Stochastic Analysis of Life Insurance Surplus

by

Natalia Lysenko

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Name: Natalia Lysenko

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Examining Committee: Dr. Carl J. Schwarz
Chair

____

Dr. Gary Parker
Senior Supervisor
Simon Fraser University

____

Dr. Cary Chi-Liang Tsai
Simon Fraser University

____

Dr. Richard Lockhart
External Examiner
Simon Fraser University

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Abstract

The behaviour of insurance surplus over time for a portfolio of homogeneous life policies in an environment of stochastic mortality and rates of return is examined. We distinguish between stochastic and accounting surpluses and derive their first two moments. A recursive formula is proposed for calculating the distribution function of the accounting surplus. We then examine the probability that the surplus becomes negative in any given insurance year. Numerical examples illustrate the results for portfolios of temporary and endowment life policies assuming an AR(1) process for the rates of return.

Keywords: insurance surplus, stochastic rates of return, AR(1) process, stochastic mortality, distribution function

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Contents

List of Tables

List of Figures

Chapter 1

Introduction

Understanding stochastic properties of life insurance surplus is essential for insurance companies to make business decisions that will guarantee a high probability of solvency. Life insurers face a variety of risks. How many death benefits have to be paid out in any given year? Uncertain timing of contingent cash flows gives rise to mortality or insurance risk. A life insurance policy is typically purchased by a series of periodic payments called (contract) premiums. This series of payments contingent on policyholders survival to the time when each payment comes due constitutes a life annuity of premiums. The premiums are invested in the market to earn interest. But what interest rates will prevail in the market in the future? Many insurance contracts have a fairly long term (think, for example, of a whole life insurance issued to someone aged 30), in which case ignoring the stochastic nature of rates of return will lead to a significant understatement of the true riskiness of these contracts. There are other sources of uncertainty arising in the life insurance context (future expenses, lapses, etc.), but the environment of stochastic mortality and rates of return already presents many challenges for analyzing life insurance contracts.

The theory of life contingencies evolved from a deterministic treatment of various risks to the introduction of a methodology for the stochastic treatment of decrements at first and later of rates of return. Traditionally, in order to take into account different sources of risk, including decrements due to mortality, disability, etc. as well as interest rates, deterministic discounting for each source of risk was used (see Jordan

(1967)). However, under this approach it was impossible to obtain information about the likelihood and magnitude of random deviations from the mean discounted values. In the text by Bowers et al. (1986), the theory of life contingencies was extended to incorporate the random nature of decrements. The concept of a random survivorship group was introduced by relating the survival function and the life table, which allowed for the full use of probability theory; whereas, the so-called deterministic survivorship group approach, previously used in actuarial science and based on the rates (as opposed to probabilities) of decrement summarized in the life table, did not have this flexibility and thus ignored the stochastic nature of mortality. Under this framework the mortality risk can be quantified by considering such summary measures as standard deviation, median and percentiles of actuarial functions' distributions.

Although mortality risk is definitely an important risk component of the life insurance business, in many circumstances it can be at least partially diversified by increasing the size of the business. On the other hand, investment risk cannot be diversified and in some cases its relative size can be quite large. Thus, models for stochastic rates of return must be incorporated in the theory of life contingencies.

Search for a useful model for rates of return that can be employed in actuarial applications can be traced back to the 1970's. Since by now there is a very extensive literature on this topic, no attempt is made here to provide a complete list of related papers. Instead, some of the key papers are mentioned to demonstrate what kind of models have been considered and in what applications they were used.

Before choosing a model for stochastic rates of return, one needs to decide, first of all, what exactly has to be modeled since there are several possibilities including an effective interest rate, force of interest and force of interest accumulation function. Other questions that have to be addressed are related to the dependence structure of rates of return in successive time intervals (i.e., should the rates be assumed to be independent or correlated?) and the type of model (i.e., should a continuous or discrete time framework be used?). This fairly wide range of possibilities for modeling interest rate randomness led researchers to consider a variety of models.

In a number of early papers on the subject it was assumed that the forces of interest in successive years were uncorrelated and identically distributed (i.e., the force of interest is generated by a white noise process). Waters (1978) used this assumption to calculate the first four moments of the compound interest and actuarial functions and to obtain the limiting distribution of the average sums of actuarial functions by fitting Pearson curves. This assumption about rates of return was also made by Boyle(1976) and Dufresne (1990) among others.

A more realistic assumption is to assume that the forces of interest in successive years are correlated. Various time series models have been employed for this purpose. For example, Pollard (1971) used an autoregressive process of order two.

Panjer and Bellhouse (1980) developed a general theory for both discrete and continuous stochastic interest models for determining the moments of the present value of deterministic and contingent cash flows. Then the authors specifically considered discrete time autoregressive models of order one and two (with real roots of characteristic polynomial only) as well as their continuous time analogue and applied their results to a whole life insurance policy and a life annuity. A shortcoming of this paper is that by considering stationary autoregressive models, future rates of return are assumed to be independent of past and current rates. In Bellhouse and Panjer (1981), the results were extended to models in which forces of interest depend on a number of past and current rates. This was achieved by using discrete time conditional autoregressive processes. Numerical illustrations for the price of a pure discount bond, an annuity certain, a whole life insurance and a life annuity were provided assuming a conditional autoregressive process of order one.

Dhaene (1989) further extended the work done by Panjer and Bellhouse as well as by Giaccotto (1986) to the case when the force of interest follows an autoregressive integrated moving average process of order (p, d, q), ARIMA (p, d, q). The paper presented a methodology for efficient computation of the moments of present value functions.

Stochastic differential equations (SDE) also found their use in the actuarial literature. For example, Beekman and Fuelling (1990) used the Ornstein-Uhlenbeck process (a first order linear SDE, also known as the Vasicek model in the finance literature) to model the force of interest accumulation function. As an application, they derived formulas for the mean values and standard deviations of future payment streams, both deterministic (an annuity certain) and contingent (a life annuity). Numerical examples illustrated the results for different values of parameters. In a series of publications by Parker (e.g., Parker (1993), Parker (1994a), Parker (1996) and Parker (1997)), the author also used the Ornstein-Uhlenbeck process but for the force of interest rather than for the force of interest accumulation function. In addition, Parker (1995) studied a second order linear stochastic differential equation for the force of interest process. Three cases for the roots of the characteristic equation were considered (real and distinct roots, real and equal roots, and complex roots). It was demonstrated in the paper that this model is able to combine the effects of a tendency to continue a recent trend and of a mean reverting property. This indicates that a second order process is more flexible compared to a first order process, which could only have one of those properties, usually the mean reversion. Numerical examples were given for the expected value and variance of a discounting function and an annuity certain.

A couple of remarks can be made at this point. In some cases, whether a discrete model or a continuous one is used does not alter the dynamic of the process. For example, a conditional AR(1) process is the discrete analogue of the Ornstein-Uhlenbeck process; also a discrete representation of a second order SDE is the ARMA(2,1) process. So, a choice of a discrete or a continuous model can be a personal preference. Refer to the text by Pandit and Wu (1983) for a discussion of the principle of covariance equivalence that can be used to establish parametric relations between discrete and continuous representations of a process.

It can be noted that some of the researcher chose to model the force of interest and others the force of interest accumulation function. In the paper by Parker (1994b) the difference between these two modeling approaches was discussed. Numerical illustrations of the expected value, standard deviation and skewness of an annuity immediate for a number of processes under each of the two approaches were presented, which demonstrated a different stochastic behavior of present value functions under the two modeling methods. Further, to provide more insight into the implicit behavior of the force of interest process under the two approaches, the conditional expected value of the force of interest accumulation function up to time t given its value up to time s $(s < t)$ and the force of interest at time s was examined. It was revealed that this conditional expectation does not depend on the value of the force of interest at time s when modeling the force of interest accumulation function. It is more realistic to assume that this conditional expectation depends on the force of interest at time s, which is the case when modeling the force of interest. A numerical example indicated that one possible implication of modeling the force of interest accumulation function is that the expected value of the force of interest in the immediate future can be significantly away from its current value. This illustration shows that modeling the force of interest accumulation function has limited practical value.

Most of the papers mentioned above gave the first two or three moments of present value and actuarial functions when only one stream of payments or only one policy was considered. Some generalizations to these applications include studying the whole distribution (either using the density function or the cumulative distribution function) for a portfolio of identical contracts or a general portfolio.

Frees (1990) presented the first two moments of the net single premium of a single insurance contract and an annuity. Premium determination under the equivalence principle and explicit extension to reserves including the second moment of the loss function were presented. The results were derived at first assuming that the forces of interest in each time interval are independent and identically distributed (i.i.d.) each following the same normal distribution and then using a moving average process of order one, MA(1). In the second part of the paper, the author considered a block of business. He proposed to approximate the distribution of the average loss random variable for a block of identical policies by another random variable, which is equal to the expected value of the loss random variable for one policy where expectation is taken over time-of-death random variable and follows the limiting distribution of the average loss when the number of policies in the block approaches infinity. A suggestion for a recursive calculation of the distribution function of the average loss under the i.i.d. assumption for interest rates was given. Finally, the limiting distribution of surplus, defined as the excess of assets over liabilities, for the case of full matching of assets and projected liabilities was presented.

Norberg (1993) derived the expected value and variance of the liability associated with one contract and then of the total liability for a portfolio of policies assuming the Ornstein-Uhlenbeck process for the force of interest. He proposed a simple solvency criterion, which requires an insurer to maintain a reserve equal to the expected value plus a multiple of the standard deviation of the loss random variable. Numerical results for an authentic portfolio were provided.

The expected value, standard deviation and coefficient of skewness as well as an approximate distribution of the present value of future deterministic cash flows when the force of interest is modeled by the white noise process, the Wiener process and the Ornstein-Uhlenbeck process were presented in Parker (1993). The approximation of the cumulative distribution function was based on a recursive integral equation. An n-year certain annuity-immediate was used for numerical illustrations. The same approximation technique for the distribution function was then applied to the limiting portfolios of identical temporary and endowment insurance contracts in Parker (1994a) and Parker (1996) with the Ornstein-Uhlenbeck process for the force of interest used for illustrative purposes (see Coppola et al. (2003) for an application of the method to large annuity portfolios). In Parker (1997), these results were further generalized to a general limiting portfolio of different life insurance policies such as temporary, endowment and whole life. In addition, this paper discussed a way of splitting the riskiness of the portfolio into an insurance risk and an investment risk (see also Bruno et al. (2000)). Although high accuracy of the approximation used by the author in the above-mentioned papers was justified, Parker (1998) presented a method for obtaining the exact distribution of the discounted and accumulated values of deterministic cash flows based on recursive double integral equations. The two results that will be derived in Chapter 5 use a variation of this method.

Marceau et al.(1999) studied the prospective loss random variable for general portfolios of life insurance contacts and compared its first two moments as well as the distribution functions obtained via Monte Carlo simulation method for portfolios of different sizes (including a limiting portfolio) and different composition. Numerical examples for portfolios of temporary, endowment and a combination of temporary and endowment life policies, in which the force of interest assumed to follow a conditional AR(1) process, were presented. It was observed that the convergence rate of the variance of the loss random variable for a portfolio of temporary contracts is much slower compared to the other two portfolios. This indicates that the mortality risk component is large compared to the non-diversifiable investment risk in any portfolio of a realistic size, and as a result of this a use of the distribution of the limiting portfolio to approximate the distribution of a finite portfolio is questionable in a context of temporary life insurances.

Another approximation for the distribution of the loss random variable for a single life annuity and for a homogeneous portfolio is given in Hoedemakers (2005). This approximation is based on the concept of comonotonicity. Upper and lower bounds on quantiles of a distribution were obtained and their convex combination was demonstrated to be a very accurate approximation of the true distribution. The authors chose to model the force of interest accumulation function by a Brownian motion with a drift and an Ornstein-Uhlenbeck process.

The focus of the papers mentioned above is on the stochastically discounted value of future deterministic or contingent cash flows with the cash flows being viewed and valued at the same point in time. For example, the net single premium of a life insurance policy or a life annuity is viewed and valued at the issue date. In the reserve calculation only cash flows that will be incurred by the inforce policies at a given valuation date are taken into account and the experience of the portfolio prior to that valuation date is ignored. In the case of a single policy, if the policyholder does not survive to a given valuation date, no reserve is allocated for that policy.

We will develop a model and perform our analysis in a different framework. To illustrate our approach consider a closed block of life insurance business at its initiation. All the quantities of interest that we study are measured at some given dates in the future but are viewed at the initiation date. This framework allows assessing adequacy of, for example, initial surplus level, pricing and future reserving method *before* the block of business is launched.

Let us follow this block of business in time. Fix one of the valuation dates and refer to it as time r. Prior to time r, the insurer collects premiums and pays death benefits according to the terms of the contract. So, by time r, the insurer's assets from this block of business are equal to the accumulated value of past premiums net of death benefits paid. After time r, the insurer will continue paying benefits as they come due and receive periodic premiums. The discounted value at time r of all future benefits net of all future premiums to be collected constitutes the insurer's liabilities.

The value of assets in excess of the value of liabilities represents the surplus. It is this quantity that we attempt to study. It is worth mentioning at this point that in the context of a portfolio of life insurance policies we will distinguish between two types of surplus. One of them we call a stochastic surplus, which was briefly described in general terms above. The other type of surplus we refer to as an accounting surplus. Although the name may suggest a deterministic nature of the accounting surplus, in fact it is a stochastic quantity. The difference between the two types of surplus lies in how liabilities are defined. In the case of accounting surplus the liability is the actuarial reserve, which is typically some summary measure (e.g., expected value) or a statistic of the prospective loss random variable, whereas in the case of stochastic surplus it is the prospective loss random variable itself.

For insurance regulators it is important that insurance companies maintain an adequate surplus level. To represent actuarial liabilities, the insurers are required to report their actuarial reserves calculated in accordance with regulations. So, when monitoring insurance companies, the regulators actually look at what we call the accounting surplus. We propose a formula for obtaining the distribution function of accounting surplus at any given valuation date. One piece of information that is readily available from this distribution function is the probability that the surplus falls below zero at any given time r. If this probability is too high, say above 5%, then perhaps the insurer should make some adjustments to the terms of the contract such as, for example, increasing the premium rate or raising additional initial surplus.

Assumptions regarding the model for rates of return and decrements due to mortality are presented in Chapter 2. In Chapter 3 we develop a methodology for studying a single life insurance policy. The ideas for one policy are further extended to study a portfolio of homogeneous policies in Chapters 4 and 5. In Chapter 4 we define two types of insurance surpluses and derive their first two moments. A method for computing the distribution function of the accounting surplus is discussed in Chapter 5. Concluding remarks and areas for future research are provided in Chapter 6.

Chapter 2

Model Assumptions

2.1 Stochastic Rates of Return

For illustrative purposes, we choose to model the force of interest by a conditional autoregressive process of order one, AR(1). A similar model was used, for example, by Bellhouse and Panjer (1981) and Marceau et al. (1999). However, the results that will be presented in the later chapters may also allow the use of other more general Gaussian models.

Let $\delta(k)$ be the force of interest in period $(k-1, k]$, $k = 1, 2, \ldots, n$, with a possible realization denoted by δ_k. The forces of interest $\{\delta(k); k = 1, 2, \ldots, n\}$ satisfy the following autoregressive model:

$$\delta(k) - \delta = \phi\,[\delta(k-1) - \delta] + \varepsilon_k, \tag{2.1}$$

where $\varepsilon_k \sim \mathrm{N}(0, \sigma^2)$ and δ is the long-term mean of the process. We assume that $|\phi| < 1$ to ensure stationarity of the process.

For our further discussion, it is convenient to introduce a notation for the force of interest accumulation function, which is then used to study both discounting and accumulation processes.

Let $I(s, r)$ denote the force of interest accumulation function between times s and

r, $(0 \leq s \leq r)$ [1]. It is given by

$$I(s,r) = \begin{cases} \sum_{j=s+1}^{r} \delta(j), & s < r, \\ 0, & s = r. \end{cases} \tag{2.2}$$

The AR(1) model has already been extensively studied and many results about it are readily available in the literature. In the rest of this section we will extend some of the known results to obtain the distribution of the force of interest accumulation function between any two given times, both unconditional and conditional.

For our further analysis, when two or more force of interest accumulation functions are involved, it will be necessary to distinguish three cases for the times between which the accumulation occurs. Suppose we are interested in obtaining the value at some given time r of two cash flows occuring at times s and t.

- If $s < t < r$ (i.e., both cash flows occur prior to time r), the values at time r of these cash flows need to be accumulated using $I(s,r)$ and $I(t,r)$;
- If $r < s < t$ (i.e., both cash flows occur after time r), the values at time r of these cash flows need to be discounted using $I(r,s)$ and $I(r,t)$;
- If $s < r < t$ (i.e., one cash flow occurs before time r and the other one occurs after time r), the values at time r of these cash flows need to be accumulated and discounted using $I(s,r)$ and $I(r,t)$ respectively.

When $I(s,r)$ follows a Gaussian process, only the first two moments are necessary to completely determine its distribution. In particular, we will derive the expected value, the variance and the autocovariance for $I(s,r)$ conditional on both the starting value of the process at time 0, $\delta(0)$, as well as on the terminal value, $\delta(r)$, for $0 < s < r$.

Similar steps can be taken to derive analogous results for the force of interest accumulation function between times r and t, $0 < r < t$, conditional on the starting value of the process $\delta(0)$ and the value of the process at time r, $\delta(r)$; i.e., for $\{I(r,t) \mid \delta(0), \delta(r)\}$, $r < t$. However, note that in this case we can use the results for $I(r,t)$, $r < t$ conditional only on $\delta(r)$, since $I(r,t)$ satisfies the Markovian property. (See Cairns and Parker (1997) for similar derivations).

[1]This notation was motivated by the notation used in Marceau et al. (1999).

To calculate the moments of $I(s,r)$, $0 < s < r$, conditional on $\delta(0)$ and $\delta(r)$, it is convenient first to derive the unconditional moments of $I(s,r)$, $0 < s < r$ (i.e., moments for the stationary distribution of $I(s,r)$, $0 < s < r$), which are then used to obtain the moments of $I(s,r)$, $0 < s < r$, conditional only on the starting value, $\delta(0)$, and consequently on both the starting and the terminal values of the process.

It is well known that for the stationary AR(1) process defined in (2.1)

$$\begin{aligned} \mathrm{E}[\delta(j)] &= \delta, \\ \mathrm{Cov}[\delta(i), \delta(j)] &= \frac{\sigma^2}{1-\phi^2}\phi^{|i-j|}. \end{aligned}$$

Then, for $s < r$,

$$\mathrm{E}[I(s,r)] = \mathrm{E}\Big[\sum_{j=s+1}^{r} \delta(j)\Big] = \sum_{j=s+1}^{r} \mathrm{E}[\delta(j)] = (r-s)\delta \tag{2.3}$$

and

$$\begin{aligned} \mathrm{Var}[I(s,r)] &= \mathrm{Var}[\sum_{j=s+1}^{r} \delta(j)] \\ &= \sum_{j=s+1}^{r}\sum_{i=s+1}^{r} \mathrm{Cov}[\delta(j), \delta(i)] \\ &= (r-s)\,\mathrm{Var}[\delta(j)] + 2\sum_{j=s+1}^{r-1}\sum_{i=j+1}^{r} \mathrm{Cov}[\delta(j), \delta(i)] \\ &= (r-s)\frac{\sigma^2}{1-\phi^2} + 2\sum_{j=s+1}^{r-1}\sum_{i=j+1}^{r} \frac{\sigma^2}{1-\phi^2}\phi^{i-j} \\ &= \frac{\sigma^2}{1-\phi^2}[r-s+2\frac{\phi}{1-\phi}(r-s-1-\frac{\phi}{1-\phi}(1-\phi^{r-s-1}))]. \end{aligned} \tag{2.4}$$

The covariance terms, corresponding to the three cases for the force of interest accumulation functions mentioned earlier, are given by the following formulas:

Case 1: $s < t < r$

$$\begin{aligned}
\mathrm{Cov}[I(s,r), I(t,r)] &= \sum_{i=s+1}^{r} \sum_{j=t+1}^{r} \mathrm{Cov}[\delta(i), \delta(j)] \\
&= \sum_{i=t+1}^{r} \sum_{j=t+1}^{r} \mathrm{Cov}[\delta(i), \delta(j)] + \sum_{i=s+1}^{t} \sum_{j=t+1}^{r} \mathrm{Cov}[\delta(i), \delta(j)] \\
&= \mathrm{Var}[I(t,r)] + \sum_{i=s+1}^{t} \sum_{j=t+1}^{r} \frac{\sigma^2}{1-\phi^2} \phi^{j-i} \\
&= \mathrm{Var}[I(t,r)] + \frac{\sigma^2}{1-\phi^2} \frac{\phi}{(1-\phi)^2} (\phi^t - \phi^r)(\phi^{-t} - \phi^{-s}).
\end{aligned}$$

Case 2: $r < s < t$

$$\begin{aligned}
\mathrm{Cov}[I(r,s), I(r,t)] &= \sum_{i=r+1}^{s} \sum_{j=r+1}^{t} \mathrm{Cov}[\delta(i), \delta(j)] \\
&= \sum_{i=r+1}^{s} \sum_{j=r+1}^{s} \mathrm{Cov}[\delta(i), \delta(j)] + \sum_{i=r+1}^{s} \sum_{j=s+1}^{t} \mathrm{Cov}[\delta(i), \delta(j)] \\
&= \mathrm{Var}[I(r,s)] + \sum_{i=r+1}^{s} \sum_{j=s+1}^{t} \frac{\sigma^2}{1-\phi^2} \phi^{j-i} \\
&= \mathrm{Var}[I(r,s)] + \frac{\sigma^2}{1-\phi^2} \frac{\phi}{(1-\phi)^2} (\phi^s - \phi^t)(\phi^{-s} - \phi^{-r}).
\end{aligned}$$

Case 3: $s < r < t$

$$\begin{aligned}
\mathrm{Cov}[I(s,r), I(r,t)] &= \sum_{i=s+1}^{r} \sum_{j=r+1}^{t} \mathrm{Cov}[\delta(i), \delta(j)] \\
&= \sum_{i=s+1}^{r} \sum_{j=r+1}^{t} \frac{\sigma^2}{1-\phi^2} \phi^{j-i} \\
&= \frac{\sigma^2}{1-\phi^2} \frac{\phi}{(1-\phi)^2} (\phi^r - \phi^t)(\phi^{-r} - \phi^{-s}).
\end{aligned}$$

Next we consider the moments of $I(s,r)$ when the force of interest follows a conditional AR(1) process. Two approaches can be used in this case. One of them involves directly applying the definition of $I(s,r)$ as being a sum of $\delta(j)$'s each of which follows a conditional AR(1) process. Another approach is to use the fact that $\{\delta(j); j = 0, 1, ...\}$ and any linear combinations of $\delta(j)$'s have a multivariate normal distribution, in which case known results from the multivariate normal theory can be applied (e.g., see Johnson and Wichern (2002)).

Either approach produces

$$\mathrm{E}[I(s,r)|\delta(0)=\delta_0] = (r-s)\delta + \frac{\phi}{1-\phi}(\phi^s - \phi^r)(\delta_0 - \delta). \tag{2.5}$$

Note that, when applying the second approach, we use the following formula:

$$\mathrm{E}[I(s,r)|\delta(0)=\delta_0] \;=\; \mathrm{E}[I(s,r)] + \frac{\mathrm{Cov}[I(s,r),\delta(0)]}{\mathrm{Var}[\delta(0)]}(\delta_0 - \mathrm{E}[\delta(0)]),$$

where $\mathrm{Cov}[I(s,r),\delta(0)]$ can be derived as follows:

$$\begin{aligned}
\mathrm{Cov}[I(s,r),\delta(0)] &= \mathrm{Cov}\Big[\sum_{j=s+1}^{r}\delta(j),\delta(0)\Big] \\
&= \sum_{j=s+1}^{r}\mathrm{Cov}[\delta(j),\delta(0)] \\
&= \sum_{j=s+1}^{r}\frac{\sigma^2}{1-\phi^2}\phi^j \\
&= \frac{\sigma^2}{1-\phi^2}\frac{\phi}{1-\phi}(\phi^s-\phi^r). \qquad (2.6)
\end{aligned}$$

Refer to Appendix A.1 for more details on the derivation of the above results.

The second approach is more general and, therefore, it is more convenient for numerical calculations.

Conditional variance and covariance are given by

$$\begin{aligned}
\mathrm{Var}[I(s,r)|\delta(0)] &= \mathrm{Var}[I(s,r)] - \frac{\mathrm{Cov}[I(s,r),\delta(0)]^2}{\mathrm{Var}[\delta(0)]} \\
&= \frac{\sigma^2}{1-\phi^2}\Bigg[r-s+\frac{2\phi}{1-\phi}\Big(r-s-1-\frac{\phi}{1-\phi}(1-\phi^{r-s-1})\Big) - \\
&\quad -(\frac{\phi}{1-\phi})^2(\phi^s-\phi^r)^2\Bigg] \qquad (2.7)
\end{aligned}$$

and, for $s<r<t$,

$$\mathrm{Cov}[I(s,r),I(r,t)|\delta(0)] = \mathrm{Cov}[I(s,r),I(r,t)] - \frac{\mathrm{Cov}[I(s,r),\delta(0)]\cdot\mathrm{Cov}[I(r,t),\delta(0)]}{\mathrm{Var}[\delta(0)]}. \tag{2.8}$$

Notice that Equation (2.8) with $s < r < t$ corresponds to case 3 described for unconditional covariances between force of interest accumulation functions. Formulas for the other two cases, namely $\mathrm{Cov}[I(s,r), I(t,r)|\delta(0)]$ for $s < t < r$ and $\mathrm{Cov}[I(r,s), I(r,t)|\delta(0)]$ for $r < s < t$, are analogous.

Finally, when conditioning on both the current force of interest and the force of interest at some given time r in the future, the expected value and variance of $I(s,r)$, $s < r$, can be calculated from the following formulas:

$$\begin{aligned} \mathrm{E}[I(s,r)|\delta(0)=\delta_0, \delta(r)=\delta_r] &= \mathrm{E}[I(s,r)|\delta(0)] + \\ &+ \frac{\mathrm{Cov}[I(s,r), \delta(r)|\delta(0)]}{\mathrm{Var}[\delta(r)|\delta(0)]}(\delta_r - \mathrm{E}[\delta(r)|\delta(0)]) \end{aligned}$$

and

$$\mathrm{Var}[I(s,r)|\delta(0), \delta(r)] = \mathrm{Var}[I(s,r)|\delta(0)] - \frac{\mathrm{Cov}[I(s,r), \delta(r)|\delta(0)]^2}{\mathrm{Var}[\delta(r)|\delta(0)]}.$$

Similarly, for $I(r,t)$, $r < t$,

$$\begin{aligned} \mathrm{E}[I(r,t)|\delta(0)=\delta_0, \delta(r)=\delta_r] &= \mathrm{E}[I(r,t)|\delta(0)] + \\ &+ \frac{\mathrm{Cov}[I(r,t), \delta(r)|\delta(0)]}{\mathrm{Var}[\delta(r)|\delta(0)]}(\delta_r - \mathrm{E}[\delta(r)|\delta(0)]) \\ &= (t-r)\delta + \frac{\phi}{1-\phi}(1-\phi^{t-r})(\delta_r - \delta) \end{aligned}$$

and

$$\mathrm{Var}[I(r,t)|\delta(0), \delta(r)] = \mathrm{Var}[I(r,t)|\delta(0)] - \frac{\mathrm{Cov}[I(r,t), \delta(r)|\delta(0)]^2}{\mathrm{Var}[\delta(r)|\delta(0)]}.$$

But because the process is Markovian, we also have

$$\mathrm{E}[I(r,t)|\delta(r)=\delta_r, \delta(0)=\delta_0] = \mathrm{E}[I(0,t-r)|\delta(0)=\delta_r]$$

and

$$\mathrm{Var}[I(r,t)|\delta(0), \delta(r)] = \mathrm{Var}[I(0,t-r)|\delta(0)],$$

and Equations (2.5), (2.7) and (2.8) can be applied.

In our notation, a discount function from time t to time r and an accumulation function from time s to time r for $s < r < t$ are given by $e^{-I(r,t)}$ and $e^{I(s,r)}$ respectively.

Since each $\delta(k)$ is normally distributed, so is any linear combination of $\delta(k)$'s. This implies that $-I(r,t) \sim N\big(-\mathrm{E}[I(r,t)], \mathrm{Var}[I(r,t)]\big)$ and

$I(r,s) \sim N\big(\mathrm{E}[I(r,s)],\, \mathrm{Var}[I(r,s)]\big)$, and that both discount and accumulation functions follow a lognormal distribution.

If $Y \sim N\big(\mathrm{E}[Y],\, \mathrm{Var}[Y]\big)$, then the m^{th}-moment of e^Y is

$$\mathrm{E}\big[e^{mY}\big] = e^{m\,\mathrm{E}[Y] + \frac{m^2}{2}\,\mathrm{Var}[Y]}. \tag{2.9}$$

We can use Equation (2.9) to find moments of $e^{-I(r,t)}$ and $e^{I(s,r)}$.

In our numerical examples, we use the following arbitrarily chosen values of the parameters:

Parameter	Value
ϕ	0.90
σ	0.01
δ	0.06
δ_0	0.08

2.2 Decrements

Following the notation developed in Bowers et al. (1986), let T_x be the future lifetime of a person aged x years, also referred to as a *life-age-x* and denoted by (x).
$\mathbf{P}(T_x \leq t) = {}_tq_x$ and $\mathbf{P}(T_x > t) = {}_tp_x$ are the distribution function and survival function of the continuous random variable T_x respectively.
Let K_x be the *curtate-future-lifetime* of (x); that is, K_x is a discrete random variable representing the number of complete years remaining until the death of (x). Its probability mass function and distribution function are given by

$$\mathbf{P}(K_x = k) = \mathbf{P}(k \leq T_x < k+1) = {}_{k|}q_x\,, \qquad k = 0,1,2,\ldots$$

and

$$\mathbf{P}(K_x \leq k) = \mathbf{P}(T_x < k+1) = \mathbf{P}(T_x \leq k+1) = {}_{k+1}q_x\,, \qquad k = 0,1,2,\ldots\,.$$

A nonparametric life table is used to determine the distribution of K_x. If l_x denotes the number of lives aged x from the initial survivorship group, then the probability that (x) will survive for k years is ${}_kp_x = \frac{l_{x+k}}{l_x}$ and the probability that (x) will survive for $k-1$ years and then die in the next year (i.e., (x) will die in the k^{th} year) is ${}_{k-1|}q_x = \frac{l_{x+k-1} - l_{x+k}}{l_x}$.

In our numerical examples we use the Canada 1991, age nearest birthday (ANB), male, aggregate, population mortality table.

The future lifetimes of policyholders in an insurance portfolio are assumed to be independent and, in a portfolio of homogeneous policies, also identically distributed.

