f again a ternary function symbol, and p a binary predicate symbol. We define $p_{\mathcal{B}}$ by $p_{\mathcal{B}}((a_0, a_1), (a_2, a_3)) \leftrightarrow a_0 = a_2$, and $f_{\mathcal{B}}$ by $f_{\mathcal{B}}((a_0, a_0'), (a_1, a_1'), (a_2, a_2')) =$

$$\begin{cases} (a_0, F(a_0', a_1', a_2') & \text{if } a_0 = a_1 = a_2, \\ (a_1, F(a_1, a_1', a_2)) & \text{otherwise.} \end{cases}$$

Intuitively $\mathcal{B}$ is obtained by replacing each point in $\mathcal{A}$ by a copy of $\mathcal{A}$. Th($\mathcal{B}$) is $\aleph_1$-categorical.

Let $\mathcal{B}'$ be obtained from $\mathcal{B}$ by adjoining unary relations naming two equivalence classes of p. $\mathcal{B}'$ is $\aleph_1$-categorical. There are infinitely many 1-types in Th($\mathcal{B}'$) so Th($\mathcal{B}'$) is not $\aleph_0$-categorical. But there are only two algebraic types.

§ 5 Proof of Vaught's Conjecture

In this section we prove Vaught's conjecture that modulo isomorphism an $\aleph_1$-categorical theory has either just one or exactly $\aleph_0$ models of power $\aleph_0$. It was shown in Morley [8] that the number of isomorphism types of countable models is always $\leq \aleph_0$ for $\aleph_1$-categorical theories. Thus we have only to demonstrate the impossibility of an $\aleph_1$-but not $\aleph_0$-categorical theory having only a finite number of isomorphism types of countable models. In this endeavor we rely on the results of Sections 1 and 3 and Lemma 17 from Section 4.

Let T be an $\aleph_1$-but not $\aleph_0$-categorical theory in a countable first order language L. Let L' be an extension of L by constants $c^1, \ldots, c^m$ and suppose T' is a principal extension of T in L' which has a strongly minimal formula D'. Then D' has the form $D(v_0, c^1, \ldots, c^m)$ where $D \in S_{m+1}(L)$.

If $\mathcal{A}_0'$ is a prime model of T' then $\dim(D'(\mathcal{A}_0'))$ is finite for otherwise by Lemma 10 T' and hence T is $\aleph_0$-categorical. By forming a principal extension of T' if necessary we may assume $\dim(D'(\mathcal{A}_0')) = 0$. For each positive integer n let $\mathcal{A}_n'$ be a minimal prime extension of $\mathcal{A}'_{n-1}$ and let $\mathcal{A}_n = \mathcal{A}_n'|L$. By Lemma 10 for each n $\dim(D'(\mathcal{A}_n')) = n$. An unsaturated model $\mathcal{B}$ of T can be expanded to an unsaturated model $\mathcal{B}'$ of T'. Clearly $\mathcal{B}' \cong \mathcal{A}_n'$

for some n whence $\mathcal{B} \cong \mathcal{A}_n$. Conversely each $\mathcal{A}_n$ is an unsaturated model of T.

We now assume that T is a counter example to Vaught's conjecture. Thus we may choose N such that every unsaturated model of T is isomorphic to one of $\mathcal{A}_0$, $\mathcal{A}_1, \ldots \mathcal{A}_N$. Clearly for each n there exists an elementary embedding f_n of $\mathcal{A}_n$ into $\mathcal{A}_N$. For each n let X_n be a basis for $D'(\mathcal{A}'_n)$ and let $Y_n = \{(c^1)_{\mathcal{A}'_n}, \ldots, (c^m)_{\mathcal{A}'_n}\}$. Fix n and let M be an arbitrary positive integer. When we write "cl" we shall mean closure with respect to L rather than L'. We show that $f_M(X_M) \subseteq cl(Z)$ where $Z = X_N \cup Y_N \cup f_M(Y_M)$. Suppose for contradiction that $x \in f_M(X_M) - cl(Z)$. Notice that $\mathcal{A}_N$ is prime over Z because some elementary submodel C of $\mathcal{A}_N$ is prime over Z and if $|C| \neq |\mathcal{A}_N|$ then T' satisfies the hypothesis of the two cardinal theorem. Hence the type p in S(Z) realized by x in $\mathcal{A}_N$ is a principal type. If c^i_M is the name of $f_M((c^i)_{\mathcal{A}'_M})$ in $\mathcal{A}_N$, i.e. if $f_M((c^i)_{\mathcal{A}'_M}) = (c^i_M)_{\mathcal{A}_N}$ then then clearly $D(v_0, c'_M, \ldots, c^m_M)$ is in p. Let X be **the set** of all **members of** $|\mathcal{A}_N|$ which realize p. Then X is infinite since $x \notin cl(Z)$ and $X \subseteq D(v_0, c^1_M, \ldots c^m_M)(\mathcal{A}_N)$. Recall that by the choice of T' $cl(Y_M) \supseteq D'(\mathcal{A}'_0)$. Hence $D(v_0, c^1_M, \ldots, c^m_M)(\mathcal{A}_N) - X$, being a superset of $f_M(D'(\mathcal{A}'_0))$, is infinite. Consider the expansion $\mathcal{A}''_N$ of $\mathcal{A}_N$ obtained by letting $(c^i)_{\mathcal{A}''_N} = (c^i_M)_{\mathcal{A}_N}$ for $1 \leq i \leq m$. Clearly, $\mathcal{A}''_N$ is a model of T', X is definable in $\mathcal{A}''_N$, and both $D'(\mathcal{A}''_N) \cap X$ and $D'(\mathcal{A}''_N) - X$ are infinite. This contradicts the strong minimality of D' in T'; hence $f_M(X_M) \subseteq cl(Z)$. Now $f_M(X_M)$ is an independent subset of $D'(\mathcal{A}''_N)$ and the members of $f_M(Y_M)$ are named in $\mathcal{A}''_N$. Hence, now with respect to L', $cl(X_N \cup Y_N)$ contains an independent subset of $D'(\mathcal{A}''_N)$ of power M. Note that the cardinality of $X_N \cup Y_N$ is m + N and does not depend on M.

Let k = m + N and apply the construction of Lemma 17 to T'. Let $T'^{(k)}$ be T". Let $\mathcal{B}$ be a model of T" such that $p(\mathcal{B})$ is isomorphic to $\mathcal{A}''_N$ by an isomorphism g. Let b be the element of $\sim p(\mathcal{B})$ whose components are the elements in $p(\mathcal{B})$ which are identified with the elements of $X_N \cup Y_N$ by g. Then in $\mathcal{B}$, $cl(\{b\}) \supseteq cl(g(X_N \cup Y_N))$ contains an independent subset of $D''(\mathcal{B})$ of power M where D" is the relativization of D' to p. Thus we have established

Proposition. If Vaught's conjecture is false there exists an $\aleph_1$-but not $\aleph_0$-categorical theory T" and a formula D" strongly minimal in T" such that for every M there exists a model $\mathcal{B}$ of T" and $b \in |\mathcal{B}|$ with $cl(\{b\})$ containing an independent subset of $D''(\mathcal{B})$ with M elements while $b \notin D(\mathcal{B})$.

We now show no theory T" can satisfy the conclusion of this proposition. We again exploit Vaught's two cardinal theorem. Let T be a complete theory in a first order language L and p a unary predicate symbol in L such that if $\mathcal{A}$ is a model of T $p(\mathcal{A})$ is infinite. Let T be extended to T' in a language L' by adjunction of a new constant c but no new axioms. Introduce special (Henkin) constants one for each closed instantiation exactly as in [20], Chapter 4. Let A, B, B' be the least subsets of the set of Henkin constants satisfying the following conditions where C denotes A ∪ B ∪ B':

(i) If $\exists v_0 A$ is closed and contains only symbols from L' and C **and if A has the form**

$$pv_0 \;.\wedge.\; \exists v_0(pv_0 \wedge B) \rightarrow B$$

then the constant for $\exists v_0 A$ is in A.

(ii) If $\exists v_0 A$ is closed and contains at most symbols from L and A∪B, and if A has the form

$$\sim pv_0 \;.\wedge.\; \exists v_0 (\sim pv_0 \wedge B) \rightarrow B$$

then the constant for $\exists v_0 A$ is in B.

(iii) If $\exists v_0 A$ is closed and contains at most symbols from L' and C, and if A has the form

$$\sim pv_0 \;.\wedge.\; \exists v_0 (\sim pv_0 \wedge B) \rightarrow B$$

then the constant for $\exists v_0 A$ is in B' if it is not in B.

It is easy to check that A, B, B' are indeed well defined by redefining them in terms of the "levels" where the constants are introduced. Let Γ be the set of nonlogical axioms of T. Let Δ be the special axioms for all the special constants in C. Let Θ be the set of formulas

$$\Gamma \cup \Delta \cup \{\sim pc\} \cup \{\sim c = r \mid r \in B\}.$$

Suppose that T(Θ), (the set of logical consequences of Θ) is consistent and C* is a model of T(Θ). It is easily shown that there are substructures $\mathcal{B}$, $\mathcal{B}'$ of C such that

$|\mathcal{B}| = \{(r)_{C*} \mid r \in A \cup B\}$ and $|\mathcal{B}'| = \{(r)_{C*} \mid r \in A \cup B'\}$. It is further easily shown that $\mathcal{B} \prec \mathcal{B}' \prec \mathcal{C}$, that $(c)_{\mathcal{C}} \in |\mathcal{B}'| - |\mathcal{B}|$, and that $p(\mathcal{B}) = p(\mathcal{B}') = \{(r)_{\mathcal{C}} \mid r \in A\}$. Thus if $T(\theta)$ is consistent T satisfies the hypothesis of the two cardinal theorem. The converse is almost immediate so we have shown

Lemma 18. Let T be a complete theory in a language L and p a unary predicate symbol in L such that $p(\mathcal{A})$ is infinite for each model $\mathcal{A}$ of T. Then T satisfies the hypothesis of the two cardinal theorem with p for D if and only if $T(\theta)$ is consistent.

Below we shall apply lemma 18 to an $\aleph_1$-categorical theory T in which p is strongly minimal. Before we can do this we need the following technical lemma.

