

MODIFICATIONS TO THE THEORY OF THE FLOW OF EVENTS -
A STUDY OF THE EFFECT OF THE GROUPING OF EVENTS IN FINITE
INTERVALS OF TIME

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ABSTRACT

This paper is concerned with the relationship between two distributions in time series which are renewal processes; the intervals between successive events and the number of events in successive intervals of fixed length.

Chapter I surveys the theory of random sequences of events in time, and discusses modifications which allow for the occurrence of simultaneous events. The Compound Poisson distribution arising from Feller's "lightning damage" model is found to be the model with the widest applicability, and a discrete equivalent is given. The choice between the intervals and the number of events distributions as the starting point of a model for series of events is examined, and it is shown that, if simultaneous events are to be allowed, neither distribution is sufficient to characterize the series.

Chapter II considers the approach to equilibrium of a renewal process. By considering special cases of the renewal theorem which are important in applications, it is shown that the approach to equilibrium is exponentially fast in these cases.

Chapter III is concerned with the simulation of a Poisson process, using discrete approximations to a negative exponential distribution. The method examined is based on the replacement of the uniform distribution over $(0,1)$ by a discrete equivalent,

but the approximations are very much coarser than would normally be used, and it is shown that the technique is tolerant of very coarse approximations as far as the number of events distribution is concerned. A remark is made concerning the wide variations between different realizations of a stochastic process, based on the counts made in different ways on the same series of events.

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Introduction

This investigation arose from a study of the paper, "Characterization of the flow of events - a problem of Simulation", by E.A.G. Knowles and D.S. Stewart.¹ The first problem to be considered is the extent to which a model for a series of events in time can be characterized either by the distribution of intervals between events, or by the distribution of the number of events occurring in successive intervals of fixed length. The answer given to this by Knowles and Stewart is that both distributions are required to characterize the series. Their assertion refers to a rather specialized model which is examined in the context of similar models of rather more generality, both discrete and continuous.

The Knowles and Stewart paper was also concerned with the problems arising from the fact that, because of the existence in any real situation of a minimum observable unit of time, "all observable intervals distributions are histograms ...". Generalizing from their special model, they came to the conclusion that simple random selection from an intervals distribution always yields a number of events distribution of a geometric type. In particular, "... no simple process of random selection from an exponential interval distribution will ever produce a Poisson distribution, as is often suggested in the literature."

This paper examines the practical problem of simulating a Poisson process by repeated selection of intervals from a

distribution which is a discrete approximation to the negative exponential. Some very coarse approximations are chosen, to discover how far one can go and still preserve a number of events distribution which is recognizably close to the Poisson distribution.

Chapter I

Some Models for Series of Events

1. We wish to consider suitable models to account for series of events where the only observations are the counts of the number of events occurring in successive intervals of fixed length. The investigations of Knowles and Stewart¹ were concerned with the arrival of orders at a factory, the observations being the number of orders arriving per day over an extended period. For convenience, we shall frequently use the day as our basic unit of time. One of the questions raised by the paper of Knowles and Stewart is whether current theories take account of the fact that there exists in every situation occurring in practice a minimum observable unit of time. It may be orders of magnitude smaller than a day, but the mathematical device of considering an interval of length δt , say, and then calculating the limit of some function as $\delta t \rightarrow 0$, has no parallel in the world of real measurements.

We begin by considering some straightforward models for series of events, and investigating the relationship between the distributions of the intervals between events, and the distributions of counts of events in successive intervals of fixed length, which we shall refer to as "census distributions" after Skellam and Shenton.²

2. We begin with the simplest of models in discrete time.

Let N_t be the number of events occurring up to and including

time t (t real > 0), and suppose:

- (a) $\text{Prob}(N_{t+1} - N_t = 0) = q$, a constant for all t ;
- (b) $\text{Prob}(N_{t+1} - N_t = 1) = p$, also constant;
- (c) $\text{Prob}(N_{t+1} - N_t > 1) = 0$.

Thus $p + q = 1$, and

$\text{Prob}(N_{t+1} - N_t = k | N_1 = k_1, N_2 = k_2 \dots N_{t-1} = k_{t-1})$
is equal to $\text{Prob}(N_{t+1} - N_t = k)$.

These conditions may be summed up by saying that the stream of events is stationary (p, q , constant), simple ($\text{Prob}(N_{t+1} - N_t > 1) = 0$) and without after effect (conditional and absolute probabilities identical). This is the language used by Khintchine³ in discussing the corresponding model in continuous time.

It follows easily (Feller⁴, p.305) that, if T_r is the time at which the r th event occurs,

$$\text{Prob}(T_r - T_{r-1} = n) = pq^{n-1} \quad r = 1, 2, \dots$$

taking for convenience the 0th event at $t = 0$.

The counting distribution is quite straightforward. The number of events at time $t = n$ has the binomial $(1, p)$ distribution, and hence the number of events in an interval of length k units is binomially distributed with parameters (k, p) .

The model is simple, but very restrictive, the counting distribution being binomial in each case. Furthermore, the unit of time for counts must be at least k times the unit for intervals if as many as k events per counting unit is to be possible. It would of course be possible to adjust the parameters to obtain approximations to the Poisson distribution (k large, p small) or the normal (k large), but the advantages, if any, of a discrete

model are going to disappear if k gets too big.

3. We now summarise the specification of the corresponding model in continuous time.

Let N_t be the number of events occurring up to and including time t . Suppose:

- (a) Prob $\{ (N_{t+\delta t} - N_t) = 0 \} = \lambda \delta t + o(\delta t)$, λ constant.
- (b) Prob $\{ N_{t+\delta t} - N_t = 1 \} = 1 - \lambda \delta t + o(\delta t)$.
- (c) Prob $\{ N_{t+\delta t} - N_t > 1 \} = o(\delta t)$.

This is the Khintchine³ model. If T_r is the time at which the r th event occurs,

$$\text{Prob } (T_r - T_{r-1} \leq t) = 1 - e^{-\mu t}, \text{ where } \mu = \frac{1}{\lambda}.$$

The counting distribution for the number of events in an interval of length τ is Poisson ($\lambda\tau$), so once again only one kind of counting distribution is obtained, though if τ is large enough the distribution will be approximately normal.

These two models describe a sequence of events occurring "randomly in time", i.e. at any instant the waiting time to the next event is completely uninfluenced by the sequence of events up to that instant. This property of the intervals between events is referred to by Feller⁴ (pp. 304, 412) as "lack of memory", or Feller⁵ (p. 8) as the "Markov property". The geometric and exponential distributions are unique among discrete and continuous distributions respectively in having this property.

Of course, in formulating a model to fit a real situation one must make observations on the sequence in order to make an estimate of the parameters. To this extent information about

the waiting time to the next event is influenced by a knowledge of the past. However, the whole sequence is used and not its detailed variability.

4. In order to obtain models of wider applicability, the Khintchine model can be modified by relaxing some of the conditions, for instance, by allowing for simultaneous events. If we consider first the discrete time model (p. 3), it is a random walk with N_t , the number of events up to time t , subject to increase of 0 and 1 with probabilities $1-p$ and p respectively. A more general model would allow increases of $0, 1, 2, \dots$ with probabilities p_0, p_1, p_2, \dots , where $\sum_{r=0}^{\infty} p_r = 1$. The intervals between events would still have the geometric distribution $\{(1-p_0)p_0^{n-1}\}$, and the number of events at "points of occurrence" would have the arbitrary distribution $\{p_r\}$.

The advantages of this model are that it will deal with the multiple events situation and at the same time it preserves the Markov property (p. 5) of the intervals between groups of events. The equivalent model in continuous time is Feller's "lightning damage" model,⁴ p. 398. The intervals between bursts of events (lightning flashes) have the negative exponential distribution, and hence the Markov property, while the number of events at each burst (number of dollars worth of damage) has an arbitrary distribution over the set of positive integers.

So far we have been considering N_t as the basic random variable, subject to increase at random points in time. An

alternative approach is to consider the interval between successive events, or bursts of events, as the basic random variable, and to derive the census distribution from assumptions about the distribution of intervals. This was the approach of Knowles and Stewart, and it may have been influenced by their wish to simulate a series of events by generating a sequence of successive intervals. We return to the simulation problem in Chapter III.

For the simple model (p. 3) the same result is achieved whether we postulate the random variable N_t subject to increases of 0 or 1, with probabilities q and p respectively, or an interval T_k between events $k-1$ and k , where $\text{Prob}(T_k = n) = pq^{n-1}$. Similarly, the continuous time Poisson process is equivalently characterized by the Khintchine assumptions (p. 5) about the increases in N_t , or by the assumption of a distribution of intervals between events which is negative exponential.

The method by which Knowles and Stewart modify the basic model to include multiple events is to postulate an intervals distribution which allows for a zero interval. They then raise the question whether this modified version can still be characterized equally by either the intervals distribution or the census distribution. Their model had previously been analysed by Skellam and Shenton² who provided an account of many of its properties. We propose to examine particularly the derivation of the census distribution, in order to compare the results with the discrete version of the lightning damage model of page 6.

5. Consider a series of events in discrete time beginning at the epoch $t = 0$, with the r th event occurring at time

$$\sum_{k=1}^r T_k \quad (r = 1, 2, \dots)$$

Suppose the T_k are independent random variables each having the distribution specified by $\text{Prob}(T=n) = p_n$ ($n = 0, 1, 2, \dots$).

Let $E(T) = \sum_{n=1}^{\infty} n p_n = \tau$ ($\tau < \infty$) and denote the probability

generating function $\sum_{n=0}^{\infty} p_n s^n$ by $G(s)$. We find first the

distribution of the number of events at time t and the limiting distribution as $t \rightarrow \infty$.

Let $\text{Prob}(k \text{ events occur at time } t) = f(k, t)$ ($k = 0, 1, 2, \dots$) and denote the probability of at least n events at time t by

$$\mathcal{F}(n, t) = \sum_{k=n}^{\infty} f(k, t)$$

We also require the probability generating function

$$\Omega(z, t) = \sum_{k=0}^{\infty} f(k, t) z^k$$

Now $\mathcal{F}(n, t) = p_0 \mathcal{F}(n-1, t) \quad n > 1$

and $f(n, t) = (1-p_0) \mathcal{F}(n, t) \quad n \geq 1$

Thus $\mathcal{F}(n, t) = p_0^{n-1} \mathcal{F}(1, t)$

and $f(n, t) = (1-p_0) p_0^{n-1} \mathcal{F}(1, t) \quad n \geq 1$

Hence $\Omega(z, t) = (1-p_0) \mathcal{F}(1, t) \frac{z}{1-p_0 z} + f(0, t)$

and $\frac{\partial \Omega}{\partial z}(z, t) = \frac{(1-p_0) \mathcal{F}(1, t)}{(1-p_0 z)^2}$

Since $\Omega(1,t) = 1 = \mathcal{F}(1,t) + f(0,t)$

and $\frac{\partial \Omega}{\partial z}(1,t) = \frac{\mathcal{F}(1,t)}{1-p_0} =$ expected number of events at time t , $\mu(t)$, say

$$\begin{aligned} \Omega(z,t) &= 1 - \mathcal{F}(1,t) + \frac{(1-p_0)\mathcal{F}(1,t)z}{1-p_0z} \\ &= 1 - \mathcal{F}(1,t) \left[1 - \frac{(1-p_0)z}{1-p_0z} \right] \\ &= 1 - \mu(t)(1-p_0) \left[\frac{1-z}{1-p_0z} \right] \end{aligned} \quad (5.1)$$

The existence of a limiting, or equilibrium, distribution for the number of events at time t , as $t \rightarrow \infty$, depends on the behaviour of $\mu(t)$ as $t \rightarrow \infty$. We assume first that the distribution of the T_k is aperiodic, i.e. values of k other than zero for which $p_k \neq 0$ have a greatest common divisor of unity. If this were not the case, the time units could be rescaled to make it so. It can then be proved that $\lim_{t \rightarrow \infty} \mu(t) = \frac{1}{\tau}$.