2.3 Summary of Assumptions

In this section we more formally restate the main assumptions of our model.
$K_x^{(i)}$ is the curtate-future-lifetime of individual i aged x.
We will consider a class of functions, denoted $\mathscr{G}_i$, which depend on $K_x^{(i)}$ and a sequence of forces of interest $\{\delta(k),\, k = 0, 1, \ldots\}$.

1. The random variables $\{K_x^{(i)}\}$ are independent and identically distributed.
2. The random variables $\{K_x^{(i)}\}$ and $\{\delta(k),\, k = 0, 1, \ldots\}$ are independent.
3. Conditional on $\{\delta(k),\, k = 0, 1, \ldots\}$, $\{\mathscr{G}_i\}$ are independent and identically distributed.

In our model, we consider two random processes. One process is related to the mortality experience of a portfolio and the other one is a sequence of future stochastic rates of return. In order to study these two processes simultaneously, we assume that future lifetimes and future rates of return are independent. This is stated in Assumption 2. Assumption 3 implies that there is one type of insurance policies being sold to a group of independent policyholders with similar characteristics. Note, however, that the values of those policies are not independent because they are invested in the same financial instruments.

Chapter 3

Single Life Insurance Policy

3.1 Methodology

Consider a single life policy issued to a person aged x, which pays a death benefit b at the end of the year of death if death occurs within n years since the policy issue date and a pure endowment benefit c if the person survives to time n. Note that if c is equal to zero then the policy is referred to as an n-year *temporary* insurance, and if c is nonzero than we deal with an n-year *endowment* insurance. In our examples, we will consider a special case of the endowment contract when c is equal to b, since this is the most basic design in practice. The net level premium for this policy is payable at the beginning of each year as long as the policy remains in force and is denoted π.

In this chapter we study the retrospective gain, prospective loss and surplus random variables for one policy. Typically, the prospective loss is defined only if a policy is still in force at a given valuation date. So, we will at first derive the first two moments of the prospective loss random variable conditional on the survival to a given time r. However, in order to define the surplus for a *policy at issue* based on the retrospective gain and the prospective loss valued at time r and viewed at time 0, we will then extend the definition of the prospective loss as being unconditional on the survival to time r.

The *retrospective gain* is the difference between the accumulated values of past premiums collected and benefits paid. Let RG_r denote the retrospective gain random variable at time r. For $r > 0$,

$$RG_r = \begin{cases} \pi \sum_{s=0}^{K_x} e^{I(s,r)} - b \cdot e^{I(K_x+1,r)}, & K_x = 0, 1, \ldots, r-1, \\ \\ \pi \sum_{s=0}^{r-1} e^{I(s,r)}, & K_x = r, r+1, \ldots. \end{cases}$$

The m^{th} moment of RG_r can be calculated directly from the definition of RG_r using the formula for computing expectations by conditioning (e.g., see equations (3.3) and (3.4) p.106 in Ross (2003)).

$$\begin{aligned} \mathrm{E}\Big[(RG_r)^m\Big] &= \mathrm{E}_{K_x}\Big[\mathrm{E}\Big[(RG_r)^m \mid K_x\Big]\Big] \\ &= \sum_{k=0}^{r-1} \mathrm{E}\left[\left(\pi \sum_{s=0}^{k} e^{I(s,r)} - b \cdot e^{I(k+1,r)}\right)^m\right] \cdot {}_{k|}q_x + \\ &\quad + \sum_{k=r}^{\infty} \mathrm{E}\left[\left(\pi \sum_{s=0}^{r-1} e^{I(s,r)}\right)^m\right] \cdot {}_{k|}q_x. \end{aligned}$$

Note that for $m = 1$ we have

$$\begin{aligned} \mathrm{E}\big[RG_r\big] &= \mathrm{E}_{K_x}\big[\mathrm{E}\big[RG_r \mid K_x\big]\big] \\ &= \sum_{k=0}^{r-1} \left(\pi \sum_{s=0}^{k} \mathrm{E}\big[e^{I(s,r)}\big] - b \cdot \mathrm{E}\big[e^{I(k+1,r)}\big]\right) \cdot {}_{k|}q_x + \\ &\quad + \sum_{k=r}^{\infty} \left(\pi \sum_{s=0}^{r-1} \mathrm{E}\big[e^{I(s,r)}\big]\right) \cdot {}_{k|}q_x \\ &= \pi \sum_{k=0}^{r-1} \mathrm{E}\big[e^{I(k,r)}\big] \cdot {}_{k}p_x - b \sum_{k=0}^{r-1} \mathrm{E}\big[e^{I(k+1,r)}\big] \cdot {}_{k|}q_x, \end{aligned} \tag{3.1}$$

where the last line follows from Theorem 3.2 given in Bowers et al. (1986) (see Appendix A.2 for the statement of this theorem) with

$$\psi(k) = \begin{cases} \sum_{s=r}^{k} \mathrm{E}\big[e^{I(s,r)}\big], & k = 0, 1, \ldots, n-1, \\ \\ \sum_{s=r}^{n-1} \mathrm{E}\big[e^{I(s,r)}\big], & k = n, n+1, \ldots, \end{cases}$$

$$\Delta\psi(k) = \begin{cases} \mathrm{E}\big[e^{I(k+1,r)}\big], & k = 0, 1, \ldots, n-2, \\ \\ 0, & k = n-1, n, \ldots \end{cases}$$

and

$$1 - G(k) = {}_{k+1}p_x.$$

The *prospective loss* is the difference between the discounted values of future benefits to be paid and premiums to be received. Let PL_r^{cond} be the prospective loss random variable valued at time r and conditional on the event that the policyholder has survived to time r.

$$PL_r^{cond} = \begin{cases} b \cdot e^{-I(r,\,r+J_{x,r}+1)} - \pi \sum_{s=r}^{r+J_{x,r}} e^{-I(r,\,s)}, & J_{x,r} = 0, 1, \ldots, n-r-1, \\ \\ c \cdot e^{-I(r,\,n)} - \pi \sum_{s=r}^{n-1} e^{-I(r,\,s)}, & J_{x,r} = n-r, n-r+1, \ldots, \end{cases}$$

where $J_{x,r}$ is the remaining future lifetime of (x) provided that (x) has survived to time r. That is, $\{J_{x,r} = j\} \equiv \{K_x - r = j | K_x \geq r\}$.

Based on the above definition of PL_r^{cond},

$$\begin{aligned} \mathrm{E}\big[PL_r^{cond}\big] &= \mathrm{E}_{J_{x,r}}\big[\mathrm{E}\big[PL_r^{cond} \,\big|\, J_{x,r}\big]\big] \\ &= \sum_{j=0}^{n-r-1} \Big(b \cdot \mathrm{E}\big[e^{-I(r,\,r+j+1)}\big] - \pi \sum_{s=r}^{r+j} \mathrm{E}\big[e^{-I(r,\,s)}\big]\Big) \cdot {}_{j|}q_{x+r} + \\ &\quad + \Big(c \cdot \mathrm{E}\big[e^{-I(r,\,n)}\big] - \pi \sum_{s=r}^{n-1} \mathrm{E}\big[e^{-I(r,\,s)}\big]\Big) \cdot {}_{n-r}p_{x+r}. \end{aligned}$$

Alternatively, the prospective loss can be rewritten as a difference of two random variables Z and Y, where Z represents the present value of future benefits and Y represents the present value of future premiums of \$1 valued at time r and conditional on survival to time r. That is,

$$PL_r^{cond} = Z - \pi Y, \tag{3.2}$$

where

$$Z = \begin{cases} b \cdot e^{-I(r,\,r+J_{x,r}+1)}, & J_{x,r} = 0, 1, \ldots, n-r-1, \\ \\ c \cdot e^{-I(r,\,n)}, & J_{x,r} = n-r, n-r+1, \ldots, \end{cases}$$

and

$$Y = \begin{cases} \sum_{s=r}^{r+J_{x,r}} e^{-I(r,\,s)}, & J_{x,r} = 0, 1, \ldots, n-r-1, \\ \\ \sum_{s=r}^{n-1} e^{-I(r,\,s)}, & J_{x,r} = n-r, n-r+1, \ldots. \end{cases}$$

Using the above definitions of Z and Y, we find that

$$\begin{aligned} \mathrm{E}[Z] &= \mathrm{E}_{J_{x,r}}[\mathrm{E}[Z \mid J_{x,r}]] \\ &= b \sum_{j=0}^{n-r-1} \mathrm{E}\big[e^{-I(r,r+j+1)}\big]\, {}_{j|}q_{x+r} + c \sum_{j=n-r}^{\infty} \mathrm{E}\big[e^{-I(r,n)}\big]\, {}_{j|}q_{x+r} \\ &= b \sum_{j=0}^{n-r-1} \mathrm{E}\big[e^{-I(r,r+j+1)}\big]\, {}_{j|}q_{x+r} + c\,\mathrm{E}\big[e^{-I(r,n)}\big]\, {}_{n-r}p_{x+r} \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \mathrm{E}[Y] &= \mathrm{E}_{J_{x,r}}[\,\mathrm{E}[\,Y \mid J_{x,r}\,]\,] \\ &= \sum_{j=0}^{n-r-1} \Big(\sum_{s=r}^{r+j} \mathrm{E}\big[e^{-I(r,s)}\big]\Big)\, {}_{j|}q_{x+r} + \Big(\sum_{s=r}^{n-1} \mathrm{E}\big[e^{-I(r,s)}\big]\Big)\, {}_{n-r}p_{x+r} \\ &= \sum_{j=0}^{n-r-1} \mathrm{E}\big[e^{-I(r,r+j)}\big]\, {}_{j}p_{x+r}. \end{aligned} \tag{3.4}$$

The last line can be obtained from the theorem in Appendix A.2 with

$$\psi(j) = \begin{cases} \sum_{s=r}^{r+j} \mathrm{E}\big[e^{-I(r,\,s)}\big], & j = 0, 1, \ldots, n-r-1, \\ \\ \sum_{s=r}^{n-1} \mathrm{E}\big[e^{-I(r,\,s)}\big], & j = n-r, n-r+1, \ldots, \end{cases}$$

$$\Delta\psi(j) = \begin{cases} \mathrm{E}\big[e^{-I(r,\,r+j+1)}\big], & j = 0, 1, \ldots, n-r-2, \\ 0, & j = n-r-1, n-r, \ldots \end{cases}$$

and

$$1 - G(j) = {}_{j+1}p_{x+r}.$$

So,

$$\begin{aligned}
\mathrm{E}[Y] &= \mathrm{E}[\psi(J_{x,r})] \\
&= \mathrm{E}\big[e^{-I(r,r)}\big] + \sum_{j=0}^{n-r-2} \mathrm{E}\big[e^{-I(r,r+j+1)}\big] \cdot {}_{j+1}p_{x+r} + \sum_{j=n-r-1}^{\infty} 0 \cdot {}_{j+1}p_{x+r} \\
&= \mathrm{E}\big[e^{-I(r,r)}\big] \cdot {}_0p_{x+r} + \sum_{j=1}^{n-r-1} \mathrm{E}\big[e^{-I(r,r+j)}\big] \cdot {}_jp_{x+r} \\
&= \sum_{j=0}^{n-r-1} \mathrm{E}\big[e^{-I(r,r+j)}\big] \cdot {}_jp_{x+r}. \qquad \square
\end{aligned}$$

Combining (3.3) and (3.4), we obtain

$$\begin{aligned}
E[PL_r^{cond}] &= \mathrm{E}[Z] - \pi\, \mathrm{E}[Y] \\
&= b \sum_{j=0}^{n-r-1} E[e^{-I(r,r+j+1)}]\, {}_{j|}q_{x+r} + c\, E[e^{-I(r,n)}]\, {}_{n-r}p_{x+r} - \\
&\quad - \pi \sum_{j=0}^{n-r-1} E[e^{-I(r,r+j)} | \delta(r)]\, {}_jp_{x+r}\,.
\end{aligned}$$

The second raw moment of PL_r^{cond} is

$$\begin{aligned}
\mathrm{E}\Big[\big(PL_r^{cond}\big)^2\Big] &= \sum_{j=0}^{n-r-1} \mathrm{E}\Big[\big(b \cdot e^{-I(r,\,r+j+1)} - \pi \sum_{s=r}^{r+j} e^{-I(r,\,s)}\big)^2\Big] \cdot {}_{j|}q_{x+r} + \\
&\quad + \mathrm{E}\Big[\big(c \cdot e^{-I(r,\,n)} - \pi \sum_{s=r}^{n-1} e^{-I(r,\,s)}\big)^2\Big] \cdot {}_{n-r}p_{x+r},
\end{aligned}$$

where, for example,

$$\begin{aligned}
\mathrm{E}\Big[\big(\sum_{s=r}^{r+j} e^{-I(r,\,s)}\big)^2\Big] &= \sum_{s=r}^{r+j}\sum_{t=r}^{r+j} \mathrm{E}\big[e^{-I(r,\,s)-I(r,\,t)}\big] \\
&= \sum_{s=r}^{r+j} \mathrm{E}\big[e^{-2I(r,\,s)}\big] + 2 \sum_{s=r}^{r+j-1} \sum_{t=s+1}^{r+j} \mathrm{E}\big[e^{-I(r,\,s)-I(r,\,t)}\big].
\end{aligned}$$

Let us now define the (unconditional) prospective loss random variable:

$$PL_r = \begin{cases} 0, & \text{if } K_x \le r-1, \\ PL_r^{cond}, & \text{if } K_x \ge r. \end{cases}$$

Since the m^{th} moment of the (unconditional) prospective loss random variable is equal to

$$\begin{aligned}
&\mathrm{E}\Big[\big(PL_r\big)^m\Big] \\
=\;& \mathrm{E}\Big[\big(PL_r\big)^m \,\big|\, K_x \le r-1\Big]\cdot \mathbf{P}[K_x \le r-1] + \mathrm{E}\Big[\big(PL_r\big)^m \,\big|\, K_x \ge r\Big]\cdot \mathbf{P}[K_x \ge r] \\
=\;& 0\cdot {}_rq_x + \mathrm{E}\Big[\big(PL_r\big)^m \,\big|\, K_x \ge r\Big]\cdot {}_rp_x = \mathrm{E}\Big[\big(PL_r^{cond}\big)^m\Big]\cdot {}_rp_x\,,
\end{aligned}$$

to calculate the moments of PL_r, we simply have to multiply the corresponding moments of the conditional prospective loss by the survival probability ${}_rp_x$. In the rest of this chapter whenever we refer to the prospective loss random variable, we mean the unconditional one.

The *insurance surplus* at time r, $1 \le r \le n$, for a single life policy is defined to be the difference between the retrospective gain and prospective loss random variables, valued at time r and viewed at time 0; i.e., $S_r = RG_r - PL_r$. As a function of K_x it is thus given by

$$S_r = \begin{cases}
\pi \sum_{s=0}^{K_x} e^{I(s,r)} - b\cdot e^{I(K_x+1,r)}, & K_x = 0,1,\ldots,r-1, \\[2ex]
\pi\Big(\sum_{s=0}^{r-1} e^{I(s,r)} + \sum_{s=r}^{K_x} e^{-I(r,s)}\Big) - b\cdot e^{-I(r,K_x+1)}, & K_x = r, r+1,\ldots,n-1, \\[2ex]
\pi\Big(\sum_{s=0}^{r-1} e^{I(s,r)} + \sum_{s=r}^{n-1} e^{-I(r,s)}\Big) - c\cdot e^{-I(r,n)}, & K_x = n, n+1,\ldots.
\end{cases}$$

The m^{th} moment of S_r can be calculated from:

$$\begin{aligned}
\mathrm{E}\Big[(S_r)^m\Big] \;&=\; \mathrm{E}_{K_x}\Big[\mathrm{E}\Big[(S_r)^m \,|\, K_x\Big]\Big] \\
&=\; \sum_{k=0}^{r-1} \mathrm{E}\Big[\Big(\pi\sum_{s=0}^{k} e^{I(s,r)} - b\cdot e^{I(k+1,r)}\Big)^m\Big]\cdot {}_{k|}q_x + \\
&+\; \sum_{k=r}^{n-1} \mathrm{E}\Big[\Big(\pi\Big(\sum_{s=0}^{r-1} e^{I(s,r)} + \sum_{s=r}^{k} e^{-I(r,s)}\Big) - b\cdot e^{-I(r,k+1)}\Big)^m\Big]\cdot {}_{k|}q_x + \\
&+\; \mathrm{E}\Big[\Big(\pi\Big(\sum_{s=0}^{r-1} e^{I(s,r)} + \sum_{s=r}^{n-1} e^{-I(r,s)}\Big) - c\cdot e^{-I(r,n)}\Big)^m\Big]\cdot {}_np_x\,.
\end{aligned}$$

3.2 Numerical Illustrations

Consider a single life policy with \$1000 benefit issued to a person aged 30. The premium for this policy is determined according to the equivalence principle (see Appendix A.4 for more details on the equivalence principle). The results derived in the previous section are illustrated for the temporary and endowment insurance contracts with the term equal to 5 years (see Tables 3.1 and 3.2), 10 years (see Figure 3.1) and 25 years (see Figure 3.2).

We begin by analyzing the behaviour of the expected values of the retrospective gain, prospective loss and surplus conditional on the force of interest at time r; the values for three possible realizations of $\delta(r)$ (4%, 6% - the long-term mean of the process and 8% - the starting value of the process) are given in the middle three columns of Tables 3.1 and 3.2. Observe that when δ_r decreases, the conditional expected values of RG_r decrease but the conditional expected values of PL_r increase. This is due to the effects of accumulating and discounting at lower rates of return. Since $S_r = RG_r - PL_r$, the expected value of the surplus decreases by the amount equal to the sum of the gain decrease and the loss increase.

Comparing the unconditional expected values of the retrospective gain and prospective loss, we can see that, for the temporary insurance contracts, the expected values at first rise but then begin to decline as r approaches the term of the contract, which is clearly demonstrated in the upper left panels of Figures 3.1 and 3.2. This is consistent with building up a small reserve in early years of the contract and then spending it since no benefit has to be paid under the terms of this contract if a policyholder survives up to the contract maturity. In the case of the endowment contracts, the expected values gradually increase to the amount of the benefit, which would have to be paid with certainty at time n if the policy is in force at time $(n-1)$ (death in year n would result in the death benefit payment and survival to time n would result in the pure endowment benefit).

We can also observe that the expected value of the surplus increases with r. Even though in these examples pricing of the contracts is done according to the equivalence principle, the mean value of the surplus is positive at all valuation dates we considered. This can be attributed to the Gaussian nature of the rates of return. The asymmetry of the process with nonzero variance results in a larger accumulation effect compared to the discounting effect. If the variance of the process was zero or, in other words, if the rates of return were deterministic, then an accumulation factor would be exactly the inverse of the corresponding discount factor. But in the environment of stochastic rates of return the product of the expected values of the accumulation and discount factors is always greater than one:

$$\begin{aligned} \mathrm{E}\big[e^{I(s,r)}\big] \cdot \mathrm{E}\big[e^{-I(s,r)}\big] &= e^{\mathrm{E}[I(s,r)]+\frac{1}{2}\mathrm{Var}[I(s,r)]} \cdot e^{-\mathrm{E}[I(s,r)]+\frac{1}{2}\mathrm{Var}[I(s,r)]} \\ &= e^{\mathrm{Var}[I(s,r)]} > 1 \qquad \text{if } \mathrm{Var}[I(s,r)] > 0. \end{aligned}$$

As r increases, so does the variability of the retrospective gain, since there is both a larger uncertainty about future cash flows and rates of return. Note that in general the variability of the prospective loss depends on the number of deaths up to time r, the death pattern after time r and the randomness of the future rates of return. The combination of these three factors may cause the overall variability to either increase or decrease with r depending on the relative importance of each of them. For the contracts we considered the behaviour of the standard deviation of PL_r varies with the type and the term of the contract. For the 5-year and 10-year temporary policies, the prospective loss becomes less volatile for larger values of r; for the 25-year policy, the standard deviation slightly increases initially and then declines. This decline in the variability for larger values of r is due to a smaller uncertainty about the death pattern after time r and the fact that the mortality component of temporary policies usually dominates the investment one. The standard deviation of the prospective loss for endowment policies at first declines but then increases for values of r approaching the term of the contract. In the case of endowment policies, it is the increase in the uncertainty about future rates of return that drives the overall variability of the prospective loss up for larger values of r.

δ_r:	.04	.06	.08	
r	$\mathrm{E}[RG_r \mid \delta(r) = \delta_r]$			$\mathrm{E}[RG_r]$
1	0.0209	0.0475	0.0748	0.0721
2	0.0504	0.0923	0.1356	0.1275
3	0.0425	0.1007	0.1612	0.1453
4	-0.0142	0.0603	0.1387	0.1128
r	$\mathrm{E}[PL_r \mid \delta(r) = \delta_r]$			$\mathrm{E}[PL_r]$
1	0.2259	0.1422	0.0633	0.0716
2	0.2452	0.1777	0.1133	0.1260
3	0.2236	0.1757	0.1295	0.1423
4	0.1493	0.1241	0.0994	0.1080
r	$\mathrm{E}[S_r \mid \delta(r) = \delta_r]$			$\mathrm{E}[S_r]$
1	-0.2051	-0.0947	0.0115	0.0005
2	-0.1948	-0.0854	0.0222	0.0015
3	-0.1811	-0.0751	0.0317	0.0030
4	-0.1635	-0.0638	0.0393	0.0048
r	$\mathrm{SD}[RG_r \mid \delta(r) = \delta_r]$			$\mathrm{SD}[RG_r]$
1	36.0321	36.0321	36.0321	36.0321
2	52.1823	52.7301	53.2943	53.1910
3	65.9218	67.1683	68.4635	68.1308
4	78.7362	80.8513	83.0687	82.3566
r	$\mathrm{SD}[PL_r \mid \delta(r) = \delta_r]$			$\mathrm{SD}[PL_r]$
1	66.8353	64.2498	61.7977	62.0645
2	59.6696	57.7330	55.8769	56.2546
3	50.2845	49.0007	47.7559	48.1110
4	36.7656	36.1097	35.4655	35.6944
r	$\mathrm{SD}[S_r \mid \delta(r) = \delta_r]$			$\mathrm{SD}[S_r]$
1	75.9255	73.6613	71.5339	71.7644
2	79.2600	78.1831	77.2133	77.4156
3	82.8997	83.1334	83.4672	83.3982
4	86.8871	88.5402	90.3162	89.7519

Table 3.1: Expected values and standard deviations of retrospective gain, prospective loss and surplus for 5-year temporary insurance contract.

δ_r:	.04	.06	.08	
r	$\mathrm{E}[RG_r\|\delta(r)=\delta_r]$			$\mathrm{E}[RG_r]$
1	165.4803	168.8495	172.2867	171.9485
2	340.9904	349.7112	358.6631	356.9976
3	525.9515	542.8011	560.2265	555.6283
4	720.9330	748.9411	778.1309	768.4117
r	$\mathrm{E}[PL_r\|\delta(r)=\delta_r]$			$\mathrm{E}[PL_r]$
1	237.4258	201.5997	168.2494	171.7691
2	416.2425	382.2705	350.0646	356.4355
3	601.9631	573.9670	546.9549	554.4756
4	794.3808	777.3667	760.6562	766.4814
r	$\mathrm{E}[S_r\|\delta(r)=\delta_r]$			$\mathrm{E}[S_r]$
1	-71.9455	-32.7502	4.0374	0.1794
2	-75.2521	-32.5593	8.5985	0.5622
3	-76.0116	-31.1659	13.2717	1.1527
4	-73.4478	-28.4257	17.4747	1.9303
r	$\mathrm{SD}[RG_r\|\delta(r)=\delta_r]$			$\mathrm{SD}[RG_r]$
1	36.0321	36.0321	36.0321	36.0737
2	56.6342	57.3125	58.0101	58.2001
3	77.2717	78.9073	80.6023	81.3451
4	99.2467	102.1808	105.2453	107.3234
r	$\mathrm{SD}[PL_r\|\delta(r)=\delta_r]$			$\mathrm{SD}[PL_r]$
1	42.0424	40.6415	39.4018	42.7368
2	37.2222	35.6113	34.1390	40.5180
3	42.8343	41.0689	39.3771	45.0867
4	59.1968	57.9345	56.6947	58.9303
r	$\mathrm{SD}[S_r\|\delta(r)=\delta_r]$			$\mathrm{SD}[S_r]$
1	49.4829	49.2533	49.1260	52.3192
2	46.7823	48.1719	49.6357	56.4664
3	44.7365	47.6706	50.6774	60.8835
4	44.9518	49.1162	53.4076	65.6090

Table 3.2: Expected values and standard deviations of retrospective gain, prospective loss and surplus for 5-year endowment insurance contract.

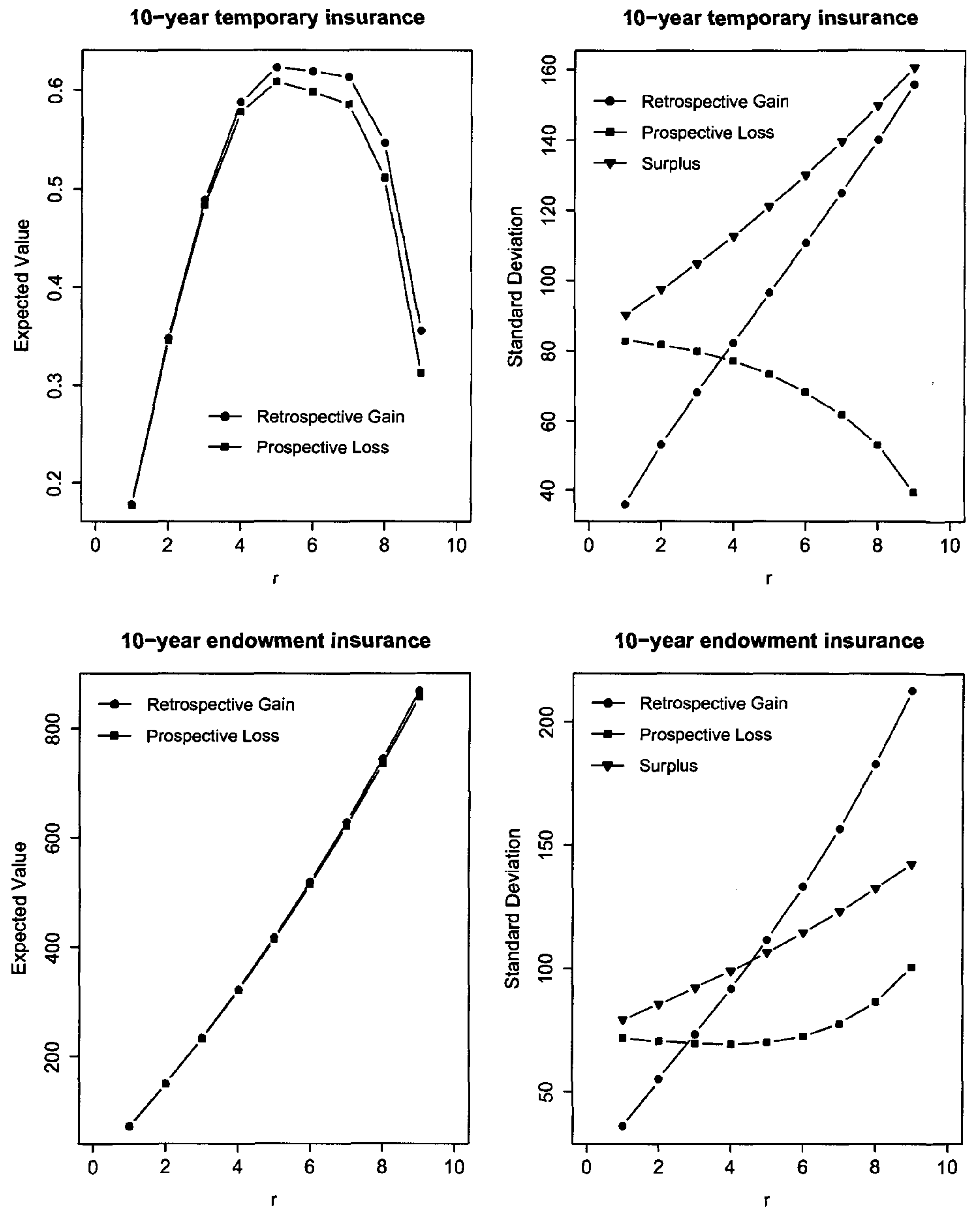

Figure 3.1: Expected values and standard deviations of retrospective gain, prospective loss and surplus for 10-year temporary and endowment contracts.

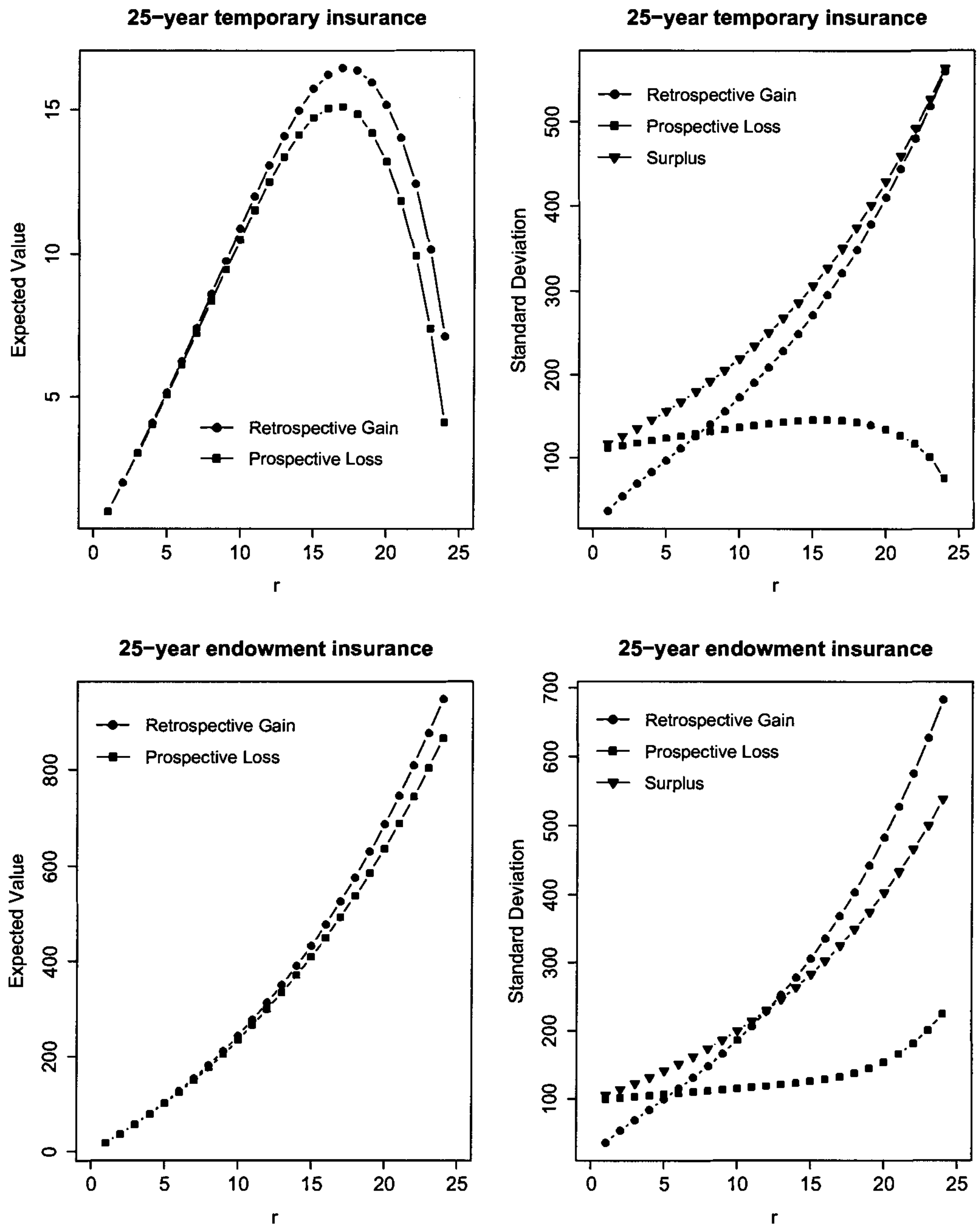

Figure 3.2: Expected values and standard deviations of retrospective gain, prospective loss and surplus for 25-year temporary and endowment contracts.

