THE CHEO ASYMPTOTIC DENSITY

IN n-DIMENSIONS

by

Fook-bun Wong

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Examining Committee:

Chairman: C. Kim

A. R. Freedman
Senior Supervisor

A. Das

N. R. Reilly

D. Eaves

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ABSTRACT.

The main object of this thesis is to discuss the "Cheo" (or "C") lower asymptotic density, upper asymptotic and natural density for sets of n-tuples of non-negative integers. These densities are closely related to the K-asymptotic density of A. R. Freedman.

Chapter 0 states the main results of Schnirelmann density, asymptotic density and natural density for subsets of the sequence of non-negative integers (i.e., in the case of dimension one). The basic properties of Cheo (lower) asymptotic density in n-dimensions are obtained in chapter 1. Chapter 2 is concerned with additive questions involving the Cheo (lower) asymptotic density. Chapter 3 consists of the structure results similar to the additivity theorem. Finally, we compare the C-asymptotic density and the K-asymptotic density of Freedman in chapter 4.

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CHAPTER 0.

INTRODUCTION.

The purpose of this thesis is to develop an asymptotic type density called the Cheo asymptotic density for sets of n-tuples of nonnegative integers. This Cheo asymptotic density is similar to the asymptotic density of Freedman [4]. We shall present the basic definitions in chapter 1.

By way of introduction we shall first present a brief discussion of the subject in the case of dimension one, i.e., in the case of non-negative integers $0,1,2,\cdots$.

Let A be a set of nonnegative integers, for any positive integer n, let A(n) be the number of positive integers not greater than n in A. Then the <u>Schnirelmann density</u> α of A is defined as

$$\alpha = \text{glb}\{\frac{A(n)}{n} \mid n\geq 1\} \quad .$$

From this definition we immediately see that

(i) If $A\subset I$ (I = the set of all nonnegative integers), then $0\leq\alpha\leq 1$ and

(ii) $\alpha=1$ if and only if $A=I$ or $A=I \setminus \{0\}$.

If A and B are subsets of the nonnegative integers, then the <u>sum A+B</u> is defined to be the set

$$\{ a+b \mid a \in A,\ b \in B \}.$$

Schnirelmann and Landau [6,P3] proved that if $0 \in A \cap B$ and if α, β, γ are the Schnirelmann densities of A, B, C=A+B respectively, then

$$\gamma \geq \alpha+\beta-\alpha\beta$$

and if $\alpha+\beta \geq 1$, then $\gamma=1$.

The question of the relation of γ to α and β has been the subject of much investigation. In 1942, H.B. Mann [7] proved the famous $\alpha+\beta$ theorem, which states that, if A and B each contain 0, then

$$\gamma \geq \min \{ 1, \alpha+\beta \}.$$

The lower asymptotic density $\delta(A)$ or, briefly, <u>the asymptotic density</u> of A is defined to be

$$\delta(A) = \liminf_{n} \frac{A(n)}{n} .$$

From this definition we can easily see that the asymptotic density of any subset of I is invariant under tranlation, i.e.

$$\delta(x+A) = \delta(A)$$

for $A \subset I$ and $x \in I$.

For any subset A,B of I, A is said to be asymptotic to B (denoted by $A \sim B$) if there is an integer $N \geq 0$ such that $A \cap \{N, N+1, \cdots\cdots\} = B \cap \{N, N+1, \cdots\}$.

It can be seen that if $A \sim B$, then $\delta(A) = \delta(B)$.

Also, in analogy to one of the Schnirelmann-Landau results, we have if $\delta(A) + \delta(B) > 1$, then $C \sim I$.

The analogue of the $\alpha+\beta$ theorem of Mann for asymptotic density is not true; consider, for example, the case, where both A and B are composed of all nonnegative even integers, so that A+B is the same set. Then we have

$$\delta(A) = \delta(B) = \delta(A+B) = \tfrac{1}{2}.$$

However, Erdös [3] proved that if $0 \in A$, $0, 1 \in B$, $\delta(B) \leq \delta(A)$ and $\delta(A)+\delta(B) \leq 1$, then

$$\delta(C) \geq \delta(A)+\tfrac{1}{2}\delta(B).$$

We now state the remarkable result of M. Kreser which amounts to the best possible for asymptotic density along the lines of the $\alpha+\beta$ theorem. The language is that of Halberstam and Roth [6, P51]:

A "system" (A', B') is said to be "worse" than (A,B) if $A \subset A'$, $B \subset B'$ and $A+B \sim A'+B'$.

A "system" is said to be "degenerate mod g" if both A and B are unions of (entire) Congruence classes mod g.

A "system" (A,B) is said to be "<u>degenerate</u>" , if there exists an g such that (A,B) is degenerate mod g.

Kreser proved that, assuming $A,B,\subset I$, $C=A+B$, $\delta(A)+\delta(B)\leq 1$:

(i) If no system worse than (A,B) is degenerate, then $\delta(C)\geq\delta(A)+\delta(B)$.

(ii) If there is a system worse than (A,B), which is degenerate mod g and g is minimal, than $\delta(C)\geq\delta(A)+\delta(B)-1/g$.

We continue our discussion by defining the <u>upper asymptotic density</u> $\overline{\delta}(A)$ of A to be

$$\overline{\delta}(A) = \limsup_{n} \frac{A(n)}{n} .$$

For any $A\subset I$ if the lower asymptotic density and the upper asymptotic have same value, then we say that the <u>natural density</u> of A exists and write $\nu(A) = \delta(A) = \overline{\delta}(A)$. In this case

$$\nu(A) = \lim_{n\to\infty} \frac{A(n)}{n} .$$

Evidently, ν is a <u>finitely additive</u> set function, i.e. if $A_1, A_2, \cdots, A_n$ are subsets of I such that $A_i \cap A_j = \phi$ for $i\neq j$ and each A_i possesses natural density, then $\bigcup_{i=1}^{n} A_i$ has natural density and

$$\nu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \nu(A_i) .$$

The function f on sets is <u>countably additive,</u> if $A_1, A_2, \cdots, A_n, \cdots$ is a sequence of sets such that $f(A_i)$ is defined for each i and $A_i \cap A_j = \phi$ for $i \neq j$, then $f(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} f(A_i)$.

An example shows that ν is not countably additive. Let $A_1=\{1\}$, $A_2=\{2\}, \cdots, A_n=\{n\}, \cdots$. Then $\nu(A_i)=0$ for each i, therefore, $\sum_{i=1}^{\infty} \nu(A_i)=0$. But $\bigcup_{i=1}^{\infty} (A_i)=I \setminus \{0\}$, hence $1 = \nu(\bigcup_{i=1}^{\infty} A_i) \neq \sum_{i=1}^{\infty} \nu(A_i)=0$.

Although ν is not countably additive, we can prove that ν is "almost" countably additive. This result called the additivity theorem for natural density, states that, if $A_1, A_2, \cdots, A_n, \cdots$ are subsets of I such that $A_i \cap A_j = \phi$ for $i \neq j$ and each A_i possesses natural density, then there exist $B_1, B_2, \cdots, B_n, \cdots$ such that $B_i \sim A_i$, $i=1,2,\cdots$, the natural density of $\bigcup_{i=1}^{\infty} B_i$ exists and

$$\nu(\bigcup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} \nu(B_i) = \sum_{i=1}^{\infty} \nu(A_i).$$

R.C. Buck [1] has defined (lower) asymptotic density, upper asymptotic density for subsets of a measure space X. Briefly, the procedure is this: Take a countable increasing sequence K(i) of subset X which covers X and a sequence μ_i of measures defined on the same class of sets which includes the sets K(i). The following properties are assumed:

(i) $\mu_i(X) = 1$ for each i,

(ii) $\mu_i(K(j)) \to 0$ as $i \to \infty$ (fixed j);

(iii) For each i there exists $\alpha(i)$ such that, if $A \cap K(\alpha(i)) = \phi$ then $\mu_i(A) = 0$.

Then define the (lower) asymptotic density of A to be

$$\underline{D}(A) = \varliminf_{i\to\infty} \mu_i(A)$$

and upper asymptotic density of A to be

$$\overline{D}(A) = \varlimsup_{i\to\infty} \mu_i(A)$$

and the natural density D(A) as usual. Furthermore, a set $A\subset X$ is called bounded if $A\subset K(i)$ for some i. We shall write $A\dot{\subset}B$ when $A\setminus B$ is bounded; there is then a value of j for which $A\setminus K(j)\subset B\setminus K(j)$. R. C. Buck proved the following theorem: If $A_1\dot{\subset}A_2\dot{\subset}\cdots\dot{\subset}A_n\dot{\subset}\cdots$, $\lim \overline{D}\ (A_n) = \Delta$ and $\lim \underline{D}\ (A_n) = \delta$, then there exists a set A with $\overline{D}(A) = \Delta$ and $\underline{D}(A) = \delta$ such that $A_n\dot{\subset}A$ for all n. From this theorem we immediately see that if the sets A_n have natural density, and $A_1\dot{\subset}A_2\dot{\subset}\cdots\dot{\subset}A_n\dot{\subset}\cdots$. Then, there is a set A, unique up to sets of zero natural density, such that $A_n\dot{\subset}A$ for all n and $D(A) = \lim D(A_n)$. Furthermore, if C_1, $C_2,\cdots$ are disjoint sets having natural density, there is a set C, unique up to sets of zero natural density, such that $C\dot{\supset}\bigcup_{k=1}^{n} C_k$ for all n and with $D(c) = \sum_{k=1}^{\infty} D(C_k)$. The reader may later wish to compare this result of Buck's with our theorem 3.19 and 3.21 in chapter 3.

To conclude this discussion we mention that it has been noticed that

$$\delta(A) = \lim_{N\to\infty} d\ (A\cup\{1,2,\cdots,N\})$$

This is the basis for our definition of (lower) Cheo asymptotic density in chapter 1.

Chapter 1 follows closely to the paper of Freedman [4, section 2]. We define the Cheo or C-asymptotic density in n-dimensional space. To do this we first generalize the Schnirelmann density given above to n-dimensions. We also give some equivalent forms of the C-asymptotic density. To conclude the chapter we reduce the C-asymptotic density to the usual asymptotic density in the case n=1.

In chapter 2,we discuss some additive questions involving the C-asymptotic density.

In chapter 3, which we consider the main part of this paper, we define the upper C-asymptotic density and the C-natural density. We prove the "additivity theorem" for C-natural density and some related results for lower and upper C-asymptotic density (Theorems 3.25, 3.19 and 3.21 respectively) which are entirely new in this setting.

In chapter 4, we attempt to compare the C-asymptotic density and the K- asymptotic density of Freedman. Finally, we will see that the lower C-asymptotic and K-asymptotic densities are in fact different.

CHAPTER 1.

THE CHEO ASYMPTOTIC DENSITY IN n-DIMENSION.

Let n be a positive integer and S the set of all n-tuples of non-negative integers, the element $(0, 0, \cdots\cdots, 0)$ is denoted by $\underset{\sim}{0}$ and generally the element $(x_1, x_2, \cdots\cdots, x_n)$ by $\underset{\sim}{x}$.

Definition (1.1). For any $\underset{\sim}{x} \in S$ define

$$L(\underset{\sim}{x}) = \{ \underset{\sim}{y} \mid \underset{\sim}{y} \in S,\ y_i \leq x_i \ (i=1,2,\cdots\cdots,n) \}$$

and

$$U(\underset{\sim}{x}) = \{ \underset{\sim}{y} \mid \underset{\sim}{x} \in L(\underset{\sim}{y}) \}.$$

We let $\mathcal{C} = \{ L(\underset{\sim}{x}) \mid \underset{\sim}{x} \in S \setminus \underset{\sim}{0} \}$. The class $\mathcal{C}$ is called the class of Cheo sets or the C-class on S.

Definition (1.2). For any set $A \subset S$ and a finite set $\tilde{X} \subset S$, $A(\tilde{X})$ is the cardinality of the set $(A \cap X) \setminus \underset{\sim}{0}$.

Definition (1.3). For any $A \subset S$ the Cheo or C-density of A is defined to be

$$d(A) = \text{glb} \left\{ \frac{A(L(\underset{\sim}{x}))}{S(L(\underset{\sim}{x}))} \;\middle|\; \underset{\sim}{x} \in S \setminus \underset{\sim}{0} \right\}.$$

In the paper of L. Cheo [2], he has defined the C-density of A in the case n=2. In the case n=1, above definition reduces to the ordinary Schnirelmann density of the set A.

Notation: For a nonnegative integer N, let

$$J(N) = \{ \underset{\sim}{x} \mid \underset{\sim}{x} \in S, \min \{x_1, \cdots, x_n\} \le N \}.$$

It can be noted that $J(N) = S \backslash U((N+1, \cdots, N+1))$.

Definition (1.4). For any $A \subset S$ the lower Cheo asymptotic density, referred to henceforth as C-asymptotic density, is defined to be

$$\delta(A) = \lim_{n\to\infty} d\,(A \cup J(N)).$$

Remark (1.5). For all $N \ge 0$ we have

$$d(A \cup J(N)) = \text{glb} \left\{ \frac{[A \cup J(N)](L(\underset{\sim}{x}))}{S(L(\underset{\sim}{x}))} \;\middle|\; \underset{\sim}{x} \in S \backslash J(N) \right\}.$$

Since for all $\underset{\sim}{x} \in J(N)$, we have $L(\underset{\sim}{x}) \subset J(N)$ and

$$\frac{[A \cup J(N)](L(\underset{\sim}{x}))}{S(L(\underset{\sim}{x}))} = 1 \ge d(A \cup J(N)).$$

Definition (1.6). For integers M, N, n with $M > N \ge 0$, $n > 0$, let

$$g(M,N,n) = \min \left\{ \frac{(M+2)^m - 1}{(M+2)^m - (M-N+1)^m} \;\middle|\; m = 1, \cdots, n \right\}.$$

We note that for fixed N,n, $g(M,N,n) \to \infty$ as $M \to \infty$. Also, for fixed M,N, if $n_1 > n_2$ then $g(M,N,n_1) \le g(M,N,n_2)$. The second statement is obvious. The first statement follows from the fact that for fixed N,n

$$\frac{(M+2)^n - 1}{(M+2)^n - (M-N+1)^n} = \frac{(M+2-1)[(M+2)^{n-1} + (M+2)^{n-2} + \cdots + 1]}{(M+2-M+N-1)[(M+2)^{n-1} + (M-N+1)(M+2)^{n-2} + \cdots + (M-N+1)^{n-1}]}$$

$$= \frac{(M+1)[(M+2)^{n-1} + (M+2)^{n-2} + \cdots + 1]}{(N+1)[(M+2)^{n-1} + (M-N+1)(M+2)^{n-2} + \cdots + (M-N+1)^{n-1}]}$$

$\to\infty$ as $M \to\infty$

Lemma (1.7). Let $0 \le N < M$ and $\underset{\sim}{x} \in S$ such that $M+1 \le x_i$ $(i=1,\cdots,n)$. If

$$f(M,N,n,\underset{\sim}{x}) = \frac{\prod_{i=1}^{n} (x_i + 1) - \prod_{i=1}^{n} (x_i - M)}{\prod_{i=1}^{n} (x_i + 1) - \prod_{i=1}^{n} (x_i - N)}$$

then $f(M,N,n,\underset{\sim}{x}) \ge g(M,N,n)$

Proof. For $n=1$,

$$f(M,N,n,\underset{\sim}{x}) = \frac{(x_1 + 1) - (x_1 - M)}{(x_1 + 1) - (x_1 - N)} = \frac{1+M}{1+N} = g(M,N,1).$$

Also, for any n, if $\underset{\sim}{x} = (M+1,\cdots,M+1)$, then

$$f(M,N,n,\underset{\sim}{x}) = \frac{(M+2)^n - 1}{(M+2)^n - (M-N+1)^n} \geq g(M,N,n).$$

Now, we perform a multiple induction. Let $k>1$ and assume the lemma true for all $M,N,n,\underset{\sim}{x}$ with $n<k$. Let $\underset{\sim}{x} = (x_1,\cdots,x_k)$ be such that $x_i > M(i=1,\cdots,k)$ and, for some j, $x_j > M+1$, and assume for each $\underset{\sim}{y} = (y_1,\cdots,y_k)$ with $M<y_i\leq x_i$ $(i=1,\cdots,k)$ and, for some j, $y_j<x_j$, that $f(M,N,k,\underset{\sim}{y})\geq g(M,N,k)$. Without loss of generality we may assume that $x_1>M+1$. And

$$f(M,N,k,\underset{\sim}{x}) = \frac{\prod_{i=1}^{k} (x_i + 1) - \prod_{i=1}^{k} (x_i - M)}{\prod_{i=1}^{k} (x_i + 1) - \prod_{i=1}^{k} (x_i - N)}$$

$$= \frac{[(x_1-1+1) \prod_{i=2}^{k} (x_i+1)-(x_1-1-M) \prod_{i=2}^{k} (x_i-M)] + [\prod_{i=2}^{k} (x_i+1)- \prod_{i=2}^{k}(x_i-M)\}}{[(x_1-1+1) \prod_{i=2}^{k} (x_i+1)-(x_1-1-N) \prod_{i=2}^{k} (x_i-N)] + [\prod_{i=2}^{k} (x_i+1)- \prod_{i=2}^{k}(x_i-N)]}$$

$$\geq \min \{f(M,N,k,(x_1-1, x_2,\cdots,x_k)), f(M,N,k-1,(x_2,\cdots,x_k))\}$$

$$\geq \min \{g(M,N,k), g(M,N,k-1)\}$$

$= g(M,N,k)$.

