OPTIMIZATION OF GEOGRAPHIC MAP PROJECTIONS
FOR
CANADIAN TERRITORY

by

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A Thesis Submitted in Partial Fulfillment

of the Requirements for the Degree of

Doctor of Philosophy

by

Special Arrangements

Simon Fraser University

November, 1982

APPROVAL

Name: Kresho Frankich

Degree: Doctor of Philosophy

Title of Thesis: Optimization of Geographic Map Projection for Canadian Territory

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Date Approved: March 3, 1983

ABSTRACT

One of the main tasks of mathematical cartography is to determine a projection of a mapped territory in such a way that the resulting deformations of angles, areas and distances are objectively minimized. Since the transformation process will generally change the original distances it is appropriate to adopt the deformation of distances as the basic parameter for the evaluation of map projections. As the qualitative measure of map projections the author decided to use the Airy-Kavraiskii criterion

$$E_{AK}^2 = \frac{1}{2A} \int_A \left(\ln^2 a + \ln^2 b\right) dA ,$$

where A is the area of the mapping domain, a and b semi-axes of the indicatrix of Tissot, and the integration is extended to the whole domain. Until now all optimization of map projections were referred to domains with analytically defined boundaries, for example, a spherical trapezoid, spherical cap or a hemisphere, and for those map projections in which the analytical evaluation of the integral was possible. The author expands the optimization process to irregular domains with boundaries consisting of a series of discrete points. The minimization of the criterion leads to a least square adjustment problem.

The main purpose of the project was to develop a uniform method to optimize the standard and most frequently used mapping systems in geography for Canadian territory. The scope of optimization was enlarged by the inclusion of the optimization of modified equiareal projections as well as the determination of the Chebyshev conformal projections.

Almost all small scale maps of the territory of Canada have been based on the normal aspect of the Lambert Conformal Conic projection with standard parallels at latitudes of 49° and 77°. Every optimized projection in the research yielded a smaller value for the Airy-Kavraiskii criterion. Thus, it was proven that any standard map projection with properly selected metagraticule and constant parameters of the projection is much better than the official projection. The best result was achieved with the optimized equidistant projection. Since the projection equations for the equidistant conic projection are very simple and the projection gives the best result with respect to the Airy-Kavraiskii criterion, the author highly recommends its application for small scale maps of Canada.

Dedicated to JASNA,

who encouraged the author
more than anybody but, at
the same time, suffered from
his research more than anybody.

ACKNOWLEDGEMENT

There are many people who to a smaller or larger extent influenced the author and thus contributed to this research. The author received his introduction to mathematical cartography from the late professor, Dr. Branko Borcic, at the Geodetic University of Zagreb. Dr. Sybren de Jong at the University of British Columbia revitalized the interest for the subject. The greatest influence on the author's knowledge however was made by a Russian geodesist, V.V. Kavraiskii, whom the author has unfortunately never met, but whose book, unsurpassed in mathematical cartography, has influenced the author more than he is probably aware of.

The author is grateful to his supervisors, Dr. Thomas Poiker and Dr. Robert Russell; the former for his constant encouragement in the research and the latter for his guidance through the complicated world of numerical analysis and for his careful proofreading of the thesis. The importance of Dr. Waldo Tobler for the present form of the thesis must also be acknowledged. He made many good suggestions and supplied the author with a list of helpful articles in the field of mathematical cartography.

The colleagues at the British Columbia Institute of Technology, Dave Martens and Ken Gysler helped the author; the former in computer programming and the latter in proofreading the text.

The illustrations for various map projections were made by Wayne Luscombe at the World Bank in Washington, D.C.

Mrs. Bernice Tong typed the manuscript overloaded with mathematical formulation in a remarkable manner. The author is grateful to her.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

0. INTRODUCTION

1. GRIMMS' INTRODUCTION

Song For Five Dollars

Five learned scholars
were each paid a dollar
to see if they could find out something new
but they met with some resistance
for according to the distance
they noticed things got smaller or they grew.

A passion in them burned
they left no worm unturned
they flattened all the bumps to fill in holes
and from Leicester to East Anglia
made continents rectangular
and evenly distributed the poles.

2. INTRODUCTION TO CARTOGRAPHIC PROBLEMS

The mathematical aspect of cartographic mapping is a process which establishes a unique connection between points of the earth's sphere and their images on a plane. It was proven in differential geometry (Eisenhart, 1960), (Goetz, 1970), (Taschner, 1977) that an isometric mapping of the sphere onto a plane with all corresponding distances on both surfaces remaining identical can never be achieved since the two surfaces do not possess the same Gaussian curvature. In other words, it is impossible to derive transformation formulae which will not alter distances in the mapping process. Cartographic transformations will always cause a certain deformation of the original surface. These deformations are reflected in changes of distances, angles and areas.

The main task of mathematical cartography is to determine projection formulae to transform a mapped territory onto a plane with a minimum deformation of the original sphere. It is possible to derive transformation equations which have no deformations in either angles or areas (Richardus and Adler, 1972), (Frankich, 1977). These projections are called conformal and equiareal, respectively. The transformation processes, however, always change distances and therefore the deformation of distances must be used as the basic parameter for the evaluation of map projections.

Criteria for the selection of an appropriate cartographic system for small scale geographic mappings are versatile (Frančula, 1974). It has been usually stated that the choice of map projection depends on the position, geometrical shape of the mapping domain, and the purpose of the map. An applied cartographic representation must be a reliable image of the mapping territory. In other words, the overall deformation of distances must be as small as possible. The distribution of distortions should be the essential governing factor for the selection of a map projection.

It remains to define a measure of deformation of distances related to a point of a mapping domain and a measure of deformation for the whole demain. The ratio of a differentially small distance on the mapping plane and its counterpart on the sphere is generally used to express the change of distances (Biernacki, 1965), (Fiala, 1957). This ratio is called the scale factor. The ideal value of the scale factor is unity. In that case there is no deformation of a distance. Chebyshev suggested (Kavraiskii, 1959) use of the natural logarithm of the scale factor as the measure of deformation. The author in this research has adopted Chebyshev's definition of deformation. Before the definition of the measure of quality can be expanded to a mapping domain it must be realized that the scale factor and also its logarithm at a point in an arbitrary non-conformal projection varies as a function of the direction (Biernacki, 1965), (Richardus and Adler, 1972). Cartographers usually consider the extreme scale factors at the point only. The extreme value of the scale occur in two orthogonal directions, called the principal directions (Kavraiskii, 1959). Airy (1861) recommended the analytical integration of the sum of squares of the principal scales for the whole territory as the measure of quality of a mapping system. He assumed that the boundary of the territory is analytically defined and that the practical integration process is possible. It is appropriate to mention that there are very few map projections and even fewer analytically defined boundaries (spherical cap, spherical trapezoid, hemisphere or the whole sphere) where the analytical integration is achievable.

3. OBJECTIVES OF RESEARCH

The two main objectives of the research were:
1) theoretical formulations of cartographic mapping, ideas and methods of optimization, and an explicit development of a new optimization process suggested by the author; 2) the practical application of the derived method for the optimization of various cartographic systems for Canadian territory. The first part is strictly theoretical and the second is practical.

The theoretical part discusses general mapping theory. It was developed in the eighteenth and nineteenth century with a small contribution introduced at the beginning of this century. There are several books in foreign languages in mathematical cartography which cover these theoretical foundations (Biernacki, 1973), (Driencourt et Laborde, 1932), (Fiala, 1957), (Hoschek, 1969), (Kavraiskii, 1959), (Meshcheryakov, 1968), (Wagner, 1962). The only English language publications

(Maling, 1973), (Richardus and Adler, 1972), are elementary books and they lack a serious mathematical treatment of map projections. The first chapter therefore presents an abbreviated overview of mathematical cartography including all necessary mathematical expressions, given without proofs and derivations. There are, however, two explicit derivations of the fundamental differential equations of mappings.

The Russian cartographic school developed in the last few decades a series of interesting approaches to map projections. In particular, Urmaev (1953) and Meshcheryakov (1968) introduced the concept of a system of two partial differential equations which can be called the fundamental system. The system involves two partial differential equations with four characteristics of distortions, thus the system is undetermined. If two of the characteristics are predefined, or if two conditions with the characteristics are superimposed on the mapping equations, the fundamental differential equations can be at least theoretically integrated, leading to the final expressions of cartographic mappings. The development is, from a mathematical point of view, of great interest since it opens an avenue for derivation of many new map projections in which the starting criterion is a distribution of characteristics of distortions over the mapping domain. This development is known to very few North Americans and therefore the author gave the detailed derivations of both Meshcheryakov's and Urmaev's system of fundamental differential equations.

Classification of mapping is given with respect to the characteristics of deformations (Kavraiskii, 1959). It treats conformal, equiareal and equidistant projections only. All other classification schemes are neglected. The classes of conformal and equiareal projections are defined by their corresponding differential equations. The possibility of integration of the differential equation has been proven by the author. The author developed from the non-linear partial differential equation two map projections. One of them is a well-known Lambert's equiareal cylindric projection and the other is a new equiareal projection.

The second chapter starts with concepts of ideal and best map projections (Meshcheryakov, 1968) and the introduction of qualitative measures for mapping systems. The author synthesized the ideas of Airy (1861), Kavraiskii (1959), Meshcheryakov (1968), Frančula (1971), and Young (1920) and decided to use for the optimization criterion the Airy measure of quality modified by Kavraiskii. The measure is the integral of the sum of squares of logarithms of the principal scales and the integration was extended to the whole mapping domain. The Airy-Kavraiskii measure formally resembles variance in statistics and its optimization, as the author proved, leads to the least squares adjustment problem. One could use some other measures of quality, for example, the sum of principal scales, the sum of absolute values of logarithms of principal scales, and others. Their optimizations, however, lead to the solution of a system of non-linear equations while the least squares adjustments are reduced to the solution of linear equations. This was the main reason that the Airy-Kavraiskii measure was selected as the basis of optimization.

Very few mathematical cartographers have seriously tackled the optimization problem. Airy (1861) was the first to use his criterion to develop a map projection of a hemisphere. He applied calculus of variations to minimize the norm in the Hilbert space. Young (1920) expanded the approach including some other projection systems and extending the integration to the whole sphere. In the two cases of Airy and Young the selected functions were integrable for domains inclosed by analytically defined curves. Kavraiskii (1959) explained the optimization of conic projections for domains bounded by two parallels. His method was reduced to the determination of the best standard parallels satisfying the Airy-Kavraiskii criterion. Frančula (1971) determined the best modified projections of the whole world by applying the same criterion, but instead of an analytical minimization of the norm he used a numerical minimization process.

The final generalization of the optimization for the Airy-Kavraiskii measure of quality was introduced by the author. A mapping domain was approximated by a series of discrete points whose density is a matter of personal choice and it is usually governed by the size of the computer. The higher the density of points the more reliable are the results of optimization.

Every mapping domain can be approximated by discrete points regardless of its shape and size. The boundary does not need to be analytically defined. In other words, every mapping of an arbitrary domain can be realistically optimized. The author further improves the determination of the best map projection by including the most optimal metagraticule, as named by Wray (1974). The metagraticule represents a coordinate system similar to the geographic graticule except that it is generally displaced on the sphere with respect to the graticule. The metagraticule represents an invariant frame for a projection system. The optimization process, developed by the author, leads to the optimal metagraticule in addition to the best determination of constants for individual map projections. The optimized mathematical model is non-linear with respect to the unknown parameters (metagraticule and constants of projections) and it must be made linear by the Newton method. The author has linearized the optimization mathematical models for almost all important conic, cylindric and azimuthal map projections.

The spectrum of optimized projections was enlarged by the inclusion of modified equiareal map projections. The major study of modified map projections has been made by Wagner (1962). He empirically selected modification constants. The author, however, developed a method where the constants were determined through least squares optimization, minimizing the Airy-Kavraiskii measure of distortion. Such determined modified equiareal projections can be used for small-scale mappings of arbitrary domains when the property of equivalency is essential to users.

Conformal map projections have generally little importance for small-scale geographic transformation. They are, however, fundamental for large-scale topographic maps. The author believes no optimization of mappings is complete without Chebyshev's projections which are the best conformal projections (Meshcheryakov, 1969). They are defined as projections in which the changes of scale are minimized. Chebyshev's theorem (Biernacki, 1965) states that the necessary and sufficient condition of the best conformal projection is to have a constant scale factor along the boundary contour of the domain. The Russian cartographic school (Urmaev, 1953) has solved the problem of obtaining the best conformal projections for symmetric boundaries. The author showed in the first part of the third chapter the suggested solutions of Urmaev using several methods (method of Ritz, method of finite differences, and method of least squares). The completion of the determination of Chebyshev's projections for most general non-symmetric domains was developed by the author. He used a complex polynomial as a mapping analytic function and computed the coefficients of the polynomial by the method of least squares. The resultant line of constant scale approximated closely the boundary contour.

4. PRACTICAL OPTIMIZATIONS FOR CANADIAN MAPS

The second part of the research was the practical application of the theoretical optimization developed in the first part of the thesis. All optimization routines were applied for various maps of Canada. Although the theoretical portion of the research contains several optimization approaches attributed to Urmaev (1953), these approaches are not implemented because of their restricted value for the Canadian territory, which can hardly be approximated by a spherical trapezoid or a symmetric domain.

The author devoted a major part of the last chapter to the optimization of conic, cylindric, azimuthal and modified equiareal map projections. The optimization criterion was minimization of the Airy-Kavraiskii measure of distortions. In other words, the author calculated the logarithms of the principal scales at a finite number of discrete points which approximate the Canadian territory. A numerical integration of the squares of these logarithms extended to the whole domain was minimized by the method of least squares. The author developed mathematical models for main cartographic mappings. All formulae were given in an explicit form suitable for further optimization of any other territory.

When the author decided to test the quality of present small scale mappings of Canada and subsequently investigate possible improvements using an objective criterion of deformations it was expected that the optimization would amount to a small refinement to the present system resulting in a marginally better graphical representation of Canada. The numerical results of the research surpassed these hopes to an unexpected amount. Every optimized map projection leads to a much better mapping system than the presently used Lambert Conformal Conic projection with standard parallels of 49° and 77°. In other words all projections optimized by the author give better representation of Canada with less distortions than the system in use. Particularly good results are obtained with an oblique aspect of an equidistant conic projection whose transformation formulae are very simple and therefore suitable for providing the basis of a new small-scale map of Canada. The overall deformations with this projection are 70 percent smaller compared to the official projection.

The author believes that from theoretical point of view this research for all practical purposes completes the optimization of small-scale mapping. However, the determination of better geodetic mapping (projections of the ellipsoid of rotation onto a plane) remains to be tackled. This research also indicated that more studies could be done in the integration of the fundamental differential equations of geographic mappings.

I. GENERAL THEORY OF CARTOGRAPHIC MAPPINGS

1. INTRODUCTION TO THEORY OF SURFACES

Let us consider a surface S and on it a closed domain D. The surface is defined by a set of curvilinear parametric coordinates u^i, where i = 1,2. The position vector of the surface is

$$\mathfrak{r} = \mathfrak{r}(u^1, u^2). \tag{I-1-1}$$

The surface is called regular if for every point, which belongs to the domain D the following condition is satisfied

$$\mathfrak{r}_1 \times \mathfrak{r}_2 \neq 0, \tag{I-1-2}$$

where

$$\mathfrak{r}_1 = \frac{\partial \mathfrak{r}}{\partial u^1} \text{ and } \mathfrak{r}_2 = \frac{\partial \mathfrak{r}}{\partial u^2}. \tag{I-1-3}$$

A point in which the vector product of the two tangent vectors $\mathfrak{r}_1$ and $\mathfrak{r}_2$ is equal to zero is called a singular point of the parametrization (u^1, u^2) and it is excluded from the domain D (Goetz, 1970).

Every regular surface S can have an infinite number of different parametrization systems (u^i). When one of the parametric coordinates u^i is kept constant and the other varies we obtain a family of curves dependent on one parameter only, called the coordinate lines.

Let us take a curve on the surface defined by the equations

$$u^1 = u^1(t) \text{ and } u^2 = u^2(t), \qquad \text{(I-1-4)}$$

where t is an arbitrary parameter. The square of a differentially small length of the curve is known as the first fundamental or the metric form of the surface and is obtained by the equation

$$ds^2 = g_{ij}\, du^i du^j, \qquad \text{(I-1-5)}$$

where ds is the differentially small length and g_{ij} is the first fundamental or metric tensor. For real surfaces and parametrizations $g_{ij} = g_{ji}$. The formula (I-1-5) uses the standard summation convention, i.e. the summation is always applied when the same indices appear twice in the same monomial, once as a superscript and once as a subscript.

Thus, the equation (I-1-5) explicitly written becomes

$$ds^2 = g_{11}(du^1)^2 + 2g_{12}\, du^1 du^2 + g_{22}(du^2)^2. \qquad \text{(I-1-6)}$$

For a given surface and a selected parametric coordinate system we obtain a metric tensor

$$g_{ij} = f(S, u^i), \qquad \text{(I-1-7)}$$

with components

$$g_{ij} = \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix}. \qquad \text{(I-1-8)}$$

The individual components of the metric tensor are obtained by the scalar products of the corresponding tangent vectors $\mathfrak{r}_1$ and $\mathfrak{r}_2$.

$$g_{11} = \mathfrak{r}_1 \cdot \mathfrak{r}_1, \quad g_{12} = \mathfrak{r}_1 \cdot \mathfrak{r}_2, \text{ and } \quad g_{22} = \mathfrak{r}_2 \cdot \mathfrak{r}_2. \qquad \text{(I-1-9)}$$

The discriminant of the first fundamental form, denoted by g, is

$$g = \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} = g_{11}\, g_{22} - g^2{}_{12}. \qquad \text{(I-1-10)}$$

A surface is regular for a chosen parametrization (u^i) when

$$g_{11}>0,\ g_{22}>0,\ \text{and } g>0. \qquad \text{(I-1-11)}$$

The tangent vector $\mathfrak{a}$ to the curve (I-1-4) can be defined as a linear combination of the tangent vectors to parametric curves

$$\mathfrak{a} = a^i\, \mathfrak{r}_i ,$$

where a^i are the components of $\mathfrak{a}$ with respect to the coordinate system (u^i). If we take another curve which intersects the first and denote its tangent vector at the common point of the two curves by $\mathfrak{b}$, then the angle ω between the two curves is obtained from the scalar product of the tangent vectors

$$\mathfrak{a}\,\mathfrak{b} = g_{ij}\, a^i b^j ,$$

which yields

$$\cos \omega = \frac{g_{ij}a^i b^j}{\sqrt{g_{ij}a^i a^j}\ \sqrt{g_{ij}b^i b^j}} \tag{I-1-12}$$

The angle between the coordinate lines is a special case of the last formula

$$\cos \theta = g_{12}/\sqrt{g_{11}\ g_{22}}\ . \tag{I-1-13}$$

The orthogonality of the parametric coordinate lines is achieved when the last formula takes on a value of zero at every point of the surface. Thus, the condition of orthogonality of parametric curves is

$$g_{12} = 0, \tag{I-1-14}$$

and the corresponding metric form becomes

$$ds^2 = g_{11}(du^1)^2 + g_{22}(du^2)^2. \tag{I-1-15}$$

The area of a differentially small element of the surface S is

$$dp = \sqrt{g}\ \ du^1 du^2, \tag{I-1-16}$$

and its integration for the whole domain

$$A_D = \iint \sqrt{g}\ du^1 du^2. \qquad \text{(I-1-17)}$$

The elements of surfaces (a differentially small distance, ds, an angle between two curves of the surface, ω , and the area of the domain, A_D) are known as the intrinsic elements of the surface since they are invariant no matter which parametrization is selected. In other words, the coordinates can be changed but the values of intrinsic elements remain unaltered.

To simplify the developments of formulae in certain types of mappings, we can make a specific parametrization of the surface which leads to a particular type of the first fundamental form.

For example, if $g_{11} = 1$ and $g_{12} = 0$ we obtain the, so-called, semigeodesic coordinates with the first fundamental form

$$ds^2 = (du^1)^2 + G(u^1,u^2)(du^2)^2. \qquad \text{(I-1-18)}$$

When $g_{11} = g_{22}$ and $g_{12} = 0$ the resulting coordinates are called isometric in geodesy and isothermic in mathematics. The metric form in isothermic coordinates becomes

$$ds^2 = [\lambda(u^1,u^2)]^2[(du^1)^2 + (du^2)^2]. \qquad (I\text{-}1\text{-}19)$$

The isothermic coordinates are extremely important in cartography since the determination of these coordinates on surfaces leads directly to conformal mappings probably the most important type of representation from a practical, as well as a theoretical, point of view.

In addition to the already mentioned intrinsic elements of a surface: distance, angle, and area, there is another intrinsic element of the utmost importance and that is the Gaussian curvature. It is obtained by the formula (Goetz, 1970)

$$K = b/g, \qquad (I\text{-}1\text{-}20)$$

where b is the discriminant of the second fundamental form whose components are computed by the vector equation

$$b_{ij} = \frac{\mathfrak{r}_{ij}\,\mathfrak{r}_1\,\mathfrak{r}_2}{\sqrt{g}}, \qquad (I\text{-}1\text{-}21)$$

and then

$$b = b_{11}b_{22} - b_{12}^{\,2}. \qquad (I\text{-}1\text{-}22)$$

Mathematical cartography generally deals with three different surfaces: plane, sphere and spheroid.

The plane is a surface whose Gaussian curvature is equal to zero. If we adopt an orthogonal Cartesian coordinate system (x^i) the first fundamental form on the plane becomes

$$ds^2 = (dx^1)^2 + (dx^2)^2. \tag{I-1-23}$$

The sphere and the spheroid, on the other hand, belong to surface of rotation, which are defined by the rotation of a planar curve about an axis. The axis of rotation lies in the plane of the curve. Various positions of the rotating curve are called meridians of the surface.

A sphere is a special case of a surface of rotation generated by the rotation of a semicircle of radius R, whose vector is

$$\mathfrak{r} = R \cos u^1 \cos u^2 \,\mathfrak{i} + R \cos u^1 \sin u^2 \,\mathfrak{j} + R \sin u^2 \,\mathfrak{k}, \tag{I-1-24}$$

with $\mathfrak{i}$, $\mathfrak{j}$, $\mathfrak{k}$ being mutually orthogonal vectors.

The first fundamental form on the sphere is

$$ds^2 = R^2[(du^1)^2 + \cos^2 u^1 (du^2)^2], \tag{I-1-25}$$

and the Gaussian curvature becomes

$$K = 1/R^2, \tag{I-1-26}$$

since

$$\left.\begin{aligned} g_{11} &= R^2, & g_{12} &= 0, & g_{22} &= R^2 \cos^2 u^1, \\ b_{11} &= R, & b_{12} &= 0, & b_{22} &= R \cos^2 u^1. \end{aligned}\right\} \tag{I-1-27}$$

A spheroid is obtained by the rotation of a meridian ellipse about its semi-minor axis. Its surface is used to approximate the actual surface of the earth for geodetic positioning and, subsequently, for geodetic mappings, but these are outside the scope of this work. Thus it suffices to give the fundamental formula of a spheroid

$$\mathfrak{r} = N \cos u^1 \cos u^2 \, \mathfrak{i} + N \cos u^1 \sin u^2 \, \mathfrak{j} + N(1-e^2) \sin u^1 \, \mathfrak{k}, \tag{I-1-28}$$

where N is the maximal radius of curvature at a point (u^1, u^2) obtained by the expression

$$N = \frac{a}{(1-e^2 \sin^2 u^1)^{1/2}}, \tag{I-1-29}$$

a is the semi-major axis of the meridian ellipse and e^2 is the square of the eccentricity

$$e^2 = (a^2 - b^2)/a^2, \tag{I-1-30}$$

and b is the semi-minor axis of the meridian ellipse.

The components of the fundamental metric tensor are

$$g_{11} = M^2, \quad g_{12} = 0, \quad g_{22} = N^2 \cos^2 u^1, \qquad \text{(I-1-31)}$$

with M being the radius of curvature of the meridian ellipse

$$M = a(1-e^2)/(1-e^2 \sin^2 u^1)^{3/2}. \qquad \text{(I-1-32)}$$

Thus, the first fundamental form is

$$ds^2 = M^2(du^1)^2 + N^2 \cos^2 u^1 (du^2)^2, \qquad \text{(I-1-33)}$$

and the Gaussian curvature is

$$K = 1/MN. \qquad \text{(I-1-34)}$$

2. CARTOGRAPHIC MAPPINGS

Let us consider two regular surfaces, S and P, and on the first surface S, a closed domain D. Both surfaces are defined by a corresponding set of curvilinear parametric coordinates: (u^i) on S, and (x^i) on P, for $i = 1,2$.

Then the relationship

$$x^1 = x^1(u^1,u^2) \text{ and } x^2 = x^2(u^1,u^2), \qquad \text{(I-2-1)}$$

defined in the domain, $D \epsilon S$, establishes a connection between points of the first surface $A(u^1,u^2) \epsilon D$ and points of the second surface $B(x^1,x^2)$, which belong to some domain Δ. In other words, the domain D of the first surface is transformed into the domain, Δ of the second surface, or the domain D is projected onto the second surface in the domain Δ. To make projections or mappings meaningful in practice, the class of transformation functions (I-2-1) is restricted to those functions which are unique, twice differentiable and continuous up to the second derivative, finite, and where in all points the Jacobian determinant must be different from zero, i.e.

$$\partial(x^1,x^2)/\partial(u^1,u^2) \neq 0. \qquad \text{(I-2-2)}$$

Such an established one-to-one correspondence between the points of the two domains, which is continuous in both directions, is called a homeomorphism.

The first surface is commonly called the original surface and the second surface is then the projection surface.
Under homeomorphism, however, the transformation direction is reversible and the mapping can be also performed from the second onto the first surface

$$u^1 = u^1(x^1,x^2) \text{ and } u^2 = u^2(x^1,x^2). \qquad \text{(I-2-3)}$$

Commonly, the first set of formulae (I-2-1) is known as the direct or forward solution, while the second set (I-2-3) represents the inverse solution of the mapping problem.

Cartography assumes the earth to be a sphere of radius R and the projection surface a plane. Thus, the subject of mathematical cartography for geographers is mainly restricted to various types of transformations of the sphere onto the plane.

Differential geometry (Goetz, 1970; Taschner, 1977) shows that a mutual projection of two surfaces is explicitly defined by the metrics of the surfaces. The metric tensor depends upon a surface and a selected parametrization, as was stated in (I-1-7). The fundamental set of parametric curvilinear coordinates on the sphere consists of the geographic latitude, ϕ, and the geographic longitude λ. The latitude of a point is the angle between the radial line through the point and the equatorial plane. In its magnitude, the latitude can be between 0° and 90° with, conventionally, the positive algebraic sign for the latitudes of the northern hemisphere and the negative for the southern hemisphere. The longitude of a point is the angle reckoned from the initial meridian plane, also called the Greenwich meridian, eastwardly or westwardly to the meridian of the point in question. In its magnitude, the longitudes can be between 0° and 180°. Eastwardly measured longitudes are conventionally taken to be positive and western longitudes, negative. The initial or central meridian of a mapped territory seldom coincides with the initial meridian of the geographic system, the Greenwich meridian. Therefore, cartographers, instead of using longitudes use the differences of longitudes between the longitude of a point, λ, and the longitude of the selected central meridian, λ_o.

$$l = \lambda - \lambda_o. \qquad \text{(I-2-4)}$$

Thus, in cartography each point on the sphere is defined by the latitude, ϕ, and the difference in longitude, l. The coordinate lines consist of meridians, lines of constant l, and parallels, lines of constant ϕ.

The projection surface is a plane with either an orthogonal Cartesian coordinate system (x,y) or a polar coordinate system (γ,ρ). Since the change from rectangular into polar coordinates and vice versa is accomplished by simple transformation formulae, it will be assumed, at least in the preliminary considerations, that points of the mapping plane are defined by their rectangular Cartesian coordinates.

Thus, a mapping system establishes a law of transformation of curvilinear spherical coordinates (ϕ,l) into the plane coordinates (x,y). To each point of the sphere the transformation functions assign a unique point on the plane. The general transformation formulae for the direct and inverse computations (I-2-1) and (I-2-3) can be transcribed for cartographic mappings into the following parametric equations

$$x = x(\phi,l) \ , \ y = y(\phi,l), \qquad \text{(I-2-5)}$$

and

$$\phi = \phi(x,y), \quad l = l(x,y), \tag{I-2-6}$$

where the first set of equations defines the direct mapping and the second set the inverse mapping. The transformation functions must satisfy the same conditions as the general formulae (I-2-1) of continuity, differentiability up to the second derivative, uniqueness, finiteness and the Jacobian determinant must differ from zero, $\partial(x,y)/\partial(\phi,l) \neq 0$. In other words, the mapping functions must be homeomorphic.

The fundamental metric on the sphere, according to the formula (I-1-15) is

$$ds^2 = R^2(d\phi^2 + \cos^2\phi \; dl^2), \tag{I-2-7}$$

and that on the plane

$$dS^2 = dx^2 + dy^2, \tag{I-2-8}$$

where dS is a differentially small linear element on the plane, obtained as an image of the corresponding linear element ds on the sphere, and the image is realized by the transformation functions (I-2-5). Since the rectangular coordinates (x,y) are functions of spherical coordinates (ϕ,l), the differentials in the last formula (I-2-8) are

$$dx = x_\phi d\phi + x_l dl \text{ and } dy = y_\phi d\phi + y_l dl. \tag{I-2-9}$$

When the differentials (I-2-9) are substituted into the equation (I-2-8) we obtain the well known expression for the square of a differentially small distance in the plane as a

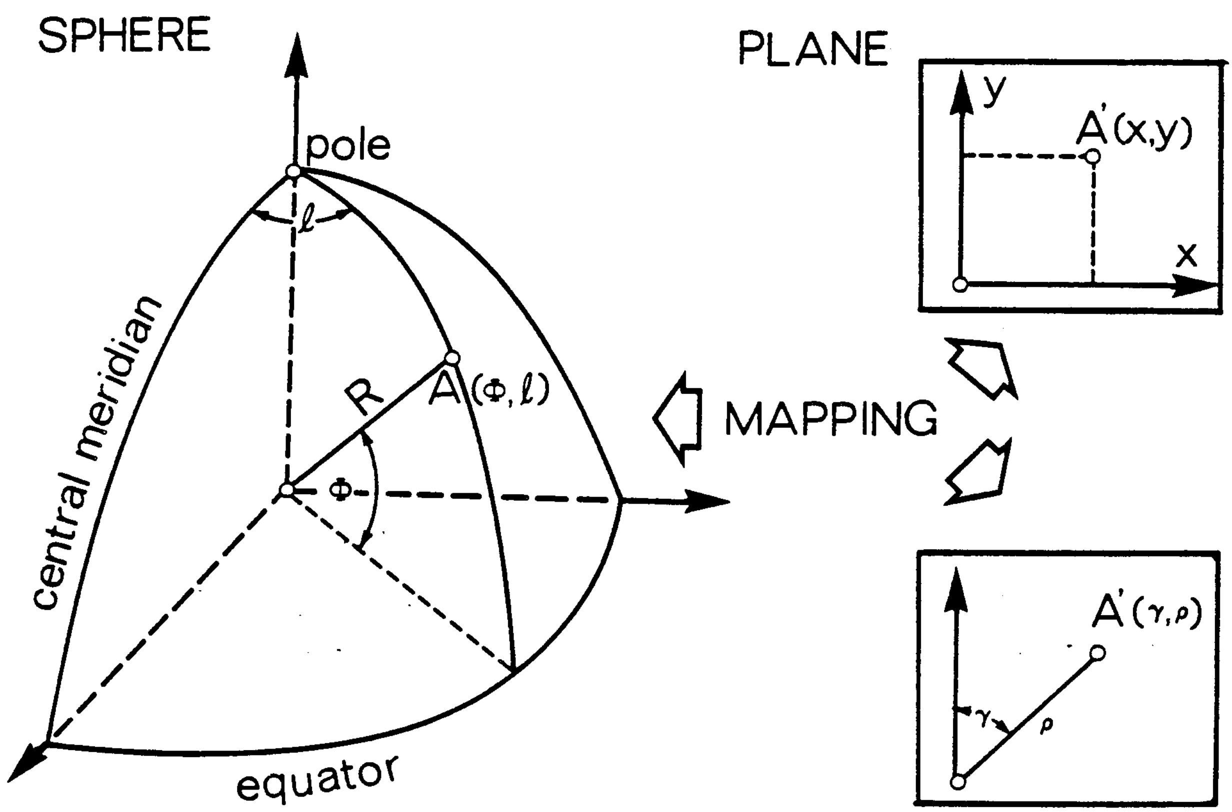


Figure I-2-1 Coordinate systems in cartographic mappings

function of spherical coordinates

$$dS^2 = g_{11}d\phi^2 + 2g_{12}d\phi dl + g_{22}dl^2, \qquad \text{(I-2-10)}$$

where

$$g_{11} = x_{\phi}^{2} + y_{\phi}^{2}, \quad g_{12} = x_{\phi}x_{l} + y_{\phi}y_{l}, \quad g_{22} = x_{l}^{2} + y_{l}^{2}. \quad \text{(I-2-11)}$$

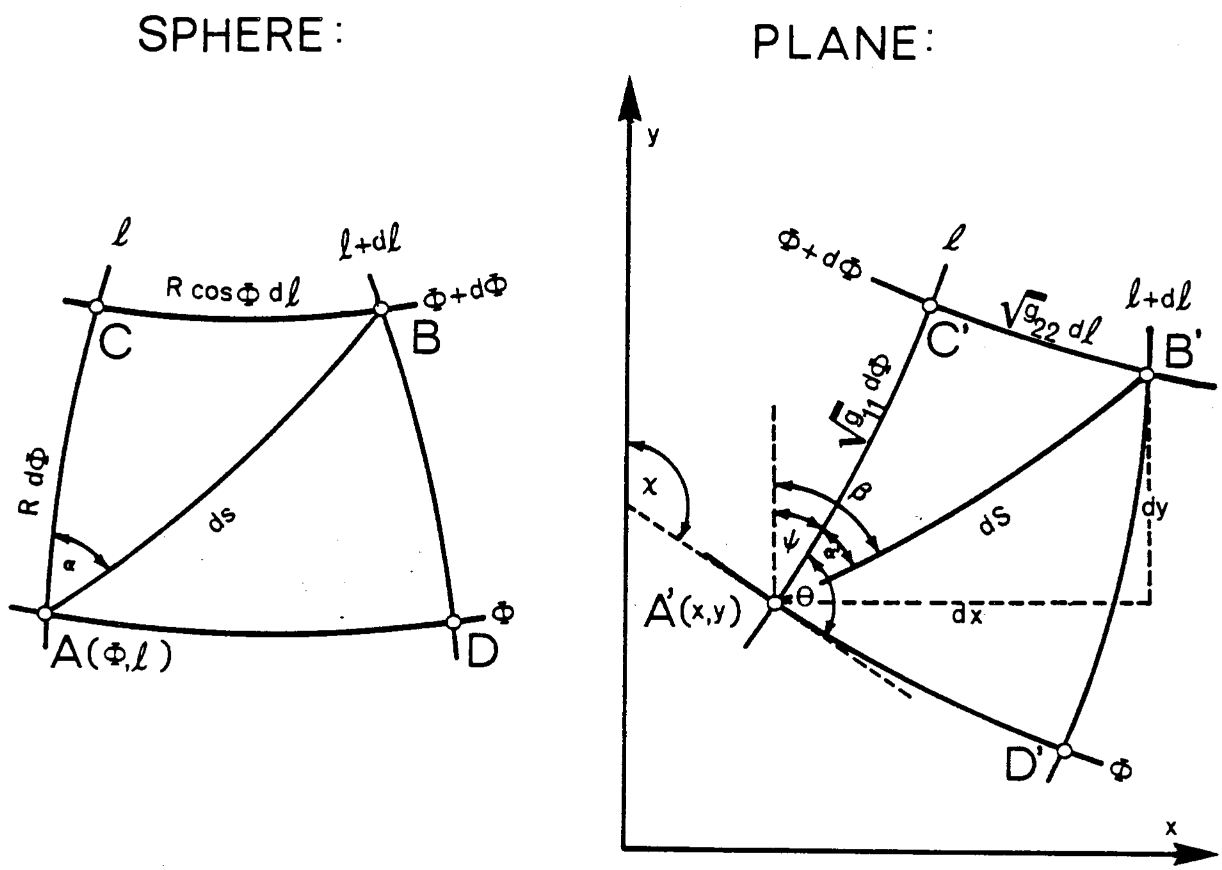


Figure I-2-2 Differentially small surface element of the sphere and its projection in the plane

The azimuth of a differentially small line segment ds on the sphere is defined as the angle reckoned clockwise from the positive direction of the meridian to the segment and is denoted by α. Its tangent function is obtained from the small right angle triangle ABC on the figure I-2-2, namely

$$\tan \alpha = \cos \phi \, dl/d\phi. \qquad (I\text{-}2\text{-}12)$$

The grid bearing, β, is the angle on the projection plane between the direction of the y-axis of the plane coordinate system, measured clockwise, and the projection of the arc segment dS, namely

$$\tan \beta = \frac{dx}{dy} = \frac{x_\phi d\phi + x_l dl}{y_\phi d\phi + y_l dl}. \qquad (I\text{-}2\text{-}13)$$

From the above equation one can also determine the bearing of the projection of meridian (l = const., $dl = 0$) and parallel (ϕ = const., $d\phi = 0$)

$$\tan \psi = x_\phi / y_\phi \text{ and } \tan \chi = x_l / y_l. \qquad (I\text{-}2\text{-}14)$$

The projection of the azimuth α, denoted by α' is then

$$\alpha' = \beta - \psi. \qquad (I\text{-}2\text{-}15)$$

Thus

$$\tan \alpha' = \tan(\beta - \psi) = \frac{\tan \beta - \tan \psi}{1 + \tan \beta \tan \psi}. \qquad \text{(I-2-16)}$$

Substituting the values for $\tan \beta$ and $\tan \psi$ from (I-2-14) respectively and rearranging the terms, we have

$$\tan \alpha' = \frac{\sqrt{g}\, dl}{g_{11} d\phi + g_{12} dl},$$

or

$$\cot \alpha' = g_{11}/\sqrt{g}\ \frac{d\phi}{dl} + g_{12}/\sqrt{g}, \qquad \text{(I-2-17)}$$

where

$$g = g_{11} g_{22} - g_{12}{}^2. \qquad \text{(I-2-18)}$$

The angle between the projections of parametric curves is computed by the formula (I-1-13)

$$\cos \theta = g_{12}/\sqrt{g_{11} g_{12}}, \qquad \text{(I-2-19)}$$

The sine function of angle θ is then

$$\sin \theta = \sqrt{1 - \cos^2 \theta} = \sqrt{g}/\sqrt{g_{11} g_{22}}. \qquad \text{(I-2-20)}$$

A differentially small area on the sphere limited by two close meridians and parallels is

$$dp = R^2 \cos \phi \; d\phi \; dl, \qquad \text{(I-2-21)}$$

and its projection on the plane becomes

$$dP = \sqrt{g} \; d\phi \; dl. \qquad \text{(I-2-22)}$$

Thus we have defined all the important intrinsic elements of the sphere: a differentially small distance, ds, and its azimuth, α, a differentially small area, dp, and the corresponding projections on the plane dS, α', dP, as functions of the differentials $d\phi$ and dl. The appropriate logical comparison of the intrinsic elements provides us with measures of quality for various projection systems.