Chapter 4

Portfolio of Homogeneous Life Insurance Policies

In practice, insurers deal not with just one insurance policy but rather with a collection of policies forming an insurance portfolio. Therefore, our next objective is to extend the results for one insurance contract developed in the previous chapter to study portfolios of insurance contracts. One way to proceed is to define the retrospective gain, prospective loss and surplus for a portfolio by aggregating the corresponding random variables for one policy over the number of policies in the portfolio. However, for large portfolios, say 100,000 policies, this approach might not be very efficient. Alternatively, we could model portfolio's cash flows in every year, in which case the maximum number of terms to add up is equal to the duration of the policies in the portfolio. This is the approach we will adopt.

Consider a portfolio of identical life policies issued to a group of m policyholders all aged x with the same mortality profile. Similar to a single policy discussed in the previous chapter, each contract pays a death benefit b at the end of the year of death if death occurs within n years and a pure endowment benefit c if a policyholder survives to the end of year n. π is the annual level premium payable at the beginning of each year as long as the contract remains in force. This portfolio is referred to as being *homogeneous*.

4.1 Retrospective Gain

Let RC_j^r denote the net cash flow at time j prior to time r, $0 \leq j \leq r$ (i.e., it is a retrospective cash flow for valuation at time r).

$$
\begin{aligned}
RC_j^r &= \sum_{i=1}^{m} \left[\pi \cdot \mathscr{L}_{i,j}(x) \cdot \mathbf{1}_{\{j<r\}} - b \cdot \mathscr{D}_{i,j}(x) \cdot \mathbf{1}_{\{j>0\}} \right] \\
&= \pi \cdot \left(\sum_{i=1}^{m} \mathscr{L}_{i,j}(x) \right) \cdot \mathbf{1}_{\{j<r\}} - b \cdot \left(\sum_{i=1}^{m} \mathscr{D}_{i,j}(x) \right) \cdot \mathbf{1}_{\{j>0\}} \\
&= \pi \cdot \mathscr{L}_j(x) \cdot \mathbf{1}_{\{j<r\}} - b \cdot \mathscr{D}_j(x) \cdot \mathbf{1}_{\{j>0\}}, \qquad (4.1)
\end{aligned}
$$

where

$$
\mathscr{L}_{i,j}(x) = \begin{cases} 1 & \text{if policyholder } i \text{ aged } x \text{ survives for } j \text{ years,} \\ 0 & \text{otherwise,} \end{cases}
$$

$$
\mathscr{D}_{i,j}(x) = \begin{cases} 1 & \text{if policyholder } i \text{ aged } x \text{ dies in year } [j-1, j), \\ 0 & \text{otherwise,} \end{cases}
$$

and $\mathbf{1}_{\{\mathcal{A}\}}$ is an indicator; it is equal to 1 if condition $\mathcal{A}$ is true and 0 otherwise.

Indicator $\mathbf{1}_{\{j>0\}}$ multiplying the second term in Equation (4.1) reinforces the fact that no death benefit is paid at the beginning of the first year of the contract (i.e., when $j = 0$). So, RC_0^r is the sum of all the premiums collected at the issue date. Now consider what happens at time $j = r$. Death benefits are paid at the *end* of year r to everyone who dies during that year. So, this cash outflow becomes a part of the retrospective cash flow RC_r^r. However, premiums are collected at the *beginning* of the next year ($(r+1)^{st}$ year) and therefore they contribute to the prospective cash flow (more formally defined in the next section). This is why indicator $\mathbf{1}_{\{j<r\}}$ multiplies the first term in Equation (4.1), which makes the premiums inflow disappear for $j = r$.

$\mathscr{L}_j(x) = \sum_{i=1}^{m} \mathscr{L}_{i,j}(x)$ denotes the number of people from the initial group of m policyholders aged x who survive to time j (i.e. it is the number of inforce policies at time j) and $\mathscr{D}_j(x) = \sum_{i=1}^{m} \mathscr{D}_{i,j}(x)$ denotes the number of deaths in year $[j-1, j)$.

Observe that $\mathscr{L}_j(x)$ and $\mathscr{D}_j(x)$ have binomial distributions with
$\mathscr{L}_j(x) \sim \mathrm{BIN}(m, {}_jp_x)$ and $\mathscr{D}_j(x) \sim \mathrm{BIN}(m, {}_{j-1|}q_x)$.

This can be used to calculate

$$\begin{aligned}
\mathrm{E}[RC_j^r] &= \pi \cdot \mathrm{E}[\mathscr{L}_j(x)] \cdot \mathbf{1}_{\{j<r\}} - b \cdot \mathrm{E}[\mathscr{D}_j(x)] \cdot \mathbf{1}_{\{j>0\}} \\
&= \pi \cdot (m \cdot {}_jp_x) \cdot \mathbf{1}_{\{j<r\}} - b \cdot (m \cdot {}_{j-1|}q_x) \cdot \mathbf{1}_{\{j>0\}}\,,
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}[RC_j^r] &= \pi^2 \cdot \mathrm{Var}(\mathscr{L}_j(x)) \cdot \mathbf{1}_{\{j<r\}} + b^2 \cdot \mathrm{Var}(\mathscr{D}_j(x)) \cdot \mathbf{1}_{\{j>0\}} - \\
&\quad - 2 \cdot \pi \cdot b \cdot \mathrm{Cov}(\mathscr{L}_j(x), \mathscr{D}_j(x)) \cdot \mathbf{1}_{\{0<j<r\}}
\end{aligned}$$

and, for $i < j$,

$$\begin{aligned}
\mathrm{Cov}[RC_i^r, RC_j^r] &= \pi^2 \cdot \mathrm{Cov}[\mathscr{L}_i(x), \mathscr{L}_j(x)] \cdot \mathbf{1}_{\{j<r\}} + b^2 \cdot \mathrm{Cov}[\mathscr{D}_i(x), \mathscr{D}_j(x)] \cdot \mathbf{1}_{\{i>0\}} \\
&\quad - \pi \cdot b \cdot \mathrm{Cov}[\mathscr{L}_i(x), \mathscr{D}_j(x)] \cdot \mathbf{1}_{\{0<i<j<r\}} \\
&\quad - b \cdot \pi \cdot \mathrm{Cov}[\mathscr{D}_i(x), \mathscr{L}_j(x)] \cdot \mathbf{1}_{\{0\le i<j\le r\}}\,.
\end{aligned}$$

The covariance between cash flows arises due to the fact that, if a person belongs to $\mathscr{L}_i(x)$ (i.e., he or she was alive at time i), the same person might also belong to either $\mathscr{L}_j(x)$ or $\mathscr{D}_j(x)$ at some later time $j > i$. For different policyholders, it was assumed earlier that their lifetimes are independent (Assumption 1).

The formulas for various variance and covariance terms used above are given in the next section; see Equations (4.6)-(4.12) with r set to zero and $\mathscr{L}_0(x)$ equal to m.

The retrospective gain at a given time r is equal to the accumulated value to time r of all net cash flows that occur prior to that time. So, using the cash flow approach, we can express the retrospective gain in terms of RC_j^r as follows:

$$RG_r = \textstyle\sum_{j=0}^r RC_j^r \cdot e^{I(j,r)}.$$

Then, under Assumption 2 (i.e., assuming independence between future lifetimes and interest rates) we obtain

$$\mathrm{E}[RG_r] = \sum_{j=0}^{r} \mathrm{E}[RC_j^r] \cdot \mathrm{E}[e^{I(j,r)}] \tag{4.2}$$

and

$$\mathrm{E}\Big[\big(RG_r\big)^2\Big] = \sum_{i=0}^{r}\sum_{j=0}^{r} \mathrm{E}[RC_i^r \cdot RC_j^r] \cdot \mathrm{E}[e^{I(i,r)+I(j,r)}]. \tag{4.3}$$

One might also be interested in the retrospective gain random variable conditional on the force of interest at time r and/or the number of policies remaining in the portfolio at that time. This allows to further investigate properties of RG_r with respect to changes in rates of return and the mortality experience of the portfolio by considering various scenarios. Appendix A.3 provides details on how to calculate the expected value, variance and covariance of retrospective cash flows conditional on the size of the portfolio at a particular valuation date. These results, combined with conditional moments of the accumulation function, can then be used to calculate $\mathrm{E}\big[RG_r \,|\, \mathscr{L}_r(x),\, \delta(r)\big]$ and $\mathrm{E}\Big[\big(RG_r\big)^2 \big|\, \mathscr{L}_r(x),\, \delta(r)\Big]$ as follows:

$$\mathrm{E}[RG_r \,|\, \mathscr{L}_r(x),\, \delta(r)] = \sum_{j=0}^{r} \mathrm{E}[RC_j^r \,|\, \mathscr{L}_r(x)] \cdot \mathrm{E}[e^{I(j,r)} \,|\, \delta(r)] \tag{4.4}$$

and

$$\mathrm{E}\Big[\big(RG_r\big)^2\Big] = \sum_{i=0}^{r}\sum_{j=0}^{r} \mathrm{E}[RC_i^r \cdot RC_j^r \,|\, \mathscr{L}_r(x)] \cdot \mathrm{E}[e^{I(i,r)+I(j,r)} \,|\, \delta(r)]. \tag{4.5}$$

4.2 Prospective Loss

Similar to RC_j^r defined in the previous section to study the retrospective gain RG_r, we let PC_j^r denote the net cash flow that occurs j time units after time r, $0 \le j \le n-r$, (i.e., it is the prospective cash flow for valuation at time r), which will be used to study the prospective loss random variable, PL_r.

Since we want to express a prospective *loss* in terms of PC_j^r, a prospective cash flow at any given time j is the difference between benefits paid and premiums collected at that time. Using the notation introduced in the previous section, we have

$$\begin{aligned}
PC_j^r &= \sum_{i=1}^{m_r} \big[b \cdot \mathscr{D}_{i,j}(x+r) \cdot \mathbf{1}_{\{j>0\}} + c \cdot \mathscr{L}_{i,(n-r)}(x+r) \cdot \mathbf{1}_{\{j=n-r\}} - \\
&\qquad - \pi \cdot \mathscr{L}_{i,j}(x+r) \cdot \mathbf{1}_{\{j<n-r\}}\big] \\
&= b \cdot \Big(\sum_{i=1}^{m_r} \mathscr{D}_{i,j}(x+r)\Big) \cdot \mathbf{1}_{\{j>0\}} + c \cdot \Big(\sum_{i=1}^{m_r} \mathscr{L}_{i,(n-r)}(x+r)\Big) \cdot \mathbf{1}_{\{j=n-r\}} \\
&\quad - \pi \cdot \Big(\sum_{i=1}^{m_r} \mathscr{L}_{i,j}(x+r)\Big) \cdot \mathbf{1}_{\{j<n-r\}} \\
&= b \cdot \mathscr{D}_j(x+r) \cdot \mathbf{1}_{\{j>0\}} + c \cdot \mathscr{L}_{n-r}(x+r) \cdot \mathbf{1}_{\{j=n-r\}} - \pi \cdot \mathscr{L}_j(x+r) \cdot \mathbf{1}_{\{j>0\}},
\end{aligned}$$

where m_r is the size of the portfolio at time r (i.e., the realization of $\mathscr{L}_r(x)$).

Since for $0 < j \leq n-r$

$\{\mathscr{L}_j(x+r)|\mathscr{L}_r(x) = m_r\} \sim \mathrm{BIN}(m_r,\, {}_jp_{x+r})$ and

$\{\mathscr{D}_j(x+r)|\mathscr{L}_r(x) = m_r\} \sim \mathrm{BIN}(m_r,\, {}_{j-1|}q_{x+r})$,

we can use the known moments of binomially distributed random variables to find the moments of PC_j^r, which are [1]

$$\begin{aligned}
\mathrm{E}[PC_j^r|\mathscr{L}_r] &= b\cdot \mathrm{E}[\mathscr{D}_j(x+r)|\mathscr{L}_r]\cdot \mathbf{1}_{\{j>0\}} + c\cdot \mathrm{E}[\mathscr{L}_{n-r}(x+r)|\mathscr{L}_r]\cdot \mathbf{1}_{\{j=n-r\}} \\
&\quad - \pi\cdot \mathrm{E}[\mathscr{L}_j(x+r)|\mathscr{L}_r]\cdot \mathbf{1}_{\{j<n-r\}} \\
&= b\cdot \left(\mathscr{L}_r\cdot {}_{j-1|}q_{x+r}\right)\cdot \mathbf{1}_{\{j>0\}} + c\cdot \left(\mathscr{L}_r\cdot {}_jp_{x+r}\right)\cdot \mathbf{1}_{\{j=n-r\}} \\
&\quad - \pi\cdot \left(\mathscr{L}_r\cdot {}_jp_{x+r}\right)\cdot \mathbf{1}_{\{j<n-r\}}
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Var}[PC_j^r|\mathscr{L}_r] &= b^2\cdot \mathrm{Var}[\mathscr{D}_j(x+r)|\mathscr{L}_r]\cdot \mathbf{1}_{\{j>0\}} \\
&+ c^2\cdot \mathrm{Var}[\mathscr{L}_{n-r}(x+r)|\mathscr{L}_r]\cdot \mathbf{1}_{\{j=n-r\}} \\
&+ \pi^2\cdot \mathrm{Var}[\mathscr{L}_j(x+r)|\mathscr{L}_r]\cdot \mathbf{1}_{\{j<n-r\}} \\
&+ 2\cdot b\cdot c\cdot \mathrm{Cov}[\mathscr{D}_j(x+r),\mathscr{L}_j(x+r)|\mathscr{L}_r]\cdot \mathbf{1}_{\{j=n-r\}} \\
&- 2\cdot b\cdot \pi\cdot \mathrm{Cov}[\mathscr{D}_j(x+r),\mathscr{L}_j(x+r)|\mathscr{L}_r]\cdot \mathbf{1}_{\{0<j<n-r\}}\,.
\end{aligned}$$

For $0 \leq i < j \leq n-r$,

$$\begin{aligned}
\mathrm{Cov}[PC_j^r, PC_i^r|\mathscr{L}_r] &= b^2\cdot \mathrm{Cov}[\mathscr{D}_j(x+r),\, \mathscr{D}_i(x+r)\,|\,\mathscr{L}_r]\cdot \mathbf{1}_{\{i>0\}} \\
&+ \pi^2\cdot \mathrm{Cov}[\mathscr{L}_j(x+r),\, \mathscr{L}_i(x+r)\,|\,\mathscr{L}_r]\cdot \mathbf{1}_{\{j<n-r\}} \\
&+ c\cdot b\cdot \mathrm{Cov}[\mathscr{L}_j(x+r),\, \mathscr{D}_i(x+r)\,|\,\mathscr{L}_r]\cdot \mathbf{1}_{\{0<i<j=n-r\}} \\
&- b\cdot \pi\cdot \mathrm{Cov}[\mathscr{D}_j(x+r),\, \mathscr{L}_i(x+r)\,|\,\mathscr{L}_r]\cdot \mathbf{1}_{\{0\leq i<j\leq n-r\}} \\
&- \pi\cdot b\cdot \mathrm{Cov}[\mathscr{L}_j(x+r),\, \mathscr{D}_i(x+r)\,|\,\mathscr{L}_r]\cdot \mathbf{1}_{\{0<i<j<n-r\}} \\
&- c\cdot \pi\cdot \mathrm{Cov}[\mathscr{L}_j(x+r),\, \mathscr{L}_i(x+r)\,|\,\mathscr{L}_r]\cdot \mathbf{1}_{\{j=n-r\}},
\end{aligned}$$

[1] For simplicity of notation, in some cases we will use $\mathscr{L}_r$ instead of $\mathscr{L}_r(x)$ when there is no ambiguity about age x.

where

$$\mathrm{Var}[\mathscr{L}_j(x+r)\,|\,\mathscr{L}_r] = \mathscr{L}_r \cdot {}_jp_{x+r} \cdot (1 - {}_jp_{x+r}); \tag{4.6}$$

$$\mathrm{Var}[\mathscr{D}_j(x+r)\,|\,\mathscr{L}_r] = \mathscr{L}_r \cdot {}_{j-1|}q_{x+r} \cdot (1 - {}_{j-1}q_{x+r}); \tag{4.7}$$

$$\mathrm{Cov}[\mathscr{D}_j(x+r),\, \mathscr{L}_j(x+r)\,|\,\mathscr{L}_r] = -\mathscr{L}_r \cdot {}_{j-1|}q_{x+r} \cdot {}_jp_{x+r}; \tag{4.8}$$

$$\mathrm{Cov}[\mathscr{D}_j(x+r),\, \mathscr{D}_i(x+r)\,|\,\mathscr{L}_r] = -\mathscr{L}_r \cdot {}_{j-1|}q_{x+r} \cdot {}_{i-1|}q_{x+r}; \tag{4.9}$$

$$\mathrm{Cov}[\mathscr{L}_j(x+r),\, \mathscr{L}_i(x+r)\,|\,\mathscr{L}_r] = \mathscr{L}_r \cdot ({}_jp_{x+r} - {}_ip_{x+r} \cdot {}_jp_{x+r}); \tag{4.10}$$

$$\mathrm{Cov}[\mathscr{D}_j(x+r),\, \mathscr{L}_i(x+r)\,|\,\mathscr{L}_r] = \mathscr{L}_r \cdot {}_{j-1|}q_{x+r} \cdot (1 - {}_ip_{x+r}); \tag{4.11}$$

$$\mathrm{Cov}[\mathscr{L}_j(x+r),\, \mathscr{D}_i(x+r)\,|\,\mathscr{L}_r] = -\mathscr{L}_r \cdot {}_jp_{x+r} \cdot {}_{i-1|}q_{x+r}. \tag{4.12}$$

Now we can rewrite PL_r in terms of PC_j^r and calculate its first two raw moments using the results developed above. We get

$$PL_r = \sum_{j=0}^{n-r} PC_j^r \cdot e^{-I(r,r+j)}, \tag{4.13}$$

$$\mathrm{E}[PL_r\,|\,\mathscr{L}_r,\, \delta(r)] = \sum_{j=0}^{n-r} \mathrm{E}[PC_j^r\,|\,\mathscr{L}_r] \cdot \mathrm{E}[e^{-I(r,r+j)}\,|\,\delta(r)] \tag{4.14}$$

and

$$\mathrm{E}[(PL_r)^2\,|\,\mathscr{L}_r,\, \delta(r)] = \sum_{i=0}^{n-r}\sum_{j=0}^{n-r} \Big(\mathrm{E}[PC_i^r \cdot PC_j^r\,|\,\mathscr{L}_r] \cdot \mathrm{E}[e^{-I(r,r+i)-I(r,r+j)}\,|\,\delta(r)]\Big)$$

$$= \sum_{j=0}^{n-r} \Big(\mathrm{E}[(PC_j^r)^2\,|\,\mathscr{L}_r] \cdot \mathrm{E}[e^{-2I(r,r+j)}\,|\,\delta(r)]\Big) +$$

$$+\,2 \cdot \sum_{i=0}^{n-r-1}\sum_{j=i+1}^{n-r} \Big(\mathrm{E}[PC_i^r \cdot PC_j^r\,|\,\mathscr{L}_r] \cdot \mathrm{E}[e^{-I(r,r+i)-I(r,r+j)}\,|\,\delta(r)]\Big), \tag{4.15}$$

since $\{K_x^{(i)}\}$ are independent of $\{\delta(j),\, j = 0,\, 1,\, \ldots\}$ (Assumption 2).

We can calculate $\mathrm{E}[PL_r]$ and $\mathrm{Var}[PL_r]$ either using Equations (4.14) and (4.15) and taking double expectations over $\mathscr{L}_r(x)$ and $\delta(r)$, or directly from Equation (4.13).

4.3 Insurance Surplus

4.3.1 Introduction

In general, we define insurance surplus to be the difference between assets and liabilities at a given valuation date. Recall that the retrospective gain is the accumulated value of past premiums collected net of past benefits paid and, thus, in our context, it can be viewed as the value of assets. In turn, the liabilities associated with a portfolio of life policies are based on the prospective loss, which is the discounted value of future obligations net of future premiums. So, the liabilities can simply be represented by the prospective loss random variable. In this case the surplus is referred to as the *net stochastic surplus* or just stochastic surplus and is denoted S_r^{stoch}.

In practice, at each valuation date, an insurer is required to set aside an actuarial reserve based on the number of policies in force as well as on the current interest rate. This reserve is a liability item on the balance sheet of the insurance company. So, an alternative definition of the surplus is the difference between the value of assets and the actuarial reserve, in which case we call it the *accounting surplus* and denote it S_r^{acct}.

The reserve is intended to cover the future liabilities of the insurer. Therefore, the amount needed to be set as a reserve at time r should be at least the expected value of PL_r conditional on the number of inforce policies in the portfolio $\mathscr{L}_r(x)$ and the force of interest $\delta(r)$. If, instead, it is required to have a conservative reserve that will cover net future obligations with a high probability, one can use a p^{th} percentile of the prospective loss random variable with p between 70% and 95%, for example. However, this reserve can be fairly difficult to incorporate in the model, since we need to know the distribution function of PL_r, which is not easy to obtain. Alternatively, a reserve could be set equal to the expected value plus a multiple of the standard deviation of PL_r (see Norberg (1993)).

In the rest of this section we derive the first two moments of the stochastic and accounting surpluses, assuming that the reserve is given by the conditional expected value of the prospective loss random variable.

4.3.2 Methodology

Let ${}_rV(\mathscr{L}_r(x),\, \delta(r))$ or simply ${}_rV$ denote the reserve at time r. It is a function of $\mathscr{L}_r(x)$ and $\delta(r)$ and thus, when viewed from time 0, is a random quantity whose value at time r depends on the realizations of $\mathscr{L}_r(x)$ and $\delta(r)$.

$$\{S_r^{acct}|\mathscr{L}_r(x),\, \delta(r)\} = \{RG_r|\mathscr{L}_r(x),\, \delta(r)\} - {}_rV(\mathscr{L}_r(x),\, \delta(r)) \tag{4.16}$$

is the accounting surplus at time r conditional on the number of policies in force and the force of interest at that time.

The stochastic surplus is given by

$$S_r^{stoch} = RG_r - PL_r\,. \tag{4.17}$$

Assume that

$${}_rV \equiv {}_rV(\mathscr{L}_r(x),\, \delta(r)) = \mathrm{E}\big[PL_r \,|\, \mathscr{L}_r,\, \delta(r)\big]. \tag{4.18}$$

We then have

$$\begin{aligned}
\mathrm{E}[S_r^{acct}] &= \mathrm{E}_{\delta(r)}\Big[\mathrm{E}_{\mathscr{L}_r}\Big[\mathrm{E}\big[S_r^{acct} \,|\, \mathscr{L}_r,\, \delta(r)\big]\Big]\Big] \\
&= \mathrm{E}_{\delta(r)}\Big[\mathrm{E}_{\mathscr{L}_r}\Big[\mathrm{E}\big[RG_r \,|\, \mathscr{L}_r,\, \delta(r)\big]\Big]\Big] - \mathrm{E}_{\delta(r)}\Big[\mathrm{E}_{\mathscr{L}_r}\Big[\mathrm{E}\big[PL_r \,|\, \mathscr{L}_r,\, \delta(r)\big]\Big]\Big] \\
&= \mathrm{E}[RG_r] - \mathrm{E}[PL_r] \\
&= \mathrm{E}[RG_r - PL_r] \\
&= \mathrm{E}[S_r^{stoch}].
\end{aligned}$$

That is, with our particular choice of the reserve, stochastic and accounting surpluses have the same expected value.

Let us next consider the variance of the accounting surplus when the reserve is given by Equation (4.18).

Result 4.3.1.

$$\begin{aligned}
Var[S_r^{acct}] &= \Big(Var_{\delta(r)}\, E[PL_r \,|\, \delta(r)] + E_{\delta(r)}\, Var_{\mathscr{L}_r(x)}\big(E[PL_r \,|\, \mathscr{L}_r(x),\, \delta(r)]\big)\Big) \\
&\quad + Var[RG_r] - 2Cov(RG_r,\, PL_r),
\end{aligned}$$

where

$$\begin{aligned}
&Var_{\delta(r)}\, E[PL_r \mid \delta(r)] + E_{\delta(r)}\, Var_{\mathscr{L}_r(x)}\big(E[PL_r \mid \mathscr{L}_r(x),\, \delta(r)]\big) = \\
&= \sum_{i=0}^{n-r}\sum_{j=0}^{n-r}\Big\{ E_{\mathscr{L}_r(x)}\Big(E[PC_i^r \mid \mathscr{L}_r(x)]\cdot E[PC_j^r \mid \mathscr{L}_r(x)]\Big)\cdot \\
&\qquad \cdot E_{\delta(r)}\Big(E\big[e^{-I(r,r+i)} \mid \delta(r)\big]\cdot E\big[e^{-I(r,r+j)} \mid \delta(r)\big]\Big)\Big\} - \\
&\qquad - \Big(E[PL_r]\Big)^2.
\end{aligned}$$

A proof of Result 4.3.1 is given in Appendix A.5.

The variance calculation for the stochastic surplus is straightforward as

$$\begin{aligned}
\mathrm{Var}\big[S_r^{stoch}\big] &= \mathrm{Var}\big[RG_r - PL_r\big] \\
&= \mathrm{Var}\big[RG_r\big] + \mathrm{Var}\big[PL_r\big] - 2\cdot \mathrm{Cov}\big[RG_r,\, PL_r\big].
\end{aligned}$$

Note that to calculate $\mathrm{Cov}\big[RG_r,\, PL_r\big]$, we need to know $\mathrm{E}\big[RC_j^r\cdot PC_i^r\big]$, since

$$\begin{aligned}
\mathrm{Cov}\big[RG_r,\, PL_r\big] &= \mathrm{E}\big[RG_r\cdot PL_r\big] - \mathrm{E}\big[RG_r\big]\cdot \mathrm{E}\big[PL_r\big] \\
&= \sum_{j=0}^{r}\sum_{i=0}^{n-r} \mathrm{E}[RC_j^r\cdot PC_i^r]\cdot \mathrm{E}[e^{I(j,r)-I(r,r+i)}] - \mathrm{E}[RG_r]\cdot \mathrm{E}[PL_r],
\end{aligned}$$

where the last line follows from Assumption 2.

$\mathrm{E}\big[RC_j^r\cdot PC_i^r\big]$ can be obtained in one of the two ways. First, conditioning on the number of policies in force,

$$\begin{aligned}
\mathrm{E}\big[RC_j^r\cdot PC_i^r\big] &= \mathrm{E}_{\mathscr{L}_r(x)}\Big[\mathrm{E}\big[RC_j^r\cdot PC_i^r|\mathscr{L}_r(x)\big]\Big] \\
&= \mathrm{E}_{\mathscr{L}_r(x)}\Big[\mathrm{E}\big[RC_j^r|\mathscr{L}_r(x)\big]\cdot \mathrm{E}\big[PC_i^r|\mathscr{L}_r(x)\big]\Big] \\
&= \sum_{m_r=0}^{m} \mathrm{E}\big[RC_j^r|\mathscr{L}_r(x)\big]\cdot \mathrm{E}\big[PC_i^r|\mathscr{L}_r(x)\big]\cdot \mathbf{P}(\mathscr{L}_r(x)=m_r).
\end{aligned}$$

Alternatively, a computationally more efficient approach is to use $\mathrm{E}\big[RC_j^r\cdot PC_i^r\big]$ $=\mathrm{Cov}\big[RC_j^r,\, PC_i^r\big]+\mathrm{E}\big[RC_j^r\big]\cdot\mathrm{E}\big[PC_i^r\big]$, where the covariance is based directly on the definitions of RC_j^τ and PC_i^τ.

$$\begin{aligned}
\mathrm{Cov}\big[RC_j^\tau,\, PC_i^\tau\big] &= \pi \cdot b \cdot \mathrm{Cov}\big[\mathscr{L}_j(x),\, \mathscr{D}_i(x+r)\big] \cdot \mathbf{1}_{\{j<r,\, i>0\}} \\
&+ \pi \cdot c \cdot \mathrm{Cov}\big[\mathscr{L}_j(x),\, \mathscr{L}_i(x+r)\big] \cdot \mathbf{1}_{\{j<r,\, i=n-r,\, i>0\}} \\
&- \pi^2 \cdot \mathrm{Cov}\big[\mathscr{L}_j(x),\, \mathscr{L}_i(x+r)\big] \cdot \mathbf{1}_{\{j<r,\, i<n-r\}} \\
&- b^2 \cdot \mathrm{Cov}\big[\mathscr{D}_j(x),\, \mathscr{D}_i(x+r)\big] \cdot \mathbf{1}_{\{j>0,\, i>0\}} \\
&- b \cdot c \cdot \mathrm{Cov}\big[\mathscr{D}_j(x),\, \mathscr{L}_i(x+r)\big] \cdot \mathbf{1}_{\{j>0,\, i=n-r,\, i>0\}} \\
&+ b \cdot \pi \cdot \mathrm{Cov}\big[\mathscr{D}_j(x),\, \mathscr{L}_i(x+r)\big] \cdot \mathbf{1}_{\{j>0,\, i<n-r\}}\,,
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{Cov}\big[\mathscr{L}_j(x),\, \mathscr{D}_i(x+r)\big] &= m \cdot \big({}_{r+i-1|}q_x - {}_jp_x \cdot {}_{r+i-1|}q_x\big) \\
&= m \cdot {}_{r+i-1|}q_x \cdot (1 - {}_jp_x)\,,
\end{aligned}$$

$$\begin{aligned}
\mathrm{Cov}\big[\mathscr{L}_j(x),\, \mathscr{L}_i(x+r)\big] &= m \cdot \big({}_{r+i}p_x - {}_jp_x \cdot {}_{r+i}p_x\big) \\
&= m \cdot {}_{r+i}p_x \cdot (1 - {}_jp_x)\,,
\end{aligned}$$

$$\mathrm{Cov}\big[\mathscr{D}_j(x),\, \mathscr{D}_i(x+r)\big] = -m \cdot {}_{j-1|}q_x \cdot {}_{r+i-1|}q_x \qquad \text{and}$$

$$\mathrm{Cov}\big[\mathscr{D}_j(x),\, \mathscr{L}_i(x+r)\big] = -m \cdot {}_{j-1|}q_x \cdot {}_{r+i}p_x.$$

4.4 A Note on Variance for Limiting Portfolio

A limiting portfolio is an abstract concept and is not achievable in practice. However, its characteristics such as variability can serve as benchmarks for portfolios of finite sizes and can provide some useful information for insurance risk managers.