The third ineqality follows from the fact that, if a,b,c,d, are positive integers, then

$$\frac{a+b}{c+d} \geq \min \{ \frac{a}{c}, \frac{b}{d} \}.$$

Hence, the lemma is proved.

By simple calculations we can get the following formulas:

(1.a) For each $\underset{\sim}{x} \in S$,

$$S(L(\underset{\sim}{x})) = \prod_{i=1}^{n} (x_i + 1) - 1.$$

(1.b) For $\underset{\sim}{x}$, $\underset{\sim}{y} \in S$, $\underset{\sim}{y} \in L(\underset{\sim}{x})$, we have

$$S(L(\underset{\sim}{x}) \cap U(\underset{\sim}{y})) = \prod_{i=1}^{n} (x_i - y_i + 1) - \lambda(\underset{\sim}{y}).$$

where $\lambda(\underset{\sim}{y}) = 0$ if $\underset{\sim}{y} \neq \underset{\sim}{0}$ and $\lambda(\underset{\sim}{y}) = 1$ if $\underset{\sim}{y} = \underset{\sim}{0}$.

(1.c) For $M \geq 0$, $\underset{\sim}{x} \notin J(M)$, then

$$S(L(\underset{\sim}{x}) \cap J(M)) = \prod_{i=1}^{n} (x_i + 1) - \prod_{i=1}^{n} (x_i - M) - 1.$$

To see (1.c), since

$$L(\underset{\sim}{x}) \cap J(M) = L(\underset{\sim}{x}) \setminus [L(\underset{\sim}{x}) \cap U((M+1, \cdots, M+1))].$$

Thus,

$$S(L(\underset{\sim}{x}) \cap J(M)) = [\prod_{i=1}^{n} (x_i + 1) - 1] - \prod_{i=1}^{n} [(x_i - (M + 1)) + 1]$$

$$= \prod_{i=1}^{n} (x_i + 1) - \prod_{i=1}^{n} (x_i - M) - 1.$$

Lemma (1.8). If $0<N<M$ and $\underset{\sim}{x} \notin J(M)$, then

$$\frac{S(L(\underset{\sim}{x}))}{S(L(\underset{\sim}{x})\cap J(N))} > \frac{S(L(\underset{\sim}{x})\cap J(M))}{S(L(\underset{\sim}{x})\cap J(N))} > g(M,N,n).$$

Proof. The first inequality is obvious.

$$\frac{S(L(\underset{\sim}{x})\cap J(M))}{S(L(\underset{\sim}{x})\cap J(N))} = \frac{\prod_{i=1}^{n} (x_i + 1) - \prod_{i=1}^{n} (x_i - M) - 1}{\prod_{i=1}^{n} (x_i + 1) - \prod_{i=1}^{n} (x_i - N) - 1}$$

$$> f(M,N,n,\underset{\sim}{x}) \geq g(M,N,n).$$

Definition (1.9). For any integer $N\geq 0$, define

$$\mathcal{L}(N) = \{ L(\underset{\sim}{x}) \mid L(\underset{\sim}{x}) \in \mathcal{C}, \underset{\sim}{x} \in S \setminus J(N) \}.$$

Definition (1.10). Let $\mathcal{S}$ be the class of all sequences $\{L(\underset{\sim}{x}^i)\}$ in $\mathcal{C}$ which satisfy the property that for each integer $N>0$

$$\lim_{i\to\infty} \frac{S(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i)\cap J(N))} = \infty .$$

Lemma (1.11). If $\{ L(\underset{\sim}{x}^i) \}$ is a sequence such that $L(\underset{\sim}{x}^i) \in \mathcal{L}$ (i), then $\{ L(\underset{\sim}{x}^i) \} \in \underset{\sim}{S}$.

Proof. Lemma (1.8) shows that for i sufficiently large we have

$$\frac{S(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i)\cap J(N))} > g(i,N,n)$$

and $g(i,N,n) \to \infty$ as $i \to \infty$.

Theorem (1.12). If $\{L(\underset{\sim}{x}^i)\} \in \underset{\sim}{S}$ and $A \subset S$, then

$$\delta(A) \le \varliminf_{i\to\infty} \frac{A(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} .$$

Proof. Let $N>0$. Then

$$d(A\cup J(N)) \le \frac{[A\cup J(N)]L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} \le \frac{A(L(\underset{\sim}{x}^i)) + S(J(N)\cap L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))}$$

Hence

$$d(A\cup J(N)) \leq \underline{\lim}_{i\to\infty} \left[\frac{A(L(\underset{\sim}{x}^{i}))}{S(L(\underset{\sim}{x}^{i}))} + \frac{S(J(N)\cap L(\underset{\sim}{x}^{i}))}{S(L(\underset{\sim}{x}^{i}))} \right]$$

$$\leq \underline{\lim}_{i\to\infty} \frac{A(L(\underset{\sim}{x}^{i}))}{S(L(\underset{\sim}{x}^{i}))} + \overline{\lim}_{i\to\infty} \frac{S(J(N)\cap L(\underset{\sim}{x}^{i}))}{S(L(\underset{\sim}{x}^{i}))}$$

$$= \underline{\lim}_{i\to\infty} \frac{A(L(\underset{\sim}{x}^{i}))}{S(L(\underset{\sim}{x}^{i}))} \quad .$$

Letting $N\to\infty$ we have the result.

The following theorem shows that $\delta(A)$ can be always obtained as a limit of quotients $A(L(\underset{\sim}{x}^{i}))/\ S(L(\underset{\sim}{x}^{i}))$ where $\{L(\underset{\sim}{x}^{i})\}$ is a sequence in $\mathcal{S}$.

<u>Theorem (1.13)</u>. For any $A\subset S$ there exists $\{L(\underset{\sim}{x}^{i})\}\in\mathcal{S}$ such that

$$\delta(A) = \lim_{i\to\infty} \frac{A(L(\underset{\sim}{x}^{i}))}{S(L(\underset{\sim}{x}^{i}))} \quad .$$

Furthermore, we may choose $L(\underset{\sim}{x}^{i})$ so that $\underset{\sim}{x}^{i}\in S\setminus J(i)$.

<u>Proof.</u> If $\delta(A) = 1$, then for any sequence $\{L(\underset{\sim}{x}^i)\} \in \mathcal{S}$ we have

$$1 = \delta(A) \leq \underline{\lim}_{i\to\infty} \frac{A(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} \leq \overline{\lim}_{i\to\infty} \frac{A(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} \leq 1$$

and the theorem is proved in this case.

Suppose that $\delta(A)<1$. For any $i\geq 1$, let $M(i)$ be such $M(i)>i$ and

$$g(M(i),i,n) > 2^i$$

and choose $L(\underset{\sim}{x}^i) \in \mathcal{C}$ such that $\underset{\sim}{x}^i \in S \setminus J(M(i))$ and

$$\frac{[\,A \cup J(M(i))\,](L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} \leq d(A \cup J(M(i)) + \frac{1}{2^i} \quad .$$

The existence of $L(\underset{\sim}{x}^i)$ follows from the remark 1.5.

It follows that $L(\underset{\sim}{x}^i) \in \mathcal{L}(M(i)) \subset \mathcal{L}(i)$ so that by lemma 1.11 $\{L(\underset{\sim}{x}^i)\} \in \mathcal{S}$.

It remains only to show that $\delta(A)$ is the limit of the quotients $A(L(\underset{\sim}{x}^i)) \,/\, S(L(\underset{\sim}{x}^i))$.

From the inequalities

$$0 \leq \frac{[A \cup J(i)](L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} - \frac{A(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} \leq \frac{J(i)(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))}$$

$$= \frac{S(L(\underset{\sim}{x}^i) \cap J(i))}{S(L(\underset{\sim}{x}^i))} \leq \frac{1}{g(M(i),i,n)} \leq \frac{1}{2^i}$$

it follows that

$$\lim_{i\to\infty} \left[\frac{[A \cup J(i)](L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} - \frac{A(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} \right] = 0.$$

But also

$$\lim_{i\to\infty} \frac{[A \cup J(i)](L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} = \delta(A)$$

for

$$d(A \cup J(i)) \leq \frac{[A \cup J(i)](L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} \leq \frac{[A \cup J(M(i))](L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))}$$

$$\leq d\,[A \cup J(M(i))] + \frac{1}{2^i} ,$$

where both ends approach $\delta(A)$ as $i \to \infty$

Therefore,

$$\lim_{i \to \infty} \frac{A(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} = \delta(A).$$

<u>Theorem (1.14).</u> For any $A \subset S$ we have

$$\delta(A) = \underset{\{L(\underset{\sim}{x}^i)\} \in \mathcal{S}}{\text{glb}} \ \overline{\lim_{i \to \infty}} \ \frac{A(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} .$$

<u>Proof.</u> By theorem 1.12 we have for any $\{L(\underset{\sim}{x}^i)\} \in \mathcal{S}$

$$\delta(A) \leq \overline{\lim_{i \to \infty}} \frac{A(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} .$$

Thus,

$$\delta(A) \leq \underset{\{L(\underset{\sim}{x}^i)\} \in \mathcal{S}}{\text{glb}} \ \overline{\lim_{i \to \infty}} \ \frac{A(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} .$$

On the other hand, there exists a sequence $\{L(\underset{\sim}{x}^i)\} \in \mathcal{S}$ such that

$$\delta(A) = \lim_{i\to\infty} \frac{A(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} = \lim_{i\to\infty} \frac{A(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))}.$$

Therefore,

$$\delta(A) \geq \underset{\{L(\underset{\sim}{x}^i)\}\in\mathcal{S}}{\text{glb}} \ \underline{\lim}_{i\to\infty} \frac{A(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))}.$$

It follows that

$$\delta(A) = \underset{\{L(\underset{\sim}{x}^i)\}\in\mathcal{S}}{\text{glb}} \ \underline{\lim}_{i\to\infty} \frac{A(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))}.$$

<u>Definition (1.15).</u> For $N\geq 0$ and $A\subset S$ we define

$$d^N(A) = \text{glb}\left\{\frac{A(L(\underset{\sim}{x})) + S(L(\underset{\sim}{x})\cap J(N))}{S(L(\underset{\sim}{x}))} \;\middle|\; \underset{\sim}{x}\in S\setminus\underset{\sim}{0}\right\}.$$

<u>Theorem (1.16)</u>. For any $A\subset S$,

$$\delta(A) = \lim_{N\to\infty} d^N(A).$$

<u>Proof.</u> Since, for each $L(\underset{\sim}{x})\in\mathcal{C}$ and for each N,

$$\frac{[A \cup J(N)](L(\underset{\sim}{x}))}{S(L(\underset{\sim}{x}))} \le \frac{A(L(\underset{\sim}{x})) + S(J(N) \cap L(\underset{\sim}{x}))}{S(L(\underset{\sim}{x}))} ,$$

it follows that $d(A \cup J(N)) \le d^N(A)$.

On the other hand, let $\{L(\underset{\sim}{x}^i)\} \in \mathcal{S}$ such that

$$\lim_{i\to\infty} \frac{A(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} = \delta(A) .$$

Then

$$d^N(A) \le \frac{A(L(\underset{\sim}{x}^i)) + S(L(\underset{\sim}{x}^i) \cap J(N))}{S(L(\underset{\sim}{x}^i))}$$

$$= \frac{A(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} + \frac{S(L(\underset{\sim}{x}^i) \cap J(N))}{S(L(\underset{\sim}{x}^i))} ,$$

the right hand side of the inequality tends to $\delta(A)$ as i tends to infinite. Hence, for each N we have

$$d(A \cup J(N)) \le d^N(A) \le \delta(A).$$

Therefore,

$$\delta(A) = \lim_{N\to\infty} d^N(A).$$

Theorem (1.17). For any $A \subset S$,

$$\delta(A) = \lim_{N\to\infty} d^N(A \cup J(N)).$$

Proof. As in the proof of theorem 1.16,

$$d(A\cup J(N)) = d(A\cup J(N)\cup J(N)) \le d^N(A\cup J(N))$$

$$\le \delta(A\cup J(N)) = \delta(A).$$

The last equality follows easily from the definition of δ.

Theorem (1.18). For any $A \subset S$,

(i) If $n \ge 2$ and $A \cap J(N)$ is finite for each $N \ge 0$, then $\delta(A) = 0$

(ii) If $S \setminus A \subset J(N)$ for some N, then $\delta(A)=1$.

Proof. (i) For $N \ge 0$, let $x_{1,N}, x_{2,N}, \cdots, x_{n-1,N}$ be chosen so large that $x_{i,N} > N$ $(i=1,\cdots,n-1)$ and $S(L((x_{1,N},\cdots,x_{n-1,N},N))) > N\cdot S(A\cap J(N))$. Let $L(x_{1,N},\cdots,x_{n-1,N},N) = L(\underset{\sim}{x}^N)$ so that $L(\underset{\sim}{x}^N) \in \mathcal{C}$ and $\{L(\underset{\sim}{x}^N)\} \in \mathcal{S}$ (since $L(\underset{\sim}{x}^N) \notin \mathcal{L}(N-1)$) and $L(\underset{\sim}{x}^N) \subset J(N)$.