3. THEORY OF DISTORTIONS

Differential geometry (Goetz, 1970) shows that an isometric mapping of two surfaces, a mapping where all corresponding distances on both surfaces remain identical, can be obtained if and only if the Gaussian curvatures of both surfaces are identical. Since the Gaussian curvature of a sphere is equal to the inverse of the square of the radius, and that of the plane is equal to zero, it is impossible to derive transformation formulae which, generally, will not alter distances.

In other words, the mapping process will always cause a certain deformation of the original intrinsic elements. Although some of the intrinsic elements may be preserved in the mapping process, the complete identity of the original surface elements and their projected counterparts can never be achieved in cartographic projections.

One of the main tasks of mathematical cartography is to determine a projection of a mapped territory in such a way that the resulting deformations of the original intrinsic elements are objectively minimized. Thus, distances, angles and areas will generally be changed in the transformation process. However, the changes of projected distances, angles and areas, and in particular the variations of these changes must be made as small as possible by the appropriate choice of transformation formulae. Distortions of surfaces in cartographic mappings are infinitely versatile, but, considered locally, the variations of the same distortions around an arbitrary point of a projected domain are governed by the general laws valid for all analytically defined projection systems. These laws are the subject of the theory of distortions. They were developed by a French mathematician and cartographer, M. Tissot (1824-1897) in 1859, with their final version being published in 1881 in Tissot's 'Memoirs'.

The theory of distortion is relatively well covered in textbooks of mathematical cartography, e.g. (Biernacky, 1965), (Fiala, 1957), (Kavraisky, 1959), (Richardus, Adler, 1972) and many others. In the English language it still remains to be treated rigorously; however, there is no need to redevelop all the formulae of the theory of distortions. It will suffice to list and explain them, since they are used later when evaluated numerically.

The comparison of a differentially small distance on the sphere and its projection on the plane is made by the scale factor, k.

$$k = \frac{dS}{ds}, \qquad \text{(I-3-1)}$$

where ds is the spherical distance and dS is its planar counterpart. The ideal value of the scale is unity, in which case, a distance on the sphere and its projection on the plane are identical. The distortion of distances is then defined by the equation

$$V_s = k - 1. \qquad \text{(I-3-2)}$$

From the fundamental definition of the scale (I-3-1) and by using the expressions (I-2-7) and (I-2-8) we have the value for the square of the scale

$$k^2 = \frac{dx^2 + dy^2}{R^2(d\phi^2 + \cos^2\phi \; dl^2)},$$

which can easily be transformed into

$$k^2 = \frac{g_{11}}{R^2} \cos^2\alpha + \frac{g_{12}}{R^2 \cos\phi} \sin 2\alpha + \frac{g_{22}}{R^2 \cos^2\phi} \sin^2\alpha, \qquad \text{(I-3-3)}$$

where α is the azimuth of the original distance ds on the sphere and g_{ij} are the elements of the metric tensor. It is obvious from the last formula that the scale in general depends on the position of a point (g_{ij}) and the direction of the line segment at the point (α), i.e.

$$k = k(g_{ij}, \alpha). \qquad \text{(I-3-4)}$$

The scales along the parametric coordinate lines, meridians and parallels are derived directly from the equation (I-2-25) knowing that for meridians $\alpha = 0$, and for parallels $\alpha = \pi/2$, thus

$$m = \sqrt{g_{11}}/R, \qquad \text{(I-3-5)}$$

and

$$n = \sqrt{g_{22}}/R \cos\phi, \qquad \text{(I-3-6)}$$

where m is the scale along meridians and n along parallels.

The equation (I-3-3) can also be expressed in terms of m,n and the parametric angle θ, i.e.

$$k^2 = m^2 \cos^2\alpha + mn \cos\theta \sin 2\alpha + n^2\sin^2\alpha. \qquad \text{(I-3-7)}$$

The extreme values of the scale are obtained from the equation $d(k^2)/d\alpha = 0$ which yields

$$\tan 2\alpha_o = \frac{2g_{12} \cos\phi}{g_{11} \cos^2\phi - g_{22}}, \qquad \text{(I-3-8)}$$

where α_o and $\alpha_o + \pi/2$ indicate two angles that satisfy the above trigonometric equation and represent the directions of the extreme scale changes. These two orthogonal directions are called the principal directions and their main characteristic is that they remain orthogonal on the projection plane as well. The definition and meaning of the principal directions are formulated by Tissot in his 'Memoirs' (Kavraiskii, 1959) in the first theorem of mappings.

> ...In every non-conformal representation of a regular surface onto another there is one and only one pair of corresponding orthogonal directions which are the principal directions and they represent the directions of the extreme scales....

The distortion of areas v_p is the difference between unity and the scale of area p

$$v_p = 1 - p, \qquad \text{(I-3-9)}$$

where p is defined as the ratio of a differentially small area on the plane and its original value on the sphere, namely

$$p = dP/dp. \tag{I-3-10}$$

Substituting in the above formula the expressions for areas dp, dP (I-2-21) and (I-2-22) we have

$$p = \frac{\sqrt{g}}{R^2 \cos\phi}, \tag{I-3-11}$$

or

$$p = mn \sin\theta. \tag{I-3-12}$$

The projection of the parametric angle, θ, is computed either by the equations (I-2-19) and (I-2-20) or

$$\tan\theta = \sqrt{g}/g_{12}. \tag{I-3-13}$$

The deformation of the parametric angle is defined by

$$\varepsilon = \pi/2 - \theta, \tag{I-3-14}$$

and its tangent function is

$$\tan\varepsilon = - g_{12}/\sqrt{g}\ . \tag{I-3-15}$$

The angular deformation ω is defined as the difference between an azimuth α and its projection α', i.e.

$$\omega = \alpha - \alpha' . \tag{I-3-16}$$

Its numerical value can be obtained from the expression

$$\tan \omega = \frac{g_{11} \cos \phi + g_{12} \tan \alpha - \sqrt{g}}{g_{11} \cos \phi \cot \alpha + g_{12} + \sqrt{g} \tan \alpha} , \tag{I-3-17}$$

but the same formula will be given later in a form more suitable for numerical computations.

At the end of this section it must be emphasized that a great majority of the formulae from the theory of distortions were already known to L. Euler but their complete and final form was elaborated only a century ago by Tissot. However, for reasons unknown to the author, cartographers in English speaking countries have been reluctant either to adopt them or to develop them further. Only in the last decade have we been experiencing a certain interest in mathematical problems of cartography (Milnor, 1969).

4. INDICATRIX OF TISSOT

In his study of general cartographic transformations Tissot introduced an ellipse of distortion or the indicatrix of projection, which found a particularly important place in mathematical cartography. Tissot's indicatrix, as a geometric characteristic of a mapping system, explained the fundamental questions of deformations of intrinsic elements and gave the distortions a more natural character and a more readily applicable visible form. The indicatrix of Tissot, with all its elements defined, completely describes the cartographic transformation system, or in other words, every measure of distortion can be expressed as a function of parameters of the indicatrix of Tissot (Biernacki, 1965).

The ellipse of distortion at a point of a projected domain is obtained by the transformation of a differentially small circle of unit radius from the original surface of the sphere onto the projection plane. The circle is generally transformed into an ellipse whose semi-axes are projected in the principal directions and in their magnitude they are equal to the extreme scale factors. The semi-major axis a is the maximal scale at the point P' and the semi-minor axis b is the minimal scale. The most suitable coordinate systems are the orthogonal local systems with the principal directions on both surfaces as the coordinate axes (ξ,η) on the sphere and (x,y) on the plane.

The semi-axes of the indicatrix are computed from known scales along the parametric curves, m and n, and the projected parametric angle, θ.

$$a = \frac{1}{2}(A + B) \quad \text{and} \quad b = \frac{1}{2}(A - B) \qquad \text{(I-4-1)}$$

where

$$\left.\begin{aligned} A^2 &= m^2 + n^2 + 2mn \sin\theta , \\ B^2 &= m^2 + n^2 - 2mn \sin\theta . \end{aligned}\right\} \qquad (I-4-2)$$

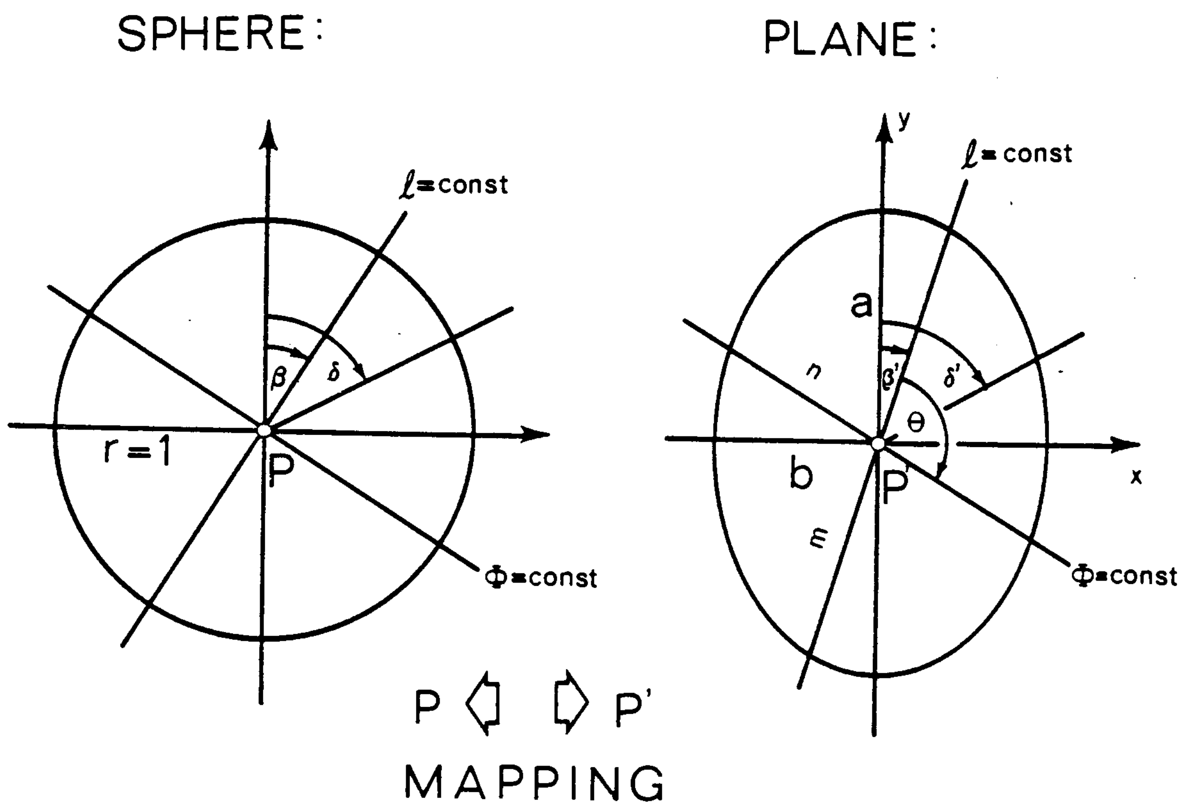


Figure I-4-1 Unit circle and the indicatrix of Tissot

The orientation angle of the meridian with respect to the first principal direction, β, is

$$1/\sin\beta = \sqrt{\frac{a^2 - b^2}{a^2 - m^2}}\,, \tag{I-4-3}$$

and its projection on the plane, β', is

$$\tan\beta' = \frac{b}{a}\tan\beta. \tag{I-4-4}$$

If we take an arbitrary direction, δ, with respect to the first principal direction, then the scale factor in its direction, k_δ, can be expressed in terms of the extreme scales and the direction angle, δ.

$$k_\delta{}^2 = a^2\cos^2\delta + b^2\sin^2\delta. \tag{I-4-5}$$

The angular distortion, defined again as the difference between the original direction, δ, and its projection, δ', is

$$\omega = \delta - \delta' \tag{I-4-6}$$

and

$$\sin\omega = \frac{a - b}{a + b}\sin(\delta + \delta'), \tag{I-4-7}$$

where the projection angle, δ', is computed by the formula (I-4-4).

The maximal angular distortion, ω_o, is

$$\sin \omega_o = \frac{a - b}{a + b}, \qquad \text{(I-4-8)}$$

and it occurs when $\sin(\delta + \delta') = 1$, or $\delta + \delta' = \pi/2$.

The direction, δ_o, in which the maximal angular distortion takes place is computed from the relationship

$$\delta_o = \pi/4 + \omega_o/2, \qquad \text{(I-4-9)}$$

and its projection

$$\delta_o' = \pi/4 - \omega_o/2. \qquad \text{(I-4-10)}$$

The scale of areas, p, can also be expressed in terms of the parameters of the indicatrix of Tissot by

$$p = ab. \qquad \text{(I-4-11)}$$

When the fundamental equations of a geographic mapping (I-2-5) are given, various quantities can be computed that fully characterize distortions. These quantities then specify the properties of the transformation system. They are: the scale factor k, the scales along parametric curves, m and n, the projection of the parametric angle, θ, the extreme scales, a and b, the bearing of the meridian, ψ, and that of the parallel, χ, the scale of areas, p, and many others. Since these quantities completely describe a projection, they are called the characteristics of mapping. Each of the characteristics can be defined as a function of eight variables

$$X_i = X_i(\phi, 1, x, y, x_\phi, x_1, y_\phi, y_1), \qquad \text{(I-4-12)}$$

where X_i is an arbitrary characteristic. From the totality of all characteristics we can select different vectors of four independent characteristics and then all others can be expressed in terms of the chosen basic vector. As a basic vector we may take, for example, (a, b, α_o, ψ), (m, n, ψ, χ), (m, n, ψ, θ), etc. The independence of a set of four characteristics can be determined by the analysis of the corresponding formulae. A more appropriate and a more rigorous approach is to prove that the set of four independent characteristics satisfies the following inequality

$$\frac{\partial(X_1, X_2, X_3, X_4)}{\partial(x_\phi, x_1, y_\phi, y_1)} \neq 0 \ . \qquad \text{(I-4-13)}$$

If the Jacobian (I-4-13) is nonsingular, and thus the four characteristics are independent, we can determine a unique set of transformation formulae (I-2-5) from the differentials (I-2-9).

5. FUNDAMENTAL DIFFERENTIAL EQUATIONS

As shown in the preceding section, any combination of four independent characteristics of map projections, X_i ($i = 1, 2, 3, 4$) may serve as the basis of the vector space of all deformation parameters. Therefore, the specification of the four basic characteristics as functions of ϕ and l at every point of the mapping domain fully determines the mapping system. In other words, it must be theoretically possible to derive the final transformation functions $x = x(\phi, l)$ and $y = y(\phi, l)$ directly from a specified distribution of distortions defined by the basis vector.

Let us now take two suggestions for the basis vectors made by Russian cartographers G.A. Meshcheryakov and N.A. Urmaev. The former (Meshcheryakov, 1968) recommended the basis vector (m, n, θ, ψ) and the latter (Urmaev, 1953) (m, n, ε, p). We shall develop both systems in order to obtain the fundamental differential equations of map projections with respect to both bases.

Meshcheryakov's suggestion of (m, n, θ, ψ) uses the known expressions for scales along parametric curves (I-3-5) and (I-3-6), and the bearings of the projections of parametric curves, whose tangent functions were given by the formulae (I-2-14), bearing in mind that the parametric angle θ is defined by the equation

$$\theta = \chi - \psi. \qquad \text{(I-5-1)}$$

When the elements of the metric tensor g_{ij} are substituted into the equations (I-3-5) and (I-3-6) and the equations are squared we have

$$\left.\begin{aligned} m^2 &= \frac{1}{R^2}\,(x_\phi^{\,2} + y_\phi^{\,2}), \\ n^2 &= \frac{1}{R^2\cos^2\phi}\,(x_1^{\,2} + y_1^{\,2}). \end{aligned}\right\} \qquad \text{(I-5-2)}$$

The formulae (I-2-14) can be rewritten in the form

and

$$\left.\begin{aligned} x_\phi &= y_\phi \tan\psi \quad , \quad y_\phi = x_\phi \cot\psi \\ x_1 &= y_1 \tan\chi \quad , \quad y_1 = x_1 \cot\chi. \end{aligned}\right\} \qquad \text{(I-5-3)}$$

When the last results are substituted into the equations (I-5-2) and employing the trigonometric identities

$$1 + \tan^2\psi = \sec^2\psi \quad , \quad 1 + \cot^2\psi = \csc^2\psi$$

we obtain

$$y_\phi = Rm \cos\psi \quad , \quad y_l = R \cos\phi \; n \cos\chi \; ,$$

and (I-5-4)

$$x_\phi = Rm \sin\psi \quad , \quad x_l = R \cos\phi \; n \sin\chi \; .$$

The integration of equations (I-5-4) leads to the transformation expressions (I-2-5), providing the conditions of integrability are satisfied, namely

$$\frac{\partial(x_\phi)}{\partial l} = \frac{\partial(x_l)}{\partial\phi} \text{ and } \frac{\partial(y_\phi)}{\partial l} = \frac{\partial(y_l)}{\partial\phi} \; . \qquad \text{(I-5-5)}$$

With the introduction of new abbreviations

$$m^* = Rm \text{ and } n^* = Rn \cos\phi \qquad \text{(I-5-6)}$$

the equation (I-5-4) becomes

$$y_\phi = m^* \cos\psi \quad , \quad y_l = n^* \cos\chi \; ,$$

(I-5-7)

$$x_\phi = m^* \sin\psi \quad , \quad x_l = n^* \sin\chi \; .$$

The conditions of integrability (I-5-5) are then

$$\begin{aligned} m^*_1 \sin\psi + m^* \cos\psi\, \psi_1 &= n^*_\phi \sin\chi + n^* \cos\chi\, \chi_\phi, \\ m^*_1 \cos\psi - m^* \sin\psi\, \psi_1 &= n^*_\phi \cos\chi - n^* \sin\chi\, \chi_\phi\,. \end{aligned} \tag{I-5-8}$$

From the definition of parametric angle, θ, (I-5-1) we have that

$$\chi = \theta + \psi\,, \tag{I-5-9}$$

and substituted into equations (I-5-8) we obtain

$$\begin{aligned} &\sin\psi(m^*_1 - n^*_\phi \cos\theta + n^* \sin\theta\, \psi_\phi + n^* \sin\theta\, \theta_\phi) = \\ &= -\cos\psi(m^*\, \psi_1 - n^*_\phi \sin\theta - n^* \cos\theta\, \psi_\phi - n^* \cos\theta\, \theta_\phi)\,, \end{aligned}$$

and

$$\begin{aligned} &\cos\psi(m^*_1 - n^*_\phi \cos\theta + n^* \sin\theta\, \psi_\phi + n^* \sin\theta\, \theta_\phi) = \\ &= \sin\psi(m^*\, \psi_1 - n^*_\phi \sin\theta - n^* \cos\theta\, \psi_\phi - n^* \cos\theta\, \theta_\phi), \end{aligned}$$

or simplified

$$\left.\begin{aligned} \Omega \sin\psi &= -\,\omega \cos\psi\,, \\ \Omega \cos\psi &= \omega \sin\psi\,, \end{aligned}\right\} \tag{I-5-10}$$

where

$$\left.\begin{aligned}\Omega &= m^*_1 - n^*_\phi \cos\theta + n^* \sin\theta\, \psi_\phi + n^* \sin\theta\, \theta_\phi,\\ &\text{and}\\ \omega &= m^*\, \psi_1 - n^*_\phi \sin\theta - n^* \cos\theta\, \psi_\phi - n^* \cos\theta\, \theta_\phi.\end{aligned}\right\} \quad \text{(I-5-11)}$$

The equations (I-5-10) then can be satisfied only if the expressions Ω and ω are equal to zero, namely

$$\left.\begin{aligned}m^*_1 - n^*_\phi \cos\theta + n^* \sin\theta\, \psi_\phi + n^* \sin\theta\, \theta_\phi &= 0,\\ m^*\psi_1 - n^*_\phi \sin\theta - n^* \cos\theta\, \psi_\phi - n^* \cos\theta\, \theta_\phi &= 0.\end{aligned}\right\} \quad \text{(I-5-12)}$$

The system of partial differential equations (I-5-12) may be called the fundamental system of differential equations of map projections, since the system must be satisfied at every point of the mapping domain. The system involves four characteristics m,n,θ,ψ and their partial derivatives of the first order with respect to the parametric coordinates $(\phi,1)$. The equations are quasilinear with respect to any combination of two characteristics from the basis vector m,n,θ,ψ. The system of two equations connects four functions and, thus, it is undetermined. For practical applications of the fundamental system we must predefine the values of two characteristics, or establish two relations between them at every point of the mapping domain. Only then is there a theoretical possibility of integration of the equations (I-5-12).

The basis vector of Urmaev (m,n,ε,p) requires the equations (I-5-2)

$$\left.\begin{aligned} x_\phi{}^2 + y_\phi{}^2 &= m^2R^2 , \\ x_1{}^2 + y_1{}^2 &= n^2R^2\cos^2\phi . \end{aligned}\right\} \qquad \text{(I-5-13)}$$

In addition to them, the tangent function of the deformation of parametric angle, ε, is given by the equation (I-5-15)

$$\tan \varepsilon = - \frac{g_{12}}{\sqrt{g}} ,$$

and from (I-3-11)

$$\sqrt{g} = R^2 \cos \phi \; p.$$

Thus

$$\tan \varepsilon = - \frac{1}{R^2 \cos \phi \; p} \; g_{12},$$

or finally

$$\tan \varepsilon = - \frac{1}{R^2 \cos \phi \; p} \left(x_\phi x_1 + y_\phi y_1\right) . \qquad \text{(I-5-14)}$$

The selected transformation functions $x = x(\phi,l)$ and $y = y(\phi,l)$ must satisfy at every point of the domain the equations (I-5-13) and (I-5-14). The establishment of new coordinate systems requires the integration of the same three equations. However much we may wish to carry out the optimization process and, thus, reduce distortions to any desirable level we cannot go too far since the relations between distortion elements (I-5-13) and (I-5-14) must always exist.

Without a loss of generality, Urmaev assumed the unit radius of the earth and has replaced in (Urmaev, 1953) the equations (I-5-13) and (I-5-14) by four expressions

$$\left.\begin{aligned} x_\phi &= -m \sin(\varepsilon + \beta) \quad , \quad x_l = \nu \cos \beta , \\ y_\phi &= m \cos(\varepsilon + \beta) \quad , \quad y_l = \nu \sin \beta , \end{aligned}\right\} \qquad \text{(I-5-15)}$$

where

$$\nu = \eta \cos \phi. \qquad \text{(I-5-16)}$$

and β is an unknown arbitrary function of the parametric coordinates (ϕ,l)

$$\beta = \beta(\phi,l). \qquad \text{(I-5-17)}$$

In the case of equations (I-5-15) the conditions of integrability (I-5-5) become

$$\left.\begin{aligned} -m_1 \sin(\varepsilon+\beta) - m\cos(\varepsilon+\beta)(\varepsilon_1+\beta_1) &= \nu_\phi \cos\beta - \nu \sin\beta\ \beta_\phi, \\ m_1 \cos(\varepsilon+\beta) - m\sin(\varepsilon+\beta)(\varepsilon_1+\beta_1) &= \nu_\phi \sin\beta + \nu \cos\beta\ \beta_\phi. \end{aligned}\right\} \quad \text{(I-5-18)}$$

The solution of these two equations with respect to the variable, β_ϕ, can be obtained by multiplying the first equation by $\sin(\varepsilon + \beta)$, the second by $\cos(\varepsilon + \beta)$ and then subtracting one from the other. The solution with respect to β_1 is derived by multiplying the first equation by $\cos\beta$, the second by $\sin\beta$ and then adding the resulting equations together. After some simple rearrangement of terms we obtain

$$\left.\begin{aligned} \beta_\phi &= \frac{m_1 + \nu_\phi \sin\varepsilon}{\nu\cos\varepsilon}, \\ -\beta_1 &= \frac{\nu_\phi}{m\cos\varepsilon} + \frac{\tan\varepsilon}{m}\, m_1 + \varepsilon_1. \end{aligned}\right\} \quad \text{(I-5-19)}$$

The equations (I-5-19) represent a second version of the fundamental differential equations of map projections. In the most general case they are partial differential equations with partial derivatives of the first and second order and in some specific cases they are ordinary differential equations of the first and second order. The determination of the function β from the fundamental system leads to a partial differential equation of the Monge-Amper type whose analytical solution is, very often, impossible not only with elementary functions but also with special functions. However, a numerical solution which yields the rectangular coordinates (x,y) of a point can often be determined: not the analytical expressions for coordinates but their numerical values. In this way a number of new transformation systems can be designed, so that the rectangular coordinates of a regular graticule net are computed as a numerical solution of the fundamental system with some additional conditions. Some authors, (Meshcheryakov, 1968), call the differential equations (I-5-19) the Euler-Urmaev fundamental equations.

6. CLASSIFICATION OF MAPPINGS

There are, theoretically, an infinite number of conceivable transformation systems that can be used in mathematical cartography. To study this totality of map projections requires some reasonable grouping; it demands a suitable classification scheme.

To classify a set of objects, whose number can be finite or infinite, means to design smaller groups of the objects so that, from a certain point of view, each group has distinct common characteristics. This point of view is called the basis of classification. The selection of the basis must lead to groupings of map projections that are either practically or theoretically important. An ideal classification must consist of mutually exclusive and collectively exhaustive groups which will contain an approximately equal number of practically significant map projections. When the basis of classification is changed, it is quite natural that the distribution and grouping should also change.

The problem of classification of cartographic projections is one of the fundamental tasks of mathematical cartography. The classical bases for classification were suggested by Tissot and later elaborated by a Russian cartographer, V.V. Kavraiskii (1959). These widely accepted divisions of map projections were developed for a certain group of transformations only. They do not comprise all existing and conceivable projections and they lead to an uneven aggregation of projections in various classes. The bases of classical groupings are:

(i) character of distortions expressed by the relationship of the semi-axes of the indicatrix of Tissot, and;

(ii) property of the normal grid, i.e. the image of the normal graticule of meridians and parallels in the mapping plane.

In addition to these two fundamental bases there are others that are used with more or less success. For example, maps are classified according to:

(iii) the character of projection equations (I-2-5) (the parametric classification of Tobler, 1962),

(iv) the aspect of the metagraticule, i.e. the position of the metapole of the used metagraticule with respect to the geographic pole, (Wray, 1974),

(v) the character of the differential equations whose solutions define the transformation functions (the genetic classification of Meshcheryakov, (Meshcheryakov, 1968),

(vi) the value of the metric tensor to the second order, suggested by B.H. Chovitz (Chovitz, 1952, 1954).

There are, naturally, other ways to classify map projections. However, the scope of this work (the optimization of cartographic projections with respect to the distribution of distortions) suffices to consider no other classification scheme than the very first one: classification according to the character of distortions.

The first classical grouping of cartographic projections according to the character of distortions leads normally to four distinct classes of projections:

a) conformal or orthomorphic projections,

b) equiareal or equivalent projections,

c) equidistant projections, and

d) arbitrary or aphylactic projections.

The Russian cartographers, G.A. Ginzburg and T.D. Salmanova, (Pavlov 1964), suggested a slightly modified version of the above classification. They took the first three classes of conformal, equiareal and equidistant projections as they were and then split the fourth class and created:

a) conformal projections,

b) projections with small deformation of angles,

c) equidistant projections,

d) projections with small deformations of areas, and

e) equiareal projections.

Although the suggested scheme allows a slightly better and more methodical arrangement of arbitrary projections in two classes b) and d), the classification does not help in the study of the arbitrary projections since their characteristics are so diverse that they can not be expressed by any reasonable common denominator, whether they belong to a single class or to two distinct classes. Let us now briefly define individual classes.

Conformal mappings are those in which at every point of the mapped domain the semi-axes of the indicatrix of Tissot are identical, that is

$$a = b, \qquad \text{(I-6-1)}$$

or the scale factor, k, at each point has a constant value independent of the direction, but generally changes from point to point. In others words, the scale factor is a function of the position of the point only,

$$k = k(\phi, l). \tag{I-6-2}$$

As a result of the above property, angles at a point in conformal projections are preserved, or, the similarity of differentially small surface elements in the conformal projections is maintained.

In equiareal projections the scale of area, p, has a constant value that, without loss of generality, can be assumed to be equal to unity. Then, the areas obtained from the plane coordinates are identical to the corresponding areas on the sphere. The condition of equiareal mappings is satisfied on the whole domain if at every point the product of the principal scales is equal to unity,

$$ab = 1. \tag{I-6-3}$$

Since the condition of conformality requires that a=b, it is obvious that a projection cannot satisfy both conditions of conformality and equivalency simultaneously on the whole projected domain.

When one of the axes of the indicatrix of Tissot has a value of unity for the whole transformation domain, the projection is called equidistant. It preserves distances along one specific direction, i.e. one of the principal directions is the direction of no linear deformations.

Thus, the conditions of equidistancy are

$$a = 1 \qquad \text{or} \qquad b = 1. \qquad \text{(I-6-4)}$$

Equidistant projections are, according to their properties of distortion elements, somewhere between conformal and equiareal mappings.

The class of arbitrary or aphilactic map projections comprises all projection systems which are neither conformal, equiareal nor equidistant. This class is theoretically much larger than the others but in practice the distribution of mappings according to the character of distortion in the four classes is relatively even. That is, the number of aphylactic map projections in use is not particularly large.

7. CONFORMAL MAPPINGS

Conformal transformations constitute a specific class of projections with a series of remarkable properties of great theoretical and practical significance. In cartography, they are at the same time the simplest and the most elaborate projections. The theory of conformal mapping of a sphere onto a plane was elaborated independently by Lambert (1728 - 1777) and Euler (1707 - 1783). The conformal projections of the surfaces of rotation onto a plane were developed by Lagrange (1736 - 1813), but the general theory of conformal representations of regular surfaces was formulated by Gauss (1777 - 1855), (Gauss, 1825).

A conformal mapping of regular surfaces is defined as a transformation where the scale factor at every point of the projected domain is independent of the direction and is therefore a function of the position only, i.e.

$$m = m(\phi, l). \tag{I-7-1}$$

As a result of the independence of the scale on the direction, the angles at every point are preserved. Sometimes the fundamental definition is made in the reverse order, i.e. a conformal mapping is defined as a transformation in which angles remain unchanged and therefore the linear scale is a function of position only. Gauss combined these two properties stating that in every conformal transformation the similarity of differentially small shapes is retained.

Conformal mappings are directly connected to the establishment of isothermic coordinates on the surfaces involved in the projection. The metric form in isothermic coordinates (I-1-19) $ds^2 = [\lambda(u^1, u^2)^2] \cdot [(du^1)^2 + (du^2)^2]$ indicates that for isothermic coordinates the elements of the metric tensor satisfy the condition

$$g_{11} = g_{22} \quad \text{and} \quad g_{12} = 0. \tag{I-7-2}$$

The fundamental metric on a sphere of radius R (I-2-7) expressed in terms of geographic coordinates (latitude and difference in longitude) is not isothermic,

$$ds^2 = R^2(d\phi^2 + \cos^2\phi \; dl^2).$$

However the quadratic form can easily be transformed into an isothermic form by the introduction of a new, so-called isothermic latitude, q, whose differential is defined by

$$dq = \sec \phi \; d\phi. \qquad \text{(I-7-3)}$$

Then the quadratic form (I-2-7) becomes

$$ds^2 = R^2(dq^2 + dl^2). \qquad \text{(I-7-4)}$$

The integration of the equation (I-7-3) yields the expression for the isothermic latitude

$$q = \ln \tan(\pi/4 + \phi/2). \qquad \text{(I-7-5)}$$

Gauss has proved that a conformal transformation is established when the following relation exists:

$$Z = F(\omega) \qquad \text{(I-7-6)}$$

where

$$Z = y + ix \quad , \quad \omega = q + il, \tag{I-7-7}$$

and F is an analytic function, i.e. a function of the complex variable ω whose first derivative does exist and is continuous at every point of the mapped domain. The differentiability of the function F is proven by the Cauchy-Riemann equations, which are the necessary and sufficient condition that the complex function F is analytic and the mapping performed by the function is conformal,

$$\left.\begin{aligned} x_l &= y_q = \operatorname{Re}\frac{dF(\omega)}{d\omega} , \\ &\text{and} \\ -x_q &= y_l = -\operatorname{Im}\frac{dF(\omega)}{d\omega} . \end{aligned}\right\} \tag{I-7-8}$$

The condition of conformality can also be expressed in terms of scales along parametric lines and the deformation of the parametric angle,

$$m = n \quad , \quad \varepsilon = 0. \tag{I-7-9}$$

In this case the fundamental differential equations of map projections (I-5-19) become

$$\left.\begin{aligned} \beta_\phi &= \frac{m_l}{\nu}\,, \\ -\beta_l &= \frac{\nu_\phi}{m}\,, \end{aligned}\right\} \tag{I-7-10}$$

where

$$\nu = m \cos \phi. \tag{I-7-11}$$

The transformation of geographic latitude into isothermic latitude is performed by the equation (I-7-3), where

$$dq/d\phi = \sec \phi = \cosh q, \tag{I-7-12}$$

and then (I-7-11) becomes

$$\nu = m \operatorname{sech} q. \tag{I-7-13}$$

The fundamental differential equations of conformal mappings (I-7-10) can now be expressed in terms of isothermal coordinates

$$\beta_q = \frac{\partial \ln \nu}{\partial l}\,, \quad -\beta_l = \frac{\partial \ln \nu}{\partial q}\,. \tag{I-7-14}$$

The condition of integrability $\beta_{ql} = \beta_{lq}$ yields

$$\frac{\partial^2 \ln \nu}{\partial l^2} = - \frac{\partial^2 \ln \nu}{\partial q^2}$$

or finally (I-7-15)

$$\frac{\partial^2 \ln \nu}{\partial l^2} + \frac{\partial^2 \ln \nu}{\partial q^2} = 0.$$

The above formula is the well known Laplace equation of mathematical physics. The equation can also be expressed as a function of the scale. From (I-7-13) we have

$$\ln \nu = \ln m - \ln \cosh q \tag{I-7-16}$$

and differentiating

$$\frac{\partial \ln \nu}{\partial l} = \frac{\partial \ln m}{\partial l} \quad ; \quad \frac{\partial \ln \nu}{\partial q} = \frac{\partial \ln m}{\partial q} - \tanh q$$

with the second derivatives

$$\frac{\partial^2 \ln \nu}{\partial l^2} = \frac{\partial^2 \ln m}{\partial l^2} \ , \quad \frac{\partial^2 \ln \nu}{\partial q^2} = \frac{\partial^2 \ln m}{\partial q^2} - \text{sech}^2 q. \tag{I-7-17}$$

When the results of differentiation (I-7-17) are substituted into the Laplace equation (I-7-15) we obtain

$$\frac{\partial^2 \ln m}{\partial q^2} + \frac{\partial^2 \ln m}{\partial l^2} = \text{sech}^2 q, \tag{I-7-18}$$

which is the Poisson equation of mathematical physics.

The Laplace and Poisson equations (I-7-15) and (I-7-18) have great significance in the optimization process of conformal mappings.

The solutions of the Laplace equations are called harmonic functions and they determine the value of $\ln \nu$ at every point of the mapped domain. In that manner the value of $\ln \nu$ can be determined also at the central meridian where $l = 0$. The rectangular coordinate y on the central meridian is then

$$y_o = c + \int \nu_o dq = F(q) \tag{I-7-19}$$

where c is an arbitrary constant and ν_o is the value of ν at the central meridian. The analytical continuation leads to the general formula

$$y + ix = F(q + il), \tag{I-7-20}$$

where F is an analytic function assumed to be unique and one-to-one.

The great majority of conformal map projections used in practical cartography can be developed easily and directly from the Laplace equation (I-7-15). For example, by assuming the quantity ν to be a constant we have

$$\ln \nu = \text{const.}\ , \text{ or } \quad \nu = c, \tag{I-7-21}$$

and (I-7-21)

$$y_o = K + c \int dq = K + cq,$$

where the constant of integration is equal to zero since for $q = 0$ also $y = 0$. Thus, the last equation becomes

$$y_o = cq,$$

and then applying the analytic continuation we have

$$y + ix = c(q + il), \tag{I-7-22}$$

which is the expression for the Mercator projection.