If the variance of the surplus per policy for a given portfolio is much larger than the corresponding variance for the limiting portfolio, then it can be concluded that a large portion of the total risk is due to the insurance risk. In other words, there is a great uncertainty about future cash flows. One implication of this is that, if the insurer decides to hedge the financial risk, for instance, by purchasing bonds whose cash flows will match those of the portfolio's liabilities, this strategy will not be very efficient and the cost incurred to implement it might not be justified. In this case, selling more policies, sharing the mortality risk or buying reinsurance are better strategies to mitigate the risk.

For a limiting portfolio, the calculation of the moments is done similarly to the case when the size of the portfolio is finite, except that the random cash flows per policy, RC_j^r/m and PC_i^r/m, are replaced by their expected values. For example, the second raw moment of RG_r/m and PL_r/m and the covariance between them become

$$\lim_{m\to\infty} \mathrm{E}\Big[\big(RG_r/m\big)^2\Big] = \sum_{i=0}^{r}\sum_{j=0}^{r} \mathrm{E}[RC_i^r/m]\cdot \mathrm{E}[RC_j^r/m]\cdot \mathrm{E}[e^{I(i,r)+I(j,r)}],$$

$$\lim_{m\to\infty} \mathrm{E}\Big[\big(PL_r/m\big)^2\Big] = \sum_{i=0}^{n-r}\sum_{j=0}^{n-r} \mathrm{E}[PC_i^r/m]\cdot \mathrm{E}[PC_j^r/m]\cdot \mathrm{E}[e^{-I(r,r+i)-I(r,r+j)}]$$

and

$$\lim_{m\to\infty} \mathrm{Cov}\big(RG_r/m,\, PL_r/m\big) = \sum_{j=0}^{r}\sum_{i=0}^{n-r} \mathrm{E}[RC_j^r/m]\cdot \mathrm{E}[PC_i^r/m]\cdot \mathrm{Cov}\big(e^{I(j,r)}, e^{-I(r,r+i)}\big).$$

4.5 Numerical Illustrations

Consider homogeneous portfolios of life policies with \$1000 benefit issued to people aged 30 and with premiums determined under the equivalence principle (see Appendix A.4). Note that the expected values of the retrospective gain and prospective loss per policy as well as the two types of surplus [2] per policy are the same as for a single policy which we discussed in the previous chapter. Here, we would like to see how the riskiness of the portfolio, as measured by the standard deviation, changes with respect to changes in the initial portfolio size. The results are presented for portfolios of size 100, 10,000, 100,000 and the infinite size (limiting portfolio). To compare portfolios of different sizes, all quantities are calculated on the per policy basis.

[2]This is true for the accounting surplus only with our particular choice of the reserve equal to the expected value of the prospective loss.

Tables 4.1 and 4.2 give standard deviations of the retrospective gain at time r per policy conditional on the force of interest $\delta(r)$ for the portfolios of 5-year temporary and 5-year endowment insurance contracts respectively. Three scenarios of possible realizations of $\delta(r)$ (4%, 6% and 8%) are considered. Comparing the standard deviations for portfolios of different sizes, we see that they decrease as m increases. This is due to the diversification of the mortality risk. However, notice that this effect is much larger for portfolios of temporary policies than for portfolios of endowment policies. For example, in the case of temporary insurances when $m = 10,000$, $\mathrm{SD}[RG_1/m]$ is almost 8 times larger than $\mathrm{SD}[RG_1/m]$ for the limiting portfolio (0.1148 vs. 0.0137) and at $r = 4$ the ratio is almost 3.5 (.2721 vs. 0.0788). But for portfolios of endowment insurances, even when there are only 100 policies, the ratio is around 2 at $r = 1$ (3.9981 vs. 1.7325) and just over 1 at $r = 4$ (29.1919 vs. 27.2842). Increasing the size of the portfolio of endowment contracts to 10,000 almost entirely eliminates the insurance risk.

This can be explained by the relative size of the mortality and investment risks. For short term (such as 5 years) temporary policies most of the risk comes from the uncertainty about how many deaths occurs during the duration of the contract. Therefore, for a small portfolio, when the size of the portfolio increases by a factor of, say, 100, one would expect the standard deviation to go down by a factor of about 10 (the square root of 100). We can see that for m increasing from 1 to 100 and from 100 to 10,000. But an endowment policy is essentially an investment product that pays the benefit at the end of the term with a very high probability (e.g., probability that a 30 year old male survives for 5 years is 0.9931488, which is the probability of paying the pure endowment benefit) and so the small mortality risk gets quickly diversified for portfolios of even moderate size leaving only the nondiversifiable investment risk. Another way to see this is to compare conditional standard deviations to the corresponding unconditional ones. In the case of temporary contracts, there is a fairly small difference between them (e.g., for $m = 10,000$, $\mathrm{SD}[RG_4/m]$ and $\mathrm{SD}[RG_4/m \mid \delta(4)]$ are all around 0.8), but in the case of endowment contracts, for some parameters, unconditional standard deviations are almost three times as large (e.g., for $m = 10,000$, $\mathrm{SD}[RG_4/m]$=27.3 vs. $\mathrm{SD}[RG_4/m \mid \delta(4)] \approx 9.6$). We can see that unconditional standard deviations are always larger than the corresponding conditional ones confirming the conditional variance formula

$$\mathrm{Var}\big(RG_r/m\big) = \mathrm{Var}\big(\mathrm{E}\big[RG_r/m \mid \delta(r)\big]\big) + \mathrm{E}\big(\mathrm{Var}\big[RG_r/m \mid \delta(r)\big]\big).$$

Also, note that the conditional standard deviations for the limiting portfolios at $r = 1$ are equal to zero due to the absence of the investment risk in our model and a full diversification of the insurance risk. Hence, the corresponding unconditional deviation represents pure investment risk at time $r = 1$.

Results for the prospective loss random variable, similar to those presented in Tables 4.1 and 4.2, are summarized in Tables 4.3 and 4.4.

Tables 4.5, 4.6, 4.7 and 4.8 give standard deviations of the accounting and stochastic surpluses for portfolios of 5-year temporary and 5-year endowment insurance policies. Table 4.9 shows correlation coefficients between retrospective gain and prospective loss random variables for portfolios of 5-year endowment policies.

Observe that as r increases, so do the conditional and unconditional standard deviations of the accounting surplus. For the stochastic surplus, although there is a reduction in the variability of the prospective loss for larger r, it might or might not be sufficient to offset an increase in the uncertainty of the retrospective gain.

Comparing standard deviations corresponding to the same r but for different values of m, we can see that as m increases, there is a reduction in variability of the accounting surplus. As we already noted above, this reduction is attributed to the diversification of the mortality risk. However, even in the limiting case of endowment contracts, variability does not reduce to zero, since investment risk is nondiversifiable and remains present regardless of the size of the portfolio.

It is interesting to note that the standard deviation of the stochastic surplus at valuation dates close to maturity for portfolios of endowment policies increases for larger portfolio sizes. Recall that

$$\mathrm{Var}\big[S_r^{stoch}/m\big] = \mathrm{Var}\big[RG_r/m\big] + \mathrm{Var}\big[PL_r/m\big] - 2 \cdot \mathrm{Cov}\big[RG_r/m,\, PL_r/m\big].$$

We saw that the variability of both retrospective gain and prospective loss per policy random variables decreases as m increases. This suggests that the increase comes from the covariance component, which has to decrease to make $\mathrm{Var}\big[S_r^{stoch}/m\big]$ larger because of the minus sign. By looking at the correlation coefficients between RG_r/m and PL_r/m, both conditional and unconditional, we can see that as m increases the correlation coefficients decrease. At earlier valuation dates, decrease in the variability of RG_r/m and PL_r/m seems to be sufficient to compensate for decrease in the covariance, but eventually, for r close to n, the reduction in the covariance slightly outweighs reduction in the variances of RG_r/m and PL_r/m.

It is easy to see from Figure 4.1 that the stochastic surplus is more volatile than the accounting surplus. In the former case, the uncertainty in liabilities arises from the uncertainty in the complete future path of rates of return and mortality experience; whereas in the latter case, the randomness of liabilities comes from the uncertainty in the number of inforce policies remaining in the portfolio and the rate of return at the valuation date only.

Also notice that as r increases, the difference in the volatilities of stochastic and accounting surpluses diminishes. In fact, at r equal to n (the term of the contract), conditional on the number of policyholders who survive to time n, there is no uncertainty about the liabilities and all the variation comes from the retrospective gain, which represents the asset side and is the same for the stochastic and accounting surpluses.

Figures 4.2 and 4.3 display the standard deviations of accounting and stochastic surpluses per policy conditional on the number of inforce policies at time r (referred to as 'inforce size' on the axes labels), $\mathscr{L}_r(x)$, and the force of interest in year r, $\delta(r)$, plotted against possible realizations of $\mathscr{L}_r(x)$ and $\delta(r)$. The plots are shown for portfolios of 100 10-year temporary and 10-year endowment life insurance policies at r equal to 5 and 8. For the portfolio of temporary policies, observe a steep increase in the variability of the accounting surplus when the inforce size decreases from 100 policies to 99 policies and a less rapid increase for further decreases in the inforce size. The shape of these plots is difficult to explain because of the different factors affecting the variability of the surplus. One would expect the variability to be high when the probability of death in the time interval from 0 to r is about 0.5 and when the variability of the accumulation and discounting factors is high.

δ_r:	.04	.06	.08	
m=1				
r	SD[$RG_r/m\|\delta(r) = \delta_r$]			SD[RG_r/m]
1	36.0321	36.0321	36.0321	36.0321
2	52.1823	52.7301	53.2943	53.1910
3	65.9218	67.1683	68.4635	68.1308
4	78.7362	80.8513	83.0687	82.3566
m=100				
r	SD[$RG_r/m\|\delta(r) = \delta_r$]			SD[RG_r/m]
1	3.6032	3.6032	3.6032	3.6032
2	5.2182	5.2730	5.3294	5.3192
3	6.5922	6.7169	6.8464	6.8133
4	7.8737	8.0852	8.3070	8.2360
m=10,000				
r	SD[$RG_r/m\|\delta(r) = \delta_r$]			SD[RG_r/m]
1	0.3631	0.3631	0.3631	0.3633
2	0.5295	0.5352	0.5411	0.5361
3	0.6725	0.6856	0.6992	0.6875
4	0.8088	0.8312	0.8546	0.8345
m=100,000				
r	SD[$RG_r/m\|\delta(r) = \delta_r$]			SD[RG_r/m]
1	0.1139	0.1139	0.1139	0.1148
2	0.1653	0.1671	0.1689	0.1711
3	0.2096	0.2136	0.2178	0.2220
4	0.2516	0.2584	0.2657	0.2721
$m \to \infty$				
r	SD[$RG_r/m\|\delta(r) = \delta_r$]			SD[RG_r/m]
1	0	0	0	0.0137
2	0.0104	0.0107	0.0111	0.0314
3	0.0218	0.0227	0.0236	0.0534
4	0.0359	0.0378	0.0397	0.0788

Table 4.1: Standard deviations of retrospective gain per policy for portfolios of 5-year temporary insurance contracts.

δ_r:	.04	.06	.08	
m=1				
r	SD$[RG_r/m\|\delta(r) = \delta_r]$			SD$[RG_r/m]$
1	36.0321	36.0321	36.0321	36.0737
2	56.6342	57.3125	58.0101	58.2001
3	77.2717	78.9073	80.6023	81.3451
4	99.2467	102.1808	105.2453	107.3234
m=100				
r	SD$[RG_r/m\|\delta(r) = \delta_r]$			SD$[RG_r/m]$
1	3.6032	3.6032	3.6032	3.9981
2	5.8129	5.8880	5.9654	8.5022
3	8.7604	8.9793	9.2067	16.5528
4	13.4286	13.9300	14.4554	29.1919
m=10,000				
r	SD$[RG_r/m\|\delta(r) = \delta_r]$			SD$[RG_r/m]$
1	0.3603	0.3603	0.3603	1.7696
2	1.4330	1.4724	1.5132	6.2560
3	4.2193	4.3783	4.5436	14.5109
4	9.1450	9.5695	10.0141	27.3040
m=100,000				
r	SD$[RG_r/m\|\delta(r) = \delta_r]$			SD$[RG_r/m]$
1	0.1139	0.1139	0.1139	1.7363
2	1.3285	1.3684	1.4096	6.2319
3	4.1553	4.3141	4.4790	14.4910
4	9.0968	9.5207	9.9646	27.2862
$m \to \infty$				
r	SD$[RG_r/m\|\delta(r) = \delta_r]$			SD$[RG_r/m]$
1	0	0	0	1.7325
2	1.3164	1.3564	1.3976	6.2292
3	4.1481	4.3069	4.4718	14.4888
4	9.0914	9.5153	9.9591	27.2842

Table 4.2: Standard deviations of retrospective gain per policy for portfolios of 5-year endowment insurance contracts.

δ_r:	.04	.06	.08	
		m=1		
r	SD$[PL_r/m \mid \delta(r) = \delta_r]$			SD$[PL_r/m]$
1	66.8353	64.2498	61.7977	62.0645
2	59.6696	57.7330	55.8769	56.2546
3	50.2845	49.0007	47.7559	48.1110
4	36.7656	36.1097	35.4655	35.6944
		m=100		
r	SD$[PL_r/m \mid \delta(r) = \delta_r]$			SD$[PL_r/m]$
0	6.6409	6.6409	6.6409	6.6409
1	6.6839	6.4253	6.1800	6.2069
2	5.9672	5.7735	5.5879	5.6258
3	5.0285	4.9002	4.7757	4.8113
4	3.6766	3.6110	3.5466	3.5695
		m=10,000		
r	SD$[PL_r/m \mid \delta(r) = \delta_r]$			SD$[PL_r/m]$
0	0.6681	0.6681	0.6681	0.6681
1	0.6716	0.6455	0.6208	0.6247
2	0.5986	0.5791	0.5605	0.5659
3	0.5038	0.4909	0.4784	0.4833
4	0.3679	0.3614	0.3549	0.3578
		m=100,000		
r	SD$[PL_r/m \mid \delta(r) = \delta_r]$			SD$[PL_r/m]$
0	0.2224	0.2224	0.2224	0.2224
1	0.2215	0.2126	0.2041	0.2086
2	0.1947	0.1882	0.1820	0.1882
3	0.1619	0.1577	0.1536	0.1589
4	0.1171	0.1150	0.1130	0.1157
		$m \to \infty$		
r	SD$[PL_r/m \mid \delta(r) = \delta_r]$			SD$[PL_r/m]$
0	0.0731	0.0731	0.0731	0.0731
1	0.0664	0.0625	0.0588	0.0707
2	0.0481	0.0458	0.0437	0.0614
3	0.0302	0.0292	0.0283	0.0460
4	0.0141	0.0139	0.0136	0.0254

Table 4.3: Standard deviations of prospective loss per policy for portfolios of 5-year temporary insurance contracts.

δ_r:	.04	.06	.08	
m=1				
r	SD$[PL_r/m \mid \delta(r) = \delta_r]$			SD$[PL_r/m]$
1	42.0424	40.6415	39.4018	42.7368
2	37.2222	35.6113	34.1390	40.5180
3	42.8343	41.0689	39.3771	45.0867
4	59.1968	57.9345	56.6947	58.9303
m=100				
r	SD$[PL_r/m \mid \delta(r) = \delta_r]$			SD$[PL_r/m]$
0	31.3961	31.3961	31.3961	31.3961
1	32.4592	30.3646	28.4031	32.8984
2	26.1491	24.8393	23.5945	31.9973
3	18.2940	17.6610	17.0498	27.1985
4	9.5378	9.3677	9.2006	17.1437
m=10,000				
r	SD$[PL_r/m \mid \delta(r) = \delta_r]$			SD$[PL_r/m]$
0	31.1793	31.1793	31.1793	31.1793
1	32.3606	30.2531	28.2783	32.7912
2	26.1022	24.7857	23.5339	31.9532
3	18.2796	17.6443	17.0308	27.1869
4	9.5378	9.3677	9.2006	17.1437
m=100,000				
r	SD$[PL_r/m \mid \delta(r) = \delta_r]$			SD$[PL_r/m]$
0	31.1773	31.1773	31.1773	31.1773
1	32.3597	30.2521	28.2771	32.7902
2	26.1018	24.7852	23.5333	31.9528
3	18.2795	17.6442	17.0306	27.1868
4	9.5378	9.3677	9.2006	17.1437
$m \to \infty$				
r	SD$[PL_r/m \mid \delta(r) = \delta_r]$			SD$[PL_r/m]$
0	31.1771	31.1771	31.1771	31.1771
1	32.3596	30.2520	28.2770	32.7901
2	26.1017	24.7851	23.5333	31.9528
3	18.2794	17.6441	17.0306	27.1868
4	9.5378	9.3677	9.2006	17.1437

Table 4.4: Standard deviations of prospective loss per policy for portfolios of 5-year endowment insurance contracts.

δ_r:	.04	.06	.08	
m=100				
r	$\text{SD}[S_r^{acct}/m \mid \delta(r) = \delta_r]$			$\text{SD}[S_r^{acct}/m]$
1	3.6024	3.6027	3.6030	3.6033
2	5.2170	5.2721	5.3289	5.3190
3	6.5908	6.7158	6.8456	6.8127
4	7.8726	8.0843	8.3062	8.2354
m=10,000				
r	$\text{SD}[S_r^{acct}/m \mid \delta(r) = \delta_r]$			$\text{SD}[S_r^{acct}/m]$
1	0.3602	0.3603	0.3603	0.3641
2	0.5218	0.5273	0.5330	0.5368
3	0.6594	0.6720	0.6850	0.6868
4	0.7881	0.8093	0.8316	0.8293
m=100,000				
r	$\text{SD}[S_r^{acct}/m \mid \delta(r) = \delta_r]$			$\text{SD}[S_r^{acct}/m]$
1	0.1139	0.1139	0.1139	0.1253
2	0.1653	0.1671	0.1689	0.1833
3	0.2096	0.2136	0.2178	0.2324
4	0.2515	0.2584	0.2656	0.2782
$m \to \infty$				
r	$\text{SD}[S_r^{acct}/m \mid \delta(r) = \delta_r]$			$\text{SD}[S_r^{acct}/m]$
1	0	0	0	0.0523
2	0.0104	0.0107	0.0111	0.0729
3	0.0218	0.0227	0.0236	0.0873
4	0.0359	0.0378	0.0397	0.0979

Table 4.5: Standard deviations of accounting surplus per policy for portfolios of 5-year temporary insurance contracts.

δ_r:	.04	.06	.08	
m=100				
r	SD$[S_r^{acct}/m \mid \delta(r) = \delta_r]$			SD$[S_r^{acct}/m]$
1	2.7466	2.8759	2.9962	18.1917
2	3.7808	4.0226	4.2583	27.7153
3	5.7585	6.1130	6.4739	35.4137
4	10.0558	10.6243	11.2189	41.3715
m=10,000				
r	SD$[S_r^{acct}/m \mid \delta(r) = \delta_r]$			SD$[S_r^{acct}/m]$
1	0.2747	0.2876	0.2996	17.9478
2	1.3633	1.4083	1.4544	27.4310
3	4.1673	4.3287	4.4962	35.1172
4	9.1016	9.5270	9.9725	41.0675
m=100,000				
r	SD$[S_r^{acct}/m \mid \delta(r) = \delta_r]$			SD$[S_r^{acct}/m]$
1	0.0869	0.0909	0.0947	17.9455
2	1.3211	1.3617	1.4034	27.4284
3	4.1500	4.3091	4.4742	35.1145
4	9.0924	9.5164	9.9605	41.0647
$m \to \infty$				
r	SD$[S_r^{acct}/m \mid \delta(r) = \delta_r]$			SD$[S_r^{acct}/m]$
1	0	0	0	17.9453
2	1.3164	1.3564	1.3976	27.4281
3	4.1481	4.3069	4.4718	35.1142
4	9.0914	9.5153	9.9591	41.0644

Table 4.6: Standard deviations of accounting surplus per policy for portfolios of 5-year endowment insurance contracts.

δ_r:	.04	.06	.08	
m=1				
r	SD$[S_r^{stoch}/m\|\delta(r)=\delta_r]$			SD$[S_r^{stoch}/m]$
1	75.9255	73.6613	71.5339	71.7644
2	79.2600	78.1831	77.2133	77.4156
3	82.8997	83.1334	83.4672	83.3982
4	86.8871	88.5402	90.3162	89.7519
m=100				
r	SD$[S_r^{stoch}/m\|\delta(r)=\delta_r]$			SD$[S_r^{stoch}/m]$
1	7.5928	7.3664	7.1536	7.1769
2	7.9261	7.8184	7.7215	7.7420
3	8.2901	8.3134	8.3468	8.3403
4	8.6888	8.8541	9.0317	8.9757
m=10,000				
r	SD$[S_r^{stoch}/m\|\delta(r)=\delta_r]$			SD$[S_r^{stoch}/m]$
1	0.7621	0.7393	0.7178	0.7220
2	0.7941	0.7832	0.7734	0.7788
3	0.8298	0.8322	0.8355	0.8390
4	0.8697	0.8863	0.9041	0.9030
m=100,000				
r	SD$[S_r^{stoch}/m\|\delta(r)=\delta_r]$			SD$[S_r^{stoch}/m]$
1	0.2491	0.2412	0.2337	0.2403
2	0.2554	0.2517	0.2483	0.2592
3	0.2648	0.2655	0.2665	0.2793
4	0.2775	0.2829	0.2887	0.3006
$m \to \infty$				
r	SD$[S_r^{stoch}/m\|\delta(r)=\delta_r]$			SD$[S_r^{stoch}/m]$
1	0.0664	0.0625	0.0588	0.0790
2	0.0492	0.0471	0.0451	0.0852
3	0.0373	0.0370	0.0368	0.0918
4	0.0386	0.0402	0.0419	0.0989

Table 4.7: Standard deviations of stochastic surplus per policy for portfolios of 5-year temporary insurance contracts.

δ_r:	.04	.06	.08	
m=1				
r	$\text{SD}[S_r^{stoch}/m \mid \delta(r) = \delta_r]$			$\text{SD}[S_r^{stoch}/m]$
1	49.4829	49.2533	49.1260	52.3192
2	46.7823	48.1719	49.6357	56.4664
3	44.7365	47.6706	50.6774	60.8835
4	44.9518	49.1162	53.4076	65.6090
m=100				
r	$\text{SD}[S_r^{stoch}/m \mid \delta(r) = \delta_r]$			$\text{SD}[S_r^{stoch}/m]$
1	32.5640	30.4918	28.5543	33.9113
2	26.3348	25.0865	23.9085	36.5228
3	18.7982	18.3339	17.9072	39.2274
4	12.5683	12.9595	13.3874	42.0229
m=10,000				
r	$\text{SD}[S_r^{stoch}/m \mid \delta(r) = \delta_r]$			$\text{SD}[S_r^{stoch}/m]$
1	32.3617	30.2544	28.2798	33.6824
2	26.1369	24.8249	23.5781	36.3144
3	18.7447	18.1639	17.6109	39.1077
4	13.1707	13.3488	13.5569	42.0990
m=100,000				
r	$\text{SD}[S_r^{stoch}/m \mid \delta(r) = \delta_r]$			$\text{SD}[S_r^{stoch}/m]$
1	32.3598	30.2523	28.2773	33.6803
2	26.1351	24.8225	23.5751	36.3125
3	18.7442	18.1624	17.6082	39.1066
4	13.1761	13.3523	13.5584	42.0997
$m \to \infty$				
r	$\text{SD}[S_r^{stoch}/m \mid \delta(r) = \delta_r]$			$\text{SD}[S_r^{stoch}/m]$
1	32.3596	30.2520	28.2770	33.6800
2	26.1349	24.8222	23.5747	36.3123
3	18.7442	18.1622	17.6079	39.1065
4	13.1767	13.3527	13.5586	42.0998

Table 4.8: Standard deviations of stochastic surplus per policy for portfolios of 5-year endowment insurance contracts.

δ_r:	.04	.06	.08	
		m=100		
r	$\text{Corr}[RG_r/m, PL_r/m \mid \delta(r) = \delta_r]$			$\text{Corr}[RG_r/m, PL_r/m]$
1	0.0264	0.0240	0.0214	-0.1965
2	0.0791	0.0763	0.0734	-0.4371
3	0.1811	0.1778	0.1745	-0.5831
4	0.4424	0.4362	0.4300	-0.6193
		m=10,000		
r	$\text{Corr}[RG_r/m, PL_r/m \mid \delta(r) = \delta_r]$			$\text{Corr}[RG_r/m, PL_r/m]$
1	0.0026	0.0024	0.0021	-0.4835
2	0.0032	0.0031	0.0029	-0.6468
3	0.0038	0.0037	0.0035	-0.7347
4	0.0065	0.0064	0.0062	-0.7829
		m=100,000		
r	$\text{Corr}[RG_r/m, PL_r/m \mid \delta(r) = \delta_r]$			$\text{Corr}[RG_r/m, PL_r/m]$
1	0.0008	0.0008	0.0007	-0.4931
2	0.0003	0.0003	0.0003	-0.6498
3	0.0004	0.0004	0.0004	-0.7364
4	0.0007	0.0006	0.0006	-0.7845
		$m \to \infty$		
r	$\text{Corr}[RG_r/m, PL_r/m \mid \delta(r) = \delta_r]$			$\text{Corr}[RG_r/m, PL_r/m]$
1	0	0	0	-0.4942
2	0	0	0	-0.6501
3	0	0	0	-0.7366
4	0	0	0	-0.7847

Table 4.9: Correlation coefficients between retrospective gain and prospective loss per policy for portfolios of 5-year endowment insurance contracts.

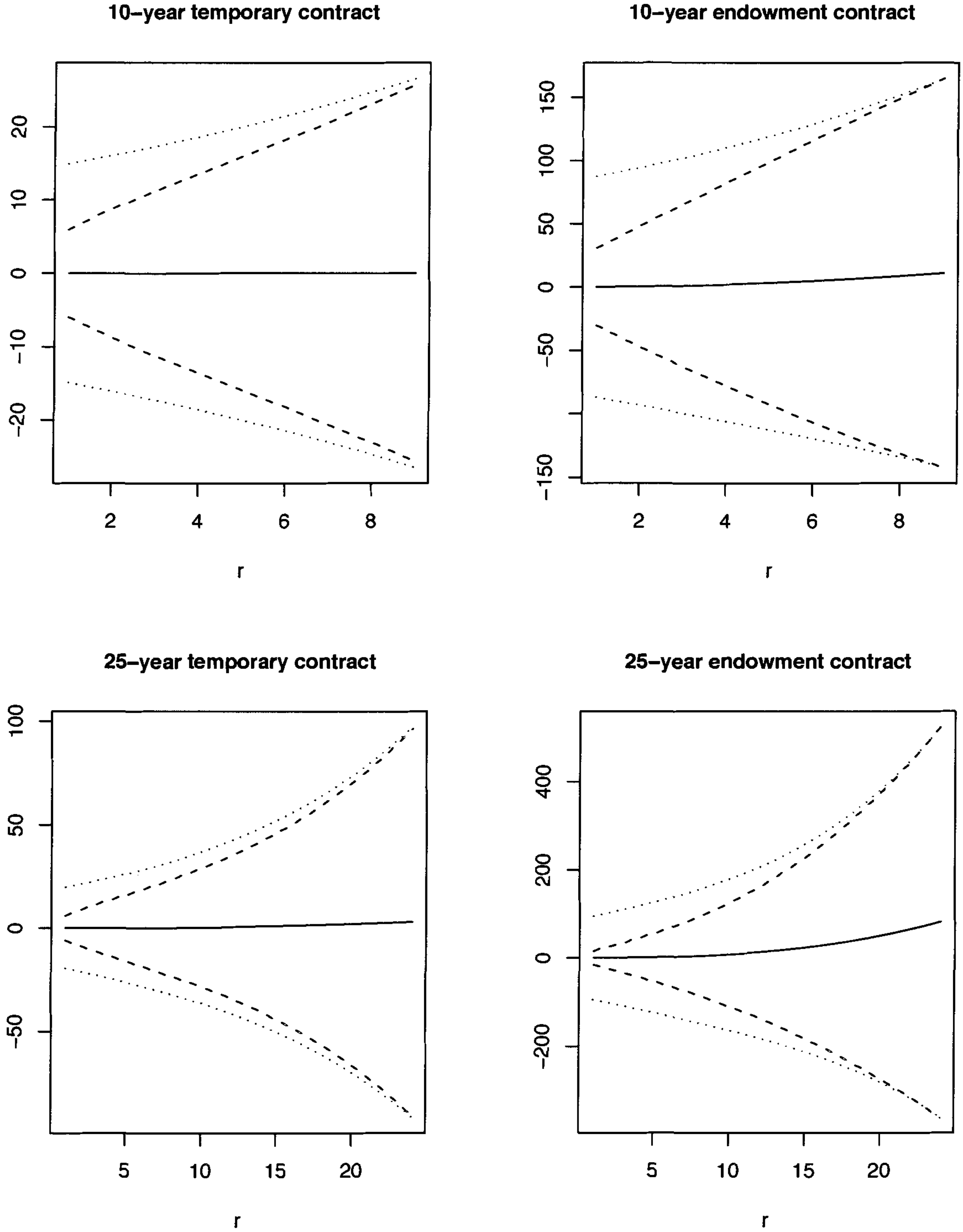

Figure 4.1: Expected value of surplus per policy, $\mathrm{E}[S_r/m]$ (solid line); $\mathrm{E}[S_r/m] \pm 1.65\sqrt{\mathrm{Var}[S_r^{acct}/m]}$ (dashed line) and $\mathrm{E}[S_r/m] \pm 1.65\sqrt{\mathrm{Var}[S_r^{stoch}/m]}$ (dotted line) for portfolios of 100 10-year and 25-year temporary and endowment contracts.