Hence

$$0 \le \delta(A) \le \lim_{N\to\infty} \frac{A(L(\chi^N))}{S(L(\chi^N))} \le \lim_{N\to\infty} \frac{[A\cap J(N)](L(\chi^N))}{N\cdot S(A\cap J(N))}$$

$$\le \lim_{N\to\infty} \frac{1}{N} = 0.$$

(ii) If $S\setminus A\subset J(N)$, then $A\cup J(M) = S$ for $M\ge N$. Thus $d(A\cup J(M)) = d(S) = 1$ and the result follows.

Remark (1.19). The part (i) of theorem 1.18 is not true for n=1. For example, let A=I the set of all nonnegative integers. Then for each $N\ge 0$, $A\cap J(N)$ is finite, but $\delta(A) = 1 \ne 0$.

To conclude this chapter we prove that δ generalizes the usual asymptotic density.

Theorem (1.20). In the case n=1, $\delta(A)$ is the usual asymptotic density of A.

Proof. When n=1, S=I. Then $S(L(i)) = i$ for each $i\in I\setminus 0$. Hence by theorem 1.12 we have

$$\delta(A) \le \lim_{i\to\infty} \frac{A(i)}{i} \quad .$$

On the other hand, by theorem 1.13, there is a sequence $\{n_i\}$ such that $n_i \to \infty$ as $i \to \infty$ corresponding to $\{L(n_i)\} \in \mathfrak{g}$ such that

$$\delta(A) = \lim_{i\to\infty} \frac{A(n_i)}{n_i} \geq \lim_{i\to\infty} \frac{A(i)}{i} \quad .$$

This completes the proof.

CHAPTER 2.

THE C-ASYMPTOTIC DENSITY OF THE SUM OF TWO SETS.

In this chapter, we discuss some proterties of the C-asymptotic density of the sum of two sets.

Definition (2.1). Let A and B be subsets of S, we define

$$A+B = \{ \underset{\sim}{a} + \underset{\sim}{b} \mid \underset{\sim}{a} \in A, \underset{\sim}{b} \in B \}.$$

Here $\underset{\sim}{a} + \underset{\sim}{b} = (a_1 + b_1, a_2 + b_2, \cdots, a_n + b_n)$ where $\underset{\sim}{a} = (a_1, a_2, \cdots, a_n)$ and $\underset{\sim}{b} = (b_1, b_2, \cdots, b_n)$. Furthermore, if $A \subset U(\underset{\sim}{y})$, we define

$$A-\underset{\sim}{y} = \{\underset{\sim}{x} \mid \underset{\sim}{x} \in S,\ \underset{\sim}{y} + \underset{\sim}{x} \in A\} .$$

Note! for any $A \subset S$, $\overline{A}$ is defined to be the set $S \setminus A$.

Lemma (2.2). If $\underset{\sim}{0} \in A \cap B$ and $A(L(\underset{\sim}{x})) + B(L(\underset{\sim}{x})) \geq S(L(\underset{\sim}{x}))$, then $\underset{\sim}{x} \in A + B$.

Proof. Suppose that $\underset{\sim}{x} \notin A+B = C$, then $\underset{\sim}{x} \in S \setminus C = \overline{C}$. Let the elements of $A \cap L(\underset{\sim}{x})$ be enumerated

$$\{ 0 = \underset{\sim}{a}_0, \underset{\sim}{a}_1, \cdots, \underset{\sim}{a}_n \}$$

For each i $(0 \leq i \leq n)$, we have $\underset{\sim}{x} - \underset{\sim}{a}_i \in (L(\underset{\sim}{x}) \cap \overline{B}) \setminus \underset{\sim}{0}$ since $\underset{\sim}{x} - \underset{\sim}{a}_i \in L(\underset{\sim}{x}) \setminus \underset{\sim}{0}$, and if $\underset{\sim}{x} - \underset{\sim}{a}_i \in B$ then $\underset{\sim}{x} = \underset{\sim}{a}_i + (\underset{\sim}{x} - \underset{\sim}{a}_i) \in C$ contrary to assumption. Thus the set

$$D = \{\underset{\sim}{x} - \underset{\sim}{a}_i \mid i=1,2,\cdots,n\} \subseteq (\overline{B} \cap L(\underset{\sim}{x})) \setminus \underset{\sim}{0}$$

and so

$$A(L(\underset{\sim}{x})) + 1 = n+1 = S(D) \leq \overline{B}(L(\underset{\sim}{x})) = S(L(\underset{\sim}{x})) - B(L(\underset{\sim}{x})).$$

Therefore,

$$A(L(\underset{\sim}{x})) + B(L(\underset{\sim}{x})) \leq S(L(\underset{\sim}{x})) - 1 < S(L(\underset{\sim}{x})).$$

This is a contradiction.

Theorem (2.3). If $\underset{\sim}{0} \in A \cap B$ and $\delta(A) + \delta(B) > 1$, then $S \setminus (A+B) \subset J(N)$ for some N. This last condition implies that $\delta(A+B) = 1$.

Proof. The last statement is just theorem (1.18(ii)).

Let $\lambda = \delta(A) + \delta(B) - 1$, then $\lambda > 0$. Since $\delta(A) = \lim_{N\to\infty} d(A \cup J(N))$ and $\delta(B) = \lim_{N\to\infty} d(B \cup J(N))$, thus for some N_0 we have

(1) $$1 + \frac{\lambda}{2} < d(A \cup J(N_0)) + d(B \cup J(N_0)).$$

Since, for any $L(\underset{\sim}{x}) \in \mathcal{L}$

$$d(A \cup J(N_0)) \leq \frac{[A \cup J(N_0)](L(\underset{\sim}{x}))}{S(L(\underset{\sim}{x}))}$$

Then

(2) $$S(L(\underset{\sim}{x}))\cdot d(A\cup J(N_0)) \le [A\cup J(N_0)](L(\underset{\sim}{x})).$$

Similarly,

(3) $$S(L(\underset{\sim}{x}))\cdot d(B\cup J(N_0)) \le [B\cup J(N_0)](L(\underset{\sim}{x})).$$

Combining (1), (2), (3) we obtain

$$S(L(\underset{\sim}{x})) + \frac{\lambda}{2}S(L(\underset{\sim}{x})) < [A\cup J(N_0)](L(\underset{\sim}{x})) + [B\cup J(N_0)](L(\underset{\sim}{x})).$$

Let M be so large that $g(M,N_0,n) > \frac{4}{\lambda}$. By lemma 1.8, if $\underset{\sim}{x}\notin J(M)$, then

$$\frac{S(L(\underset{\sim}{x}))}{S(L(\underset{\sim}{x})\cap J(N_0))} > g(M,N_0,n) > \frac{4}{\lambda}\ .$$

Hence, for $\underset{\sim}{x}\notin J(M)$,

$$\begin{aligned} A(L(\underset{\sim}{x})) + B(L(\underset{\sim}{x})) &\ge [A\cup J(N_0)](L(\underset{\sim}{x})) + [B\cup J(N_0)](L(\underset{\sim}{x})) - 2S(L(\underset{\sim}{x}))\cap J(N_0)) \\ &\ge S(L(\underset{\sim}{x})) + \frac{\lambda}{2}S(L(\underset{\sim}{x})) - 2\cdot\frac{\lambda}{4}S(L(\underset{\sim}{x})) \\ &= S(L(\underset{\sim}{x})). \end{aligned}$$

Therefore, by lemma 2.2, $\underset{\sim}{x}\in A+B$ so that $S\setminus(A+B)\subset J(M)$.

<u>Remark (2.4).</u> If $\delta(A) + \delta(B) = 1$, the conclusion of theorem 2.3 is not necessarily true. For example, let $S = I^2$ and let

$$A = B = \{(a,b) \mid (a,b)\in S,\quad a \text{ is even}\}\ .$$

We are going to show that $\delta(A) = \delta(B) = \frac{1}{2}$. It is sufficient to show that

$$\delta(A) = \lim_{i\to\infty} \frac{A(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} = \frac{1}{2}$$

for all sequence $\{L(\underset{\sim}{x}^i)\} \in \mathcal{S}$ (see theorem 3.12 below).

Let $\{L(\underset{\sim}{x}^i)\}$ be any sequence in $\mathcal{S}$ and let $\underset{\sim}{x}^i = (a_i, b_i)$.

<u>Case I</u>. If $\underset{\sim}{x}^i \notin A$, then

$$\frac{A(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} = \frac{\frac{1}{2}(a_i + 1)(b_i + 1) - 1}{(a_i + 1)(b_i + 1) - 1}$$

$$= \frac{\frac{1}{2}(a_i + 1)(b_i + 1)}{(a_i + 1)(b_i + 1) - 1} - \frac{1}{(a_i + 1)(b_i + 1) - 1}$$

$$= \frac{1}{2 - \frac{2}{(a_i + 1)(b_i + 1)}} - \frac{1}{(a_i + 1)(b_i + 1) - 1}$$

<u>Case II</u>. If $\underset{\sim}{x}^i \in A$, then

$$\frac{A(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} = \frac{\frac{1}{2}(a_i)(b_i + 1) - 1 + (b_i + 1)}{(a_i + 1)(b_i + 1) - 1}$$

$$= \frac{\frac{1}{2}(a_i + 1)(b_i + 1) - 1 + \frac{1}{2}(b_i + 1)}{(a_i + 1)(b_i + 1) - 1}$$

$$= \frac{\frac{1}{2}(a_i + 1)(b_i + 1)}{(a_i + 1)(b_i + 1) - 1} - \frac{1}{(a_i + 1)(b_i + 1) - 1} + \frac{1}{2}\,\frac{b_i + 1}{(a_i + 1)(b_i + 1) - 1}$$

$$= \frac{1}{2 - \frac{2}{(a_i + 1)(b_i + 1)}} - \frac{1}{(a_i + 1)(b_i + 1) - 1} - \frac{1}{2}\,\frac{1}{(a_i + 1) - \frac{1}{b_i + 1}}$$

Thus we have

$$\lim_{i\to\infty} \frac{A(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} = \tfrac{1}{2} .$$

Therefore,

$$\delta(A) + \delta(B) = \tfrac{1}{2} + \tfrac{1}{2} = 1 .$$

But $A+B = A$ and $\delta(A+B) = \delta(A) = \frac{1}{2}$.

The following theorem shows that the C-asymptotic density of a set is invariant under translation. We note that, if $\underset{\sim}{0} \notin A$ then $A(L(\underset{\sim}{y})) = (x+A)(L(\underset{\sim}{x} + \underset{\sim}{y}))$ for any $\underset{\sim}{x}, \underset{\sim}{y} \in S$.

<u>Theorem (2.5)</u>. $\delta(\underset{\sim}{x} + A) = \delta(A)$ for each $A \subset S$ and $\underset{\sim}{x} \in S$.

<u>Proof.</u> If $\underset{\sim}{x} = \underset{\sim}{0}$, then $\underset{\sim}{x} + A = A$ and the theorem is trivial. Hence it is assumed that $\underset{\sim}{x} \neq \underset{\sim}{0}$. Furthermore, since $\underset{\sim}{x} + A$ and $\underset{\sim}{x} + (A \setminus \underset{\sim}{0})$

differ in at most one point, it may be assumed that $\underset{\sim}{0} \notin A$.

It is first shown that $\delta(\underset{\sim}{x} + A) \geq \delta(A)$. Let $N = \max\{x_1, \cdots, x_n\}$, then for any $\underset{\sim}{y} \in S \setminus J(N)$ we have

$$(*) \qquad A(L(\underset{\sim}{y})) = (\underset{\sim}{x} + A)(L(\underset{\sim}{x} + \underset{\sim}{y}))$$

$$= (\underset{\sim}{x} + A)(L(\underset{\sim}{x} + \underset{\sim}{y}) \setminus L(\underset{\sim}{y})) + (\underset{\sim}{x} + A)(L(\underset{\sim}{y}))$$

$$\leq J(N)(L(\underset{\sim}{y})) + (\underset{\sim}{x} + A)(L(\underset{\sim}{y})) + 1.$$

The last inequality follows from the fact that

$$(\underset{\sim}{x} + A)(L(\underset{\sim}{x} + \underset{\sim}{y}) \setminus L(\underset{\sim}{y})) \leq J(N)(L(\underset{\sim}{y})) + 1.$$

To see this, for any $\underset{\sim}{y} \notin J(N)$

$$J(N)(L(\underset{\sim}{y})) = \prod_{i=1}^{n} (y_i + 1) - \prod_{i=1}^{n} (y_i - N) - 1 \qquad \text{(by formula 1.c)}$$

and

$$(\underset{\sim}{x} + A)(L(\underset{\sim}{x} + \underset{\sim}{y}) \setminus L(\underset{\sim}{y})) \leq \prod_{i=1}^{n} [(x_i + y_i + 1) - x_i] - \prod_{i=1}^{n} (y_i - x_i)$$

$$= \prod_{i=1}^{n} (y_i + 1) - \prod_{i=1}^{n} (y_i - x_i) \quad ,$$

but $N \geq x_i$ $(i = 1, \cdots, n)$ then $\prod_{i=1}^{n} (y_i - N) \leq \prod_{i=1}^{n} (y_i - x_i)$ and so

$$(\underset{\sim}{x} + A)(L(\underset{\sim}{x} + \underset{\sim}{y}) \setminus L(\underset{\sim}{y})) \leq J(N)(L(\underset{\sim}{y})) + 1.$$

Now, if we let $\{L(\underset{\sim}{y}^i)\} \in \mathcal{S}$ such that

$$\delta(\underset{\sim}{x} + A) = \lim_{i\to\infty} \frac{(\underset{\sim}{x} + A)(L(\underset{\sim}{y}^i))}{S(L(\underset{\sim}{y}^i))}$$

Then, we have, by (*)

$$\delta(\underset{\sim}{x} + A) \geq \lim_{i\to\infty} \frac{A(L(\underset{\sim}{y}^i)) - J(N)(L(\underset{\sim}{y}^i)) - 1}{S(L(y^i))}$$

$$= \lim_{i\to\infty} \frac{A(L(\underset{\sim}{y}^i)) - S(L(\underset{\sim}{y}^i) \cap J(N)) - 1}{S(L(\underset{\sim}{y}^i))}$$

$$= \lim_{i\to\infty} \frac{A(L(\underset{\sim}{y}^i))}{S(L(\underset{\sim}{y}^i))} = \underline{\lim}_{i\to\infty} \frac{A(L(\underset{\sim}{y}^i))}{S(L(\underset{\sim}{y}^i))} \geq \delta(A).$$

It remains to show that $\delta(\underset{\sim}{x} + A) \leq \delta(A)$. Clearly, for each $L(\underset{\sim}{y}) \in \mathcal{C}$,

$$(\underset{\sim}{x} + A)(L(\underset{\sim}{y})) \leq A(L(\underset{\sim}{y})).$$

Let $\{L(\underset{\sim}{y}^i)\} \in \mathcal{S}$ such that

$$\lim_{i\to\infty} \frac{A(L(\underset{\sim}{y}^i))}{S(L(\underset{\sim}{y}^i))} = \delta(A).$$

Then

$$\delta(\underset{\sim}{x} + A) \leq \varliminf_{i\to\infty} \frac{(\underset{\sim}{x} + A)(L(\underset{\sim}{y}^i))}{S(L(\underset{\sim}{y}^i))} \leq \varliminf_{i\to\infty} \frac{A(L(\underset{\sim}{y}^i))}{S(L(\underset{\sim}{y}^i))} = \delta(A)$$

and the theorem is proved.