Let us now assume that the scale is a function of the isothermic latitude, q, only: i.e.

$$m = m(q), \tag{I-7-23}$$

which reduces the Laplace equation (I-7-15) to one term only

$$\frac{d^2 \ln \nu}{dq^2} = 0. \tag{I-7-24}$$

The first integration yields

$$\frac{d \ln \nu}{dq} = -c,$$

where the negative sign of the constant, c, is used simply for convenience. The second integration then gives

$$\ln \nu = \ln cK - cq, \qquad \text{(I-7-25)}$$

where the constant of integration is expressed as the natural logarithm of the product of the first constant, c, and the second constant, K. The quantity ν is then

$$\nu = cKe^{-cq}.$$

The ordinate on the central meridian is

$$y_o = cK \int e^{-cq}\, dq = Q - Ke^{-cq},$$

and with the analytic continuation we obtain

$$y + ix = Q - Ke^{-c(q + il)}, \qquad \text{(I-7-26)}$$

which is the Lambert conformal conic projection. A special case of the Lambert conformal conic projection for c = 1 is the stereographic projection. In the same way we can develop many more standard conformal projections.

In the optimization process of conformal map projections there is another special solution of the Laplace equation (I-7-15) which has considerable interest and importance. The solution is given by the equation

$$\ln \nu = (q + il)^n, \tag{I-7-27}$$

which is a harmonic polynomial whose first five values are:

$$\left.\begin{array}{lll} n = 1 & q & + \; il, \\ n = 2 & q^2 & + \; i2ql - l^2, \\ n = 3 & q^3 & + \; i3q^2l - 3ql^2 - il^3, \\ n = 4 & q^4 & + \; i4q^3l - 6q^2l^2 - i4ql^3 + l^4, \\ n = 5 & q^5 & + \; i5q^4l - 10q^3l^2 - i10q^2l^3 + 5ql^4 + il^5. \end{array}\right\} \tag{I-7-28}$$

In each of these expressions there are two groups of uniform polynomials with coefficients related to real and imaginary terms. Let us denote a polynomial with real coefficients by ψ_n and that with imaginary coefficients by θ_n.

$$\left.\begin{array}{ll} \psi_0 = 1, & \theta_0 = 1, \\ \psi_1 = q, & \theta_1 = \ell, \\ \psi_2 = q^2 - l^2, & \theta_2 = 2ql, \\ \psi_3 = q^3 - 3ql^2, & \theta_3 = 3q^2l - l^3, \\ \psi_4 = q^4 - 6q^2l^2 + l^4, & \theta_4 = 4q^3l - 4ql^3, \\ \psi_5 = q^5 - 10q^3l^2 + 5ql^4, & \theta_5 = 5q^4l - 10q^2l^3 + l^5. \end{array}\right\} \tag{I-7-29}$$

Each one of the polynomials ψ_n and θ_n is a special solution of the Laplace equation, thus an arbitrary linear combination of the two polynomials is also a solution, i.e.

$$\ln \nu = \sum_{j=1}^{n} (a_j \psi_j + b_j \theta_j). \tag{I-7-30}$$

This solution yields an infinite series of conformal map projections whose coefficients a_j and b_j can be determined so that the scale factor m, and thus the function ν is optimized for a particular mapping domain. It can also be shown that some well known projections like the Lambert conformal conic, Lagrange's projection, Littrov's projection and the Transverse Mercator projection can be directly derived from the equation (I-7-30). In addition to these known projections we can develop many unknown but useful map projections.

8. EQUIAREAL PROJECTIONS

Equiareal, or equivalent map projections are those in which the scale of areas at every point of the mapping domain has one and the same value. For reasons of simplicity the constant is assumed to be equal to unity. Since the scale of areas was defined by the equations (I-3-11), (I-3-12) and (I-4-11) it is easy to prove that the mathematical condition of equiareal projection is

$$x_1 y_\phi - x_\phi y_1 = R^2 \cos \phi. \tag{I-8-1}$$

This is a nonlinear partial differential equation with two dependent variables x,y and two independent variables ϕ,l. Since we have only one equation and two unknowns, the system (I-8-1) is undetermined and has an infinite number of solutions. Particular solutions can be obtained theoretically by the integration of the above differential equation if some additional conditions are formulated, the conditions which clarify the relationship between dependent variables or their connections to the independent variables. The totality of all these conditions is again infinite, thus leading to a totality of equiareal projections which is very difficult to derive directly from the initial differential equation (I-8-1) in an organized manner. However, the integration process is theoretically, as well as practically, possible. Two map projections will be developed to prove the possibility of obtaining cartographic transformation systems directly from the fundamental condition of equiareal mappings. One of these projections is a new equiareal projection developed by the author and the other is a well-known map projection, Lambert's equiareal cylindric projection.

Let us assume that the transformation formulae $x = x(\phi, l)$ and $y = y(\phi, l)$ can be expressed as products of four functions in the following way:

$$\left.\begin{aligned} x &= f_1(\phi)\ f_2(l), \\ y &= g_1(\phi)\ g_2(l). \end{aligned}\right\} \qquad \text{(I-8-2)}$$

Thus, we are using the standard separation of variables. The required partial derivatives are then

$$\left.\begin{aligned} x_\phi &= f'_1(\phi)\; f_2(1)\;,\; x_1 = f_1(\phi)\; f'_2(1), \\ y_\phi &= g'_1(\phi)\; g_2(1)\;,\; y_1 = g_1(\phi)\; g'_2(1), \end{aligned}\right\} \qquad (\text{I-8-3})$$

and the fundamental condition of equiareal mapping becomes

$$f_1(\phi)f_2'(1)g_2(1)g_1'(\phi) - f_1'(\phi)f_2(1)g_2'(1)g_1(\phi) = R^2 \cos \phi. \qquad (\text{I-8-4})$$

In order to develop specific solutions, let us introduce an additional condition

$$\left.\begin{aligned} f_2'(1)\; g_2(1) &= 1, \\ f_2(1)\; g_2'(1) &= 1. \end{aligned}\right\} \qquad (\text{I-8-5})$$

If we combine the above equations we have

$$f_2'(1)/f_2(1) = g_2'(1)/g_2(1),$$

and the integration yields

$$\ln f_2(1) = \ln g_2(1) + \ln c,$$

or

$$f_2(l) = c\ g_2(l).$$

Assuming that the constant of integration, c, is equal to unity we finally obtain

$$f_2(l) = g_2(l). \qquad \text{(I-8-6)}$$

The equality of functions $f_2(l)$ and $g_2(l)$ results in a simple form of the first equation of (I-8-5)

$$f_2 df_2 = dl,$$

or integrating

$$\frac{1}{2} f_2{}^2 = l$$

and finally

$$f_2 = \sqrt{2l}. \qquad \text{(I-8-7)}$$

The fundamental equation of equiareal mappings (I-8-4) in this particular case, with the assumptions (I-8-5) is

$$f_1(\phi)\ g_1'(\phi) - f_1'(\phi)\ g_1(\phi) = R^2 \cos \phi,$$

or

$$\frac{d}{d\phi}\,[f_1(\phi)\; g_1(\phi)] = R^2 \cos\phi. \tag{I-8-8}$$

Let us now apply the method of undetermined coefficients. Then the required functions $f_1(\phi)$ and $g_1(\phi)$ have the form

$$\left.\begin{aligned} f_1(\phi) &= a_1 \cos\phi + b_1 \sin\phi + c_1, \\ g_1(\phi) &= a_2 \cos\phi + b_2 \sin\phi + c_2, \end{aligned}\right\} \tag{I-8-9}$$

and the corresponding derivatives are

$$\left.\begin{aligned} f_1'(\phi) &= -a_1 \sin\phi + b_1 \cos\phi, \\ g_1'(\phi) &= -a_2 \sin\phi + b_2 \cos\phi, \end{aligned}\right\} \tag{I-8-10}$$

where a_1, a_2, b_1, b_2, c_1, c_2 are unknown coefficients.

Substituting the equations (I-8-9) and (I-8-10) into (I-8-8), after some rearrangement of terms we obtain

$$a_1 b_2 - b_1 a_2 + c_1(b_2\cos\phi - a_2\sin\phi) + c_2(a_1\sin\phi - b_1\cos\phi) = R^2\cos\phi.$$

At least one of the parameters c_i must be different from zero. The last equation will be satisfied if

$$\left.\begin{aligned} a_1 b_2 - b_1 a_2 &= 0 \\ a_1 c_2 - a_2 c_1 &= 0 \\ b_2 c_1 - b_1 c_2 &= R^2, \end{aligned}\right\} \tag{I-8-11}$$

yielding three equations with six unknowns. If we assume $c_1 = \alpha$ and $c_2 = 0$, from the second equation we obtain that $a_2 = 0$ from the third equation $b_2 = R^2/\alpha$ and from the first that $a_1 = 0$. In addition, the parameter b_1 can take any value and therefore can be set to be equal to zero. Thus the only non-zero coefficients are

$$b_2 = \frac{R^2}{\alpha} \text{ and } c_1 = \alpha, \tag{I-8-12}$$

which lead to a specific version of the equations (I-8-3)

$$f_1(\phi) = \alpha \qquad , \; f_2(l) = \sqrt{2l},$$

$$g_1(\phi) = \frac{R^2}{\alpha} \sin\phi \; , \; g_2(l) = \sqrt{2l},$$

and finally to the transformation formulae

$$\left.\begin{aligned} x &= \alpha\sqrt{2l}, \\ y &= \frac{R^2}{\alpha}\sin\phi\,\sqrt{2l}. \end{aligned}\right\} \tag{I-8-13}$$

It can easily be proven that the last equations satisfy the fundamental differential equation of equiareal mappings (I-8-1), thus the transformation (I-8-13) is really equiareal.

Another equiareal projection can be obtained if we again assume the validity of the first equation of (I-8-5) and also that

$$f_2{}'(l) = \alpha, \tag{I-8-14}$$

where α is an arbitrary non-zero constant. Then

$$g_2(1) = 1/\alpha. \qquad \text{(I-8-15)}$$

Integrating the equation (I-8-14) and differentiating (I-8-15) we obtain

$$f_2(1) = \alpha \cdot 1 \text{ and } g_2'(1) = 0, \qquad \text{(I-8-16)}$$

and then the fundamental differential equation becomes

$$f_1(\phi)\ g_1'(\phi) = R^2 \cos\phi. \qquad \text{(I-8-17)}$$

The functions $f_1(\phi)$ and $g_1(\phi)$ are again expressed as a linear combination (I-8-9) leading to the following form of the fundamental differential equation (I-8-17)

$$a_1a_2\sin\phi\cos\phi - a_1b_2\sin^2\phi + b_1a_2\cos^2\phi - b_1b_2\sin\phi\cos\phi + \\ + c_1a_2\cos\phi - c_1b_2\sin\phi = R^2\cos\phi,$$

or

$$\begin{aligned} a_1a_2 &= 0 \quad , \quad a_1a_2 - b_1b_2 = 0, \\ a_1b_2 &= 0 \quad , \quad b_1a_2 = 0 \\ c_1b_2 &= 0 \quad , \quad c_1a_2 = R^2. \end{aligned}$$

Thus

$$f_1(\phi) = c_1 = c$$

$$g_1(\phi) = \frac{R^2}{c} \sin \phi, \; g_1'(\phi) = \frac{R^2}{c} \cos \phi,$$

with the transformation formulae

$$\left.\begin{aligned} x &= k \cdot l, \\ y &= \frac{R^2}{k} \sin \phi, \end{aligned}\right\} \quad \text{(I-8-18)}$$

where

$$k = \alpha \cdot c \qquad \text{(I-8-19)}$$

The equations (I-8-18) define Lambert's equiareal cylindric projection.

The most important group of equiareal projections, from the practical point of view, is Euler's. Euler's projections are the equiareal mappings in which the image of the metagraticule constitutes an orthogonal grid. Without a loss of generality let us take direct aspects only. Thus, in Euler's projections the image of meridians and parallels is an orthogonal net of lines in the projection plane. The conditions of Euler's projections are

$$p = 1 \quad \text{and} \quad \varepsilon = 0. \qquad \text{(I-8-20)}$$

The orthogonality of parametric lines in the projection indicates that the scales along meridians and parallels are the principal scales. Thus, the conditions (I-8-20) can be written as

$$m = 1/n \quad , \quad \varepsilon = 0. \qquad \text{(I-8-21)}$$

Let us now introduce a new parameter, t, defined from the expression (I-5-16) by

$$t = \nu^2 = n^2 \cos^2\phi, \qquad \text{(I-8-22)}$$

which gives

$$n = \frac{\sqrt{t}}{\cos \phi} \quad \text{and} \quad m = \frac{\cos \phi}{\sqrt{t}}. \qquad \text{(I-8-23)}$$

Instead of the independent variable ϕ we shall use another variable, s defined as

$$s = \sin \phi \qquad \text{(I-8-24)}$$

and then

$$ds = \cos \phi \; d\phi. \qquad \text{(I-8-25)}$$

Substituting the equations (I-8-23), (I-8-24) and (I-8-25) into the formulae (I-5-15) we obtain

$$\left.\begin{aligned} x_s &= \frac{\cos \beta}{\sqrt{t}} \quad , \quad x_l = \sqrt{t} \sin \beta \\ y_s &= -\frac{\sin \beta}{\sqrt{t}} \quad , \quad y_l = \sqrt{t} \cos \beta \end{aligned}\right\} \qquad \text{(I-8-26)}$$

Conditions of differentiability $x_{sl} = x_{ls}$ and $y_{sl} = y_{ls}$ yield

$$
\begin{aligned}
-\frac{\sin\beta}{\sqrt{t}}\,\beta_l - \frac{\cos\beta}{2\sqrt{t^3}}\,t_l &= \sqrt{t}\cos\beta\,\beta_s + \frac{\sin\beta}{2\sqrt{t}}\cdot t_s,\\
-\frac{\cos\beta}{\sqrt{t}}\,\beta_l - \frac{\sin\beta}{2\sqrt{t^3}}\,t_l &= -\sqrt{t}\sin\beta\,\beta_s + \frac{\cos\beta}{2\sqrt{t}}\cdot t_s.
\end{aligned}
\tag{I-8-27}
$$

In order to eliminate β_l and t_s from the equation (I-8-27) the first equation is multiplied by cos β, the second by sin β and then they are subtracted giving

$$-\frac{1}{t^2}\cdot t_l = 2\,\beta_s. \tag{I-8-28}$$

In the same way, one can eliminate β_s and t_l by multiplying the first equation of (I-8-27) by sin β, the second by cos β and then adding them together

$$2\beta_l = -t_s. \tag{I-8-29}$$

The differentiation of the equation (I-8-28) with respect to l and of (I-8-29) with respect to s yields

$$2\beta_{sl} = -\frac{1}{t^2}\,t_{ll} + \frac{2}{t^3}\,(t_l)^2,$$

$$2\beta_{ls} = -t_{ss},$$

or

$$t^3 t_{ss} - tt_{11} + 2(t_1)^2 = 0. \tag{I-8-30}$$

The partial differential equation of the second order (I-8-30) represents the fundamental differential equation of Euler's projections. Superimposing some additional conditions on the function t, we can develop, by the integration of the fundamental equation, various types of Euler's projections.

For example, if we take a special case where t is a function of latitude only, or in our case $t = t(s)$, then

$$\left.\begin{array}{l} t_{ss} = 0 \quad , \quad t_s = -2c \quad , \quad t = -2cs + 2cK \\ \text{or finally} \\ t = 2c(K - s), \end{array}\right\} \tag{I-8-31}$$

where c and K are arbitrarily selected constants. The derived projection is Lambert's equiareal conic projection.

If we take another special case where t is a function of difference in longitude only, i.e.

$$t = t(1)$$

then the equation (I-8-30) becomes

$$-tt_{11} + 2(t_1)^2 = 0$$

with the solution

$$t = \frac{1}{c - Kl} . \qquad \text{(I-8-32)}$$

Projections which satisfy the last equation have the parametric grid consist of circular arcs. In a special limited case, when the constant K is equal to zero, we obtain Lambert's equiareal cylindric projection, the same projection that has already been developed in (I-8-18).

Generalization of developments of equiareal projections from the fundamental differential equation (I-8-1) requires further systematic study of the problem. The main difficulty is the lack of an organized series of different solutions of the undetermined system (I-8-1). Two of many solutions using the equations (I-8-2), derived by the author, were obtained by a more or less trial and error approach. It requires a mathematician with a deep insight into partial differential equations to develop a great number of practically important solutions.

II. OPTIMAL MAP PROJECTIONS

1. IDEAL AND BEST MAP PROJECTIONS

The problem of the determination of an optimal map projection of the sphere, or its portion onto a plane, can be extremely ambiguous unless the criteria for qualitative assessments of transformation formulae are clearly and rigorously defined. The definition of the problem is always the most reasonable starting point for its solution.

Meshcheryakov (1968) has suggested that the optimized map projection can belong to two distinct categories. They can be either ideal or the best transformation systems.

The concept and definition of ideal map projections was introduced to cartography by the most important of the Russian cartographers, V.V. Kavraiskii with the following words (Kavraiskii, 1959):

> "It is possible to find a map projection under a unique condition where, for example, the maximum deformation of distances for the whole mapped domain is as small as possible."

The problem of ideal map projections has not been solved. Even its mathematical definition has not yet been clearly given. Generally, for cartographers, the problem is too difficult from the mathematical point of view, and mathematicians only recently rediscovered mathematical cartography (Milnor, 1969). The problem of minimization of maximal distortion, which leads to ideal map projections, will be simply called the minimax problem. Although the existence and uniqueness of an ideal map projection was introduced by Kavraiskii and the existence mathematically proven by an American mathematician, J. Milnor (1969), the determination of a real ideal projection for an arbitrary domain of the sphere cannot be practically solved unless an infinite number of conceivable transformation systems are optimized and then compared, and this, in actuality, is an impossible task.

Meshcheryakov (1968) recommended an alternative solution for the problem: instead of trying to find an ideal map projection of a certain domain, cartographers should concentrate on the best projections. The best projection of a given class is the one in which deformations are at a minimum. In other words, knowing the requirements of map users and the shape and extent of the mapped territory, a cartographer can predetermine the class of transformation systems and then optimize it. The result will be the best map projection of the particular class. By comparing several of the easily optimized classes, or the best projections, one can make the final decision as to which of one of the best projections will be adopted for the mapping of the domain.

The criteria for the qualitative assessment of map projections, and thus the basis of the optimization process, will be explained in the subsequent sections. For a large number of transformation systems the optimization consists of the solution of a variational problem under the condition of extremum. The criteria leading to such solutions are called the criteria of the variational type (Meshcheryakov, 1968).

The optimization of conformal mappings was suggested by various authors, but the criterion which was adopted as the most realistic was defined by a Russian mathematician, Chebyshev (Kavraiskii, 1959). Chebyshev stated that the best conformal projections of a closed domain is the one for which the logarithm of the maximal scale is minimized. Thus, the Chebyshev conformal projections belong to the minimax type. The rigorous analytical determination of the Chebyshev projection for a domain defined by an arbitrary contour line is unknown. However, in practice, approximate solutions are feasible and for map users they are sufficiently accurate, as the author will show.

2. LOCAL QUALITATIVE MEASURES

Every transformation process of a closed domain of a regular surface onto another surface whose Gaussian curvatures differ will result in deformation of distances. By selecting special transformation functions, areas or angles may be preserved, but distances which are also intrinsic elements of the original surface will always undergo a certain change. Thus it is quite appropriate to adopt the deformation of distances as the basic parameter for the evaluation of map projections. Changes of angles and areas will be expressed as functions of deformation of distances in the principal directions.

The deformation of distances at a point, thus locally defined, was given by the formula (I-3-2)

$$v_s = k - 1, \qquad \text{(II-2-1)}$$

where k is the scale factor, a function of the position and the direction $k = (\phi, l, \alpha)$. There are however, some additional measures of deformations in mathematical cartography, some of which are more and the others less important. For example, the deformation can also be defined by the expression

$$v'_s = 1 - 1/k \qquad \text{(II-2-2)}$$

Chebyshev, Weber and Markov (Kavraiskii, 1959) have used the natural logarithm of the scale factor as the definition of deformation, i.e.

$$v_s'' = \ln k. \qquad \text{(II-2-3)}$$

The linear deformation for conical equiareal projections is sometimes defined by the expression

$$v_s''' = \frac{1}{2}(k^2 - 1), \qquad \text{(II-2-4)}$$

but this definition appears to be of lesser practical importance.

All measures of distortions of distances are functions of the scale factor, and are thus linearly dependent, since

$$k = 1 + v_s = \frac{1}{1 - v_s'} = e^{v_s''} = \sqrt{1 + 2v_s'''}\,, \qquad \text{(II-2-5)}$$

and thus they differ among themselves by the quantities of second order only. In other words, for the first order term it is irrelevant which definition of distortion is used.

From the author's point of view, the most natural measure of distortion is $v'' = \ln k$ since its optimization in the cases of conformal or equidistant projections automatically leads to a minimal distortion of areas, and in the case of equidistant and equiareal projections the optimization process guarantees minimal deformations of angles. Analogous results are not obtained with other definitions of linear distortion.

From the theory of distortions and the indicatrix of Tissot we know that at every regular point of a mapping domain the principal directions represent the directions of the extreme scales a and b and that every element of distortion can be easily expressed in terms of the principal scales. For example, the extreme angular distortion at a point in a non-conformal projection is given by the formula

$$\omega_o = \arc \sin ((a - b)/(a + b)). \qquad \text{(II-2-6)}$$

The deformation of areas, v_p, is defined by the equation

$$v_p = ab - 1. \qquad \text{(II-2-7)}$$

In 1861 an English astronomer, G.B. Airy, made the first significant attempt in cartography to introduce a qualitative measure for a combination of disortions. His measure of quality was designed to be an equivalent to the variance in statistics. It was defined (Airy, 1861), at first, by the expression

$$\varepsilon_A^2 = \left(\frac{a}{b} - 1\right)^2 + (ab - 1)^2, \qquad \text{(II-2-8)}$$

but later in the optimization process he used another version which may be called the mean quadratic deformation of distances

$$\varepsilon_A^2 = \frac{1}{2}\left(v_a^2 + v_b^2\right), \qquad \text{(II-2-9)}$$

where

$$v_a = a - 1 \quad \text{and} \quad v_b = b - 1. \qquad \text{(II-2-10)}$$

Airy's two definitions of the mean quadratic deformation (II-2-8) and (II-2-9) differ in the terms of third order only. We can take, for example, the first definition and transform it in the following way:

$$\left(\frac{a}{b} - 1\right)^2 + (ab - 1)^2 = \left(\frac{1 + v_a}{1 + v_b} - 1\right)^2 + [(1 + v_a)(1 + v_b) - 1]^2 =$$

$$\left(\frac{v_a - v_b}{1 + v_b}\right)^2 + (v_a + v_b + v_a v_b)^2 = 2\,[v_a^2 + v_b^2 + O\,(v_a^3 + v_b^3)].$$

The first term of the middle row in the last formula approximates the square of the maximal angular deformation (II-2-6) and the second term is the square of the deformation of areas (II-2-7).

In 1897, the German cartographer A. Klingatsch generalized the first mean quadratic deformation of Airy (II-2-8) by the introduction of arbitrarily selected weights for the accuracy of angles and areas. The mean quadratic deformation of Klingatsch is then computed by the weighted mean

$$\varepsilon_k^2 = \frac{\left(\frac{a}{b} - 1\right)^2 p_w + (ab - 1)^2 p_p}{p_w + p_p}, \qquad \text{(II-2-11)}$$

where p_w and p_p are positive dimensionless quantities called weights. Their numerical values can be arbitrarily varied to satisfy the specific requirements of users (Kavraiskii, 1959).

The measure of the quality of map projections by Airy (II-2-9) and the subsequent generalization by Klingatsch (II-2-11) uses only the principal scales and their deformations. A more realistic evaluation of the deformations of distances at a point was suggested by a German geodesist, W. Jordan, in 1896, whose mean square deformation was defined by the formula

$$\varepsilon_J^2 = \frac{1}{2\pi} \int_0^{2\pi} (k - 1)^2 \, d\alpha, \qquad \text{(II-2-12)}$$

where α is the direction angle usually reckoned from the first principal direction (Meshcheryakov, 1969).

Kavraiskii (1959) recommended a small modification of the mean square deformations of Airy and Jordan by the logarithmic definition of linear deformation (II-2-3). Such altered mean square deformations are called Airy-Kavraiskii and Jordan-Kavraiskii,

$$\varepsilon_{AK}^2 = \frac{1}{2} (\ln^2 a + \ln^2 b) \qquad \text{(II-2-13)}$$

and

$$\varepsilon_{JK}^2 = \frac{1}{2\pi} \int_0^{2\pi} \ln^2 k \, d\alpha \, . \qquad \text{(II-2-14)}$$

From a theoretical point of view there is a distinct difference between Airy's and Jordan's approaches. However, the optimization process using either measure of quality for mapping systems will lead to similar results whose differences scarcely justify Jordan's much more complicated measure (Kavraiskii, 1959).

3. QUALITATIVE MEASURES FOR DOMAINS

So far the qualitative measures of map projections have been locally defined, since when we use one of the formulae (II-2-9), (II-2-11), (II-2-12), (II-2-13) and (II-2-14) we can compute the mean square deformation of distances at a point. The evaluation and comparison of map projections of a closed domain point by point in applying one of the above mentioned expressions will generally lead to too many ambiguities with respect to the selection of points and their distribution over the mapped area. To alleviate these difficulties Airy (1861) introduced the mean square error of a domain

$$E_A^2 = \frac{1}{2A} \int_A (V_a^2 + V_b^2)\, dA, \qquad \text{(II-3-1)}$$

where the integration is extended over the whole area A of the domain.

With the logarithmic definition of distortion (II-2-3) the criterion of Airy becomes the criterion of Airy-Kavraiskii

$$E_{AK}^2 = \frac{1}{2A} \int_A (\ln^2 a + \ln^2 b)\, dA. \qquad \text{(II-3-2)}$$

The optimization process leading to the minimization of equation (II-3-2) will be called the optimization according to the Airy-Kavraiskii criterion.

If, instead of Airy's version of the mean square deformation, we take Jordan's formula (II-2-12) or the Jordan-Kavraiskii (II-2-14) and evaluate them for the whole domain we obtain

$$E_J^2 = \frac{1}{2\pi A} \int_A \int_0^{2\pi} (k - 1)^2 \, d\alpha \, dA, \qquad \text{(II-3-3)}$$

or

$$E_{JK}^2 = \frac{1}{2\pi A} \int_A \int_0^{2\pi} \ln^2 k \, d\alpha \, dA. \qquad \text{(II-3-4)}$$

The optimization process, using the last formula as the basis, is known as the optimization according to the Jordan-Kavraiskii criterion.

In his numerical minimizations of distortions the author will use the criterion of Airy-Kavraiskii (II-3-2).

From a theoretical point of view, the criterion of Jordan-Kavraiskii is certainly superior to the criterion of Airy-Kavraiskii since it takes deformations in all directions around a point and not only in the principal directions as in the Airy-Kavraiskii formula. Practically, however, the analytical evaluation of the integral (II-2-12) is often very difficult or even impossible and we have to approximate it by a numerical integration.

The integration of the mean square deformation for the whole area in both criteria has the same problem. The boundary contour line of a domain is very seldom defined by a 'nice' analytic expression. In the optimization of mappings of a hemisphere, for example, Airy (1861) and Young (1920) had an analytical definition of the boundary, but it is more usual to define the boundary by a polygon of discrete points. Even in the cases of an analytical definition of the boundary the analytical integration may be extremely difficult or impossible. To reduce the problem of integration, the author will use the practical procedure of numerical integration throughout the research.

Kavraiskii (1959) suggested a very simple summation of the individual mean square deformations evaluated at regular mesh points on the sphere. Young (1920) insisted on a 'better' summation taking a regular mesh on the projection plane. For relatively small sizes of mapping domains both meshes will produce more or less identical results. For larger domains the differences can be considerable. The author does not see why Young's mesh on the projection plane will yield more realistic results, and because its formation is numerically more complicated the preference will be given to a regular mesh on the original surface of the sphere.

4. OPTIMIZATION OF CONICAL PROJECTIONS

Conical map projections include those cartographic transformations in which the metagraticule on the sphere, a system of orthogonal coordinate lines, is projected onto a plane in the following way. The metameridians (η = const.) are transformed either into straight lines intersecting at a point or parallel straight lines. Metaparallels (ξ = const.) become either concentric circles with the centre at the intersection point of the projection of the metameridians, or they become parallel straight lines orthogonal to the projection of metameridians.

To clarify the exact meaning of the metagraticule, as named by Wray (1974), and its connection to the graticule, the set of geographic coordinates (latitude ϕ and difference in longitude l), let us refer to the figure II-4-1. The metapole, O, is usually selected to be the central point of a mapping domain or is determined so that the central line of the mapping territory becomes either metaparallel or the metaequator. Great circles passing through the metapole, O, and its antipodal point, O', are called metameridians. The position of a metameridian is fixed by an angle, η, called the metalongitude, reckoned clockwise from the geographical central meridian. Thus, the geographic meridian through the metapole is the initial metameridian. Orthogonal circular curves to metameridians are called metaparallels. They are defined by the metalatitude, ξ, an arc length on the unit sphere between the metaequator and the metaparallel in question.

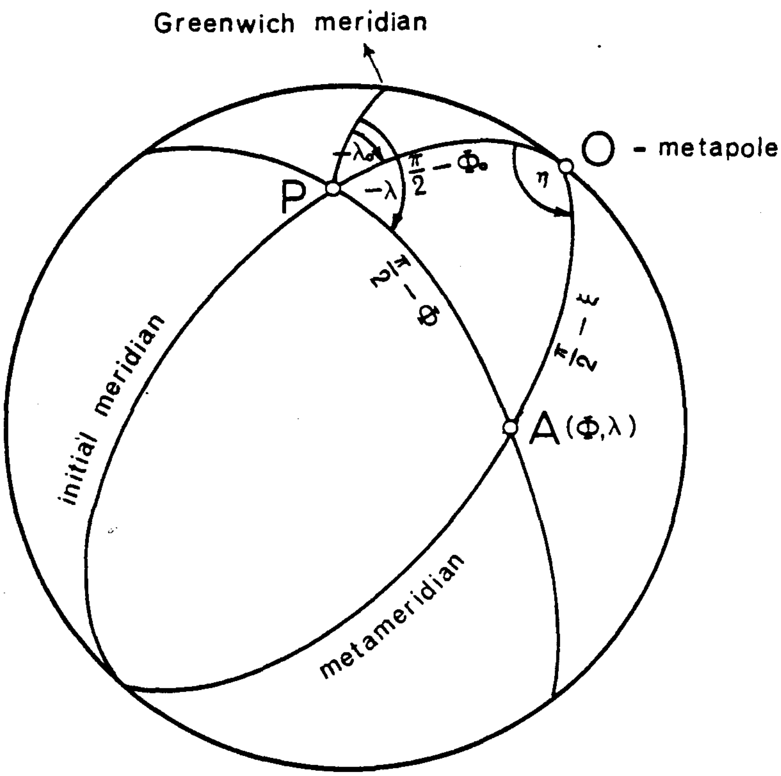


Figure II-4-1 Graticule and metagraticule

It is clear from the figure that the metagraticule represents a coordinate system similar to the geographic graticule except that it is generally displaced on the sphere with respect to the graticule. According to the geographic latitude of the metapole, ϕ_o, in conical projections we can distinguish three different aspects:

(1) $\phi_o = {}^{\pi}/2$ - direct aspect,

(2) $0 < \phi_o < {}^{\pi}/2$ - oblique aspect, and

(3) $\phi_o = 0$ - transverse aspects.

Because the metagraticule represents an invariant frame for a projection system, the first step in computation is the transformation of the geographic coordinates into the metacoordinates. The second step is the computation of plane coordinates.

$$(\phi,l) \rightarrow (\xi,\eta) \rightarrow (x,y)$$

The metacoordinates (ξ,η) of a point defined by its geographic coordinates (ϕ,l) are obtained by the formulae of spherical trigonometry in the spherical triangle OPA.

$$\sin\xi = \sin\phi_o \sin\phi + \cos\phi_o \cos\phi \cos(\lambda_o-\lambda),$$

and (II-4-1

$$\tan\eta = \frac{\cos\phi \sin(\lambda_o - \lambda)}{\sin\phi \cos\phi_o - \sin\phi_o \cos\phi \cos(\lambda_o-\lambda)}$$

For the transverse aspect ($\phi_o = 0$) the last formulae are reduced to:

$$\left.\begin{aligned} \sin\xi &= \cos\phi\,\cos(\lambda_o-\lambda), \\ \tan\eta &= \frac{\sin(\lambda_o-\lambda)}{\tan\phi}. \end{aligned}\right\} \quad \text{(II-4-2)}$$

The final transformation in rectangular coordinates

$$x = x(\xi,\eta)\;,\; y = y(\xi,\eta), \quad \text{(II-4-3)}$$

or in polar coordinates

$$\gamma = \gamma(\xi,\eta),\; \rho = \rho(\xi,\eta). \quad \text{(II-4-4)}$$

yields the required cartographic projection.

The conical projections are subdivided into three subgroups of conic, azimuthal and cylindric map projections. The general formulae of conic projections are

$$\left.\begin{aligned} x &= \rho\sin\gamma\;,\; y = -\rho\cos\gamma, \\ \gamma &= c\eta\quad,\quad \rho = \rho(\xi) \end{aligned}\right\} \quad \text{(II-4-5)}$$

where

The quantity c is an arbitrary positive constant whose numerical value is usually smaller than one. When the constant c is exactly equal to one, we obtain azimuthal projections. Cylindric map projections are defined by the general expressions

$$x = k\eta \ , \ y = y(\xi), \tag{II-4-6}$$

where k is an arbitrary positive constant.

The principal scales coincide with the scales along the parametric curves and for conic projections they are

$$m = \frac{d\rho}{d\phi} \ , \ n = c \frac{\rho}{\cos\phi}. \tag{II-4-7}$$

For cylindric projections they become

$$m = \frac{dy}{d\phi} \ , \ n = k/\cos\phi. \tag{II-4-8}$$

For a reader unfamiliar with mathematical cartography the appendix 1 contains typical grids of various conical map projections, i.e. conic, cylindric and azimuthal mappings.

The general formulae of conic and cylindric projections, (II-4-5) and (II-4-6) respectively, clearly indicate that there is an infinite number of conceivable conic projections. With a proper choice of functions $\rho = \rho(\xi)$ and $y = y(\xi)$ we can derive conformal, equiareal or equidistant or an arbitrary conical projection. All these derivations include the determination of two parameters. One of them is the initial constant, c or k, and the second parameter is the constant of integration in the solution of a particular differential equation: the equation which specifies the character of linear distortion.

The selection of the two parameters is made in such a way that the deformations of scale are as small as possible. In other words, an optimization process which satisfies one of the criteria, Airy-Kavraiskii (II-3-2) or Jordan-Kavraiskii (II-3-4), will determine the best choice of projection constants.

The most important conical projections are:

Lambert conformal conic projection:

$$\gamma = c_1 \eta \ , \ \rho = c_2 e^{-c_1 q}, \qquad \text{(II-4-9)}$$

where c_2 is the integration constant and q is the isothermic latitude obtained by the expression (I-7-5), which, in the case of metalatitude becomes

$$q = \ln \tan (\pi/4 + \xi/2). \qquad \text{(II-4-10)}$$

Lambert equiareal conic projection:

$$\gamma = c_1\eta \quad , \quad \rho = \sqrt{\frac{2}{c_1}(c_2 - \sin\xi)}. \qquad \text{(II-4-11)}$$

Equidistant conic projection:

$$\gamma = c_1\eta \quad , \quad \rho = c_2 - \xi. \qquad \text{(II-4-12)}$$

Azimuthal projections may be considered as a special case of conic projections in which the first constant, c, has a value of unity. Thus, the formula (II-4-9), (II-4-11) and (II-4-12) with $c_1 = 1$ represent azimuthal conformal, equiareal and equidistant projections respectively.

The most important cylindric projections are:

Mercator projection:

$$x = k\eta \quad , \quad y = kq. \qquad \text{(II-4-13)}$$

Lambert equiareal cylindric projection:

$$x = k\eta \quad , \quad y = \frac{1}{k}\sin\xi. \qquad \text{(II-4-14)}$$

Plate carree projection (equidistant):

$$x = k\eta \quad , \quad y = \xi. \qquad \text{(II-4-15)}$$

Urmaev's cylindric projection:

$$x = k\eta \quad , \quad y = a_1\xi + a_2\xi^3 + a_3\xi^5, \qquad \text{(II-4-16)}$$

where a_1, a_2, a_3 are constants which can be directly computed by defining the scale on three different metalatitudes ξ_1, ξ_2, ξ_3.

$$\left.\begin{aligned} a_1 + 3\, a_2\xi_1^2 + 5\, a_3\xi_1^4 &= m_1, \\ a_1 + 3\, a_2\xi_2^2 + 5\, a_3\xi_2^4 &= m_2, \\ a_1 + 3\, a\, \xi_3^2 + 5\, a_3\xi_3^4 &= m_3. \end{aligned}\right\} \qquad \text{(II-4-17)}$$

The constants a_i $(i = 1,2,3)$ can also be determined through optimization by applying one of the criteria of the best projections.

Generally speaking, the optimization of conical projections is a relatively simple problem that was solved at the end of the nineteenth century. The detailed description of various optimization methods can be found in (Kavraiskii, 1959), where a particular emphasis was given to the optimization of Lambert's conformal conic projection by the method of least squares. The optimization of one or two constants leads to a system of one or two equations with the same number of unknowns. The only small problem is the mathematical definition of an optimization criterion in terms of the required constants. The author expands the optimization process including the simultaneous determination of the best metagraticule for an arbitrary shaped territory and a specific map projection. This will enlarge the mathematical structure by two more unknowns, the latitude and longitude of the metapole.

Thus, in the most generalized optimization of map projections, one of the selected optimization criteria would have to be expressed as a function of the transformation constants and the metagraticule. Then the numerical approximation process will yield the best values of constants and the best metagraticule for the particular domain and the selected transformation system.