This result is a particular case of the Renewal Theorem, which is proved in Feller, Vol.I, pp.306-7,⁴ for discrete distributions, and in Vol.II, pp.346-51,⁵ for the general case. We return to the question of the approach to equilibrium in Chapter II. In the meantime we obtain the equilibrium census distribution of the Knowles and Stewart model by replacing $\mu(t)$ by $\frac{1}{\tau}$ in (5.1), (giving

$$\lim_{t \rightarrow \infty} \Omega(z,t) = 1 - \frac{(1-p_0)}{\tau} \left[\frac{1-z}{1-p_0z} \right]$$

$$\begin{aligned} \text{and hence, } f(0) &= 1 - \frac{1 - p_0}{\tau} \\ f(k) &= \frac{(1 - p_0)^2 p_0^{k-1}}{\tau} \quad (k > 0) \end{aligned} \tag{5.2}$$

This result agrees with that of Knowles and Stewart, although they did not make it clear that it was an equilibrium distribution. What they calculated were frequency ratios within a single realization of the series, and these will tend to the values of the probabilities when the equilibrium state is reached, provided such a state exists.

6. A number of questions can now be answered about this model. The first question, already raised on page 7, is whether the intervals distribution and the census distribution would each be sufficient by itself to characterize the series. The answer is clearly not, since the census distribution depends only on the two parameters p_0 and τ , the same values of which could belong to an infinite set of intervals distributions.

The next question is how does this model compare with the lightning damage model (p. 6)? Firstly, the Markov property no longer holds in general. If, for instance, k units of time have elapsed since the last event, the probability of an event during the next time unit will be at least p_{k+1} , and if $k + 1$ is the largest value of r for which $p_r \neq 0$, then an event in the next time unit is certain. Then also, the census distribution is a very specialized one: it is always of the geometric type with

$$f(k) = p_0 f(k - 1) \quad \text{for } k > 1.$$

Evidently it is at a disadvantage on both counts; the intervals

distribution is arbitrary when it could usefully be geometric, and the census distribution is geometric (except for the first term) when it might be arbitrary.

7. We now consider the consequences of making the census distribution the basis of the model. We take the simple model of page 3 as our starting point and, to accommodate multiple events, we modify condition (c) to allow increases in the number of events of greater than unity. Because both the intervals and the census distributions are discrete, the discussion of paras. 4 and 5 is easily modified to give the properties of the reversed model.

We suppose the number of events at time t to be N_t , where

$$\text{Prob} \{N_t = n\} = \pi_n \quad (n = 0, 1, 2, \dots) \text{ for all } t > 0.$$

$$\text{Let } E(N_t) = \sum_{n=0}^{\infty} n \pi_n = \mu \text{ and denote the p.g.f.}$$

$$\sum_{n=0}^{\infty} \pi_n s^n \text{ by } H(s)$$

If the probability that the interval between the $(r-1)$ th event and the r th event is t is given by $g(t, r)$, then

$$\sum_{i=t}^{\infty} g(i, r) = G(t, r), \text{ say,}$$

is the probability that the interval is of length at least t .

$$\text{Then } G(t, r) = \bar{\pi}_0 G(t-1, r) \quad t \geq 1$$

$$\text{and } g(t, r) = (1 - \bar{\pi}_0) G(t, r) \quad t \geq 1$$

$$\text{Thus } G(t, r) = \bar{\pi}_0^{t-1} G(1, r)$$

$$\text{and } g(t, r) = (1 - \bar{\pi}_0) \bar{\pi}_0^{t-1} G(1, r)$$

$$\begin{aligned} \text{Hence } \bar{\Lambda}(z, r) &= \sum_{k=0}^{\infty} g(k, r) z^k \\ &= 1 - \tau(r)(1 - \bar{\kappa}_0) \left[\frac{1 - z}{1 - \bar{\kappa}_0 z} \right] \end{aligned} \quad (7.1)$$

by the same argument as before, where $\tau(r)$ is the expected value of the interval terminating at the r th event, and $\tau(r) = \frac{\partial \bar{\Lambda}(1, r)}{\partial z}$.

As before, the existence of a limiting distribution depends on the existence of $\lim_{r \rightarrow \infty} \tau(r)$. This follows from the renewal theorem. So long as $\bar{\kappa}_1 \neq 0$, which ensures that the distribution of N_t is aperiodic, we have

$$\lim_{r \rightarrow \infty} \tau(r) = \frac{1}{\mu} \quad (7.2)$$

The census distribution, i.e. that of N_t , is now arbitrary (provided it is not periodic), and the derived intervals distribution in the equilibrium state can be obtained from (7.1) by replacing $\tau(r)$ by $\frac{1}{\mu}$. We obtain,

$$\begin{aligned} g(0) &= 1 - \frac{1 - \bar{\kappa}_0}{\mu} \\ g(t) &= \frac{(1 - \bar{\kappa}_0)^2 \bar{\kappa}_0^{t-1}}{\mu} \quad t > 0 \end{aligned} \quad (7.2)$$

Now, the concept of a zero interval no longer serves any useful purpose. What we are concerned with is the distribution of (non-zero) intervals between bursts of events, and if we define

$$p(t) = \text{Prob}(T = t | T > 0) = \frac{g(t)}{1 - g(0)}$$

then,
$$p(t) = (1 - \bar{\kappa}_0) \bar{\kappa}_0^{t-1} \quad t > 0$$

and the intervals distribution is pure geometric.

This model is plainly the discrete equivalent of Feller's lightning damage model, having its desirable properties of an arbitrary census distribution with a geometric intervals distribution preserving the Markov property.

8. We have already remarked (para. 6, p.10) that, when multiple events are admitted, the model cannot, in general, be characterized either by the intervals distribution, or by the census distribution. However, there is one model for which the specification of either distribution would be sufficient: it is the one in which both distributions are geometric. It is not then necessary to limit consideration to the equilibrium state either, since the Markov property of geometric intervals implies that equilibrium is immediately attained.

To establish this result we take an intervals distribution with probability of a zero interval, p_0 , and probability of an interval of length k , $(1-p_0)pq^{k-1}$, $k > 0$, where $p + q = 1$.

The expectation, τ , is then

$$\begin{aligned} & \sum_{k=1}^{\infty} (1-p_0)pq^{k-1}k \\ &= (1-p_0)p \sum_{k=1}^{\infty} \frac{\partial}{\partial q} q^k \\ &= (1-p_0)p \frac{\partial}{\partial q} \left(\frac{q}{1-q} \right) \\ &= (1-p_0)p \frac{1}{(1-q)^2} \\ &= \frac{(1-p_0)}{p} \end{aligned}$$

Using equations (5.2) (p.101), we obtain the census distribution,

$$\begin{aligned} f(0) &= 1 - \frac{1 - p_0}{\tau} = 1 - p = q \\ f(k) &= \frac{(1-p_0)^2 p_0^{k-1}}{\tau} \\ &= p(1-p_0)p_0^{k-1} \quad (k > 0) \end{aligned}$$

This is, of course, a geometric distribution, except for $f(0)$. We must now show that, beginning with this census distribution, we should arrive back at the intervals distribution above, using equations (7.2) (p.12).

It is easily verified that the expectation of the census distribution is

$$\frac{p}{1 - p_0} = \frac{1}{\tau} = \mu .$$

Thus,
$$g(0) = 1 - \frac{1 - \bar{p}_0}{\mu} = 1 - p\tau = p_0$$

and
$$g(t) = \frac{(1 - \bar{p}_0)^2 \bar{p}_0^{t-1}}{\mu} \quad t > 0$$

$$= \tau p^2 q^{t-1} = (1 - p_0) p q^{t-1},$$

which is indeed the original distribution of intervals.

9. The models considered so far all come within the realm of Renewal Theory. A renewal process is characterized by the fact that the intervals between successive pairs of events are independent identically distributed random variables. The language derives from industrial replacement theory, where the events are failures of components, which are then replaced by

new components, assumed to have the same lifetime distribution.

The time to the n th renewal is the sum of n independent identically distributed random variables, $T_1 + T_2 + \dots + T_n$. A renewal process can also be considered as a random walk with positive increments (in N_t , the number of renewals up to time t). The two aspects of the process are related by the equivalence of the probabilities of

$$(a) \quad T_1 + T_2 + \dots + T_n > t$$

and
$$(b) \quad N_t < n .$$

Discrete renewal processes are discussed by Feller⁴, Chapter XIV, under the heading of recurrent events. The renewal process in continuous time is dealt with by Cox⁶ and by Feller⁵, Chapter XI. The survey paper by Smith⁷ gives a number of applications, and Cox and Lewis⁸ give statistical significance tests for renewal processes. The particular problem of renewal theory to which we intend to devote our attention in the next chapter is rapidity of approach to equilibrium. Except for those special cases where the Markov property holds, the renewal density, or expected rate of occurrence of events, will not be constant. It does have a constant limiting value though; this is the Renewal Theorem referred to on page 9. Now, if one wishes to count the number of events occurring in an interval of fixed length starting at an arbitrary point in time, the waiting time to the first event in the interval will have a distribution which is conditional on the choice of the starting point of the interval. This distribution of initial waiting times

will also settle down to a common limiting distribution as the counting interval becomes more remote in time from the start of the process. The approach to this limiting situation is our next concern.

Chapter II

The Approach to Equilibrium

1. We begin by formulating the "elementary renewal theorem" in discrete and in continuous time, and investigate proofs of the two versions in certain specialized cases which have particular importance in practical applications. General proofs exist, and are not always difficult (see, for instance, Smith⁷, p.246). However, our reason for considering special proofs, rather than a general one, is that they throw some light on the rapidity with which the limiting situation is reached in these practical applications.

2. In discrete time first, an event is held to have occurred at time $t = 0$, and the r th event occurs at time $\sum_{k=1}^r T_k$, $r = 1, 2, \dots$

The T_k are positive, integer valued random variables, all having the same distribution, with $E(T_k) = \tau$.

If $\text{Prob}(T_k = n) = p_n$, ($n = 1, 2, \dots$), the probability generating function $G(s)$ may be defined by

$$G(s) = \sum_{n=1}^{\infty} p_n s^n$$

The probability of the first event occurring at time t is, of course, p_t . Now let the probability of an event at time t be u_t . u_t can also be interpreted as the expected number of events at time t . These definitions hold for $t > 0$ and we define

$$u_0 = 1 \quad \text{and} \quad p_0 = 0.$$

The elementary renewal theorem states simply that

$$\lim_{t \rightarrow \infty} u_t = \frac{1}{\tau} .$$

This is Feller's⁴ Theorem 3 (p.286) proved on pages 306-7.