Lemma 19. Let T be a complete theory in a first order language L. Suppose D is in $S_1(L)$ and D is strongly minimal in T. Let $\mathcal{A}$ be a model of T, $a_1, \ldots, a_n \in |\mathcal{A}|$, and $X \subseteq D(\mathcal{A})$ such that $\langle a_1, \ldots, a_n \rangle$ realizes a principal type in $Th(\mathcal{A}, X)$. If $X \subseteq X' \subseteq D(\mathcal{A})$, $X' - X$ is finite, and $D(\mathcal{A}) \cap cl(\emptyset)$ is infinite then the type in $Th(\mathcal{A}, X')$ which is realized by $\langle a_1, \ldots, a_n \rangle$ is also a principal type.

Proof. By adjoining names it suffices to consider the case where X is empty. By induction we may further suppose $X' - X$ is a singleton, say $\{a\}$. There are two cases depending on whether or not $a \in cl(\emptyset)$.

Suppose $a \notin cl(\emptyset)$ and let $A \in S_n(L)$ generate the type in T realized by $< a_1, \ldots, a_n >$ in $\mathfrak{A}$. We claim that A also generates the type in $Th(\mathfrak{A}, \{a\})$ realized by $< a_1, \ldots, a_n >$. If not, the theory T' obtained by adjoining to $L(\mathfrak{A})$ the constants $a_1, \ldots a_n$ and extending T by the nonlogical axiom $A(a_1, \ldots, a_n)$ is incomplete. Thus there exists a formula $B \in S_{n+1}(L)$ such that $B(a, a_1, \ldots, a_n)$ is neither provable nor refutable in T'. Without loss of generality assume $B(v_0, a_1, \ldots, a_n) \wedge D(\mathfrak{A})$ is infinite. $D(\mathfrak{A}) \cap cl(\emptyset)$ is a disjoint union of sets $A_i(\mathfrak{A})$, each of which is finite and minimal in the sense that there is no formula $C \in S_1(L)$ such that $(C \wedge A_i)(\mathfrak{A})$ is a proper nonempty subset of $A_i(\mathfrak{A})$. For each formula A_i either

(i) $\vdash_{T'} \forall v_0 \forall v_1, \ldots, \forall v_n$

$$((A(v_1, \ldots, v_n) \wedge A_i(v_0)) \rightarrow B(v_0, v_1, \ldots, v_n) \wedge D(v_0))$$

or

(ii) $\vdash_{T'} \forall v_1, \ldots, \forall v_n \; (A(v_1, \ldots, v_n) \rightarrow$

$$\exists v_0 \; (A_i(v_0) \wedge \sim B(v_0, v_1, \ldots, v_n) \wedge D(v_0)))$$

Since $(B(v_0, a_1, \ldots, a_n) \wedge D)(\mathfrak{A})$ is infinite and $D(\mathfrak{A})$ is strongly minimal there can be only finitely many A_i such that (ii) holds. Thus for each k

$$\vdash_{T'} \exists^{\geq k} v_0 \; \forall v_1, \ldots \forall v_n \; (A(v_1, \ldots, v_n) \rightarrow B \wedge D).$$

Since D is strongly minimal, for some m

$\vdash_{T'} \exists^{\leq m} v_0 \exists v_1, \ldots, \exists v_n (A(v_1, \ldots, v_n) \wedge \sim B \wedge D)$.
Since $a \notin cl(\varphi)$ by assumption $A(a_1, \ldots a_n) \wedge$
$\sim B(a, a_1, \ldots, a_n) \wedge D(a)$ is refutable in T', whence
$\vdash_{T'} B(a, a_1, \ldots, a_n)$.

Now suppose that $a \in cl(\emptyset)$ and choose $C \in S_{n+1}(L)$ such that $\mathcal{C}(v_0, a_1, \ldots, a_n)(\mathfrak{A}) = C$ and is the smallest definable subset of $|\mathfrak{A}|$ containing a. Clearly C is finite and C is contained in $D(\mathfrak{A})$. Extend L to L' by adjoining constants $a, a_1, \ldots, a_n$ to L. Form T' by adding the nonlogical axiom $A(a_1, \ldots, a_n) \wedge C(a, a_1, \ldots a_n)$ to $Th(\mathfrak{A}, \{a\})$. It suffices to show T' is complete. To show this let $B(a, a_1, \ldots, a_n)$ be an arbitrary sentence in L' where $B \in S_{n+1}(L)$. Let E_0, E_1, E_2 be the formulas

$$A(v_1, \ldots v_n) \rightarrow \exists v_0 (C \wedge B) \wedge \exists v_0 (C \wedge \sim B)$$

$$A(v_1, \ldots, v_n) \rightarrow . \quad C \rightarrow B$$

$$A(v_1, \ldots, v_n) \rightarrow C \rightarrow \sim B$$

Since $A(v_1, \ldots v_n)$ generates a principal type in T one of E_0, E_1, E_2 is a theorem of T'. Now C is the least subset of $|\mathfrak{A}|$ definable in $(\mathfrak{A}, \{a_1, \ldots a_n\})$ which contains a. Hence E_0 cannot be a theorem of T'. Thus either E_1 or E_2 is a theorem of T' so either $B(a, a_1, \ldots, a_n)$ or its negation is a theorem of T' so T' is complete. This proves the lemma.

We are now in a position to apply Lemma 18 to an $\aleph_1$-categorical theory which has a strongly minimal formula.

Lemma 20. Let T be a complete $\aleph_1$-categorical theory in which pv_0 is strongly minimal. There exists a positive integer N such that if $\mathcal{B}$ is a model of T, b ε $|\mathcal{B}|$, and $D(\mathcal{B}) \cap cl(\emptyset)$ is infinite, then there exists $\mathcal{A} \prec \mathcal{B}$ and $X \subseteq p(\mathcal{A})$ such that b ε $|\mathcal{A}|$, $\kappa(X) \leq N$ and $\mathcal{A}$ is prime over X.

Proof. Assume that T, $\mathcal{B}$, b are given satisfying the hypothesis of the lemma. Assume also that $b \notin p(\mathcal{B})$ because otherwise the lemma is trivial. By Lemma 18 T(Θ) is inconsistent. By the compactness theorem there is a finite subset Δ_0 of Δ such that replacing Δ by Δ_0 in the definition of Θ would leave T(Θ) inconsistent. Choose such a Δ_0 with the additional property that if the special constant r appears in a member of Δ_0 then the special axiom for r is in Δ_0. Let r_1, ... r_N be the special constants occurring in Δ_0 so that if r_i is in a lower level than r_j then $i < j$. We expand $\mathcal{B}$ to C by interpreting c and $r_1, \ldots, r_N$ in $|\mathcal{B}|$ so that all the sentences in Δ_0 are true in C. We first let $(c)_C$ be b and then we choose $(r_i)_C$ for $i = 1, \ldots, N$ in that order. In choosing $(\kappa_j)_C$ when r_j ε A ∪ B' we are concerned only to ensure the truth in C of the special axiom for r_j. However, if r_j ε B we choose $(r_j)_C$ not only to satisfy the special axiom for r_j but also such that the type of $(r_j)_C$ in $Th(\mathcal{B}, Z_j)$ is principal where $Z_j = \{(r_i)_C \mid r_i \in A \cup B \wedge 1 \leq i \leq j\}$. This is possible because, as we pointed out in the proof of Lemma 6, in any model $\mathcal{A}$ of a totally transcendental theory the principal types are dense in $S(\mathcal{A})$. From Lemma 19 it is clear that if we let $X = \{(r_i)_C \mid r_i \in A \wedge 1 \leq i \leq N\}$, $Y = \{(r_i)_C \mid r_i \in B \wedge 1 \leq i \leq N\}$, and if $b_1, \ldots, b_k$ is an enumeration of Y, then the k-type of $b_1, \ldots, b_k$ in $Th(\mathcal{B},X)$ is principal. Let $\mathcal{A} \preccurlyeq \mathcal{B}$ be chosen prime over $X \cup Y$ and $\mathcal{A}' \preccurlyeq \mathcal{B}$ be chosen prime over X. Since principal types are always realized in models there exists $b_1', \ldots, b_k'$ in $|\mathcal{A}'|$ such that $\langle b_1', \ldots, b_k' \rangle$ and $\langle b_1, \ldots, b_k \rangle$ realize the same type in $Th(\mathcal{B},X)$. Since $\mathcal{A}$ is prime over $X \cup Y$ there is an elementary embedding f of $\mathcal{A}$ into $\mathcal{A}'$ which takes b_i to b_i' $1 \leq i \leq k$ and is the identity on X. Hence $\mathcal{A}$ is prime over X since $\mathcal{A}'$ is. Also, since the special axioms in Δ_0 are a sufficient subset of Δ to make $T(\Theta)$ inconsistent not all of the axioms $\sim c = r_i$, $r_i \in B$ and $1 \leq i \leq N$, are true in C. Thus $b \in Y$ and since $Y \subseteq |\mathcal{A}|$ the lemma is proved.

We can now prove Vaught's conjecture.

Theorem 7. If T is an $\aleph_1$-but not $\aleph_0$-categorical theory then there are $\aleph_0$ isomorphism types of models of T of power $\aleph_0$.

Proof. If not, from the proposition proved at the beginning of this section there must be an $\aleph_1$-but not $\aleph_0$-categorical theory T" such that for every M there exists a model $\mathcal{B}$ of T" and $b \in |\mathcal{B}|$ such that $cl(\{b\})$ contains an independent subset of $D''(\mathcal{B})$ of power M, where D" is strongly minimal in T". Without loss we can suppose that D" is pv_0, and that in any model $\mathcal{B}$ of T", $p(\mathcal{B}) \cap cl(\emptyset)$ is infinite. This follows because $\dim(p(\mathcal{B}^0))$ is finite if $\mathcal{B}^0$ is the prime model of T". Now we apply Lemma 20 to T". Suppose the value of N for T" is N". Choose M > N" and corresponding $\mathcal{B}$, b so that cl({b}) contains an independent subset of $p(\mathcal{B})$ of power M. Then there exist $\mathcal{A} \prec \mathcal{B}$ and $X \subseteq p(\mathcal{A})$ such $\kappa(X) \leq N''$, $b \in |\mathcal{A}|$ and $\mathcal{A}$ is prime over X. Since $p(\mathcal{B}) \cap cl(\emptyset)$ is infinite any principal 1-type in $Th(\mathcal{B},X)$ which contains the formula pv_0 is realized by only a finite number of members of $|\mathcal{B}|$. Otherwise pv_0 would not be strongly minimal in T". Thus $p(\mathcal{A}) \subseteq cl(X)$ which shows that $\dim(p(\mathcal{A})) \leq \kappa(X)$. But cl({b}) contains an independent subset of $p(\mathcal{B})$ of power M so $p(\mathcal{A})$ contains an independent subset of power M. Hence $M \leq \kappa(X)$. Since $\kappa(X) \leq N''$ we have contradicted the choice of M.