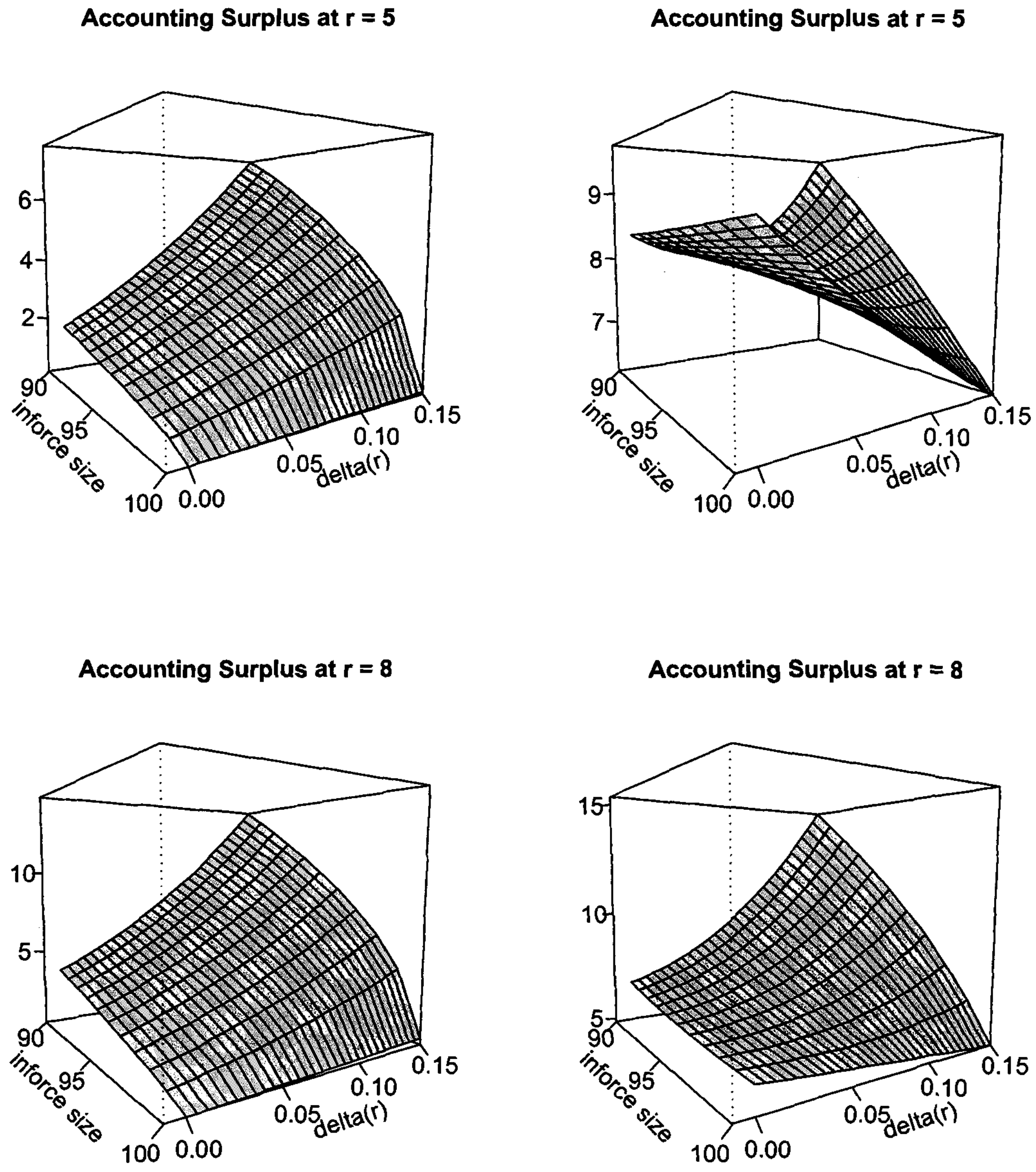

Figure 4.2: Conditional standard deviation of surplus per policy for a portfolio of 100 10-year temporary contracts given the inforce size $\mathscr{L}_r(x)$ and the force of interest $\delta(r)$.

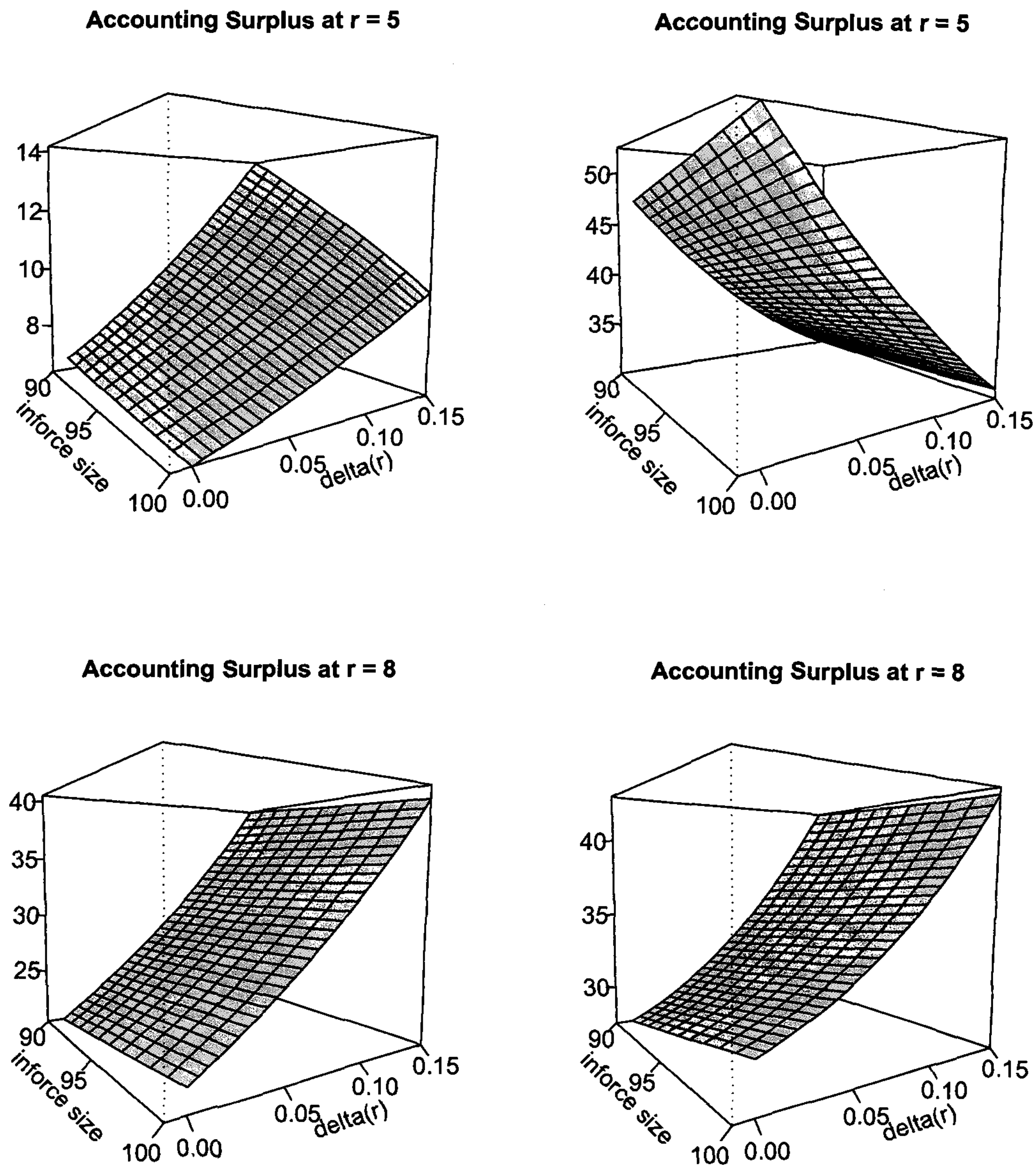

Figure 4.3: Conditional standard deviation of surplus per policy for a portfolio of 100 10-year endowment contracts given the inforce size $\mathscr{L}_r(x)$ and the force of interest $\delta(r)$.

Chapter 5

Distribution Function of Accounting Surplus

In the previous chapter we derived and studied the first two moments of the stochastic and accounting surpluses for a homogeneous portfolio of life insurance policies. Although the analysis of the moments certainly helped us gain better understanding of the stochastic properties of the insurance surplus, it can be viewed only as a first step towards exploring the surplus' random behaviour. The standard deviation as a risk measure is unable to provide meaningful information when dealing with asymmetric distributions. Also, in the insurance context, usually only one of the tails of the distribution is of concern. So, nowadays commonly used risk measures are the Value-at-Risk (VaR) and the expected shortfall or conditional tail expectation (CTE), calculation of which requires the knowledge of the distribution function. One of the objectives of this study was to assess the probability of insolvency; i.e., the probability that the surplus will fall below zero. This chapter is, thus, devoted to the calculation of the distribution function of the accounting surplus at a given valuation date, which in turn allows to obtain the probability of insolvency.

5.1 Distribution Function of Accounting Surplus

Recall that the accounting surplus at time r, conditional on the number of inforce policies and the force of interest at that time, is given by

$$\{S_r^{acct}|\mathscr{L}_r(x),\, \delta(r)\} = \{RG_r|\mathscr{L}_r(x),\, \delta(r)\} - {}_rV(\mathscr{L}_r(x),\, \delta(r)),$$

where ${}_rV(\mathscr{L}_r(x),\, \delta(r)) \equiv {}_rV$ is the reserve at time r.

Notice that, given the values of $\mathscr{L}_r(x)$ and $\delta(r)$, ${}_rV$ is constant. Therefore, we can obtain the distribution function (df) of $\{S_r^{acct}\,|\,\mathscr{L}_r(x),\, \delta(r)\}$ from the df of $\{RG_r\,|\,\mathscr{L}_r(x),\, \delta(r)\}$ as follows:

$$\mathbf{P}[S_r^{acct} \leq \xi\,|\,\mathscr{L}_r = m_r,\, \delta(r) = \delta_r] = \mathbf{P}[RG_r \leq \xi + {}_rV(m_r,\, \delta_r)\,|\,\mathscr{L}_r = m_r,\, \delta(r) = \delta_r], \tag{5.1}$$

with $\mathscr{L}_r(x)$ replaced by $\mathscr{L}_r$ for simplicity of notation.

Since it is not trivial to get the distribution function of $\{RG_r\,|\,\mathscr{L}_r,\, \delta(r)\}$ directly, we propose a recursive approach.

For the valuation at a given time r, let $G_t = \sum_{j=0}^{t} RC_j^r \cdot e^{I(j,t)}$ denote the accumulated value to time t of the *retrospective* cash flows that occured up to and including time t, $0 \leq t \leq r$. Observe that G_r is equal to RG_r.

We can relate G_t and G_{t-1} as follows:

$$\begin{aligned}
G_t &= \sum_{j=0}^{t} RC_j^r \cdot e^{I(j,t)} \\
&= \sum_{j=0}^{t-1} RC_j^r \cdot e^{I(j,t-1)+I(t-1,t)} + RC_t^r \cdot e^{I(t,t)} \\
&= \Big(\sum_{j=0}^{t-1} RC_j^r \cdot e^{I(j,t-1)}\Big) \cdot e^{\delta(t)} + RC_t^r \\
&= G_{t-1} \cdot e^{\delta(t)} + RC_t^r. \qquad (5.2)
\end{aligned}$$

Equation (5.2) can be used to build up the df of G_t from the df of G_{t-1} and thus the df of RG_r recursively from G_t for $t = 0, 1, \ldots, r-1$.

Note that [1]

$$\mathbf{P}[G_t \le \lambda \,|\, \mathscr{L}_t = m_t,\ \delta(t) = \delta_t] = \frac{\mathbf{P}[\mathscr{L}_t = m_t,\ \delta(t) = \delta_t \,|\, G_t \le \lambda] \cdot \mathbf{P}[G_t \le \lambda]}{\mathbf{P}[\mathscr{L}_t = m_t,\ \delta(t) = \delta_t]}$$

$$= \frac{\mathbf{P}[\mathscr{L}_t = m_t \,|\, G_t \le \lambda] \cdot f_{\delta(t)}(\delta_t \,|\, G_t \le \lambda) \cdot \mathbf{P}[G_t \le \lambda]}{\mathbf{P}[\mathscr{L}_t = m_t] \cdot f_{\delta(t)}(\delta_t)}, \tag{5.3}$$

where the last line follows from the independence of $\mathscr{L}_t$ and $\delta(t)$.

Next we consider a function $g_t(\lambda,\ m_t,\ \delta_t)$ given by

$$g_t(\lambda,\ m_t,\ \delta_t) \quad = \quad \mathbf{P}[G_t \le \lambda \,|\, \mathscr{L}_t(x) = m_t,\ \delta(t) = \delta_t] \cdot \mathbf{P}[\mathscr{L}_t(x) = m_t] \cdot f_{\delta(t)}(\delta_t)$$

and motivated by Equation (5.3).

The following result gives a way for calculating g_t from g_{t-1}, $1 < t \le r \le n$.

Result 5.1.1.

$$g_t(\lambda,\ m_t,\ \delta_t) = \sum_{m_{t-1}=m_t}^{m} \mathbf{P}\Big[\mathscr{L}_t = m_t \,\Big|\, \mathscr{L}_{t-1} = m_{t-1}\Big] \times$$
$$\times \int_{-\infty}^{\infty} g_{t-1}\Big(\frac{\lambda - \eta_t}{e^{\delta_t}},\ m_{t-1},\ \delta_{t-1}\Big) \cdot f_{\delta(t)}(\delta_t \,|\, \delta(t-1) = \delta_{t-1})\, d\delta_{t-1},$$

where η_t is the realization of RC_t^r for given values of m_{t-1} and m_t,

$$\eta_t = \begin{cases} \pi \cdot m_t - b \cdot (m_{t-1} - m_t), & 1 \le t \le r-1, \\ -b \cdot (m_{t-1} - m_t), & t = r, \end{cases}$$

with the starting value for g_t

$$g_1(\lambda,\ m_1,\ \delta_1) = \begin{cases} \mathbf{P}[\mathscr{L}_1(x) = m_1] \cdot f_{\delta(1)}(\delta_1) & \text{if } G_1 \le \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

[1] $f(\cdot)$ denotes the probability density function (pdf). Under our assumption for the rates of return, $f_{\delta(t)}(\cdot)$ is the pdf of a normal random variable with mean $\mathrm{E}[\delta(t)|\delta(0) = \delta_0]$ and variance $\mathrm{Var}[\delta(t)|\delta(0) = \delta_0]$, and $f_{\delta(t)}(\cdot|\mathscr{A})$, where $\mathscr{A} \equiv \{\delta(t-1) = \delta_{t-1}\}$, is the pdf of a normal random variable with mean $\mathrm{E}[\delta(t)|\delta(0) = \delta_0,\ \mathscr{A}]$ and variance $\mathrm{Var}[\delta(t)|\delta(0) = \delta_0,\ \mathscr{A}]$.

Proof:
From Equation (5.3) we have

$$g_t(\lambda,\, m_t,\, \delta_t) = \mathbf{P}[\mathscr{L}_t = m_t,\, \delta(t) = \delta_t \,|\, G_t \le \lambda] \cdot \mathbf{P}[G_t \le \lambda]$$

$$\begin{aligned}
&= \sum_{m_{t-1}=m_t}^{m} \mathbf{P}[\mathscr{L}_{t-1} = m_{t-1},\, \mathscr{L}_t = m_t,\, \delta(t) = \delta_t \,|\, G_t \le \lambda] \cdot \mathbf{P}[G_t \le \lambda] \\
&= \sum_{m_{t-1}=m_t}^{m} \mathbf{P}[G_t \le \lambda \,|\, \mathscr{L}_{t-1} = m_{t-1},\, \mathscr{L}_t = m_t,\, \delta(t) = \delta_t] \times \\
&\qquad \times \mathbf{P}[\mathscr{L}_{t-1} = m_{t-1},\, \mathscr{L}_t = m_t,\, \delta(t) = \delta_t] \qquad \text{(by the Bayes' rule).}
\end{aligned}$$

Using Equation (5.2), which implies that $\Big\{G_t \le \lambda\Big\} \equiv \Big\{G_{t-1} \le \frac{\lambda - RC_t^T}{e^{\delta(t)}}\Big\}$, and the assumption of independence of $\mathscr{L}_{t-1}$ and $\mathscr{L}_t$ from $\delta(t)$, we get

$$\begin{aligned}
g_t(\lambda,\, m_t,\, \delta_t) &= \sum_{m_{t-1}=m_t}^{m} \mathbf{P}\Big[G_{t-1} \le \frac{\lambda - \eta_t}{e^{\delta_t}} \,\Big|\, \mathscr{L}_{t-1} = m_{t-1},\, \mathscr{L}_t = m_t,\, \delta(t) = \delta_t\Big] \times \\
&\qquad \times \mathbf{P}[\mathscr{L}_{t-1} = m_{t-1},\, \mathscr{L}_t = m_t] \cdot f_{\delta(t)}(\delta_t) \\
&= \sum_{m_{t-1}=m_t}^{m} \mathbf{P}\Big[\mathscr{L}_{t-1} = m_{t-1},\, \mathscr{L}_t = m_t \,\Big|\, G_{t-1} \le \frac{\lambda - \eta_t}{e^{\delta_t}}\Big] \cdot f_{\delta(t)}\Big(\delta_t \,\Big|\, G_{t-1} \le \frac{\lambda - \eta_t}{e^{\delta_t}}\Big) \times \\
&\qquad \times \mathbf{P}\Big[G_{t-1} \le \frac{\lambda - \eta_t}{e^{\delta_t}}\Big] \\
&= \sum_{m_{t-1}=m_t}^{m} \mathbf{P}\Big[\mathscr{L}_t = m_t \,\Big|\, \mathscr{L}_{t-1} = m_{t-1},\, G_{t-1} \le \frac{\lambda - \eta_t}{e^{\delta_t}}\Big] \times \\
&\qquad \times \mathbf{P}\Big[\mathscr{L}_{t-1} = m_{t-1} \,\Big|\, G_{t-1} \le \frac{\lambda - \eta_t}{e^{\delta_t}}\Big] \cdot \mathbf{P}\Big[G_{t-1} \le \frac{\lambda - \eta_t}{e^{\delta_t}}\Big] \times \\
&\times \int_{-\infty}^{\infty} f_{\delta(t)}\Big(\delta_t \Big| \delta(t-1) = \delta_{t-1}, G_{t-1} \le \frac{\lambda - \eta_t}{e^{\delta_t}}\Big) \cdot f_{\delta(t-1)}\Big(\delta_{t-1} | G_{t-1} \le \frac{\lambda - \eta_t}{e^{\delta_t}}\Big)\, d\delta_{t-1}.
\end{aligned}$$

By the Markovian property of $\mathscr{L}_t$ and $\delta(t)$ and the definition of $g_{t-1}\Big(\frac{\lambda - \eta_t}{e^{\delta_t}},\, m_{t-1},\, \delta_{t-1}\Big)$, $g_t(\lambda,\, m_t,\, \delta_t)$ becomes

$$\begin{aligned}
g_t(\lambda,\, m_t,\, \delta_t) &= \sum_{m_{t-1}=m_t}^{m} \mathbf{P}\Big[\mathscr{L}_t = m_t \,\Big|\, \mathscr{L}_{t-1} = m_{t-1}\Big] \times \\
&\times \int_{-\infty}^{\infty} g_{t-1}\Big(\frac{\lambda - \eta_t}{e^{\delta_t}},\, m_{t-1},\, \delta_{t-1}\Big) \cdot f_{\delta(t)}(\delta_t \,|\, \delta(t-1) = \delta_{t-1})\, d\delta_{t-1}. \qquad \square
\end{aligned}$$

Once $g_r(\lambda,\, m_r,\, \delta_r)$ is obtained using Result 5.1.1, the cumulative distribution function of S_r^{acct} can be calculated as follows:

$$\begin{aligned}
&\mathbf{P}[S_r^{acct} \leq \xi] = \\
&= \int_{-\infty}^{\infty} \sum_{m_r=0}^{m} \mathbf{P}[S_r^{acct} \leq \xi \mid \mathscr{L}_r = m_r,\, \delta(r) = \delta_r] \cdot \mathbf{P}[\mathscr{L}_r = m_r] \cdot f_{\delta(r)}(\delta_r)\, d\delta_r \\
&= \int_{-\infty}^{\infty} \sum_{m_r=0}^{m} \mathbf{P}[RG_r \leq \xi + {}_rV \mid \mathscr{L}_r = m_r,\, \delta(r) = \delta_r] \cdot \mathbf{P}[\mathscr{L}_r = m_r] \cdot f_{\delta(r)}(\delta_r)\, d\delta_r \\
&= \int_{-\infty}^{\infty} \sum_{m_r=0}^{m} \mathbf{P}[G_r \leq \xi + {}_rV \mid \mathscr{L}_r = m_r,\, \delta(r) = \delta_r] \cdot \mathbf{P}[\mathscr{L}_r = m_r] \cdot f_{\delta(r)}(\delta_r)\, d\delta_r \\
&= \int_{-\infty}^{\infty} \sum_{m_r=0}^{m} g_r(\xi + {}_rV,\, m_r,\, \delta_r)\, d\delta_r. \qquad (5.4)
\end{aligned}$$

Note that the reserve value, ${}_rV$, depends on $\mathscr{L}_r$ and $\delta(r)$ and so is different for different realizations of $\mathscr{L}_r$ and $\delta(r)$, m_r and δ_r respectively.

Another approach that is easier to understand but which requires keeping track of more information is given in Appendix D.

5.2 Distribution Function of Accounting Surplus per Policy for a Limiting Portfolio

For a very large insurance portfolio, the actual mortality experience follows very closely the life table. In this case we can approximate the true distribution of the surplus by its limiting distribution, which takes into account the investment risk but treats cash flows as given and equal to their expected values.

The limiting distribution can be derived similarly to the case of random cash flows. Define $\mathcal{G}_t = \sum_{j=0}^{t} \mathrm{E}[RC_j^r/m] \cdot e^{I(j,t)}$.

It can easily be shown that $\mathcal{G}_t = \mathcal{G}_{t-1} \cdot e^{\delta_t} + \mathrm{E}[RC_t^r/m]$ (cf. Equation (5.2)).

Now, let $h_t(\lambda,\, \delta_t) = \mathbf{P}[\mathcal{G}_t \leq \lambda \mid \delta(t) = \delta_t] \cdot f_{\delta(t)}(\delta_t)$. This function can be used to calculate the df of $\mathcal{G}_t$ recursively similar to the way $g_t(\lambda,\, m_t,\, \delta_t)$ was used for obtaining the df of G_t. A recursive relation for $h_t(\lambda,\, \delta_t)$ is given in the following result.

Result 5.2.1.

$$h_t(\lambda,\,\delta_t) \;=\; \int_{-\infty}^{\infty} f_{\delta(t)}(\delta_t\,|\,\delta(t-1)=\delta_{t-1})\cdot h_{t-1}\Big(\frac{\lambda - E[RC_t^r/m]}{e^{\delta_t}},\,\delta_{t-1}\Big)\,d\delta_{t-1}$$

with the starting value for h_t

$$h_1(\lambda,\,\delta_1) = \begin{cases} f_{\delta(1)}(\delta_1) & \text{if } \mathcal{G}_1 \le \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\lim_{m\to\infty}\mathbf{P}[S_r^{acct}/m \le \xi\,|\,\delta(r)=\delta_r] = \mathbf{P}[\mathcal{G}_r \le \xi + {}_r\mathcal{V}\,|\,\delta(r)=\delta_r]$,

$$\lim_{m\to\infty}\mathbf{P}[S_r^{acct}/m \le \xi] = \int_{-\infty}^{\infty} h_r(\xi + {}_r\mathcal{V},\,\delta_r)\,d\delta_r\,, \tag{5.5}$$

where ${}_r\mathcal{V} \equiv {}_r\mathcal{V}(\delta(r))$ denotes the benefit reserve at time r per policy for the limiting portfolio.

5.3 Numerical Illustrations of Results

For numerical illustrations, we assume that

$${}_rV \equiv {}_rV(\mathscr{L}_r(x),\,\delta(r)) = \mathrm{E}\big[PL_r\,|\,\mathscr{L}_r,\,\delta(r)\big]$$

and

$${}_r\mathcal{V} \equiv {}_r\mathcal{V}(\delta(r)) = \mathrm{E}\big[PL_r/m\,|\,\delta(r)\big].$$

5.3.1 Example 1: Portfolio of Endowment Life Insurance Policies

Consider a portfolio of 100 10-year endowment life insurance policies with \$1000 death and endowment benefits issued to a group of people aged 30 with the same mortality profile. Table 5.1 provides estimates of the probability of insolvency in any given year for different premium rates. The first column corresponds to the premium determined under the equivalence principle and the second column corresponds to the premium with a 10% loading factor. We can see that when $\theta = 0\%$, all probabilities are slightly less than 50%. This can be expected since no profit or contingency margin is built into the premium when pricing is done under the equivalence principle. The fact that these probabilities are not exactly 50% is due to the asymmetry of the discount function. With the 10% loading factor, the probability of insolvency sharply decreases compared to the case of $\theta = 0\%$ in the first few years but this reduction is not as large in the later years of the contract. The probability that the accounting surplus falls below zero increases from 0.23% at $r = 1$ to 14.57% at $r = 10$. A 20% loading factor appears to be sufficient to ensure that the probability of insolvency in any given year is less than 5% .

Cumulative distribution functions of accounting surplus per policy for different values of r are displayed in Figure 5.1 for three cases of $\theta = 0\%$, $\theta = 10\%$ and $\theta = 20\%$. It can be observed that applying a loading factor to the benefit premium leads to an almost parallel shift in the distribution. We saw earlier that the variability of surplus increases with r. This is confirmed by the shape of the curves, which seem to be tilting to the right and look more spread out for larger values of r . Another interesting feature of the surplus distribution is a change in its skewness over time. Estimates of the skewness coefficients are summarized in Table 5.2. We can see that as r increases, the distribution changes from being negatively skewed to fairly positively skewed.

Based on the analysis of the variability of accounting surplus per policy in the previous chapter, there is little difference between portfolios of size 10,000 or more and the limiting portfolio. So, let us also look at the accounting surplus per policy for the limiting portfolio. Table 5.3 contains estimates of insolvency probabilities. In addition to the benefit premium, we consider premiums with 10% and 20% loading factors as well as the case when nonzero initial surplus is included [2]. Our arbitrary choice of the amount of initial surplus is based on the 70^{th} percentile of the surplus distribution at time $r = n = 10$. Probabilities when $\theta = 0\%$ and $\theta = 10\%$ are very close to the corresponding probabilities for the 100-policy portfolio we studied

[2]To calculate the df of S_r^{acct}/m with nonzero initial surplus per policy S_0, one simply has to adjust the cash flow at time 0 from $RC_0^r = \pi \cdot m$ to $RC_0^r = (\pi + S_0) \cdot m$, and then apply Result 5.1.1 or Result 5.2.1.

above. A 20% loading seems to be adequate to ensure no more than 5% probability of insolvency for all r. Instead of charging the premium with a 20% loading, the insurer can start this block of business with some initial surplus, say \$61.74 per policy, and a lower premium. This initial surplus combined with the premium with only a 10% loading results in similar (slightly lower) insolvency probabilities.

Cumulative distribution functions are plotted in Figure 5.2 and estimates of the skewness coefficients are given in Table 5.4. The results are very similar to the case of the 100-policy portfolio.

r	$\theta = 0\%$ $\pi = 67.90$	$\theta = 10\%$ $\pi = 74.69$	$\theta = 20\%$ $\pi = 81.48$
1	0.4799	0.0023	0.0000
2	0.4872	0.0224	0.0001
3	0.4760	0.0540	0.0011
4	0.4724	0.0817	0.0042
5	0.4708	0.1042	0.0090
6	0.4696	0.1208	0.0143
7	0.4688	0.1329	0.0190
8	0.4683	0.1405	0.0225
9	0.4681	0.1445	0.0243
10	0.4680	0.1457	0.0254

Table 5.1: Estimates of probabilities that accounting surplus falls below zero for a portfolio of 100 10-year endowment policies.

r	$\theta = 0\%$ $\pi = 67.90$	$\theta = 10\%$ $\pi = 74.69$	$\theta = 20\%$ $\pi = 81.48$
1	-0.1968	-0.2062	-0.2178
2	-0.2108	-0.2200	-0.2182
3	-0.1922	-0.1878	-0.1850
4	-0.1515	-0.1446	-0.1384
5	-0.0947	-0.0856	-0.0772
6	-0.0248	-0.0145	-0.0052
7	0.0564	0.0681	0.0770
8	0.1474	0.1570	0.1641
9	0.2457	0.2523	0.2560
10	0.3460	0.3489	0.3486

Table 5.2: Estimates of skewness coefficients of accounting surplus distribution for a portfolio of 100 10-year endowment policies.

r	$\theta = 0\%$ $\pi = 67.90$ $S_0 = 0$	$\theta = 10\%$ $\pi = 74.69$ $S_0 = 0$	$\theta = 20\%$ $\pi = 81.48$ $S_0 = 0$	$\theta = 0\%$ $\pi = 67.90$ $S_0 = 61.74$	$\theta = 10\%$ $\pi = 74.69$ $S_0 = 61.74$
1	0.4891	0.0019	0.0000	0.0006	0.0000
2	0.4664	0.0221	0.0001	0.0113	0.0000
3	0.4723	0.0513	0.0010	0.0322	0.0007
4	0.4717	0.0796	0.0040	0.0536	0.0028
5	0.4701	0.1022	0.0086	0.0721	0.0062
6	0.4688	0.1190	0.0138	0.0865	0.0100
7	0.4680	0.1313	0.0185	0.0969	0.0136
8	0.4675	0.1386	0.0219	0.1037	0.0161
9	0.4673	0.1427	0.0240	0.1072	0.0178
10	0.4672	0.1439	0.0245	0.1082	0.0185

Table 5.3: Estimates of probabilities that accounting surplus per policy falls below zero for the limiting portfolio of 10-year endowment policies. Initial surplus per policy $S_0 = 61.74$ is the 70^{th} percentile of the S_{10}^{acct}/m distribution.