5. OPTIMIZATION OF MODIFIED PROJECTIONS

A modification of map projections is the process of obtaining new cartographic systems from already existing map projections. Let us assume that the rectangular coordinates (x,y) are expressed in terms of the metagraticule (ξ,η).

$$x = x(\xi,\eta) \quad , \quad y = y(\xi,\eta). \qquad \text{(II-5-1)}$$

This transformation represents the initial map projection which will be modified by the equations

$$X = A\ x(u,v) \ , \ Y = B\ y(u,v) \qquad \text{(II-5-2)}$$

where

$$u = u(\xi,\eta) \quad , \quad v = v(\xi,\eta), \tag{II-5-3}$$

and A and B are arbitrary constants. The functions x and y in (II-5-2) are identical to the initial functions (II-5-1) except that they are expressed in terms of new variables (u,v).

To simplify the modification process, the most general modification of the metagraticule (II-5-3) will be restricted to those modifications in which only the original metameridians (η = const.) are transformed into modified metameridians (v = const.) and the original metaparallels (ξ = const.) become modified metaparallels (u = const.). In this case, the transformation equations (II-5-3) are simplified to

$$u = u(\xi) \quad , \quad v = v(\eta). \tag{II-5-4}$$

Both of these transformations must be regular, i.e. the Jacobian determinant of the modified transformation must be different from zero at every point of the mapping domain,

$$\frac{\partial(u,v)}{\partial(\xi,\eta)} \neq 0. \tag{II-5-5}$$

In order to investigate the distortion parameters of the modified map projections, we must compute the elements of the metric tensor

$$G_{11} = X_\xi^2 + Y_\xi^2,\ G_{12} = X_\xi X_\eta + Y_\xi Y_\eta,\ G_{22} = X_\eta^2 + Y_\eta^2. \qquad \text{(II-5-6)}$$

The partial derivatives of the new coordinates (X,Y) with respect to the independent variables (ξ,η) can be expressed in the following way

$$\left.\begin{aligned} X_\xi &= A \cdot x_u \cdot \frac{du}{d\xi}, & X_\eta &= A \cdot x_v \cdot \frac{dv}{d\eta}, \\ Y_\xi &= B \cdot y_u \cdot \frac{du}{d\xi}, & Y_\eta &= B \cdot y_v \cdot \frac{dv}{d\eta}. \end{aligned}\right\} \qquad \text{(II-5-7)}$$

Then the elements of the metric tensor (II-5-6) become

$$\left.\begin{aligned} G_{11} &= (A^2 x_u^2 + B^2 y_u^2)\left(\frac{du}{d\xi}\right)^2, \\ G_{12} &= (A^2 x_u x_v + B^2 y_u y_v)\ \frac{du}{d\xi} \cdot \frac{dv}{d\eta}, \\ G_{22} &= (A^2 x_v^2 + B^2 y_v^2)\left(\frac{dv}{d\eta}\right)^2, \end{aligned}\right\} \qquad \text{(II-5-8)}$$

with the Jacobian determinant

$$G = A^2 B^2 (x_v y_u - x_u y_v)^2 \left(\frac{du}{d\xi}\right)^2 \left(\frac{dv}{d\eta}\right)^2. \qquad \text{(II-5-9)}$$

Taking again a unit sphere as the original surface the distortion parameters (the scale along metameridian m, scale along metaparallel n, parametric angle Θ, and the scale of areas p) are

$$m = \sqrt{G_{11}} , \tag{II-5-10}$$

$$n = \sec u \sqrt{G_{22}} , \tag{II-5-11}$$

$$\sin \Theta = \sqrt{G} / \sqrt{G_{11} G_{22}} , \tag{II-5-12}$$

$$p = \sec u \sqrt{G} . \tag{II-5-13}$$

When the original mapping (II-5-1) is conformal then its modification would preserve conformality if $m = n$ and $\Theta = \pi/2$. From the expressions (II-5-10), (II-5-11), (II-5-12) it can easily be seen that only an identity mapping with constants $A = B$ will result again in a conformal map projection. In other words, a real modification of a transformation system cannot retain conformality. If the original projection is conformal its modified version can never be conformal.

When the original mapping is equiareal, its modification, in order to be also equiareal, must satisfy the required condition that the scale of the area is equal to unity, i.e.

$$AB(x_v y_u - x_u y_v) \frac{du}{d\xi} \cdot \frac{dv}{d\eta} \sec \xi = 1,$$

or

$$\frac{dv}{d\eta} = \frac{\cos\xi}{AB(x_v y_u - x_u y_v)} \frac{d\xi}{du}$$

Since for the original equiareal mapping $x_v y_u - x_u y_v = \cos u$ the last differential equation becomes

$$\frac{dv}{d\eta} = \frac{1}{AB} \frac{\cos\xi \, d\xi}{\cos u \, du} \qquad \text{(II-5-14)}$$

The resulting differential equation can be satisfied at every point of the mapping domain only if both sides have a constant value. Thus

$$\frac{dv}{d\eta} = C_n$$

or

$$v = C_n \eta + K_1 \qquad \text{(II-5-15)}$$

At the same time the transformation of equation (II-5-14) yields

$$\cos u \, du = \frac{1}{ABC_n} \cos\xi \, d\xi \ ,$$

and its integration

$$\sin u = \frac{1}{ABC_n} \sin \xi + K_2. \tag{II-5-16}$$

Assuming that both coordinate systems (ξ,η) and (u,v) have the same origin, i.e. for $\xi = 0$ also $u = 0$ and for $\eta = 0$ also $v = 0$, both integration constants K_1 and K_2 become zero and the resulting modified curvilinear coordinates become

$$\sin u = \frac{1}{ABC_n} \sin \xi, \; v = C_n\eta. \tag{II-5-17}$$

The first affine transformation of an equiareal conic projection had already been suggested in 1913 by Zinger. His modified conic equiareal projection was computed by the formulae (Kavraiskii, 1959)

$$X = \mu x \quad , \quad Y = \frac{1}{\mu} y \; , \tag{II-5-18}$$

where (x,y) are the coordinates of the equiareal conic projection and μ is an arbitrary constant. A more serious investigation of modified equiareal projections was initiated by Siemon in 1938. However, the most thorough study of modifications has been made by Wagner, (Wagner, 1941 and Wagner, 1962). The author has adopted his theory of modifications with the notation suggested by Franvula in his dissertation, (Franvula, 1971).

Since constants A and B in the first equation of (II-5-17) are arbitrarily selected values, Wagner (1962) recommended that they be composed of three constants C_k, C_m, C_n in the following way

$$A = C_k/\sqrt{C_m C_n} \quad , \quad B = 1/C_k\sqrt{C_m C_n} \; . \qquad \text{(II-5-19)}$$

Then the transformation formulae (II-5-2) and (II-5-4) become

$$X = \frac{C_k}{\sqrt{C_m C_n}}\, x(u,v) \quad , \quad Y = \frac{1}{C_k\sqrt{C_m C_n}}\, y(u,v) \; , \qquad \text{(II-5-20)}$$

and

$$\sin u = C_m \sin\xi \quad , \quad v = C_n \eta . \qquad \text{(II-5-21)}$$

Thus, if the original mapping (II-5-1) is equiareal the modification by the expressions (II-5-20) and (II-5-21) also yields an equiareal transformation.

The quantities C_k, C_m, C_n are positive constants whose numerical values were subjectively selected in Wagner's work. Frančula (1971), however, optimized several map projections according to the criterion of Airy by varying the values of constants. He used electronic computers and a trial-and-error method to determine the constants in such a way that Airy's measure of the quality E_A was minimized for domains of the whole earth and the individual hemispheres. The result of his investigations are given in (Francula, 1971). When the original equiareal property of the projection (II-5-1) is not to be retained by the modified system, Wagner introduced a new constant, C_a, which regulates the distortion of areas. The transformation equations are then

$$X = \frac{C_a C_k}{\sqrt{C_m C_n}} \, x(u,v) \quad , \quad Y = \frac{1}{C_k \sqrt{C_m C_n}} \, y(u,v) \ , \qquad \text{(II-5-22)}$$

For non-equiareal projections the modification of the metagraticule can be simplified, and instead of equations (II-5-21) we can use another set of expressions

$$u = C_m \xi \quad , \quad v = C_n \eta \ . \qquad \text{(II-5-23)}$$

From many equiareal projections the author has selected the following transformations which can be modified and their constants optimized by the criterion of Airy-Kavraiskii.

Sanson's projection:

Sanson's projection is defined as a sinusoidal equiareal projection which satisfies three conditions: (i) scale along the symmetric central metameridian is constant, (ii) all other meridians are sinusoidal curves, and (iii) the pole is projected as a point. The original equations of the projection are:

$$X = \eta \cos \xi \quad \text{and} \quad Y = \xi \ , \qquad \text{(II-5-24)}$$

and the modified versions become

$$X = \frac{C_k C_n}{\sqrt{C_m C_n}} \eta \sqrt{1 - C_m^2 \sin^2 \xi} \ ,$$

$$Y = \frac{1}{C_k \sqrt{C_m C_n}} \text{arc} \sin(C_m \sin \xi) \ . \qquad \text{(II-5-25)}$$

Mollweide's projection:

The projection is equiareal and maps the hemisphere into a circle of radius of $\sqrt{2}$, metameridians are symmetric ellipses with respect to the central meridian which is a straight line. The pole is projected as a point.

$$X = \frac{2\sqrt{2}}{\pi} \eta \cos \Psi \ , \quad Y = \sqrt{2} \sin \Psi \ ,$$

where (II-5-26)

$$2 \Psi + \sin \Psi = \pi \sin \xi \ .$$

The modified equations of the projection are

$$X = \frac{2\sqrt{2}}{\pi} \frac{C_k C_n}{\sqrt{C_n C_m}} \eta \cos\psi \quad , \quad Y = \frac{\sqrt{2}}{C_k \sqrt{C_n C_m}} \sin\psi \; , \qquad \text{(II-5-27)}$$

where

$$2\psi + \sin\psi = \pi\, C_m \sin\xi \; . \qquad \text{(II-5-28)}$$

<u>Hammer's projection</u>:

This projection was obtained by an affine transformation of the Lambert azimuthal equiareal projection.

$$x = 2\,\delta \sin\alpha \; , \; y = 2 \sin\frac{\sigma}{2} \cos\alpha \; , \qquad \text{(II-5-29)}$$

where

$$\cos\sigma = \cos\frac{\eta}{2} \cos\xi \; , \; \cos\alpha = \frac{\sin\xi}{\sin\sigma} \; . \qquad \text{(II-5-30)}$$

The modified equations become

$$X = \frac{2 C_k C_n}{\sqrt{C_n C_m}} \sigma \sin\alpha \; , \; Y = \frac{2}{C_k \sqrt{C_n C_m}} \sin\frac{\sigma}{2} \cos\alpha \; , \qquad \text{(II-5-31)}$$

where

$$\cos\delta = \cos\frac{C_n \eta}{2} \sqrt{1 - C_m^2 \sin^2\xi} \quad , \quad \cos\alpha = \frac{C_m \sin\xi}{\sin\delta} \; . \qquad \text{(II-5-32)}$$

<u>Eckert's IV projection (elliptical)</u>:

The original equations of the projection are

$$x = \eta\,(1 + \cos\psi) \; , \; y = \pi \sin\psi \; , \qquad \text{(II-5-33)}$$

where

$$\Psi + 2 \sin\Psi + \frac{1}{2} \sin 2\Psi = \left(2 + \frac{\pi}{2}\right) \sin\xi \; . \qquad \text{(II-5-34)}$$

The modified version of the equation is

$$X = C_k \sqrt{\frac{C_n}{C_m}}\, \eta\,(1 + \cos\psi) \quad \text{and} \quad Y = \frac{\pi}{C_k \sqrt{C_m C_n}} \sin\psi, \qquad \text{(II-5-35)}$$

where

$$\psi + 2 \sin\psi + \frac{1}{2} \sin 2\psi = \left(2 + \frac{\pi}{2}\right) C_m \sin\xi \; . \qquad \text{(II-5-36)}$$

6. OPTIMIZATION AND THE METHOD OF LEAST SQUARES

Let us adopt the criterion of Airy-Kavraiskii (II-3-2) as the basis for the optimization of map projections,

$$E_{AK}^2 = \frac{1}{2A} \int_A (\ln^2 a + \ln^2 b)\, dA = \min.$$

To simplify the writing we can introduce the abbreviations

$$\ln a = v_a \quad \text{and} \quad \ln b = v_b \ , \tag{II-6-1}$$

and then the criterion becomes

$$E_{AK}^2 = \frac{1}{2A} \int_A (v_a^2 + v_b^2)\, dA = \min \ , \tag{II-6-2}$$

where A is the total area of the mapping domain and the integration is extended to the whole domain. There are very few domains in cartography which can be analytically defined, but even for these cases it is often difficult, if not impossible, to perform integration analytically. To alleviate this problem the integral in the last formula will be approximated by the finite summation

$$\hat{E}^2{}_{AK} = \frac{1}{2A} \sum_{i=1}^{n} [(v_a)_i^2 + (v_b)_i^2] \, \Delta A_i = \min \, , \qquad (II-6-3)$$

where $\hat{E}_{AK}$ is the approximation of E_{AK}, ΔA is a small but finite portion of the domain, n is the number of area elements covering the whole domain, and the distortion parameters (v_a, v_b) are numerically evaluated at the central point of each element of the area, ΔA. To introduce regularity into the computation, the mapping domain is covered by the mesh of meridians and parallels which subdivide the domain into a large number of spherical trapezoids. The area of such a trapezoid limited by two parallels, ϕ_1 and ϕ_2, and two meridians, λ_1 and λ_2, is obtained by the formula (Sigl, 1977)

$$\Delta A = 2R^2(\lambda_2 - \lambda_1)\sin\frac{1}{2}(\phi_2 - \phi_1)\cos\frac{1}{2}(\phi_2 + \phi_1) \, , \qquad (II-6-4)$$

where R is the radius of the sphere. For a small distance between parallels we can assume that

$$\sin \frac{1}{2}(\phi_2 - \phi_1) = \frac{1}{2}(\phi_2 - \phi_1) \, , \qquad (II-6-5)$$

and with the abbreviations

$$\phi_2 - \phi_1 = \Delta\phi \text{ and } \lambda_2 - \lambda_1 = \Delta\lambda \, , \qquad (II-6-6)$$

for a unit sphere (R=1) formula (II-6-4) becomes

$$\Delta A_i = \Delta\phi \ \Delta\lambda \ \cos\phi_i \ , \qquad \text{(II-6-7)}$$

where ϕ_i is the latitude of the central point of ΔA_i.

Further simplification is achieved if the differences between meridians and parallels are kept constant for the domain. In that case

$$\Delta\phi \cdot \Delta\lambda = K \qquad \text{(II-6-8)}$$

and the area becomes

$$\Delta A_i = K \cos\phi_i \ . \qquad \text{(II-6-9)}$$

Substituting the last expression into the criterion (II-6-3) we have

$$E^2{}_{AK} = \frac{K}{2A} \sum_{i=1}^{n} [(v_a)_i^2 + (v_b)_i^2] \cos\phi_i = \min \ , \qquad \text{(II-6-10)}$$

and since K/2A is a constant, the most suitable non-conformal projection of a closed domain is obtained by the optimization of the expression

$$\sum_{i=1}^{n} \left[(v_a)_i^2 + (v_b)_i^2 \right] \cos\phi_i = \min , \qquad \text{(II-6-11)}$$

with the summation sign including all trapezoids which approximate the domain. The minimization of the above expression is nothing but a least squares problem. The unknown parameters of a projection, metagraticule and various constants, are to be determined in such a way that the sum of the squares of distortions multiplied by the cosine functions of the corresponding latitudes is at a minimum.

The method of least squares has been known to mathematical cartography for some time. The first suggestion that the theory for the determination of constants in the Lambert conformal projection be applied, was made in 1913 by Zinger (Kavraiskii, 1959). In 1934 Kavraiskii elaborated the evaluation of all constants of integration in the normal aspect of conical, i.e. conic, azimuthal and cylindric projections by the method of least squares. In 1953 Urmaev solved the problem of a symmetric Chebyshev projection by least squares. The result was a conformal map projection for a domain symmetric with respect to the central meridian, the projection which did not perfectly satisfy the boundary conditions, but where the deviations from the boundary requirements squared and added together gave a minimum. Finally, in 1977 Tobler suggested a new map projection with minimized distance errors. As a result he did not obtain a pair of analytic transformation functions $x = x(\xi,\eta)$, $y = y(\xi,\eta)$ but numerical values of plane coordinates (x,y) for points whose positions on the sphere were known. If the known points are nicely and evenly distributed over the mapping domain it is not difficult to introduce a numerical interpolation procedure to transform additional points of the sphere into the plane coordinate system. The determination of plane coordinates by the method of least squares in Tobler's approach corresponds to the determination of the most probable values of coordinates in the least squares adjustment of geodetic control survey nets by the method of trilateration (Mikhail, 1976).

In addition to the briefly described applications of the method of least squares in mathematical cartography, the author is suggesting further generalization of the optimization process including the determination of the most suitable metagraticule as well as other constants and parameters of projections. The attempt will also be made to optimize modified map projections according to the criterion of Airy-Kavraiskii (II-6-11). The method of least squares will be applied in the following way. The distortion elements v_a and v_b must be expressed as functions of unknown parameters:

metagraticule and modification constants. If we adopt the most suitable notation, the matrix notation, for the method of least squares, the functional connection between the distortion elements and unknown parameters becomes

$$\mathfrak{v} = \mathfrak{F} (\mathfrak{c}) , \qquad \text{(II-6-12)}$$

where $\mathfrak{v}$ is a column vector of 2n distortion elements, $\mathfrak{F}(\mathfrak{c})$ is a column vector of 2n functions and c is a vector of unknown parameters whose number must be smaller than 2n, i.e. smaller than twice the number of approximation trapezoids of the domain.

$$\mathfrak{v} = \begin{bmatrix} (v_a)_1 \\ (v_b)_1 \\ \vdots \\ \vdots \\ (v_a)_n \\ (v_b)_n \end{bmatrix} \text{ and } \mathfrak{c} = \begin{bmatrix} \phi_o \\ \lambda_o \\ c_a \\ c_k \\ c_m \\ c_n \end{bmatrix} . \qquad \text{(II-6-13)}$$

If we introduce a strictly diagonal matrix $\mathfrak{G}$ of 2n by 2n dimensions whose elements are the cosines of the latitudes listed in the following way,

$$\mathcal{P} = \begin{bmatrix} \cos\phi_1 & 0 & . & 0 & 0 \\ 0 & \cos\phi_1 & . & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & . & \cos\phi_n & \\ 0 & 0 & . & 0 & \cos\phi_n \end{bmatrix}, \qquad \text{(II-6-14)}$$

the basic condition of an optimized map projection according to the criterion of Airy-Kavraiskii (II-6-11) becomes

$$\boldsymbol{v}^T \mathcal{P} \boldsymbol{v} = \min. \qquad \text{(II-6-15)}$$

In this form, the equation (II-6-15) is the easily recognizable fundamental requirement of least squares. In the classical application of least squares for the adjustment of physical measurements, the vector $\boldsymbol{v}$ represents an array of residual corrections of measurements and $\mathcal{P}$ is the weight matrix. In our case the weight matrix can also be modified. If we feel that certain points of the mapping domain are more important than others, we can assign higher weights to the more valuable points according to some empirical rule. For example, the density of population can serve as the weighting basis. In that case more populated areas will have smaller deformations than the less populated portions of the domain. The most appropriate selection of weights is a very difficult problem and requires special attention and very careful research. The author will not dwell further on the problem since it does not directly belong to an objective optimization process. In this work the weight matrix will always be defined by the matrix (II-6-14), except in one case where the Lambert Conformal Conic projection is optimized according to the population density as the weighting bases.

In order to apply the method of least squares the distortion elements $v_a = \ln(a)$ and $v_b = \ln(b)$ must be expressed in terms of the unknown parameters, $\mathcal{C}$. This is probably the main hindrance in the whole process. From the theory of distortions in the first chapter, it is known that the semi-axes of the indicatrix of Tissot are computed in the following way:

$$\left.\begin{array}{l} g_{11} = x_\xi^2 + y_\xi^2,\ g_{12} = x_\xi x_\eta + y_\xi y_\eta,\ g_{22} = x_\eta^2 + y_\eta^2 , \\[2ex] \sqrt{g} = x_\eta y_\xi - x_\xi y_\eta,\ \sin\theta = \dfrac{\sqrt{g}}{\sqrt{g_{11} g_{22}}} ,\ m^2 = g_{11} ,\ n^2 = g_{22}\sec^2\xi , \\[2ex] A^2 = m^2 + n^2 + 2mn\sin\theta ,\ B^2 = m^2 + n^2 - 2mn\sin\theta , \\[2ex] a = \dfrac{1}{2}(A + B) ,\ b = \dfrac{1}{2}(A - B) . \end{array}\right\} \quad \text{(II-6-16)}$$

The mathematical model (II-6-12) is generally nonlinear and to be used in the least squares process we shall linearize it by taking the Taylor series

$$\mathcal{V} = \mathcal{F}(\mathcal{C}^\circ + \Delta\mathcal{C}) = \mathcal{F}(\mathcal{C}^\circ) + \left.\frac{\partial \mathcal{F}}{\partial \mathcal{C}}\right|_\circ \Delta\mathcal{C} + \ldots. \quad \text{(II-6-17)}$$

where $\mathcal{C}^\circ$ is the vector of approximate values of unknowns. Taking only the first two terms in this expansion, the vector of corrections to the approximations, $\Delta\mathcal{C}$, will be obtained by the method of least squares. The vector of unknown parameters, $\mathcal{C}$, is then

$$\mathcal{C} = \mathcal{C}^\circ + \Delta\mathcal{C} \quad \text{(II-6-18)}$$

When the approximate values of parameters are close to the solution, the correction vector is small and all terms of second or higher order of $\Delta\mathcal{C}$ in (II-6-17) can be neglected as being practically insignificant. The vector of approximations is usually determined from previous experience. It is particularly important to establish the position of the metagraticule relatively well. The better the approximations the better the end results will be. The method of least squares can also be used iteratively. After the first computation, the vector $\mathcal{C}$ can be reentered into the computation as the first approximation, and the method of least squares will then result in smaller corrections, $\Delta\mathcal{C}$, assuming the convergence of the process. Divergence indicates too much an error in the initial approximation vector or the incorrect formulation of the mathematical model (II-6-17).

Denoting by

$$\mathcal{V}^{\circ} = \mathcal{F}(\mathcal{C}^{\circ}) \quad \text{and} \quad \mathcal{B} = \left.\frac{\partial \mathcal{F}}{\partial \mathcal{C}}\right|_{\circ} \tag{II-6-19}$$

the mathematical model (II-6-17) becomes a typical case of Newton's method

$$\mathcal{V} = \mathcal{B} \cdot \Delta\mathcal{C} + \mathcal{V}^{\circ} , \tag{II-6-20}$$

and the fundamental condition of least squares (II-6-15) $\mathcal{V}\,\mathcal{P}\,\mathcal{V}$ = min is obtained when

$$\frac{\partial\, \mathcal{V}^{T}\mathcal{P}\mathcal{V}}{\partial \mathcal{C}} = 0 . \tag{II-6-21}$$

Since

$$\mathscr{v}^T \mathscr{P} \mathscr{v} = (\mathscr{B} \cdot \Delta \mathscr{c} + \mathscr{v}^\circ)^T \mathscr{P} (\mathscr{B} \cdot \Delta \mathscr{c} + \mathscr{v}) =$$

$$= \Delta \mathscr{c}^T \mathscr{B}^T \mathscr{P} \cdot \mathscr{B} \cdot \Delta \mathscr{c} + 2 \Delta \mathscr{c}^T \mathscr{B}^T \mathscr{P} \cdot \mathscr{v}^\circ + \mathscr{v}^{T\circ} \mathscr{P} \mathscr{v}^\circ ,$$

the partial differentiation of the above matrix equation with respect to C yields

$$\frac{\partial \mathscr{v}^T \mathscr{P} \mathscr{v}}{\partial \mathscr{c}} = 2 \mathscr{B}^T \mathscr{P} \mathscr{B} \Delta \mathscr{c} + 3 \mathscr{B}^T \mathscr{P} \mathscr{v}^\circ = 0 ,$$

or

$$\mathscr{N} \cdot \Delta \mathscr{c} + \mathscr{u} = 0 , \qquad \text{(II-6-22)}$$

where

$$\mathscr{N} = \mathscr{B}^T \mathscr{P} \cdot \mathscr{B} \quad \text{and} \quad \mathscr{u} = \mathscr{B}^T \mathscr{P} \cdot \mathscr{v}^\circ . \qquad \text{(II-6-23)}$$

Linear equations (II-7-22) are called the normal equations with N being a non-singular symmetric matrix with respect to the principal diagonal.

The solution of the system (II-6-22), found using Cholesky method, yields the vector of corrections

$$\Delta \mathfrak{c} = -\mathfrak{N}^{-1} \cdot \mathfrak{u}, \qquad \text{(II-6-24)}$$

and then, finally, by equations (II-6-18) the required unknowns.

After the determination of the unknowns, the measure of distortion Airy-Kavraiskii is obtained by the formula

$$E_{AK}^2 = \frac{K}{2A} \, \mathfrak{v}^T \mathfrak{P} \mathfrak{v}, \qquad \text{(II-6-25)}$$

where for a regular mesh the quantity K is the product of the differences in latitude and longitude of the mesh points, i.e.

$$K = \Delta\phi \cdot \Delta\lambda \, . \qquad \text{(II-6-26)}$$

The method of least squares has been by far the most predominant approximation technique in many scientific fields. It yields a unique solution by a very general computation algorithm which can be applied even in the most complicated mathematical models. If the mathematical models are linear or linearized, the approximation algorithm consists of the solution of linear equations whose number is equal to the number of unknowns. If the resulting vector of corrections $\Delta\mathcal{C}$, is small, i.e. we are dealing with close approximations, the first application of the method will give sufficiently good results. For larger values of $\Delta\mathcal{C}$, the computed unknowns serve only as improved approximations and the method of least squares is iteratively applied until the difference between two successive computations of $\Delta\mathcal{C}$ are practically negligible.

The main difficulty in the application of least squares for the optimization of cartographic mappings is the linearization of the mathematical model (II-6-12) by the Newton method. Analytical expressions for the deformation elements $v_a = \ln a$ and $v_b = \ln b$ in terms of the vector of unknown parameters $\mathcal{C}$ are relatively complicated and their partial differentiation with respect to the unknowns are cumbersome. The problem can be largely simplified by the usage of one of the available packages for a non-linear least squares fit. Although the author has not used a single non-linear least squares approach it must be emphasized that under certain circumstances they are the most elegant and efficient solutions of the optimization problem. Difficulties with a non-linear least squares method mainly arise from the difficult questions which must be dealt with in programming of a non-linear iterative method when to stop and either admit failure or declare acceptable approximation (Dennis, 1977). In order to ensure convergence of iterative solutions the initial approximations must be reasonably good. Since the author could not fulfill this important initial requirement to any meaningful degree of accuracy the optimization was performed by the linear Gauss-Newton method only.

7. CRITERION OF CHEBYSHEV

Chebyshev formulated in 1856 a theorem about the best projection from a class of conformal projections (Meshcheryakov, 1969). Conformal transformation systems in which the changes of scale $m=m(\phi,1)$ are minimized are called Chebyshev's projections. In other words, the ratio of the maximum and minimum scale factor for the whole mapping domain bounded by a closed contour line will be smaller than in any other conformal mapping of the same domain. Since the scales in Chebyshev's projections will deviate as little as possible from unity the logarithms of scales will deviate as little as possible from zero.

Chebyshev's theorem (Biernacki, 1965) states that the necessary and sufficient condition for a conformal projection to belong to the Chebyshev projections is that, along the boundary contour of the domain the mapping shall yield a constant scale factor.

Thus, the determination of a Chebyshev projection for a closed domain consists of a search for an analytic function of the isothermic variable (q+il) which will produce a constant scale factor along the boundary of the domain. Poisson's equation (I-7-18), which was developed earlier, must have a constant value on the boundary contour

$$\frac{\partial^2 \ln m}{\partial q^2} + \frac{\partial^2 \ln m}{\partial l^2} = \operatorname{sech}^2 q \ ,$$

where for the boundary contour Γ

$$\operatorname{sech}^2 q = \text{const.} \tag{II-7-1}$$

The generality of the solution is not restricted if we assume the constant in the equation (II-7-1) to be zero. In this case, the determination of the Chebyshev projections is reduced to the solution of Dirichlet's problem with zero boundary values (Urmaev, 1953).

Chebyshev did not present a proof of his theorem. Much later, in 1896, another Russian mathematician and cartographer, D.A. Grave, rigorously proved the theorem. From that time many authors have called the theorem the Chebyshev-Grave theorem of conformal mappings. The proof for the theorem can be found in (Meshcheryakov 1968).

The determination of a real Chebyshev projection of a closed domain on the sphere can be subdivided into three parts. First the earth's surface is conformally mapped onto a plane by the isothermic coordinates

$$z = \omega \ , \qquad \text{(II-7-2)}$$

where

$$z = y + ix \text{ and } \omega = q + il \ . \qquad \text{(II-7-3)}$$

This is actually the Mercator projection of a unit sphere onto a plane with the scale factor $m_1 = \sec\phi$.

The second step is the definition of a harmonic function which maps the domain into a unit circle. The transformation must be normalized, so that a certain point (q_o, l_o) of the domain becomes the centre of the unit circle. The definition of an analytic function which conformally maps the domain into the circle is performed by either of two groups of methods. In the first group the result is an approximate analytic function, i.e. an approximation of the harmonic transformation function which rigorously satisfies the boundary condition. The best example of this kind of method is the approximation method developed in 1908 by a German engineer, W. Ritz (Courant and Hilbert, 1937). The second group of methods consists of finding a rigorous analytic function which does not perfectly satisfy the boundary condition. Thus, the found mapping will be fully conformal but the Poisson equation will not be completely fulfilled along the boundary contour. Both groups of methods will yield a function $u=u(\omega)$ with the scale factor

$$|u'| = \left|\frac{du}{d\omega}\right| . \qquad \text{(II-7-4)}$$

The third and final step is the establishment of a transformation $z=z(u)$ of the unit circle into the closed domain of the z-plane satisfying the fundamental condition of Chebyshev's map projections of the constant scale factor along the boundary contour. The scale factor combining all three transformation steps is

$$m = \sec\phi \; |\frac{du}{d\omega}|\frac{dz}{du}| = c \; , \qquad (II-7-5)$$

where c is a constant.

The derivative dz/du is defined on the unit circle. At the same time we know that

$$\ln |\frac{dz}{du}| = \nu \text{ and } \arg \frac{dz}{du} = \beta \; . \qquad (II-7-6)$$

Both functions, ν and β, are harmonic functions and they appear to be conjugated. Therefore Cauchy-Riemann equations (I-7-8) will yield the function β and after that, on the unit circle, they will also yield dz/du. By integrating the function dz/du along the unit circle we finally obtain z=z(u) on the contour line Γ, and its transformation, for example by the method of Kantorovich (Kantorovich, Krylov, 1958), yields a conformal projection of the unit circle onto the initial domain bounded by the contour Γ.

A more detailed description of the development of Chebyshev's projections will be given in the third chapter.

An additional remark concerning the quality of Chebyshev's projections must be made. The scale of areas in conformal projections is obtained by the formula

$$p = m^2 \ . \tag{II-7-7}$$

Because the Chebyshev projections optimize the distortion, defined as the natural logarithm of the scale factor, $\ln m$, they automatically optimize the logarithm of the scale of areas

$$\ln p = 2 \ln m \ . \tag{II-7-8}$$

Thus, Chebyshev's projections, of all conformal projections, are closest to equiareal projections. Among other conformal projections they occupy a position similar to Euler's projections among equiareal transformations. Euler's projections, as is well known, satisfy one of the conditions of conformality, $\varepsilon=0$, and are therefore closest to conformal mappings. There is also, however, the second condition of conformality, $m=n$, and if we define a class of equiareal mappings which satisfy the second condition of conformality as the group of equiareal projections closest to conformal, then we have an ambiguity which does not exist with Chebyshev's projections. Thus the uniqueness of Chebyshev's projections among conformal projections not only does not correspond to the position of Euler's projections among equiareal mappings, but is in fact far greater.

III. THE CHEBYSHEV MAP PROJECTIONS

1. INTRODUCTION

The best conformal map projections, or Chebyshev's projections, of a closed domain by the contour line Γ is the projection in which the ratio of the maximal scale factor and the minimal scale factor has the smallest possible value. In order to belong to the class of Chebyshev projections, a conformal mapping must produce a constant scale factor along the boundary contour line. Thus, the determination of the best conformal transformation of a closed domain consists of solving the Laplace equation (I-7-15)

$$\frac{\partial^2 \nu}{\partial q^2} + \frac{\partial^2 \nu}{\partial l^2} = 0,$$

where $\nu = m \cdot \cos \xi$, for an analytic function of the isothermic complex variable $(q+il)$ which will yield a constant scale factor along the boundary, i.e.

$$m_\Gamma = \text{const.} \qquad \text{(III-1-1)}$$

The criterion of Chebyshev can also be defined mathematically by the Poisson equation (I-7-18)

$$\frac{\partial^2 \ln m}{\partial q^2} + \frac{\partial^2 \ln m}{\partial l^2} = \operatorname{sech}^2 q \qquad \text{(III-1-2)}$$

with

$$\operatorname{sech}^2 q_r = \text{const.} \qquad \text{(III-1-3)}$$

The integration of the Poisson equation (III-1-2) with boundary conditions (II-1-3) yields the best conformal projection.

It must be emphasized at the beginning of this chapter that very few mapping domains in cartography have a regular and mathematically defined contour line. The majority of mapping areas have irregular boundaries and the rigorous determination of Chebyshev's projections for such irregular contours is theoretically impossible. The boundary is usually approximated by a series of discrete points at which the boundary condition (III-1-3) can be rigorously satisfied.

Instead of the isothermic latitude, q, we shall introduce a new variable, the difference of latitude, q, and the latitude of the central point, q_o, i.e.

$$\Delta q = q - q_o \qquad \text{(III-1-4)}$$

In this way the isothermic coordinates (q,l) are normalized and the Poisson equation (III-1-2) becomes

$$\frac{\partial^2 \ln m}{\partial q^2} + \frac{\partial^2 \ln m}{\partial l^2} = \operatorname{sech}^2 (\Delta q + q_o) \qquad \text{(III-1-5)}$$

If we introduce abbreviations

$$\ln m = u, \quad \frac{\partial u}{\partial q} = u_q, \quad \frac{\partial u}{\partial l} = u_l, \quad \operatorname{sech}^2 (\Delta q + q_o) = f \qquad \text{(III-1-6)}$$

the Poisson equation is transformed into

$$\frac{\partial u_q}{\partial q} + \frac{\partial u_l}{\partial l} = f. \qquad \text{(III-1-7)}$$

The boundary condition (III-1-3) can easily be replaced by the zero boundary condition, i.e.

$$\operatorname{sech}^2 (\Delta q + q_o)_\Gamma = 0 \qquad \text{(III-1-8)}$$

This simplification of the boundary condition is generally achieved by the construction of a function h (q,l) that agrees with the values of function u on the boundary. Then

$$u = z + h\ (q,l), \qquad \text{(III-1-9)}$$

where z is an unknown function which also satisfies the Poisson equation, however, with another free term. Thus in all further discussions and developments we shall take the Poisson equation (III-1-7) with the zero boundary condition.

The calculus of variation shows that the solution of the Poisson equation with the zero boundary condition is the solution of the Dirichlet problem and is equivalent to the minimization of the Dirichlet integral

$$I(u) = \iint_D \left[\left(\frac{\partial u}{\partial q}\right)^2 + \left(\frac{\partial u}{\partial l}\right)^2 + 2uf\right] dq\ dl, \qquad \text{(III-1-10)}$$

where D is the plane region bounded by the contour Γ. In other words, the solution will provide us with a function, u(q,l), which is continuous in the domain D, together with its partial derivatives of the first and second orders and vanishes along the contour Γ.

Most solution methods of the Dirichlet problem can be subdivided into two fundamentally different groups (Urmaev, 1953). The first group consists of solutions in which the harmonic function, u(q,l), is approximated, but where the approximation perfectly satisfies the boundary conditions. Because the solution is only an approximation of a harmonic function, the Poisson equation for an arbitrary point of the domain will not be exactly satisfied. The best known among these methods was developed by W. Ritz in 1908 and is known today as the Ritz method. The second group of methods yields a rigorous harmonic function, therefore the Poisson equation is completely satisfied at every pont of the domain; however, the boundary conditions are not perfectly fulfilled. Since the boundary in cartography is usually approximated by a series of discrete points, the boundary will be a closed polygon and the line of constant scale deformation will be a smooth curve which approximates the polygon. Lines of constant deformations are called isocols. Thus the second group of methods yields conformal mappings whose isocols only approximate the real boundary. The method of least squares is the most suitable method if the required harmonic function is expressed by harmonic polynomials. The method ascertains the best conformal projection with the boundary isocol approximately following the real boundary.

When the mapping domain resembles an ellipse and is relatively small, the determination of the Chebyshev projection for the domain is considerably simpler. The solution was suggested by various authors (Lagrange, Laborde, Schols, Kavraiskii, Vahramaeva) in slightly different ways but they can all be brought to the same denominator. All these methods are based on the property of a differentially small isocol around the origin of the plane coordinate system that is also the central point of the domain, and where the scale factor in the origin is equal to unity. Such an isocol is an ellipse expressed by the formula

$$v = m-1 = Ax^2 - 2Bxy + (\frac{1}{2} - A)y^2 = 0,$$

or

$$Ax^2 + 2Bxy + A'y^2 = 0, \qquad \text{(III-1-11)}$$

in which the coefficients A and B depend upon the size and orientation of the boundary ellipse.

$$\begin{aligned} A &= \frac{1}{4}(1 - C \cos 2\alpha), \\ B &= \frac{1}{4} C \sin 2\alpha, \\ A' &= \frac{1}{2} - A, \\ C &= \frac{a^2 - b^2}{a^2 + b^2}. \end{aligned} \qquad \text{(III-1-12)}$$

The parameters a, b and α are the semi-major axis, the semi-minor axis and the orientation angle of the semi-major axis of the boundary ellipse, respectively.