In the continuous time model, an event is held to have occurred at time $t = 0$ as before, and the r th event at time $\sum_{k=1}^r T_k$, $r = 1, 2, \dots$. The T_k are now independent continuous random variables having the same distribution, defined by the density function $f(t)$, ($t > 0$) with $E(T_k) = \tau$ as before. Instead of a probability generating function we employ the Laplace transform of $f(t)$, denoted by $f^*(s)$, and defined by

$$f^*(s) = \int_0^{\infty} f(t) e^{-st} dt$$

We note here that $f^*(s)$ is a moment generating function in the

sense that $f^*(s) = \sum_{r=0}^{\infty} \frac{(-1)^r \mu_r' s^r}{r!}$, where μ_r' is the r th

moment of T_k about the origin. Thus, for instance,

$\tau = -f^{*'}(0)$, i.e. the first derivative of $f^*(s)$ evaluated at $s = 0$.

If we now define the number of events up to and including time t to be N_t , then the equivalent function to u_t in the discrete case is the renewal density $h(t)$, defined by

$$h(t) = \lim_{\delta t \rightarrow 0^+} \frac{E(N_{t+\delta t} - N_t)}{\delta t}$$

The elementary renewal theorem in continuous time states that

$$\lim_{t \rightarrow \infty} h(t) = \frac{1}{\tau} .$$

This is not quite the result proved by Smith, quoted on page 15, but is equivalent to it. (See, for instance, Cox⁶, p. 55.)

A further modification is required to deal with multiple events models, and thus to establish the result $\lim \mu(t) = \frac{1}{\tau}$ quoted on page 9. We deal with this first.

For the distribution of intervals $\{p_n\}$, $n = 0, 1, 2, \dots$ with expectation τ , defined in I.4 (p. 6), the occurrence of bursts of events is a renewal process having an intervals

distribution $\left\{ \frac{p_n}{1-p_0} \right\}$, $n = 1, 2, \dots$ with expectation $\frac{\tau}{1-p_0}$;

by the elementary renewal theorem, if u_t is the probability of at least one event at time t , $\lim_{t \rightarrow \infty} u_t = \frac{1-p_0}{\tau}$.

The conditional probability of k events at time t given at least one event is $p_0^{k-1} (1-p_0)$. Thus the unconditional probability of k events is $u_t(1-p_0)p_0^{k-1}$. The expected number of events at time t is therefore

$$\begin{aligned} E(N_t) &= u_t(1-p_0) \sum_{k=1}^{\infty} k p_0^{k-1} \\ &= u_t(1-p_0) \frac{1}{(1-p_0)^2} \\ &= \frac{u_t}{1-p_0} \end{aligned}$$

But $u_t \rightarrow \frac{1-p_0}{\tau}$, so $E(N_t) \rightarrow \frac{1}{\tau}$:

3. The first special case we consider is a discrete time model where the intervals distribution is finite, i.e. there exists a positive integer m , such that:

$$\begin{aligned} \text{Prob}(T_k = m) &= p_m > 0 \\ \text{and} \quad \text{Prob}(T_k > m) &= 0 \end{aligned}$$

The probability generating function $G(s)$ is then a polynomial. If the intervals distribution is to be estimated from observations of a realization of a process, a finite distribution will often be the result. Assuming that no parametric form of distribution suggests itself, a natural estimator for the distribution function $F(t)$ of the T_k is the empirical distribution function $F_n(t)$, based on n observations, where $F_n(t) = \frac{1}{n}$ (number of observed intervals $\leq t$). This is an unbiased and consistent estimator of $F(t)$. (Cox and Lewis⁸, pp.142-3.) With this procedure, m is the value of the largest observed interval.

In addition to the definitions of para. 2 (p. 17), we define the "generating function" $U(s)$ by

$$U(s) = \sum_{t=0}^{\infty} u_t s^t$$

The following relations then hold:

$$\begin{aligned} u_n &= u_0 p_n + u_1 p_{n-1} + \dots + u_n p_0 & 1 \leq n \leq m \\ u_{n-m} p_m + u_{n-m+1} p_{m-1} + \dots + u_n p_0 & & n > m \end{aligned}$$

These relations express the fact that an event at $t = n$ is the result of the compound event, an event at $t = n - k$ followed by

an interval of length k , the two components being independent; further, the different values of k are mutually exclusive and exhaustive.

Multiplying by s^n and summing from $n = 1$ to $n = \infty$, we have

$$U(s) = \frac{1}{1 - G(s)} .$$

Since $G(s)$ is a polynomial of degree m the right-hand side can be expressed as a sum of partial fractions. If we suppose that the roots, s_1, s_2, \dots, s_m of $1 - G(s)$ are all distinct, the expansion is of the form

$$U(s) = \sum_{j=1}^m \frac{a_j}{s_j - s}$$

and the numerators are given by the formula $a_j = \frac{1}{G'(s_j)}$.

A p -fold repeated root s_r will introduce terms

$$\frac{a_r}{s_r - s}, \frac{a_{r+1}}{(s_r - s)^2}, \dots, \frac{a_{r+p-1}}{(s_r - s)^p} .$$

One root of $1 - G(s)$ is $s = 1$, and this is not repeated, for if it were, it would imply $-G'(1) = 0$, i.e. $\tau = 0$. Furthermore, since $|s| < 1$ implies $|G(s)| < 1$, and hence $G(s) \neq 1$, the root $s = 1$ is the smallest in absolute value.

Having established this we now consider the expansion of each partial fraction as a power series in s . The coefficient of s^n in $U(s)$ will be the sum of the contributions from m such expansions. For example,

$$\frac{a_j}{s_j - s} = a_j \cdot \frac{1}{s_j} \left(1 - \frac{s}{s_j}\right)^{-1},$$

giving a contribution of $\frac{a_j}{s_j^{n+1}}$, and

$$\frac{a_{r+k-1}}{(s_r - s)^k} = a_{r+k-1} \cdot \frac{1}{s_r^k} \left(1 - \frac{s}{s_r}\right)^{-k}$$

which gives a contribution of $\frac{a_{r+k-1}}{s_r^{k+n}} \binom{k+n-1}{n}$

We may label the root $s = 1$ as s_1 with the corresponding value of a_1 equal to $\frac{1}{G'(s_1)} = \frac{1}{\tau}$. It is clear then that the contribution of the corresponding partial fraction to the coefficient of s^n in $U(s)$ is simply $\frac{1}{\tau}$. All the other contributions contain at least the n th power of $\frac{1}{s_j}$, and, since in each case $|s_j| > 1$, these contributions will diminish exponentially as n increases.

Thus, firstly, $\lim_{n \rightarrow \infty} u_n = \frac{1}{\tau}$, and secondly, the limit is approached exponentially.

4. Our next example is a special case of the renewal theorem in continuous time. We consider the situation when $f^*(s)$, the Laplace transform of the density function of intervals (see p.18), is a rational function. Included in this class of density functions are the gamma distributions for which

$$f(t) = \frac{\alpha^\nu t^{\nu-1} e^{-\alpha t}}{\Gamma(\nu)} \quad \text{with } \nu > 1 \text{ integral,}$$

and $f^*(s) = \left(\frac{\alpha}{\alpha + s}\right)^\nu$.

In queuing theory the gamma distribution with integral is known as the special Erlang distribution. It is widely used in the theory of queues and in replacement theory where the lack of memory or lack of aging implied by the use of the negative exponential distribution is too unrealistic. A random variable having the gamma distribution with parameter ν can be interpreted as the sum of ν independent random variables, each having a negative exponential distribution. Some of the pleasanter properties of the latter distribution carry over, and some "aging" is introduced. In replacement theory the model is known as "failure by stages", the failure of an item being supposed to occur as a result of a sequence of minor failures at the points of a Poisson process. It should be noted that, even if the Poisson process changed its parameter α after each minor failure, the transform $f^*(s)$ would still be a rational function.

We begin by establishing a relation between $f^*(s)$ and $h^*(s)$, the Laplace transform of $h(t)$. We pick up the exposition from page 19.

Let $T_1 + T_2 + \dots + T_r = S_r$, where the random variable S_r has the distribution function $K_r(t)$ and density function $k_r(t) = K_r'(t)$.

$$\text{Now Prob } \{N_t < r\} = \text{Prob } S_r > t = 1 - K_r(t).$$

$$\begin{aligned} \text{Hence Prob } \{N_t = r\} &= K_r(t) - K_{r+1}(t) \quad (\text{we define } K_0(t) = 1) \\ &= P_{N_t}(r), \text{ say.} \end{aligned}$$

If we now define the probability generating function of N_t as

$$G(t, z) = \sum_{r=0}^{\infty} P_{N_t}(r) z^r$$

we have

$$\begin{aligned} G(t, z) &= \sum_{r=0}^{\infty} [K_r(t) - K_{r+1}(t)] z^r \\ &= 1 + \sum_{r=1}^{\infty} (z-1) z^{r-1} K_r(t) \end{aligned}$$

Since $E\{N_t\} = \frac{\partial G}{\partial z}(t, 1) = \sum_{r=1}^{\infty} K_r(t)$,

$$h(t) = \frac{d}{dt} [E\{N_t\}] = \sum_{r=1}^{\infty} k_r(t) .$$

Now, taking Laplace transforms,

$$h^*(s) = \sum_{r=1}^{\infty} k_r^*(s)$$

and since $k_r^*(s) = [f^*(s)]^r$,

$$h^*(s) = \frac{f^*(s)}{1 - f^*(s)} .$$

From this relation it is clear that when $f^*(s)$ is a rational function so is $h^*(s)$. Furthermore, s is a factor of its denominator, since $f^*(0) = 1$. Remembering that $f^*(s)$ is a moment generating function, $f^{*'}(0) = -\tau$, which is not zero (p.18), so that $s = 0$ is a simple root of $1 - f^*(s)$.

The object of examining the roots of $1 - f^*(s)$ is to consider the expansion of $h^*(s)$ as a sum of partial fractions. We can then consider the approach to equilibrium in a manner similar to the discrete case (pp.21,22). Having expressed

$h^*(s)$ as the ratio of two polynomials, we can label the roots of the denominator s_j ; $j = 1, 2, \dots$. Also, we may specify $s_1 = 0$. Then for $j > 1$, we assert that s_j has a negative real part. For if $\Re(s_j) > 0$, then

$$|f^*(s_j)| \leq \int_0^{\infty} |\exp(-s_j t)| f(t) dt < \int_0^{\infty} f(t) dt = 1$$

which contradicts $f^*(s_j) = 1$.

The other possibility is that there exists a pure imaginary root $s_j = ip$, say, where p is real. This would imply that

$$f^*(ip) = 1 = \int_0^{\infty} \exp(-ipt) f(t) dt = \int_0^{\infty} f(t) dt$$

which is not possible for a continuous $f(t)$.