Corollary (2.6). If $A\subset S$, $\underset{\sim}{x}\in S$ we have $\delta(A-\underset{\sim}{x}) = \delta(A)$.

Proof. Since $A - \underset{\sim}{x} = \{\underset{\sim}{a} - \underset{\sim}{x} \mid \underset{\sim}{a}\in A,\ x_i\leq a_i\ (i = 1,\cdots,n)\}$ and $(A - \underset{\sim}{x}) + \underset{\sim}{x} = \{\underset{\sim}{a} \mid \underset{\sim}{a}\in A,\ x_i\leq a_i\ (i = 1,\cdots,n)\}\subset A$. Then by theorem 2.5 we have

$$\delta(A - \underset{\sim}{x}) = \delta((A - \underset{\sim}{x}) + \underset{\sim}{x}) \leq \delta(A).$$

Furthermore, if we let $N = \max\{x_1,\cdots,x_n\}$ then

$$A\setminus J(N) \subset (A - \underset{\sim}{x}) + \underset{\sim}{x} = \{\underset{\sim}{a} \mid \underset{\sim}{a}\in A,\ x_i\leq a_i\ (i = 1,\cdots,n)\}.$$

Hence

$$\delta((A - \underset{\sim}{x}) + \underset{\sim}{x}) \geq \delta(A\setminus J(N)) = \delta(A).$$

This completes the proof.

Theorem (2.7). If $A\cap B\subset J(N)$ for some N, then $\delta(A\cup B) \geq \delta(A) + \delta(B)$. In particular, if $\underset{\sim}{0}\in A\cap B\subset J(N)$, then $\delta(A+B)\geq \delta(A) + \delta(B)$.

Proof. The second statement follows easily from the first since $A+B \supset A \cup B$ if $0 \in A \cap B$.

Let $C = A \cup B$. If $M \geq N$, then $A \cap B \subset J(M)$ and so for any $L(x) \in \mathcal{C}$

$$[C \cup J(M)](L(x)) = A(L(x) \setminus J(M)) + B(L(x) \setminus J(M)) + S(L(x) \cap J(M))$$

$$= [A \cup J(M)](L(x)) + [B \cup J(M)](L(x)) - S(L(x) \cap J(M)).$$

Thus, for $M \geq N$, $L(x) \in \mathcal{C}$,

$$\frac{[C \cup J(M)](L(x)) + S(L(x) \cap J(M))}{S(L(x))}$$

$$= \frac{[A \cup J(M)](L(x))}{S(L(x))} + \frac{[B \cup J(M)](L(x))}{S(L(x))}$$

$$\geq d(A \cup J(M)) + d(B \cup J(M)).$$

Hence, by definition of d^N,

$$d^M(C \cup J(M)) \geq d(A \cup J(M)) + d(B \cup J(M)).$$

Letting $M \to \infty$, we obtain by theorem 1.17

$$\delta(C) \geq \delta(A) + \delta(B).$$

Remark(2.8) The inequality sign of theorem 2.7 can not be omitted. For instance, if we let $S = I^2$ and let

$$A = \{(x,y) \mid (x,y) \in S,\ y \geq x\},$$

$$B = S \setminus A.$$

Then $A \cup B = S$ and so $\delta(A \cup B) = 1$. But $\delta(A) + \delta(B) = 0 + 0 = 0$. To see this recall, for any $\{L(\underset{\sim}{x}^i)\} \in \mathcal{S}$,

$$\delta(A) \leq \underline{\lim}_{i\to\infty} \frac{A(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} .$$

Letting $L(\underset{\sim}{x}^i) = L((2^i, i))$, $i = 1, 2, \cdots$, we have $\{L(\underset{\sim}{x}^i)\} \in \mathcal{S}$ and

$$0 \leq \delta(A) \leq \underline{\lim}_{i\to\infty} \frac{A(L(2^i, i))}{S(L(2^i, i))}$$

$$= \underline{\lim}_{i\to\infty} \frac{\frac{(i+1)(i+2)}{2} - 1}{(i+1)(2^i+1) - 1}$$

$$\leq \underline{\lim}_{i\to\infty} \frac{(i+1)(i+2)}{2(i+1)(2^i+1)} = \underline{\lim}_{i\to\infty} \frac{i+2}{2(2^i+1)} = 0.$$

Similarly, we have $\delta(B) = 0$.

CHAPTER 3.

THE UPPER C-ASYMPTOTIC DENSITY, THE C-NATURAL DENSITY AND SOME STRUCTURE RESULTS.

In this chapter, we define the upper C-asymptotic density and the C-natural density. We give analogies to theorems 1.12--1.17 for the upper C-asymptotic density. We prove the "additivity" theorem for C-natural density and some related results for lower and upper C-asymptotic density (Theorem 3.25, 3.19 and 3.21 respectively) which are entirely new in this setting.

Definition (3.1). For a set $A \subset S$, we define the upper C-density of A to be

$$\overline{d}(A) = \text{lub} \left\{ \frac{A(L(\underset{\sim}{x}))}{S(L(\underset{\sim}{x}))} \;\middle|\; \underset{\sim}{x} \in S \setminus \underset{\sim}{0} \right\}$$

and the upper C-asymptotic density of A to be

$$\overline{\delta}(A) = \lim_{N \to \infty} \overline{d}\,(A \setminus J(N)).$$

It is easy to see that

$$\overline{d}(A \setminus J(N)) = \text{lub}\{ \frac{[A \setminus J(N)](L(\underset{\sim}{x}))}{S(L(\underset{\sim}{x}))} \Big| \underset{\sim}{x} \in S \setminus J(N)\} .$$

Since $\overline{d}(A \setminus J(N))$ forms a nonincreasing sequence as $N \to \infty$ it follows that $\overline{\delta}(A)$ always exists.

Theorem (3.2). If $\{L(\underset{\sim}{x}^i)\} \in \mathcal{S}$ and $A \subset S$, then

$$\overline{\delta}(A) \geq \overline{\lim_{i \to \infty}} \frac{A(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} .$$

Proof. For any $N > 0$ and $\{L(\underset{\sim}{x}^i)\} \in \mathcal{S}$, we have

$$\overline{d}(A \setminus J(N)) \geq \frac{[A \setminus J(N)](L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))}$$

$$\geq \frac{A(L(\underset{\sim}{x}^i)) - S(J(N) \cap L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} .$$

Thus

$$\overline{d}(A \setminus J(N)) \geq \overline{\lim_{i \to \infty}} \left[\frac{A(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} - \frac{S(J(N) \cap L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} \right]$$

$$\geq \overline{\lim_{i\to\infty}} \left[\frac{A(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} \right] + \underline{\lim_{i\to\infty}} \left[- \frac{S(J(N)\cap L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} \right]$$

$$= \overline{\lim_{i\to\infty}} \left[\frac{A(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} \right] \quad .$$

Letting $N\to\infty$ we obtain

$$\overline{\delta}(A) \geq \overline{\lim_{i\to\infty}} \frac{A(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} \quad .$$

<u>Corallary (3.3)</u>. For any $A\subset S$ we have $\delta(A) \leq \overline{\delta}(A)$.

<u>Proof</u>. For any $\{L(\underset{\sim}{x}^i)\}\in \mathcal{S}$ we have

$$\delta(A) \leq \underline{\lim_{i\to\infty}} \leq \frac{A(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} \leq \overline{\lim_{i\to\infty}} \frac{A(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} \leq \overline{\delta}(A).$$

<u>Theorem (3.4)</u>. For any $A\subset S$ there is a sequence $\{L(\underset{\sim}{x}^i)\}\in \mathcal{S}$ such that

$$\overline{\delta}(A) = \lim_{i\to\infty} \frac{A(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} \quad .$$

<u>Proof</u>. For any $i\geq 1$, let $M(i)$ be such that $M(i)>i$ and

$$g(M(i), i, n) > 2^i$$

and choose $L(\underset{\sim}{x}^i) \in \mathcal{C}$ such that $\underset{\sim}{x}^i \in S \setminus J(M(i))$ and

$$\frac{[A \setminus J(M(i))](L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} \geq \overline{d}(A \setminus J(M(i))) - \frac{1}{2^i}.$$

Since, $L(\underset{\sim}{x}^i) \in \mathcal{L}(M(i)) \subset \mathcal{L}(i)$ we have $\{L(\underset{\sim}{x}^i)\} \in \mathcal{S}$. From the inequalities

$$0 \leq \frac{A(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} - \frac{[A \setminus J(i)](L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))}$$

$$\leq \frac{[(A \setminus J(i)) \cup J(i)](L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} - \frac{[A \setminus J(i)](L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))}$$

$$\leq \frac{S(J(i) \cap L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} \leq \frac{1}{g(M(i), i, n)} \leq \frac{1}{2^i},$$

it follows that

$$\lim_{i \to \infty} \left[\frac{A(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} - \frac{[A \setminus J(i)](L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} \right] = 0.$$

But also

$$\lim_{i \to \infty} \frac{[A \setminus J(i)](L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} = \overline{\delta}(A),$$

for

$$\overline{d}(A\setminus J(i)) \geq \frac{[A\setminus J(i)](L(\underset{\sim}{x}^{i}))}{S(L(\underset{\sim}{x}^{i}))}$$

$$\geq \frac{[A\setminus J(M(i))](L(\underset{\sim}{x}^{i}))}{S(L(\underset{\sim}{x}^{i}))} \geq \overline{d}(A\setminus J(M(i))) - \frac{1}{2^{i}}$$

where both ends approach $\overline{\delta}(A)$ as $i\to\infty$. This shows that

$$\overline{\delta}(A) = \lim_{i\to\infty} \frac{A(L(\underset{\sim}{x}^{i}))}{S(L(\underset{\sim}{x}^{i}))} \quad .$$

The following three theorems give the other equivalent forms for the upper C-asymptotic density.

<u>Theorem(3.5)</u>. For each $A\subset S$,

$$\overline{\delta}(A) = \underset{\{L(\underset{\sim}{x}^{i})\}\in\mathcal{S}}{\text{lub}} \ \overline{\lim_{i\to\infty}} \frac{A(L(\underset{\sim}{x}^{i}))}{S(L(\underset{\sim}{x}^{i}))}.$$

<u>Proof</u>. By theorem 3.2, for any $\{L(\underset{\sim}{x}^{i})\}\in\mathcal{S}$

$$\overline{\delta}(A) \geq \overline{\lim_{i\to\infty}} \frac{A(L(\underset{\sim}{x}^{i}))}{S(L(\underset{\sim}{x}^{i}))} \quad .$$

Thus

$$\overline{\delta}(A) \geq \underset{\{L(\underset{\sim}{x}^i)\}\in\mathcal{S}}{\text{lub}} \ \overline{\lim_{i\to\infty}} \ \frac{A(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} \quad .$$

On the other hand, by theorem 3.4 there exists a sequence $\{L(\underset{\sim}{x}^i)\}\in\mathcal{S}$ such that

$$\overline{\delta}(A) = \lim_{i\to\infty} \frac{A(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} = \overline{\lim_{i\to\infty}} \ \frac{A(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))}$$

$$\leq \underset{\{L(\underset{\sim}{x}^i)\}\in\mathcal{S}}{\text{lub}} \ \overline{\lim_{i\to\infty}} \ \frac{A(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))}.$$

Therefore,

$$\overline{\delta}(A) = \underset{\{L(\underset{\sim}{x}^i)\}\in\mathcal{S}}{\text{lub}} \ \overline{\lim_{i\to\infty}} \ \frac{A(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} \quad .$$

<u>Notation.</u> For $N\geq 0$ and $A\subset S$ define

$$\overline{d}^N(A) = \text{lub}\left\{ \frac{A(L(\underset{\sim}{x})) - S(J(N)\cap L(\underset{\sim}{x}))}{S(L(\underset{\sim}{x}))} \ \middle| \ L(\underset{\sim}{x})\in\mathcal{C} \right\} \quad .$$

<u>Theorem (3.6)</u>. For any $A\subset S$,

$$\overline{\delta}(A) = \lim_{N\to\infty} \overline{d}^N(A) \ .$$

<u>Proof</u>. Since, for each $L(\underset{\sim}{x})\in\mathcal{C}$

$$\frac{[A \setminus J(N)](L(\underset{\sim}{x}))}{S(L(\underset{\sim}{x}))} \geq \frac{A(L(\underset{\sim}{x})) - S(J(N) \cap L(\underset{\sim}{x}))}{S(L(\underset{\sim}{x}))} .$$

Then

$$\text{lub}\left\{ \frac{[A \setminus J(N)](L(\underset{\sim}{x}))}{S(L(\underset{\sim}{x}))} \,\middle|\, L(\underset{\sim}{x}) \in \mathcal{C} \right\} \geq \text{lub}\left\{ \frac{A(L(\underset{\sim}{x})) - S(J(N) \cap L(\underset{\sim}{x}))}{S(L(\underset{\sim}{x}))} \,\middle|\, L(\underset{\sim}{x}) \in \mathcal{C} \right\}.$$

Therefore,

$$\overline{d}(A \setminus J(N)) \geq \overline{d}^N(A).$$

Furthermore, let $\{L(x^i)\} \in \mathcal{S}$ such that

$$\overline{\delta}(A) = \lim_{i \to \infty} \frac{A(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} .$$

Thus

$$\overline{d}^N(A) \geq \frac{A(L(\underset{\sim}{x}^i)) - S(J(N) \cap L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))}$$

$$= \frac{A(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} - \frac{S(J(N) \cap L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} .$$

The right hand side of the inequality tends to $\overline{\delta}(A)$ as $i \to \infty$. Then

$$\overline{d}^N(A) \geq \overline{\delta}(A).$$

Therefore, for each N we have

$$\overline{\delta}(A) \le \overline{d}^N(A) \le \overline{d}(A\setminus J(N))$$

and so

$$\lim_{N\to\infty} \overline{\delta}^N(A) = \overline{\delta}(A).$$

<u>Theorem (3.7)</u>. For any $A \subset S$,

$$\overline{\delta}(A) = \lim_{N\to\infty} \overline{d}^N(A\setminus J(N)).$$

<u>Proof.</u> As in the proof of theorem 3.6

$$\begin{aligned}\overline{d}(A\setminus J(N)) &= \overline{d}([(A\setminus J(N))]\setminus J(N)) \\ &\ge \overline{d}^N(A\setminus J(N)) \\ &\ge \overline{\delta}\,(A\setminus J(N)) = \overline{\delta}(A).\end{aligned}$$

The last equality follows from the definition of $\overline{\delta}$.

<u>Theorem (3.8)</u>. In the case n=1, $\overline{\delta}(A)$ is the usual upper asymptotic density of A.

<u>Proof.</u> When n = 1, S = I the set of all nonnegative integers, then for any $i \in I \setminus 0$, $S(L(i)) = i$. Hence by theorem 3.2, we we have

$$\overline{\delta}(A) \ge \overline{\lim_{i\to\infty}} \frac{A(i)}{i} .$$

On the other hand, by theorem 3.4, there exists a sequence $\{n_i\}$ such that $n_i \to \infty$ as $i \to \infty$ corresponding to $\{L(n_i)\} \in \mathcal{S}$ such that

$$\overline{\delta}(A) = \lim_{i\to\infty} \frac{A(n_i)}{n_i} \leq \overline{\lim_{i\to\infty}} \frac{A(i)}{i}$$

and the result follows.