The rectangular coordinates, according to Kavraiskii, (1959), are then obtained by the expressions

$$
\begin{aligned}
X &= x + \frac{1}{3} By^3 + Axy^2 - Bx^2y - \frac{1}{3} Ay^3 , \\
Y &= y + \frac{1}{3} Ay^3 - Bxy^2 - Ax^2y + \frac{1}{3} Bx^3 ,
\end{aligned}
\qquad \text{(III-1-13)}
$$

where x, y are the Gaussian coordinates with respect to the central point (ϕ_0, λ_0).

The elliptic conformal projection, i.e., a conformal projection with elliptic isocols (III-1-13) is only one of several examples which are elaborated in the literature of mathematical cartography (Pavlov, 1964). It is given purely as an illustration of a special type of Chebyshev's projections which have rather limited practical significance, although their determination is very simple. Unfortunately, the boundary contour lines can seldom be approximated by an ellipse and therefore the merits of such a projection are reduced whenever the boundary fluctuates considerably about the adopted ellipse.

Various solutions of Chebyshev projections for generally symmetric domains can be mostly found in the Russian cartographic literature. For example, in 1953 N.A. Urmaev investigated different ways of minimizing the Dirichlet integral by the method of Ritz, finite difference method and the method of least squares for conformal mapping of a symmetric spherical trapezoid. Although a great majority of mapping domains in geography can be seldom approximated by a symmetric spherical trapezoid, Urmaev's work (1953) is an interesting contribution to mathematical cartography. Since it is largely unknown to North American cartographers the author decided to describe in detail the work of Urmaev in the next three sections. The notation is slightly modified, particularly in the method of least squares, but the essense of Urmaev's work is preserved.

In addition to the minimization of Dirichlet's integral for symmetric domains the author proposes the method of least squares for a non-symmetric boundary consisting of a series of discrete points. The development of the method is given in the fifth section of the chapter.

It must be also mentioned that in 1973 a geodesist from New Zealand, W.I. Reilly, developed a conformal map projection whose isocols closely approximate the actual shape of New Zealand. This projection to the author's knowledge is the only real approximation of a Chebyshev projection in use at present.

A sketchy development of formulae for the projection of New Zealand can be found in (Reilly, 1973).

2. METHOD OF RITZ

In the introduction of this chapter it was indicated that the solution of the Poisson equation (III-1-7) is equivalent to the minimization of the Dirichlet integral (III-1-10),

$$I(u) = \iint_D [(\frac{\partial u}{\partial q})^2 + (\frac{\partial u}{\partial l})^2 + 2uf]\, dq\, dl,$$

where D is the domain bounded by the contour Γ, and the required function $u(\Delta q,l)$ vanishes on the boundary contour. Let us denote by $u^*(\Delta q,l)$ the exact solution of the Poisson equation with zero boundary conditions and by $I(u^*)=m$ the exact value of the Dirichlet integral. It is logical to expect that a constructed approximation $\overline{u}(\Delta q,l)$ of the required function u^* which satisfies the boundary conditions and for which the value of the Poisson integral is close to m would be a relatively close approximation of the unknown solution. If, moreover, one can construct a sequence of approximate solutions $\overline{u}_n(\Delta q,l)$ so that

$$\lim_{n\to\infty} I(\overline{u}_n) = m \qquad \text{(III-2-1)}$$

then we know that

$$\lim_{n\to\infty} \overline{u}_n = u^*, \qquad \text{(III-2-2)}$$

or the constructed sequence converges to the real solution of the problem, (Rectorys, 1977).

For the minimization of the Dirichlet integral, Ritz suggested a sequence of functions depending on several parameters

$$u_n = \Phi(\Delta q, l, c_1, c_2, \ldots, c_n), \qquad \text{(III-2-3)}$$

so that for all the values of the parameters the boundary conditions are fulfilled. The value of the Dirichlet integral is then a function of the unknown parameters, c_i, i.e.

$$I(u_n) = I(c_1, c_2, \ldots, c_n), \qquad \text{(III-2-4)}$$

and the minimum of the integral is achieved when

$$\frac{\partial I(u_n)}{\partial c_i} = 0 \quad \text{for } i = 1,2,\ldots,n. \qquad \text{(III-2-5)}$$

Solving the system of n equations (III-2-5) with n unknowns, we obtain particular values of parameters $\bar{c}_i$. Substituting into the Dirichlet integral the function $I(\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_n)$ gives the absolute minimum. Finally, selecting the function in the family (III-2-3) corresponding to the computed values of the parameters, $\bar{c}_i$, we construct the required approximate solution

$$\bar{u}_n = \Phi(\Delta q, l, \bar{c}_1, \bar{c}_2, \ldots, \bar{c}_n). \qquad \text{(III-2-6)}$$

In 1953 the Russian geodesist N.A. Urmaev was the first to recommend the application of the Ritz method for conformal projections of symmetric domains with respect to the y-axis of the plane coordinate system. He suggested that the sequence (III-2-3) be defined as a linear combination of independent functions ψ_i (i=0,1,2,...,n), i.e.

$$\bar{u}_n = \psi_o + a_1\psi_1 + a_2\psi_2 + \ldots + a_n\psi_n = \psi_o + \sum_{i=1}^{n} a_i\psi_i \qquad \text{(III-2-7)}$$

Functions ψ_i must be continuous inside the domain and they must vanish along the contour line. When the linear combination (III-2-7) is substituted into the Dirichlet integral (III-2-4) it yields a quadratic function of coefficients a_i whose partial differentiation (III-2-5) results in a set of n linear equations with n unknowns which can easily be solved. The simplicity of the solution of the system (III-2-5) appears to be the main reason for the suggested formulation of the function u_n.

The domain of Urmaev's investigations was a spherical trapezoid bounded by the isothermic latitudes q_1 and q_2, and the difference in longitude l. If we define the auxiliary quantities

$$q_o = \frac{1}{2}(q_1+q_2), \ a = \frac{1}{2}(q_2-q_1), \ b = \frac{1}{2}l, \qquad \text{(III-2-8)}$$

then the function, u, must vanish for values of $\bar{q}=\pm a$ and $l=\pm b$. Among many, the following family of functions will satisfy the boundary conditions for any values of parameters c,

$$\left.\begin{aligned} &(a^2-\Delta q^2)(b^2-l^2)c_1, \\ &(a^2-\Delta q^2)(b^2-l^2)(c_1+c_2\Delta q), \\ &(a^2-\Delta q^2)(b^2-l^2)(c_1+c_2\Delta q+c_3\Delta q^3), \\ &(a^2-\Delta q^2)(b^2-l^2)(c_1+c_2\Delta q+c_3\Delta q^3+c_4 l^2), \text{ etc.} \end{aligned}\right\} \qquad \text{(III-2-9)}$$

The above series assumes symmetry with respect to the l-axis of the curvilinear coordinate system and thus all odd-power terms of l disappear.

Let us take the second of these expressions as our approximation,

$$u = (a^2-\Delta q^2)(b^2-l^2)(c_1+c_2\Delta q) \qquad \text{(III-2-10)}$$

The partial derivatives of the selected function with respect to the independent variables (q,l) are

$$u_q = (b^2-l^2)\left(-2c_1\Delta q+c_2(a^2-3\Delta q^2)\right),$$
$$u_l = -2l(a^2-\Delta q^2)(c_1+c_2\Delta q),$$

and the corresponding squares

$$\left.\begin{aligned}(u_q)^2 &= (b^2-l^2)^2\left(4c_1^2\Delta q^2-4c_1c_2\Delta q(a^2-3\Delta q^2)+c_2^2(a^2-3\Delta q^2)^2\right),\\ (u_l)^2 &= 4l^2(a^2-\Delta q^2)(c_1+2c_1c_2\Delta q+c_2^2\Delta q^2).\end{aligned}\right\} \quad \text{(III-2-11)}$$

Let us split the Dirichlet integral into two parts

$$I(u) = I_1 + 2I_2, \qquad \text{(III-2-12)}$$

where

$$I_1 = \iint_D [(u_q)^2 + (u_l)^2]\; dq\; dl, \qquad \text{(III-2-13)}$$

and

$$I_2 = \iint_D uf\; dq\; dl. \qquad \text{(III-2-14)}$$

Then the first part, I_1, with equations (III-2-11) becomes

$$I_1 = c_1^2\, A_{11} + c_1c_2\, A_{12} + c_2^2\, A_{22}, \qquad \text{(III-2-15)}$$

where

$$A_{11} = 4\iint_D \left((b^2-l^2)^2\Delta q^2+(a^2-\Delta q^2)l^2\right)\, dq\; dl, \qquad \text{(III-2-16)}$$

$$A_{12} = 4\iint_D \left(2(a^2-\Delta q^2)^2\Delta q l^2-\Delta q(a^2-3\Delta q^2)(b^2-l^2)^2\right)\, dq\; dl, \qquad \text{(III-2-17)}$$

$$A_{22} = \iint_D \left(4\Delta q^2 l^2(a^2-\Delta q^2)+(b^2-l^2)^2(a^2-3\Delta q^2)^2\right)\, dq\; dl. \qquad \text{(III-2-18)}$$

The second part of the Dirichlet integral, I_2, can also be expressed as the sum of two integrals M_1 and M_2.

$$I_2 = \iint_D ((a^2-\Delta q^2)(b^2-l^2)(c_1+c_2\Delta q)f \, dg \, dl = c_1 M_1 + c_2 M_2, \qquad \text{(III-2-19)}$$

where

$$M_1 = \iint_D ((a^2-\Delta q^2)(b^2-l^2) \; f \, dq \, dl, \qquad \text{(III-2-20)}$$

and

$$M_2 = \iint_D ((a^2-\Delta q^2)(b^2-l^2)\Delta q \; f \, dq \, dl. \qquad \text{(III-2-21)}$$

The evaluations of integrals A_{ij} $(i,j=1,2)$ in limits from $-a$ to $+a$ and from $-b$ to $+b$ can easily be performed. See, for example (Kantovovich-Krylov, 1958). Their values are

$$A_{11} = \frac{128}{45} a^3 b^3 (a^2+b^2), \quad A_{12}=0, \quad A_{22} = \frac{128}{15} a^3 b^3 \left(\frac{a^2 b^2}{5} + \frac{a^4}{21}\right). \qquad \text{(III-2-22)}$$

In both integrals, M_1 and M_2, we have the evaluation of

$$\int_{-b}^{+b} (b^2-l^2) dl = [b^2 l - \frac{l^3}{3}]_{-b}^{+b} = 2b^3 - \frac{2}{3}b^3 = \frac{4}{3}b^3; \qquad \text{(III-2-23)}$$

therefore M_1 and M_2, taking into account (III-2-23), are transformed into

$$M_1 = \frac{4}{3}a^2b^3\int_{-a}^{+a} f\ dq - \frac{4}{3}a^2b^3\int_{-a}^{+b} \frac{\Delta q^2}{a^2}\ f\ dq, \tag{III-2-24}$$

and

$$M_2 = \frac{4}{3}a^3b^3\int_{-a}^{+a} \frac{\Delta q}{a}\ f\ dq - \frac{4}{3}a^3b^3\int_{-a}^{+b} \frac{\Delta q^3}{a^3}\ f\ dq. \tag{III-2-25}$$

Let us now introduce new symbols

$$\left.\begin{aligned} K_o &= \int_{-a}^{+a} f\ dq,\quad K_1 = \frac{1}{a}\int_{-a}^{+a} f\ \Delta q\ dq,\quad K_2 = \frac{1}{a^2}\int_{-a}^{+a} f\ \Delta q^2\ dq, \\ K_3 &= \frac{1}{a^3}\int_{-a}^{+b} f\ \Delta q^3\ dq. \end{aligned}\right\} \tag{III-2-26}$$

With these abbreviations the integrals, M_1 and M_2, become

$$M_1 = \frac{4}{3}\ a^2b^3(K_o - K_2), \tag{III-2-27}$$

and

$$M_2 = \frac{4}{3}\ a^3b^3\ (K_1 - K_3). \tag{III-2-28}$$

The only remaining problem is the computation of integrals K_i (i=0,1,2,3). Their determination is done in the following way

$$K_o = \int_{-a}^{+a} \operatorname{sech}^2(q_o+\Delta q)dq = [\tanh(q_o+\Delta q)]_{-a}^{+a} =$$

$$\tanh q_2 - \tanh q_1, \tag{III-2-29}$$

or, knowing the relationship between the metalatitude and the corresponding isothermic latitude we have

$$K =_o \sin\xi_2 - \sin\xi_1. \tag{III-2-30}$$

Other values of K_i (i=1,2,3) are derived from the integration by parts

$$K_1 = \frac{1}{a}\int_{-a}^{+a} \operatorname{sech}^2(q_o+\Delta q)\Delta q \; dq = \frac{1}{a}\Big([\Delta q \tanh(q_o+\Delta q)]_{-a}^{+a} -$$

$$- \int_{-a}^{+a} \tanh(q_o+\Delta q)dq\Big),$$

$$K_1 = \tanh q^2 + \tanh q_1 + \frac{1}{a} \ln \frac{\cosh q_1}{\cosh q_2}, \tag{III-2-31}$$

or in terms of the metalatitude

$$K_1 = \sin\xi_2 + \sin\xi_1 + \frac{1}{a} \ln \frac{\cos\xi_2}{\cos\xi_1}. \tag{III-2-32}$$

$$K_2 = \frac{1}{a^2}\int_{-a}^{+a} \mathrm{sech}^2(q_o+\Delta q)\Delta q^2 dq = \frac{1}{a^2}\Big(\big[\Delta q^2 \tan(q_o+\Delta q)\big]_{-a}^{+a} -$$

$$-2\int_{-a}^{+a} \tanh(q_o+\Delta q)\ \Delta q\ dq\Big),$$

$$K_2 = \frac{1}{a^2}\Big(a^2(\tanh q_2 - \tanh q_1) - 2\big[\Delta q \ln \cosh(q_o+\Delta q)\big]_{-a}^{+a} +$$

$$+2\int_{-a}^{+a} \ln \cosh(q_0+\Delta q)\ dq\Big),$$

$$K_2 = \tanh q_2 - \tanh q_1 - \frac{2}{a} \ln \frac{\cosh q_2}{\cosh q_1} + \frac{2}{a^2} N_o, \qquad \text{(III-2-33)}$$

or

$$K_2 = \sin\xi_2 - \sin\xi_1 + \frac{2}{a} \ln \frac{\cos\xi_2}{\cos\xi_1} + \frac{2}{a^2} N_o, \qquad \text{(III-2-34)}$$

where

$$N_o = \int_{-a}^{+a} \ln \cosh\ (q_o+\Delta q)\ dq. \qquad \text{(III-2-35)}$$

The final integral K_3 is solved by a similar way, and

$$K_3 = \frac{1}{a^3}\int_{-a}^{+a} \mathrm{sech}^2(q_o+\Delta q)\Delta q^3 dq = \frac{1}{a^3}\Big(\big[\Delta q^3 \tanh(q_o+\Delta q)\big]_{-a}^{+a} -$$

$$-3\int_{-a}^{+a} \tanh(q_o+\Delta q)\Delta q^2\ dq,$$

$$K_3 = \tanh q_2 + \tanh q_1 + \frac{3}{a} \ln \frac{\cosh q_1}{\cosh q_2} + \frac{6}{a^3} N_1, \qquad \text{(III-2-36)}$$

where

$$N_1 = \int_{-a}^{+a} \ln \cosh (q_o + \Delta q)\, \Delta q\, dq. \qquad \text{(III-2-37)}$$

With the metalatitude the integral K_3 becomes

$$K_3 = \sin\xi_2 + \sin\xi_1 + \frac{3}{a} \ln \frac{\cos\xi_2}{\cos\xi_1} + \frac{6}{a^3} N_1 \qquad \text{(III-2-38)}$$

The values of integrals N_1 and N_2 can be determined by one of the numerical procedures. It is sufficient to evaluate the functions Δq, $\ln(\cosh q)$, where $\Delta q = q - q_o$, in limits between q_1 and q_2 and then apply an integration formula.

The Dirichlet integral (III-2-12) with expressions (III-2-15) and (III-2-19), knowing that $A_{12} = 0$, becomes

$$I(u) = c_1^2 A_{11} + c_2^2 A_{22} + 2c_1 M_1 + 2c_2 M_2, \qquad \text{(III-2-39)}$$

and its minimum occurs when

$$\frac{\partial I(u)}{\partial c_1} = 2c_1 A_{11} + 2M_1 = 0,$$

and

$$\frac{\partial I(u)}{\partial c_2} = 2c_2 A_{22} + 2M_2 = 0,$$

or

$$c_1 = -M_1/A_{11} \quad \text{and} \quad c_2 = -M_2/A_{22}. \qquad \text{(III-2-40)}$$

Formula (III-2-10) will provide us with values of the required function, $u=\ln(m)$, for various points of the mapping domain.

The whole described approach, suggested and developed by Urmaev, is applicable when the mapping domain is a spherical trapezoid. The most general definition of the family of functions equivalent to (III-2-9) cannot be formulated for an arbitrary boundary consisting of a series of discrete points.

For a circular domain with the centre at the origin of the isothermic coordinate system, $(\Delta q, l)$, the author suggests another family of functions

$$\left.\begin{array}{l} (r^2 - \Delta q^2 - l^2)\, c_1, \\ (r^2 - \Delta q^2 - l^2)(c_1 + c_2\Delta q), \\ (r^2 - \Delta q^2 - l^2)(c_1 + c_2\Delta q + c_3\Delta q^2), \ \ldots \text{ etc.,} \end{array}\right\} \text{(III-2-41)}$$

where r is the radius of the circular sperical domain expressed in radian measure.

When the boundary consists of a convex, non-symmetric polygon of m discrete points, the family of functions is

$$\left.\begin{aligned}
&(a_1\Delta q+b_1 l+d_1)\ldots(a_m\Delta q+b_m l+d_m)c_1, \\
&(a_1\Delta q+b_1 l+d_1)\ldots(a_m\Delta q+b_m l+d_m)(c_1+c_2\Delta q+c_3 l), \\
&(a_1\Delta q+b_1 l+d_1)\ldots(a_m\Delta q+b_m l+d_m)\cdot \\
&\qquad (c_1+c_2\Delta q+c_3 l+c_4\Delta q^2+c_5\Delta q l+c_6 l^2).
\end{aligned}\right\} \quad \text{(III-2-42)}$$

Thus it is obvious that, for a circular cap, (III-2-41) has a similar symmetric form as the system (III-2-9) and derivations with evaluations of required integrals can be performed in a similar manner. However, the system (III-2-42) with a slightly larger number of points is very difficult to solve. In addition to the computational disadvantage, the polygon must be strictly convex, i.e., every straight line which defines a section of the boundary between any two successive points of the boundary must be completely outside the mapping domain. Because of all these complications cartographers have never applied the Ritz method, at least to the author's knowledge, for a non-symmetric arbitrary boundary. The complexity of integration discourages even the most enthusiastic cartographer. The integration of M_1 and M_2 could be performed numerically rather than analytically which would introduce a certain reduction of problems. It is also important to emphasize that better approximations to the true conformality are achieved when more terms in (III-2-9) are used, which again increases the computational difficulties.

3. THE METHOD OF FINITE DIFFERENCES

The Ritz method is not the only method of finding a function, $u(q,l)$, as an approximate solution of the Poisson equation with zero boundary conditions. Urmaev (1953) also suggested the application of the finite difference method. The method is basically very simple but it also assumes symmetry about the q-axis, which certainly decreases its range of application.

Let us assume that the domain is subdivided in the direction of isometric latitude into $(q_2-q_1)/h$ rows, where h is the width of a row, and in the direction of longitude into $(l_2-l_1)/k$ columns, where k is the width of a column.

The function u depends upon two variables, q and l. At an arbitrary point of the domain the Taylor expansion yields

$$u(q+h,l+k) = u(q,l) + \frac{\partial u}{\partial q}h + \frac{\partial u}{\partial l}k + \frac{1}{2}[\frac{\partial^2 u}{\partial q^2}h^2 + 2\frac{\partial^2 u}{\partial q \partial l}hk +$$

$$+ \frac{\partial^2 u}{\partial l^2}k^2] + \frac{1}{6}[\frac{\partial^2 u}{\partial q^2}h^3 + 3\frac{\partial^3 u}{\partial q^2 \partial l}h^2 k + 3\frac{\partial^3 u}{\partial q \partial l^2}hk^2 + \frac{\partial^3 u}{2 l^3}k^3] + ... \quad \text{(III-3-1)}$$

Let us take five neighbouring points, as shown in figure III-3-1, and express the values of the function u at four points in terms of the central point.

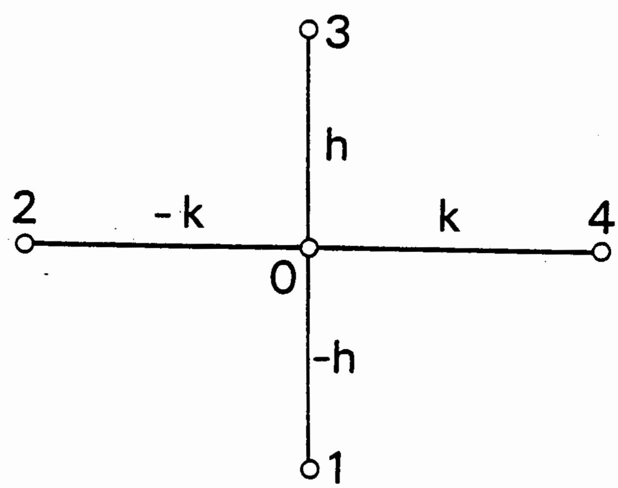


Figure III-3-1 A section of the grid

$$\left.\begin{aligned}
u_1 &= u_o - \frac{\partial u}{\partial q}h + \frac{1}{2}\frac{\partial^2 u}{\partial q^2}h^2 - \frac{1}{6}\frac{\partial^3 u}{\partial q^3}h^3 + \frac{1}{24}\frac{\partial^4 u}{\partial q^4}h^4 + \dots \\
u_2 &= u_o - \frac{\partial u}{\partial l}k + \frac{1}{2}\frac{\partial^2 u}{\partial l^2}k^2 - \frac{1}{6}\frac{\partial^3 u}{\partial l^3}k^3 + \frac{1}{24}\frac{\partial^4 u}{\partial l^4}k^4 + \dots \\
u_3 &= u_o + \frac{\partial u}{\partial q}h + \frac{1}{2}\frac{\partial^2 u}{\partial q^2}h^2 + \frac{1}{6}\frac{\partial^3 u}{\partial q^3}h^3 + \frac{1}{24}\frac{\partial^4 u}{\partial q^4}h^4 + \dots \\
u_4 &= u_o + \frac{\partial u}{\partial l}k + \frac{1}{2}\frac{\partial^2 u}{\partial l^2}k^2 + \frac{1}{6}\frac{\partial^3 u}{\partial l^3}k^3 + \frac{1}{24}\frac{\partial^4 u}{\partial l^4}k^4 + \dots
\end{aligned}\right\} \quad (III\text{-}3\text{-}2)$$

When the above equations are added in pairs, i.e., the first to the third and the second to the fourth we obtain

$$\left.\begin{aligned}
u_1 + u_3 &= 2u_o + \frac{\partial^2 u}{\partial q^2}h^2 + \frac{1}{12}\frac{\partial^4 u}{\partial q^4}h^4 + \dots \\
u_2 + u_4 &= 2u_o + \frac{\partial^2 u}{2l^2}k^2 + \frac{1}{12}\frac{\partial^4 u}{\partial l^4}k^4 + \dots
\end{aligned}\right\} \quad (III\text{-}3\text{-}3)$$

The first equation of (III-3-3) is divided now by h^2, the second by k^2 and the results are added

$$\frac{u_1+u_3}{h^2} + \frac{u_2+u_4}{k^2} = 2u_o\left(\frac{1}{h^2} + \frac{1}{k^2}\right) + \left(\frac{\partial^2 u}{\partial q^2} + \frac{\partial^2 u}{2l^2}\right) + R, \qquad (III\text{-}3\text{-}4)$$

where R is the remainder which contains the terms of the fourth and higher orders of h and k.

From the Poisson equation (III-1-2) we have

$$\frac{\partial^2 u}{\partial q^2} + \frac{\partial^2 u}{\partial 1^2} = \text{sech}^2\, q$$

and therefore equation (III-3-4) becomes

$$u_2 + u_4 = 2u_o\left(\frac{k^2}{h^2}+1\right) + k^2\text{sech}^2\, q - \frac{k^2}{h^2}(u_1+u_3),$$

or

$$u_2 + u_4 = 2u_o(p+1) - p(u_1+u_3) + f', \qquad \text{(III-3-5)}$$

where

$$p = \frac{k^2}{h^2} \qquad \text{and} \qquad f' = k^2\text{sech}^2\, q. \qquad \text{(III-3-6)}$$

In this manner the rigorous solution of the Poisson equation is substituted by an approximate numerical solution. At every point of the symmetric mapping domain, starting from the boundary where the values of function u(q,1) are known, we can write a computer program with ease, but for a symmetric boundary with respect to one coordinate axis only, the program can be very complex.

4. SYMMETRIC CHEBYSHEV PROJECTIONS BY LEAST SQUARES

The Ritz and finite difference methods solve the Poisson equation (I-7-18) with zero boundary conditions by approximating a true harmonic function. Thus, strictly speaking, the results of the two methods are not exact conformal mappings but approximations only. Using the results in the Poisson equation we will obtain some discrepancies, i.e., the Poisson equation will not be exactly satisfied except along the boundary contour of the domain. Let us now take the opposite approach and develop exact harmonic functions which yield true conformal mappings but only approximately satisfy the boundary conditions. In order to eliminate large discrepancies in the zero boundary conditions, the method of least squares will be applied. Once the exact harmonic function is selected, the method of least squares will yield coefficients related to the selected harmonic function determined in such a way that the discrepancies of the zero boundary conditions squared and added together will give us a minimum. The easiest harmonic function for these purposes is a harmonic polynomial (I-7-27). We shall assume symmetry of the boundary contour in the first case and non-symmetric cases will be dealt with in the subsequent section.

The quantity ν as defined by the equation (I-7-13) is a complex function, thus the natural logarithm can be expressed by harmonic polynomial (I-7-29)

$$\ln(m \operatorname{sech} q) = a_o + a_i\psi_1 + a_2\psi_2 + \ldots + a_k\psi_k,$$

or

$$\ln m = a_o + a_1\psi_1 + a_2\psi_2 + \ldots + a_k\psi_k + \ln \cosh q. \qquad \text{(III-4-1)}$$

The coefficients a_i, for $i=0,1,\ldots\ldots,k$, are determined by the method of least squares from the fundamental condition of least squares

$$\sum_{i=1}^{n} (\ln m_i)^2 = \min, \qquad \text{(III-4-2)}$$

where n is the number of discrete points which approximate the boundary contour. The number of fixed boundary points must be larger than the number of unknown coefficients, i.e.,

$$n > k + 1.$$

Let us again use matrix notation, the most suitable type of notation for the method of least squares, in which the fundamental equation of the mathematical model (III-4-1) obtains the form

$$\mathfrak{u} = \mathfrak{K} \cdot \mathfrak{a} + \mathfrak{l}, \qquad \text{(III-4-3)}$$

where

$$\mathfrak{u} = \begin{bmatrix} \ln\ m_1 \\ \ln\ m_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \ln\ m_n \end{bmatrix}, \quad \mathfrak{K} = \begin{bmatrix} 1 & \psi_1^{(1)} & \cdots\cdots & \psi_k^{(1)} \\ 1 & \psi_1^{(2)} & \cdots\cdots & \psi_k^{(2)} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 1 & \psi_1^{(n)} & \cdots\cdots & \psi_k^{(n)} \end{bmatrix},$$

$$\mathfrak{a} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_k \end{bmatrix}, \quad \mathfrak{l} = \begin{bmatrix} \ln\ \cosh\ q_1 \\ \ln\ \cosh\ q_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \ln\ \cosh\ q_n \end{bmatrix}. \qquad \text{(III-4-4)}$$

The condition of least squares (III-4-2) then becomes

$$\mathfrak{u}^T \mathfrak{u} = \min, \qquad \text{(III-4-5)}$$

which takes place when the first derivative of $\mathfrak{u}^T\mathfrak{u}$ with respect to the vector of the unknowns, $\mathfrak{a}$, is equal to zero, i.e.,

$$\partial(\mathfrak{u}^T\mathfrak{u})/\partial\mathfrak{a} = 0. \qquad \text{(III-4-6)}$$

Taking the mathematical model (III-4-3) we can write

$$\mathfrak{u}^T\mathfrak{u} = (\mathfrak{A}\mathfrak{a}+\mathfrak{l})^T(\mathfrak{A}\mathfrak{a}+\mathfrak{l}) = \mathfrak{a}^T\mathfrak{A}^T\mathfrak{A}\mathfrak{a} + \mathfrak{a}^T\mathfrak{A}^T\mathfrak{l} +$$

$$+\mathfrak{l}^T\mathfrak{A}\mathfrak{a} + \mathfrak{l}^T\mathfrak{l},$$

or

$$\mathfrak{u}^T\mathfrak{u} = \mathfrak{a}^T\mathfrak{A}^T\mathfrak{A}\mathfrak{a} + 2\,\mathfrak{a}^T\mathfrak{A}^T\mathfrak{l} + \mathfrak{l}^T\mathfrak{l}, \qquad \text{(III-4-7)}$$

and the derivative of the above expression is

$$\partial(\mathfrak{u}^T\mathfrak{u})/\partial\mathfrak{a} = 2\,\mathfrak{a}^T\mathfrak{A}^T\mathfrak{A} + 2\mathfrak{A}^T\mathfrak{l} = 0,$$

or finally,

$$\mathfrak{N}\mathfrak{a} + \mathfrak{b} = 0, \qquad \text{(III-4-8)}$$

where

$$\mathfrak{N} = \mathfrak{A}^T\mathfrak{A} \quad \text{and} \quad \mathfrak{b} = \mathfrak{A}^T\mathfrak{l}. \qquad \text{(III-4-9)}$$

The solution of the system of normal equations (III-4-8) yields the vector of unknowns

$$\mathfrak{a} = -\mathfrak{N}^{-1}\mathfrak{b}. \qquad \text{(III-4-10)}$$

As a measure of quality of the least squares fit, we can use the variance from stochastic mathematical models

$$\sigma^2 = (\mathfrak{u}^T\mathfrak{u})/(n-(k+1)). \qquad \text{(III-4-11)}$$

The higher the degree of harmonic polynomials, the better the fit the solution will have. If we keep n, the number of fixed boundary points, constant, the increase of the order of the polynomial, k, results in a smaller value of the denominator in (III-4-11) and also, naturally, in a smaller value of $\mathfrak{u}^T\mathfrak{u}$. Thus, there must be an optimal value of k which will yield the smallest variance. To optimize the order k, the author sees no other way than a trial-and-error approach. Several orders of k should be taken and the solutions evaluated. When the differences in the scale factor, m, become practically insignificant using k-th and (k+1)-st order of the harmonic polynomial, further increases of the order are meaningless. If, however, the boundary conditions are weighted more, the smallest value of variance, σ^2, will indicate the best order of the polynomial.

With known coefficients, a_i, we can compute the logarithm of the scale factor and, thus, the scale factor at every point of the mapping domain

$$m = e^{(a_o + a_1\psi_1 + \ldots a_k\psi_k + \ln \cosh q)}. \qquad \text{(III-4-12)}$$

The rectangular coordinates (x,y) of a Chebyshev projection determined in this way for a symmetric domain are also computed by a harmonic polynomial

$$y + ix = F\ (q + il), \qquad \text{(III-4-13)}$$

where

$$F = \sum_{i=0}^{k} (A_i\psi_i + A_i\Theta_i). \qquad \text{(III-4-14)}$$

If we separate the real and imaginary parts we obtain the formulae

$$x = \sum_{i=1}^{k} A_i\Theta_i\ , \quad y = \sum_{i=0}^{k} A_i\psi_i. \qquad \text{(III-4-15)}$$

The unknown coefficients A_i, for i=0,1,...,k, are obtained from the expression (I-7-19)

$$y_0 = \int \nu_0\ dq + c$$

or

$$y_o = A_o + A_1\Delta q + A_2\Delta q^2 + \ldots + A_k\Delta q^k. \qquad \text{(III-4-16)}$$

The first derivative of the above equation with respect to the isothermic latitude yields the function ν_o, thus

$$\nu_o = A_1 + 2A_2\Delta q + 3A_3\Delta q^2 + \ldots + kA_k\Delta q^{k-1}. \qquad \text{(III-4-17)}$$

Since function ν_o can also be determined from (III-4-1) by the formula

$$\nu_o = e^{(a_o+a_1\Delta q+a_2\Delta q^2 + \ldots + a_k\Delta q^k)}, \qquad \text{(III-4-18)}$$

it suffices to determine ν_o values at k different points along the central meridian and substitute them into (III-4-17). The result is k linear equations with k unknowns A_i $(i=1,2,\ldots,k)$. In the matrix form we have

$$\mathcal{Q}\mathcal{A} = \mathcal{d}, \qquad \text{(III-4-19)}$$

where

$$\mathcal{Q} = \begin{bmatrix} 1 & 2\Delta q_{(1)} \cdots\cdots & k\Delta q_{(1)}^{k-1} \\ 1 & 2\Delta q_{(2)} \cdots\cdots & k\Delta q_{(2)}^{k-1} \\ \cdot & \cdot & \\ \cdot & \cdot & \\ \cdot & \cdot & \\ \cdot & \cdot & \\ 1 & 2\Delta q_{(k)} \cdots\cdots & k\Delta q_{(k)}^{k-1} \end{bmatrix}, \mathcal{A} = \begin{bmatrix} A_1 \\ A_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ A_k \end{bmatrix}, \mathcal{d} = \begin{bmatrix} \nu_o^{(1)} \\ \nu_o^{(2)} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \nu_o^{(k)} \end{bmatrix} \qquad \text{(III-4-20)}$$

The solution of the system (III-4-19) yields the final unknowns, A_i.

The elements of the matrix $\mathcal{K}$ can be determined directly from the equations (I-7-29) which in our case are

$$\left.\begin{aligned} \psi_1^{(i)} &= \Delta q_i, \\ \psi_2^{(i)} &= \Delta q_i^2 - 1_i^2, \\ \psi_3^{(i)} &= \Delta q_i^3 - 3\Delta q_i\ 1_i^2, \\ \psi_4^{(i)} &= \Delta q_i^4 - 6\Delta q_i^2\ 1_i^2 + 1_i^4, \\ \psi_5^{(i)} &= \Delta q_i^5 - 10\Delta q_i^3\ 1_i^2 + 5\Delta q_i\ 1_i^4. \end{aligned}\right\} \qquad \text{(III-4-22)}$$

In the same way we can compute (for $j=1,2,\ldots,k$)

$$\left.\begin{aligned}
\Theta_1^{(i)} &= l_i, \\
\Theta_2^{(i)} &= 2\Delta q_i\, l_i, \\
\Theta_3^{(i)} &= 3\Delta q_i^2\, l_i - l_i^3, \\
\Theta_4^{(i)} &= 4\Delta q_i^3\, l_i - 4\Delta q_i\, l_i^3, \\
\Theta_5^{(i)} &= 5\Delta q_i^4\, l_i - 10\Delta q_i^2\, l_i^3 + l_i^5 .
\end{aligned}\right\} \qquad (III-4-23)$$

However, with electronic computers it is much more suitable to use the recursion relations. With the initial values

$$\psi_o^{(i)} = 1 \;, \quad \Theta_o^{(i)} = 0,$$

other elements are

$$\left.\begin{aligned}
\psi_{j+1}^{(i)} &= \Delta q_i \psi_j^{(i)} - l_i \Theta_j^{(i)}, \\
\Theta_{j+1}^{(i)} &= \Delta q_i \Theta_j^{(i)} + l_i \psi_j^{(i)},
\end{aligned}\right\} \qquad (III-4-24)$$

where j=1,2,...,k.

The Chebyshev projection defined by the equations (III-4-15) will give a symmetric isocol which smoothly approximates the real boundary contour.

5. NON-SYMMETRIC CHEBYSHEV PROJECTIONS

The adaptability of the symmetric Chebyshev projections, as described in the previous section, is restricted to domains whose boundary can be approximated by a symmetric isocol, a smooth curve along which the scale distortion takes on a constant value. For non-symmetric territories, the application of the symmetric Chebyshev projections may lead to too large deviations of scale from the desired constant value along the boundary contour and thus the effort of the optimization process can become almost meaningless. The problem can be solved by the development of a non-symmetric analytic function which yields a non-symmetric isocol whose shape resembles the mapping domain.

In the modern Russian cartographic literature the problem is clearly stated. For example, Meshcheryakov (1968) discusses all aspects of the Chebyshev projections without giving the actual solution of a non-symmetric case. Pavlov (1974) listed the required formulae for a symmetric Chebyshev projection and mentioned non-symmetric generalizations by the method of Vahramaeva, without describing the actual process or giving any mathematical expression. The author was unsuccessful in originally obtaining any references with a detailed explanation of the method used by Vahramaeva. With the assumption that Vahramaeva applied non-symmetric harmonic polynomials (I-7-30) the author has developed all the required formulae which, he had believed, constitute the method of Vahramaeva. However, after reading the first version of the thesis, Dr. Tobler sent to the author the English translation of an article by Vahramaeva. The developed formulae by the author do not resemble the work of Vahramaeva.