Theorem 8. If $\mathcal{A}$ is a model of an $\aleph_1$-categorical complete theory T, then $\mathcal{A}$ is homogeneous.

Proof. By Theorem 4 (iv) and (v) it suffices to consider an unsaturated model of T. Let $a_1, \ldots, a_k$ and $a_1', \ldots a_k'$ be elements of $|\mathcal{A}|$ such that $a_1, \ldots, a_k$ and $a_1', \ldots, a_k'$ have the same type in T. We shall show there is an automorphism f of $\mathcal{A}$ such that $f(a_i) = a_i'$ for $1 \leq i \leq k$. Since any model of T of power $\aleph_1$ is saturated it is easy to see that $Th(\mathcal{A}, \{a_1, \ldots a_k\})$ is $\aleph_1$-categorical. Let T* be principal extension of $Th(\mathcal{A}, \{a_1, \ldots, a_k\})$

in which D is a strongly minimal formula. Without loss we may assume that T* comes from Th($\mathcal{A}$, $\{a_1, \ldots, a_k\}$) by adjoining names $a_{k+1}, \ldots, a_{k+m}$ for $a_{k+1}, \ldots, a_{k+m} \in |\mathcal{A}|$ and a nonlogical axiom $A(a_1, \ldots, a_{k+m})$ which is true in $\mathcal{A}$. Let $\mathcal{B} = (\mathcal{A}, \{a_1, \ldots a_{k+m}\})$, then $\mathcal{B}$ is a model of T*. Since principal types are always realized in models we can choose $a'_{k+1}, \ldots, a'_{k+m} \in |\mathcal{A}|$ such that $\mathcal{A} \models A(a'_1, \ldots, a'_{k+m})$. Hence $\mathcal{A}$ can be expanded to a model $\mathcal{B}'$ of T* by letting $(a_i)_{\mathcal{B}'} = a'_i$ for $1 \leq i \leq k+m$. Now let X, X' be bases of $D(\mathcal{B})$, $D(\mathcal{B}')$ respectively. Let f be a 1-1 map of X into X' and onto if possible. Since $\mathcal{B}$ is prime over X by Lemma 10, f can be extended to an elementary embedding g of $\mathcal{B}$ into $\mathcal{B}'$. Now if f is onto and $\mathcal{B}''$ is the image of $\mathcal{B}$ under g then $D(\mathcal{B}'') = D(\mathcal{B})$, whence $|\mathcal{B}''| = |\mathcal{B}|$ by Lemma 7, so g is the required automorphism. If g is not onto $\mathcal{A}$ is isomorphic to a proper elementary submodel of $\mathcal{A}$. Let $\langle C_n \mid n < \omega \rangle$ be a sequence of models of T such that C_0 is a prime model and C_n is a minimal prime extension of C_{n-1}. Let h_0 be an elementary embedding of C_0 in $g(\mathcal{A})$. For each $n > 0$ such that $h_{n-1}(C_{n-1})$ is a proper elementary submodel of $g(\mathcal{A})$ we extend h_{n-1} to an elementary embedding h_n of C_n in $g(\mathcal{A})$. Since $g(\mathcal{A})$ is not saturated by Theorem 4 (v) we have, for some N, $h_N(C_N) = g(\mathcal{A})$. We now extend h_N to an elementary embedding h_{N+1} of C_{N+1} into $\mathcal{A}_1$ and so on. Again by Theorem 4 (v) since $\mathcal{A}$ is unsaturated there must exist $M > N$ such that $h_M(C_M) = \mathcal{A}$. Since $C_M \cong C_N$ it follows that $C_{M+j} \cong C_{N+j}$ for all $j > 0$. That is at most M of $C_0, C_1, \ldots$ are nonisomorphic. But just as we proved that $g(\mathfrak{A})$ is isomorphic to some C_j we can prove each unsaturated model is isomorphic to some C_j. Since modulo isomorphism there is only one countable saturated model it now follows from Theorem 7 that T is $\aleph_0$-categorical. Hence $\mathfrak{A}$ is saturated contrary to hypothesis so g must be onto. This completes the proof.

Notice that the reasoning given is adequate for a stronger result in that we may replace "homogeneous" by "countably homogeneous", i.e. the finite sequences $(a_1, \ldots a_k)$ and $(a_1', \ldots a_k')$ by sequences of length ω. Theorem 8 is more or less equivalent to Theorem 7 since each can be deduced from the other purely on the basis of Marsh's results.

Morley has observed [6; p. 16] that if some 1-type is not realized in a model $\mathfrak{A}$ of an $\aleph_1$-categorical theory then $\mathfrak{A}$ cannot contain an infinite set of indiscernibles. We can sharpen this to: _a model of an $\aleph_1$-categorical theory is saturated if and only if it contains an infinite set of indiscernibles_. The "only if" part is immediate. For the "if" part we continue the line of reasoning begun in the last proof where it was shown that $C_0, C_1, \ldots$ are all distinct and that none of them is isomorphic to a proper submodel of itself. Let $X \subseteq |\mathfrak{A}|$ be an infinite set of indiscernibles. Without loss suppose X is maximal in the sense that there exists no proper superset of X which is a set of indiscernibles. Let X_0 be any proper subset of X with $\kappa(X) = \kappa(X_0)$. Let $\mathcal{B} \preccurlyeq \mathcal{A}$ be prime over X and h a 1-1 map from X onto X_0. Since $\mathcal{B}$ is prime over X h can be extended to an elementary embedding h' of $\mathcal{B}$ in itself. Since X is maximal $h'(|\mathcal{B}|) \cap (X - X_0) = \emptyset$. Thus $\mathcal{B}$ is isomorphic to one of its elementary submodels. Hence $\mathcal{B}$ is saturated whence, by Theorem 4 (vi), $\mathcal{A}$ is saturated.

§6 Proof of Morley's Conjecture

In this section we prove Morley's conjecture that for an $\aleph_1$-categorical theory T, α_T is finite. Our first step in this project is to introduce a concept of the rank of a formula in a model of a theory. We will compare this notion with two other sorts of rank.

If $\mathcal{A}$ is an L-structure and A is an element of $S_1(L(\mathcal{A}))$ then in §1 we defined A to be minimal in $\mathcal{A}$ if $A(\mathcal{A})$ is infinite and for each formula $B \in S_1(L(\mathcal{A}))$ $B \wedge A(\mathcal{A})$ or $\sim B \wedge A(\mathcal{A})$ is finite. We will define a notion of rank of a formula in a model such that minimal formulas have rank one.

Well order the class X consisting of {-1} and the direct product of the class of all ordinals with the positive integers by placing -1 first in the order and then following the natural lexiocographic order. For each L-structure $\mathcal{A}$ define $f_{\mathcal{A}} : X \to 2^{S_1(L(\mathcal{A}))}$ by induction

$$f_{\mathcal{A}}(-1) = \{A \in S_1(L(\mathcal{A})) \mid A(\mathcal{A}) = \phi\}$$

$A \in f_{\mathcal{A}}(<\alpha,k>)$ if and only if $A \notin f_{\mathcal{A}}(x)$ for any $x << \alpha,k>$ and if for any set of k + 1 formulas $B_1, \ldots B_{k+1}$ from $S_1(L(\mathcal{A}))$ such that the sets $B_i(\mathcal{A})$ partition $A(\mathcal{A})$ there exists an $x < <\alpha,1>$ with one of the $B_i \in f(x)$.

Let T be totally transcendental, $\mathcal{A}$ a model of T, and $A \in S_1(L(\mathcal{A}))$. Call a formula A <u>rankless</u> if A not in the range of $f_{\mathcal{A}}$. Let A be a rankless formula and let $< B_\alpha >_{\alpha<\lambda}$

list the formulas in $S_1(L(\mathfrak{A}))$ such that $B_\alpha(\mathfrak{A})$ and $\sim B_\alpha(\mathfrak{A})$ partition $A(\mathfrak{A})$ where $\lambda = \kappa(\mathfrak{A}) + \aleph_0$. There exist an $x \in X$ such that if B_α is not rankless $B_\alpha \in f_{\mathfrak{A}}(y)$ for $y < X$, because λ is not cofinal with the class of all ordinals. For some $\alpha < \lambda$ both B_α and $\sim B_\alpha$ must be rankless; otherwise A would be in $f_{\mathfrak{A}}(x)$. Now by an argument similar to that in Lemma 6 it can be deduced that if there is a rankless formula A in a model of T then T is not totally transcendental.

Thus if $\mathfrak{A}$ is a model of a totally transcendental theory we may define for each $A \in S_1(L(\mathfrak{A}))$ the rank of $A(\mathfrak{A})$ (the rank of A in $\mathfrak{A}$) which we denote by $R_{\mathfrak{A}}(A)$. $R_{\mathfrak{A}}(A)$ is -1 if $A \in f_{\mathfrak{A}}(-1)$. $R_{\mathfrak{A}}(A)$ is $\langle\alpha,k\rangle$ if $A \in f_{\mathfrak{A}}(\langle\alpha,k\rangle)$.

In [6], Morley introduced for a countable first order theory T, $X \in N(T)$, and $p \in S(X)$ the concept of the transcendental rank of p. In [4] Lachlan interprets this notion in terms of the rank of a formula A in $S_1(L(\mathfrak{A}))$ as follows

$$r_{\mathfrak{A}}(A) = \begin{cases} -1 \text{ if } A(\mathfrak{A}) = \phi \\ \sup\{\alpha \mid (\exists p)\ p \in U_A \wedge p \in Tr^\alpha(a)\} \text{ otherwise} \end{cases}$$

We relate $r_{\mathfrak{A}}(A)$ to $R_{\mathfrak{A}}(A)$ in the following theorem.

Theorem 9. Let $\mathfrak{A}$ be a model of a totally transcendental theory T and $A \in S_1(L(\mathfrak{A}))$.

(i) $r_{\mathfrak{A}}(A) \geq \sup\{\alpha \mid \exists \mathfrak{B}\ \exists k\ \ \mathfrak{B} \succeq \mathfrak{A} \wedge R_{\mathfrak{B}}(A) = \langle\alpha,k\rangle\}$

(ii) For some $\mathcal{B}$ an elementary extension of $\mathcal{A}$ and some integer k, $R_{\mathcal{B}}(A) = (r_{\mathcal{A}}(A),k)$

(iii) For some $\mathcal{B}$ an elementary extension of $\mathcal{A}$ $R_{\mathcal{B}}(A) = \sup \{R_C(A) \mid C \succcurlyeq \mathcal{A}\}$

To prove this theorem we need the following extension of a lemma in [4].