Theorem (3.9). If $A \cap B \subset J(N)$ for some N, then

$$\overline{\delta}(A \cup B) \leq \overline{\delta}(A) + \overline{\delta}(B).$$

Proof. For $M \geq N$ we have $A \cap B \subset J(M)$, then for any $L(\underset{\sim}{x}) \in \mathcal{C}$

$$\frac{[(A \cup B) \setminus J(M)](L(\underset{\sim}{x}))}{S(L(\underset{\sim}{x}))} = \frac{([A \setminus J(M)] \cup [B \setminus J(M)])(L(\underset{\sim}{x}))}{S(L(\underset{\sim}{x}))}$$

$$\leq \frac{[A \setminus J(M)](L(\underset{\sim}{x})) + [B \setminus J(M)](L(\underset{\sim}{x}))}{S(L(\underset{\sim}{x}))}$$

$$\leq \overline{d}(A \setminus J(M)) + \overline{d}(B \setminus J(M)).$$

Thus

$$\overline{d}[(A \cup B) \setminus J(M)] \leq \overline{d}(A \setminus J(M)) + \overline{d}(B \setminus J(M)).$$

Letting $M \to \infty$ we obtain

$$\overline{\delta}(A \cup B) \leq \overline{\delta}(A) + \overline{\delta}(B).$$

Remark (3.10). The inequality sign of the theorem 3.9 can not be removed. For example, let $S = I^2$, A and B defined as in remark 2.8. Then we can see that

$$\overline{\delta}(A \cup B) = \overline{\delta}(S) = 1.$$

But

$$\overline{\delta}(A) = \overline{\delta}(B) = 1.$$

To see this recall, for any $\{L(\underset{\sim}{x}^i)\} \in \mathcal{S}$

$$\overline{\delta}(A) \geq \overline{\lim_{i \to \infty}} \frac{A(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} .$$

Letting $L(\underset{\sim}{x}^i) = L((i, 2^i))$, $i = 1,2,\cdots$, we have $\{L(x^i)\} \in \mathcal{S}$ and

$$1 \geq \overline{\delta}(A) \geq \overline{\lim_{i \to \infty}} \frac{A(L((i, 2^i)))}{S(L((i, 2^i)))}$$

$$= \overline{\lim_{i \to \infty}} \frac{\frac{(i + 1)(i + 2)}{2} - 1 + (2^i - i)(i + 1)}{(i + 1)(2^i + 1) - 1}$$

$$= \overline{\lim_{i \to \infty}} \frac{\frac{(i+1)(i+2)}{2(i+1)(2^i+1)} - \frac{1}{(i+1)(2^i+1)} + \frac{(2^i-i)(i+1)}{(i +1)(2^i+1)}}{1 - \frac{1}{(i+1)(2^i+1)}}$$

$$= \overline{\lim_{i \to \infty}} \frac{\frac{i+2}{2(2^i+1)} - \frac{1}{(i+1)(2^i+1)} + \frac{2^i-i}{2^i+1}}{1 - \frac{1}{(i+1)(2^i+1)}}$$

$$= 1.$$

Similarly, we have $\overline{\delta}(B) = 1$. Therefore

$$\overline{\delta}(A) + \overline{\delta}(B) = 1 + 1 > \overline{\delta}(A \cup B) = 1.$$

We begin now our study of the some structure results.

Definition (3.11). For any $A \subset S$ if $\underline{\delta}(A) = \overline{\delta}(A)$, then we say that the C-natural density of A exists and write $\nu(A) = \underline{\delta}(A) = \overline{\delta}(A)$.

Note. The C-natural density of a set A does not always exist. For example, if we let $S = I^2$ and $A = \{(x,y) \mid (x,y) \in S,\ y \geq x\}$, then from remark 2.8 and remark 3.10, we have $\underline{\delta}(A) = 0$ and $\overline{\delta}(A) = 1$. Therefore, the C-natural density of A does not exist.

Theorem (3.12). The C-natural density of $A \subset S$ exists if and only if, for each $\{L(\underset{\sim}{x}^i)\} \in \mathcal{S}$, the quotients $A(L(\underset{\sim}{x}^i))/S(L(\underset{\sim}{x}^i))$ form a convergent sequence. In this case

$$\nu(A) = \lim_{i\to\infty} \frac{A(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))}$$

for each sequence $\{L(\underset{\sim}{x}^i)\} \in \mathcal{S}$.

Proof. (i) Suppose that the $\nu(A)$ exists, then for each sequence $\{L(\underset{\sim}{x}^i)\} \in \mathcal{S}$, by theorem 1.12 and theorem 3.2,

$$\nu(A) = \underline{\delta}(A) \leq \underline{\lim}_{i\to\infty} \frac{A(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} \leq \overline{\lim}_{i\to\infty} \frac{A(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))}$$

$$\leq \overline{\delta}(A) = \nu(A) .$$

Thus

$$\nu(A) = \lim_{i\to\infty} \frac{A(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))}.$$

(ii) Suppose $A(L(x^i))/S(L(x^i))$ is convergent for each $\{L(x^i)\} \in \mathcal{S}$. Then all limits must be the same, for, if $\{L(x^i)\}$ and $\{L(y^i)\}$ are two sequences in $\mathcal{S}$ such that $A(L(x^i))/S(L(x^i))$ and $A(L(y^i))/S(L(y^i))$ converge to different limits, then the sequence $\{L(z^i)\}$ defined by

$$z^i = \begin{cases} x^i & \text{for } i \text{ odd} \\ y^i & \text{for } i \text{ even} \end{cases}$$

is in $\mathcal{S}$ and $\lim A(L(z^i))/S(L(z^i))$ does not exist. This is a contradiction. Therefore, all the limits are the same. By theorem 1.13 and theorem 3.4. there exist $\{L(x^i)\}$ and $\{L(y^i)\}$ in $\mathcal{S}$ such that

$$\underline{\delta}(A) = \lim_{i\to\infty} \frac{A(L(x^i))}{S(L(x^i))} = \lim_{i\to\infty} \frac{A(L(y^i))}{S(L(y^i))} = \overline{\delta}(A).$$

Hence, $\nu(A)$ exists and the last statement of the theorem is obvious.

Theorem (3.13). For any $A \subset S$, if the C-natural density $\nu(A)$ exists, then the C-natural density $\nu(\overline{A})$ also exists and

$$\nu(A) + \nu(\overline{A}) = 1.$$

Proof. Since the C-natural density $\nu(A)$ exists, theorem 3.12 shows that for each $\{L(x^i)\} \in \mathcal{S}$ the quotients $A(L(x^i))/S(L(x^i))$ form a convergent sequence and

$$\nu(A) = \lim_{i\to\infty} \frac{A(L(x^i))}{S(L(x^i))}.$$

Hence, for each $\{L(\underset{\sim}{x}^i)\} \in \mathcal{S}$

$$\frac{\bar{A}(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} = \frac{S(L(\underset{\sim}{x}^i)) - A(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))}$$

$$= 1 - \frac{A(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} \rightarrow 1 - \nu(A)$$

as $i \to \infty$, it follows from the theorem 3.12 that $\nu(\bar{A})$ will exist and

$$\nu(\bar{A}) = \lim_{i\to\infty} \frac{\bar{A}(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} = 1 - \nu(A).$$

Corollary (3.14). For any $A \subset S$, the C-natural density exists if and only if

$$\delta(A) + \delta(\bar{A}) = 1.$$

Proof. (i) Suppose that $\nu(A)$ exists. Theorem 3.13 shows that $\nu(\bar{A})$ exists and

$$\nu(A) + \nu(\bar{A}) = 1.$$

But also

$$\nu(A) = \delta(A) \text{ and } \nu(\bar{A}) = \delta(\bar{A}).$$

Hence

$$\delta(A) + \delta(\bar{A}) = 1.$$

(ii) Suppose that $\delta(A) + \delta(\bar{A}) = 1$. Let $\{L(\underset{\sim}{x}^i)\}$ be a sequence such that $\{L(\underset{\sim}{x}^i)\} \in \mathcal{S}$ and

$$\overline{\delta}(A) = \lim_{i\to\infty} \frac{A(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))}.$$

Then

$$1 - \delta(A) = \delta(\overline{A}) \le \underline{\lim}_{i\to\infty} \frac{\overline{A}(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} = \underline{\lim}_{i\to\infty} 1 - \frac{A(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))}$$

$$= \lim_{i\to\infty} 1 - \frac{A(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))}$$

$$= 1 - \lim_{i\to\infty} \frac{A(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))}$$

$$= 1 - \overline{\delta}(A).$$

Hence, $\delta(A) \ge \overline{\delta}(A)$.

On the other hand, we have, by corollary 3.3, $\delta(A) \le \overline{\delta}(A)$. Therefore, $\delta(A) = \overline{\delta}(A)$ and $\nu(A)$ exists.

Corollary (3.15). For any $A \subset S$, the C-natural density exists if and only if

$$\overline{\delta}(A) + \overline{\delta}(\overline{A}) = 1.$$

The proof is similar to corollary 3.14.

From the remark 2.8 and remark 3.10, we know that, for set A, B with $A \cap B \subset J(N)$, the equalities $\delta(A \cup B) = \delta(A) + \delta(B)$ and $\overline{\delta}(A \cup B) = \overline{\delta}(A) + \overline{\delta}(B)$ need not be true in general. But if the C-natural densities of A and B exist, then we can prove the following result which amounts to finite additivity for C-natural density.

Theorem (3.16). Let $A_1, A_2, \cdots, A_n$ be sets with C-natural density such that, for each pair i, j with $i \neq j$ there is an N_{ij} such that $A_i \cap A_j \subset J(N_{ij})$. Then $A = A_1 \cup A_2 \cup \cdots A_n$ has C-natural density and

$$\nu(A) = \sum_{i=1}^{n} \nu(A_i).$$

Proof. Let $N = \max_{i,j} \{N_{ij}\}$. Clearly $B_i = A_i \setminus J(N)$ has natural density and

$$\nu(B_i) = \nu(A_i) \qquad i = 1,2,\cdots,n.$$

Since the B_i are disjoint, for any sequence $\{L(\underset{\sim}{x}^i)\} \in \mathcal{S}$

$$\frac{A_1(L(\underset{\sim}{x}^i)) + A_2(L(\underset{\sim}{x}^i)) + \cdots + A_n(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} \geq \frac{A(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))}$$

$$\geq \frac{(B_1 \cup B_2 \cup \cdots \cup B_n)(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))}$$

$$= \frac{B_1(L(\underset{\sim}{x}^i)) + B_2(L(\underset{\sim}{x}^i)) + \cdots + B_n(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))}$$

where both ends converge to $\sum_{i=1}^{n} \nu(A_i)$ as $i \to \infty$. By theorem 3.12 we conclude that $\nu(A)$ exists and

$$\nu(A) = \sum_{i=1}^{n} \nu(A_i).$$

The remainder of the chapter will be devoted to the proofs of some structure results for lower and upper C-asymptotic density and the "additivity" theorem. Before discussing those theorems, let us give the following definition:

Definition (3.17). For any $A,B\subset S$, A is said to be asymptotic to B (denoted by $A\sim B$) if there is an integer N such that $A\setminus J(N) = B\setminus J(N)$.

It is easily seen that $A\sim B$ if and only if there exists an N such that $A\cup J(N) = B\cup J(N)$.

Lemma (3.18). For any $A,B\subset S$, if $A\sim B$ then $\delta(A) = \delta(B)$ and $\overline{\delta}(A) = \overline{\delta}(B)$. Furthermore, $\nu(A)$ exists if and only if $\nu(B)$ exists and $\nu(A) = \nu(B)$ if both exist.

Proof. Since $A\sim B$, there exists an N_1 such that $A\cup J(N_1) = B\cup J(N_1)$. Thus for $N \geq N_1$ we have

$$A\cup J(N) = B\cup J(N).$$

Therefore

$$d(A\cup J(N)) = d(B\cup J(N)) \quad \text{for all } N\geq N_1.$$

Letting $N\to\infty$ we obtain

$$\delta(A) = \delta(B).$$

The remainder of the proof is similar and left to the reader.

We now proceed to prove some structure results. We consider a sequence of sets $\{A_i\}$ on S such that $A_1\subset A_2\subset\cdots\subset A_n\subset\cdots$, then $\delta(A_1)\leq \delta(A_2) \leq\cdots\leq\delta(A_n)\leq \cdots$. Thus $\{\delta(A_n)\}$ is an increasing sequence bounded above by 1.

Therefore, the limit of $\{\delta(A_i)\}$ exists.

A question raises "suppose $A_1 \subset A_2 \subset \cdots \subset A_n \subset \cdots$ does it imply that

$$\delta(\bigcup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \delta(A_n)?".$$

A simple example shows that this is not true. Let $S = I^2$, $A_1 = J(1)$, $A_2 = J(2), \cdots, A_n = J(n), \cdots$. Then $A_1 \subset A_2 \subset \cdots \subset A_n \subset \cdots$, it is easy to see that $\delta(A_n) = 0$ for each n. Thus

$$\lim_{n\to\infty} \delta(A_n) = 0.$$

But
$$\delta(\bigcup_{n=1}^{\infty} A_n) = \delta(S) = 1.$$

However, we can find a sequence of sets $\{B_i\}$ such that $B_i \sim A_i$ $(i = 1,2,\cdots)$ and

$$\delta(\bigcup_{i=1}^{\infty} B_i) = \lim_{i\to\infty} \delta(A_i).$$

In this case we could take $B_i = \phi$ for each i. In general we have the following theorem.

<u>Theorem (3.19)</u>. Let $A_1, A_2, \cdots, A_k, \cdots$ be sets with C-asymptotic density δ_k $(k = 1,2,\cdots)$ such that $A_1 \subset A_2 \subset \cdots \subset A_k \subset \cdots$. Then there exist $B_1, B_2, \cdots, B_k \cdots$ such that $B_k \sim A_k$ $(k = 1,2,\cdots)$ and

$$\delta(\bigcup_{k=1}^{\infty} B_k) = \lim_{k\to\infty} \delta(A_k).$$

<u>Proof</u>. (i) If $B_k \sim A_k$ $(k = 1,2,\cdots)$, let

$$\mathcal{B} = \bigcup_{k=1}^{\infty} B_k .$$

Then $B_k \subset \mathcal{B}$ for any k. Therefore, $\delta(B_k) \le \delta(\mathcal{B})$ for any k and $\delta(A_k) \le \delta(\mathcal{B})$ for any k. This implies $\lim_{k\to\infty} \delta(A_k) \le \delta(\mathcal{B})$.

(ii) We are going to choose the B_k such that

$$\delta(\mathcal{B}) \le \lim_{k\to\infty} \delta(A_k).$$

Since, $\delta(A_k) = \delta_k$ ($k = 1,2,\cdots$ there exists a sequence $\{L(\underset{\sim}{x}^{i,1})\} \in \mathcal{S}$ such that

$$\delta_1 = \delta(A_1) = \lim_{i\to\infty} \frac{A_1(L(\underset{\sim}{x}^{i,1}))}{S\ (L(\underset{\sim}{x}^{i,1}))}.$$

Thus, there is an integer $N_1 > 0$ such that for all $\underset{\sim}{x}^{i,1} \in S \setminus J(N_1)$

$$\frac{A_1(L(\underset{\sim}{x}^{i,1}))}{S(L(\underset{\sim}{x}^{i,1}))} \le \delta_1 + 1.$$

Similarly, there exists a sequence $\{L(\underset{\sim}{x}^{i,2})\} \in \mathcal{S}$ such that

$$\delta_2 = \delta(A_2) = \lim_{i\to\infty} \frac{A_2(L(\underset{\sim}{x}^{i,2}))}{S\ (L(\underset{\sim}{x}^{i,2}))}.$$

Choose N_2 big enough such that

(i) $N_2 > N_1$;

(ii) There exists an $\underset{\sim}{x}^{i,1} \in J(N_2) \setminus J(N_1)$;

(iii) $\dfrac{A_2(L(\underset{\sim}{x}^{i,2}))}{S\ (L(\underset{\sim}{x}^{i,2}))} \le \delta_2 + \tfrac{1}{2}$ for all $\underset{\sim}{x}^{i,2} \in S \setminus J(N_2)$.