Let us take the harmonic polynomial (I-7-30) and write it in the form

$$\begin{aligned}\ln \nu = a_o + a_1\psi_1 + \ldots +& \\ + a_k\psi_k + b_1\Theta_1 + \ldots + b_k\Theta_k.&\end{aligned} \tag{III-5-1}$$

Using the definition of function ν in (I-7-13) the last formula can be transformed into

$$\begin{aligned}\ln m = a_o + a_1\psi_1 + \ldots + a_k\psi_k +& \\ + b_1\Theta_1 + \ldots + b_k\Theta_k + \ln \cosh q,&\end{aligned} \tag{III-5-2}$$

where a_i $(i=0,1,...,k)$ and b_i $(i=1,2,...,k)$ are unknown coefficients to be determined by the method of least squares thus satisfying the fundamental requirement (III-4-2)

$$\sum_{i=1}^{n} (\ln m_i)^2 = \min,$$

where n is the number of discrete points which approximate the boundary. The number n must be larger than the number of unknown coeffients, i.e.,

$$n > 2k + 1 \qquad \text{(III-5-3)}$$

Let us again introduce a similar matrix notation to that in the previous section

$$\mathcal{U}^T = [\ln m_1 \ \ln m_2 \ \ldots\ldots \ \ln m_n], \qquad \text{(III-5-4)}$$

$$\mathcal{C}^T = [a_0, \ a_1, \ \ldots, \ a_k \ b_1 \ b_2 \ \ldots \ b_k], \qquad \text{(III-5-5)}$$

$$\mathcal{L}^T = [\ln\cosh q_1 \ \ln\cosh q_2 \ \ldots \ \ln\cosh q_n] \qquad \text{(III-5-6)}$$

$$\mathcal{K} = \begin{bmatrix} 1 & \psi_1^{(1)} \ldots \psi_k^{(1)} & \Theta_1^{(1)} \ldots \Theta_k^{(1)} \\ 1 & \psi_1^{(2)} \ldots \psi_k^{(2)} & \Theta_1^{(2)} \ldots \Theta_k^{(2)} \\ \cdots & \cdots\cdots\cdots & \cdots\cdots\cdots \\ 1 & \psi_1^{(n)} \ldots \psi_k^{(n)} & \Theta_1^{(n)} \ldots \Theta_k^{(n)} \end{bmatrix} , \qquad \text{(III-5-7)}$$

where $\psi_j^{(i)}$ and $\Theta_j^{(i)}$, for j=1,...,k and i=1,...,n are computed by the expressions (III-4-22) and (III-4-23) respectively or by the recursion formulae (III-4-24).

The vector of unknowns $\mathcal{C}$ is then

$$\mathcal{C} = -\mathcal{N}^{-1}\mathcal{d} , \qquad \text{(III-5-8)}$$

where

$$\mathcal{N} = \mathcal{K}^T\mathcal{K} \quad \text{and} \quad \mathcal{d} = \mathcal{K}^T\ell . \qquad \text{(III-5-9)}$$

Compared to the determination of the symmetric Chebyshev projections, the development of non-symmetric formulae up to now is, more or less, identical to the former except that the matrix $\mathcal{K}$ also includes the terms of the imaginary part of the harmonic polynomial and the vector of the unknown coefficients $\mathcal{C}$ consists of parameters a_i and b_i where i=1,...,k. Theoretically the order of polynomials ψ_i and Θ_j do not need to be the same, but throughout this work, for reasons of simplicity, the author assumes that i,j=1,...,k.

With known coefficients a_i, b_i, ascertained from the boundary conditions, the analytic continuation enables us to compute the logarithm of the scale and, thus, the scale itself at every point of the mapping domain by the formula (III-5-2). Let us now express the same formula as the sum of two vector products and the logarithm of the cosine hyperbolic function of the isothermic latitude,

$$\ln m = \mathfrak{a}^T \mathfrak{Q} + \mathfrak{b}^T \mathfrak{S} + a_0 + \ln \cosh q \qquad \text{(III-5-10)}$$

where

$$\begin{aligned} \mathfrak{a}^T &= [a_1 \; a_2 \; . \; . \; . \; . \; . \; . \; . \; . \; a_k], \\ \mathfrak{b}^T &= [b_1 \; b_2 \; . \; . \; . \; . \; . \; . \; . \; . \; b_k], \\ \mathfrak{Q}^T &= [\psi_1 \; \psi_2 \; . \; . \; . \; . \; . \; . \; . \; . \; \psi_k], \\ \mathfrak{S}^T &= [\theta_1 \; \theta_2 \; . \; . \; . \; . \; . \; . \; . \; . \; \theta_k]. \end{aligned} \qquad \text{(III-5-11)}$$

In the same way the formula (III-5-1) becomes

$$\ln \nu = \mathfrak{a}^T \mathfrak{Q} + \mathfrak{b}^T \mathfrak{S} + a_0 \qquad \text{(III-5-12)}$$

and

$$\nu = e^{(\mathfrak{a}^T \mathfrak{Q} + \mathfrak{b}^T \mathfrak{S} + a_0)} \qquad \text{(III-5-13)}$$

We have already seen that the plane rectangular coordinates (x,y) obtained by a conformal mapping are conjugated harmonic functions of the isothermic coordinates which, thus, satisfy the Cauchy-Riemann equations (I-7-8). Therefore, if the selected analytic function of conformal mapping is a harmonic polynomial, the rectangular coordinates will be computed by the pair of functions

$$\left.\begin{aligned} x &= \mathcal{A}^T \mathbf{Q} + \mathfrak{B}^T \mathcal{S} \\ y &= \mathcal{A}^T \mathbf{Q} - \mathfrak{B}^T \mathcal{S} \ , \end{aligned}\right\} \qquad \text{(III-5-14)}$$

where

$$\left.\begin{aligned} \mathcal{A}^T &= [A_1 \ A_2 \ . \ . \ . \ . \ . \ A_k] \ , \\ \mathfrak{B}^T &= [B_1 \ B_2 \ . \ . \ . \ . \ . \ B_k] \ . \end{aligned}\right\} \qquad \text{(III-5-15)}$$

The coefficients A_i, B_i, for i=1,...,k, are unknown and will be obtained from the fundamental equations of map projections (I-5-15) and (I-5-19). Since we are dealing with a conformal mapping in which there is no distortion of the parametric angle, $\varepsilon=0$, and the scale factor is constant at a point, m=n, the equations (I-5-15) become

$$\left.\begin{aligned} x_\phi &= -m \sin\beta , & x_l &= \nu \cos\beta , \\ y_\phi &= m \cos\beta , & y_l &= \nu \sin\beta , \end{aligned}\right\} \qquad \text{(III-5-16)}$$

where

$$\nu = m \cos\phi = m \operatorname{sech} q. \qquad \text{(III-5-17)}$$

The relationship between the geographic and the isothermic latitude (I-7-12), where $dq/d\phi = \sec\phi = \cosh q$, enables us to transform the expressions (III-5-16) into

$$\left.\begin{aligned} x_q &= -\nu \sin\beta , & x_l &= \nu \cos\beta , \\ y_q &= \nu \cos\beta , & y_l &= \nu \sin\beta . \end{aligned}\right\} \qquad \text{(III-5-18)}$$

Let us now differentiate the rectangular coordinates (x,y) as expressed by (III-5-14) with respect to the isothermic latitude q,

$$\left.\begin{aligned} x_q &= \mathcal{A}^T \frac{\partial}{\partial q} (\mathcal{Q}) + \mathcal{B}^T \frac{\partial}{\partial q} (\mathcal{S}) , \\ y_q &= \mathcal{A}^T \frac{\partial}{\partial q} (\mathcal{S}) - \mathcal{B}^T \frac{\partial}{\partial q} (\mathcal{S}) , \end{aligned}\right\} \qquad \text{(III-5-19)}$$

bearing in mind that $\mathfrak{Q}$ and $\mathfrak{S}$ are vectors of conjugated harmonic polynomials which must satisfy the Cauchy-Riemann equations

$$\frac{\partial}{\partial q}(\mathfrak{Q}) = \frac{\partial}{\partial l}(\mathfrak{S}) \quad \text{and} \quad \frac{\partial}{\partial q}(\mathfrak{S}) = -\frac{\partial}{\partial l}(\mathfrak{Q}). \qquad \text{(III-5-20)}$$

Let us denote the derivatives of vectors $\mathfrak{Q}$ and $\mathfrak{S}$ with respect to the isothermic latitude by vectors $\mathfrak{T}$ and $\mathfrak{Y}$, respectively,

$$\frac{\partial}{\partial q}(\mathfrak{Q}) = \mathfrak{T}, \quad \frac{\partial}{\partial q}(\mathfrak{S}) = \mathfrak{Y}, \qquad \text{(III-5-21)}$$

where

$$\left.\begin{aligned} \mathfrak{T}^T &= [\,1 \quad 2\psi_1 \quad 3\psi_2 \ \ldots \ k\psi_{k-1}]\,, \\ \mathfrak{Y}^T &= [\,1 \quad 2\Theta_1 \quad 3\Theta_2 \ \ldots \ k\Theta_{k-1}]\,. \end{aligned}\right\} \qquad \text{(III-5-22)}$$

Thus the individual elements of vectors $\mathfrak{T}$ and $\mathfrak{Y}$ are obtained by the formulae

$$\mathcal{T} = \{t_j = j\psi_{j-1}\}, \quad \mathcal{V} = \{v_j = j\Theta_{j-1}\} \quad \text{for } j=1,\ldots,k \qquad \text{(III-5-23)}$$

where

$$\psi_o = \Theta_o - 1. \qquad \text{(III-5-24)}$$

The convergence of meridians, β, is also a harmonic function of the isothermic coordinates since it satisfies the equations (I-7-14),

$$\beta_q = \frac{\partial \ln \nu}{\partial l}, \qquad -\beta_l = \frac{\partial \ln \nu}{\partial q}.$$

The integration of one of these differential equations will give the convergence. For example,

$$\beta = \int [\frac{\partial \ln \nu}{\partial l}]\, dq, \qquad \text{(III-5-25)}$$

or with (III-5-12)

$$\beta = \int [\frac{\partial}{\partial l} (\mathbf{a}^T \mathbf{Q} + \mathbf{b}^T \mathcal{S} + a_o)\, dq,$$

which finally yields

$$\beta = \int [\, \mathfrak{a}^T \frac{\partial}{\partial l} (\mathfrak{Q}) + \mathfrak{b}^T \frac{\partial}{\partial l} (\mathfrak{S})] \, dq. \qquad \text{(III-5-26)}$$

The substitution of the Cauchy-Riemann equations (III-5-20) into the last integral gives

$$\beta = \int [- \mathfrak{a}^T \frac{\partial}{\partial q} (\mathfrak{S}) + \mathfrak{b}^T \frac{\partial}{\partial q} (\mathfrak{Q})] \, dq,$$

or

$$\beta = - \mathfrak{a}^T \mathfrak{S} + \mathfrak{b}^T \mathfrak{Q} + c,$$

where c is the constant of integration whose value is equal to zero if we decide to have no convergence of meridians at the origin for q=0 and l=0. Thus the final version of the last equation is

$$\beta = - \mathfrak{a}^T \mathfrak{S} + \mathfrak{b}^T \mathfrak{Q} \,. \qquad \text{(III-5-27)}$$

The formula for convergence of meridians (III-5-27) permits us to determine the numerical value of the convergence at every point of the mapping domain.

Let us now combine the obtained results for ν and β in the right-hand side of the equation (III-5-18)

$$\left.\begin{aligned} x_q &= -e^{(\mathfrak{a}^T \mathfrak{Q} + \mathfrak{b}^T \mathfrak{S} + a_o)} \sin(-\mathfrak{a}^T \mathfrak{S} + \mathfrak{b}^T \mathfrak{Q}), \\ y_q &= e^{(\mathfrak{a}^T \mathfrak{Q} + \mathfrak{b}^T \mathfrak{S} + a_o)} \cos(-\mathfrak{a}^T \mathfrak{S} + \mathfrak{b}^T \mathfrak{Q}). \end{aligned}\right\} \quad \text{(III-5-28)}$$

Since all elements defining the derivatives x_q and y_q are known, we can compute the derivatives at an arbitrary point of the domain. At the same time the derivatives are expressed in terms of unknown coefficients A_i and B_i in the equations (III-5-19),

$$\left.\begin{aligned} &\mathfrak{A}^T \mathfrak{T} + \mathfrak{B}^T \mathfrak{Y} = -e^{(\mathfrak{a}^T \mathfrak{Q} + \mathfrak{b}^T \mathfrak{S} + a_o)} \\ &\quad \sin(-\mathfrak{a}^T \mathfrak{S} + \mathfrak{b}^T \mathfrak{Q}), \\ &\mathfrak{A}^T \mathfrak{Y} - \mathfrak{B}^T \mathfrak{T} = e^{(\mathfrak{a}^T \mathfrak{Q} + \mathfrak{b}^T \mathfrak{S} + a_o)} \\ &\quad \cos(-\mathfrak{a}^T \mathfrak{S} + \mathfrak{b}^T \mathfrak{Q}). \end{aligned}\right\} \quad \text{(III-5-29)}$$

This is a linear system of two equations with 2k unknown coefficients, A_i, B_i for i=1,...,k. Therefore, if we take k different points of the mapping domain the system will be fully determined. At each selected point with known isothermic coordinates (q,l), for i=1,...,k, we must evaluate vectors $\mathfrak{Q}$, $\mathfrak{S}$, $\mathfrak{T}$ and $\mathfrak{V}$. At each point formulae (III-5-28) will yield two linear equations with 2k unknowns. The totality of all the necessary 2k equations can be given in the matrix form by the expression

$$\mathfrak{W} \cdot \mathfrak{f} = \mathfrak{g}\,, \qquad \text{(III-5-30)}$$

where is the vector of unknowns

$$\mathfrak{f}^T = [A_1 \ A_2 \ \dots \ A_k \ B_1 \ B_2 \ \dots \ B_k]\,, \qquad \text{(III-5-31)}$$

$\mathfrak{g}$ is the vector of the right-hand side of the equations (III-5-28)

$$\mathfrak{g} = (g_i) = \begin{cases} -e^{(\mathfrak{a}^T\mathfrak{Q} + \mathfrak{b}^T\mathfrak{S} + a_o)} \\ \sin(-\mathfrak{a}^T\mathfrak{S} + \mathfrak{b}^T\mathfrak{Q}) \text{ for } i=1,\dots,k \\ e^{(\mathfrak{a}^T\mathfrak{Q} + \mathfrak{b}^T\mathfrak{S} + a_o)} \\ \cos(-\mathfrak{a}^T\mathfrak{S} + \mathfrak{b}^T\mathfrak{Q}) \text{ for } i=k+1,\dots,2k \end{cases} \qquad \text{(III-5-32)}$$

and $\mathcal{W}$ is the coefficient matrix whose individual elements ij are derived from the following expressions

$$\mathcal{W} = (w_{ij}) = \left\{ \begin{array}{ll} j\ \psi_{j-1}^{(i)} & \text{for } i=1,\ldots,k;\ j=1, \\ & \ldots,k \\ (j-k)\ \Theta_{j-k-1}^{(i)} & \text{for } i=1,\ldots,k;\ j=k+1, \\ & \ldots,2k \\ j\ \Theta_{j-1}^{(i-k)} & \text{for } i=k+1,\ldots,2k;\ j=1, \\ & \ldots,k \\ -(j-k)\ \psi_{j-k-1}^{(i-k)} & \text{for } i=k+1,\ldots,2k;\ j=k+1, \\ & \ldots,2k \end{array} \right\} \quad \text{(III-5-33)}$$

where $\psi_o^{(i)} = 1$ and $\Theta_o^{(i)} = 0$ for all points from i=1 to i=k.

The solution of the system (III-5-30) yields the unknown coefficients A_i, B_i and thus completes the determination of a non-symmetric Chebyshev projection.

$$f = \mathcal{W}^{-1} \cdot g \ . \qquad \text{(III-5-34)}$$

The higher the order of the harmonic polynomial, the better the fit of the isocol to the selected approximation of the boundary contour. However, some unreasonable increase of the order may lead to too many computational difficulties for a very small increase in accuracy which, in the author's opinion, is too small to justify the additional computational costs. The optimization of the order of the harmonic polynomial can be made by an investigation of the deviations of the boundary contour from the boundary isocol, but the author sees no other way than the trial-and-error approach.

IV. OPTIMAL CARTOGRAPHIC PROJECTIONS FOR CANADA

1. INTRODUCTION AND HISTORICAL BACKGROUND

Early maps of Canadian territories made by various explorers and surveyors contain no indications of the transformation formulae applied. Even the first atlas from 1875, compiled by H.F. Walling and published in London under the title "Atlas of the Dominion of Canada", has no descriptions of the map projections used. According to the grid of meridians and parallels it seems that a polyconic projection was used as the basis of maps.

At the beginning of the twentieth century three different organizations have been mainly involved in the mapping process. They were the Topographic Survey of the Department of the Interior, the Bureau of Geology and Topography of the Department of Mines and the Geographical Section of Militia and Defence. There was apparently no coordination between the agencies. They sometimes mapped the same territory in three different versions using different scales and different mapping systems. In order to unify efforts and make the topographic mapping universal, the Geographic Board of Canada was established in 1923. Two years later the Board suggested the Simple Polyconic Projection, a suggestion that was changed after two additional years to the Transverse Mercator Projection. Thus, from 1927 medium and large scale topographic mappings used the transverse aspect of the conformal cylindric projection with 8° zones. Finally after the Second World War Canadian topographers adopted the Universal Transverse Mercator Projection with 6° zones.

For small scale maps, particularly those which covered the whole Canadian territory, cartographers selected, under the influence of the military, the normal aspect of the Lambert Conformal Conic Projection. At that time the military considered Canada to be a country long and narrow around the 49th parallel and, thus, the Lambert Conformal Conic Projection seemed as the most suitable projection for such a mapping domain elongated in an east-west direction. Map deformations were functions of latitude only and they were minimized around the 49th parallel by an adequate choice of standard parallels. That the northern part of the country was tremendously distorted was of a little concern. The North had no value at that time.

In 1944 for a new map series MCR 8 in the scale of 1" = 64 miles a Canadian cartographer, Parry suggested a conformal conic projection with standard parallels at 47° and 70°. However, the Surveyor General, F.H. Peters, in his memorandum from October 20, 1945 to Mr. Murdie simply changes the recommendation with the following words:

> "Mr. Parry in his memorandum of May 5, 1944, on files 1177, 21027 and 20788, recommends for the 64-mile map of Canada a conical orthomorphic projection with two standard parallels. For the two standard parallels he recommends 47° and 70°, but I think it would be more desirable to adopt 47° and 69°, and you may proceed accordingly in preparing the projection co-ordinates."

The reasons for the change are neither stated nor obviously evident. The difference of a degree in the northern standard latitude can hardly be significant for overall distribution of map distortions. As a matter of fact, the distribution of distortions is for the first time mentioned in a memorandum of M. Grieve to M.G. Cameron from March 26, 1949. For the new proposed series of maps which will represent all the land area of Canada in the scale of 1" = 100 miles, M. Grieve suggested again the Lambert Conformal Conic Projection this time with standard parallels at 49° and 77°. M. Grieve writes:

> "For a map of Canada extending from the southern tip of Lake Michigan to the north of Ellesmere Island and from the remote part of Newfoundland to that of Queen Charlotte Island, a Lambert Conformal Conic Projection with standard parallels at 49° and 77° has the best distribution of scale error. Between these latitudes the scale is too small and beyond them, too great."

The recommendations by M. Grieve about the standard parallels were adopted and they have been serving as the basis for mapping of the whole country ever since.

Before any conclusions about the choice of projection and its parameters for Canada is made let us discuss and evaluate criteria which govern the choice.

For large scale topographic mappings advantages of conformal projections which locally preserve the shape are clear. Almost all countries in the world have adopted one or the other conformal mapping systems as the basis of their integrated survey coordinates and the topographic mapping. Conformal transformations preserve angles and their scale factor is a function of position only. Thus, for the surveying profession, mainly concerned with measurements of angles and distances, conformal map projections require a minimal alteration of measured quantities.

For small scale geographic mappings, however, the criteria for the selection of an appropriate cartographic system are not so clearly defined. It has been usually stated that the choice of map projection depends on the position and geometrical shape of the mapping domain and the purpose of the map. A cartographic representation must be a reliable image of the mapping territory. In other words, the overall deformations of intrinsic elements must be as small as possible. The distribution of distortions and their character should be the most essential governing factor for the selection of a map projection. Only conical map projections (conic, cylindric and azimuthal) have the property that the curves of constant deformations, isocols, for areas, angles and distances have the same shape and are functions of distances from the selected metapole (ϕ_o, λ_o), only. this is the reason that conical map projections are used more than any other class of projections in standard geographic atlases.

Until now the assessment of individual mapping systems and their parameters for Canadian territory has been made subjectively. It is evident from the two mentioned memorandums that even the selection of appropriate standard parallels was decided without too much investigation in the resulting distribution of distortions. To restrict the choice of applicable map projections for small scale maps to conformal mappings only is a priori an unreasonable thing. In addition to claim and believe that conformality preserves the shape of the mapping domain is a widely spread misconception and a sign of ignorance. Gauss (1825) in his famous general solution of conformal projections of regular surfaces stated already in the title that conformality means that "die Abbildung dem Abgebildeten in den kleinsten Theilen ahnlich wird" (the projection is similar to the original in its smallest parts). Thus, conformality preserves shape only locally but not globally. In other words, for large scale mappings where a map sheet covers a small portion of the mapping domain only, the shapes are preserved in their first approximations. However, in small scale mappings the local preservation of shape from the practical point of view is meaningless. To give conformal mappings a preference in atlas cartography is a more or less subjective decision which can hardly be justified by a realistic and objective criterion. On the other hand, to adopt equiareal map projections for certain types of maps in geography can easily be explained and understood.

The author in his research decided that all recommendations for the choice of a projection and its parameters will be based on the Airy-Kavraiskii criterion (II-3-2). The parameters of the projection will be numerically optimized using the method of least squares where the optimization model is defined by the equation (II-6-11). The distortion elements, $v_a = \ln a$ and $v_b = \ln b$, are numerically evaluated at a finite number of points which approximate the Canadian territory. The mapping domain is represented by 75 relatively evenly distributed points. Their number can vary and the more points that are used the more reliable answers may be expected. In the author's research the number of points was strictly governed by the size of computer memory. The whole research was performed in a Horizon microcomputer, with a restricted memory of 32 K, at the British Columbia Institute of Technology. All optimization and computation routines were written in the BASIC language.

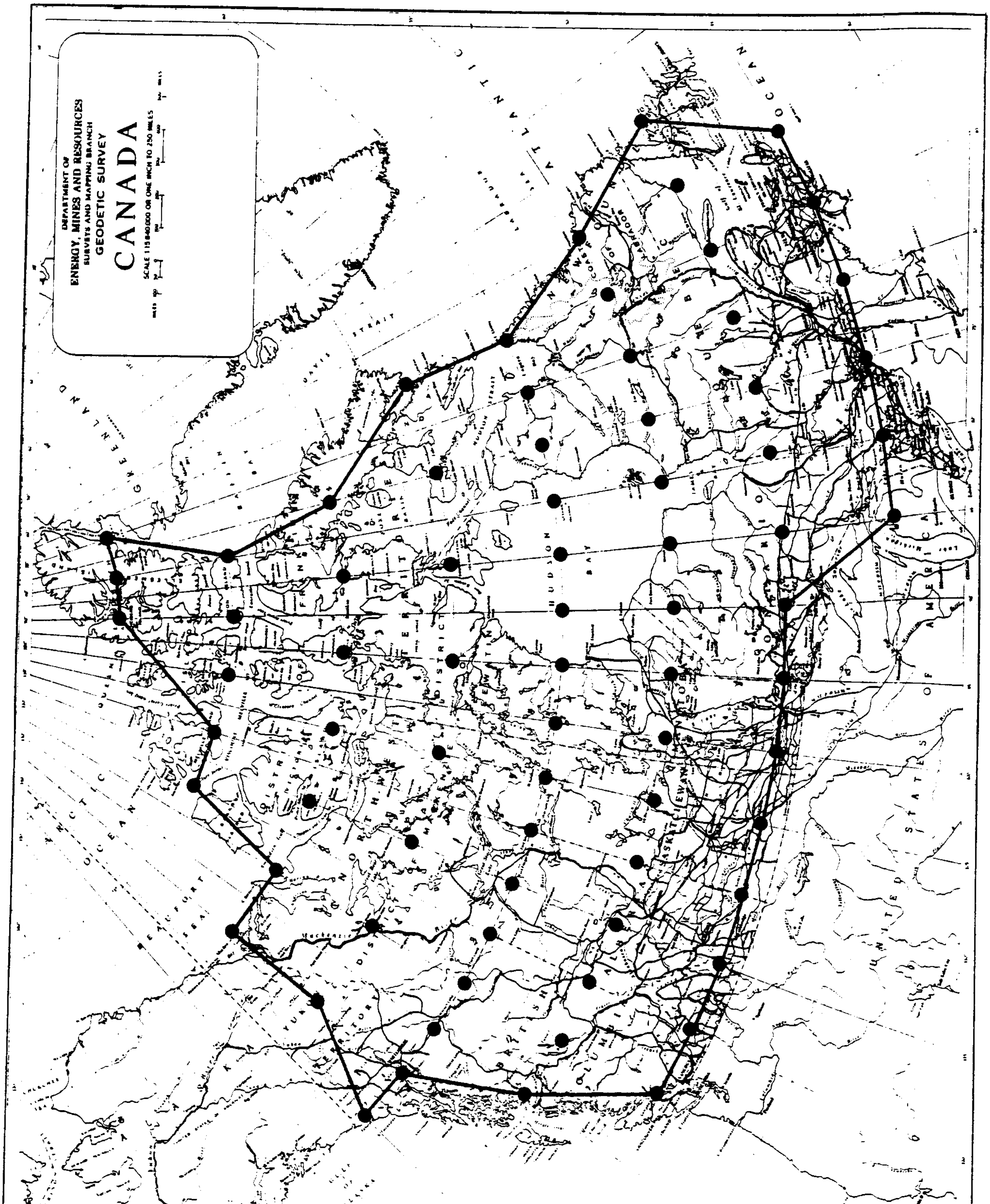


Figure IV-1-1 Distribution of points which approximate Canadian Territory

2. OPTIMAL CONIC PROJECTIONS FOR CANADA

Three standard conic projections are optimized: the Lambert conformal, Lambert equiareal and equidistant conic projection.

The Lambert Conformal Conic projection of a unit sphere is defined by the equations

$$\left.\begin{aligned} & x = \rho \sin \gamma \quad , \quad y = C_2 - \rho \cos \gamma \ , \\ & \text{where} \\ & \gamma = C_1 \eta \quad \text{and} \quad \rho = C_2 e^{-C_1 q} \ . \end{aligned}\right\} \qquad \text{(IV-2-1)}$$

The isothermal latitude, q, is given by the expression (II-4-10)

$$q = \ln \tan (\pi/4 + \xi/2) \ , \qquad \text{(IV-2-2)}$$

and the metacoordinates are computed by equations (II-4-1)

$$\left.\begin{aligned} & \sin \xi = \sin \phi_o \sin \phi + \cos \phi_o \cos \phi \cos (\lambda_o - \lambda), \\ & \text{and} \\ & \tan \eta = \frac{\cos \phi \sin (\lambda_o - \lambda)}{\sin \phi \cos \phi_o - \sin \phi_o \cos \phi \cos (\lambda_o - \lambda)} \ . \end{aligned}\right\} \qquad \text{(IV-2-3)}$$

Thus, the optimization process will determine four unknown parameters: the geographic coordinates of the metapole (ϕ_o, λ_o) and the projection constants C_1 and C_2.

Since the first projection is conformal one scale factor per point must be evaluated. The scale factor in the Lambert conformal cone projection is given by the formula

$$k = C_1 C_2 \frac{e^{-C_1 q}}{\cos \xi} . \qquad \text{(IV-2-4)}$$

Defining the distortion as the natural logarithm of the scale factor

$$v = \ln k , \qquad \text{(IV-2-5)}$$

the optimization model (II-6-11) in the case of conformal projections becomes

$$\sum_{i=1}^{n} v_i^2 \cos \xi_i = \min, \qquad \text{(IV-2-6)}$$

where n is the number of points which approximate the domain.

To linearize the mathematical model (IV-2-4) the Newton method is applied. Then the elements of $\mathcal{B}$ matrix (II-6-20) are defined by the partial derivatives

$$\left.\begin{aligned} b(i,1) &= \frac{\partial v_i}{\partial \phi_o}\,, \\ b(i,2) &= \frac{\partial v_i}{\partial \lambda_o}\,, \\ b(i,3) &= \frac{\partial v_i}{\partial C_1}\,, \\ b(i,4) &= \frac{\partial v_i}{\partial C_2}\,, \text{ for } i=1,\ldots,n. \end{aligned}\right\} \qquad \text{(IV-2-7)}$$

When the equation (IV-2-4) is substituted into (IV-2-5) we obtain

$$v = \ln C_1 + \ln C_2 - C_1 q - \ln \cos \xi \,. \qquad \text{(IV-2-8)}$$

The partial differentiation of the above expression with respect to the unknown vector $[\phi_o \; \lambda_o \; C_1 \; C_2]$ yields

$$b(i,1) = \left(\tan \xi_i - C_1 \cdot \frac{dq_i}{d\xi}\right) \cdot \frac{d\xi_i}{d\phi_o}\,,$$

or since

$$\frac{dq}{d\xi} = \frac{1}{\cos \xi} \quad \text{and} \quad \frac{d\xi}{d\phi_o} = t\,, \qquad \text{(IV-2-9)}$$

we obtain

$$b(i,1) = \left(\tan \xi_i - \frac{C_1}{\cos \xi_i}\right) t_i \; , \qquad \text{(IV-2-10)}$$

where

$$t_i = \frac{\cos \phi_o \sin \phi_i - \sin \phi_o \cos \phi_i \cos (\lambda_o - \lambda_i)}{\cos \xi_i} \; . \qquad \text{(IV-2-11)}$$

The elements of the second column of $\mathcal{B}$ matrix are similarly

$$b(i,2) = \left(\tan \xi_i - \frac{C_1}{\cos \xi_i}\right) u_i \; , \qquad \text{(IV-2-12)}$$

where

$$u_i = \frac{d\xi}{d\lambda_o} = - \frac{\cos \phi_o \cos \phi_i \sin (\lambda_o - \lambda_i)}{\cos \xi_i} \; . \qquad \text{(IV-2-13)}$$

The elements of the final two columns are

$$b(i,3) = \frac{1}{C_1} - q_i \; , \qquad \text{(IV-2-14)}$$

and

$$b(i,4) = \frac{1}{C_2} \; . \qquad \text{(IV-2-15)}$$

The normal aspect of Lambert conformal conic projection presently used in Canada for all small scale mappings has the standard parallels at latitudes

$$\phi' = 49° \quad \text{and} \quad \phi'' = 77° \; .$$

In order to compare optimized map projections with the system presently used the Airy-Kavraiskii measure of quality (II-6-10) will be evaluated for the optimal versions as well as for the official projection. The numerical values of constants C_1 and C_2 for the official projection are calculated from the expressions for the scale factors (IV-2-4) along the standard parallels.

$$k' = k'' = 1 \; ,$$

or

$$C_1 C_2 \, \frac{e^{-C_1 q'}}{\cos \phi'} = C_1 C_2 \, \frac{e^{-C_1 q''}}{\cos \phi''}$$

and then from here

$$C_1 = \frac{\ln \cos \phi' - \ln \cos \phi''}{q'' - q'} = .900\ 745 \; .$$

When the first constant is known the second is obtained either from k' = 1 or k" = 1 yielding

$$C_2 = \frac{e^{C_1 q'} \cos\phi'}{C_1} = 1.766\ 833\ .$$

With such calculated constants the distortions of the linear scale factor were determined at 75 points which approximate the country yielding the following result for the Airy-Kavraiskii measure of quality

$$\hat{E}_{AK} = .021\ 65\ .$$

The initial approximations are very important to ensure convergence of the optimization process. Thus, the unknown vector $[\phi_o\ \lambda_o\ C_1\ C_2]$ must be determined relatively well. In order to compute the first approximation for the metapole (ϕ_o, λ_o) the author has measured from a globe the geographic coordinates of three points which approximate the central line of Canadian territory. Then

$$\tan \lambda_o = \frac{A - B}{C - D}\ , \qquad \text{(IV-2-16)}$$

where

$$\left.\begin{aligned}
A &= (\sin \phi_3 - \sin \phi_2)(\cos \phi_2 \cos \lambda_2 - \cos \phi_1 \cos \lambda_1), \\
B &= (\sin \phi_2 - \sin \phi_1)(\cos \phi_3 \cos \lambda_3 - \cos \phi_2 \cos \lambda_2), \\
C &= (\sin \phi_2 - \sin \phi_1)(\cos \phi_3 \sin \lambda_3 - \cos \phi_2 \sin \lambda_2), \\
D &= (\sin \phi_3 - \sin \phi_2)(\cos \phi_2 \sin \lambda_2 - \cos \phi_1 \sin \lambda_1),
\end{aligned}\right\} \quad (\text{IV-2-17})$$

with known longitude of the metapole its latitude is computed by the formula

$$\tan \phi_o = \frac{\cos \phi_1 \cos (\lambda_1 - \lambda_o) - \cos \phi_2 \cos (\lambda_2 - \lambda_o)}{\sin \phi_2 - \sin \phi_1} \,. \quad (\text{IV-2-18})$$

Having determined the first approximation of the metapole (ϕ_o, λ_o) the author has calculated the meta latitude, ξ, by the first equation (IV-2-3), for several boundary points to obtain the range of metalatitude. The first approximations of constants C_1 and C_2 were derived from the conditions that

$$\left.\begin{aligned}
&k_{\xi_{max}} = k_{\xi_{min}} \,, \\
&\text{and} \\
&k_{\xi_M} = 1 \,,
\end{aligned}\right\} \quad (\text{IV-2-19})$$

where

$$\xi_M = \frac{1}{2}\left(\xi_{max} + \xi_{min}\right) . \qquad \text{(IV-2-20)}$$

The two conditions (IV-2-19) when applied to the formula for the scale factor (IV-2-4) yield

$$C_1 = \frac{\ln \cos \xi_{min} - \ln \cos \xi_{max}}{q_{max} - q_{min}} , \qquad \text{(IV-2-21)}$$

and

$$C_2 = \frac{e^{C_1 q_{max}} \cos \xi_{max}}{C_1} . \qquad \text{(IV-2-21)}$$

Convergence was fast, having thus determined the approximate values of unknowns. The results of optimization and the subsequent Airy-Kavraiskii measure of quality are given in the last section of the chapter.

The Lambert Equiareal Conic projection of a unit sphere is defined by the equations

$$\left.\begin{array}{l} x = \rho \sin \gamma \quad , \quad y = C_2 - \rho \cos \gamma , \\ \\ \text{where} \\ \\ \gamma = C_1 \eta \quad \text{and} \quad \rho = \sqrt{\dfrac{2}{C_1} \left(C_2 - \sin \xi\right)} \end{array}\right\} \qquad \text{(IV-2-22)}$$

and the principal scale factors are

$$n = \frac{\sqrt{2C_1(C_2 - \sin\xi)}}{\cos\ \xi} \quad \text{and} \quad m = \frac{1}{n} . \qquad \text{(IV-2-23)}$$

Distortions of the principal scales are then

$$\left.\begin{aligned} v(2i-1) &= \ln\cos\xi_i - \tfrac{1}{2}\left(\ln C_1 + \ln(C_2 - \sin\xi_i) + \ln 2\right) \\ v(2i) &= -\, v(2i-1) \ . \end{aligned}\right\} \text{(IV-2-24)}$$

Thus every point of the mapping domain yields two of the above equations. In other words the size of the $\mathcal{B}$ matrix is 2n x 4, where n is again the number of points which approximate Canada. The elements of the $\mathcal{B}$ matrix are

$$\left.\begin{aligned} b(2i-1,1) &= \left(-\tan\xi(i) + \frac{\cos\xi(i)}{2(C_2-\sin\xi(i))}\right) t(i) \ , \\ b(2i\ \ ,1) &= -b(2i-1) \ , \end{aligned}\right\} \text{(IV-2-25)}$$

where i=1,....,n and t is defined by the equation (IV-2-11).

$$\left.\begin{aligned} b(2i-1,2) &= \left(-\tan\xi(i) + \frac{\cos\xi(i)}{2\left(C_2-\sin\xi(i)\right)}\right) u(i) \ , \\ b(2i\ \ ,2) &= -b(2i-1,2) \ , \end{aligned}\right\} \text{(IV-2-26)}$$

where u is defined by the expression (IV-2-13).

$$\left.\begin{aligned} b(2i-1,3) &= -\frac{1}{C_1}\,, \\ b(2i\ \ \ ,3) &= -b(2i-1,3)\,, \end{aligned}\right\} \quad (IV\text{-}2\text{-}27)$$

$$\left.\begin{aligned} b(2i-1,4) &= -\frac{1}{2\left(C_2 - \sin\xi(i)\right)}\,, \\ b(2i\ \ \ ,4) &= -\,b(2i-1,4)\,. \end{aligned}\right\} \quad (IV\text{-}2\text{-}28)$$

The initial approximations for constants C_1 and C_2 are again obtained from the range of metalatitudes, ξ_{max} and ξ_{min}, assuming that the scales at the extreme values of metalatitudes will be identical and equal to unity. Then from the first equation of (IV-2-23) we have two equations

$$\cos\xi_{min} = \sqrt{2C_1(C_2 - \sin\xi_{min})}\,,$$

$$\cos\xi_{max} = \sqrt{2C_1(C_2 - \sin\xi_{max})}\,,$$

whose solution gives the unknowns.

$$\left.\begin{aligned} C_2 &= \frac{\cos^2\xi_{min}\,\sin\xi_{max} - \cos^2\xi_{max}\,\sin\xi_{min}}{\cos^2\xi_{min} - \cos^2\xi_{max}}\,, \\ C_1 &= \frac{\cos^2\xi_{min}}{2(C_2 - \sin\xi_{min})}\,. \end{aligned}\right\} \quad (IV\text{-}2\text{-}29)$$

The approximations for the geographic coordinates of the metapole (ϕ_o, λ_o) are derived from the equations (IV-2-16) and (IV-2-18). However in this research the author has simply adopted the values obtained from the optimization process of the Lambert conformal conic projection.