The partial fraction expansion of $h^*(s)$ is of the form:

$$h^*(s) = \sum_{j=1}^m \frac{a_j}{s - s_j},$$

where m is the degree of the denominator of $f^*(s)$. This assumes that the s_j are all different, in which case the numerators are given by the formula

$$a_j = \frac{-f^*(s_j)}{f^{*'}(s_j)}.$$

Modifications similar to that on page 22 apply in the case of repeated roots. $s_1 = 0$ is not a repeated root, as we have seen, so that

$$a_1 = \frac{f^*(0)}{-f^{*'}(0)} = \frac{1}{2}.$$

We now need some standard results of Laplace transform theory which can be expressed in the following way. Denote the inverse transform of a function $g(s)$ by $L^{-1}(g)$. Then,

$$L^{-1} \left(\frac{1}{s} \right) = 1$$

$$L^{-1} \left(\frac{1}{s-a} \right) = \exp(at)$$

$$L^{-1} \left(\frac{1}{(s-a)^n} \right) = \frac{t^{n-1}}{(n-1)!} \exp(at)$$

Hence, if the s_j are all different,

$$h(t) = \frac{1}{t} + \sum_{j=2}^m a_j \exp(s_j t)$$

and, since the non-zero s_j all have negative real parts, the limit $\frac{1}{t}$ of $h(t)$ as $t \rightarrow \infty$ is approached exponentially.

The presence of a k -fold root s_r in the denominator of $h^*(s)$ will give rise to terms like

$$\frac{a_p}{(s-s_r)^p} \quad p = 1, 2, \dots, k$$

in the partial fraction expansion of $h^*(s)$, and on inversion to terms like

$$\frac{t^p}{p!} \exp(s_r t)$$

in $h(t)$. The approach to zero is again comparable to exponential decay.

5. So far we have considered the way in which the renewal density approaches its limiting value with increasing time. Our final special case considers the way in which equilibrium is approached with increasing serial number of events.

Let us suppose that our basic time unit is one day, and each day is divided into k fractions, a fraction being the smallest observable unit of time. The distribution of intervals between events is assumed discrete, each interval being a whole number of fractions, with one fraction the smallest possible interval. Suppose an event occurs during the r th fraction of some particular day ($1 \leq r \leq k$), then we can define the process as being in state E_r until the next event.

Thus, a transition occurs at each event, and since the transition probability p_{rs} of changing from state E_r to state E_s as a result of an event is independent of the previous history of the process, the sequence of states will form an embedded Markov chain. If the process is in state E_r , and the interval between the event which brought it there and the subsequent event is p fractions, then the next state is E_j , where $j = r + p \pmod{k}$, unless this is zero, in which case $j = k$.

Since we have specified that an interval of one fraction is possible, all states are accessible. The transition probability p_{rs} is obtained by adding the probabilities of all intervals t for which $t = s - r \pmod{k}$. Hence, if the values of p_{1j} , $j = 1, 2, \dots, k$, are denoted by q_j , then the transition matrix is

$$P = \begin{pmatrix} q_1 & q_2 & \cdot & \cdot & q_k \\ q_k & q_1 & \cdot & \cdot & q_{k-1} \\ q_{k-1} & q_k & \cdot & \cdot & q_{k-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ q_2 & q_3 & \cdot & \cdot & q_1 \end{pmatrix}$$

The matrix is circulant, and hence doubly stochastic (Feller⁴, p.358), and in the limit all states have equal probability of $\frac{1}{k}$. We propose to obtain an explicit formula for the probability of each state after n events, and hence show that the approach to equilibrium proceeds at a rate comparable with $\exp(-n)$. Some numerical results will be obtained from this result in Chapter III.

If initially the probabilities of the process being in state j are a_j , $j = 1, 2, \dots, k$, and A is the row matrix defined by $A = (a_1 \ a_2 \ a_3 \ \cdot \ \cdot \ \cdot \ a_k)$, then after n transitions ("events") the state probabilities are given by the row matrix AP^n . To calculate P^n we diagonalize P , noting that if $P = UDU^{-1}$, where D is a diagonal matrix, then $P^n = UD^nU^{-1}$.

Let $\theta = \exp\left(\frac{2\pi i}{k}\right)$ be a k th root of unity.

Then if U_r is a column vector defined by

$$U_r = (1 \ \theta^r \ \theta^{2r} \ \cdot \ \cdot \ \cdot \ \theta^{(k-1)r})$$

the j th element of the column vector (PU_r) is

$$\sum_{m=1}^k q_m \theta^{(m+j-2)r}$$

$$= \left(\sum_{m=1}^k q_m \theta^{(m-1)r} \right) \theta^{(j-1)r}$$

Hence, if $\lambda_r = \sum_{m=1}^k q_m \theta^{(m-1)r}$

$$PU_r = \lambda_r U_r \quad \text{for } r = 1, 2, \dots, k$$

If, therefore, D is the diagonal matrix $\{d_{ij}\}$, where

$$d_{ii} = \lambda_i$$

$$d_{ij} = 0, \quad i \neq j \quad i, j = 1, 2, \dots, k$$

and U is the $k \times k$ matrix with r th column U_r as already defined,

then $PU = UD$

$$P = UDU^{-1} .$$

The n -step transition probabilities, $p_{is}^{(n)}$, are obtained as the elements of P^n , and the probability of each state after n events is given by the appropriate element of the matrix $AP^n = AUD^nU^{-1}$. The rapidity with which these probabilities approach their limiting values is determined by the nature of the latent roots λ_r of P, which form the non-zero elements of the matrix D.

Now, $\lambda_r = \sum_{m=1}^k q_m \theta^{(m-1)r}$

so that $\lambda_k = 1$

and $|\lambda_r| \leq \sum_{m=1}^k |q_m \theta^{(m-1)r}|$
 $= \sum_{m=1}^k q_m |\theta^{(m-1)r}| = 1$

and the inequality will be strictly less than, unless all

$\theta^{(m-1)r}$ have the same argument, which is only true for $r = 0$. Thus $(\lambda_r)^n \rightarrow 0$ as $n \rightarrow \infty$ for $r \neq k$, and the limiting form of D^n is $\{d_{ij}\}$ where $d_{ij} = 1$ for $i = j = k$, and 0 otherwise.

The calculation of U^{-1} follows from the result

$$\sum_{m=1}^k \theta^{im} \theta^{mj} = \sum_{m=1}^k \theta^{(i+j)m} = \begin{cases} k & \text{if } i+j = 0 \pmod{k} \\ 0 & \text{otherwise.} \end{cases}$$

Hence the r th column of U^{-1} has elements

$$\frac{1}{k} \theta^{(k-r+1)j}, \quad j = 1, 2, \dots, k.$$

The limiting form of the matrix P^n is therefore,

$$\begin{aligned} & \frac{1}{k} \begin{pmatrix} 1 & 1 & \dots & \dots & \dots & 1 \\ \theta & \theta^2 & \dots & \dots & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \theta^{k-1} & \dots & \dots & \dots & \theta & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & \dots & \dots & 0 \\ 0 & 0 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & 1 \end{pmatrix} \begin{pmatrix} 1 & \theta^{k-1} & \dots & \dots & \theta \\ 1 & \theta^{k-2} & \dots & \dots & \theta^2 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \dots & \dots & \dots & 1 \end{pmatrix} \\ &= \frac{1}{k} \begin{pmatrix} 0 & 0 & \dots & \dots & 0 & 1 \\ 0 & 0 & \dots & \dots & 0 & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \theta^{k-1} & \dots & \dots & \theta \\ 1 & \theta^{k-2} & \dots & \dots & \theta^2 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & \dots & 1 \end{pmatrix} \\ &= \frac{1}{k} \begin{pmatrix} 1 & 1 & \dots & \dots & \dots & 1 \\ 1 & 1 & \dots & \dots & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & \dots & \dots & 1 \end{pmatrix} \end{aligned}$$

Hence $AP^n \rightarrow \frac{1}{k}$ since $\sum_{m=1}^k a_m = 1$.

This confirms the result quoted on page 28 that in the limit all states have equal probability of $\frac{1}{k}$.

The explicit formula for the state matrix will be obtained after assigning numerical values to the initial state probabilities. This simplifies the formulae a little without sacrificing any generality of technique.

We choose $a_1 = a_2 = \dots = a_{k-1} = 0$, and $a_k = 1$, so that initially the process is in state E_k with probability one. Then the probability that the process is in state E_j after n further events is $(a_j)_n$, the j th element in the row matrix.

$\frac{1}{k} (0 \ 0 \ \dots \ 0 \ 1) UDU^{-1}$, where U and D are as defined on page 29. Thus,

$$\begin{aligned} (a_j)_n &= \frac{1}{k} \sum_{m=1}^k \theta^{k-m} \lambda_m^n \theta^{(k-j+1)m} \\ &= \frac{1}{k} \sum_{m=1}^k \theta^{(k-j)m} \lambda_m^n \\ &= \frac{1}{k} \sum_{m=1}^k [\theta^{(k-j)m} (\sum_{i=1}^k q_i \theta^{(i-1)m})^n] \end{aligned}$$

If $n = 1$ this formula gives for the coefficient of q_i the value

$$\frac{1}{k} \sum_{m=1}^k \theta^{(k-j)m} \theta^{(i-1)m} = \frac{1}{k} \sum_{m=1}^k \theta^{(k-j+i-1)m}$$

and this sum is zero unless $k-j+i-1 = 0 \pmod{k}$, in which case it is unity. Thus, the formula gives for $(a_j)_1$ the value q_{j+1} for $j < k$, q_1 for $j = k$.

This agrees with the last row of the transition matrix P, and is a useful check on the formula.

For values of n greater than one the formula for $(a_j)_n$ is not a particularly simple one from which to compute. However, it does show quite clearly that, if the largest value of $|\lambda_m|$ for $m \neq k$ is a, say, then the difference, between the value of $(a_j)_n$ and $\frac{1}{k}$, is certainly less than a^n . Thus, the approach to equilibrium is exponential as asserted on page 28.

Chapter III

Simulating a Poisson Series

1. We now come to the second problem mentioned in the introduction: how to simulate a series in continuous time when all intervals distributions are discrete because of the minimum observable unit of time. We take the Poisson process because it is generally reckoned to be a straightforward matter to simulate the negative exponential distribution, and also because of the statement in the paper by Knowles and Stewart¹ (p.123) that a Poisson distribution never occurs as a result of simple random selection from an exponential intervals distribution. (There is unfortunately a reference at this point in the paper to a non-existent entry in the bibliography.) We reproduce their argument in brief.