In general, there exists a sequence $\{L(\underset{\sim}{x}^{i,k})\} \in \mathcal{S}$ such that

$$\delta_k = \delta(A_k) = \lim_{i\to\infty} \frac{A_k(L(\underset{\sim}{x}^{i,k}))}{S\ (L(\underset{\sim}{x}^{i,k}))} .$$

Choose N_k big enough such that

(i) $N_k > N_{k-1}$

(ii) there exists an $\underset{\sim}{x}^{i,k-1} \in J(N_k) \setminus J(N_{k-1})$

(iii) $\dfrac{A_k(L(\underset{\sim}{x}^{i,k}))}{S\ (L(\underset{\sim}{x}^{i,k}))} \leq \delta_k + \dfrac{1}{k}$ for all $\underset{\sim}{x}^{i,k} \in S \setminus J(N_k)$.

Let

$$B_k = A_k \setminus J(N_k) \qquad (k = 1,2,\cdots).$$

Then

$$B_k \sim A_k \qquad (k = 1,2,\cdots).$$

Since, $B_j \subset A_j$ $(j = 1,2,\cdots)$ and $A_1 \subset A_2 \subset \cdots \subset A_k \subset \cdots$, then $\bigcup_{j=1}^{k} B_j \subset \bigcup_{j=1}^{k} A_j = A_k$.

Now, let $\mathcal{B} = \bigcup_{k=1}^{\infty} B_k$, by theorem 1.12, for any $\{L(\underset{\sim}{y}^i)\} \in \mathcal{S}$

$$\delta(\mathcal{B}) \leq \underline{\lim}_{i \to \infty} \frac{\mathcal{B}(L(\underset{\sim}{y}^i))}{S(L(\underset{\sim}{y}^i))} \quad .$$

Take

$$\underset{\sim}{y}^1 = \text{some } \underset{\sim}{x}^{i,1} \in J(N_2) \setminus J(N_1),$$

$$\underset{\sim}{y}^2 = \text{some } \underset{\sim}{x}^{i,2} \in J(N_3) \setminus J(N_2),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$\underset{\sim}{y}^k = \text{some } \underset{\sim}{x}^{i,k} \in J(N_{k+1}) \setminus J(N_k),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots.$$

Then, the sequence $\{L(\underset{\sim}{y}^k)\} \in \mathcal{S}$. It follows from our chosen B_k that

$$\frac{\mathcal{B}(L(\underset{\sim}{y}^k))}{S\ (L(\underset{\sim}{y}^k))} = \frac{((\bigcup_{j=1}^{k} B_j) \cup B_{k+1} \cup \cdots)(L(\underset{\sim}{y}^k))}{S(L(\underset{\sim}{y}^k))}$$

$$= \frac{(\bigcup_{j=1}^{k} B_j)(L(\underset{\sim}{y}^k))}{S(L(\underset{\sim}{y}^k))}$$

$$\leq \frac{A_k(L(\underset{\sim}{y}^k))}{S\ (L(\underset{\sim}{y}^k))} \leq \delta_k + \frac{1}{k}$$

(since, $B_j(L(\underset{\sim}{y}^k)) = 0$ for all $j \geq k+1$). Therefore

$$\overline{\lim_{k\to\infty}} \frac{\mathcal{B}(L(\underset{\sim}{y}^k))}{S\ (L(\underset{\sim}{y}^k))} \leq \overline{\lim_{k\to\infty}} (\delta_k + \frac{1}{k}) = \overline{\lim_{k\to\infty}} \delta_k = \lim_{k\to\infty} \delta_k = \lim_{k\to\infty} \delta(A_k)$$

(since, $\{\delta_k\}$ is an increasing sequence bounded above by 1).

Hence

$$\delta(\mathcal{B}) \leq \lim_{k\to\infty} \delta(A_k).$$

This completes the proof.

We also prove the analogue of theorem 3.19 in terms of upper C-asymptotic density. We require the following lemma:

Lemma (3.20). If $A \subseteq S$ and $\overline{\delta}(A) = \overline{\delta}$, then for any $\varepsilon > 0$ there exists an integer $N > 0$ such that for all $\underset{\sim}{x} \in S \setminus J(N)$ we have

$$\frac{A(L(\underset{\sim}{x}))}{S(L(\underset{\sim}{x}))} \leq \overline{\delta} + \varepsilon.$$

Proof. We suppose that this lemma is not true. Then there is an $\varepsilon > 0$ and for any integer $N > 0$ there exists an $\underset{\sim}{x}^N$ such that $\underset{\sim}{x}^N \in S \setminus J(N)$ with

$$\frac{A(L(\underset{\sim}{x}^N))}{S(L(\underset{\sim}{x}^N))} > \overline{\delta} + \varepsilon > \overline{\delta}$$

It follows since $\{L(\underset{\sim}{x}^N)\} \in \underset{\sim}{g}$ that

$$\overline{\lim_{N\to\infty}} \frac{A(L(\underset{\sim}{x}^N))}{S(L(\underset{\sim}{x}^N))} > \overline{\delta}$$

which contradicts to the theorem 3.2.

Theorem (3.21). Let $A_1, A_2, \cdots, A_k \cdots$ be sets with upper C-asymptotic density $\overline{\delta}_k$ $(k = 1,2,\cdots)$ such that $A_1 \subset A_2 \subset \cdots \subset A_k \subset \cdots$. Then there exist $B_1, B_2, \cdots, B_k, \cdots$ such that $B_k \sim A_k$ $(k = 1,2,\cdots)$ and

$$\overline{\delta}\left(\bigcup_{k=1}^{\infty} B_k\right) = \lim_{k\to\infty} \overline{\delta}(A_k).$$

Proof. (i) If $B_k \sim A_k$ $(k = 1,2,\cdots)$, let

$$\mathcal{B} = \bigcup_{k=1}^{\infty} B_k \quad .$$

Then $B_k \subset \mathcal{B}$ for any k. Therefore, $\overline{\delta}(B_k) \leq \overline{\delta}(\mathcal{B})$ for any k and $\overline{\delta}(A_k) \leq \overline{\delta}(\mathcal{B})$ for any k. This implies $\lim_{k\to\infty} \overline{\delta}(A_k) \leq \overline{\delta}(\mathcal{B})$.

(ii) We want to choose the B_k such that

$$\delta(\mathcal{B}) \leq \lim_{k\to\infty} \overline{\delta}(A_k).$$

By lemma 3.20, there exists an integer $N_1 > 0$ such that for any $\underset{\sim}{x}^i \in S \setminus J(N_1)$

$$\frac{A_1(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} \leq \overline{\delta}_1 + 1$$

and there exists $N_2 \geqslant N_1$ such that for any $\underset{\sim}{x}^i \in S \setminus J(N_2)$

$$\frac{A_2(L(\underset{\sim}{x}^i))}{S\,(L(\underset{\sim}{x}^i))} \leq \bar{\delta}_2 + \tfrac{1}{2},$$

......................,

there exists N_k such that $N_k \geqslant N_{k-1}$ and for any $\underset{\sim}{x}^i \in S \setminus J(N_k)$

$$\frac{A_k(L(\underset{\sim}{x}^i))}{S\,(L(\underset{\sim}{x}^i))} \leq \bar{\delta}_k + \frac{1}{k},$$

....

Let

$$B_k = A_k \setminus J(N_k) \qquad (k = 1,2,\cdots).$$

Then

$$B_k \sim A_k \qquad (k = 1,2,\cdots).$$

Since, $B_i \subset A_i$ $(i = 1,2,\cdots)$ and $A_1 \subset A_2 \subset \cdots \subset A_k \subset \cdots$, then $\bigcup_{i=1}^{k} B_i \subset \bigcup_{i=1}^{k} A_i = A_k$.

Now, let

$$\mathcal{B} = \bigcup_{k=1}^{\infty} B_k .$$

By theorem 3.4, there exists a sequence $\{L(\underset{\sim}{x}^i)\} \in \underset{\sim}{S}$ such that

$$\bar{\delta}(\mathcal{B}) = \lim_{i \to \infty} \frac{\mathcal{B}(L(\underset{\sim}{x}^i))}{S\,(L(\underset{\sim}{x}^i))} .$$

For any $\underset{\sim}{x}^i \neq \underset{\sim}{0}$, there exists an k such that $\underset{\sim}{x}^i \notin J(N_k)$ but $\underset{\sim}{x}^i \in J(N_{k+1})$.

Therefore

$$\frac{\mathcal{B}(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} = \frac{((\bigcup_{j=1}^{k} B_j) \cup B_{k+1} \cdots)(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))}$$

$$= \frac{(\bigcup_{j=1}^{k} B_j)(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} \leq \frac{A_k(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} \leq \overline{\delta}_k + \frac{1}{k}$$

(since, $B_j(L(\underset{\sim}{x}^i)) = 0$ for all $j \geq k+1$).

Since $\{L(\underset{\sim}{x}^i)\} \in \mathcal{S}$, as $i \to \infty$, $k \to \infty$ also. Therefore,

$$\lim_{i\to\infty} \frac{\mathcal{B}(L(\underset{\sim}{x}^i))}{\mathcal{S}(L(\underset{\sim}{x}^i))} \leq \lim_{i\to\infty} (\overline{\delta}_k + \frac{1}{k}) = \lim_{k\to\infty} (\overline{\delta}_k + \frac{1}{k}) = \lim_{k\to\infty} \overline{\delta}_k$$

$$= \lim_{k\to\infty} \overline{\delta}\,(A_k).$$

Hence

$$\overline{\delta}(\mathcal{B}) \leq \lim_{k\to\infty} \overline{\delta}(A_k).$$

This proves the theorem.

<u>Corollary (3.22).</u> Let $A_1, A_2, \cdots, A_K, \cdots$ be sets with C-asymptotic density δ_k and upper C-asymptotic density $\overline{\delta}_k$($k = 1,2,\cdots$) such that $A_1 \subset A_2 \subset \cdots \subset A_k \subset \cdots$. Then there exist $B_1, B_2, \cdots, B_K, \cdots$ such that $B_k \sim A_k$ ($k = 1,2,\cdots$) and such that both

$$\delta(\bigcup_{k=1}^{\infty} B_k) = \lim_{k\to\infty} \delta(A_k) \quad \text{and} \quad \overline{\delta}(\bigcup_{k=1}^{\infty} B_k) = \lim_{k\to\infty} \overline{\delta}(A_k).$$

<u>Proof</u>. From theorem 3.19, there exist $B_1', B_2', \cdots, B_k', \cdots$ and $N_1', N_2', \cdots, N_k', \cdots$ such that $B_k' = A_k \setminus J(N_k')$ and

(1) $$\delta(\bigcup_{k=1}^{\infty} B_k') = \lim_{k\to\infty} \delta(A_k) .$$

Furthermore, theorem 3.21 shows that there also exist $B_1'', B_2'', \cdots, B_k'', \cdots$

and $N_1'', N_2'', \cdots, N_k'', \cdots$ such that $B_k'' = A_k \setminus J(N_k'')$ and

(2) $$\overline{\delta}\left(\bigcup_{k=1}^{\infty} B_k''\right) = \lim_{k\to\infty} \overline{\delta}(A_k).$$

If we take $B_k = A_k \setminus J(\max\{N_k', N_k''\})$, then $B_k \sim A_k$ $(k = 1,2,\cdots)$ and $B_k \subset B_k'$, $B_k \subset B_k''$ for each k. By the first part of proof of theorem 3.19, we have

(3) $$\lim_{k\to\infty} \delta(A_k) \leq \delta\left(\bigcup_{k=1}^{\infty} B_k\right).$$

Combining (1) and (3), we obtain

$$\lim_{k\to\infty} \delta(A_k) \leq \delta\left(\bigcup_{k=1}^{\infty} B_k\right) \leq \delta\left(\bigcup_{k=1}^{\infty} B_k'\right) = \lim_{k\to\infty} \delta(A_k)$$

and so

$$\delta\left(\bigcup_{k=1}^{\infty} B_k\right) = \lim_{k\to\infty} \delta(A_k) .$$

Similarly, we have

$$\overline{\delta}\left(\bigcup_{k=1}^{\infty} B_k\right) = \lim_{k\to\infty} \overline{\delta}(A_k) .$$

Now, we are going to prove the "additivity" theorem for the C-natural density.

Definition (3.23). A set function f is said to be countably additive if the sets $A_1, A_2, \cdots, A_k, \cdots$ such that $A_i \cap A_j = \phi$ for $i \neq j$, then

$$f\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} f(A_k) .$$

Remark (3.24). There is a simple example to show that ν is not a countably additive set function. Let $S = I^2$, $A_1 = J(1)$, $A_2 = J(2)\setminus J(1),\cdots$, $A_n = J(n)\setminus J(n-1),\cdots$. Then $\bigcup_{n=1}^{\infty} A_n = I^2 = S$ and so $\nu(\bigcup_{n=1}^{\infty} A_n) = 1$. But for each n, $\nu(A_n) = \delta(A_n) = \overline{\delta}(A_n) = 0$. Therefore,

$$\sum_{n=1}^{\infty} \nu(A_n) = 0 \neq \nu(\bigcup_{n=1}^{\infty} A_n) = 1.$$

Although ν is not countably additive, we can prove that ν is "almost" countably additive. This result is called the additivity theorem for C-natural density.

Theorem (3.25). (Additivity Theorem). Let A_i (i =1, 2,$\cdots$) be sets with C-natural density such that for each pair i,j with $i \neq j$, $A_i \cap A_j = \phi$. Then there exist B_i (i = 1,2,$\cdots$) such that $B_i \sim A_i$ (i = 1,2,$\cdots$) and

$$\nu(\bigcup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} \nu(B_i) = \sum_{i=1}^{\infty} \nu(A_i).$$

Proof. (i) If $B_i \sim A_i$ (i = 1,2,$\cdots$), let $\mathcal{B} = \bigcup_{i=1}^{\infty} B_i$.

Then $$\delta(\mathcal{B}) = \delta(\bigcup_{i=1}^{\infty} B_i) \geq \delta(\bigcup_{i=1}^{n} B_i) = \sum_{i=1}^{n} \delta(B_i) = \sum_{i=1}^{n} \nu(B_i).$$

Letting $n \to \infty$ we have

$$\delta(\mathcal{B}) \geq \sum_{i=1}^{\infty} \nu(B_i).$$

(ii) To choose the B_i such that $\overline{\delta}(\mathcal{B}) \leq \sum_{i=1}^{\infty} \nu(B_i)$.