An <u>equidistant conic</u> projection of a unit sphere is defined by the equations

$$x = \rho \sin \gamma \quad , \quad y = C_2 \rho \cos \gamma \ ,$$

where (IV-2-30)

$$\gamma = C_1 \eta \quad \text{and} \quad \rho = C_2 - \xi \ .$$

The principal scale factors are

$$m = 1 \quad \text{and} \quad n = \frac{C_1 (C_2 - \xi)}{\cos \xi} \qquad \text{(IV-2-31)}$$

with the corresponding deformations

$$v(2i-1) = 0 \text{ and } v(2i) = \ln C_1 + \ln (C_2 - \xi_i) - \ln \cos \xi_i \qquad \text{(IV-2-32)}$$

and the subsequent elements of $\mathfrak{B}$ matrix are

$$b(i,1) = C_1 \frac{\left(C_2 - \xi(i)\right) \sin \xi(i) - \cos \xi(i)}{\cos^2 \xi(i)} \cdot t(i) \ , \qquad \text{(IV-2-33)}$$

$$b(i,2) = C_1 \frac{(C_2 - \xi(i)) \sin \xi(i) - \cos \xi(i)}{\cos^2 \xi(i)} \cdot u(i) , \qquad (IV-2-34)$$

$$b(i,3) = \frac{C_2 - \xi(i)}{\cos \xi(i)} , \qquad (IV-2-35)$$

$$b(i,4) = \frac{C_1}{\cos \xi(i)} , \qquad (IV-2-36)$$

where i=1,....,n and t and u functions are defined by the formulae (IV-2-11) and (IV-2-13) respectively.

The initial approximations for constants C_1 and C_2 are, as usual, computed from the extreme values of metalatitude ξ_{min} and ξ_{max} and the corresponding scales equated to unity.

$$\cos \xi_{min} = C_1(C_2 - \xi_{min}) ,$$

$$\cos \xi_{max} = C_1(C_2 - \xi_{max}) ,$$

and then

$$C_2 = \frac{\xi_{max} \cos \xi_{min} - \xi_{min} \cos \xi_{max}}{\cos \xi_{min} - \cos \xi_{max}} , \qquad (IV-2-37)$$

$$C_1 = \frac{\cos \xi_{min}}{C_2 - \xi_{min}} . \qquad (IV-2-38)$$

With such values determined for the constants the convergence was relatively fast.

3. OPTIMAL CYLINDRIC PROJECTIONS

Three cylindric map projections will be optimized for Canadian territory and they are the Mercator conformal, Lambert equiareal and Urmaev's projection. Since in all cylindric mappings the smallest deformations occur in the vicinity of the metaequator the position of the metapole must be selected such that the central line of Canadian territory becomes the metaequator. The author has scaled from a globe two points, (ϕ_1, λ_1) and (ϕ_2, λ_2) and then the first approximations of the geographic coordinates of the metapole (ϕ_o, λ_o) were computed by the formulae

$$\tan \lambda_o = \frac{\tan \phi_1 \cos \lambda_2 - \tan \phi_2 \cos \lambda_1}{\tan \phi_2 \sin \lambda_1 - \tan \phi_1 \sin \lambda_2} , \qquad \text{(IV-3-1)}$$

and

$$\tan \phi_o = \frac{\sin (\lambda_2 - \lambda_1) \cos \lambda_o}{\tan \phi_2 \sin \lambda_1 - \tan \phi_1 \sin \lambda_2} . \qquad \text{(IV-3-2)}$$

An oblique Mercator projection of a unit sphere is defined by the equations

$$x = C\eta \quad , \quad y = Cq , \qquad \text{(IV-3-3)}$$

where

$$q = \ln \tan (\pi/4 + \xi/2) , \qquad \text{(IV-3-4)}$$

and C is an unknown constant to be determined by the optimization process. Thus, there are three unknowns ϕ_o, λ_o, C and the dimension of the $\mathfrak{B}$ matrix is n x 3. The elements of $\mathfrak{B}$ matrix are determined by the differentiation of distortions with respect to the vector of unknowns $[\phi_o, \lambda_o, C]$. The scale factor in the Mercator projection is calculated by the formula

$$k = \frac{C}{\cos \xi} \qquad \text{(IV-3-5)}$$

and the corresponding deformation by

$$v = \ln k = \ln C - \ln \cos \xi \ . \qquad \text{(IV-3-6)}$$

Now the elements b(i,j) for i=1,....,n and j=1,2,3 are

$$b(i,1) = \tan \xi(i)\ t(i) \ , \qquad \text{(IV-3-7)}$$

$$b(i,2) = \tan \xi(i)\ u(i) \ , \qquad \text{(IV-3-8)}$$

and

$$b(i,3) = \frac{1}{C} \ , \qquad \text{(IV-3-9)}$$

where t(i) and u(i) are calculated by the expressions (IV-2-11) and (IV-2-13) respectively.

The initial approximation for the constant was taken to be equal to unity.

An <u>oblique Lambert equiareal cylindric projection</u> is defined by equations.

$$x = Cn \quad , \quad y = \frac{1}{C} \sin \xi \tag{IV-3-10}$$

with the principal scale factors

$$n = \frac{C}{\cos \xi} \quad \text{and} \quad m = \frac{\cos \xi}{C} \tag{IV-3-11}$$

Therefore the distortions are

$$v(2i-1) = \ln C - \ln \cos \xi \quad \text{and} \quad v(2i) = - v(2i-1) \tag{IV-3-12}$$

The subsequent elements of $\mathfrak{B}$ matrix, for i=1,....,n, are

$$\left.\begin{aligned} b(2i-1,1) &= \tan \xi(i)\ t(i) \ , \\ b(2i\ \ ,1) &= - b(2i-1,\ 1) \ , \end{aligned}\right\} \tag{IV-3-13}$$

$$\left.\begin{aligned} b(2i-1,2) &= \tan \xi(i)\ u(i) \ , \\ b(2i\ \ ,2) &= - b(2i-1,2) \ , \end{aligned}\right\} \tag{IV-3-14}$$

$$\left.\begin{aligned} b(2i-1,3) &= \frac{1}{C} \ , \\ b(2i\ \ ,3) &= -b(2i-1,3) \ . \end{aligned}\right\} \tag{IV-3-15}$$

In Urmaev's oblique cylindric projection the rectangular coordinates are given by the equations

$$x = C\eta \quad , \quad y = a_1\xi + a_2\xi^3 + a_3\xi^5 \qquad \text{(IV-3-16)}$$

with the principal scale factors

$$n = \frac{C}{\cos\ \xi} \quad \text{and} \quad m = a_1 + 3a_2\xi^2 + 5a_3\xi^4 \qquad \text{(IV-3-17)}$$

Thus, the expressions for deformation elements are

$$v(2i-1) = \ln n \quad , \quad v(2i) = \ln m \qquad \text{(IV-3-18)}$$

and their partial derivatives with respect to the vector of unknowns $[\phi_o, \lambda_o, a_1, a_2, a_3, c]$ become

$$\left.\begin{aligned} b(2i-1,1) &= \frac{1}{m(i)}\left(6a_2\xi(i) + 20a_3\xi^3(i)\right)\ t(i)\ , \\ b(2i\quad,1) &= \tan\ \xi(i)\ t(i)\ , \end{aligned}\right\} \qquad \text{(IV-3-19)}$$

$$\left.\begin{aligned} b(2i-1,2) &= \frac{1}{m(i)}\left(6a_2\xi(i) + 20a_3\xi^3(i)\right)\ u(i)\ , \\ b(2i\quad,2) &= \tan\ \xi(i)\ u(i)\ , \end{aligned}\right\} \qquad \text{(IV-3-20)}$$

$$\left.\begin{aligned} b(2i-1,3) &= 0\ , \\ b(2i\quad,3) &= \frac{1}{m(i)}\ , \end{aligned}\right\} \qquad \text{(IV-3-21)}$$

$$\left.\begin{aligned} b(2i-1,4) &= \frac{3\ \xi^2(i)}{m(i)}\ , \\ b(2i\ \ ,4) &= 0\ , \end{aligned}\right\} \quad \text{(IV-3-22)}$$

$$\left.\begin{aligned} b(2i-1,5) &= \frac{5\ \xi^4(i)}{m(i)}\ , \\ b(2i\ \ ,5) &= 0\ , \end{aligned}\right\} \quad \text{(IV-3-23)}$$

$$\left.\begin{aligned} b(2i-1,6) &= 0\ , \\ b(2i\ \ ,6) &= \frac{1}{C}\ , \end{aligned}\right\} \quad \text{(IV-3-24)}$$

where again i=1,....,n and functions t(i), u(i) are calculated by the equations (IV-2-11) and (IV-2-13) respectively. Constants a_1, a_2, a_3 and C were initially given the value of unity. Although the optimization process showed later that the initial guesses had been very far from their optimized values the convergence was rapid.

4. OPTIMAL AZIMUTHAL PROJECTIONS

Azimuthal projections may be considered as special cases of conical projections in which the wedge constant, c_1, is equal to unity. Therefore the optimization process includes at most three unknowns, the latitude and longitude of the metapole and a constant. In some projections, like the equidistant azimuthal and the Lambert equiareal azimuthal, only the metapole remains to be determined by the minimization of distortions. In that case the optimization yields the central point of the mapping domain only.

Since the optimization of azimuthal projections is so simple the author will determine an optimal stereographic projection only. The general formulae of an oblique stereographic projection are

$$x = K_1 + \rho \sin \gamma \quad , \quad y = K_2 - \rho \cos \gamma \ , \qquad \text{(IV-4-1)}$$

where

$$\gamma = \eta \quad , \quad \rho = ce^{-q} = c \tan\left(\frac{\pi}{4} - \frac{\xi}{2}\right) \qquad \text{(IV-4-2)}$$

and K_1, K_2 are arbitrarily selected constants. Since the stereographic projection is a conformal projection it has one scale factor per point only which is computed by the formula

$$k = \frac{c}{2 \cos^2 \left(\frac{\pi}{4} - \frac{\xi}{2}\right)} \ . \qquad \text{(IV-4-3)}$$

Thus, the distortion is

$$v = \ln k = \ln c - 2 \ln \cos\left(\frac{\pi}{4} - \frac{\xi}{2}\right) - \ln 2 \ . \qquad \text{(IV-4-4)}$$

The differentiation of the above expression with respect to the vector of unknowns $[\phi_o, \lambda_o, c]$ yields the elements of the $\mathcal{B}$ matrix.

$$b(i,1) = -\tan\left(\frac{\pi}{4} + \frac{\xi(i)}{2}\right) t(i) , \qquad \text{(IV-4-5)}$$

$$b(i,2) = -\tan\left(\frac{\pi}{4} + \frac{\xi(i)}{2}\right) u(i) , \qquad \text{(IV-4-6)}$$

$$b(i,3) = \frac{1}{c} , \qquad \text{(IV-4-7)}$$

where function t(i) and u(i) are again evaluated by the expressions (IV-2-11) and (IV-2-13) respectively and i=1,....,n.

Approximate values of unknowns can easily be determined. The geographic coordinates of the midpoint of a mapping domain serve as the first approximations of ϕ_o and λ_o. The initial approximation of the constant for a stereographic projection of a unit sphere is c = 2 and the optimization process converges quickly.

5. OPTIMAL MODIFIED EQUIAREAL PROJECTIONS

In the fifth section of the second chapter it was shown that the modification of equiareal map projections is a process in which the metacoordinates (ξ,η) are changed into a new coordinate system (u,v) by equations (II-5-21).

$$\sin u = C_m \sin \xi \quad , \quad v = C_n \eta . \qquad \text{(IV-5-1)}$$

The final modification of rectangular coordinates is accomplished by the formulae (II-5-20).

$$X = \frac{C_k}{\sqrt{C_m C_n}}\, x(u,v) \quad , \quad Y = \frac{1}{C_k \sqrt{C_m C_n}}\, y(u,v) \quad , \qquad \text{(IV-5-2)}$$

where C_m, C_n, and C_k are constants to be determined by the optimization process and $x(u,v)$, $y(u,v)$ are the original mapping functions expressed in terms of the new variables (u,v).

The elements of the metric tensor (II-5-8) and the square root of Jacobian determinant (II-5-9) for the modified equiareal mappings are

$$G_{11} = \frac{C_m}{C_n} \frac{\cos^2\xi}{\cos^2 u} K_1 \quad , \quad G_{22} = \frac{C_n}{C_m} K_2 \quad , \quad \sqrt{G} = \frac{\cos\xi}{\cos u} K_3 \quad , \qquad \text{(IV-5-3)}$$

where

$$K_1 = C_k^2\, x_u^2 + \frac{1}{C_k^2}\, y_u^2 \,, \qquad \text{(IV-5-4)}$$

$$K_2 = C_k^2\, x_v^2 + \frac{1}{C_k^2}\, y_v^2 \,, \qquad \text{(IV-5-5)}$$

and

$$K_3 = x_v y_u - x_u y_v \,. \qquad \text{(IV-5-6)}$$

The condition of equiareal mapping is

$$p = m\, n \sin\theta = 1 ,$$

or

$$\sqrt{G} = \cos u . \qquad \text{(IV-5-7)}$$

The computation of the principal scale factors, a and b, are performed by the equations

$$a = \frac{1}{2}(A + B) \quad \text{and} \quad b = \frac{1}{2}(A - B) , \qquad \text{(IV-5-8)}$$

with the corresponding distortion parameters

$$\left.\begin{aligned} v_a &= \ln a = \ln (A + B) - \ln 2 , \\ v_b &= \ln b = \ln (A - B) - \ln 2 , \end{aligned}\right\} \qquad \text{(IV-5-9)}$$

where

$$A^2 = G_{11} + \frac{1}{\cos^2 u} G_{22} + 2 ,$$

and

$$B^2 = G_{11} + \frac{1}{\cos^2 u} G_{22} - 2 . \qquad \text{(IV-5-10)}$$

Applying Newton's method for the linearization of v_a and v_b as functions of unknown parameters C_m, C_n, C_k, ϕ_o and λ_o elements of the $\mathcal{B}$ matrix are obtained. For i=1,....,n, where n is the number of points at which the distortion elements v_a and v_b are evaluated.

$$\left.\begin{aligned} b(2i-1,1) &= \frac{\partial v_a}{\partial C_m} = \frac{1}{A+B}\left(\frac{\partial A}{\partial C_m} + \frac{\partial B}{\partial C_m}\right), \\ b(2i\ \ ,1) &= \frac{\partial v_b}{\partial C_m} = \frac{1}{A-B}\left(\frac{\partial A}{\partial C_m} - \frac{\partial B}{\partial C_m}\right), \end{aligned}\right\} \quad \text{(IV-5-11)}$$

$$\left.\begin{aligned} b(2i-1,2) &= \frac{\partial v_a}{\partial C_n} = \frac{1}{A+B}\left(\frac{\partial A}{\partial C_n} + \frac{\partial B}{\partial C_n}\right), \\ b(2i\ \ ,2) &= \frac{\partial v_b}{\partial C_n} = \frac{1}{A-B}\left(\frac{\partial A}{\partial C_n} - \frac{\partial B}{\partial C_n}\right), \end{aligned}\right\} \quad \text{(IV-5-12)}$$

$$\left.\begin{aligned} b(2i-1,3) &= \frac{\partial v_a}{\partial C_k} = \frac{1}{A+B}\left(\frac{\partial A}{\partial C_k} + \frac{\partial B}{\partial C_k}\right), \\ b(2i\ \ ,3) &= \frac{\partial v_b}{\partial C_k} = \frac{1}{A-B}\left(\frac{\partial A}{\partial C_k} - \frac{\partial B}{\partial C_k}\right), \end{aligned}\right\} \quad \text{(IV-5-13)}$$

$$\left.\begin{aligned} b(2i-1,4) &= \frac{\partial v_a}{\partial \phi_o} = \frac{1}{A+B}\left(\frac{\partial A}{\partial \phi_o} + \frac{\partial B}{\partial \phi_o}\right), \\ b(2i\ \ ,4) &= \frac{\partial v_b}{\partial \phi_o} = \frac{1}{A-B}\left(\frac{\partial A}{\partial \phi_o} - \frac{\partial B}{\partial \phi_o}\right), \end{aligned}\right\} \quad \text{(IV-5-14)}$$

$$\left.\begin{aligned} b(2i-1,5) &= \frac{\partial v_a}{\partial \lambda_o} = \frac{1}{A+B}\left(\frac{\partial A}{\partial \lambda_o} + \frac{\partial B}{\partial \lambda_o}\right), \\ b(2i\ \ ,5) &= \frac{\partial v_b}{\partial \lambda_o} = \frac{1}{A-B}\left(\frac{\partial A}{\partial \lambda_o} - \frac{\partial B}{\partial \lambda_o}\right). \end{aligned}\right\} \quad \text{(IV-5-15)}$$

In order to standardize the approach in partial differentiation for individual modified equiareal map projections new abbreviations are introduced:

$$\frac{\partial A}{\partial C_m} = \frac{1}{2A}\left(E_1 + E_2\right) \quad \text{and} \quad \frac{\partial B}{\partial C_m} = \frac{1}{2B}\left(E_1 + E_2\right) , \qquad \text{(IV-5-16)}$$

where

$$E_1 = \frac{\partial G_{11}}{\partial C_m} \quad \text{and} \quad E_2 = \frac{\partial}{\partial C_m}\left(\frac{G_{22}}{\cos^2 u}\right) . \qquad \text{(IV-5-17)}$$

From the equations of transformation of coordinates (IV-5-1)

$$\frac{du}{dC_m} = \frac{\sin\,\xi}{\cos\,u} \quad \text{and} \quad \frac{dv}{dC_n} = \eta , \qquad \text{(IV-5-18)}$$

and using the definitions of elements of the metric tensor (IV-5-3) the formulae for quantities E_1 and E_2 are easily derived:

$$E_1 = \frac{\cos^2\xi}{C_n\cos^4 u}\left(K_1(\cos^2 u + 2C_m\ \sin u\ \sin\xi) + C_m K_4 \frac{\sin\xi}{\cos u}\right) , \qquad \text{(IV-5-19)}$$

$$E_2 = \frac{C_n}{C_m^2\cos^4 u}\left(K_5\ \sin u\ \cos u - K_2(\cos^2 u - 2\ \sin^2 u)\right) , \qquad \text{(IV-5-20)}$$

where

$$K_4 = \frac{\partial K_1}{\partial u} = 2 \left(C_k^2 \; x_u \; x_{uu} + \frac{1}{C_k^2} \; y_u \; y_{uu} \right) , \qquad \text{(IV-5-21)}$$

and

$$K_5 = \frac{\partial K_2}{\partial u} = 2 \left(C_k^2 \; x_v \; x_{vv} + \frac{1}{C_k^2} \; y_v \; y_{vv} \right) . \qquad \text{(IV-5-22)}$$

with calculated values of E_1 and E_2 equations (IV-5-11) become

$$b(2i-1,1) = \frac{E_1 + E_2}{2AB} , \quad b(2i,1) = -b(2i-1,1) . \qquad \text{(IV-5-23)}$$

In a similar way to equations (IV-5-16) further abbreviations are introduced.

$$\frac{\partial A}{\partial C_n} = \frac{1}{2A} \left(F_1 + F_2 \right) \quad \text{and} \quad \frac{\partial B}{\partial C_n} = \frac{1}{2B} \left(F_1 + F_2 \right) \qquad \text{(IV-5-24)}$$

where

$$F_1 = \frac{\partial G_{11}}{\partial C_n} \quad \text{and} \quad F_2 = \frac{1}{\cos^2 u} \frac{\partial G_{22}}{\partial C_n} \qquad \text{(IV-5-25)}$$

$$F_1 = \frac{C_m}{C_n^2} \frac{\cos^2 \xi}{\cos^2 u} \left(K_6 v - K_1 \right) , \qquad \text{(IV-5-26)}$$

$$F_2 = \frac{1}{C_m \cos^2 u} \left(K_2 + K_7 v \right) , \qquad \text{(IV-5-27)}$$

with

$$K_6 = \frac{\partial K_1}{\partial v} = 2 \left(C_k^2 \; x_u \; x_{uv} + \frac{1}{C_k^2} \; y_u \; y_{uv} \right) . \qquad \text{(IV-5-28)}$$

and

$$K_7 = \frac{\partial K_2}{\partial v} = 2 \left(C_k^2 \; x_v \; x_{vv} + \frac{1}{C_k^2} \; y_v \; y_{vv} \right) . \qquad \text{(IV-5-29)}$$

Thus, the elements of the second column of $\mathfrak{B}$ matrix (IV-5-12) are

$$b(2i-1,2) = \frac{F_1 + F_2}{2AB} \; , \quad b(2i,2) = -b(2i-1,2) \; . \qquad \text{(IV-5-30)}$$

For the third column of matrix $\mathfrak{B}$

$$\frac{\partial A}{\partial C_k} = \frac{1}{2A} \left(G_1 + G_2 \right) \quad \text{and} \quad \frac{\partial B}{\partial C_k} = \frac{1}{2B} \left(G_1 + G_2 \right) , \qquad \text{(IV-5-31)}$$

where

$$G_1 = \frac{\partial G_{11}}{\partial C_k} \quad \text{and} \quad G_2 = \frac{1}{\cos^2 u} \frac{\partial G_{22}}{\partial C_k} , \qquad \text{(IV-5-32)}$$

or

$$G_1 = 2 \frac{C_m}{C_n} \frac{\cos^2\xi}{\cos^2 u} \left(C_k x_u^2 - \frac{1}{C_k^3} y_u^2\right) , \qquad \text{(IV-5-33)}$$

$$G_2 = \frac{2C_n}{C_m \cos^2 u} \left(C_k x_v^2 - \frac{1}{C_k^3} y_v^2\right) , \qquad \text{(IV-5-34)}$$

with final expressions

$$b(2i-1,3) = \frac{G_1 + G_2}{2AB} , \quad b(2i,3) = -b(2i-1,3) . \qquad \text{(IV-5-35)}$$

The last two columns of $\mathfrak{B}$ matrix will be derived in a similar fashion

$$\frac{\partial A}{\partial \phi_o} = \frac{1}{2A} \left(\frac{\partial A^2}{\partial u} \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial \phi_o} + \frac{\partial A^2}{\partial v} \cdot \frac{\partial v}{\partial \eta} \cdot \frac{\partial \eta}{\partial \phi_o}\right) ,$$

or

$$\frac{\partial A}{\partial \phi_o} = \frac{1}{2A} \left(\frac{\partial A^2}{\partial u} T_1 + \frac{\partial A^2}{\partial v} T_2\right) , \qquad \text{(IV-5-36)}$$

where

$$\frac{\partial A^2}{\partial u} = H_1 + H_2 \quad \text{and} \quad \frac{\partial A^2}{\partial v} = H_3 + H_4 . \qquad \text{(IV-5-37)}$$

$$T_1 = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial \phi_o} = \frac{C_m}{\cos u}\left(\cos\phi_o \sin\phi - \sin\phi_o \cos\phi \cos(\lambda_o-\lambda)\right),$$

$$T_2 = \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial \phi_o} = C_n \frac{\sin\eta \cos\eta \sin\xi}{\sin\phi \cos\phi_o - \sin\phi_o \cos\phi \cos(\lambda_o-\lambda)}, \qquad \text{(IV-5-38)}$$

$$H_1 = \frac{\partial G_{11}}{\partial u} = \frac{1}{C_m C_n \cos^2 u}\left((C_m^2 - \sin^2 u)(K_4 + K_1 \sin 2u) - K_1 \sin 2u\right), \qquad \text{(IV-5-39)}$$

$$H_2 = \frac{\partial}{\partial u}\left(\frac{G_{22}}{\cos^2 u}\right) = \frac{C_n}{C_m \cos^2 u}\left(K5 + 2K_2 \tan u\right), \qquad \text{(IV-5-40)}$$

$$H_3 = \frac{\partial G_{11}}{\partial v} = \frac{C_m}{C_n}\frac{\cos^2\xi}{\cos^2 u} K_7, \qquad \text{(IV-5-41)}$$

$$H_4 = \frac{1}{\cos^2 u}\cdot\frac{\partial G_{22}}{\partial v} = \frac{C_n}{C_m \cos^2 u} K_7. \qquad \text{(IV-5-42)}$$

Thus, equation (IV-5-36) becomes

$$\frac{\partial A}{\partial \phi_o} = \frac{1}{2A}\left((H_1 + H_2)\, T_1 + (H_3 + H_4) T_2\right), \qquad \text{(IV-5-43)}$$

and since $\frac{\partial A^2}{\partial X_i} = \frac{\partial B^2}{\partial X_i}$, where X_i is an arbitrary variable,

$$\frac{\partial B}{\partial \phi_o} = \frac{1}{2B}\left((H_1 + H_2)\, T_1 + (H_3 + H_4)\, T_2\right) \qquad \text{(IV-5-44)}$$

$$\left.\begin{aligned} b(2i-1,4) &= \frac{1}{2AB}\left((H_1 + H_2)\, T_1 + (H_3 + H_4)\, T_2\right), \\ b(2i,4) &= -b(2i,4) = -b(2i-1,4). \end{aligned}\right\} \qquad \text{(IV-5-45)}$$

For the last column of $\mathfrak{B}$ matrix

$$\frac{\partial A}{\partial \lambda_o} = \frac{1}{2A} \left(\frac{\partial A^2}{\partial u} \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial \lambda_o} + \frac{\partial A^2}{\partial v} \frac{\partial v}{\partial \eta} \frac{\partial \eta}{\partial \lambda_o}\right) ,$$

or

$$\frac{\partial A}{\partial \lambda_o} = \frac{1}{2A} \left(\frac{\partial A^2}{\partial u} u_1 + \frac{\partial A^2}{\partial v} u_2 \right) , \qquad \text{(IV-5-46)}$$

where

$$u_1 = \frac{\partial u}{\partial \xi} \cdot \frac{\partial \xi}{\partial \lambda_o} = - \frac{C_m}{\cos u} \cos\xi \cos\phi \sin(\lambda_o - \lambda) \qquad \text{(IV-5-47)}$$

and

$$u_2 = \frac{\partial v}{\partial \eta} \cdot \frac{\partial \eta}{\partial \lambda_o} C_n \sin\eta \cos\eta \left(\cot (\lambda_o - \lambda) - \sin\phi_o \tan\eta\right). \qquad \text{(IV-5-48)}$$

$$\left.\begin{aligned} b(2i-5) &= \frac{1}{2AB} \left((H_1 + H_2) u_1 + (H_3 + H_4) u_2\right) , \\ b(2i,5) &= -b(2i-1,5) . \end{aligned}\right\} \qquad \text{(IV-5-49)}$$

The main optimization computer program for equiareal modified map projections was identical to all optimized mapping systems. The differences between various projections were given at the end of the program in subroutines in which partial derivatives x_u, y_u, x_v, y_v, x_{uv}, y_{uv}, x_{uu}, y_{uu}, x_{vv}, y_{vv} were defined.

The author optimized four modified equiareal map projections:

a) Sanson's projection,

b) Mollweide's projection,

c) Hammer's projection, and

d) Eckert's IV projection.

The partial differentiation of transformation equations for individual projections is given in the Appendix III. The inclusion of differential formulae in this section would make an already difficult section completely unreadable.

6. OPTIMIZATION RESULTS OF CONICAL PROJECTIONS

The official version of the normal aspect of the Lambert conformal conic projection with standard parallels at latitudes of 49° and 77° yielded the Airy-Kavraiskii measure of quality (II-6-10) of

$$\hat{E}_{AK} = .0216$$

The first optimization dealt with the normal aspect of the Lambert conformal conic projection. The constants of the projection, c_1 and c_2, were optimized. In other words, indirectly a better choice of standard parallels was made. The improvement resulted in a 30 percent smaller measure of distortion. Further optimization of the metagraticule gave even better results. The final measure of distortion was reduced by more than one half. The Lambert equiareal conic projection gave a slightly better result of optimization than the conformal projection, but the smallest measure of distortion was achieved with the optimized equidistant projection. Although the position of the metapole was expected to be identical in all optimized conic projections the author obtained small differences which are probably caused by numerical evaluation of linearized mathematical models.

At the suggestion of Dr. T. Poiker the author has also determined the optimized Lambert conformal projection where the weights were based on the distribution of population. Since the density of population drastically varies from one side of the country to the other, the initial approximations for unknown parameters were difficult to determine. After many hours of pure trial-and-error attempts, convergence of the optimization process was finally established leading to a reasonably good mapping system.

The optimization of cylindric and azimuthal projections was a more stable process than the optimization of conic projections. Convergence was easy to establish.

The results of the optimization process are given in the following table.

CONIC PROJECTIONS:

PROJECTION:	ϕ_o	λ_o	C_1	C_2	$\hat{E}_{AK}$
Lambert Conformal Conic (official projection)	90° 00'	-	.9007	1.7668	.02165
Lambert Conformal Conic (optimized normal)	90° 00'	-	.8869	1.7888	.01468
Lambert Conformal Conic (weights: population)	79° 20'	0° 36'	.8735	1.7855	.01232
Lambert Conformal Conic (optimized)	72° 16'	-64° 36'	.9571	1.8653	.00961
Equiareal Conic	72° 29'	-58° 48'	.9485	1.0005	.00913
Equidistant Conic	72° 28'	-59° 20'	.9505	1.5785	.00686

Table IV-6-1 Optimized parameters of conic projections

CYLINDRIC AND AZIMUTHAL PROJECTIONS:

Projection:	ϕ_o	λ_o	C	$\hat{E}_{AK}$
Mercator (Conformal Cylindric)	25° 57'	156° 27'	.9904	.1096
Equiareal Cylindric	25° 57'	156° 27'	.9904	.01096
Urmaev's Cylindric*	25° 57'	156° 27'	.9904	.00775
Stereographic	60° 15'	-91° 45'	1.9678	.01130

Table IV-6-2 Optimized parameters of cylindric and azimuthal projections

*Optimized constants of Urmaev's cylindric projection are

$$a_1 = 1.000\ 0000$$
$$a_2 = .000\ 0001$$
$$a_3 = .000\ 0009$$

Thus, the optimized version of Urmaev's projection is almost an equidistant cylindric projection. When the values of coefficients a_2 and a_3 converge to zero the projection becomes a true equidistant cylindric projection.

7. OPTIMIZATION RESULTS OF MODIFIED EQUIAREAL PROJECTIONS

The optimization of modified equiareal projections was much more susceptible to the divergence of the optimization process than was the ease with the conical projection. The determination of reasonable close approximations for unknown parameters was decisive and it required many hours of tedious trial-and-error approach. However, when the final approximations were found they led to very good results with respect to the Airy-Kavraiskii measure of quality.

MODIFIED EQUIAREAL PROJECTIONS:

Projection	ϕ_o	λ_o	c_m	c_n	c_k	$\hat{E}_{AK}$
Sanson	25° 57'	61° 27'	.9915	.8770	1.0620	.0105
Mollweide	25° 57'	61° 27'	.9918	1.0041	1.1000	.0070
Hammer	25° 57'	61° 27'	.9914	1.0000	.9992	.0069
Eckert IV	25° 57'	61° 27'	.9794	.9135	1.2320	.0068

Table IV-7-1 Optimized parameters of modified equiareal projections

8. CHEBYSHEV'S PROJECTIONS FOR CANADA

Theoretical aspects of Chebyshev's projections were discussed in the fourth chapter. Several of the possible ways to satisfy the fundamental requirement of the constant scale factor along the boundary polygon were described in detail. However, for the practical computation the author has decided to use a series of harmonic polynomials (I-7-27) only. The boundary of Canadian territory was approximated by 31 discrete points (See Figure IV-1-1). The optimization process determined the coefficients of the harmonic polynomials by satisfying the fundamental condition of least squares (III-4-2)

$$\sum_{i=1}^{n} (\ln m_i)^2 = \min , \qquad \text{(IV-8-1)}$$

where n extended to all 31 boundary points. Since the number of unknowns is 2k + 1, where k is the order of the harmonic polynomial it is obvious that in the case of 31 boundary points we can go up to the order of 15 for the harmonic polynomial. In that case the boundary isocol will pass through all 31 points. However the higher the order of the interpolating polynomial the larger the system of equations we have to solve for. It is probably the best approach to control the sum of the squares of logarithms of scale (IV-8-1) and if there is no essential improvements between i-th order and i + 1st order to stop further computations. The author has calculated the coefficients for the harmonic polynomials up to the seventh order. The minimized sum of residuals (IV-8-1) for the seventh order polynomial was only slightly better than the sum of the sixth order polynomial. Thus, according to the author's opinion, for the boundary of 31 points which define Canada, there is no need to go further than to the sixth order. It is naturally a questionable problem what one considers a significant improvement and what is an unimportant change. These questions were neither investigated nor answered. The author was rather interested in the optimization process and the inclusion of conformal mappings in that process was made for the sake of completeness. The author is fully aware that optimized conformal mappings have rather a small chance of ever being used in geographic mappings. Equiareal and particularly equidistant projections are much more applicable.

CHEBYSHEV'S PROJECTIONS:

Order:	a_i	b_i	$\sum(\ln m)^2$
1	- .11545 - .20366	- .08570	.20744
2	- .10514 - .08196 - .24218	- .01952 .01502	.04605
3	- .08241 - .06808 - .20076 - .09338	.00980 .06233 .08345	.00814
4	- .08027 - .06793 - .20144 - .11756 .03916	- .02250 .06059 .05741 .02725	.00468
5	- .07953 - .06790 - .18890 - .12520 .04069 - .00199	- .02049 .06843 - .05717 .00151 - .03519	.00419
6	- .07879 - .06919 - .18241 - .14406 .04467 - .02413 .02154	.02052 .07472 .06049 .00623 - .06743 .04119	.00358
7			.003578

Table IV-8-1 Coefficients of harmonic polynomials

9. CONCLUSIONS AND RECOMMENDATIONS

This study of optimization of cartographic projections for small scale mappings was conducted to investigate the general approaches for obtaining the best projections using the Airy-Kavraiskii measure of quality and, in particular, to derive the coefficients for various mapping systems for Canadian territory. The philosophical question whether the Airy-Kavraiskii or the Jordan-Kavaraiskii criterion should be used as the basis of the optimization process was answered by the author's subjective choice of the former. The two criteria should lead to similar results but the application of the Airy-Kavraiskii criterion in the computation process was much simpler. This was the main reason for its selection as the basis of finding the best projections for Canada.

For the sake of completeness the author has added to the research the determination of Chebyshev's projections, i.e. the best conformal mappings, although conformal projections, generally, should not be used for small scale maps in geography. From the class of all analytic functions the author optimized the complex polynomials only.

The optimization results clearly indicate that in the family of conical projections neither conformal nor equiareal mappings belong to the best system. The equidistant oblique conic projection gave the best result and since it is a very simple projection the author highly recommends its application for small scale maps of Canada. Here we have all necessary formulae and coefficients which fully define the recommended projection, namely

$$\phi_o = 72° \; 21', \; \lambda_o = -59° \; 20', \; C_1 = .9505, \; C_2 = 1.5785,$$

$$\left.\begin{aligned}
&\sin \xi = \sin \phi_o \sin \phi + \cos \phi_o \cos \phi \cos (\lambda_o - \lambda) \;, \\
&\tan \eta = \frac{\cos \phi \sin (\lambda_o - \lambda)}{\sin \phi \cos \phi_o - \sin \phi_o \cos \phi_o \cos (\lambda_o - \lambda)} \;, \\
&\gamma = C_1 \eta \;, \quad \rho = C_2 - \xi \;, \\
&x = \rho \sin \gamma \;, \quad y = C_2 - \rho \cos \gamma \;.
\end{aligned}\right\} \quad \text{(IV-9-1)}$$

Since the above equations for the rectangular Cartesian coordinates (x,y) are related to the mapping of a unit sphere the results of the last two equations must be multiplied by an average radius for Canada in the scale of mapping.

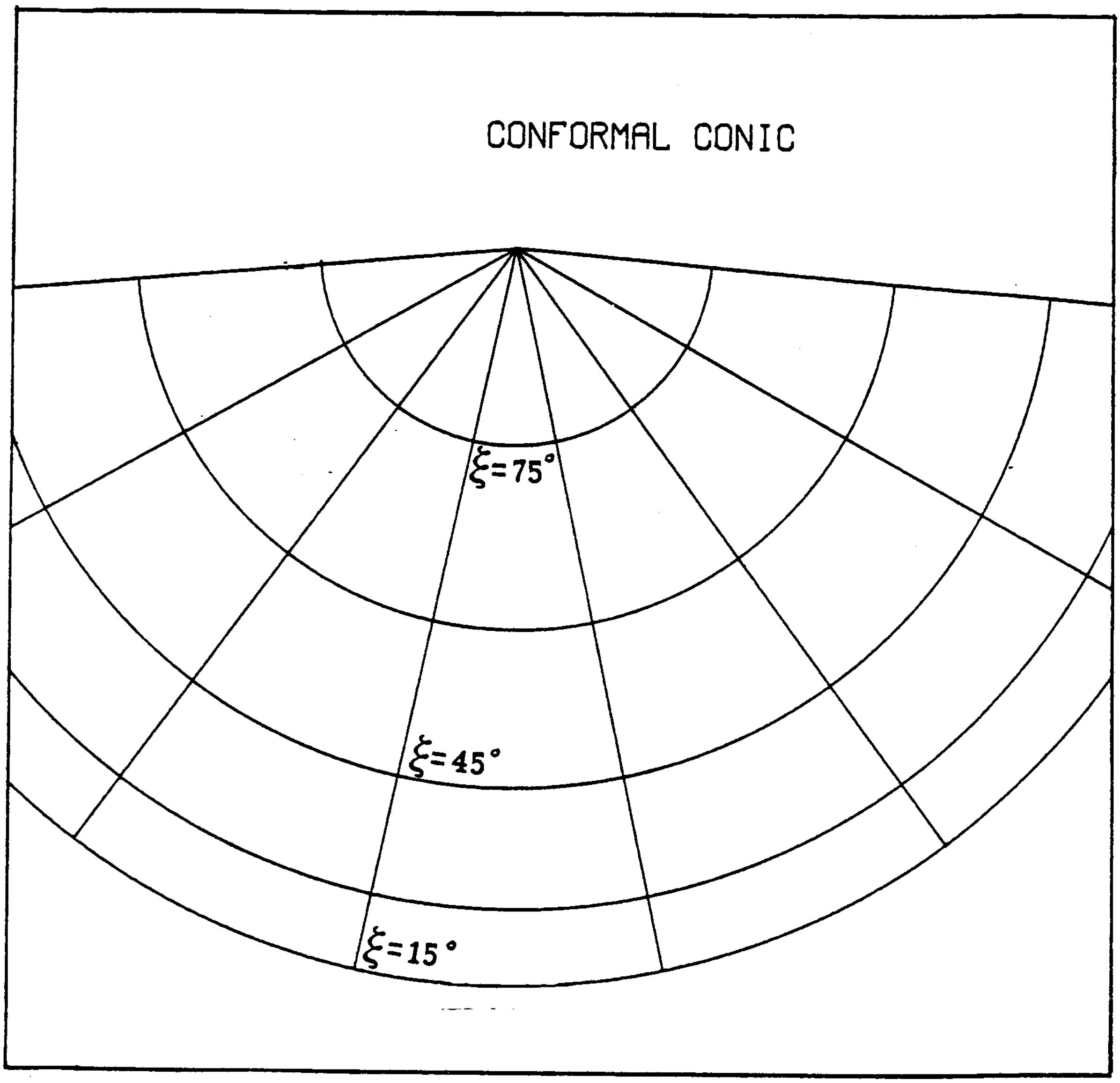
The optimized versions of modified equiareal projections also gave very good results but they are not recommended since the three best projections: Mollweide, Hammer, and Eckert IV involve a numerical solution of a transcendental equation for each mapping point. Because these solutions can only be made by a suitable iterative numerical approach, the computation process becomes too lengthy and complicated. However, when the equiareal property is of special importance and the cost of computation is unimportant then any of the last three modified equiareal projections can be used.