Lemma 21. Let T be a first order theory, $\mathcal{A}$ a model of T, $A \in S_1(L(\mathcal{A}))$ and suppose $r_{\mathcal{A}}(A) = \alpha$ then for each $\beta < \alpha$ there exist an elementary extension $\mathcal{B}$ of $\mathcal{A}$ such that $i^*_{\mathcal{A}\mathcal{B}}{}^{-1}(U_A) \cap Tr^{\beta}(\mathcal{B})$ is infinite.

Proof. If the lemma is false there exists a model of T and a formula $A \in S_1(L(\mathcal{A}))$ with $r_{\mathcal{A}}(A) = \alpha$ and some $\beta < \alpha$ such that for each $\mathcal{B} \succcurlyeq \mathcal{A}$ $i^*_{\mathcal{A}\mathcal{B}}{}^{-1}(U_A) \cap Tr^{\beta}(\mathcal{B})$ is finite. Suppose $q \in Tr^{\beta+1}(\mathcal{B})$. Then for each $C \succcurlyeq \mathcal{B}$, $i^*_{\mathcal{B}C}{}^{-1}(q) \cap S^{\beta+1}(C)$ is a set of isolated points in $S^{\beta+1}(C)$. But then if $A \in q$, $i^*_{\mathcal{B}C}{}^{-1}(q) \cap S^{\beta}(C)$ is a set of isolated points in $S^{\beta}(C)$ since $i^*_{\mathcal{A}C}{}^{-1}(U_A) = i^*_{\mathcal{B}C}{}^{-1}(U_A)$ and $i^*_{\mathcal{A}C}{}^{-1}(U_A) \cap Tr^{\beta}(C)$ is finite. Thus $q \in Tr^{\beta}(\mathcal{B})$ but q was chosen in $Tr^{\beta+1}(\mathcal{B})$ so this is impossible. Hence $i^*_{\mathcal{A}\mathcal{B}}{}^{-1}(U_A) \cap Tr^{\beta+1}(\mathcal{B})$ is empty and by induction for each $\gamma \geq \beta + 1$, for each $C \succcurlyeq \mathcal{A}$ $Tr^{\gamma}(C) \cap i^*_{\mathcal{A}C}{}^{-1}(U_A)$ is empty So $r_{\mathcal{A}}(A) \neq \alpha$.

Proof of Theorem 9.

(i) The proof proceeds by induction on $r_{\mathcal{A}}(A)$. If

$r_{\mathcal{A}}(A) = -1$ then $\vdash_T \sim \exists v_0 A$ and so the theorem holds. Suppose, as the induction hypothesis, the theorem holds for a formula A if $r_{\mathcal{A}}(A) = \gamma$ is less than α. We first prove that for each $\mathcal{B} \succcurlyeq \mathcal{A}$ $R_{\mathcal{B}}(A) < (\alpha + 1, 1)$. If not, there is some $\mathcal{B}_1 \succcurlyeq \mathcal{A}$ with $R_{\mathcal{B}_1}(A) \geq (\alpha + 1, 1)$. Then there exists a sequence of formulas $(A_i)_{i<\omega}$ each $A_i \in S_1(L(\mathcal{B}_1))$ such that $A_i(\mathcal{B}_1) \subseteq A((\mathcal{B}_1)$, $(A_i \wedge A_j)(\mathcal{B}_1) = \phi$ if $i \neq j$, and $R_{\mathcal{B}_1}(A_i) = (\alpha, 1)$. Now we show that for each natural number i there is a 1-type $p_i \in U_{A_i} \cap S^{\alpha}(\mathcal{B}_1)$.

Case 1) α is a successor ordinal, say $\alpha = \lambda + 1$. Since $R_{\mathcal{B}_1}(A_i) = (\lambda + 1, 1)$ there exists a sequence of formulas $(A_{ij})_{j<\omega}$ each $A_{ij} \in S_1(L((\mathcal{B}_1))$, such that $A_{ij}(\mathcal{B}_1) \subseteq A_i(\mathcal{B}_1)$, $(A_{ij} \wedge A_{ik})(\mathcal{B}_1) = \emptyset$ if $j \neq k$, and $R_{\mathcal{B}_1}(A_{ij}) = (\lambda, 1)$. Then by induction for each j $r_{\mathcal{B}_1}(A_{ij}) \geq \lambda$ so there exists $p_{ij} \in U_{A_{ij}} \cap Tr^{\lambda}(\mathcal{B}_1)$. Then for each i, since $S(\mathcal{B}_1)$ is compact and U_{A_i} is closed, there exists p_i, an accumulation point of the p_{ij}, such that $p_i \in U_{A_i} \cap S^{\lambda+1}(\mathcal{B}_1)$.

Case 2) α is a limit ordinal. α has cofinality ω since $\alpha < \omega_1$ [4]. Then there exists a sequence of ordinals $(\alpha_j)_{j<\omega}$ and a sequence of formulas $(A_{ij})_{j<\omega}$, each $A_{ij} \in S_1(L(\mathcal{B}_1))$, such that $A_{ij}(\mathcal{B}_1) \subseteq A_i(\mathcal{B}_1)$, $(A_{ij} \wedge A_{ik})(\mathcal{B}_1) = \emptyset$ if $j \neq k$, $R_{\mathcal{B}_1}(A_{ij}) = (\alpha_j, 1)$ for each j, and the α_j increase monotonically to α. Then by induction $r_{\mathcal{B}_1}(A_{ij}) = \alpha_j$ so there exists a type $p_{ij} \in U_{A_{ij}} \cap Tr^{\alpha_j}(\mathcal{B}_1)$. Since U_{A_i} is closed and $S(\mathcal{B}_1)$ is compact there exists p_i an accumulation point of the p_{ij} for each i. But $p_i \notin Tr^\gamma(\mathcal{B}_1)$ for any $\gamma < \alpha$ so $p_i \in U_{A_i} \cap S^\alpha(\mathcal{B}_1)$.

Since U_A is closed there exists p, an accumulation point of the p_i and $p \in U_A \cap S^{\alpha+1}(\mathcal{B}_1)$ since each $p_i \in U_A \cap Tr^\alpha(\mathcal{B}_1)$. But then $r_{\mathcal{A}}(A) \geq \alpha + 1$ so i) is proved.

(ii) Now we show that there exists $\mathcal{B} \succ \mathcal{A}$ such that for some k $R_{\mathcal{B}}(A) = (\alpha,k)$. By Lemma 22 since $r_{\mathcal{A}}(A) = \alpha$, for each $\gamma < \alpha$ there exists an elementary extension $\mathcal{A}_\gamma$ of $\mathcal{A}$ such that $i^*_{\mathcal{A}\mathcal{A}_\gamma}{}^{-1}(U_A) \cap Tr^\gamma(\mathcal{A}_\gamma)$ is infinite. Hence there exists a sequence of formulas $(A^\gamma{}_i)_{i<\omega}$, each $A^\gamma{}_i \in S_1(L(\mathcal{A}_\gamma))$, such that $(A^\gamma{}_i \wedge A^\gamma{}_j)(\mathcal{A}_\gamma) = \emptyset$ if $i \neq j$, $A^\gamma{}_i(\mathcal{A}_\gamma) \subseteq A(\mathcal{A}_\gamma)$ and $r_{\mathcal{A}_\gamma}(A^\gamma{}_i) = \gamma$. So by induction there exists $\mathcal{A}_{\gamma,i}$ such that for each γ and i $R_{\mathcal{A}_{\gamma,i}}(A^\gamma{}_i) = (\gamma,k)$ for some k. Without loss of generality we may assume $(|\mathcal{A}_{\gamma,i}|-|C|) \cap (|\mathcal{A}_{\delta,j}|-|C|) = \emptyset$ if $(\gamma,i) \neq (\delta,j)$. There exists a model C such that for each γ, i $C \succ \mathcal{A}_{\gamma,i}$ by the compactness theorem. Then for each $\gamma < \alpha$ and each $i < \omega$ there is a k such that $R_C(A^\gamma{}_k) \geq (\gamma,k)$ and $(A^\gamma{}_i \wedge A^\gamma{}_j)(C) = \emptyset$ if $i \neq j$. So $R_C(A) \geq (\alpha,1)$. Since for each $\mathcal{B} \succ \mathcal{A}$ $R_{\mathcal{B}}(A) < (\alpha + 1, 1)$ by i), for some k $R_C(A) = (\alpha,k)$ and C is the required model.

(iii) It remains to find $\mathcal{B} \succ \mathcal{A}$ such that $R_{\mathcal{B}}(A) = \sup\{R_C(A) \mid C \succ \mathcal{A}\}$. It suffices to show that the set of k such that for some $C \succ \mathcal{A}$ $R_C(A) = (\alpha,k)$ is bounded. If not there exists a sequence of models $\mathcal{B}_i$ such that for each

$k < \omega$ there exists $i < \omega$ with $R_{\mathcal{B}_i}(A) = (\alpha, k)$. We may assume $(|\mathcal{B}_i| - |\mathcal{A}|) \cap (|\mathcal{B}_j| - |\mathcal{A}|) = \emptyset$ if $i \neq j$. By the compactness theorem there exists a model $\mathcal{B}$ of T which extends each of the $\mathcal{B}_i$. Then $R_{\mathcal{B}}(A) \geq (\alpha + 1, 1)$ contrary to ii).

Note that $\mathcal{q}$ (i) and $\mathcal{q}$ (ii) together imply that $r_{\mathcal{A}}(A) = \sup \{\alpha | \exists \mathcal{B}, \exists k \ (\mathcal{B} \succ \mathcal{A} \wedge R_{\mathcal{B}}(A) = (\alpha, k))\}$.

We now restrict our attention to $\aleph_1$-categorical theories. In particular, we will deal with an $\aleph_1$-categorical theory T with a specified strongly minimal formula D such that for each model $\mathcal{B}$ of T $D(\mathcal{B}) \cap cl(\emptyset)$ is infinite.