Let $\nu(A_i) = \nu_i$ $(i = 1,2,\cdots)$, then for any $\{L(\underset{\sim}{x})\} \in \mathcal{S}$, there exists an integer $N_1 > 0$ such that for all $\underset{\sim}{x} \in S \setminus J(N_1)$

$$\frac{A_1(L(\underset{\sim}{x}))}{S\ (L(\underset{\sim}{x}))} \leq \nu_1 + 1$$

and there exists $N_2 > N_1$ such that for any $\underset{\sim}{x} \in S \setminus J(N_2)$ we have

$$\frac{A_1(L(\underset{\sim}{x})) + A_2(L(\underset{\sim}{x}))}{S(L(\underset{\sim}{x}))} \leq \nu_1 + \nu_2 + \tfrac{1}{2}$$

....

there exists $N_k > N_{k-1}$ such that for any $\underset{\sim}{x} \in S \setminus J(N_k)$ we have

$$\frac{A_1(L(\underset{\sim}{x})) + A_2(L(\underset{\sim}{x})) + \cdots + A_k(L(\underset{\sim}{x}))}{S(L(\underset{\sim}{x}))} \leq \nu_1 + \nu_2 + \cdots + \nu_k + \frac{1}{k} .$$

Now, let $B_k = A_k \setminus J(N_k)$ $(k = 1,2,\cdots)$.

Thus

$$B_k \sim A_k \qquad (k = 1,2,\cdots).$$

Let

$$\mathcal{B} = \bigcup_{k=1}^{\infty} B_k .$$

By theorem 3.4, there is a sequence $\{L(\underset{\sim}{x})\} \in \mathcal{S}$ such that

$$\overline{\delta}(\mathcal{B}) = \lim_{i\to\infty} \frac{\mathcal{B}(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} .$$

For any $\underset{\sim}{x}^i \neq \underset{\sim}{0}$, there exists an k such that $\underset{\sim}{x}^i \notin J(N_k)$ but $\underset{\sim}{x}^i \in J(N_{k+1})$.

Therefore

$$\frac{\mathcal{B}(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} = \frac{B_1(L(\underset{\sim}{x}^i)) + B_2(L(\underset{\sim}{x}^i)) + \cdots + B_k(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))}$$

$$\leq \frac{A_1(L(\underset{\sim}{x}^i)) + A_2(L(\underset{\sim}{x}^i)) + \cdots + A_k(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))}$$

$$\leq \nu_1 + \nu_2 + \cdots + \nu_k + \frac{1}{k}$$

(since, $B_j\ (L(\underset{\sim}{x}^i)) = 0$ for all $j \geq k + 1$).

Since $\{L(\underset{\sim}{x}^i)\} \in \mathcal{S}$, as $i \to \infty$, $k \to \infty$ also. Therefore

$$\overline{\delta}(\mathcal{B}) = \lim_{i\to\infty} \frac{\mathcal{B}(L(\underset{\sim}{x}^i))}{S(L(\underset{\sim}{x}^i))} \leq \lim_{i\to\infty} (\nu_1 + \nu_2 + \cdots + \nu_k + \frac{1}{k})$$

$$= \lim_{k\to\infty} (\nu_1 + \nu_2 + \cdots + \nu_k + \frac{1}{k})$$

$$= \sum_{k=1}^{\infty} \nu(A_k) = \sum_{k=1}^{\infty} \nu(B_k).$$

It follows that

$$\overline{\delta}(\mathcal{B}) \leq \sum_{k=1}^{\infty} \nu(B_k) \leq \delta(\mathcal{B}) \leq \overline{\delta}(\mathcal{B}) .$$

Hence, $\delta(\mathcal{B}) = \overline{\delta}(\mathcal{B})$ it implies that $\nu(\mathcal{B})$ exists and

$$\nu(\bigcup_{k=1}^{\infty} B_k) = \sum_{k=1}^{\infty} \nu(B_k) = \sum_{k=1}^{\infty} \nu(A_k).$$

This completes the proof.

Here we give an application for the additivity theorem. Let $S = I^2$

and let $\underset{\sim}{x}^j$ be any enumeration of all ordered pairs (r,s) (points in S) such that r and s are relatively prime and $r,s>0$. Let

$$A_j = \{ t\,\underset{\sim}{x}^j \mid t = 1,2,\cdots \} \ .$$

Then $\bigcup_{j=1}^{\infty} A_j = S \setminus$ axes and so $\nu(\bigcup_{j=1}^{\infty} A_j) = 1$. Furthermore, $A_i \cap A_j = \phi$ for $i \neq j$ and $\nu(A_j) = 0$ $(j = 1,2,\cdots)$. To see this, for any $\{L(\underset{\sim}{y}^i)\} \in \mathcal{S}$ there corresponds $\{L(t\underset{\sim}{x}^j)\}$ such that

$$\frac{A_j(L(\underset{\sim}{y}^i))}{S\ (L(\underset{\sim}{y}^i))} \leq \frac{A_j(L(t\underset{\sim}{x}^j))}{S\ (L(t\underset{\sim}{x}^j))} \ .$$

If we let $\underset{\sim}{x}^j = (a,b)$, then $t\underset{\sim}{x}^j = (ta, tb)$ and $S(L(t\underset{\sim}{x}^j)) = (ta + 1)(tb + 1) - 1 \geq t^2$ so that

$$\frac{A_j(L(\underset{\sim}{y}^i))}{S\ (L(\underset{\sim}{y}^i))} \leq \frac{t}{t^2} = \frac{1}{t} \ ,$$

Since $\{L(\underset{\sim}{y}^i)\} \in \mathcal{S}$, as $i \to \infty$, $t \to \infty$ also. Therefore

$$\nu(A_j) = \lim_{i \to \infty} \frac{A_j(L(\underset{\sim}{y}^i))}{S\ (L(\underset{\sim}{y}^i))} = 0.$$

By additivity theorem, there exist $B_j \sim A_j$ $(j = 1,2,\cdots)$ and $\overline{\nu}(\bigcup_{j=1}^{\infty} B_j) = \sum_{j=1}^{\infty} \nu(A_j) = 0$. Here we note that $A_i \setminus B_i$ is only finite.

As an example of an application of theorem 3.19 and theorem 3.21, we can prove the following theorem which differs slightly from the additivity theorem.

Theorem (3.26). Let $A_1, A_2, \cdots, A_n, \cdots$ be sets with C-natural density such that for each pair i,j with $i \neq j$, $A_i \cap A_j = \phi$. Then there exist B_k $(k = 1,2,\cdots)$ such that $B_k \sim \bigcup_{i=1}^{k} A_i$ $(k = 1,2,\cdots)$ and

$$\nu\left(\bigcup_{k=1}^{\infty} B_k\right) = \sum_{k=1}^{\infty} \nu(A_k) \quad .$$

Proof. Let

$$A_1' = A_1 \quad ;$$

$$A_2' = A_1 \cup A_2 \quad ;$$

$$\cdots\cdots\cdots\cdots$$

$$A_k' = A_1 \cup A_2 \cup \cdots \cup A_k = \bigcup_{i=1}^{k} A_i \quad ;$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots .$$

Then $A_1' \subset A_2' \subset \cdots \subset A_k' \subset \cdots$. By corollary 3.22, there exist $B_1, B_2, \cdots, B_k, \cdots$ such that $B_k \sim A_k' = \bigcup_{i=1}^{k} A_i$ $(k = 1,2,\cdots)$ and such that both

$$\underline{\delta}\left(\bigcup_{k=1}^{\infty} B_k\right) = \lim_{k\to\infty} \underline{\delta}(A_k') \text{ and } \overline{\delta}\left(\bigcup_{k=1}^{\infty} B_k\right) = \lim_{k\to\infty} \overline{\delta}(A_k').$$

But also

$$\underline{\delta}\left(\bigcup_{k=1}^{\infty} B_k\right) = \lim_{k\to\infty} \underline{\delta}(A_k') = \lim_{k\to\infty} \underline{\delta}\left(\bigcup_{i=1}^{k} A_i\right)$$

$$= \lim_{k\to\infty} \sum_{i=1}^{k} \underline{\delta}(A_i) = \sum_{i=1}^{\infty} \underline{\delta}(A_i).$$

The third equality follows from the fact that, since the C-natural density of A_i $(i = 1,2,\cdots,k)$ exist and

$$\delta\left(\bigcup_{i=1}^{k} A_i\right) = \nu\left(\bigcup_{i=1}^{k} A_i\right) = \sum_{i=1}^{k} \nu(A_i) = \sum_{i=1}^{k} \delta(A_i).$$

Similarly, we have

$$\overline{\delta}\left(\bigcup_{k=1}^{\infty} B_k\right) = \sum_{i=1}^{\infty} \overline{\delta}(A_i) \; .$$

Furthermore, since $\delta(A_i) = \overline{\delta}(A_i) = \nu(A_i)$ $(i = 1,2,\cdots)$, then

$$\delta\left(\bigcup_{k=1}^{\infty} B_k\right) = \sum_{i=1}^{\infty} \delta(A_i) = \sum_{i=1}^{\infty} \nu(A_i) = \sum_{i=1}^{\infty} \overline{\delta}(A_i) = \overline{\delta}\left(\bigcup_{k=1}^{\infty} B_k\right).$$

It follows that the C-natural density of $\bigcup_{k=1}^{\infty} B_k$ exists and

$$\nu\left(\bigcup_{k=1}^{\infty} B_k\right) = \sum_{k=1}^{\infty} \nu(A_k) \; .$$

CHAPTER 4.

COMPARISON OF C- AND K-ASYMPTOTIC DENSITIES.

A. R. Freedman [4] has defined K-density and K-asymptotic density as follows:

Definition (4.1). A set $\mathcal{K}$ is said to be a K-class on S if

$$\mathcal{K} = \{ F \mid F \cap (S \setminus \underset{\sim}{0}) \text{ is nonempty and finite; } \underset{\sim}{x} \in F \Rightarrow L(\underset{\sim}{x}) \in F \} .$$

For $F \in \mathcal{K}$, let

$$F^* = \{ \underset{\sim}{x} \mid \underset{\sim}{x} \in F;\ \underset{\sim}{x} \in L(\underset{\sim}{y}) \setminus \underset{\sim}{y} \Rightarrow \underset{\sim}{y} \notin F \}$$

F^* is just the set of maximal points of F with respect to the partial ordering $<$ determined by the equivalence

$$\underset{\sim}{x} < \underset{\sim}{y} \quad \Leftrightarrow \quad x_i \leq y_i \quad (i = 1,2,\cdots,n)$$

It is clear that, for each $F \in \mathcal{K}$

$$F = \cup \{ L(\underset{\sim}{x}) \mid \underset{\sim}{x} \in F^* \} .$$

Definition (4.2). For any $A \subset S$, the lower K-density of A is defined to be

$$d_k(A) = \text{glb} \left\{ \frac{A(F)}{S(F)} \;\middle|\; F \in \mathcal{K} \right\}$$

and the upper K-density of A is defined to be

$$\overline{d}_k(A) = \text{lub}\left\{ \frac{A(F)}{S(F)} \mid F \in \mathcal{K} \right\}.$$

In the case n = 1, $d_k(A)$ reduces to the ordinary Schnirelmann density of the set A.

Definition (4.3). For any $A \subset S$, the lower K-asymptotic density of A is defined to be

$$\delta_k(A) = \lim_{N \to \infty} d_k(A \cup J(N))$$

and the upper K-asymptotic density of A is defined to be

$$\overline{\delta}_k(A) = \lim_{N \to \infty} \overline{d}_k(A \setminus J(N)).$$

Theorem (4.4). For any $A \subset S$ we have $\delta_k(A) \le \delta_c(A)$ and $\overline{\delta}_c(A) \le \overline{\delta}_k(A)$.

Proof. If A is either empty or finite, then

$$\delta_k(A) = \delta_c(A) = 0.$$

If A is asymptotic to S , then

$$\delta_k(A) = \delta_c(A) = 1.$$

Therefore, in above cases, the theorem is trivial. Since F is a set of finite union of $L(\underset{\sim}{x})$, then for each $\underset{\sim}{x} \in S \setminus \underset{\sim}{0}$, $L(\underset{\sim}{x}) \subset F$ for some F in $\mathcal{K}$.

Hence, for any $N > 0$

$$\text{glb}\ \{ \frac{[A \cup J(N)](F)}{S(F)} \mid F \in \mathcal{K} \} \le \text{glb}\ \{ \frac{[A \cup J(N)](L(\underset{\sim}{x}))}{S(L(\underset{\sim}{x}))} \mid \underset{\sim}{x} \in S \setminus \underset{\sim}{0} \} .$$

That is,

$$d_k(A \cup J(N)) \le d_c(A \cup J(N)) .$$

Letting $N \to \infty$ we have

$$\delta_k(A) \le \delta_c(A) .$$

Similarly, we have

$$\overline{\delta}_c(A) \le \overline{\delta}_k(A) .$$

From theorem 4.4, we know that $\delta_k(A) \le \delta_c(A) \le \overline{\delta}_c(A) \le \overline{\delta}_k(A)$. Thus if the K-natural density of A exists, then so does the C-natural density and the two are equal.

Now, we are going to obtain a relationship between the C-asymptotic density and the K-asymptotic density. Before discussing this question, let us mention a result of B. Muller [8] : For any $A \subset S$ we have

$$(1) \qquad \frac{A(F) + 1}{S(F) + 1} \ge [1 - (1 - d_c(A))^{\frac{1}{n}}]^n,$$

where $F \in \mathcal{K}$ and n is the dimension of S (see [5, theorem 17]).

<u>Theorem (4.5).</u> For any $A \subset S = I^n$ we have

$$\delta_k(A) \geq [\ 1 - (1 - \delta_c(A))^{\frac{1}{n}}\ ]^n \quad .$$

<u>Proof</u>. First we show that for any integer $N > 0$

(2) $$d_k(A \cup J(N)) \geq \frac{(N+1)^n - 1}{(N+1)^n} [1 - (1 - d_c(A \cup J(N)))^{\frac{1}{n}}\ ]^n \quad .$$

We take an arbitrary $F \in \mathcal{K}$ and show that the right hand side of (2) does not exceed $[A \cup J(N)](F)/S(F)$. If the point $(N,N,\cdots,N) \notin F$, then $F \subset J(N)$ and

$$\frac{[A \cup J(N)](F)}{S(F)} = 1$$

and we are done. Hence, assuming that $(N,N,\cdots) \in F$, we have

$$[A \cup J(N)](F) \geq (N+1)^n - 1.$$

We have, applying (1)

$$\frac{[A \cup J(N)](F)}{S(F)} > \frac{[A \cup J(N)](F)}{[A \cup J(N)](F) + 1} \cdot \frac{[A \cup J(N)](F) + 1}{S(F) + 1}$$

$$\geq [\frac{(N+1)^n - 1}{(N+1)^n}][1 - (1 - d_c(A \cup J(N)))^{\frac{1}{n}}\ ]^n \ .$$

Hence,

$$d_k(A \cup J(N)) \geq [\frac{(N+1)^n - 1}{(N+1)^n}][1 - (1 - d_c(A \cup J(N)))^{\frac{1}{n}}\ ]^n \ .$$

Letting $N \to \infty$ we obtain

$$\delta_k(A) \geq [\, 1 - (1 - \delta_c(A))^{\frac{1}{n}}\,]^n \quad .$$

Corollary (4.6). For any $A \subset S$, $\delta_c(A) = 1$ if and only if $\delta_k(A) = 1$.