Canada, the United States of America and Mexico will legislate the new geodetic datum in the next few years. One of the results of the new North America Datum will be changes to large scale mappings. Almost all frames of the present large scale topographic maps of Canada will have to be changed. Although small scale atlas maps will not be effected by the change of datum it might be opportune, while undertaking these large modifications and transformations, for geographers to select a better cartographic system. The system suggested by the author has several recommendations.

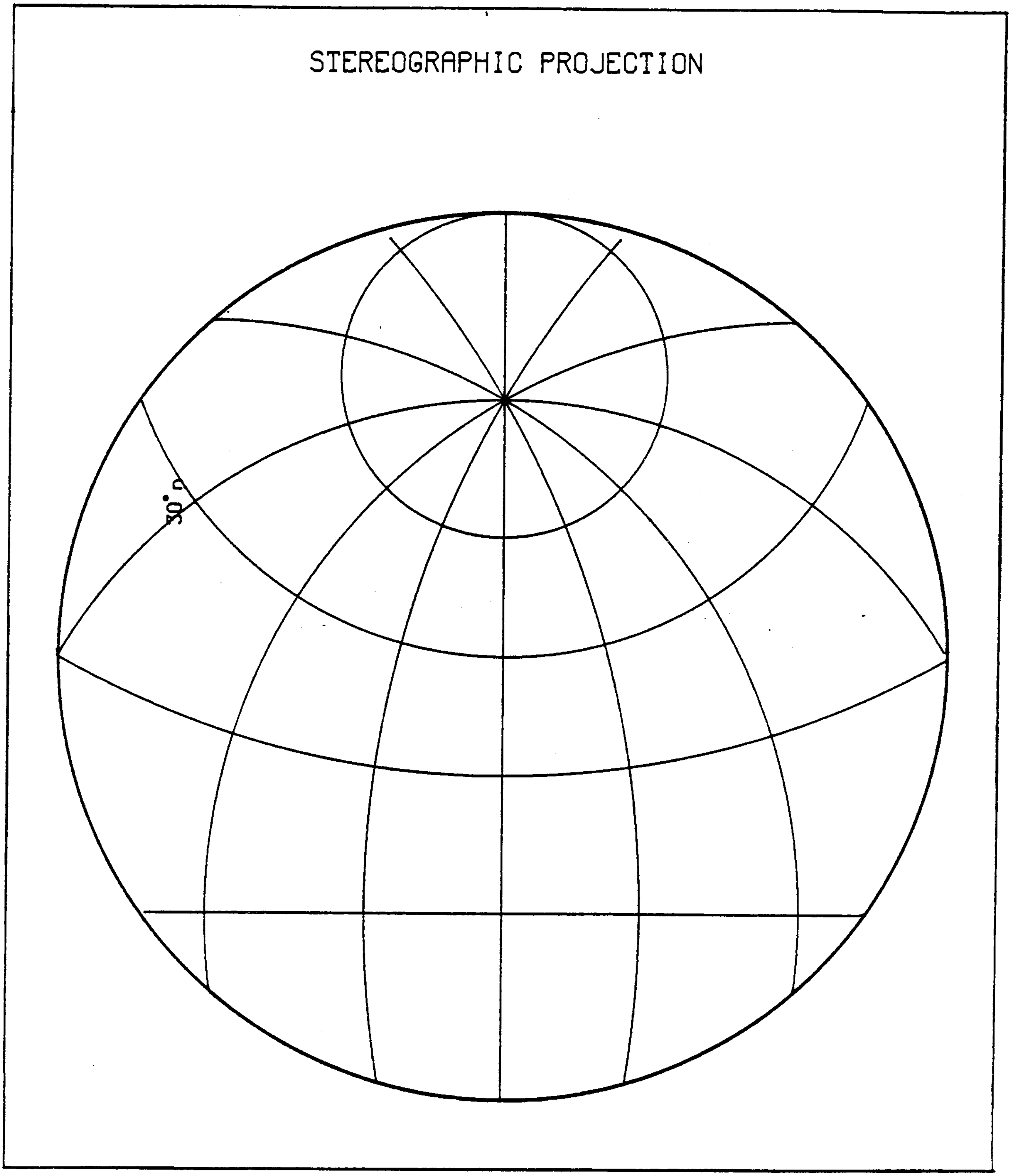
10. EPILOGUE, WRITTEN BY LEWIS CARROLL

"What's the good of Mercator's North Poles
    and Equators,
Tropics, Zones, and Meridian Lines?"
So the Bellman would cry: and the crew
    would reply
"They are merely conventional signs!
Other maps are such shapes, with their
    islands and capes!
But we've got our brave Captain to thank"
    (So the crew would protest)" that he's
    bought us
                THE BEST -
        A PERFECT AND ABSOLUTE BLANK!"

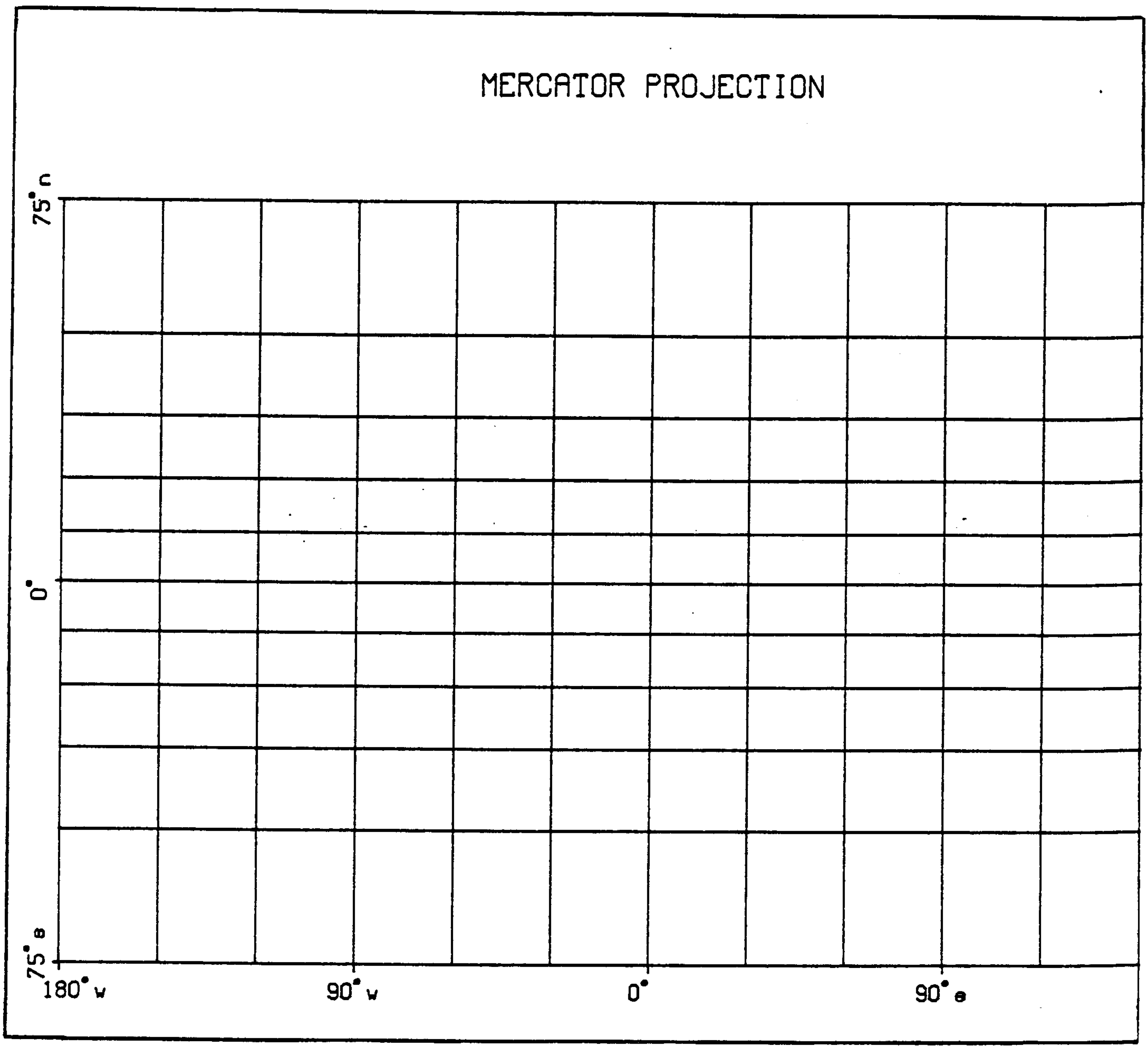
APPENDIX 1: GRAPHICAL PRESENTATIONS OF TYPICAL CONICAL PROJECTIONS



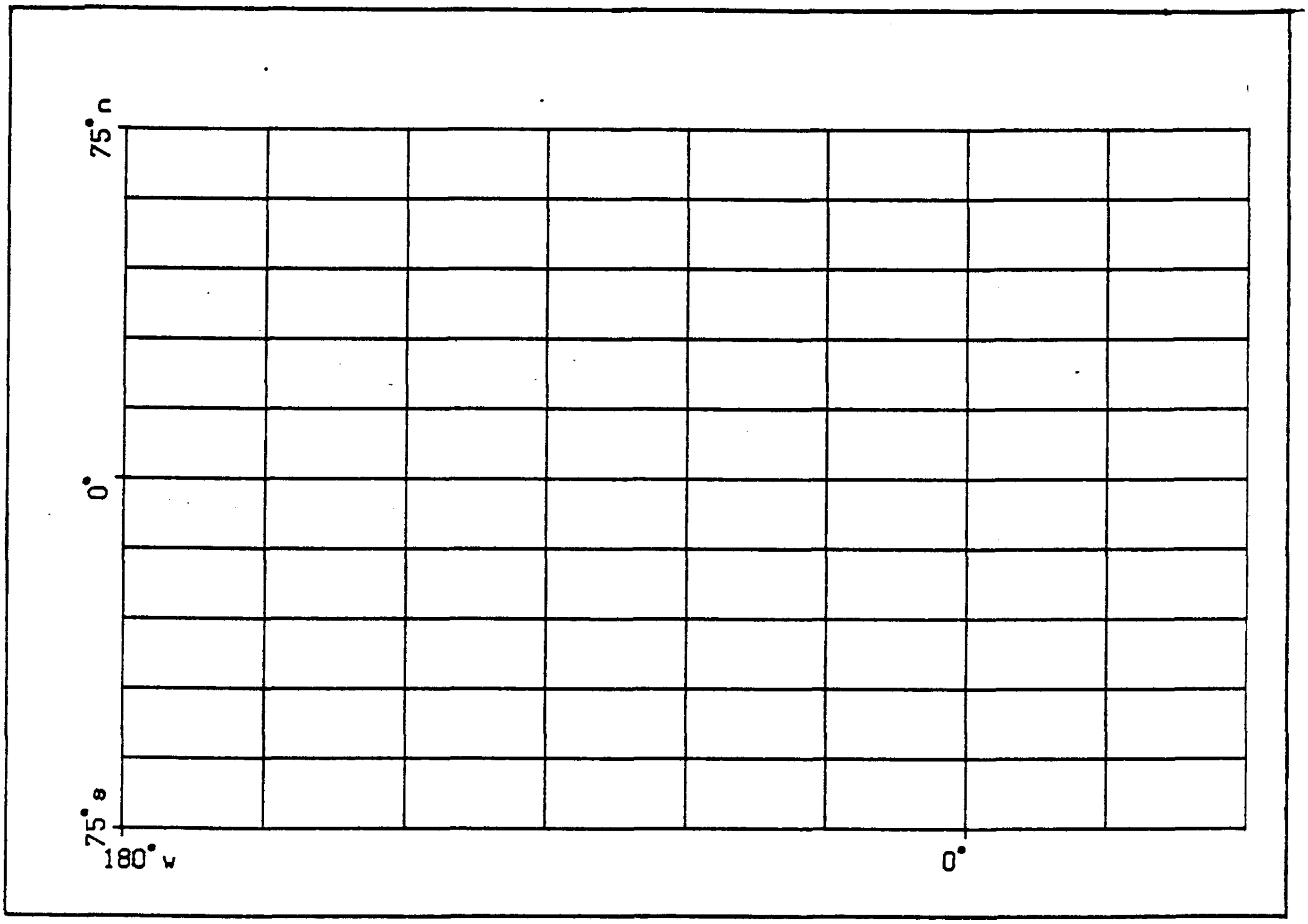
Lambert Conformal Conic Projection



Oblique Aspect of Stereographic Projection

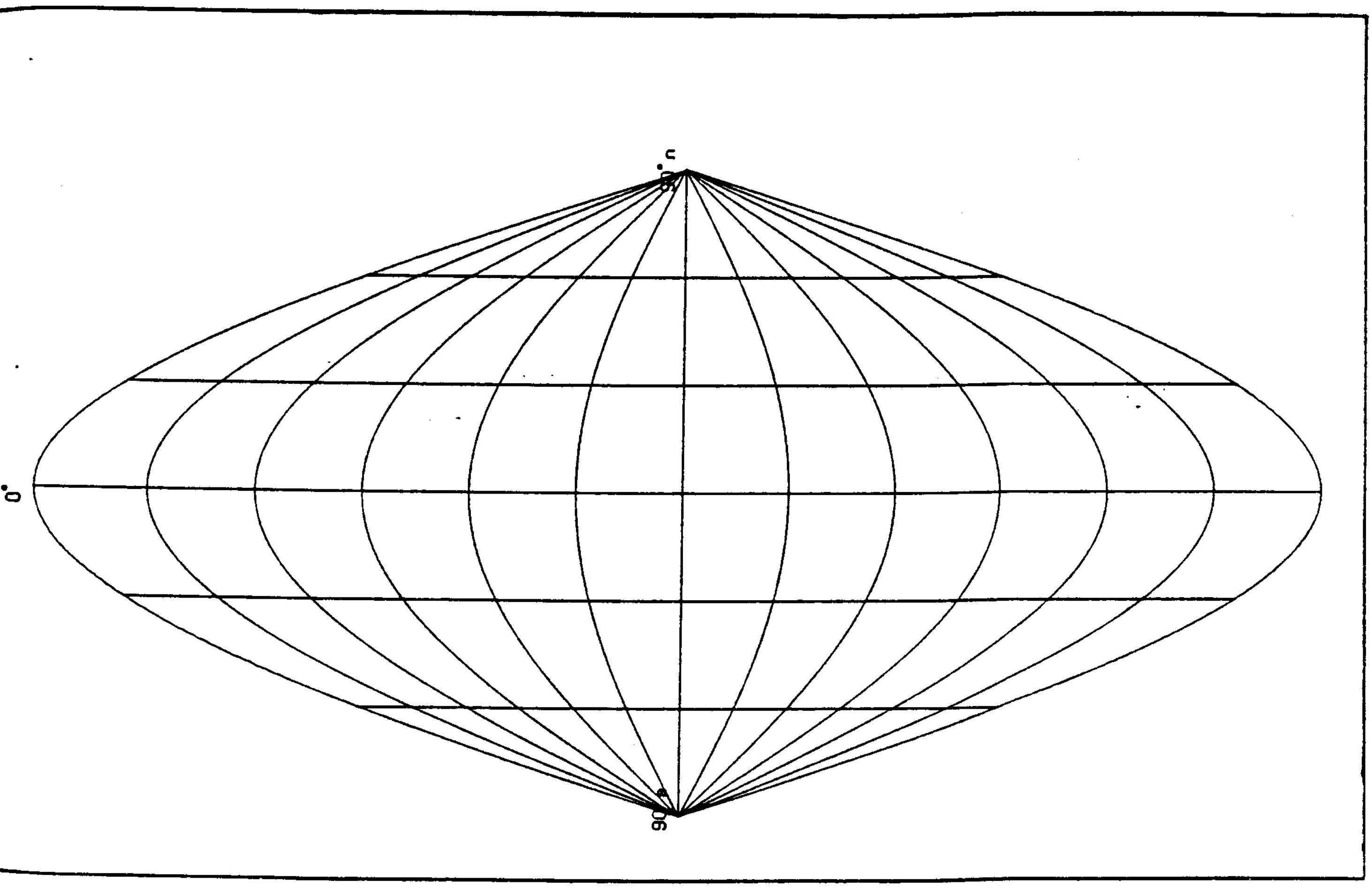


Mercator Projection

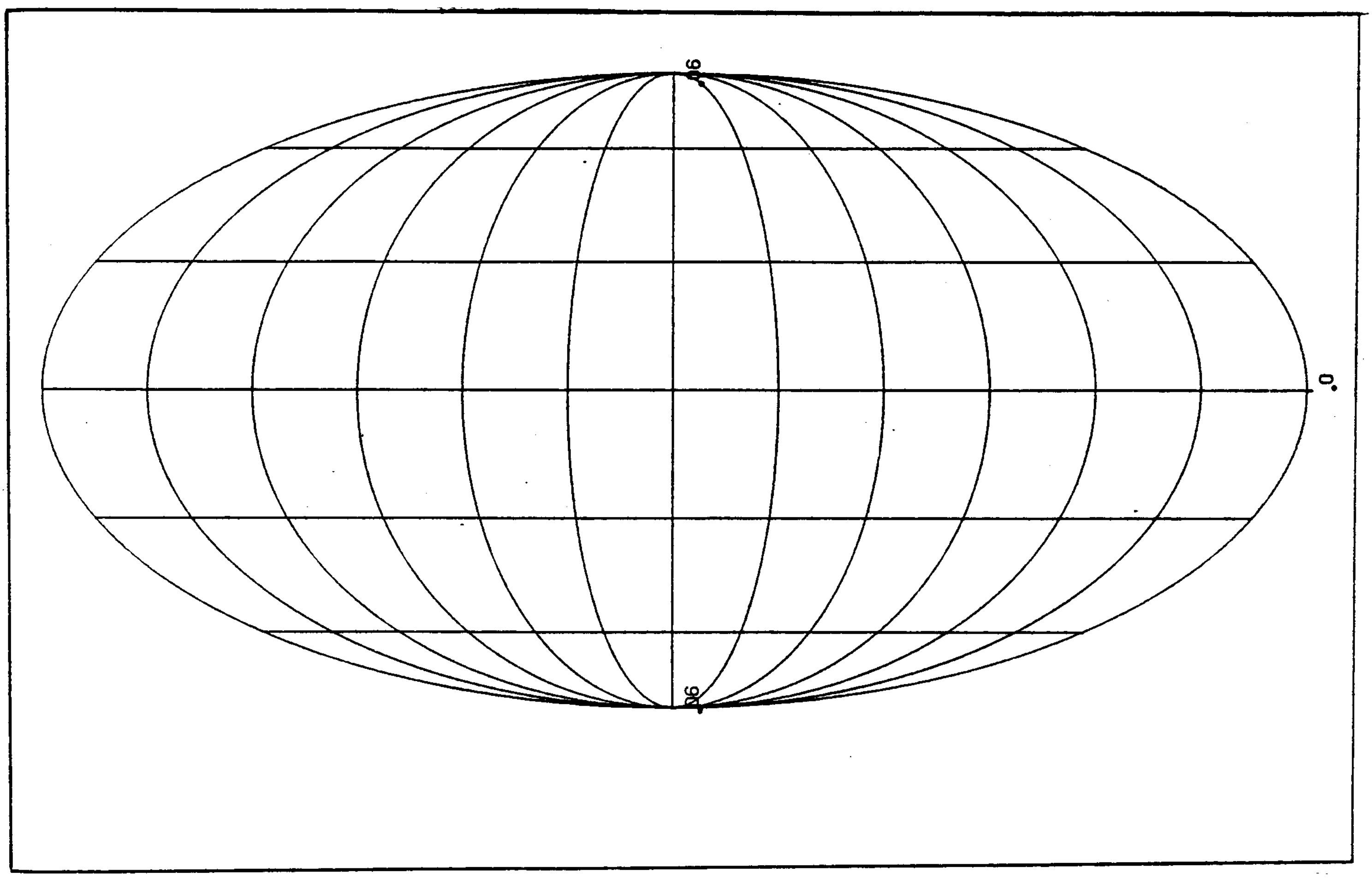


Equidistant Cylindric Projection

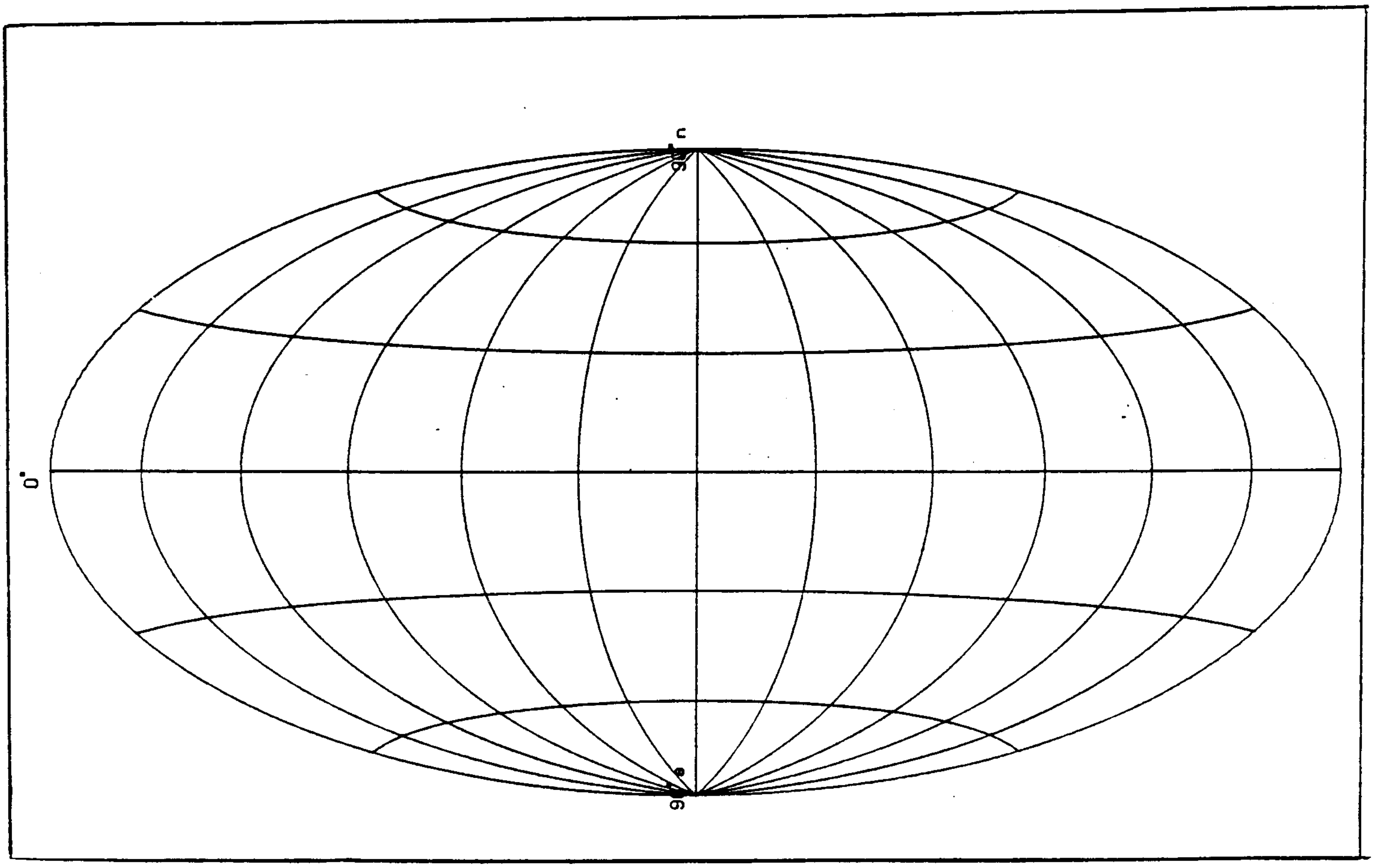
APPENDIX 2: GRAPHICAL REPRESENTATION OF TYPICAL EQUIAREAL MODIFIED PROJECTIONS



Sanson's Projection



Mollweide's Projection



Hammer's Projection

APPENDIX 3: DERIVATION OF FORMULAE FOR OPTIMIZED MODIFIED PROJECTIONS

(i) Sanson's Projection:

$$\boxed{x = v \cos u \quad , \quad y = u \,.}$$

$$x_u = - v \sin u \quad , \quad x_v = \cos u$$

$$y_u = 1 \quad , \quad y_v = 0$$

$$x_{uv} = - \sin u \quad , \quad y_{uv} = 0$$

$$x_{uu} = - v \cos u \quad , \quad y_{uu} = 0$$

$$x_{vv} = 0 \quad , \quad y \;\; = 0$$

(ii) Mollweide's Projection:

$$\boxed{x = \frac{2\sqrt{2}}{\pi} v \cos \psi \quad ; \quad y = \sqrt{2} \sin \psi}$$

$$2\psi + \sin 2\psi = \pi \sin u$$

Computation of ψ:

$$\psi(0) = \frac{\pi}{4} \sin u \qquad \text{for } i = 0 \text{ to } n$$

$$F(i) = 2\psi(i) + \sin 2\psi(i) - \pi \sin u ,$$

$$F'(i) = 2(1 + \cos 2\psi)$$

$$\psi(i+1) = \psi(i) - \frac{F}{F'}$$

The iterative process is repeated until two successive iterations are practically identical, i.e.

$$|\psi(i+1) - \psi(i)| < \varepsilon ,$$

where ε is an arbitrarily selected small number.

$$\frac{d\psi}{du} = \frac{\pi}{2} \frac{\cos u}{1 + \cos 2\psi}$$

$$x_u = \frac{2\sqrt{2}}{\pi} v \sin \psi \frac{d\psi}{du}$$

$$x_v = \frac{2\sqrt{2}}{\pi} \cos \psi$$

$$x_{u_1 v} = \frac{2\sqrt{2}}{\pi} \sin \psi \frac{d\psi}{du}$$

$$x_{uu} = \frac{2\sqrt{2}}{\pi} v \left(\left(\frac{d\psi}{du}\right)^2 \cos \psi + \sin \psi \frac{d^2\psi}{du^2}\right)$$

$$x_{vv} = 0$$

$$y_u = \sqrt{2} \cos \psi \frac{d\psi}{du}$$

$$y_v = y_{uv} = 0$$

$$y_{uu} = \sqrt{2} \left(\cos \psi \frac{d^2\psi}{du^2} - \sin \psi \left(\frac{d\psi}{du}\right)^2\right) ,$$

where

$$\frac{\partial^2\psi}{\partial u^2} = \frac{\pi}{2(1+\cos 2\psi)^2} \left(2 \cos u \sin 2\psi \frac{\partial\psi}{du} - \sin u (1+\cos 2\psi)\right)$$

(iii) Hammer's Projection:

$$\boxed{x = \frac{2 \cos u \sin \frac{v}{2}}{\cos \frac{\delta}{2}} , \quad y = \frac{\sin u}{\cos \frac{\delta}{2}}}$$

$$\cos \delta = \cos u \cos \frac{v}{2} ,$$

$$\frac{d\delta}{du} = \cos \frac{v}{2} \frac{\sin u}{\sin \delta} ,$$

$$\frac{d^2\delta}{du^2} = \cos \frac{v}{2} \frac{\sin \delta \cos u - \sin u \cos \delta \frac{d\delta}{du}}{\sin^2\delta} ,$$

$$\frac{d\delta}{dv} = \cos u \frac{\sin \frac{v}{2}}{2 \sin \delta} ,$$

$$\frac{d^2\delta}{dv^2} = \frac{\cos u}{2 \sin^2\delta} \left(\frac{1}{2} \sin \delta\phi \cos \frac{v}{2} - \sin \frac{v}{2} \cos \delta \frac{d\delta}{dv}\right) ,$$

$$\frac{d^2\delta}{du\ dv} = - \frac{\sin u}{\sin^2\delta} \left(\frac{1}{2} \sin \delta \sin \frac{v}{2} + \cos \delta \cos \frac{v}{2} \frac{d\delta}{dv}\right) ,$$

$$x_u = \frac{\sin \frac{v}{2}}{\cos^2 \frac{\delta}{2}} \left(\cos u \sin \frac{\delta}{2} \frac{d\delta}{du} - 2 \sin u \cos \frac{\delta}{2}\right) ,$$

$$x_v = \frac{\cos u}{\cos^2 \frac{\delta}{2}} \left(\cos \frac{\delta}{2} \cos \frac{v}{2} + \sin \frac{\delta}{2} \sin \frac{v}{2} \frac{d\delta}{dv}\right) ,$$

$$x_{uv} = \frac{1}{\cos^4\frac{\delta}{2}}\Big(\cos^2\frac{\delta}{2}\Big(\frac{1}{2} \cos\frac{v}{2}\Big(\cos u \sin\frac{\delta}{2} \frac{d\delta}{du} - 2 \sin u \cos \frac{\delta}{2}\Big) +$$

$$+ \sin \frac{v}{2} \Big(\cos u \Big(\frac{1}{2} \cos \frac{\delta}{2} \frac{d\delta}{du} \frac{d\delta}{dv} + \sin \frac{\delta}{2} \frac{d^2\delta}{du\ dv} +$$

$$+ \sin u \sin \frac{\delta}{2} \frac{d\delta}{dv}\Big)\Big) +$$

$$+ \sin \delta \frac{1}{2} \sin\frac{v}{2}\Big(\cos u \sin\frac{\delta}{2} \frac{d\delta}{du} - 2 \sin u \cos\frac{\delta}{2}\Big)\frac{d\delta}{dv}\Big) ,$$

$$x_{uu} = \frac{\sin\frac{v}{2}}{\cos^4\frac{\delta}{2}} \Big(\cos^2\frac{\delta}{2} \Big(- \sin u \sin\frac{\delta}{2} \frac{d\delta}{du} + \frac{1}{2} \cos u \cos \frac{\delta}{2}\Big(\frac{d\delta}{du}\Big)^2 +$$

$$+ \cos u \sin\frac{\delta}{2} \frac{d^2\delta}{du^2} - 2 \cos u \cos\frac{\delta}{2} + \sin u \sin\frac{\delta}{2} \frac{d\delta}{du}\Big) +$$

$$+ \frac{1}{2} \Big(\cos u \sin \frac{\delta}{2} \frac{d\delta}{du} - 2 \sin u \cos \frac{\delta}{2}\Big) \sin \delta \frac{d\delta}{du}\Big) ,$$

$$x_{vv} = \frac{\cos u}{2 \cos^4 \frac{\delta}{2}} \Big(\cos^2 \frac{\delta}{2} \Big(\cos\frac{\delta}{2} \sin \frac{v}{2} \Big(\frac{d\delta}{dv}\Big)^2 - \cos\frac{\delta}{2} \sin \frac{v}{2} +$$

$$+ 2 \sin \frac{\delta}{2} \sin \frac{v}{2} \frac{d^2\delta}{dv^2}\Big) +$$

$$+ \sin \delta \frac{d\delta}{dv} \Big(\cos \frac{\delta}{2} \cos \frac{v}{2} + \sin \frac{\delta}{2} \sin \frac{v}{2} \frac{d\delta}{dv}\Big)\Big) ,$$

$$y_u = \frac{1}{\cos^2 \frac{\delta}{2}} \Big(\cos \frac{\delta}{2} \cos u + \frac{1}{2} \sin u \sin \frac{\delta}{2} \frac{d\delta}{du}\Big) ,$$

$$y_v = \frac{\sin u \sin \frac{\delta}{2} \frac{d\delta}{dv}}{2 \cos^2 \frac{\delta}{2}} ,$$

$$y_{uv} = \frac{1}{2 \cos^4 \frac{\delta}{2}} \Big(\cos^2\frac{\delta}{2} \Big(\frac{1}{2} \sin u \cos\frac{\delta}{2} \frac{d\delta}{du}\cdot\frac{d\delta}{dv} - \sin\frac{\delta}{2} \cos \frac{d\delta}{dv} +$$

$$+ \sin u \sin\frac{\delta}{2} \frac{d^2\delta}{du dv}\Big) + \sin\delta\frac{d\delta}{dv}\Big(\cos\frac{\delta}{2}\cos u + \frac{1}{2}\sin u \sin\frac{\delta}{2} \frac{d\delta}{du}\Big)\Big) ,$$

$$y_{uu} = \frac{1}{2\cos^4\frac{\delta}{2}} \left(\cos^2\frac{\delta}{2}\left(\frac{1}{2}\sin u \cos\frac{\delta}{2}\left(\frac{d\delta}{du}\right)^2 - 2\cos\frac{\delta}{2}\sin u + \right.\right.$$

$$\left.\left. + \sin u \sin\frac{\delta}{2}\frac{d^2\delta}{du^2}\right) + \sin\delta\frac{d\delta}{du}\left(\cos\frac{\delta}{2}\cos u + \frac{1}{2}\sin u \sin\frac{\delta}{2}\frac{d\delta}{du}\right)\right) ,$$

$$y_{vv} = \frac{\sin u}{2\cos^4\frac{\delta}{2}} \left(\cos^2\frac{\delta}{2}\left(\frac{1}{2}\cos\frac{\delta}{2}\left(\frac{d\delta}{du}\right)^2 + \sin\frac{\delta}{2}\frac{d^2\delta}{dv^2} + \right.\right.$$

$$\left.\left. + \sin u \sin\delta \sin\frac{\delta}{2}\left(\frac{d\delta}{dv}\right)^2\right) .$$

(iv) <u>Eckert's IV Projection</u>:

$$\boxed{x = \frac{2}{\sqrt{\pi(4+\pi)}}\, v\,(1+\cos\Psi) \; , \; y = \frac{2\sqrt{\pi}}{\sqrt{4+\pi}} \sin\Psi}$$

Computation of Ψ:

$$\Psi(0) = \frac{1}{2}\left(1 + \frac{\pi}{4}\right)\sin u ,$$

$$F(i) = \Psi(i) + 2\sin\Psi(i) + \frac{1}{2}\sin 2\Psi(i) - \left(2 + \frac{\pi}{2}\right)\sin u ,$$

$$F'(i) = 1 + 2\cos\Psi(i) + \cos 2\Psi(i)$$

$$\Psi(i+1) = \Psi(i) - \frac{F(i)}{F'(i)} \qquad \text{for } i=1 \text{ to } n$$

until

$$|\Psi(i+1) - \Psi(i)| < \varepsilon \ .$$

$$\frac{d\Psi}{du} = \frac{(2 + \frac{\pi}{2}) \cos u}{1 + 2 \cos \Psi + \cos 2\Psi} ,$$

$$\frac{d^2\Psi}{du^2} = \frac{2 + \frac{\pi}{2}}{(1 + 2 \cos\Psi + \cos 2\Psi)^2}\Big(2 \cos u \frac{d\Psi}{du} (\sin\Psi + \sin 2\Psi) -$$

$$- \sin u \,(1 + 2 \cos \Psi + \cos 2\Psi)\Big) ,$$

$$x_u = - \frac{2}{\sqrt{\pi(4+\pi)}} v \sin \Psi \frac{d\Psi}{du} ,$$

$$x_v = \frac{2}{\sqrt{\pi(4+\pi)}} (1 + \cos \Psi) ,$$

$$x_{uv} = - \frac{2}{\sqrt{\pi(4+\pi)}} \sin \Psi \frac{d\Psi}{du} ,$$

$$x_{uu} = - \frac{2v}{\sqrt{\pi(4+\pi)}} \Big(\cos \Psi \left(\frac{d\Psi}{du}\right)^2 + \sin \Psi \frac{d^2\Psi}{du^2}\Big) ,$$

$$x_{vv} = 0 ,$$

$$y_u = \frac{2\sqrt{\pi}}{\sqrt{4+\pi}} \cos \Psi \frac{d\Psi}{du} ,$$

$$y_v = y_{uv} = y_{vv} = 0 ,$$

$$y_{uu} = \frac{2\sqrt{\pi}}{\sqrt{4+\pi}} \Big(\cos \Psi \frac{d^2\Psi}{du^2} - \sin \Psi \left(\frac{d\Psi}{du}\right)^2\Big) .$$

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