For each natural number ℓ, for each $A \in S_{\ell+1}(L)$ and to -1 and each natural number n assign a set of formulas as follows

$$\Gamma_A^{(-1)} = \{ \sim \exists v_0 A\}$$

$$\Phi_A^{(0)} = \{\exists v_0 A \wedge \exists^{\leq k} v_0 A \mid 0 < k < \omega\}$$

$$\Phi_A^{(n)} = \{\exists v_{\ell+1}, \ldots, \exists v_k \ (\forall v_0 (A \leftrightarrow \exists v_{k+1} (C \wedge D(v_{k+1}) \wedge C^*))$$

$$\wedge \ (\forall v_0 \ (A \rightarrow \exists^{\leq p} v_{k+1} \ (C \wedge D(v_{k+1})))) \wedge$$

$$(\exists^{\leq p} v_{k+1} \ \exists v_0 (D(v_{k+1}) \wedge C \wedge (\sim A \vee \sim C^*))) \mid$$

$$0 < p < \omega, \ \ell \leq k \leq \omega, \ C \in S_{k+2}(L), \text{ and } C^* \in \Gamma_C^{(n-1)}\}$$

$$\Theta_A^{(n)} = \{\exists v_{\ell+1}, \ldots, \exists v_k (\forall v_0 (A \leftrightarrow (A_1 \vee \ldots \vee A_s)$$

$$\wedge A_1^* \wedge \ldots A_s^*) \mid \ell \leq k \leq \omega, \ A_i \in S_{k+1}(L), \ s < \omega \text{ each}$$

$$A_i^* \in \bigcup_{r<n} \Gamma_{A_i}^{(r)} \cup \Phi_{A_i}^{(n)} \text{ and some } A_i^* \in \Phi_{A_i}^{(n)}\}$$

$\Gamma_A^{(n)} = \Phi_A^{(n)} \cup \Theta_A^{(n)}$

Note that if $A \in S_{\ell+1}(L)$ and $A^* \in \Gamma_A^{(n)}$ for some n then A^* has free variables $v_1, \ldots v_\ell$. Thus when we write $A^*(a_1, \ldots a_\ell)$ we mean the result of substituting a_i for v_i for $i = 1, 2 \ldots, \ell$. In this section we abbreviate $A_{v_1, \ldots v_\ell}(a_1, \ldots a_\ell)$ by $A(a_1, \ldots a_\ell)$. Thus $A(a_1, \ldots a_\ell) \in S_1(L(\{a_1, \ldots a_\ell\}))$.

Theorem 10. Let T be a theory of the kind described above, $\mathcal{A}$ a model of T, $m \in \{-1\} \cup \omega$ $A \in S_{\ell+1}(L)$, and $a_1, \ldots, a_\ell \in |\mathcal{A}|$. The following two propositions are equivalent.

i) There exists a formula $A^* \in \Gamma_A^{(m)}$ such that $\mathcal{A} \models A^*(a_1, \ldots, a_\ell)$.

ii) For some k $R_{\mathcal{A}}(A(v_0,a_1, \ldots, a_\ell)) = (m,k)$ if $m \geq 0$. If $m = -1$, $R_{\mathcal{A}}(A(v_0,a_1, \ldots a_\ell)) = -1$.

Notice that there is no loss of generality in this theorem because of our assumption that T has a strongly minimal formula D and that for each model $\mathcal{B}$ of T $D(\mathcal{B}) \cap cl(\emptyset)$ is infinite. For, let T be an arbitrary $\aleph_1$-categorical theory in a first order language L. Then there is a principal extension T' of T with a strongly minimal formula D'. Let $\mathcal{A}'$ be a prime model of T'. Let X be an infinite subset of $D'(\mathcal{A}')$. Then $Th(\mathcal{A}',X) = T''$ is a theory of the specified kind. Suppose $\mathcal{B}$ is a model of T", $A \in S_{\ell+1}(L)$, $A^* \in \Gamma_A^{(m)}$ for some m, and

$a_1, \ldots, a_\ell \in |\mathcal{B}|$. Then $\mathcal{B} \models A^*(a_1, \ldots, a_\ell)$ if and only if $\mathcal{B}|L \models A^*(a_1, \ldots, a_\ell)$. Moreover $R_{\mathcal{B}|L}(A(v_0, a_1, \ldots, a_\ell)) = R_{\mathcal{B}}(A(v_0, a_1, \ldots, a_\ell))$. Thus it suffices to prove the theorem for T''.

Proof of Theorem. The proof proceeds by induction on m. If m = -1 $\mathcal{A} \models A^*(a_1, \ldots, a_\ell)$ for some $A^* \in \Gamma_A^{(-1)}$ if and only if $A(v_0, a_1, \ldots, a_\ell)(\mathcal{A}) = \emptyset$ which is equivalent to $R_{\mathcal{A}}(A(v_0, a_1, \ldots, a_\ell)) = -1$. We assume the theorem is true for $m < n$ and prove (i) implies (ii) for $m = n$. Then we prove a lemma. Finally we assume the theorem holds for $m < n$ and prove (ii) implies (i) for $m = n$.

To prove (i) implies (ii) consider a formula $A \in S_{\ell+1}(L)$ and a formula $A^* \in \Gamma_A^{(n)}$ such that $\mathcal{A} \models A^*(a_1, \ldots a_\ell)$ with $a_1, \ldots a_\ell \in |\mathcal{A}|$. Notice first that it suffices to prove the case in which $A^* \in \Phi_A^{(n)}$. For, suppose the (i) implies (ii) has been shown for each integer ℓ, each $A \in S_{\ell+1}(L)$ and each $A^* \in \Phi_A^{(n)}$ and that $A^* \in \Theta_A^{(n)}$. Then since $\mathcal{A} \models A^*(a_1, \ldots a_\ell)$, $A(v_0, a_1, \ldots a_\ell)(\mathcal{A}) = \bigcup_{i=1}^{s} (A_i(v_0, a_1, \ldots a_k)(\mathcal{A}))$ for some $a_{\ell+1}, \ldots a_k$ in $|\mathcal{A}|$ and some $A_1, \ldots, A_s$. Moreover for each i $\mathcal{A}$ satisfies $A_i^*(a_1, \ldots, a_k)$ and each $A_i^* \in \bigcup_{i=1}^{n-1} \Gamma_{A_i}^{(n-1)} \cup \Phi_{A_i}^{(n)}$ and some $A_i^* \in \Phi_{A_i}^{(n)}$. So for each i there exists $n_i \le n$ and a k_i such that $R_{\mathcal{A}}(A_i(v_0, a_1, \ldots a_\ell)) = (n_i, k_i)$ and for some i there exists k such that $R_{\mathcal{A}}(A_i(a_1, \ldots a_\ell)) = (n, k)$, by induction and the assumption that the theorem holds for each $B^* \in \Phi_B^{(n)}$. But then $R_{\mathcal{A}}(A(a_1, \ldots a_\ell)) = (n, m)$ for some integer m.

Thus to prove (i) implies (ii) when m = n, let $A \in S_{\ell+1}(L)$ and suppose $\mathcal{A} \models A^*(a_1, \ldots, a_\ell)$ where $A^* \in \Phi_A^{(n)}$. Letting $A' = A(v_0, a_1, \ldots, a_\ell)$ we wish to prove that for some q $R_{\mathcal{A}}(A') = (n,q)$. From the definition of $\Phi_A^{(n)}$ we see A^* has the form

$$\exists v_{\ell+1}, \ldots \exists v_k \ (\forall v_0 \ (A \leftrightarrow \exists v_{k+1} \ (C \wedge D(v_{k+1}) \wedge C^*))$$

$$\wedge \ (\forall \ v_0(A \rightarrow \exists^{\leq} v_{k+1}^p \ (C \wedge D(v_{k+1})))) \ \wedge$$

$$\exists^{\leq p} v_{k+1} \ \exists v_0(D(v_{k+1}) \wedge C \wedge (\sim \ell \vee \sim C^*)))$$

where p is a positive integer, $\ell \leq k < \omega$ C is in $S_{k+2}(L)$ and C^* is in $\Gamma_C^{(n-1)}$. Since $\mathcal{A} \models A^*(a_1, \ldots, a_\ell)$ there exist $a_{\ell+1}, \ldots, a_k \in |\mathcal{A}|$ such that for all but p elements b of $D(\mathcal{A})$ $\mathcal{A} \models C^*(a_1, \ldots a_k, b)$. Thus for any $\mathcal{A}_1 \succ \mathcal{A}$ and $d \in D(\mathcal{A}_1) - D(\mathcal{A})$ $\mathcal{A}_1 \models C^*(a_1, \ldots, a_k, d)$ since D is strongly minimal. By induction for some s $R_{\mathcal{A}_1}(C'{}_{v_{k+1}}(d)) = (n-1, s)$ where $C' = C(v_0, a_1, \ldots a_k, v_{k+1})$. Then $R_{\mathcal{A}}(A')$ is $\leq (n,s)$. For, if not there exist L-formulas $B_1, \ldots, B_{s+1}$ where each B_i has free variables $v_0, v_{k+2}, \ldots v_m$ with the following properties. There exist constants $a_{k+2}^i, \ldots, a_m^i \in |\mathcal{A}|$ such that if $B_i' = B_i(v_0, a_{k+2}^i, \ldots, a_m^i)$, $B_i'(\mathcal{A}) \subseteq A'(\mathcal{A})$, $B_i'(\mathcal{A}) \cap B_j'(\mathcal{A}) = \emptyset$ if $i \neq j$, and $R_{\mathcal{A}}(B_i') \geq (n,1)$. We will show that this condition implies for each elementary extension $\mathcal{A}_1$ of $\mathcal{A}$, each $d \in D(\mathcal{A}_1) - D(\mathcal{A})$, and each i that $R_{\mathcal{A}_1}(B_i' \wedge C'_{v_{k+1}}(d)) \geq (n-1, 1)$. This in turn implies $R_{\mathcal{A}_1}(C'_{v_{k+1}}(d)) \geq (n-1,s)$

which is a contradiction allowing us to conclude that $R_{\mathcal{A}}(A') \leq (n,s)$.