Proof. Suppose that $\delta_k(A) = 1$. Then $1 \geq \delta_c(A) \geq \delta_k(A) = 1$ which implies $\delta_c(A) = 1$. If $\delta_c(A) = 1$, then theorem 4.5 shows that

$$1 \geq \delta_k(A) \geq [1 - (1 - \delta_c(A))^{\frac{1}{n}}\,]^n = [1 - (1 - 1)^{\frac{1}{n}}\,]^n = 1 \ .$$

Therefore,

$$\delta_k(A) = 1.$$

Theorem (4.7). For any $A \subset S = I^n$ we have

$$\overline{\delta}_k(A) \leq 1 - [1 - (\overline{\delta}_c(A))^{\frac{1}{n}}\,]^n \quad .$$

Proof. First we show that for any integer $N > 0$

$$\text{(3)} \qquad \overline{d}_k(A \setminus J(N)) \leq 1 - \left(\frac{(N+1)^n - 1}{(N+1)^n}\right)(1 - (\overline{d}_c(A \setminus J(N)))^{\frac{1}{n}})^n \quad .$$

We take an arbitrary $F \in \mathcal{K}$ and show that $[A \setminus J(N)](F)/S(F)$ does not exceed the right side of (3). If the point $(N,N,\cdots) \notin F$, then $[A \setminus J(N)](F) = 0$ and we are done. Hence, assuming that $(N,\cdots,N) \in F$, we have $[\overline{A} \cup J(N)](F) \geq (N+1)^n - 1$. We have, applying (1),

$$\frac{[\overline{A} \cup J(N)](F)}{S(F)} > \frac{[\overline{A} \cup J(N)](F)}{[\overline{A} \cup J(N)](F) + 1} \cdot \frac{[\overline{A} \cup J(N)](F) + 1}{S(F) + 1}$$

$$\geq \left(\frac{(N+1)^n - 1}{(N+1)^n}\right)[1 - (1 - d_c(\overline{A}\cup J(N)))^{\frac{1}{n}}]^n ,$$

so that

$$\frac{[A\setminus J(N)](F)}{S(F)} = 1 - \frac{[\overline{A}\cup J(N)](F)}{S(F)}$$

$$\leq 1 - \left(\frac{(N+1)^n - 1}{(N+1)^n}\right)[1 - (1 - d_c(\overline{A}\cup J(N)))^{\frac{1}{n}}]^n .$$

We obtain (3) if we can show that $d_c(\overline{A}\cup J(N)) \geq 1 - \overline{d}_c(A\setminus J(N))$. But this is easy since, for each $\underset{\sim}{x}\in S\setminus \underset{\sim}{0}$, we have

$$\frac{[\overline{A}\cup J(N)](L(\underset{\sim}{x}))}{S(L(\underset{\sim}{x}))} = 1 - \frac{[A\setminus J(N)](L(\underset{\sim}{x}))}{S(L(\underset{\sim}{x}))}$$

$$\geq 1 - \overline{d}_c(A\setminus J(N)) .$$

Hence

$$\overline{d}_k(A\setminus J(N)) \leq 1 - \left(\frac{(N+1)^n - 1}{(N+1)^n}\right)[1 - (\overline{d}_c(A\setminus J(N)))^{\frac{1}{n}}]^n .$$

Letting $N\to\infty$ we obtain

$$\overline{\delta}_k(A) \leq 1 - [1 - (\overline{\delta}_c(A))^{\frac{1}{n}}]^n .$$

<u>Corollary (4.8)</u>. For any $A\subset S$, $\overline{\delta}_c(A) = 0$ if and only if $\overline{\delta}_k(A) = 0$.

<u>Proof</u>. Suppose that $\overline{\delta}_k(A) = 0$, then $0 = \overline{\delta}_k(A) \geq \overline{\delta}_c(A) \geq 0$ which implies $\overline{\delta}_c(A) = 0$. If $\overline{\delta}_c(A) = 0$, then theorem 4.7 shows that

$$0 \le \overline{\delta}_k(A) \le 1 - (1 - (0)^{\frac{1}{n}})^n = 0 \quad .$$

Therefore

$$\overline{\delta}_k(A) = 0.$$

We conclude with an example which shows that, the C-asymptotic density and the K-asymptotic density are in fact different. In this example, some notations and properties for K-asymptotic density can be found in the Freedman's paper [4].

Example. Let S be of dimension two. For integers $a,b \ge 10$ let

$$D(a,b) = [U((a+b,\ a))\cap L((a+2b,a+b))]\cup$$

$$\cup[U((a,\ a+b))\cap L((a+b,\ a+2b))].$$

Take $a_1 = 10$, $b_1 = 10!$, $a_{i+1} = a_i + 2b_i$, $b_{i+1} = (a_{i+1})!$ and define

$$A = S \setminus \bigcup_{i=1}^{\infty} D(a_i,b_i).$$

We prove that $\delta_k(A) \le \frac{1}{3}$ and $\delta_c(A) \ge \frac{1}{2}$ so that $\delta_k(A) \ne \delta_c(A)$.

(i) Let $F_i = L((a_i+b_i,\ a_i+2b_i))\cup L((a_i+2b_i,\ a_i+b_i))$, note: $F_i \in \mathcal{K}$ for each i and $(F_i) \in \mathcal{S}$ since $F_i \subset \mathcal{K}(b_i) \subset \mathcal{K}(i)$ for each i. Thus

$$\delta_k(A) \le \underline{\lim}_{i\to\infty} \frac{A(F_i)}{S(F_i)} \quad .$$

Now

Now

$$\frac{A(F_i)}{S(F_i)} \le \frac{(a_i+b_i+1)^2 + 2a_ib_i}{(a_i+b_i+1)(a_i+2b_i+1) - 1 + (a_i+b_i+1)b_i}$$

$$= \frac{b_i^2 + a_i^2 + 4a_ib_i + 2b_i + 2a_i + 1}{3b_i^2 + 4a_ib_i + 4b_i + 2a_i + a_i^2}$$

$$= \frac{1 + \frac{a_i^2}{b_i^2} + 4\frac{a_i}{b_i} + 2\frac{1}{b_i} + 2\frac{a_i}{b_i^2} + \frac{1}{b_i^2}}{3 + 4\frac{a_i}{b_i} + 4\frac{1}{b_i} + 2\frac{a_i}{b_i} + \frac{a_i^2}{b_i^2}}$$

$$\to \frac{1}{3} \qquad (i\to\infty)$$

since the last 5 terms of the numerator and the last 4 terms of denominater all approach 0 as $i\to\infty$. Hence we have $\delta_k(A) \le \frac{1}{3}$.

(ii) We show $\delta_c(A) \ge \frac{1}{2}$ by proving that, in fact $d_c(A) \ge \frac{1}{2}$, i.e., for each $(x,y) \in S$ we have

$$\frac{A(L((x,y)))}{S(L((x,y)))} \ge \frac{1}{2} \quad .$$

First we show

$$\frac{A(L((a_{i+1}, a_{i+1})))}{S(L((a_{i+1}, a_{i+1})))} \ge \frac{1}{2} \qquad \text{for } i = 0,1,2,\cdots,$$

note: if i = 0 then

$$\frac{A(L((a_1,\ a_1)))}{S(L((a_1,\ a_1)))} = 1 > \frac{1}{2} \quad .$$

For $i \geq 1$ we have

$$\frac{A(L((a_{i+1},\ a_{i+1})))}{S(L((a_{i+1},\ a_{i+1})))} \geq \frac{2b_i^2 + 2a_i(2b_i)}{(a_i+2b_i+1)^2 - 1}$$

$$= \frac{2b_i^2 + 4a_ib_i}{4b_i^2 + 4a_ib_i + 4b_i + 2a_i + a_i^2} \quad .$$

Now

(*) $\quad 4a_ib_i > 4b_i + 4a_i + a_i^2$

since $\quad 4a_i - 4 > a_i + 4 \quad$ (since $a_i > 2$) and $b_i > a_i$ so

$(4a_i - 4)b_i > (a_i + 4)a_i$ which implies (*). Thus

$$\frac{4a_ib_i}{4a_ib_i + 4b_i + 2a_i + a_i^2} > \frac{1}{2} \quad ,$$

$$\frac{2b_i^2}{4b_i^2} = \frac{1}{2} \quad ,$$

so we conclude

$$\frac{A(L((a_{i+1},\ a_{i+1})))}{S(L((a_{i+1},\ a_{i+1})))} > \frac{1}{2} \quad .$$

We now consider an arbitrary $(x,y) \in S$. If either $x<a_1$ or $y<a_1$ we have

$$\frac{A(L((x,y)))}{S(L((x,y)))} = 1 > \frac{1}{2} .$$

Letting i be the largest index such that $a_i \le x$ and $a_i \le y$, we may assume that $i \ge 1$. If $y \ge a_{i+1}$, then $x<a_{i+1}$ and we see that

$$\frac{A(L((x,y)))}{S(L((x,y)))} \ge \frac{A(L((x,\ a_{i+1})))}{S(L((x,\ a_{i+1})))} ,$$

and similarly if $x \ge a_{i+1}$. Hence we can in fact assume that $a_i \le x,\ y \le a_{i+1}$.

We distinguish 3 cases: (see figure)

Case I. $X_1 = (x,y)$ and $x<a_i+b_i$, $y<a_i+b_i$. In this case it is clear that

$$\frac{A(L(X_1))}{S(L(X_1))} \ge \frac{A(L((a_i,\ a_i)))}{S(L((a_i,\ a_i)))} \ge \frac{1}{2} .$$

Case II. $X_2 = (x,y)$ and $x \ge a_i+b_i$, $y<a_i+b_i$. Here

$$\frac{A(L(X_2))}{S(L(X_2))} \ge \frac{(a_i+b_i)(y-a_i) + a_i(x-a_i) + A(L((a_i,\ a_i)))}{(x+1)(y-a_i) + (a_i+1)(x-a_i) + S(L((a_i,\ a_i)))} .$$

We need only to show that

$$\frac{(a_i+b_i)(y-a_i) + a_i(x-a_i)}{(x+1)(y-a_i) + (a_i+1)(x-a_i)} \ge \frac{1}{2} .$$

This is clear since $2(a_i+b_i)>x+1$ and $2a_i\geq a_i+1$. Note! By symmetry we also have the case $x<a_i+b_i$, $y\geq a_i+b_i$.

Case III. $X_3 = (x,y)$ and $x\geq a_i+b_i$, $y\geq a_i+b_i$. Let $x = a_i+b_i + w$ and $y = a_i+b_i + z$. We may assume $z\geq w$. Now we have

$$\frac{A(L(X_3))}{S(L(X_3))} = \frac{wz + b_i^2 - 1 + a_i(b_i+w) + a_i(b_i+z) + A(L((a_i,\ a_i)))}{(b_i+w)(a_i+b_i+z+1) + (b_i+z)(a_i+1) + S(L((a_i,\ a_i)))} .$$

Since $\dfrac{A(L((a_i,\ a_i)))}{S(L((a_i,\ a_i)))} \geq \dfrac{1}{2}$ we need only to show that

(**) $$2[wz + b_i^2 - 1 + a_i(b_i+w) + a_i(b_i+z)] \geq$$

$$(b_i+w)(a_i+b_i+z+1) + (b_i+z)(a_i+1).$$

Now, we clearly have

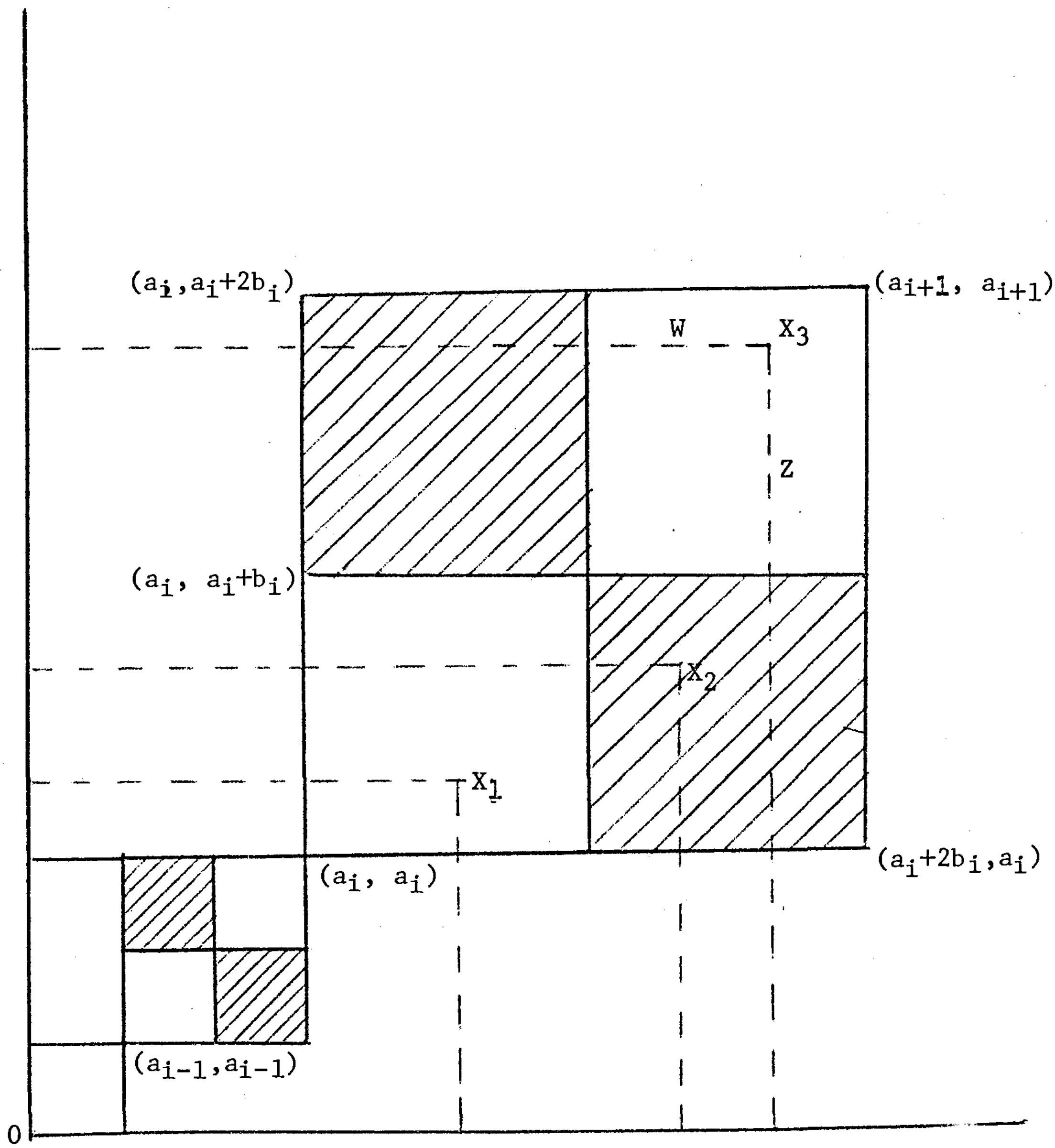
$$2[a_i(b_i+z)] \geq a_i(b_i+w) + a_i(b_i+z),$$

$$2(b_i^2+wz) \geq (b_i+w)(b_i+z) ,$$

$$2a_i(b_i+w) - 2 \geq 2b_i + w + z .$$

Adding these inequalities we get (**) above. So we conclude $d_c(A)\geq\frac{1}{2}$.

Therefore, $\delta_c(A)\geq d_c(A) \geq \frac{1}{2}$. This completes the proof.



(not drawn to scale)

▨ = points missing from A

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