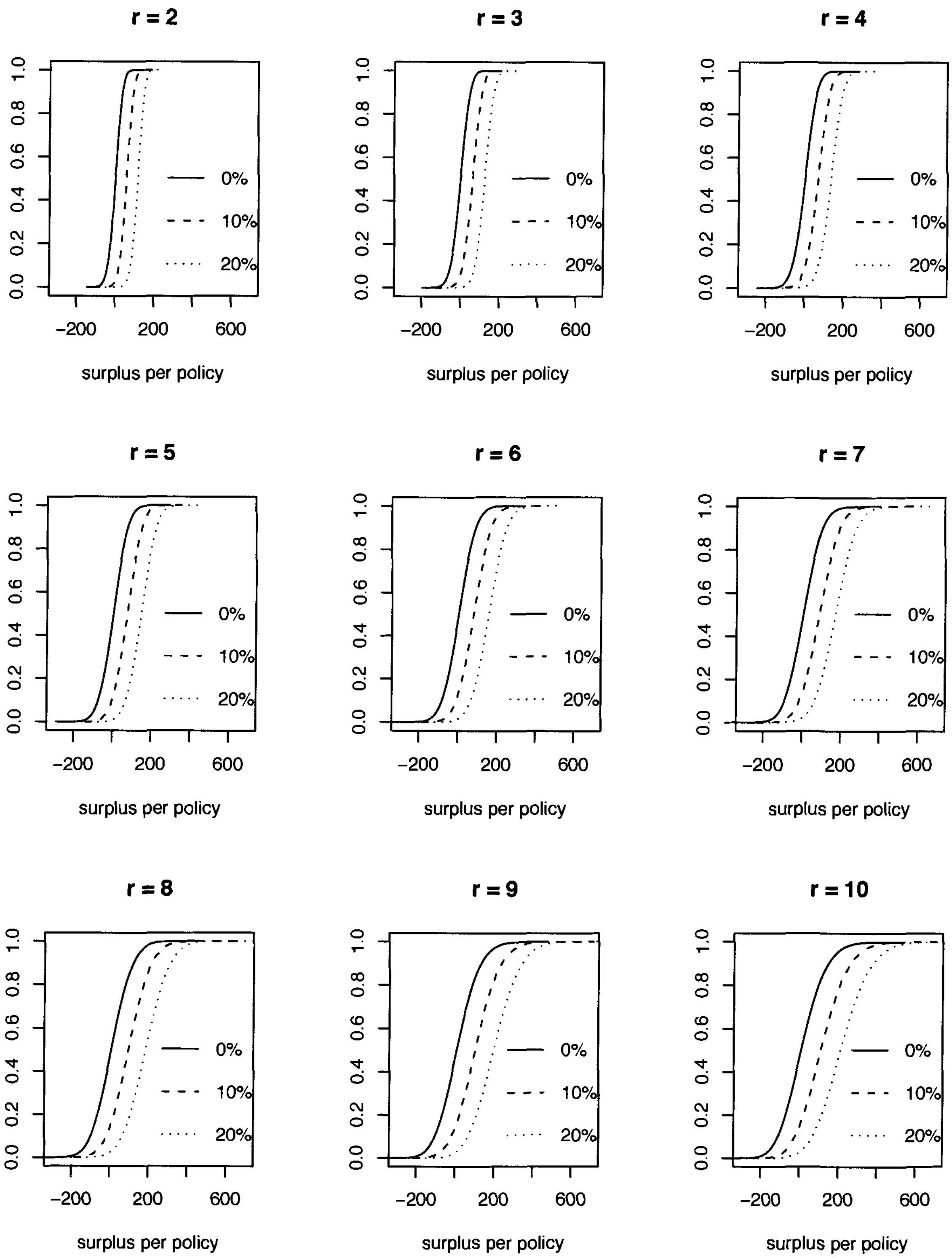

Figure 5.1: Distribution functions of accounting surplus per policy for a portfolio of 100 10-year endowment policies.

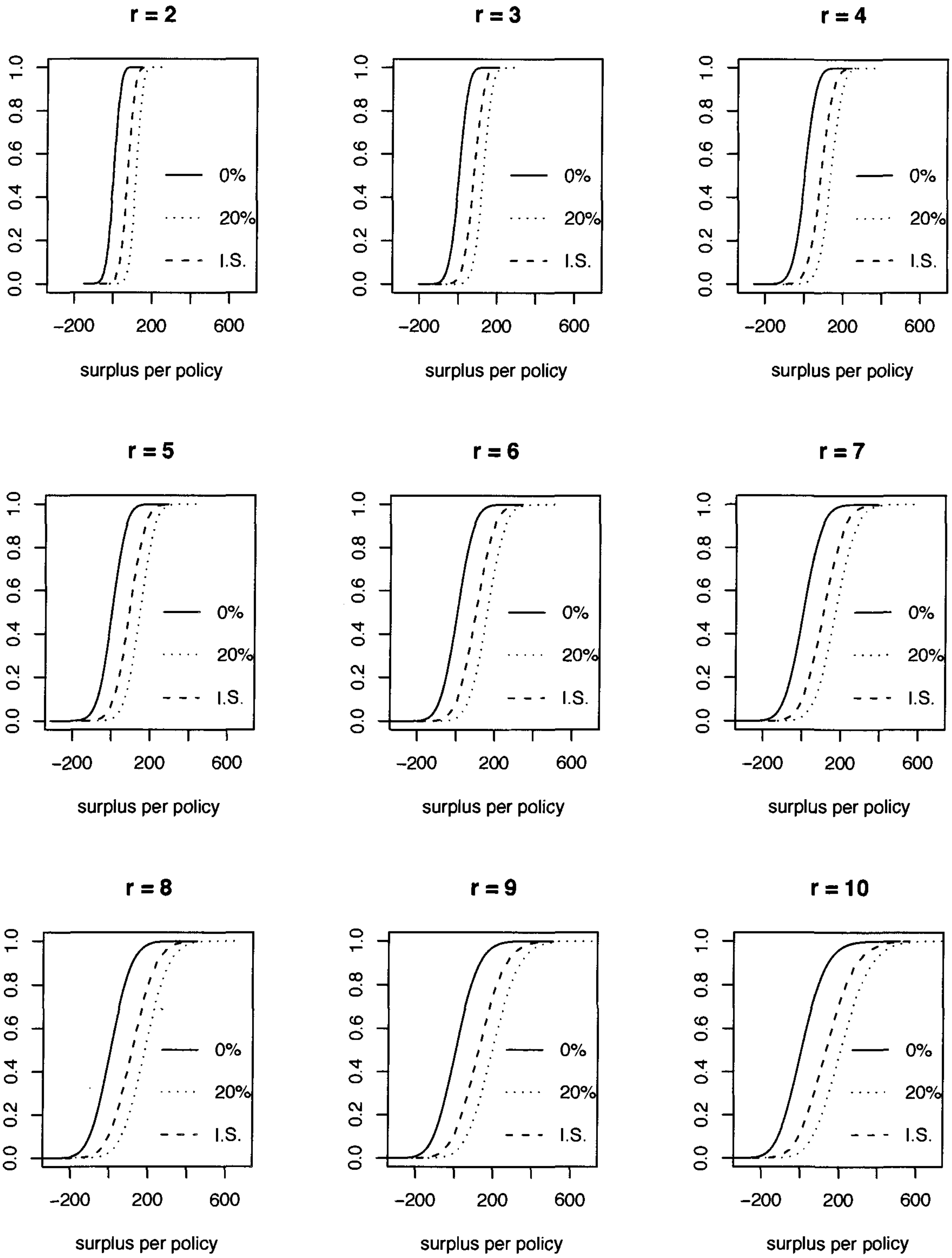

Figure 5.2: Distribution functions of accounting surplus per policy for the limiting portfolio of 10-year endowment contracts. Initial surplus per policy I.S.$\equiv S_0 = 61.74$.

r	$\theta = 0\%$ $\pi = 67.90$ $S_0 = 0$	$\theta = 10\%$ $\pi = 74.69$ $S_0 = 0$	$\theta = 20\%$ $\pi = 81.48$ $S_0 = 0$	$\theta = 0\%$ $\pi = 67.90$ $S_0 = 61.74$	$\theta = 10\%$ $\pi = 74.69$ $S_0 = 61.74$
1	-0.1936	-0.1985	-0.1980	-0.1857	-0.1899
2	-0.2156	-0.2169	-0.2178	-0.2235	-0.1915
3	-0.1991	-0.1962	-0.1934	-0.1895	-0.1852
4	-0.1598	-0.1533	-0.1471	-0.1419	-0.1356
5	-0.1034	-0.0944	-0.0857	-0.0819	-0.0734
6	-0.0333	-0.0228	-0.0129	-0.0109	-0.0016
7	0.0484	0.0591	0.0687	0.0685	0.0777
8	0.1401	0.1490	0.1568	0.1537	0.1617
9	0.2391	0.2444	0.2488	0.2414	0.2473
10	0.3418	0.3418	0.3419	0.3278	0.3301

Table 5.4: Estimates of skewness coefficients of accounting surplus per policy distribution for the limiting portfolio of 10-year endowment policies. Initial surplus per policy $S_0 = 61.74$ is the 70^{th} percentile of the S_{10}^{acct}/m distribution.

5.3.2 Example 2: Portfolio of Temporary Life Insurance Policies

In our next example, we study a homogeneous portfolio of 1000 5-year temporary insurance policies and the corresponding limiting portfolio with \$1000 death benefit issued to people aged 30. As we saw in the previous chapter, even a very large portfolio (e.g., 100,000 policies) of temporary policies is still quite far from the limiting one. This is confirmed again by the distribution function of the accounting surplus per policy. Tables 5.5 and 5.6 give estimates of the probabilities of insolvency for different premiums charged. Premiums with 2% or 3% loading factors considerably decrease the probability of insolvency over the whole term of the contract for the limiting portfolio but have essentially no impact on those probabilities for the 1000-policy portfolio. Even a 20% loading factor is not sufficient to reduce the probabilities of insolvency to a reasonably low level (e.g. 5-10%). An implication of this is that for portfolios of temporary insurances, an insurer either has to maintain a very large portfolio or use a large premium loading.

The distribution of the surplus for the 1000-policy portfolio remains negatively skewed for all values of r; see Table 5.7. In the case of the limiting portfolio, skewness coefficients are faily small in magnitude and change from being negative for small values of r to being positive for larger r; see Table 5.8.

The distribution functions of the accounting surplus per policy for the two portfolios are plotted in Figures 5.3 and 5.4. In the case of the 1000-policy portfolio, note that the plots for small values of r look more like plots of step functions for the df of a discrete random variable. This should not be a surprise. Remember that the surplus depends on the two random processes - a continuous one for the rates of return and a discrete one for the decrements. In the earlier years of the temporary contract, only a few deaths are likely to occur but each of them would have a relatively large impact on the surplus. This is reflected in the 'jumps' of the df of S_r^{acct}/m. The slightly upward sloped segments of the plots between any two 'jumps' indicate very small probabilities that the surplus realizes values in those regions. But in the later years the shape of the df gradually smoothes out due to the fact that there are more possibilities for allocating death events over the past years.

Finally, Figure 5.5 presents the probability density functions of the accounting surplus per policy in every insurance year for the limiting portfolio of 5-year temporary policies. Four different combinations of the premium loading and the initial surplus are considered. These plots reinforce many of the observations we have already made regarding the distribution of the surplus. We can clearly see that with time the distribution becomes more dispersed. The mean value of the surplus shifts to the right as r increases; these shifts are larger for the contracts with a nonzero premium loading and initial surplus.

r	$\theta = 0\%$ $\pi = 1.27$	$\theta = 2\%$ $\pi = 1.29$	$\theta = 3\%$ $\pi = 1.31$	$\theta = 20\%$ $\pi = 1.52$
1	0.3732	0.3732	0.3732	0.1428
2	0.4845	0.4769	0.4686	0.2554
3	0.4679	0.4194	0.4058	0.2286
4	0.4628	0.4476	0.4414	0.2756
5	0.4765	0.4550	0.4437	0.2839

Table 5.5: Estimates of probabilities that accounting surplus falls below zero for a portfolio of 1000 5-year temporary policies.

r	$\theta = 0\%$ $\pi = 1.27$ $S_0 = 0$	$\theta = 2\%$ $\pi = 1.29$ $S_0 = 0$	$\theta = 3\%$ $\pi = 1.31$ $S_0 = 0$	$\theta = 0\%$ $\pi = 1.27$ $S_0 = 0.06$	$\theta = 2\%$ $\pi = 1.29$ $S_0 = 0.06$
1	0.5191	0.0140	0.0005	0.1221	0.0004
2	0.4908	0.0394	0.0046	0.1675	0.0041
3	0.4884	0.0570	0.0097	0.1916	0.0088
4	0.4879	0.0650	0.0126	0.2009	0.0114
5	0.4877	0.0669	0.0135	0.2035	0.0123

Table 5.6: Estimates of probabilities that accounting surplus falls below zero for the limiting portfolio of 5-year temporary policies. Initial surplus per policy $S_0 = 0.06$ is the 70^{th} percentile of the S_5^{acct}/m distribution.

r	$\theta = 0\%$ $\pi = 1.27$	$\theta = 2\%$ $\pi = 1.29$	$\theta = 3\%$ $\pi = 1.31$	$\theta = 20\%$ $\pi = 1.52$
1	-0.8699	-0.8702	-0.8703	-0.8698
2	-0.6158	-0.6157	-0.6155	-0.6137
3	-0.4994	-0.4991	-0.4989	-0.4961
4	-0.4274	-0.4270	-0.4267	-0.4226
5	-0.3778	-0.3771	-0.3768	-0.3707

Table 5.7: Estimates of skewness coefficients of accounting surplus distribution for a portfolio of 1000 5-year temporary policies.

r	$\theta = 0\%$ $\pi = 1.27$ $S_0 = 0$	$\theta = 2\%$ $\pi = 1.29$ $S_0 = 0$	$\theta = 3\%$ $\pi = 1.31$ $S_0 = 0$	$\theta = 0\%$ $\pi = 1.27$ $S_0 = 0.06$	$\theta = 2\%$ $\pi = 1.29$ $S_0 = 0.06$
1	-0.0580	-0.0576	-0.0573	-0.0569	-0.0565
2	-0.0257	-0.0269	-0.0266	-0.0255	-0.0262
3	0.0323	0.0335	0.0336	0.0331	0.0343
4	0.1106	0.1105	0.1099	0.1093	0.1095
5	0.1985	0.1954	0.1935	0.1917	0.1903

Table 5.8: Estimates of skewness coefficients of accounting surplus distribution for the limiting portfolio of 5-year temporary policies. Initial surplus per policy $S_0 = 0.06$ is the 70^{th} percentile of the S_5^{acct}/m distribution.

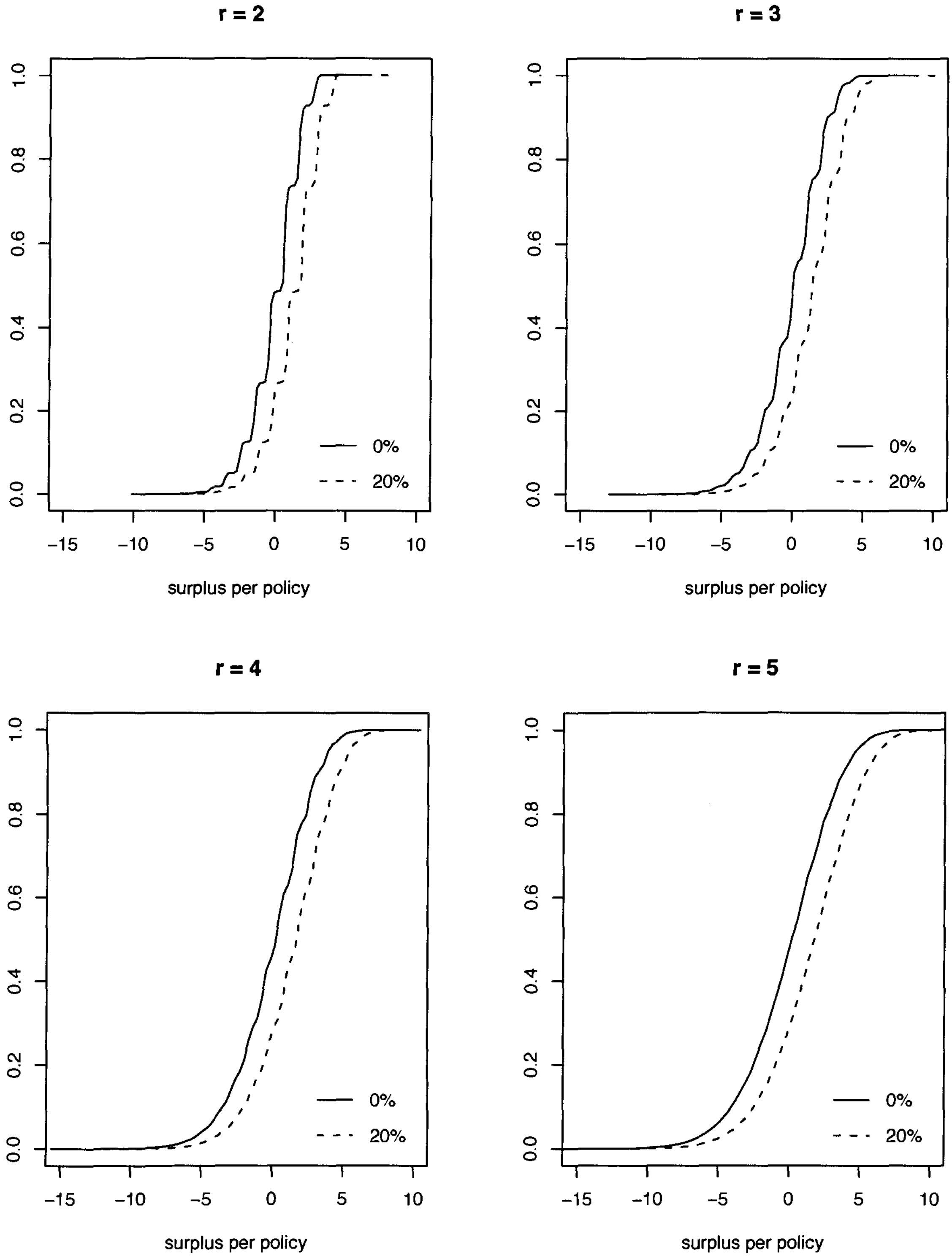

Figure 5.3: Distribution functions of accounting surplus per policy for a portfolio of 1000 5-year temporary policies.

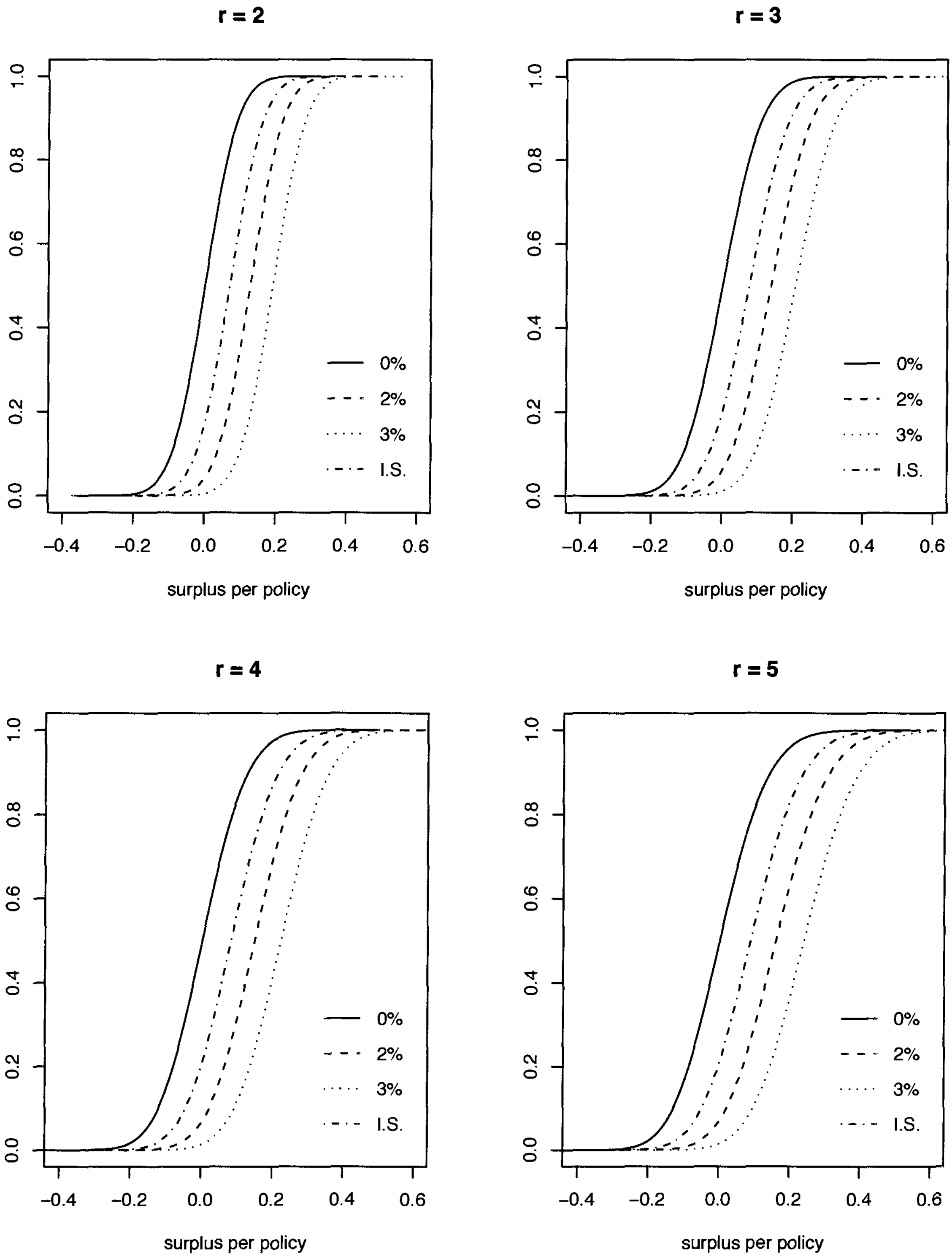

Figure 5.4: Distribution functions of accounting surplus per policy for the limiting portfolio of 5-year temporary policies. Initial surplus per policy I.S.$\equiv S_0 = 0.06$.

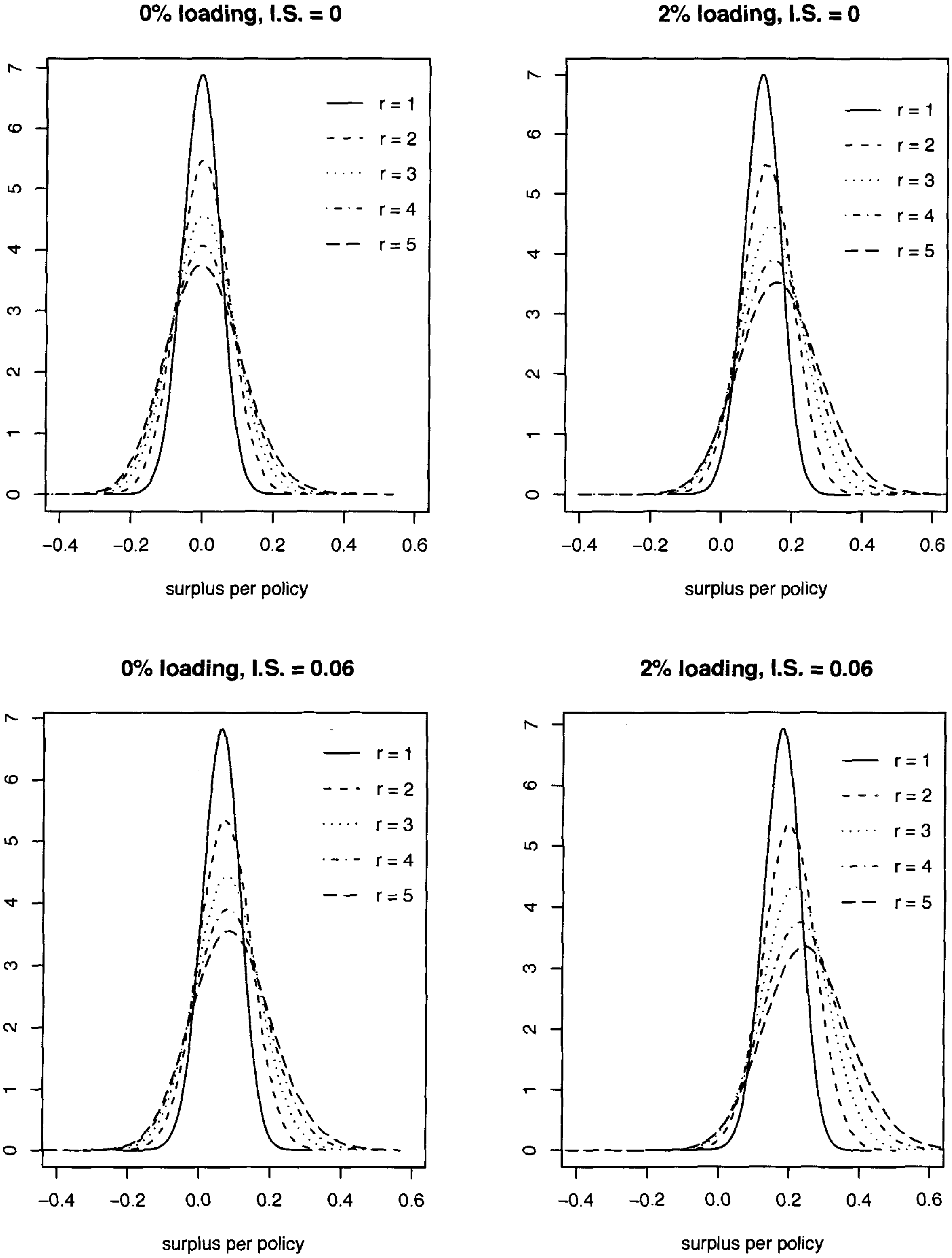

Figure 5.5: Density functions of accounting surplus per policy for the limiting portfolio of 5-year temporary policies.

Chapter 6

Conclusions

This research project explored the behavior of life insurance surplus in an environment of stochastic mortality and interest rates. The surplus was examined at different future times from the point of the contract initiation. An advantage of this framework was that it allowed to assess an insurer's position from a solvency perspective throughout the duration of a contract before its initiation and so any necessary modifications to the terms of the contract can be made.

The first two moments of the retrospective gain, prospective loss and insurance surplus for a single life insurance policy were derived. Then, these results were extended to the case of a homogeneous portfolio of life policies. It was suggested to distinguish between two types of insurance surplus, namely stochastic and accounting surpluses, each serving a slightly different purpose in addressing insurer's solvency. The accounting surpluses represent the financial results as they will be seen by shareholders and regulators at future valuation dates. When studying the stochastic surplus, one considers the range of possible portfolio values measured at a given valuation date that could become reality once all contracts in the portfolio have matured. Finally, the distribution function of the accounting surplus was numerically obtained by applying the proposed recursive formula. The precision of the numerical approach was validated by comparing the first two moments estimated from the distribution function with the exact ones. It was found that in most of the cases the relative errors were well within 1%.

We have seen that the variability of accounting surplus is less than the variability of stochastic surplus, since the accounting surplus depends on the experience only up to a given valuation date, whereas the stochastic surplus depends on the experience during the whole term of the contract. The difference in the variability between the two types of surplus diminishes with time and becomes negligible as we approach the maturity date of the contracts.

Interesting observations were made regarding the changes in the variability of surplus per policy for different types of contracts in response to changes in the portfolio size. In particular, in the case of 5-year temporary policies, the reduction in the standard deviation of surplus was roughly equal to the square root of the factor by which the portfolio size was increased. Even for a fairly large portfolio of 100,000 policies, the standard deviation was still considerably away from the corresponding standard deviation for the limiting portfolio. However, in the case of 5-year endowment policies, increasing the portfolio size over 100 policies did not have a large impact on the variability of the surplus. Several important conclusions can be drawn from these observations. First of all, (short term) temporary and endowment policies are quite different in nature. In the former case, the uncertainty about future realizations mainly comes from the *diversifiable* mortality risk; whereas, in the latter case, the uncertainty mainly comes from the *nondiversifiable* investment risk. As a result, different risk mitigation strategies should be used in each case. Temporary policies are very risky when sold to a small group of people, but for extremely large portfolios, most of the risk is diversified and reduced to just a small fraction of the fair (i.e., determined under the equivalence principle) premium charged for these policies. The limiting portfolio of endowment policies can be used as a proxy for portfolios of finite size, in which case a gain in the computing time will outweigh a relatively small loss in the accuracy.

The analysis of the probabilities of insolvency was used to comment on the adequacy of premium rates and levels of the initial surplus. In fact, the probability of insolvency can be used as a risk measure. For example, the premium loading required to ensure a sufficiently small probability of insolvency is much larger for a small portfolio than it is for a very large portfolio.

This work can be further continued and extended in a number of ways. First of all, there are still some questions that remain unanswered even within our set of assumptions. We have developed results for calculating probabilities of insolvency at any given valuation date. The next question is what is the overall probability of insolvency in a given time horizon, finite or infinite. Surplus amounts at different times are highly correlated. If surplus falls below zero in one time period because of unfavorable experience, it is more likely to remain below zero in the next period.

The recursive formula for the distribution of the accounting surplus took advantage of conditionally constant liability. Obtaining the distribution of stochastic surplus is much harder since one needs to take into account both random assets and liabilities.

The model can be made more realistic by including expenses and lapses. If lapses are considered, the assumption of independence between decrements and rates of return might not be valid.

Although only the case of life insurances was considered, the methodology can easily be extended to study life annuities and other insurance products by adjusting the cash flows.

A homogeneous portfolio can be used as a proxy for a portfolio of policies with similar risk characteristics. But of course real insurance portfolios are comprised of different policies with different durations and benefits and issued to people with different mortality profiles (e.g., gender, smoking status). So, the project could be generalized to study general portfolios of life insurances.

Appendix A

Additional Material

A.1 Interest Rate Model

Two approaches were mentioned for calculating moments of $\{I(s,r)|\delta(0)\}$. We give details of $\mathrm{E}[I(s,r)|\delta(0)=\delta_0]$ derivation under the first approach, which directly uses the definition of $I(s,r)$.

$$\begin{aligned}
\mathrm{E}[I(s,r)|\delta(0)=\delta_0] &= \mathrm{E}\left[\sum_{j=s+1}^{r} \delta(j)|\delta(0)=\delta_0\right] \\
&= \sum_{j=s+1}^{r} \mathrm{E}[\delta(j)|\delta(0)=\delta_0] \\
&= \sum_{j=s+1}^{r} (\delta+(\delta_0-\delta)\phi^j) \\
&= (r-s)\delta + \frac{\phi^{s+1}(1-\phi^{r-s})}{1-\phi}(\delta_0-\delta) \\
&= (r-s)\delta + \frac{\phi}{1-\phi}(\phi^s-\phi^r)(\delta_0-\delta), \qquad \text{(A.1)}
\end{aligned}$$

where the following facts about the conditional AR(1) process are used (see, for example, Bellhouse and Panjer (1981)):

$$\mathrm{E}[\delta(j)|\delta(0)=\delta_0] = \delta+(\delta_0-\delta)\phi^j\,, \qquad \text{(A.2)}$$

$$\mathrm{Var}[\delta(j)|\delta(0)=\delta_0] = \frac{\sigma^2}{1-\phi^2}(1-\phi^{2j}) \qquad \text{and} \qquad \text{(A.3)}$$

$$\mathrm{Cov}[\delta(i), \delta(j)|\delta(0) = \delta_0] = \frac{\sigma^2}{1-\phi^2}(\phi^{|i-j|} - \phi^{i+j}). \tag{A.4}$$

A.2 Theorem 1

We restate Theorem 3.2 given in Bowers et al. (1986) p. 64 for completeness.