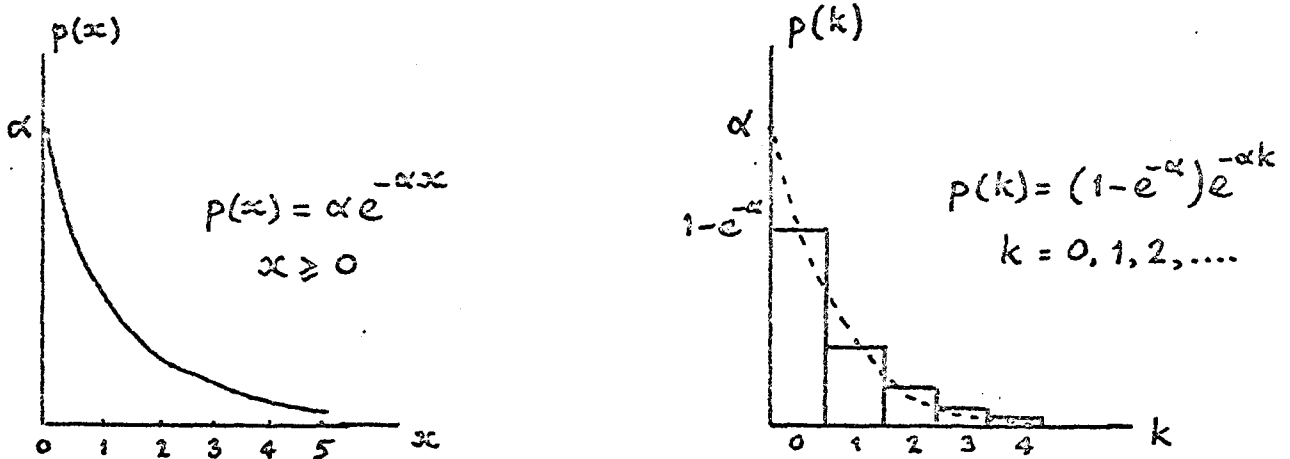


Fig. 1

Grouping the Negative Exponential Distribution

Grouping the exponential distribution to obtain the geometric on the right of Fig. 1 gives

$$p(k) = \int_k^{k+1} \alpha e^{-\alpha x} dx = pq^k$$

where $p = (1 - e^{-\alpha})$ and $q = e^{-\alpha}$.

The resulting geometric distribution of intervals $\{pq^k\}$, $k = 0, 1, 2, \dots$ (note the existence of zero intervals), is then substituted into the equivalent of equations (5.2) (p. 10), which will always yield a geometric number of events distribution.

The technique is then applied to a specific example, taking $\alpha = 1$, and drawing on a random sample of 500 "events" (intervals) from the corresponding geometric distribution. The result is illustrated by displaying the two histograms as in Figure 2.

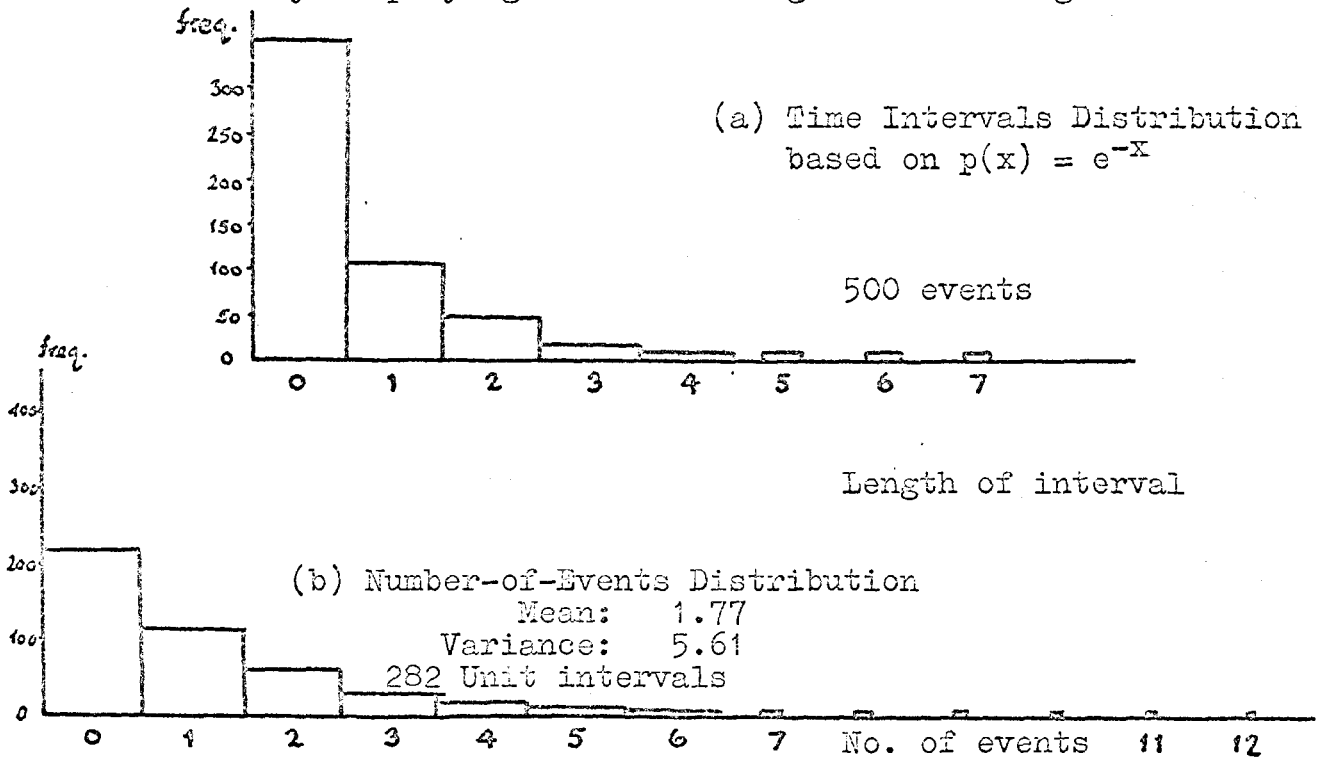


Fig. 2

Histograms of Intervals and Number-of-Events Distributions

As the authors rightly observe, the number of events histogram of Figure 2 could not be mistaken for that of a Poisson distribution.

There are several criticisms which can be levelled at this procedure. The first is that the grouping of the continuous distribution is very coarse: the first column alone represents a probability of $1 - \frac{1}{e} \approx 0.632$. Furthermore, all these intervals of less than unit length in the continuous distribution are identified as zero intervals in the grouped distribution. A sequence of three intervals of lengths 0.8, 0.9, 0.7 from the continuous distribution would represent 4 events of a Poisson process with an interval between the first and fourth of 2.4 units of time. After the grouping of intervals as described, these would be listed as 4 simultaneous events. There is one further point. After grouping the exponential distribution in the manner described, the resulting set of intervals, although having only integral values, would have no upper limit - a further approximation would have to be made in truncating the distribution at some point where the probability of a larger interval reaches an acceptably small value.

2. There is a well-known method of simulating a negative exponential distribution which avoids all these criticisms. (See, for instance, Tocher⁹, p.14.) The method is generally used for converting random numbers into sample values from a known distribution. It depends on the fact that, if X is a random variable with distribution function $F(x)$, then $Y = F(X)$

is uniformly distributed over $(0,1)$ (Fraser¹², p.85). Thus, if X has the negative exponential distribution with unit mean, its distribution function, $F(x) = 1 - e^{-x}$ and $Y = 1 - e^{-X}$, will be uniform over $(0,1)$. So will $1 - Y$.

The procedure for generating negative exponential intervals with unit mean is therefore to choose a value of $P = 1 - Y$ from the uniform $(0,1)$ distribution, and then take $X = -\ln P$. It is in the selection of the samples from the uniform distribution that the approximation to a continuous distribution by a discrete one is made.

Suppose a 2-digit random number is used. Then a decimal point is prefixed and .005 added. Thus the random number 47 yields 0.475 and 02 yields 0.025. This procedure gives a discrete random variable uniformly distributed over the numbers $0.005(2n+1)$ ($n = 0,1,2,\dots,99$). This is the approximation to the random variable P above, and corresponding values of X are obtained, for instance, by using 4 figure tables. Table 1 enables 2-digit random numbers to be translated directly into negative exponential variates.

Some trial runs were carried out using random number tables and a calculating machine. A sequence of intervals was generated by the method described above, the machine being used to sum the intervals, giving the elapsed time T after each "event". If the r th event occurs when $T = T_r$, then it is held to have occurred during day N , where $N = [T_r] + 1$, where $[x]$ denotes the integral part of x . The values of T were held to 3 decimal places. In order to eliminate the

Table I

Conversion of 2-digit random numbers into negative exponential
variates

If N is an integer randomly chosen from the set $0, 1, \dots, 99$, then $X = -\ln 0.005(2N + 1)$ has approximately the negative exponential density function e^{-X} .

N	X	N	X	N	X	N	X
00	5.298	25	1.367	50	0.683	75	0.281
01	4.200	26	1.328	51	0.664	76	0.268
02	3.689	27	1.291	52	0.644	77	0.255
03	3.352	28	1.255	53	0.626	78	0.242
04	3.101	29	1.221	54	0.607	79	0.229
05	2.901	30	1.188	55	0.589	80	0.217
06	2.733	31	1.155	56	0.571	81	0.205
07	2.590	32	1.124	57	0.553	82	0.192
08	2.465	33	1.094	58	0.536	83	0.180
09	2.354	34	1.064	59	0.519	84	0.168
10	2.254	35	1.036	60	0.503	85	0.157
11	2.163	36	1.008	61	0.486	86	0.145
12	2.080	37	0.981	62	0.470	87	0.134
13	2.002	38	0.955	63	0.454	88	0.122
14	1.931	39	0.929	64	0.439	89	0.111
15	1.864	40	0.904	65	0.423	90	0.100
16	1.802	41	0.880	66	0.408	91	0.089
17	1.743	42	0.856	67	0.393	92	0.078
18	1.687	43	0.832	68	0.378	93	0.067
19	1.635	44	0.810	69	0.364	94	0.057
20	1.585	45	0.788	70	0.350	95	0.046
21	1.537	46	0.766	71	0.336	96	0.036
22	1.492	47	0.745	72	0.322	97	0.025
23	1.448	48	0.724	73	0.308	98	0.015
24	1.407	49	0.703	74	0.294	99	0.005

transient effects due to the start of the process, a 3-digit random number was chosen to determine the start of the process in day -7, and counts were not recorded until day 1 was reached. To illustrate the method, the start of one of the trials is shown.

The initial 3-digit random number chosen was 764, so the process was started at $t = -6.764$. The process then continued as indicated in the following partial record:

Random number	Interval (from Table I)	Elapsed time	Number of events
37	0.981	-5.783	
78	0.242	-5.541	
93	0.067	-5.474	
09	2.354	-3.120	
23	1.448	-1.672	
47	0.745	-0.927	
71	0.336	-0.591	
44	0.810	0.219	1 in day 1
09	2.354	2.573	0 " 2
19	1.635	4.208	1 " 3
32	1.124	5.332	0 " 4
14	1.931	7.263	1 " 5
31	1.155	8.418	1 " 6
			0 " 7
			1 " 8

etc.

In a simulation covering 227 complete "days", 201 events were recorded, the observed distribution of the number of events per day being as follows:

Number of events	Observed	Number of days Expected (from a Poisson distribution)
0	92	84
1	84	84
2	38	42
3	11	14
4	2	3
Total	<u>227</u>	<u>227</u>

The number of events distribution is a good fit to the Poisson distribution ($\chi^2 = 2.12$), as were similar trials. Detailed tests on other simulations are given on page 52.