Suppose $R_{\mathcal{A}}(B_i^!) \geq (n,1)$ and for some $\mathcal{A}_1 \succ \mathcal{A}$ and some $d \in D(\mathcal{A}_1) \sim D(\mathcal{A})$ $R_{\mathcal{A}_1}(B_i^! \wedge C'_{v_{k+1}}(d)) < (n-1,1)$. By induction there exists a formula $(B_i \wedge C)^* \in \Gamma^{(r)}_{B_i \wedge C}$ for some $r < n-1$ such that $\mathcal{A}_1 \models (B_i \wedge C)^* (a_1, \ldots a_k, d, a^i_{k+2}, \ldots a^i_m)$. Since D is strongly minimal, there exists $p_1 \in \omega$ which may be assumed larger than p such that for all but p_1 members of $D(\mathcal{A})$ $\mathcal{A}_1 \models (B_i \wedge C)^* (a_1, \ldots a_k, b, a^i_{k+2}, \ldots a^i_m)$. Consider the formulas

$$F = \exists v_{k+1}(D(v_{k+1}) \wedge (B_i \wedge C) \wedge (B_i \wedge C)^*)$$

$$F^* = (\forall v_0(F \leftrightarrow F)) \wedge (\forall v_0(F \rightarrow \exists^{\leq p_1} v_{k+1}(D(v_{k+1}) \wedge (B_i \wedge C)))$$

$$\wedge (\exists^{\leq p_1} v_{k+1} \exists v_0(D(v_{k+1}) \wedge (B_i \wedge C) \wedge (\sim F \vee \sim (B_i \wedge C)^*))$$

Now $F^* \in \Gamma^{(r+1)}_F$ and $\mathcal{A} \models F^*(a_1, \ldots a_k, a^i_{k+2}, \ldots a^i_m)$ so if F' is the formula $F(v_0, a_1, \ldots, a_k, a^i_{k+2}, \ldots a^i_m)$ by induction there is an integer q such that $R_{\mathcal{A}}(F') = (r+1,q) < (n,1)$. For each element $c \in B_i^!(\mathcal{A})$ there exists an element b in $D(\mathcal{A})$ such that $\mathcal{A} \models B_i^!(c) \wedge C'(b,c) \wedge C^*(a_1, \ldots a_k, b)$ since $B_i^!(\mathcal{A}) \subseteq A'(\mathcal{A})$ and $\mathcal{A} \models A^*(a_1, \ldots a_k)$. Let $b_1, \ldots b_q$ be an enumeration of the elements $b \in D(\mathcal{A})$ such that

$$\mathcal{A} \models C^*(a_1, \ldots, a_k, b) \wedge \sim (B_i \wedge C^*)$$
$$(a_1, \ldots a_k, b, a^i_{k+1}, \ldots a_{k+2})$$

We know there are only finitely many such b from above.

Then $R_{\mathcal{A}}(B_i^! \wedge C'_{v_{k+1}}(b)) \leq R_{\mathcal{A}}(C'_{v_{k+1}}(b)) = (n-1,u)$ for some $u < \omega$

by induction. But $\mathfrak{A} \models \forall v_0(B_i' \leftrightarrow F' \vee \bigvee_{j=1}^{q} (B_i \wedge C_{v_{k+1}}(b_j)).$ So $B_i'(\mathfrak{A})$ is the union of a finite number of definable sets each with rank less than $(n,1)$ and thus $R_{\mathfrak{A}}(B_i') < (n,1)$ contrary to assumption. Thus we conclude as outlined above $R_{\mathfrak{A}}(A') \leq (n,s)$. Since $\forall v_0 \exists^{\leq p_1} v_{k+1}(C'), R_{\mathfrak{A}}(A') \geq (n,1)$. Therefore there exists an ℓ $1 \leq \ell \leq s$ such that $R_{\mathfrak{A}}(A') = (n,\ell)$. We have shown (i) implies (ii) when $m = n$.

Lemma 22. Let $\mathfrak{A} \models T$ $A \in S_{\ell+1}(L)$ $a_1, \ldots a_\ell \in |\mathfrak{A}|$, $A' = A(v_0, a_1, \ldots a_\ell)$ and $\alpha \leq \omega$. Suppose the Theorem holds for each $m < \alpha$ and that for each $\mathfrak{B} \succ \mathfrak{A}$ there is some k such that $R_{\mathfrak{B}}(A') = (\alpha, k)$ then there exists $r < \alpha$ and

$$A^* \in \Gamma_A^{(r+1)} \text{ such that } \mathfrak{A} \models A^*(a_1, \ldots a_\ell).$$

Proof: Adjoin a new unary predicate symbol q to L to form L' and a new constant symbol f to L' to form L'. Let Δ be the set of L' sentences which are true in an L' structure C' just if there is an elementary substructure C* of the reduct of C' to L such that $|C^*| = q(C')$. Let D^n be the L' sentence $\exists^{\geq n} v_0(D \wedge \sim q)$. Let Γ_1 be the set of sentences

$$\{\text{elementary diagram of } \mathfrak{A}\} \cup \Delta \cup \{D^n | n < \omega\}.$$

If $k < \omega$ and $F \in S_{k+2}(L)$ consider the following formulas.

Let $m = \ell + k$.

Let $F_1 \in S_{m+2}(L)$ be the formula

$$F(v_0, v_{\ell+1}, \ldots v_m, v_{m+1}) \wedge A.$$

Let $F_1{}^*$ be in $S^{m+1}(L)$.

Let $G(F,F_1{}^*) = \exists v_{m+1}(D(v_{m+1}) \wedge F_1 \wedge F_1{}^*)$.

Let $G^*(F,F_1{}^*, p)$ be

$$(\forall v_0(G(F,F_1{}^*) \leftrightarrow G(F,F_1{}^*)))$$

$$\wedge\ (\forall v_0(G(F,F_1{}^*) \rightarrow \exists^{\leq p} v_{m+1}(D(v_{m+1}) \wedge F_1))$$

$$\wedge\ \exists^{\leq p} v_{m+1}\ \exists v_0\ (D(v_{m+1}) \wedge (\sim G(F,F_1{}^*) \vee \sim F_1{}^*)).$$

Then if $F_1{}^*$ is in $\Gamma^{(s)}_{F_1}$, $G^*(F,F_1{}^*, p)$ is in $\Gamma^{(s+1}_{G(F,F_1{}^*)}$.

Let Γ_2 be the set of sentences

$$\Gamma_1 \cup \{A'(f) \wedge \sim q(f)\} \cup$$

$$\{\sim(G(F, F_1^*)(f, a_1, \ldots a_\ell, b_{\ell+1}, \ldots b_m) \wedge$$

$$G^*(F,F_1{}^*,p)(a_1, \ldots a_\ell, b_{\ell+1}, \ldots b_m))\ |\ \text{for } k \in \omega \text{ let}$$

$$F \in S_{k+2}(L),\ F_1^* \in \cup \bigcup_{\omega<\alpha} \Gamma^{(u)}_{F_1},$$

$$b_{\ell+1}, \ldots b_m, \in |\mathcal{A}|\}$$

Now we show that Γ_2 is inconsistent by finding for each L" structure C" such that $C'' \models \Gamma_1$, for each element $f \in (A' \wedge \sim q)(C')$ formulas F and $F_1{}^*$, an integer p, and constants $c_{\ell+1}, \ldots c_m$ such that

$$C'' \models G(F,F_1{}^*)(f,a_1, \ldots a_\ell, c_{\ell+1}, \ldots C_m)$$

$$\wedge G^*(F, F_1{}^*, p)(a_1, \ldots a_\ell, c_{\ell+1} \ldots c_m).$$

Let $C'' \models \Gamma_1$ and $|\mathcal{B}| = q(C'')$. Let $C = C''|L$. B is an L-structure. Let C_1 be an L-structure prime over $|\mathcal{A}| \cup \{f\}$.

Then $D(C_1) - D(B) \neq \emptyset$. For, suppose $D(C_1) \subseteq D(B)$ and let B_1 be prime over $D(C_1)$. (B_1, C_1 exist by 4.3 of [7].) Then $C_1 = B_1$ for if not $B_1 \subsetneq C_1$ while $D(B_1) = D(C_1)$. But then B_1 and C_1 are models of T which satisfy the hypothesis of the two cardinal theorem so, as in section 3, T is not $\aleph_1$ categorical. Thus there exists $d \in D(C_1) \wedge D(B)$. Let $C \in S_{k+2}(L)$ and $c_1, \ldots c_k \in |\mathcal{A}|$ such that $C(f, c_1, \ldots c_k, v_{k+1})$ generates the principal 1-type in $Th(\mathcal{C}, |\mathcal{A}| \cup \{f\})$ realized by d. Then $C(f, c_1, \ldots c_k, v_{k+1})(\mathcal{C})$ is finite. For if not, since D is strongly minimal and contains infinitely many algebraic points there exists an algebraic point $b \in |\mathcal{A}|$ such that $\mathcal{C} \models C(f, c_1, \ldots c_k, b)$. Since b is algebraic there exists a formula $B \in S_1(L)$ and an integer t such that $\mathcal{C} \models B(b) \wedge \exists^{\leq t} v_0\, B$. But since $\mathcal{C} \models C(f, c_1, \ldots c_{,}, b)$, $C(f, c_1, \ldots c_k, v_{k+1})$ generates a principal type and $C(f, c_1, \ldots c_k, v_{k+1})(\mathcal{C})$ is infinite, $B(\mathcal{C})$ is infinite. So for some $q < \omega$

$$\mathcal{C} \models C(f, c_1, \ldots, c_k, d) \wedge \exists^{\leq q} v_{k+1}\, C(f, c_1, \ldots c_k, v_{k+1}).$$

Let C_1 be the following member of $S_{m+2}(L)$.

$$C_{v_1, \ldots, v_{k+1}}(v_{\ell+1}, \ldots, v_{m+1}) \wedge$$

$$A \wedge \exists^{\leq q} v_{m+1} C_{v_1, \ldots v_{k+1}}(v_{\ell+1}, \ldots v_{m+1})$$