Theorem 1. *Let K be a discrete random variable on nonnegative integers with $g(k) = \mathbf{P}(K = k) = G(k) - G(k-1)$ and $\psi(k)$ be a nonnegative, monotonic function such that $E[\psi(K)]$ exists. Then,*

$$E[\psi(K)] = \sum_{k=0}^{\infty} \psi(k) \cdot g(k) = \psi(0) + \sum_{k=0}^{\infty} (1 - G(k)) \cdot \Delta\psi(k).$$

A.3 Retrospective Cash Flows Conditional on Number of Policies In Force

We first show how to obtain the distribution of the number of survivors at time j for $0 < j < r$, conditional on the number of survivors at time r. This distribution is then used to calculate the mean and the variance of the retrospective cash flow at time j, RC_j^r, as well as the covariance between RC_i^r and RC_j^r conditional on the number of policies in force at time r, $\mathscr{L}_r(x)$. To simplify notation, $\mathscr{L}_j(x)$ and $\mathscr{D}_j(x)$ are denoted as $\mathscr{L}_j$ and $\mathscr{D}_j$.

Probability that m_j people survive to time j given that m_r people will survive to time r, $j < r$, can be calculated as follows:

$$\begin{aligned}
\mathbf{P}[\mathscr{L}_j = m_j|\mathscr{L}_r = m_r] &= \frac{\mathbf{P}[\mathscr{L}_j = m_j \bigcap \mathscr{L}_r = m_r]}{\mathbf{P}[\mathscr{L}_r = m_r]} \\
&= \frac{\mathbf{P}[\mathscr{L}_r = m_r|\mathscr{L}_j = m_j] \cdot \mathbf{P}[\mathscr{L}_j = m_j]}{\mathbf{P}[\mathscr{L}_r = m_r]},
\end{aligned}$$

where

- $\mathbf{P}[\mathscr{L}_r = m_r|\mathscr{L}_j] = \binom{\mathscr{L}_j}{m_r} ({}_{r-j}p_{x+j})^{m_r} (1 - {}_{r-j}p_{x+j})^{\mathscr{L}_j - m_r},$
- $\mathbf{P}[\mathscr{L}_j = m_j] = \binom{m}{m_j} ({}_jp_x)^{m_j} (1 - {}_jp_x)^{m-m_j}, \qquad 0 \le j \le r,$

since $\mathscr{L}_r|\mathscr{L}_j \sim \mathrm{BIN}(\mathscr{L}_j,\ {}_{r-j}p_{x+j})$ and $\mathscr{L}_j \sim \mathrm{BIN}(m,\ {}_jp_x)$ for $0 \le j \le r$.

Now, consider the retrospective cash flows. The conditional mean, variance and covariance in terms of $\mathscr{L}_j$ and $\mathscr{D}_j$ are given by

$$\mathrm{E}[RC_j^r \,|\, \mathscr{L}_r] = \pi \cdot \mathrm{E}[\mathscr{L}_j \,|\, \mathscr{L}_r] \cdot \mathbf{1}_{\{j<r\}} - b \cdot \mathrm{E}[\mathscr{D}_j \,|\, \mathscr{L}_r] \cdot \mathbf{1}_{\{j>0\}}, \tag{A.5}$$

$$\begin{aligned}\mathrm{Var}[RC_j^r \,|\, \mathscr{L}_r] &= \pi^2 \cdot \mathrm{Var}[\mathscr{L}_j \,|\, \mathscr{L}_r] \cdot \mathbf{1}_{\{j<r\}} + b^2 \cdot \mathrm{Var}[\mathscr{D}_j \,|\, \mathscr{L}_r] \cdot \mathbf{1}_{\{j>0\}} - \\ &\quad - 2 \cdot \pi \cdot b \cdot \mathrm{Cov}[\mathscr{L}_j, \mathscr{D}_j \,|\, \mathscr{L}_r] \cdot \mathbf{1}_{\{0<j<r\}}\end{aligned} \tag{A.6}$$

and for $i < j$,

$$\begin{aligned}\mathrm{Cov}[RC_j^r, RC_i^r \,|\, \mathscr{L}_r] &= \pi^2 \cdot \mathrm{Cov}[\mathscr{L}_j, \mathscr{L}_i \,|\, \mathscr{L}_r] \cdot \mathbf{1}_{\{j<r\}} \\ &+ b^2 \cdot \mathrm{Cov}[\mathscr{D}_j, \mathscr{D}_i \,|\, \mathscr{L}_r] \cdot \mathbf{1}_{\{i>0\}} \\ &- \pi \cdot b \cdot \mathrm{Cov}[\mathscr{L}_j, \mathscr{D}_i \,|\, \mathscr{L}_r] \cdot \mathbf{1}_{\{0<i<j<r\}} \\ &- b \cdot \pi \cdot \mathrm{Cov}[\mathscr{D}_j, \mathscr{L}_i \,|\, \mathscr{L}_r] \cdot \mathbf{1}_{\{0\le i<j\le r\}}.\end{aligned} \tag{A.7}$$

The distribution of $\{\mathscr{L}_j = m_j \,|\, \mathscr{L}_r = m_r\}$ can be used to obtain the following quantities necessary to evaluate expressions (A.5) - (A.7):

$$\mathrm{E}[\mathscr{L}_j \,|\, \mathscr{L}_r = m_r] = \sum_{m_j=m_r}^{m} m_j \cdot \mathbf{P}[\mathscr{L}_j = m_j \,|\, \mathscr{L}_r = m_r];$$

$$\mathrm{E}[\mathscr{L}_j^2 \,|\, \mathscr{L}_r = m_r] = \sum_{m_j=m_r}^{m} m_j^2 \cdot \mathbf{P}[\mathscr{L}_j = m_j \,|\, \mathscr{L}_r = m_r];$$

for $i < j \le r$,

$$\begin{aligned}\mathrm{E}[\mathscr{L}_i \cdot \mathscr{L}_j \,|\, \mathscr{L}_r] &= \sum_{m_i=m_r}^{m} m_i \cdot \Big(\sum_{m_j=m_r}^{m_i} m_j \cdot \mathbf{P}[\mathscr{L}_j = m_j \,|\, \mathscr{L}_i = m_i, \mathscr{L}_r = m_r] \Big) \times \\ &\quad \times \mathbf{P}[\mathscr{L}_i = m_i \,|\, \mathscr{L}_r = m_r],\end{aligned}$$

where

$$\mathbf{P}[\mathscr{L}_j = m_j \mid \mathscr{L}_i = m_i, \mathscr{L}_r = m_r] =$$

$$= \frac{\mathbf{P}[\mathscr{L}_j = m_j, \mathscr{L}_i = m_i, \mathscr{L}_r = m_r]}{\mathbf{P}[\mathscr{L}_i = m_i, \mathscr{L}_r = m_r]}$$

$$= \frac{\mathbf{P}[\mathscr{L}_r = m_r \mid \mathscr{L}_j = m_j, \mathscr{L}_i = m_i] \cdot \mathbf{P}[\mathscr{L}_j = m_j \mid \mathscr{L}_i = m_i] \cdot \mathbf{P}[\mathscr{L}_i = m_i]}{\mathbf{P}[\mathscr{L}_r = m_r \mid \mathscr{L}_i = m_i] \cdot \mathbf{P}[\mathscr{L}_i = m_i]}$$

$$= \frac{\mathbf{P}[\mathscr{L}_r = m_r \mid \mathscr{L}_j = m_j] \cdot \mathbf{P}[\mathscr{L}_j = m_j \mid \mathscr{L}_i = m_i]}{\mathbf{P}[\mathscr{L}_r = m_r \mid \mathscr{L}_i = m_i]},$$

and using the above formulas we can calculate:

$$\mathrm{Var}[\mathscr{L}_j \mid \mathscr{L}_r = m_r] = \mathrm{E}[\mathscr{L}_j^2 \mid \mathscr{L}_r = m_r] - \mathrm{E}[\mathscr{L}_j \mid \mathscr{L}_r = m_r]^2;$$

$$\mathrm{Cov}[\mathscr{L}_i, \mathscr{L}_j \mid \mathscr{L}_r] = \mathrm{E}[\mathscr{L}_i \cdot \mathscr{L}_j \mid \mathscr{L}_r] - \mathrm{E}[\mathscr{L}_i \mid \mathscr{L}_r] \cdot \mathrm{E}[\mathscr{L}_j \mid \mathscr{L}_r].$$

Since the number of deaths in year j is the difference between the number of people alive at time $j-1$ and at time j; i.e. $\mathscr{D}_j = \mathscr{L}_{j-1} - \mathscr{L}_j$; we have

$$\mathrm{E}[\mathscr{D}_j \mid \mathscr{L}_r] = \mathrm{E}[\mathscr{L}_{j-1} \mid \mathscr{L}_r] - \mathrm{E}[\mathscr{L}_j \mid \mathscr{L}_r];$$

$$\mathrm{Var}[\mathscr{D}_j \mid \mathscr{L}_r] = \mathrm{Var}[\mathscr{L}_{j-1} \mid \mathscr{L}_r] + \mathrm{Var}[\mathscr{L}_j \mid \mathscr{L}_r] - 2 \cdot \mathrm{Cov}[\mathscr{L}_{j-1}, \mathscr{L}_j \mid \mathscr{L}_r];$$

$$\begin{aligned}
\mathrm{Cov}[\mathscr{D}_j, \mathscr{D}_i \mid \mathscr{L}_r] &= \mathrm{Cov}[\mathscr{L}_{j-1} - \mathscr{L}_j, \mathscr{L}_{i-1} - \mathscr{L}_i \mid \mathscr{L}_r] \\
&= \mathrm{Cov}[\mathscr{L}_{j-1}, \mathscr{L}_{i-1} \mid \mathscr{L}_r] - \mathrm{Cov}[\mathscr{L}_{j-1}, \mathscr{L}_i \mid \mathscr{L}_r] - \\
&\quad - \mathrm{Cov}[\mathscr{L}_j, \mathscr{L}_{i-1} \mid \mathscr{L}_r] + \mathrm{Cov}[\mathscr{L}_j, \mathscr{L}_i \mid \mathscr{L}_r];
\end{aligned}$$

$$\mathrm{Cov}[\mathscr{L}_j, \mathscr{D}_i \mid \mathscr{L}_r] = \mathrm{Cov}[\mathscr{L}_j, \mathscr{L}_{i-1} \mid \mathscr{L}_r] - \mathrm{Cov}[\mathscr{L}_j, \mathscr{L}_i \mid \mathscr{L}_r].$$

A.4 On Benefit Premium Determination

In the numerical examples unless stated otherwise, the *equivalence principle* is used to set insurance premiums. As described in Bowers et al. (1997) (see pp.167-170), this principle requires the premium to be chosen so that the expected value of the prospective loss random variable at issue is equal to zero (i.e., $\mathrm{E}[PL_0] = 0$).

From Equation (3.2) we have $\mathrm{E}[PL_0] = \mathrm{E}[Z] - \pi \cdot \mathrm{E}[Y]$, implying that the premium π determined under the equivalence principle (also referred to as the *benefit premium*) is given by

$$\pi = \frac{\mathrm{E}[Z]}{\mathrm{E}[Y]},$$

where:

- Z is the present value *at issue* of future benefits;
- Y is the present value *at issue* of future premiums of \$1.

The following table provides benefit premiums for temporary and endowment insurance contracts per \$1000 benefit issued to (30) with 5, 10 and 25-year terms.

n	Temporary insurance	Endowment insurance
5	1.2691	160.2407
10	1.3675	67.9009
25	2.0883	17.5089

Note that premiums determined under the equivalence principle are only based on the pattern of benefits and premiums. To take into account, for instance, profit and contingency margins, we introduce a premium loading factor θ. Then, the premium charged is equal to

$$\pi = (1+\theta)\,\frac{\mathrm{E}[Z]}{\mathrm{E}[Y]}. \tag{A.8}$$

If $\theta = 0$ in (A.8), then π is the benefit premium.

A.5 Proof of Result 4.3.1

In the proof, we use the formula for computing expectation by conditioning and the conditional variance formula (e.g., see Equation 3.3 p.106 and Proposition 3.1 p.118

in Ross (2003)).

$$
\begin{aligned}
\operatorname{Var}[S_r^{acct}] &= \operatorname{Var}_{\delta(r)} \mathrm{E}[S_r^{acct} \mid \delta(r)] + \mathrm{E}_{\delta(r)} \operatorname{Var}[S_r^{acct} \mid \delta(r)] \\
&= \operatorname{Var}_{\delta(r)}\Big(\mathrm{E}_{\mathscr{L}_r}\Big[\mathrm{E}[S_r^{acct} \mid \mathscr{L}_r,\, \delta(r)]\Big]\Big) + \\
&\quad + \mathrm{E}_{\delta(r)}\Big(\operatorname{Var}_{\mathscr{L}_r}\Big[\mathrm{E}[S_r^{acct} \mid \mathscr{L}_r,\, \delta(r)]\Big] + \mathrm{E}_{\mathscr{L}_r}\Big[\operatorname{Var}[S_r^{acct} \mid \mathscr{L}_r,\, \delta(r)]\Big]\Big) \\
&= \operatorname{Var}_{\delta(r)}\mathrm{E}_{\mathscr{L}_r}\Big[\mathrm{E}[RG_r \mid \mathscr{L}_r,\, \delta(r)] - \mathrm{E}[PL_r \mid \mathscr{L}_r,\, \delta(r)]\Big] + \\
&\quad + \mathrm{E}_{\delta(r)} \operatorname{Var}_{\mathscr{L}_r}\Big(\mathrm{E}\big(RG_r - \mathrm{E}[PL_r \mid \mathscr{L}_r,\, \delta(r)] \mid \mathscr{L}_r,\, \delta(r)\big)\Big) + \\
&\quad + \mathrm{E}_{\delta(r)} \mathrm{E}_{\mathscr{L}_r}\Big(\operatorname{Var}\big(RG_r - \mathrm{E}[PL_r \mid \mathscr{L}_r,\, \delta(r)] \mid \mathscr{L}_r,\, \delta(r)\big)\Big) \\
&= \operatorname{Var}_{\delta(r)}\Big(\mathrm{E}[RG_r \mid \delta(r)] - \mathrm{E}[PL_r \mid \delta(r)]\Big) + \\
&\quad + \mathrm{E}_{\delta(r)}\Big[\operatorname{Var}_{\mathscr{L}_r}(\mathrm{E}[RG_r \mid \mathscr{L}_r,\, \delta(r)]) + \operatorname{Var}_{\mathscr{L}_r}(\mathrm{E}[PL_r \mid \mathscr{L}_r,\, \delta(r)]) - \\
&\qquad\qquad -2\operatorname{Cov}_{\mathscr{L}_r}(\mathrm{E}[RG_r \mid \mathscr{L}_r,\, \delta(r)],\, \mathrm{E}[PL_r \mid \mathscr{L}_r,\, \delta(r)])\Big] + \\
&\quad + \mathrm{E}_{\delta(r)} \mathrm{E}_{\mathscr{L}_r}\Big[\operatorname{Var}\big(RG_r \mid \mathscr{L}_r,\, \delta(r)\big)\Big] \\
&= \operatorname{Var}_{\delta(r)}\Big(\mathrm{E}[RG_r \mid \delta(r)]\Big) + \mathrm{E}_{\delta(r)} \operatorname{Var}_{\mathscr{L}_r}(\mathrm{E}[PL_r \mid \mathscr{L}_r,\, \delta(r)]) + \\
&\quad + \mathrm{E}_{\delta(r)} \mathrm{E}_{\mathscr{L}_r}\Big[\operatorname{Var}\big(RG_r \mid \mathscr{L}_r,\, \delta(r)\big)\Big] \qquad \text{(A.9)} \\
&\quad + \operatorname{Var}_{\delta(r)}\Big(\mathrm{E}[PL_r \mid \delta(r)]\Big) + \mathrm{E}_{\delta(r)} \operatorname{Var}_{\mathscr{L}_r}(\mathrm{E}[PL_r \mid \mathscr{L}_r,\, \delta(r)]) \qquad \text{(A.10)} \\
&\quad -2 \cdot \Big(\operatorname{Cov}_{\delta(r)}\Big(\mathrm{E}[RG_r \mid \delta(r)],\, \mathrm{E}[PL_r \mid \delta(r)]\Big) + \\
&\qquad + \mathrm{E}_{\delta(r)} \operatorname{Cov}_{\mathscr{L}_r}(\mathrm{E}[RG_r \mid \mathscr{L}_r,\, \delta(r)],\, \mathrm{E}[PL_r \mid \mathscr{L}_r,\, \delta(r)])\Big). \qquad \text{(A.11)}
\end{aligned}
$$

Next, we simplify expressions (A.9), (A.10) and (A.11).

Applying the conditional variance formula twice, Expression (A.9) becomes

$$\begin{aligned}
&\mathrm{Var}_{\delta(r)}\Big(\mathrm{E}[RG_r \mid \delta(r)]\Big) + \mathrm{E}_{\delta(r)}\Big(\mathrm{Var}_{\mathscr{L}_r}(\mathrm{E}[RG_r \mid \mathscr{L}_r,\, \delta(r)]) + \\
&- \mathrm{E}_{\mathscr{L}_r}\Big[\mathrm{Var}\big(RG_r \mid \mathscr{L}_r,\, \delta(r)\big)\Big]\Big) \\
&= \mathrm{Var}_{\delta(r)}\Big(\mathrm{E}[RG_r \mid \delta(r)]\Big) + \mathrm{E}_{\delta(r)}\Big(\mathrm{Var}[RG_r \mid \delta(r)]\Big) \\
&= \mathrm{Var}[RG_r].
\end{aligned} \tag{A.12}$$

To numerically evaluate Expression (A.10) it can be rewritten as

$$\begin{aligned}
&\mathrm{Var}_{\delta(r)}\Big(\mathrm{E}[PL_r \mid \delta(r)]\Big) + \mathrm{E}_{\delta(r)}\,\mathrm{Var}_{\mathscr{L}_r}\Big(\mathrm{E}[PL_r \mid \mathscr{L}_r,\, \delta(r)]\Big) \\
&= \mathrm{Var}_{\delta(r)}\Big(\sum_{j=0}^{n-r}\mathrm{E}[PC_j^r]\cdot\mathrm{E}[e^{-I(r,r+j)}|\delta(r)]\Big) + \\
&+ \mathrm{E}_{\delta(r)}\,\mathrm{Var}_{\mathscr{L}_r}\Big(\sum_{j=0}^{n-r}\mathrm{E}[PC_j^r|\mathscr{L}_r]\cdot\mathrm{E}[e^{-I(r,r+j)}|\delta(r)]\Big) \\
&= \sum_{i=0}^{n-r}\sum_{j=0}^{n-r}\mathrm{E}[PC_i^r]\cdot\mathrm{E}[PC_j^r]\cdot\mathrm{Cov}_{\delta(r)}\Big(\mathrm{E}[e^{-I(r,r+i)}|\delta(r)],\,\mathrm{E}[e^{-I(r,r+j)}|\delta(r)]\Big) + \\
&+ \sum_{i=0}^{n-r}\sum_{j=0}^{n-r}\mathrm{Cov}_{\mathscr{L}_r}\Big(\mathrm{E}[PC_i^r|\mathscr{L}_r],\,\mathrm{E}[PC_j^r|\mathscr{L}_r]\Big)\times \\
&\qquad\qquad \times\mathrm{E}_{\delta(r)}\Big(\mathrm{E}[e^{-I(r,r+i)}|\delta(r)]\cdot\mathrm{E}[e^{-I(r,r+j)}|\delta(r)]\Big) \\
&= \sum_{i=0}^{n-r}\sum_{j=0}^{n-r}\mathrm{E}_{\mathscr{L}_r}\Big(\mathrm{E}[PC_i^r|\mathscr{L}_r]\cdot\mathrm{E}[PC_j^r|\mathscr{L}_r]\Big)\times \\
&\qquad\qquad \times\mathrm{E}_{\delta(r)}\Big(\mathrm{E}[e^{-I(r,r+i)}|\delta(r)]\cdot\mathrm{E}[e^{-I(r,r+j)}|\delta(r)]\Big) - \\
&- \sum_{i=0}^{n-r}\sum_{j=0}^{n-r}\mathrm{E}[PC_i^r]\cdot\mathrm{E}[PC_j^r]\cdot\mathrm{E}[e^{-I(r,r+i)}]\cdot\mathrm{E}[e^{-I(r,r+j)}] \\
&= \sum_{i=0}^{n-r}\sum_{j=0}^{n-r}\mathrm{E}_{\mathscr{L}_r}\Big(\mathrm{E}[PC_i^r|\mathscr{L}_r]\cdot\mathrm{E}[PC_j^r|\mathscr{L}_r]\Big)\times \\
&\qquad\qquad \times\mathrm{E}_{\delta(r)}\Big(\mathrm{E}[e^{-I(r,r+i)}|\delta(r)]\cdot\mathrm{E}[e^{-I(r,r+j)}|\delta(r)]\Big) - \Big(\mathrm{E}[PL_r]\Big)^2.
\end{aligned} \tag{A.13}$$

Finally, Expression (A.11) simplifies to

$$
\begin{aligned}
&\mathrm{E}_{\delta(r)}\Big(\mathrm{E}[RG_r \mid \delta(r)] \cdot \mathrm{E}[PL_r \mid \delta(r)]\Big) - \mathrm{E}[RG_r] \cdot \mathrm{E}[PL_r] + \\
&+ \mathrm{E}_{\delta(r)}\Big[\mathrm{E}_{\mathscr{L}_r}\Big(\mathrm{E}[RG_r \mid \mathscr{L}_r,\, \delta(r)] \cdot \mathrm{E}[PL_r \mid \mathscr{L}_r,\, \delta(r)]\Big)\Big] - \\
&- \mathrm{E}_{\delta(r)}\Big(\mathrm{E}[RG_r \mid \delta(r)] \cdot \mathrm{E}[PL_r \mid \delta(r)]\Big) \\
&= \sum_{j=0}^{r}\sum_{i=0}^{n-r} \mathrm{E}_{\mathscr{L}_r}\Big(\mathrm{E}[RC_j^r|\mathscr{L}_r] \cdot \mathrm{E}[PC_i^r|\mathscr{L}_r]\Big) \cdot \mathrm{E}_{\delta(r)}\Big(\mathrm{E}[e^{I(j,r)}|\delta(r)] \cdot \mathrm{E}[e^{-I(r,r+i)}|\delta(r)]\Big) \\
&\qquad\qquad - \mathrm{E}[RG_r] \cdot \mathrm{E}[PL_r] \\
&= \sum_{j=0}^{r}\sum_{i=0}^{n-r} \mathrm{E}_{\mathscr{L}_r}\Big(\mathrm{E}[RC_j^r \cdot PC_i^r|\mathscr{L}_r]\Big) \cdot \mathrm{E}_{\delta(r)}\Big(\mathrm{E}[e^{I(j,r)-I(r,r+i)}|\delta(r)]\Big) \\
&\qquad\qquad - \mathrm{E}[RG_r] \cdot \mathrm{E}[PL_r] \\
&= \sum_{j=0}^{r}\sum_{i=0}^{n-r} \mathrm{E}[RC_j^r \cdot PC_i^r] \cdot \mathrm{E}[e^{I(j,r)-I(r,r+i)}] - \mathrm{E}[RG_r] \cdot \mathrm{E}[PL_r] \\
&= \mathrm{E}[RG_r \cdot PL_r] - \mathrm{E}[RG_r] \cdot \mathrm{E}[PL_r] = \mathrm{Cov}(RG_r,\, PL_r). \qquad \text{(A.14)}
\end{aligned}
$$

In the above derivation we use the fact that $\{RC_j^r|\mathscr{L}_r\}$ and $\{PC_i^r|\mathscr{L}_r\}$ are uncorrelated as well as $\{e^{I(j,r)}|\delta(r)\}$ and $\{e^{-I(r,r+i)}|\delta(r)\}$, which can be shown as follows:

$$
\begin{aligned}
\mathrm{E}\big[e^{I(j,r)} \cdot e^{-I(r,r+i)}\big|\delta(r)\big] &= \mathrm{E}_{I(j,r)}\Big[\mathrm{E}\big[e^{I(j,r)} \cdot e^{-I(r,r+i)}\big|I(j,r),\, \delta(r)\big]\Big] \\
&= \mathrm{E}_{I(j,r)}\Big[e^{I(j,r)} \cdot \mathrm{E}\big[e^{-I(r,r+i)}\big|I(j,r),\, \delta(r)\big]\Big] \\
&= \mathrm{E}_{I(j,r)}\Big[e^{I(j,r)} \cdot \mathrm{E}\big[e^{-I(r,r+i)}\big|\delta(r)\big]\Big] \quad \because \text{Markovian property} \\
&= \mathrm{E}\big[e^{I(j,r)}\big|\delta(r)\big] \cdot \mathrm{E}\big[e^{-I(r,r+i)}\big|\delta(r)\big],
\end{aligned}
$$

$\therefore \mathrm{Cov}\big(e^{I(j,r)},\, e^{-I(r,r+i)}\big|\delta(r)\big) = 0.$

Similarly,

$$
\begin{aligned}
\mathrm{E}\big[RC_j^r \cdot PC_i^r\big|\mathscr{L}_r\big] &= \mathrm{E}_{RC_j^r}\Big[\mathrm{E}\big[RC_j^r \cdot PC_i^r\big|RC_j^r,\, \mathscr{L}_r\big]\Big] \\
&= \mathrm{E}_{RC_j^r}\Big[RC_j^r \cdot \mathrm{E}\big[PC_i^r\big|RC_j^r,\, \mathscr{L}_r\big]\Big] \\
&= \mathrm{E}_{RC_j^r}\Big[RC_j^r \cdot \mathrm{E}\big[PC_i^r\big|\mathscr{L}_r\big]\Big] \quad \because \text{Markovian property} \\
&= \mathrm{E}\big[RC_j^r\big|\mathscr{L}_r\big] \cdot \mathrm{E}\big[PC_i^r\big|\mathscr{L}_r\big],
\end{aligned}
$$

$\therefore \mathrm{Cov}\big(RC_j^r,\, PC_i^r\big|\mathscr{L}_r\big) = 0.$

Now, replacing (A.9)-(A.11) with (A.12)-(A.14) proves the result. □

Appendix B

On Numerical Computations

Results 5.1.1 and 5.2.1 were used for obtaining the distribution function of the accounting surplus for portfolios of finite size and limiting portfolios respectively. To apply these results, one needs to evaluate the improper integrals over the values of δ_t, $t = 1, \ldots, r$, where r corresponds to a valuation time. Because of the complexity of the integrand, a numerical evaluation of these integrals is required.

In this appendix, several suggestions are made regarding the implementation of the results. Only the case of the portfolio with a finite number of policies is discussed. The implementation of the method for the limiting portfolio is analogous; in fact, it is simpler since there is no randomness about the cash flows and thus there is no weighted sum over possible number of survivors in any given time period.

Each integral is approximated by a sum over a finite number of values of δ_t. Recall that in our model the force of interest in any year is normally distributed with mean $\mathrm{E}[\delta(t)|\delta_0] =: \mu_\delta$ and standard deviation $\mathrm{SD}[\delta(t)|\delta_0] =: \sigma_\delta$. This implies that the distribution of $\delta(t)|\delta_0$ is symmetric and centered around μ_δ. We thus choose to consider points in the following range: $\mu_\delta \pm k \cdot \sigma_\delta$. The number of points is denoted $pts.d$ and the values of δ_t are denoted $\{\delta_i,\, i = 1, \ldots, pts.d\}$ for any given t. It was found that $k = 5$ and $pts.d = 65$ produce fairly accurate estimates of the integrals. Note that in order to include μ_δ into the set of considered points $\{\delta_i,\, i = 1, \ldots, pts.d\}$, $pts.d$ must be an odd integer.

When the size of the portfolio is large, taking into account all possibilities for the number of inforce policies in any given year is time-consuming. In fact, some of those outcomes can be quite unlikely. For example, for a portfolio of 100 policies issued to a group of people aged 30, the probability that there will remain 90 policies in force after one year is $2.12 \cdot 10^{-16}$ and for a portfolio of 1000 policies issued to (30), the probability that there will remain 900 policies in force after one year is $4.91 \cdot 10^{-150}$. Ignoring these unlikely events will not have much impact on the accuracy of the df estimation but will definitely reduce computing time. In our numerical illustrations we choose to ignore those realizations of $\mathscr{L}_t(x)$ for which $\mathbf{P}[\mathscr{L}_t(x) = m_t] < 10^{-10}$.

The choice of values of λ for which $g(\lambda, m_t, \delta_t)$ $(t = 1, \ldots, r)$ is evaluated and ξ for which $\mathbf{P}[S_r^{acct} \leq \xi]$ is evaluated is based on the range of values we use given by $\Big(\mathrm{E}[G_t] - k_1{\cdot}\mathrm{SD}[G_t], \mathrm{E}[G_t] + k_2{\cdot}\mathrm{SD}[G_t]\Big)$ and $\Big(\mathrm{E}[S_r^{acct}] - k_1{\cdot}\mathrm{SD}[S_r^{acct}], \mathrm{E}[S_r^{acct}] + k_2{\cdot}\mathrm{SD}[S_r^{acct}]\Big)$, and the number of points denoted $pts.g$ and $pts.s$ respectively. For most portfolios, $k_1 = k_2 = 5$ and $pts.g = pts.s = 85$ seemed to work fine. However, for a small portfolio of temporary insurance contracts for which the distribution of S_r^{acct} is fairly skewed to the left, we used $k_1 = 6$ and $k_2 = 4$.

Linear interpolation is used to obtain the values of $g_{t-1}\Big(\frac{\lambda-\eta_t}{e^{\delta_t}}, m_{t-1}, \delta_{t-1}\Big)$.

Once the df of the surplus is computed, we would also like to assess the accuracy of the numerical calculations. In Chapter 4 the first two moments of the accounting surplus were derived. Since they are evaluated from exact formulas, they can be used as the exact moments. We can also estimate the moments from the df. Remember that although the proposed method for calculating the df is exact, we have to use a number of numerical approximations to evaluate the improper integrals and to reduce computing time by ignoring unlikely realizations of the processes; so, the moments estimated from this df contain numerical errors. These estimates are then compared to the exact moments. If the difference between them is small, then we can conclude that the df is accurate and use it to obtain probabilities of various events associated with the accounting surplus.