The intervals distribution is not the correct one for a Poisson process; in particular, extremely short and very long intervals are excluded. There are exactly 100 possible intervals, all having the same probability, the smallest being 0.005 and the largest 5.298. The practical man would probably regard this truncation as an advantage, likely to make his model more realistic than a true exponential distribution rather than less. However, our concern is to investigate the adequacy or otherwise of a discrete approximation to a negative exponential distribution, and it is interesting to compare the way in which the probability density curve is converted into a histogram by the method just used, with the Knowles and Stewart technique illustrated in Figure 1 (p.33). In order to make the comparison, a much coarser approximation will be considered; one in which there are only ten possible intervals, each having a probability of one tenth. This is achieved by choosing a single random digit, prefixing a decimal point and adding 0.05 to approximate $P = 1 - Y$. The following results are obtained:

P	.05	.15	.25	.35	.45	.55	.65	.75	.85	.95
$-\ln P$	3.00	1.98	1.43	1.05	0.78	0.58	0.42	0.28	0.16	0.05

The mean of these ten values of $-\ln P$ is 0.973, rather than unity. This particular discrepancy will be dealt with later (p.40) when we propose an improved method of grouping into a

small number of intervals. The basis for converting a continuous density curve into a histogram is the same, however. The resulting histogram has a finite number of columns of equal area, as illustrated in Figure 3, which contrasts with the method illustrated by Figure 1 (p.33).

3. The exact details of the procedure are perhaps more clearly revealed by Figure 4 on page 41. The unit interval (0,1) on the ordinate axis (probability) of the distribution function graph is divided into ten equal sub-intervals. If p_r is the midpoint of the r th sub-interval, then $F^{-1}(p_r)$ is taken to be the r th value of t to be assigned a non-zero probability. Thus, for instance, point B in Figure 4 identifies the 8th value of t , 1.43. An obvious improvement on this procedure would be to use the endpoints of the ten sub-intervals on the ordinate axis. Suppose q_r and q_r' are the endpoints of the r th sub-interval. Then the r th value of t to be assigned a non-zero probability would be more appropriately chosen as the mean of $F^{-1}(q_r)$ and $F^{-1}(q_r')$. This would mean, for example, instead of taking point B in Figure 4 for the 8th value, choosing the midpoint of the segment AC. Since this procedure would not give a finite value for the 10th value of t , this last value is best chosen to give a unit mean for the ten values, disposing of the discrepancy referred to on page 39.

To highlight the effect of this modification procedure, consider the extreme case of a histogram with only two columns. The first procedure would assign values 0.25, 0.75 to P,

$$f(t) = e^{-t}$$

Partitioning the area under the density curve of the negative exponential distribution into ten equal parts.

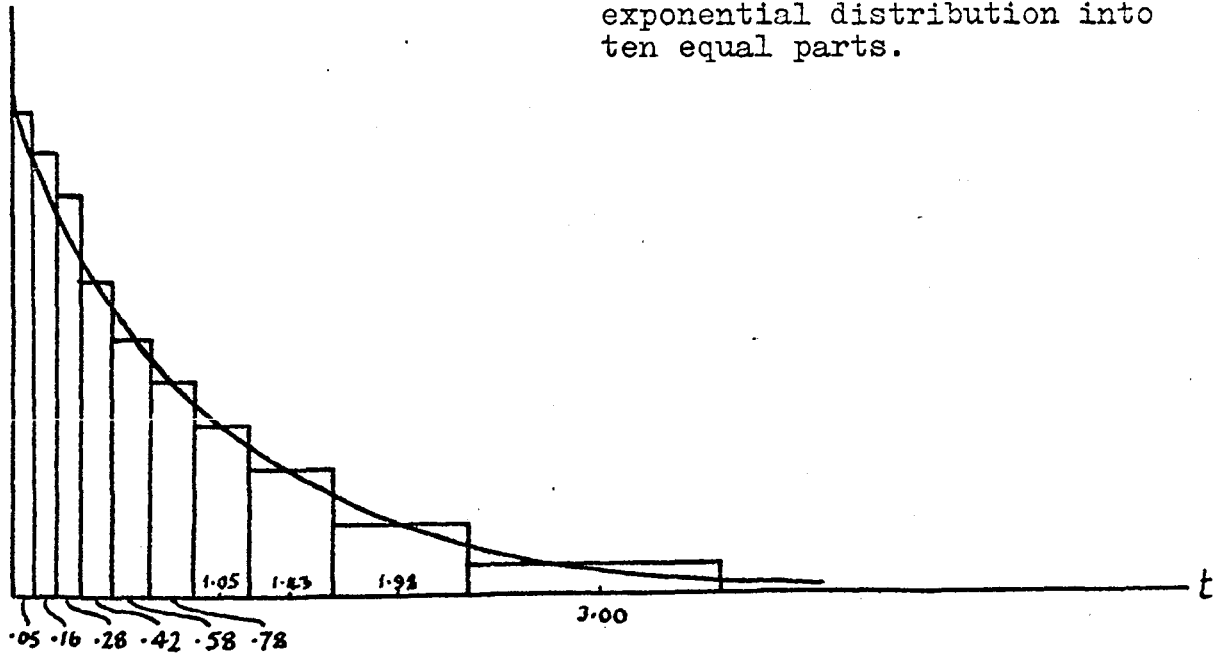
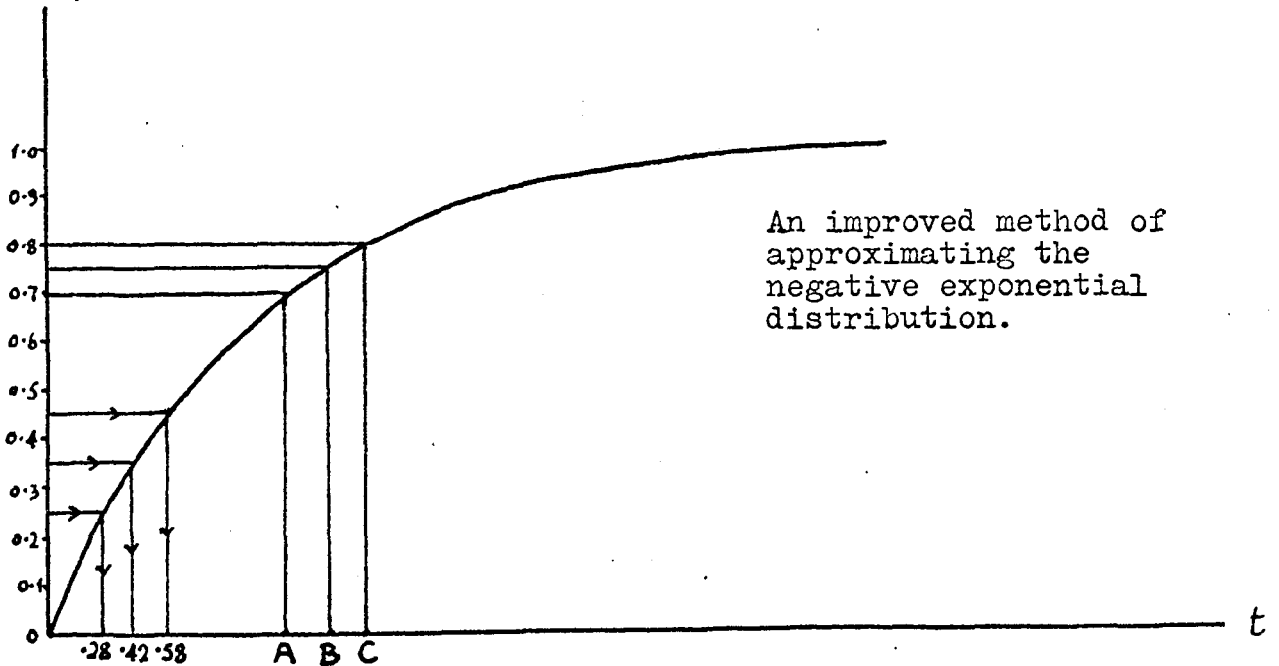


Fig. 3

$$F(t) = 1 - e^{-t}$$



An improved method of approximating the negative exponential distribution.

Fig. 4

corresponding to the random bits 0 and 1 respectively. The two possible values of t , each with probability $\frac{1}{2}$, would be $-\ln 0.25$ and $-\ln 0.75$, i.e. 1.386 and 0.288, with mean 0.837. The "improved" procedure would give for the smaller value the mean of $-\ln 0.5$ and $-\ln 1$, i.e. 0.347, and the larger value, to make up a mean of 1, would be 1.653.

Values of t , each with probability $\frac{1}{n}$, corresponding to histograms with n columns, calculated by the "improved" procedure, are given in Table II. They are rounded to 2 decimal places.

Table II

Conversion of the density curve of the exponential distribution with unit mean into a histogram with n columns each of equal area

n	Mid-points of intervals									
2	0.35	1.65								
3	0.20	0.75	2.05							
4	0.14	0.47	1.04	2.35						
5	0.11	0.41	0.76	1.26	2.46					
6	0.09	0.29	0.50	0.90	1.44	2.78				
7	0.08	0.20	0.46	0.71	1.05	1.60	2.90			
8	0.06	0.21	0.38	0.58	0.84	1.18	1.73	3.02		
9	0.06	0.18	0.33	0.50	0.70	0.95	1.30	1.85	3.13	
10	0.05	0.16	0.29	0.43	0.60	0.80	1.06	1.41	1.96	3.24

We are looking at these discrete approximations to the exponential distribution in terms of how well they approximate a Poisson process when used for the intervals distribution. If our criterion for a good approximation is the census

distribution over intervals of unit length, which should be a Poisson distribution with unit mean, then taking $n = 2$ will not be satisfactory. It will not be possible to obtain more than 3 events in a day with this choice of intervals distribution.

4. There is no simple way of determining the census distribution when any of these intervals distributions are used. It has been established in Chapter II (p. 22) that the equilibrium census distribution will have unit mean, but beyond that the straightforward method of making a comparison with the Poisson distribution is by a computer simulation of a realization of the process.

Before carrying out the simulation, it was necessary to apply the results of Chapter II on the approach to equilibrium, in order to decide how long after the start of the process the transient effect of a determined starting point could be neglected. In order to reduce some heavy computation, the case of 10 equal intervals was chosen from Table II (p.42), but the intervals were rounded to one place of decimals. The rounding was done in such a way as to avoid zero intervals (thus 0.05 was rounded up to 0.1) and to preserve a unit mean. This process gave the following ten intervals, each with a probability of $\frac{1}{10}$:

0.1 0.2 0.3 0.4 0.6 0.8 1.1 1.4 1.9 3.2

In the notation of II.2 (p.17), if we take time units of 0.1, the generating function $G(s)$ is given by

$$G(s) = \frac{1}{10} [s + s^2 + s^3 + s^4 + s^6 + s^8 + s^{11} + s^{14} + s^{19} + s^{32}]$$

with $\tau = 10$, and $\lim_{t \rightarrow \infty} u(t) = \frac{1}{10}$, as proved on pp. 20-22 for the case when $G(s)$ is a polynomial.