Let C_1' be obtained from C_1 by substituting $a_1, \ldots a_\ell$ for $v_1, \ldots v_\ell$ and $c_1, \ldots c_k$ for $v_{\ell+1}, \ldots v_m$. For any $b \in D(\mathcal{C}) - D(B)$ $R_{\mathcal{C}}(C'_{1\,v_{m+1}}(b)) = R_{\mathcal{C}}(C'_{1\,v_{m+1}}(d))$ since any such b realizes the same 1-type in $Th(\mathcal{C}, \{a_1, \ldots a_\ell, c_1, \ldots c_k\})$

as d and C is homogeneous by Theorem 8. Since D(C) - D(B) is infinite $\mathcal{A}_1 \models \forall v_0 \exists^{\leq q} v_{m+1} C'$; therefore if $R_C(C^1_{1_{v_{m+1}}}(d)) \geq (\alpha, 1)$ then $R_C(A') \geq (\alpha+1, 1)$ contrary to hypothesis. So for some $u < \alpha$ and some k $R_C(C^1_{1_{v_{m+1}}}(d)) = (u,k)$. Thus by hypothesis, there exists a formula $C^*_1 \in \Gamma^{(u)}_{C_1}$ such that $C \vdash C^*_1(a_1, \ldots, a_\ell, c_1, \ldots c_p, d)$. Let p be the maximum of q and the cardinality of $\sim C^*_1(a_1, \ldots a_\ell, c_1, \ldots c_k)(C'')$ which is a finite subset of $D(C'')$. Then

$$C'' \dagger A'(f) \wedge \sim q(f) \wedge G(C, C^*_1)(f, a_1, \ldots, a_\ell, c_1, \ldots c_k)$$
$$\wedge G^*(C, C^*_1, p)(a_1, \ldots a_\ell, c_1, \ldots c_k)$$

so C" does not model Γ_2 but C" was an arbitrary model of Γ_1 so Γ_2 is inconsistent. By the compactness theorem there exists $k \in \omega$ $F^1, \ldots, F^S$ in $S_{k+2}(L)$ and $F^{i*}_1 \in \Gamma^{(t_i)}_{F^i_1}$ for some $t_i < \alpha$ such that

$$\Gamma_1 \vdash (\forall v_0(A'(v_0) \wedge \sim q(v_0) \rightarrow \bigvee_1^S G(F^i, F^{i*}_1)(a_1 a_\ell, c_1, \ldots c_k)))$$
$$\wedge (\bigwedge_1^S G^*(F^i, F^{i*}_1, p_i)(a_1, \ldots a_\ell, c_1, \ldots c_k)).$$

$c_1, \ldots, c_k$ list the constants occurring in some F^i and are assumed to occur in each F_i for notational convenience.

$$\text{Let } B' = \bigvee_1^S G(F^i, F^{i*}_1)(v_0, a_1, \ldots, a_\ell, c_1, \ldots, c_k).$$

If $(A' \wedge \sim B')(\mathcal{A})$ is infinite then there are models of T of arbitrarily large cardinality with $(A' \wedge \sim B')(B) - (A' \wedge \sim B')(\mathcal{A}) \neq \emptyset$. Thus there is a model C of Γ_1 with $(A' \wedge \sim B')(C) - q(C) \neq \emptyset$. But this is impossible. So

$$\mathcal{A} \models (\forall v_0(A' \leftrightarrow (\bigvee_{i=1}^{s}(G(F^i, F_1^{i*})(a_1, \ldots a_\ell, c_1, \ldots c_k)) \vee (A' \wedge \sim B))))$$

$$\wedge(\bigwedge_{i=1}^{s}(G^*(F^i, F_1^{i*}, p_i))) \wedge (\exists^{\leq j} v_0(A \wedge \sim(\bigvee_{i=1}^{s} G(F^i, F_1^i))))) \in \Gamma_A^{(U+1)}$$

where $u = \max(u_i) < \alpha$.

We return to the proof of Theorem 10. The induction hypothesis asserts that (i) is equivalent to (ii) if $m < n$. We have already proved (i) implies (ii) if $m = n$ and now we wish to show (ii) implies (i) if $m = n$. Suppose $A \in S_{\ell+1}(L)$, $a_1, \ldots a_\ell \in |\mathcal{A}|$, $A' = A(a_1, \ldots a_\ell)$ and for some k $R_{\mathcal{A}}(A') = (n,k)$. The definition of $\Theta_A^{(n)}$ allows us to assume that $k = 1$. We will find a formula $A^* \in \Gamma_A^{(n)}$ such that $\mathcal{A} \models A^*(a_1, \ldots a_\ell)$.

We first assert that there is an elementary extension $\mathcal{B}$ of $\mathcal{A}$ and a formula $B' \in S_1(L(\mathcal{B}))$ such that $B'(\mathcal{B}) \subseteq A'(\mathcal{B})$ $R_{\mathcal{B}}(B') = (n,1) = \sup \{R_{\mathcal{C}}(B') \mid \mathcal{C} \succ \mathcal{B}\}$. To see this construct the following sequences of formulas and models. Let $\mathcal{B}_0 = \mathcal{A}$ and $B_0 = A'$. Given $\mathcal{B}_m$ and $B_m \in S_1(L(\mathcal{B}_m))$ choose $\mathcal{B}_{m+1}$ by theorem 9 such that $R_{\mathcal{B}_{m+1}}(B_m) = \sup \{R_{\mathcal{C}}(B_m) \mid \mathcal{C} \succ \mathcal{B}_m\}$. Choose B_{m+1} such that $B_{m+1}(\mathcal{B}_{m+1}) \subseteq B_m(\mathcal{B}_{m+1})$ and $R_{\mathcal{B}_{m+1}}(B_m) = (\lambda,\ell) > (n,1)$ for each $\mathcal{C}$ $\mathcal{B}_{m+1}$ $R_{\mathcal{C}}(B_{m+1}) < R_{\mathcal{B}_{m+1}}(B_m)$.

For, suppose not, then there exists $\mathcal{C}$ such that $R_{\mathcal{C}}(B_{m+1}) \geq R_{\mathcal{B}_{m+1}}(B_m) = R_{\mathcal{C}}(B_m)$; i.e., since $B_{m+1}(\mathcal{C}) \subseteq B_m(\mathcal{C})$, $R_{\mathcal{C}}(B_{m+1}) = R_{\mathcal{C}}(B_m) = (\lambda,\ell)$. Hence there exists $\mu < \lambda$, there exists q such that $R_C(B_m \wedge \sim B_{m+1}) = (u,q)$. But then since $R_{\mathcal{B}_{m+1}}(B_m \wedge \sim B_{m+1}) \leq (u,q)$ and $(n,1) < (\lambda,\ell)$ $R_{\mathcal{B}_{m+1}}(B_m) < (\lambda,\ell)$ which is a contradiction. Since there is no finite descending sequence in a well ordered set, for some k $R_{\mathcal{B}_{k+1}}(B_k) = (n,1)$. Let B_k be the formula $B' = B(b_1, \ldots b_s)$ where $b_1, \ldots b_s \in |\mathcal{B}_k|$ and let $\mathcal{B}_k$ be $\mathcal{B}$. Then as desired $R_{\mathcal{B}}(B') = (n,1) = \sup\{R_C(B') \mid C \succ \mathcal{B}\}$. Now B' and $\mathcal{B}$ satisfy the hypothesis of Lemma 22 so there exists $B^* \in \Gamma_B^{(k+1)}$ for some $k < n$ such that $\mathcal{B} \models B^*(k_1, \ldots b_s)$. If $k < n - 1$ by the induction hypothesis $R_{\mathcal{B}}(B') < (n,1)$ so $k = n - 1$. $\mathcal{B} \models B^*(b_1, \ldots b_s) \wedge \forall v_0(B(b_1, \ldots b_s) \rightarrow A')$ and $\mathcal{B}$ is an elementary extension of $\mathcal{A}$ so for some $c_1, \ldots c_s \in |\mathcal{A}|$, $\mathcal{A} \models B^*(c_1, \ldots c_s) \wedge \forall v_0(B(c_1, \ldots c_s) \rightarrow A')$. Since $B^* \in \Gamma_B^{(n)}$, and we have proved (i) implies (ii) for $m = n$, for some ℓ $R_{\mathcal{A}}(B(c_1, \ldots c_s)) = (n, \ell)$. ℓ must equal 1 since $B(c_1, \ldots c_s)(\mathcal{A}) \subseteq A'(\mathcal{A})$ and $R_{\mathcal{A}}(A') = (n,1)$. If $C' = C(v_0, a_1, \ldots a_\ell, c_1, \ldots c_s) = A' \wedge \sim B(v_0, c_1, \ldots c_s)$ then $R(C') < (n,1)$. So by induction there exists $C^* \in \bigcup_{j=-1}^{n-1} \Gamma_C^{(j)}$ such that $\mathcal{A} \models C^*(a_1, \ldots a_\ell, c_1, \ldots c_s)$. Hence letting

$$A^* = \exists v_{\ell+1}, \ldots \exists v_s((\forall v_0(A \leftrightarrow B(v_0, v_{\ell+1}, \ldots v_s) \vee C)) \wedge B^* \wedge C^*)$$

A^* is in $\Gamma_A^{(n)}$ and $\mathcal{A} \models A^*(a_1, \ldots a_\ell)$ proving the theorem.

Recall from Section 2 that α_T is defined to be the least ordinal such that for all $\mathcal{A} \in N(T)$ and $\beta > \alpha_T$ $S^{\alpha_T}(\mathcal{A}) = S^B(\mathcal{A})$. In [7] Morley proved α_T exists and is less than $(2^{\aleph_0})^+$ for every complete theory. In [4] Lachlan shows that $\alpha_T \leq \omega_1$ for each complete theory. We apply Theorem 10 to prove the following conjecture of Morley.

Theorem 11. If T is $\aleph_1$ categorical then α_T is finite.

Proof. If for some $\mathcal{A}$ and some $\beta \geq \omega$ there exists $p \in S^{\beta}(\mathcal{A})$, then since T is totally transcendental for some $\gamma \geq \beta$, $p \in Tr^{\gamma}(\mathcal{A})$ and by lemma 21 there exists $\mathcal{B} \succ \mathcal{A}$, $q \in Tr^{\omega}(\mathcal{B}) \cap i^{*-1}_{\mathcal{A}\mathcal{B}}(p)$ so there is a formula $A' = A(v_0, a_1, \ldots a_\ell)$ in $S_1(L(\mathcal{B}))$ with $r_{\mathcal{B}}(A') = \omega$. By Theorem 9, there exists $\mathcal{C} \succ \mathcal{B}$ and an integer k such that for every elementary extension $\mathcal{C}_1$ of $\mathcal{C}$ $\tilde{R}_{\mathcal{C}}(A) = (\omega, k)$. Now by lemma 22 with $\alpha = \omega$, there exists an $n < \omega$ and a formula $A^* \in \Gamma_A^{(n+1)}$ such that $\mathcal{C} \models A^*(a_1, \ldots a_\ell)$. By theorem 10 for some k $R_{\mathcal{C}}(A') = (n+1, k)$. This is a contradiction so there is no $\mathcal{A}$ and no $\beta \geq \omega$ and no p with $p \in S^{\beta}(\mathcal{A})$. Hence $\alpha_T < \omega$.

§7 A Note on Definability

Let L be a first order language containing a unary predicate p, $\mathcal{A}$ an L structure, and $B = p(\mathcal{A})$. Recall that if $A \in S_{n+k}(L(\mathcal{A}))$ and $a_1, \ldots, a_n \in |\mathcal{A}|$ $A(a_1, \ldots, a_n)(\mathcal{A})$ denotes the set of k-tuples $b_1, \ldots b_k$ such that $\mathcal{A} \models A(a_1, \ldots, a_n, b_1, \ldots, b_k)$. We call each relation on B of the form $A(a_1, \ldots, a_n)(\mathcal{A}) \cap B$ an A-relation. We prove under certain conditions on $Th(\mathcal{A})$ each A-relation is definable by naming a finite number of constants from B. There is no assumption in this section that L is countable.