An estimate of the k^{th} moment of S_r^{acct} is given by

$$\widehat{\mathrm{E}}\Big[\big(S_r^{acct}\big)^k\Big] = \sum_{j=1}^{pts.s-1} (\xi_j^*)^k \cdot \Big(\widehat{\mathbf{P}}[S_r^{acct} \leq \xi_{j+1}] - \widehat{\mathbf{P}}[S_r^{acct} \leq \xi_j]\Big),$$

where $\xi_j^* = \frac{1}{2}(\xi_{j+1} + \xi_j)$.

It seems to be computationally more convenient to treat the case $r = 1$ separately and use the recursive formula for $g(\lambda,\, m_t,\, \delta_t)$ $(t = 2, \ldots, r)$ starting at $t = 2$.

For $r = 1$ we have

$$\begin{aligned}
\mathbf{P}[S_1^{acct} \leq \xi] &= \int_{-\infty}^{\infty} \sum_{m_1=0}^{m} g_1(\xi + {}_1V(m_1,\, \delta_1),\, m_1,\, \delta_1)\, d\delta_1 \\
&= \sum_{m_1=0}^{m} \Big(\mathbf{P}[\mathscr{L}_1(x) = m_1] \cdot \int_{-\infty}^{\infty} f_{\delta(1)}(\delta_1) \cdot \mathbf{1}_{\{G_1 \leq \xi + {}_1V(m_1,\, \delta_1)\}} d\delta_1 \Big) \\
&= \sum_{m_1=0}^{m} \Big(\mathbf{P}[\mathscr{L}_1(x) = m_1] \cdot \int_{-\infty}^{\infty} \mathbf{1}_{\{G_1 \leq \xi + {}_1V(m_1,\, \delta_1)\}} dF_{\delta(1)}(\delta_1) \Big) \\
&\approx \sum_{m_1=0}^{m} \Big(\mathbf{P}[\mathscr{L}_1(x) = m_1] \cdot \sum_{i=1}^{pts.d-1} \mathbf{1}_{\{G_i^* \leq \xi + {}_1V(m_1,\, \delta_i^*)\}} \cdot \Delta F_{\delta(1)}(\delta_i) \Big),
\end{aligned} \tag{B.1}$$

where

- $\delta_i^* = \frac{1}{2}(\delta_{i+1} + \delta_i)$;
- $\Delta F_{\delta(1)}(\delta_i) = F_{\delta(1)}(\delta_{i+1}) - F_{\delta(1)}(\delta_i)$;
- $G_1 = m \cdot \pi \cdot e^{\delta_1} + m_1 \cdot \pi - b \cdot (m - m_1)$;
- $G_i^* = m \cdot \pi \cdot e^{\delta_i^*} + m_1 \cdot \pi - b \cdot (m - m_1)$.

The indicator function $\mathbf{1}_{\{G_i^* \leq \xi + {}_1V(m_1,\, \delta_i^*)\}}$ is a step function with the jump at $\xi = G_i^* - {}_1V(m_1,\, \delta_i^*)$. To avoid interpolation between the values of the step function, which results in a numerical error whenever the values of the function argument are on different sides of the jump point, we consider values of ξ that satisfy $\xi = G_i^* - {}_1V(m_1,\, \delta_i^*)$ for given values of m_1 and δ_i^*. Then, the k^{th} moment of S_1^{acct} can be estimated as follows:

$$\widehat{\mathrm{E}}\Big[\big(S_1^{acct}\big)^k \Big] = (\xi_1)^k \cdot \widehat{\mathbf{P}}[S_1^{acct} \leq \xi_1] + \sum_{j=2}^{pts.s} (\xi_j)^k \cdot \Big(\widehat{\mathbf{P}}[S_1^{acct} \leq \xi_j] - \widehat{\mathbf{P}}[S_1^{acct} \leq \xi_{j-1}] \Big)$$

where $pts.s$ is equal to the sum of the number of values of m_1 used in the df estimation and $pts.d - 1$.

Tables B.1, B.2, B.3 and B.4 provide exact values and estimates of the expected values and the standard deviations of the accounting surplus for portfolios of life insurance policies considered in Example 1 and Example 2 of Chapter 5. We also report relative errors. However, it should be noted that, when comparing very small numbers, relative errors might be misleading; instead one should look at the absolute errors. It can be observed that most of the relative errors are well within 1% of the corresponding exact values. For the portfolio of 1000 temporary policies with 0% premium loading (see Table B.3), the expected values of the accounting surplus are very close to zero, which produces large relative errors. But the absolute errors between the exact expected values and their estimates do not exceed \$0.002, which can be considered quite negligible in the insurance context.

r	$\mathrm{E}[S_r^{acct}/m]$	$\widehat{\mathrm{E}}[S_r^{acct}/m]$	rel. error	$\mathrm{SD}[S_r^{acct}/m]$	$\widehat{\mathrm{SD}}[S_r^{acct}/m]$	rel. error
$\theta = 0\%$, $\pi = 67.90$						
1	0.1813	0.1810	-0.0017	18.4445	18.4494	0.0003
2	0.5799	0.5975	0.0304	28.9847	28.9773	-0.0003
3	1.2282	1.2223	-0.0048	38.9385	38.9717	0.0009
4	2.1532	2.1436	-0.0045	48.7257	48.7720	0.0010
5	3.3749	3.3639	-0.0033	58.3658	58.4269	0.0010
6	4.9043	4.8912	-0.0027	67.7569	67.8404	0.0012
7	6.7403	6.7251	-0.0023	76.7586	76.8750	0.0015
8	8.8671	8.8483	-0.0021	85.2437	85.4109	0.0020
9	11.2499	11.2241	-0.0023	93.1651	93.4105	0.0026
10	13.8300	13.7951	-0.0025	102.4769	101.0684	-0.0137
$\theta = 10\%$, $\pi = 74.69$						
1	53.7584	53.7581	0.0000	17.3837	17.3884	0.0003
2	58.3794	58.3607	-0.0003	27.9597	27.9325	-0.0010
3	63.4848	63.4855	0.0000	38.3216	38.3458	0.0006
4	69.1228	69.1293	0.0001	48.8169	48.8547	0.0008
5	75.3356	75.3362	0.0000	59.4432	59.5002	0.0010
6	82.1574	82.1566	0.0000	70.0953	70.1778	0.0012
7	89.6116	89.6139	0.0000	80.6368	80.7490	0.0014
8	97.7079	97.7082	0.0000	90.9464	91.1297	0.0020
9	106.4383	106.4369	0.0000	100.9743	101.2591	0.0028
10	115.7718	115.7689	0.0000	112.5302	111.2875	-0.0110
$\theta = 20\%$, $\pi = 81.48$						
1	107.3356	107.3352	0.0000	16.3317	16.3362	0.0003
2	116.1789	116.1350	-0.0004	26.9414	26.9350	-0.0002
3	125.7415	125.7439	0.0000	37.7102	37.7357	0.0007
4	136.0925	136.0953	0.0000	48.9139	48.9544	0.0008
5	147.2963	147.2979	0.0000	60.5285	60.5887	0.0010
6	159.4104	159.4111	0.0000	72.4441	72.5320	0.0012
7	172.4828	172.4849	0.0000	84.5272	84.6519	0.0015
8	186.5488	186.5492	0.0000	96.6601	96.8607	0.0021
9	201.6267	201.6259	0.0000	108.7897	109.0995	0.0028
10	217.7137	217.7126	0.0000	122.6051	121.4694	-0.0093

Table B.1: Estimates of expected values and standard deviations of accounting surplus per policy for a portfolio of 100 10-year endowment policies.

r	$\mathrm{E}[S_r^{acct}/m]$	$\widehat{\mathrm{E}}[S_r^{acct}/m]$	rel. error	$\mathrm{SD}[S_r^{acct}/m]$	$\widehat{\mathrm{SD}}[S_r^{acct}/m]$	rel. error
			$\theta = 0\%$, $\pi = 67.90$			
1	0.1813	0.1812	-0.0006	18.1388	18.1396	0.0000
2	0.5799	0.5673	-0.0217	28.5924	28.6234	0.0011
3	1.2282	1.2269	-0.0011	38.4954	38.5285	0.0009
4	2.1532	2.1504	-0.0013	48.2473	48.2884	0.0009
5	3.3749	3.3697	-0.0015	57.8586	57.9109	0.0009
6	4.9043	4.8974	-0.0014	67.2236	67.2957	0.0011
7	6.7403	6.7304	-0.0015	76.1986	76.2996	0.0013
8	8.8671	8.8532	-0.0016	84.6528	84.7998	0.0017
9	11.2499	11.2325	-0.0015	92.5369	92.7561	0.0024
10	13.8300	13.8048	-0.0018	99.9982	100.2132	0.0022
			$\theta = 10\%$, $\pi = 74.69$			
1	53.7584	53.7584	0.0000	17.0255	17.0262	0.0000
2	58.3794	58.3699	-0.0002	27.5109	27.5425	0.0011
3	63.4848	63.4797	-0.0001	37.8251	37.8570	0.0008
4	69.1228	69.1140	-0.0001	48.2904	48.3332	0.0009
5	75.3356	75.3294	-0.0001	58.8945	58.9499	0.0009
6	82.1574	82.1481	-0.0001	69.5275	69.6061	0.0011
7	89.6116	89.5988	-0.0001	80.0499	80.1630	0.0014
8	97.7079	97.6932	-0.0002	90.3369	90.5005	0.0018
9	106.4383	106.4162	-0.0002	100.3369	100.5821	0.0024
10	115.7718	115.7397	-0.0003	110.1670	110.4057	0.0022
			$\theta = 20\%$, $\pi = 81.48$			
1	107.3356	107.3507	0.0001	15.9122	15.9380	0.0016
2	116.1789	116.1726	-0.0001	26.4296	26.4625	0.0012
3	125.7415	125.7381	0.0000	37.1560	37.1893	0.0009
4	136.0925	136.0858	0.0000	48.3372	48.3826	0.0009
5	147.2963	147.2888	-0.0001	59.9373	59.9985	0.0010
6	159.4104	159.3989	-0.0001	71.8417	71.9291	0.0012
7	172.4828	172.4680	-0.0001	83.9136	84.0382	0.0015
8	186.5488	186.5335	-0.0001	96.0319	96.2148	0.0019
9	201.6267	201.6012	-0.0001	108.1418	108.4152	0.0025
10	217.7137	217.6760	-0.0002	120.3358	120.6012	0.0022

Table B.2: Estimates of expected values and standard deviations of accounting surplus per policy for the limiting portfolio of 10-year endowment policies.

r	$\mathrm{E}[S_r^{acct}/m]$	$\widehat{\mathrm{E}}[S_r^{acct}/m]$	rel. error	$\mathrm{SD}[S_r^{acct}/m]$	$\widehat{\mathrm{SD}}[S_r^{acct}/m]$	rel. error
$\theta = 0\%,\ \pi = 1.27$						
1	0.0005	0.0007	0.4000	1.1405	1.1403	-0.0002
2	0.0015	0.0012	-0.2000	1.6834	1.6850	0.0010
3	0.0030	0.0048	0.6000	2.1560	2.1598	0.0018
4	0.0048	0.0067	0.3958	2.6059	2.6120	0.0023
5	0.0068	0.0088	0.2941	3.0540	3.0624	0.0028
$\theta = 2\%,\ \pi = 1.29$						
1	0.1187	0.1188	0.0008	1.1406	1.1403	-0.0003
2	0.1291	0.1287	-0.0031	1.6836	1.6852	0.0010
3	0.1404	0.1422	0.0128	2.1562	2.1601	0.0018
4	0.1527	0.1546	0.0124	2.6063	2.6124	0.0023
5	0.1658	0.1677	0.0115	3.0544	3.0630	0.0028
$\theta = 3\%,\ \pi = 1.31$						
1	0.1778	0.1779	0.0006	1.1407	1.1404	-0.0003
2	0.1929	0.1924	-0.0026	1.6836	1.6853	0.0010
3	0.2091	0.2108	0.0081	2.1563	2.1603	0.0019
4	0.2266	0.2284	0.0079	2.6064	2.6126	0.0024
5	0.2453	0.2471	0.0073	3.0546	3.0632	0.0028
$\theta = 20\%,\ \pi = 1.52$						
1	1.1825	1.1830	0.0004	1.1412	1.1414	0.0002
2	1.2770	1.2763	-0.0005	1.6849	1.6870	0.0012
3	1.3773	1.3784	0.0008	2.1584	2.1629	0.0021
4	1.4839	1.4850	0.0007	2.6095	2.6163	0.0026
5	1.5969	1.5980	0.0007	3.0589	3.0679	0.0029

Table B.3: Estimates of expected values and standard deviations of accounting surplus per policy for a portfolio of 1000 5-year temporary policies.

r	$\mathrm{E}[S_r^{acct}/m]$	$\widehat{\mathrm{E}}[S_r^{acct}/m]$	rel. error	$\mathrm{SD}[S_r^{acct}/m]$	$\widehat{\mathrm{SD}}[S_r^{acct}/m]$	rel. error
$\theta = 0\%$, $\pi = 1.27$						
1	0.0005	0.0005	0.0000	0.0523	0.0523	0.0000
2	0.0015	0.0015	0.0000	0.0729	0.0730	0.0014
3	0.0030	0.0030	0.0000	0.0873	0.0874	0.0011
4	0.0048	0.0048	0.0000	0.0979	0.0981	0.0020
5	0.0068	0.0068	0.0000	0.1065	0.1067	0.0019
$\theta = 2\%$, $\pi = 1.29$						
1	0.1187	0.1187	0.0000	0.0515	0.0515	0.0000
2	0.1291	0.1291	0.0000	0.0731	0.0731	0.0000
3	0.1404	0.1405	0.0007	0.0892	0.0893	0.0011
4	0.1527	0.1527	0.0000	0.1023	0.1024	0.0010
5	0.1658	0.1659	0.0006	0.1136	0.1138	0.0018
$\theta = 3\%$, $\pi = 1.31$						
1	0.1778	0.1778	0.0000	0.0510	0.0511	0.0020
2	0.1929	0.1929	0.0000	0.0732	0.0733	0.0014
3	0.2091	0.2092	0.0005	0.0902	0.0903	0.0011
4	0.2266	0.2267	0.0004	0.1044	0.1046	0.0019
5	0.2453	0.2453	0.0000	0.1171	0.1174	0.0026
$\theta = 0\%$, $\pi = 1.27$, $S_0 = 0.06$						
1	0.0655	0.0655	0.0000	0.0529	0.0529	0.0000
2	0.0716	0.0717	0.0014	0.0744	0.0744	0.0000
3	0.0785	0.0786	0.0013	0.0898	0.0899	0.0011
4	0.0861	0.0861	0.0000	0.1019	0.1021	0.0020
5	0.0942	0.0942	0.0000	0.1121	0.1124	0.0027
$\theta = 2\%$, $\pi = 1.29$, $S_0 = 0.06$						
1	0.1837	0.1837	0.0000	0.0521	0.0521	0.0000
2	0.1992	0.1992	0.0000	0.0746	0.0746	0.0000
3	0.2160	0.2160	0.0000	0.0918	0.0919	0.0011
4	0.2340	0.2341	0.0004	0.1062	0.1064	0.0019
5	0.2533	0.2533	0.0000	0.1192	0.1194	0.0017

Table B.4: Estimates of expected values and standard deviations of accounting surplus per policy for the limiting portfolio of 5-year temporary policies.

Appendix C

Mortality Table

x	q_x		x	q_x		x	q_x		x	q_x
0	0.00707		25	0.00122		50	0.00452		75	0.05248
1	0.00054		26	0.00126		51	0.00497		76	0.05723
2	0.00041		27	0.00128		52	0.00555		77	0.06224
3	0.00032		28	0.00128		53	0.00621		78	0.06756
4	0.00027		29	0.00129		54	0.00686		79	0.07343
5	0.00025		30	0.00130		55	0.00753		80	0.08016
6	0.00022		31	0.00132		56	0.00835		81	0.08800
7	0.00018		32	0.00136		57	0.00930		82	0.09693
8	0.00017		33	0.00141		58	0.01038		83	0.10659
9	0.00017		34	0.00148		59	0.01152		84	0.11657
10	0.00018		35	0.00152		60	0.01276		85	0.12679
11	0.00020		36	0.00152		61	0.01422		86	0.13748
12	0.00024		37	0.00158		62	0.01581		87	0.14883
13	0.00030		38	0.00170		63	0.01747		88	0.16078
14	0.00038		39	0.00180		64	0.01920		89	0.17305
15	0.00053		40	0.00189		65	0.02105		90	0.18513
16	0.00074		41	0.00199		66	0.02300		91	0.19670
17	0.00098		42	0.00214		67	0.02511		92	0.20775
18	0.00120		43	0.00231		68	0.02735		93	0.21843
19	0.00135		44	0.00254		69	0.02975		94	0.22877
20	0.00139		45	0.00284		70	0.03225		95	0.23869
21	0.00132		46	0.00315		71	0.03514		96	0.24783
22	0.00125		47	0.00348		72	0.03876		97	0.25580
23	0.00121		48	0.00385		73	0.04307		98	0.26246
24	0.00120		49	0.00419		74	0.04776		99	0.26783

Appendix D

Alternative Method for Computing the Distribution Function of Accounting Surplus

Note that

$$
\begin{aligned}
&\mathbf{P}[G_t \leq \lambda \,|\, \mathscr{L}_{t-1} = m_{t-1},\ \mathscr{L}_t = m_t,\ \delta(t) = \delta_t] = \\
= \ & \frac{\mathbf{P}[\mathscr{L}_{t-1} = m_{t-1},\ \mathscr{L}_t = m_t,\ \delta(t) = \delta_t \,|\, G_t \leq \lambda] \cdot \mathbf{P}[G_t \leq \lambda]}{\mathbf{P}[\mathscr{L}_{t-1} = m_{t-1},\ \mathscr{L}_t = m_t,\ \delta(t) = \delta_t]} = \\
= \ & \frac{\mathbf{P}[\mathscr{L}_{t-1} = m_{t-1},\ \mathscr{L}_t = m_t \,|\, G_t \leq \lambda] \cdot f_{\delta(t)}(\delta_t \,|\, G_t \leq \lambda) \cdot \mathbf{P}[G_t \leq \lambda]}{\mathbf{P}[\mathscr{L}_{t-1} = m_{t-1},\ \mathscr{L}_t = m_t] \cdot f_{\delta(t)}(\delta_t)}, \qquad \text{(D.1)}
\end{aligned}
$$

where the last line follows from the independence of $\mathscr{L}_{t-1}$ and $\mathscr{L}_t$ from $\delta(t)$.

Next we consider a function $g_t(\lambda,\ m_{t-1},\ m_t,\ \delta_t)$ given by

$$
\begin{aligned}
g_t(\lambda,\ m_{t-1},\ m_t,\ \delta_t) \ &= \ \mathbf{P}[G_t \leq \lambda \,|\, \mathscr{L}_{t-1} = m_{t-1},\ \mathscr{L}_t = m_t,\ \delta(t) = \delta_t] \times \\
&\times \ \mathbf{P}[\mathscr{L}_{t-1} = m_{t-1},\ \mathscr{L}_t = m_t] \times f_{\delta(t)}(\delta_t)
\end{aligned}
$$

and motivated by Equation (D.1).

The following result gives a way for calculating g_t from g_{t-1}, $1 < t \leq r \leq n$.

Result D.0.1.

$$\begin{aligned}g_t(\lambda,\, m_{t-1},\, m_t,\, \delta_t) &= \mathbf{P}[\mathscr{L}_t(x) = m_t \,|\, \mathscr{L}_{t-1}(x) = m_{t-1}] \times \\ &\times \Bigg(\int_{-\infty}^{\infty} f_{\delta(t)}(\delta_t \,|\, \delta(t-1) = \delta_{t-1}) \times \\ &\times \Bigg[\sum_{m_{t-2}=m_{t-1}}^{m} g_{t-1}\Big(\frac{\lambda - \eta_t}{e^{\delta_t}},\, m_{t-2},\, m_{t-1},\, \delta_{t-1}\Big) \Bigg] \, d\delta_{t-1} \Bigg),\end{aligned}$$

where η_t *is the realization of* RC_t^τ *for given values of* m_{t-1} *and* m_t,

$$\eta_t = \begin{cases} \pi \cdot m_t - b \cdot (m_{t-1} - m_t), & 1 < t \le r-1, \\ -b \cdot (m_{t-1} - m_t), & t = r \end{cases}$$

with the starting value for g_t

$$g_1(\lambda,\, m,\, m_1,\, \delta_1) = \begin{cases} \mathbf{P}[\mathscr{L}_1(x) = m_1] \cdot f_{\delta(1)}(\delta_1) & \text{if } G_1 \le \lambda \\ 0 & \text{otherwise.} \end{cases}$$

Proof:
Using Equation (5.2), which implies that $\Big\{G_t \le \lambda\Big\} \equiv \Big\{G_{t-1} \le \frac{\lambda - RC_t^\tau}{e^{\delta(t)}}\Big\}$, we obtain

$$\begin{aligned}& g_t(\lambda,\, m_{t-1},\, m_t,\, \delta_t) = \\ =\;& \mathbf{P}[G_t \le \lambda \,|\, \mathscr{L}_{t-1} = m_{t-1},\, \mathscr{L}_t = m_t,\, \delta(t) = \delta_t] \times \mathbf{P}[\mathscr{L}_{t-1} = m_{t-1},\, \mathscr{L}_t = m_t] \times \\ & \times f_{\delta(t)}(\delta_t) \\ =\;& \mathbf{P}\Big[G_{t-1} \le \frac{\lambda - \eta_t}{e^{\delta_t}} \,\Big|\, \mathscr{L}_{t-1} = m_{t-1},\, \mathscr{L}_t = m_t,\, \delta(t) = \delta_t\Big] \times \\ & \times \mathbf{P}[\mathscr{L}_{t-1} = m_{t-1},\, \mathscr{L}_t = m_t] \times f_{\delta(t)}(\delta_t)\end{aligned}$$

and from Equation (D.1):

$$\begin{aligned}g_t(\lambda,\, m_{t-1},\, m_t,\, \delta_t) &= \mathbf{P}\Big[\mathscr{L}_{t-1} = m_{t-1},\, \mathscr{L}_t = m_t \,\Big|\, G_{t-1} \le \frac{\lambda - \eta_t}{e^{\delta_t}}\Big] \times \\ &\times\; f_{\delta(t)}\Big(\delta_t \,\Big|\, G_{t-1} \le \frac{\lambda - \eta_t}{e^{\delta_t}}\Big) \cdot \mathbf{P}\Big[G_{t-1} \le \frac{\lambda - \eta_t}{e^{\delta_t}}\Big].\end{aligned} \tag{D.2}$$

Consider $\mathbf{P}\Big[\mathscr{L}_{t-1} = m_{t-1},\, \mathscr{L}_t = m_t \,\Big|\, G_{t-1} \le \frac{\lambda-\eta_t}{e^{\delta_t}}\Big] \cdot f_{\delta(t)}\Big(\delta_t \,\Big|\, G_{t-1} \le \frac{\lambda-\eta_t}{e^{\delta_t}}\Big) \equiv (\star)$.

It can be rewritten as follows:

$$
\begin{aligned}
(\star) &= \mathbf{P}\Big[\mathscr{L}_{t-1} = m_{t-1},\, \mathscr{L}_t = m_t \,\Big|\, G_{t-1} \le \frac{\lambda - \eta_t}{e^{\delta_t}}\Big] \cdot f_{\delta(t)}\Big(\delta_t \,\Big|\, G_{t-1} \le \frac{\lambda - \eta_t}{e^{\delta_t}}\Big) = \\
&= \int_{-\infty}^{\infty} \sum_{m_{t-2}=m_{t-1}}^{m} \mathbf{P}\Big[\mathscr{L}_{t-1} = m_{t-1},\, \mathscr{L}_t = m_t \,\Big|\, \mathscr{L}_{t-2} = m_{t-2},\, G_{t-1} \le \frac{\lambda - \eta_t}{e^{\delta_t}}\Big] \times \\
&\qquad\qquad \times \mathbf{P}\Big[\mathscr{L}_{t-2} = m_{t-2} \,\Big|\, G_{t-1} \le \frac{\lambda - \eta_t}{e^{\delta_t}}\Big] \times \\
&\times \ f_{\delta(t)}\Big(\delta_t \,\Big|\, \delta(t-1) = \delta_{t-1},\, G_{t-1} \le \frac{\lambda - \eta_t}{e^{\delta_t}}\Big) \cdot f_{\delta(t-1)}\Big(\delta_{t-1} \,\Big|\, G_{t-1} \le \frac{\lambda - \eta_t}{e^{\delta_t}}\Big)\, d\delta_{t-1}.
\end{aligned}
$$

Applying the Markovian property of $\mathscr{L}_t$ and $\delta(t)$, we get

$$
\begin{aligned}
(\star) &= \int_{-\infty}^{\infty} \sum_{m_{t-2}=m_{t-1}}^{m} \mathbf{P}[\mathscr{L}_{t-1} = m_{t-1},\, \mathscr{L}_t = m_t \,|\, \mathscr{L}_{t-2} = m_{t-2}] \times \\
&\qquad\qquad \times \mathbf{P}\Big[\mathscr{L}_{t-2} = m_{t-2} \,\Big|\, G_{t-1} \le \frac{\lambda - \eta_t}{e^{\delta_t}}\Big] \times \\
&\quad \times f_{\delta(t)}(\delta_t \,|\, \delta(t-1) = \delta_{t-1}) \cdot f_{\delta(t-1)}\Big(\delta_{t-1} \,\Big|\, G_{t-1} \le \frac{\lambda - \eta_t}{e^{\delta_t}}\Big)\, d\delta_{t-1} \\
&= \int_{-\infty}^{\infty} \sum_{m_{t-2}=m_{t-1}}^{m} \mathbf{P}[\mathscr{L}_t = m_t \,|\, \mathscr{L}_{t-1} = m_{t-1}] \cdot \mathbf{P}[\mathscr{L}_{t-1} = m_{t-1} \,|\, \mathscr{L}_{t-2} = m_{t-2}] \times \\
&\qquad\quad \times \mathbf{P}\Big[\mathscr{L}_{t-2} = m_{t-2} \,\Big|\, G_{t-1} \le \frac{\lambda - \eta_t}{e^{\delta_t}}\Big] \times f_{\delta(t-1)}\Big(\delta_{t-1} \,\Big|\, G_{t-1} \le \frac{\lambda - \eta_t}{e^{\delta_t}}\Big) \\
&\qquad\quad \times f_{\delta(t)}(\delta_t \,|\, \delta(t-1) = \delta_{t-1})\, d\delta_{t-1} \\
&= \mathbf{P}[\mathscr{L}_t = m_t \,|\, \mathscr{L}_{t-1} = m_{t-1}] \times \\
&\quad \times \Big(\int_{-\infty}^{\infty} \sum_{m_{t-2}=m_{t-1}}^{m} \mathbf{P}\Big[\mathscr{L}_{t-2} = m_{t-2},\, \mathscr{L}_{t-1} = m_{t-1} \,\Big|\, G_{t-1} \le \frac{\lambda - \eta_t}{e^{\delta_t}}\Big] \times \\
&\qquad\qquad \times f_{\delta(t-1)}\Big(\delta_{t-1} \,\Big|\, G_{t-1} \le \frac{\lambda - \eta_t}{e^{\delta_t}}\Big) \cdot f_{\delta(t)}(\delta_t \,|\, \delta(t-1) = \delta_{t-1})\, d\delta_{t-1} \Big) \\
&= \mathbf{P}[\mathscr{L}_t = m_t \,|\, \mathscr{L}_{t-1} = m_{t-1}] \times \\
&\quad \times \Big(\int_{-\infty}^{\infty} \sum_{m_{t-2}=m_{t-1}}^{m} g_{t-1}\Big(\frac{\lambda - \eta_t}{e^{e^{\delta_t}}},\, m_{t-2},\, m_{t-1},\, \delta_{t-1}\Big)\Big] / \mathbf{P}\Big[G_{t-1} \le \frac{\lambda - \eta_t}{e^{\delta_t}}\Big] \times \\
&\qquad\qquad \times f_{\delta(t)}(\delta_t \,|\, \delta(t-1) = \delta_{t-1})\, d\delta_{t-1} \Big). \qquad\qquad \text{(D.3)}
\end{aligned}
$$

Substituting (D.3) into (D.2) proves the result.

The starting value is obtained from the definition of $g_t(\lambda,\, m_{t-1},\, m_t,\, \delta_t)$ for $t = 1$ with $\mathbf{P}[\mathscr{L}_0 = m] = 1$. □

Once $g_r(\lambda, m_{r-1}, m_r, \delta_r)$ is obtained using Result D.0.1, the cumulative distribution function of S_r^{acct} can be calculated as follows:

$$
\begin{aligned}
&\mathbf{P}[S_r^{acct} \leq \xi] = \\
&= \int_{-\infty}^{\infty} \sum_{m_{r-1}=0}^{m} \sum_{m_r=0}^{m_{r-1}} \mathbf{P}[S_r^{acct} \leq \xi \,|\, \mathscr{L}_{r-1} = m_{r-1},\, \mathscr{L}_r = m_r,\, \delta(r) = \delta_r] \times \\
&\qquad\qquad \times \mathbf{P}[\mathscr{L}_{r-1} = m_{r-1},\, \mathscr{L}_r = m_r] \cdot f_{\delta(r)}(\delta_r)\, d\delta_r \\
&= \int_{-\infty}^{\infty} \sum_{m_{r-1}=0}^{m} \sum_{m_r=0}^{m_{r-1}} \mathbf{P}[RG_r \leq \xi + {}_rV \,|\, \mathscr{L}_{r-1} = m_{r-1},\, \mathscr{L}_r = m_r,\, \delta(r) = \delta_r] \times \\
&\qquad\qquad \times \mathbf{P}[\mathscr{L}_{r-1} = m_{r-1},\, \mathscr{L}_r = m_r] \cdot f_{\delta(r)}(\delta_r)\, d\delta_r \\
&= \int_{-\infty}^{\infty} \sum_{m_{r-1}=0}^{m} \sum_{m_r=0}^{m_{r-1}} \mathbf{P}[G_r \leq \xi + {}_rV \,|\, \mathscr{L}_{r-1} = m_{r-1},\, \mathscr{L}_r = m_r,\, \delta(r) = \delta_r] \times \\
&\qquad\qquad \times \mathbf{P}[\mathscr{L}_{r-1} = m_{r-1},\, \mathscr{L}_r = m_r] \cdot f_{\delta(r)}(\delta_r)\, d\delta_r \\
&= \int_{-\infty}^{\infty} \sum_{m_{r-1}=0}^{m} \sum_{m_r=0}^{m_{r-1}} g_r(\xi + {}_rV,\, m_{r-1},\, m_r,\, \delta_r)\, d\delta_r \\
&= \int_{-\infty}^{\infty} \sum_{m_{r-1}=0}^{m} \sum_{m_r=0}^{m_{r-1}} g_r(\xi + {}_rV(m_r,\, \delta_r),\, m_{r-1},\, m_r,\, \delta_r)\, d\delta_r. \qquad \text{(D.4)}
\end{aligned}
$$

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