The equilibrium distribution is therefore such that the expected number of events in 10 units, equalling 1 "day", is unity. In order to demonstrate the exponential approach to equilibrium, the first few values of $u(t)$ were calculated and plotted against t . Instead of expanding $U(s)$ as a sum of partial fractions (p.21), it was found more convenient to work directly from the relation

$$u_n = u_0 p_n + u_1 p_{n-1} + \dots + u_n p_0,$$

taking $p_k = 0$ for all values of k except 1,2,3,4,6,8,11,14,19 and 32, for each of which $p_k = 0.1$.

The calculated values of u_n are given in Table III (p.45), and the graph, Figure 5, shows u_n converging to its equilibrium value of 0.1.

The values of u_n are given to 3D, although the calculations carried 9D throughout, in order to eliminate rounding errors, bearing in mind that these build up with a recursive calculation. Taking the values in cycles of 32, it can be seen that large fluctuations due to the influence of the start of the process have almost died out by the end of the second cycle. This corresponds to 64 units of 0.1, or 6.4 "days".

The embedded Markov chain of Chapter II (p.27) can be used to check the approach to equilibrium as a function of serial number of event, rather than of time. The transition probabilities are calculated as follows:

Table III

Convergence of u_n , the expected number of events at time n

u_n has generating function $U(s) = \frac{1}{1 - F(s)}$, where $F(s)$ is the generating function of the intervals distribution and, in this case, $F(s) = \frac{1}{10} (s + s^2 + s^3 + s^4 + s^6 + s^8 + s^{11} + s^{14} + s^{19} + s^{32})$.

n	u_n	n	u_n	n	u_n
1	0.1	25	0.107	49	0.098
2	0.11	26	0.076	50	0.096
3	0.121	27	0.088	51	0.111
4	0.133	28	0.087	52	0.104
5	0.046	29	0.075	53	0.097
6	0.141	30	0.090	54	0.103
7	0.054	31	0.080	55	0.101
8	0.148	32	0.169	56	0.093
9	0.061	33	0.117	57	0.104
10	0.065	34	0.100	58	0.097
11	0.150	35	0.097	59	0.098
12	0.080	36	0.110	60	0.099
13	0.057	37	0.087	61	0.096
14	0.176	38	0.112	62	0.099
15	0.081	39	0.095	63	0.098
16	0.076	40	0.105	64	0.103
17	0.086	41	0.096	65	0.104
18	0.075	42	0.094	66	0.100
19	0.172	43	0.104	67	0.098
20	0.097	44	0.102	68	0.102
21	0.080	45	0.089	69	0.098
22	0.110	46	0.114	70	0.102
23	0.090	47	0.099	71	0.100
24	0.070	48	0.094	72	0.100

The graph shows how the probability of an event at time n varies as n increases given an initial event at time $n = 0$. Interarrivals distribution is the one specified on p. 43

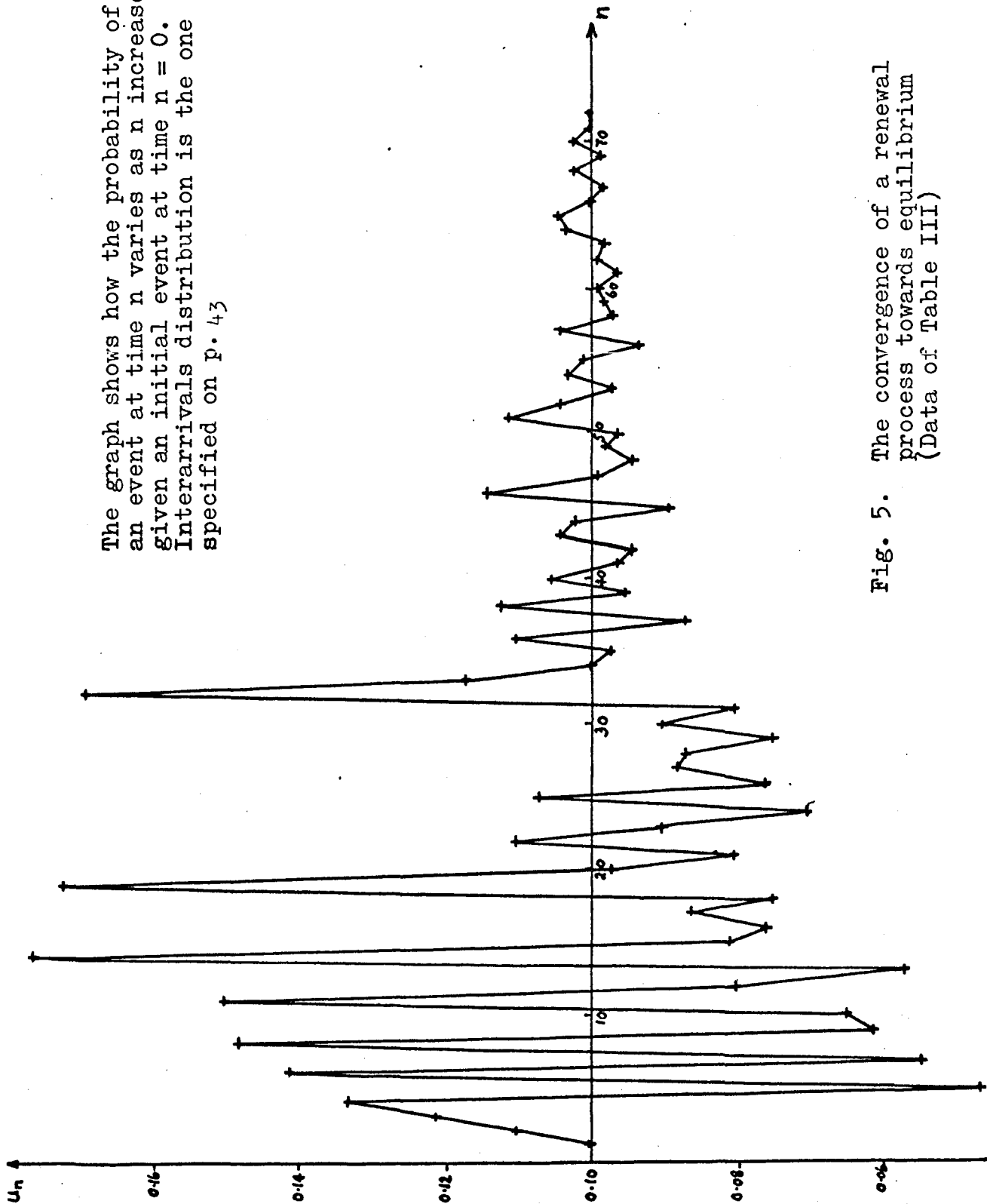


Fig. 5. The convergence of a renewal process towards equilibrium (Data of Table III)

An event at time k units + r tenths (k integral, $0 \leq r \leq 9$) is held to have occurred in the $(r+1)$ th fraction of day $k + 1$.

$q_1 = 0$ (no interval is an integral multiple of 1);

$q_2 = 0.2$ (intervals of 0.1 and 1.1 will increase the number of fractions by one).

Similarly, $q_3 = 0.2$, $q_4 = 0.1$, $q_5 = 0.2$, $q_6 = q_8 = 0$,

$q_7 = q_9 = q_{10} = 0.1$.

The latent roots have the following absolute values:

$ \lambda_1 $	=	1
$ \lambda_2 $	=	0.314
$ \lambda_3 $	=	0.138
$ \lambda_4 $	=	0.177
$ \lambda_5 $	=	0.362
$ \lambda_6 $	=	0.2
$ \lambda_7 $	=	0.362
$ \lambda_8 $	=	0.177
$ \lambda_9 $	=	0.138
$ \lambda_{10} $	=	0.314

Since $|\lambda^n| \leq |\lambda|^n$, the largest value of the diagonal matrix D^n (p.29) in absolute value (except for $d_{10,10}$) will be less than 0.362^n .

Now, $(0.362)^6 = 0.0022$ approximately, which shows that a good approximation to equilibrium has been reached after only 6 events. The start of the simulation described on page 38 was a random point in day -7, and in the computer simulation which follows it was a random point in day -10. Equilibrium in the sense of equal probability for all fractions will take

longer to achieve if each day is broken into 1,000 fractions, but it will be comparable as far as the census distribution is concerned if the number of events per day is counted in each case.

5. The computer was used to generate a sequence of pseudo-random numbers, uniform over $(0, 1)$ by the multiplicative congruence method (Tocher⁹, pp. 75-81), the first digit after the decimal point being taken as a random digit. The program listing (pp. 49-50) shows how this was converted into an interval from the table on page 42. For each set of intervals, the same sequence of digits was used, although this does not mean that there is any direct comparison between the realizations, since different proportions of the random digits are discarded in the different realizations.


```
// FOR
*NAME DFSIM
*IOCS(CARD,1132 PRINTER)
*LIST SOURCE PROGRAM
*ONE WORD INTEGERS
    DIMENSION NVENT(10),TERVL(8,10)
    READ(2,21)TERVL
21  FORMAT(8F5.2)
5   READ(2,100)M,IX,LTOP,TIME
    IF(IX)60,60,50
50  DO 20 N=1,10
20  NVENT(N)=0
    NDAYS=0
    K=-1
100 FORMAT(3I7,F5.2)
    DO 7 L=1,L TOP
C   GENERATE AN INTERVAL
30  CALL RANDD(IX,IY,ID)
    X=TERVL(M,ID)
    IF(X)30,30,32
32  TIME=TIME+X
    IF(TIME)30,31,31
31  NCOMP=TIME
    IF(K)33,33,34
33  NEMP=NCOMP
    I=NCOMP
    K=1
    GO TO 30
34  J=NCOMP-I
    I=NCOMP
C   TEST IF NEW DAY HAS BEEN ENTERED
    IF(J-1)3,4,4
C   IF NOT INCREASE NUMBER OF EVENTS BY ONE
3   K=K+1
    GO TO 7
C   IF SO INCREASE COUNT OF DAYS WITH K EVENTS ALSO EMPTY DAYS IF ANY
4   NDAYS=NDAYS+J
    NVENT(K)=NVENT(K)+1
    NEMP=NEMP+J-1
C   RECORD ONE EVENT IN NEW DAY
    K=1
7   CONTINUE
8   FORMAT('1','SIMULATION COVERED',I7,' COMPLETE DAYS' /)
    WRITE(3,8)NCOMP
    WRITE(3,9)NEMP
9   FORMAT(6X,20H EVENTS DISTRIBUTION//6X,27HNO. OF EVENTS   NO. OF DA
1YS//12X,1HO,10X,I4)
    DO 10 N=1,10
```

PAGE 2

```
10 WRITE(3,11)N,NVENT(N)
11 FORMAT(I13,I14)
   GO TO 5
60 CALL EXIT
   END
```

```
// FOR
*LIST SOURCE PROGRAM
*ONE WORD INTEGERS
  SUBROUTINE RANDD (IX,IY,ID)
    IY=IX*899
    IF(IY)5,6,6
5    IY=IY+32767+1
6    YFL=IY
    YFL=YFL/32767.
    ID=10.*YFL+1
    IX=IY
    RETURN
  END
```

6. For this series of simulations, the following statistical tests were carried out on the census distributions:

(a) to find whether the mean number of events per day was consistent with the hypothesis that the census distribution was Poisson, with parameter 1;

(b) to find whether the census distribution could be fitted by a Poisson distribution with unit mean, as measured by the χ^2 goodness-of-fit test;

(c) it being assumed that the mean number of events per day was unity (as established by the Renewal Theorem), to determine whether the variance of the number of events per day was significantly different from one. For this test, since

$\frac{(n-1)s^2}{\sigma^2}$ is a χ^2 variable, with $n - 1$ degrees of freedom,

σ^2 being taken as 1, Fisher's approximation, that $\sqrt{2\chi^2}$ is approximately normal, $(\sqrt{2n-1}, 1)$, was used. The approximation is very good for $n = 100$ (Kendall and Stewart¹⁰, p.374).