For simplicity in notation we assume $n = k = 1$.

Each A relation on an infinite set B can be represented by a pair (λ,η) where λ is a bijective map of $\kappa(B)$ into B and $\eta \in 2^{\kappa(B)}$. Thus $\lambda(i)$ is in the A relation if and only if $\eta(i) = 1$ where $i < \kappa(B)$. There is a natural equivalence relation on such representations produced by calling two representations equivalent if they represent the same relation on B. Ambiguously we denote both the equivalence class containing (λ,η) and the relation (λ,η) represents by $[(\lambda,\eta)]$. We also consider pairs (ℓ,h) where $\ell \in B^n$ and $h \in 2^n$ for some $n < \omega$. (ℓ,h) is a partial representation of $[(\lambda,\eta)]$ if some $(\lambda',\eta') \in [(\lambda,\eta)]$ extends (ℓ,h).

To simplify notation in the following formulas we assume that in addition to the sequence of variables $v_0, v_1 \ldots$ L contains an infinite set of variables $Z_{\alpha|n}$ where $\alpha \in 2^\omega$ and

$\alpha \mid n$ is the initial segment of α of length n. We order such finite sequences by length and among those of the same length lexiocographically. If q is the last sequence of length n under this order the prefix $\exists z_0 \ldots \exists z_q$ will indicate that each variable $z_{\alpha|m}$, $m \leq n$ is being existentially quantified.

If A is a formula and $\alpha \in 2^\omega$ then $A^{\alpha(i)} = A$ if $\alpha(i) = 1$ and $A^{\alpha(i)} = \sim A$ if $\alpha(i) = 0$.

Let C_n be the formula $\exists z_0, \ldots, \exists z_q$ $(\bigwedge_{\alpha\varepsilon 2^n}(\exists v_0(\bigwedge_{i=0}^{n-1} A(v_0, z_{\alpha|i})^{\alpha(i)})$ where q is the last sequence of length n in the order described above. Then if C_n is true in T for each model of $\mathfrak{a}$ of T there is a complete tree of length n of distinct A-relations.

Let ℓ be an element of B^k, X = range ℓ and $h \in 2^k$ for some positive integer k. Define $H^n_{(\ell,h)} \in S_2n(L(X))$ to be

$$\exists z_0, \ldots \exists z_q \; (\bigwedge_{\alpha\varepsilon 2^n}(\exists v_0(\bigwedge_{j=0}^{k-1} A(v_0, \ell(j))^{h(j)} \wedge \bigwedge_{i=0}^{n-1} A(v_0, z_{\alpha|i})^{\alpha(i)})$$

where q is the last sequence of length n.

Define $[(\lambda,\eta)]$ to be <u>in class</u> 1 if $\mathfrak{a} \models \sim H^1_{(\ell,h)}$ for some partial representation (ℓ,h) of $[(\lambda,\eta)]$. For $n > 0$, define $[(\lambda,\eta)]$ to be <u>in class</u> n if $[(\lambda,\eta)]$ is not in class n-1 and $\mathfrak{a} \models \sim H^n_{(\ell,h)}$ for some partial representation (ℓ,h) of $[(\lambda,\eta)]$.

Lemma 23. If $[(\lambda,\eta)]$ is in class n for $n \in \omega$ then $[(\lambda,\eta)]$ is definable in $(\mathfrak{a}, X)$ for some finite $X \subseteq B$.

Proof. Let (ℓ,h) be the partial representation of $[(\lambda,\eta)]$ such that $\mathcal{A} \models \sim H^n_{(\ell,h)}$. Suppose that $\ell \in B^k$ and $h \in 2^k$. Let X = range ℓ. If $n = 1$ let G be the formula $\bigwedge_{i=0}^{k-1} A(v_0, \ell(i))^{h(i)}$. If $n > 1$ let G be the formula

$$\bigwedge_{i=0}^{k-1} A(v_0, \ell(i))^{h(i)} \wedge \forall v_1 \, [A(v_0, v_1) \to$$

$$\exists z_0, \ldots \exists z_{q'} (\bigwedge_{\alpha \in 2^{n-1}} \exists v_0 (A(v_0,v_1) \wedge$$

$$\bigwedge_{i=0}^{k-1} A(v_0,\ell(i))^{h(i)} \wedge \bigwedge_{j=0}^{n-2} A(v_0,z_{\alpha|j})^{\alpha(j)})]$$

$$\wedge \forall v_1 \, [\sim A(v_0,v_1) \to \exists z_0, \ldots \exists z_{q'}$$

$$(\bigwedge_{\alpha \in 2^{n-1}} \exists v_0 (\sim A(v_0,v_1) \wedge \bigwedge_{i=0}^{k-1} A(v_0,\ell(i))^{h(i)} \wedge$$

$$\bigwedge_{j=0}^{k-1} A(v_0,z_{\alpha|j})^{\alpha(j)})]$$

where q' is the last sequence of length n.

Since $[(\lambda,\eta)]$ is not in class $n-1$ if $[(\lambda\ \eta)] = A(a)(\mathcal{A}) \cap B$ clearly $\mathcal{A} \models G(a)$. Thus, if we can show there is but one A-relation $[(\lambda',\eta')]$ such that $[(\lambda',\eta')] = A(a')(\mathcal{A}) \cap B$ and $\mathcal{A} \models G(a')$ we can define $[(\lambda,\eta)]$ by

$$\exists v_0 \; G(v_0) \wedge A(v_0,v_1) \wedge p(v_1).$$

Suppose there exist A-relations $[(\lambda_1,\eta_1)]$ and $[(\lambda_2,\eta_2)]$ and elements a_1, $a_2 \in |\mathcal{A}|$ such that $[(\lambda_1, \eta_1)] = A(a_1)(\mathcal{A}) \cap B$, $[(\lambda_1,\eta_1)] \neq [(\lambda_2,\eta_2)]$, and $\mathcal{A} \models G(a_1) \wedge G(a_2)$.

Then there is some element $b \in B$ such that $\mathfrak{A} \models A(a_1,b) \wedge \sim A(a_2,b)$. Then $\mathfrak{A} \models H^1_{(\ell,h)}$ which is a contradiction if $n = 1$. If $n > 1$ let $\ell_1 = \ell \cup \langle k,0\rangle$ and $h_1 = h \cup \langle k,1\rangle$. Let $\ell_2 = \ell \cup \langle k,b\rangle$ and $h_2 = h \cup \langle k,0\rangle$. Since $\mathfrak{A} \models G(a_1)$, $\mathfrak{A} \models H^{n-1}_{(\ell_1,h_1)}$. Since $\mathfrak{A} \models G(a_2)$, $\mathfrak{A} \models H^{n-1}_{(\ell_2,h_2)}$. But then $\mathfrak{A} \models H^n_{(\ell,h)}$. This contradicts our hypothesis and the lemma follows as outlined above.

In [19] Shelah defines a complete theory T to be stable if for each n + 1-ary predicate R the following set of formulas is inconsistent.

$$\Gamma_R = \{\exists v_0 (\bigwedge_{i=0}^{\ell(\alpha)-1} R(v_0, (y_1, \ldots, y_n)_{\alpha | i})^{\alpha(i)}) | \ \ell(\alpha) < \omega\} \cup T$$

$\ell(\alpha)$ denotes the length of α, a sequence of zeroes and ones. $(y_1, \ldots y_n)_{\alpha | i}$ is an n-tuple of variables indexed by $\alpha | i$.

In our context we can conclude that if T is stable Γ_A is inconsistent where

$$\Gamma_A = \{\exists v_0 (\bigwedge_{i=0}^{\ell(\alpha)-1} A(v_0, z_{\alpha|i})^{\alpha(i)}) | \ell(\alpha) < \omega\} \cup T$$

Theorem 12. Let $\mathfrak{A}$ be a model of a stable theory in a language L and B be a subset of $|\mathfrak{A}|$ definable in L. Any relation on B which is definable in $(\mathfrak{A}, |\mathfrak{A}|)$ is also definable in $(\mathfrak{A}, B)$.

Proof. Since T is stable Γ_A is inconsistent. By the compactness theorem for some n $\mathfrak{A} \models \sim C_n$. But then for every (ℓ, h) $\mathfrak{A} \models \sim H^n_{(\ell,h)}$. Hence each $[(\lambda,\eta)]$ has class $\leq n$ and the theorem follows from Lemma 23.

We can invoke the compactness theorem once more to produce the following uniformization of Theorem 12.

Corollary. If T is a stable theory in a language L, p is a unary predicate in L, and A a binary predicate in L such that $\vdash_T \exists v_0 \forall v_1 (A(v_0,v_1) \rightarrow p(v_1))$ there exist integers N, M and formulas $C_1, \ldots C_M$ in $S_{N+1}(L)$ such that

$$\vdash_T \forall v_0 ((\forall v_1 A(v_0,v_1) \rightarrow p(v_1)) \rightarrow (\bigvee_{i=1}^{M} \exists v_{i_1}, \ldots \exists v_{i_N}$$

$$(\forall v_{N+1} (A(v_0,v_{N+1}) \leftrightarrow C_i(v_{i_1}, \ldots v_{i_N}, v_{N+1})) .$$

§8 Related Results

In this section we list some recent results on problems related to the subject of this thesis.

A natural generalization of the Łos conjecture is: if T is a complete theory having infinite model in a language L such that there are κ sentences in L then T is categorical in κ^+ if and only if T is categorical in every $\lambda > \kappa$. Special cases of this theorem have been proved by Rowbottom [15] and Ressayrè [13]. The general theorem has been proven by Shelah [17].

In [4] Morley conjectured that if $\kappa(T) = \lambda > \aleph_0$ and T is categorical in λ then T has a model with power less than λ. Shelah [18] has proven this conjecture in the case $\lambda^{\aleph_0} = \lambda$.

Harnik and Ressayre [14] prove the following theorem. T is categorical in $\kappa_0 > \kappa(T)$, if and only if every set C of power κ_0, which is a proper subset of a model $\mathfrak{A}$, has a prime extension which is an elementary submodel of $\mathfrak{A}$.

The notion of stability and many other concepts related to categoricity are explored by Shelah in [18].

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