The results are displayed on page 52. They show that for a series of 1,000 events or thereabouts, a census distribution not discernibly different from a Poisson distribution is obtained with only the roughest of approximations to a negative exponential gap distribution. It would not be true, of course, that a census distribution taken over intervals of length substantially shorter than one day would be equally close to a Poisson distribution.

Table IV

Statistical tests carried out on computer simulations of series of events, using the intervals distributions of Table II

	3	4	5	6	7	8	9	10
Number of intervals	969	980	1002	979	977	975	967	1010
Number of days	999	1000	1000	1000	1000	999	999	1000
Number of events	1.03	1.02	1.00	1.02	1.02	1.03	1.03	0.99
Mean number of events per day	913.071	901.592	887.996	998.549	948.458	991.409	948.941	1015.901
$(n-1)s^2$	42.733	42.464	42.143	44.689	43.554	44.529	43.565	45.076
$\sqrt{2(n-1)s^2}$	44.011	44.261	44.755	44.238	44.193	44.147	43.966	44.933
$\sqrt{2n-1}$	-1.278	-1.797	-2.612	0.451	-0.639	0.382	-0.431	0.143
z	0.20	0.07	0.01	0.65	0.52	0.70	0.67	0.89
$2\Phi(- z)$	5.83	10.47	5.31	1.50	4.51	1.73	3.17	10.19
χ^2_S	4	4	4	4	4	4	4	4
Degrees of freedom	0.2	0.02	0.2	0.9	0.3	0.8	0.7	0.02
$P(\chi^2 > \chi^2_S)$								

$z = \sqrt{2(n-1)s^2} - \sqrt{2n-1}$ and is therefore approximately Normal (0,1) (see text, p.51). $\Phi(x)$ is the Normal distribution function; thus if Z is a random sample from a Normal (0,1) distribution, $\text{Prob}(|Z| > |z|)$ is given by $2\Phi(-|z|)$.

The results of the test carried out on the simulation runs may be summarized as follows:

(a) For a Poisson distribution with unit mean covering about 1,000 days, the standard error of the mean would be about 0.032. In each case, the mean number of events per day, shown in row 4 of Table IV, is within one standard error of the mean. These figures are therefore consistent with the hypothesis of a Poisson distribution with unit parameter.

(b) The sample values χ_s^2 obtained from the goodness-of-fit tests are shown in row 10 of Table IV, with the corresponding number of degrees of freedom in row 11. In row 12, the percentage point of the χ^2 distribution in Table IV of Fisher and Yates¹¹ which is adjacent to the value found for χ_s^2 is given. The null hypothesis of a Poisson distribution with unit parameter would be rejected at the 5% level of significance in two cases: the runs with 4 and with 10 intervals.

(c) On the null hypothesis that the variance of the census distribution is unity, the values of z in row 8 of Table IV, are samples from a Normal (0,1) distribution, whose absolute values will be exceeded with the probabilities given in row 9. The null hypothesis would be rejected at the 5% level in only one case, the run with 5 intervals.

7. Realizations of a stochastic process can vary widely (Feller⁴, Ch.III, pp.83-85, has some cautionary remarks about coin-tossing sequences). An experiment was carried out on the sequence generated by the intervals distribution of para.4 (p.43)

to discover the effect of shifting the counting grid. In the first run, an event at time $t = 0.0$ was assumed, 1,000 intervals were sampled, and the number of events recorded in the successive intervals $[0.0, 0.9]$, $[1.0, 1.9]$,, transient effects at the start of the process being ignored as irrelevant to this experiment.

In subsequent runs, using the same sequence of random digits to sample the intervals, the number of events in successive intervals spanning 1 unit of time were again recorded, but with the first such interval starting at 0.1, then 0.2, and so on up to 0.9. The counting distributions are recorded in Table V.

Table V

Showing the variation in census distributions obtained from the same series of events obtained by shifting the counting grid forward in successive steps of 0.1

First day starts at time $t =$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Number of events	Number of days									
0	360	347	351	347	353	358	356	358	357	364
1	366	392	393	403	397	379	386	380	374	364
2	200	189	180	170	163	189	181	190	198	197
3	64	58	58	65	76	59	61	54	59	62
4	8	14	18	14	8	11	13	15	12	14
5	2			1	3	4	3	2	1	
6								1		
Total	1000	1000	1000	1000	1000	1000	1000	1000	1001	1001

For a span of 1,000 days, the expected frequencies of a Poisson distribution would be:

Number of events	0	1	2	3	4	>4
Number of days	368	368	184	61	15	4

If each of the ten distributions above were subjected to a χ^2 goodness-of-fit test, the values of χ_s^2 obtained would be, respectively,

5.86 4.48 2.77 10.71 12.58 1.64 1.79 1.71 3.45 2.34

as compared with the tabular values (4 d.f.) of

χ^2	1.064	7.779	9.488	11.688	13.277
$P(\chi_s^2 > \chi^2)$	0.9	0.1	0.05	0.02	0.01

Thus, according to the initial point chosen for the counting grid, the same series of events can give rise to a census distribution judged by the χ^2 test to be consistent with a Poisson distribution, or to one which, by the same criterion, would be judged significantly different from the Poisson distribution at the 2% level of significance.

This result is not an indictment of the χ^2 test, but rather an indication of the wide variation to be found in census distributions generated by the same distribution of intervals between events. Whilst it is obviously very satisfying to obtain exact probability distributions for census distributions, it may, in some cases, obscure the fact that different realizations of a process may have widely differing appearances. If such a series of events is used as the input to the simulation of some

practical system, a queuing process, for instance, the behaviour of the system may be expected to vary widely from one run to the next.

Chapter IV

Summary and Conclusions

1. The original problem considered by Knowles and Stewart was the simulation of the arrival of orders at a factory by the daily post. Their approach began by postulating a distribution of intervals between arrivals in discrete units of one day, including zero intervals. This proved to be unsatisfactory, and they concluded that it was necessary to postulate two distributions - one of (non-zero) intervals between arrivals, and the other of the number of arrivals on days when at least one arrival occurs.

Having examined the appropriate models in Chapter I, we conclude that simply postulating a number of events distribution, which includes the possibility of zero events in a day, would cover the Knowles and Stewart problem and a wide range of similar situations. The number of events distribution would naturally be based on observations of the actual pattern of arrivals if these are available. If arrivals occur every day, the problem of the intervals between arrivals does not arise. Otherwise, it is shown that the interval between days when events do occur will be distributed geometrically, giving a pattern of bursts of arrivals at random points in time - a discrete equivalent of the Feller "lightning damage" model. All number of events distributions having the same mean, and the same probability of zero events will give the same distribution for the intervals between bursts of events.

2. An alternative method of simulating the flow of events is to formulate a model in which events occur only one at a time, and then to superimpose a counting grid on the sequence. The number of events within each division of the grid can then be interpreted as the number of arrivals in a day. If each interval is sampled from the same distribution, the basic model is a renewal process. For many purposes a model will be required for a process which has been going for some time and has reached a state of equilibrium. If a renewal process starts with an event at time $t = 0$, say, then the probability of an event at time $t = n$ (for a discrete process) will be a function of n , but will approach a limit of $\frac{1}{\tau}$ as $n \rightarrow \infty$, where τ is the mean of the intervals distribution.

Investigations on the rapidity of the approach to equilibrium are made in Chapter II for particular forms of the intervals distribution which will be widely used in practical applications. The first is a discrete distribution of intervals in which only a finite number of values have non-zero probabilities. This covers the cases where a continuous distribution is approximated by a discrete one by the use of a digital computer. It also includes intervals distributions based on an empirical distribution function applied to a sample from an observed series. The second form considered is the class of continuous distributions whose density functions have rational Laplace transforms. This includes gamma distributions with integral parameter, widely used in industrial replacement theory. The first case is then investigated from a second

point of view: the approach to equilibrium is looked at in terms of increasing serial number of event, rather than increasing time. This is done by studying a Markov chain embedded in the process. The conclusion in each case was that the approach to equilibrium was exponential. The consequence is that in a simulation exercise, only a small number of samples at the beginning of a process need to be discarded before it can be assumed that an equilibrium situation has been reached.

3. Simulating a flow of events as points in a renewal process can be performed using a digital computer by repeated sampling from an intervals distribution. The sampling will inevitably be from a discrete approximation if the intervals distribution is continuous (or discrete but not finite). Knowles and Stewart raised the question whether this approximation leads to serious discrepancies in the census distribution obtained from the counting grid.

The investigation in Chapter III concentrated on approximations to the negative exponential distribution, since it was claimed by Knowles and Stewart that this could never lead to a Poisson census distribution. Their technique for making the approximation was rejected, and a more conventional method adopted. This is based on the replacement of the uniform continuous distribution over $(0,1)$ by a set of n equal intervals of the segment $(0,1)$, each having probability $\frac{1}{n}$, before applying the probability integral transform.

There are certain difficulties in obtaining exact results for the census distributions, since within one realization, the number of events in two successive days (i.e. cells of the counting grid) are not independent, even when equilibrium has been reached. Simulation runs were therefore made, each of 1,000 events, and the census distributions sampled as a frequency table. The values of n , the number of subdivisions of the interval $(0,1)$, ranged from 3 to 10, the mean interval in each case being one day. Tests for the mean and variance of the census distributions, and for goodness-of-fit with the Poisson distribution using the χ^2 test, were made. In most cases the null hypothesis of a Poisson distribution of unit mean and variance was not rejected at the 5% level of significance. The results of Chapter II, on the approach to equilibrium of a renewal process, were used to determine the time for which the process should be allowed to run before imposing the counting grid, in order to ensure an equilibrium situation.

The conclusion drawn from the simulation experiments was that the Poisson process is highly tolerant of coarse approximations to the negative exponential gap distribution, provided these are made in the right way. There would be no necessity in practice to consider approximations anything like as crude as the ones employed in this exercise.

4. A further experiment was carried out, in which the counting grid was progressively displaced forward across a given series of points of a simulated Poisson process. At each shift, which was made through 0.1 of a day, in a process in which all intervals were multiples of 0.1 days, the census distribution was counted. There were wide variations between the frequency tables produced at each shift, so that some of them were good fits to a Poisson distribution, and others were significantly different from Poisson at the 5% level. It would appear that, if discrepancies in the census distributions do occur as a result of discrete approximations to intervals distributions, then they are likely to be masked by much larger variations between different realizations of the same process.

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