

THE UNIQUENESS OF ENVELOPES IN \mathcal{N}_0 -CATEGORICAL, \mathcal{N}_0 -STABLE STRUCTURES

by

James Gordon Loveys

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APPROVAL

Name: James Gordon Loveys
Degree: Master of Science
Title of Thesis: The uniqueness of envelopes in N_0 -categorical,
 N_0 -stable structures.

Chairman: B.S. Thomson

A.H. Lachlan
Senior Supervisor

N.R. Reilly

A. Mekler

G. Cherlin
External Examiner
Visiting Associate Professor
Department of Mathematics
Simon Fraser University

Date approved: December 8, 1982

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Author:

(signature)

James Loveys

(name)

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(date)

ABSTRACT

A strongly minimal set H is strictly minimal if it is definable without parameters, realizes only 1-type, and there are no nontrivial equivalence relations on it definable without parameters.

$(A)_H$ is by definition $H \cap \text{acl}(A)$; $\dim_H(A)$ the size of a maximal independent subset of $(A)_H$. If H is strictly minimal, it is

modular if for any $A, B \subseteq H$, $\dim_H(A) + \dim_H(B) = \dim_H(A \cup B)$

+ $\dim_H((A)_H \cap (B)_H)$. E is an H-envelope of A if E is maximal

subject to $(E \cup A)_H = (A)_H$. In the following, M is an

\aleph_0 -categorical, \aleph_0 -stable structure, $H \subseteq M$ is strictly minimal with

either H modular or $(A)_H \neq \emptyset$, and E is an H-envelope of $A \subseteq M$.

Theorem 2.7. For any $\bar{a}_1, \bar{a}_2, b_1 \in M$ with $\text{st}(\bar{a}_1|A) = \text{st}(\bar{a}_2|A)$ there

is $b_2 \in M$ with $\text{st}(\bar{a}_1 \wedge \langle b_1 \rangle | A) = \text{st}(\bar{a}_2 \wedge \langle b_2 \rangle | A)$.

Lemma 3.1. $\text{ST}(E|A \cup H) = \{\text{st}(\bar{b}|A \cup H) : \bar{b} \in M \text{ and } (A \cup \bar{b})_H = (A)_H\}$,

where $\text{ST}(A|B)$ is by definition $\{\text{st}(\bar{a}|B) : \bar{a} \in A\}$.

Theorem 4.5. (1) M is atomic over $E \cup H$.

(2) If $(A)_H$ is finite, M is atomic over E .

Corollary 4.6. If M is countable, E is unique up to an

automorphism of M fixing $A \cup H$ pointwise.

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INTRODUCTION

The notion of a theory categorical in power α was first introduced by Łos (see Łos) and Vaught (see V1) in 1954. Probably the first important result about \aleph_0 -categorical theories is the 1959 theorem that a theory T is \aleph_0 -categorical if and only if T has only finitely many types in any finite set of variables; this is attributed to Engeler and Svenonius as well as Ryll-Nardzewski (all independently; see En, R-N, Sv). Vaught (V2) gave some other equivalent conditions in 1961.

In Łos it was conjectured that any countable theory categorical in one uncountable power is categorical in every uncountable power. This conjecture was proved by Morley in 1965 (Mo); in this paper Morley introduced the notions of transcendental rank and degree and totally transcendental theory. Morley's rank is used in the present paper, and of course in countable languages totally transcendental and \aleph_0 -stable theories are the same as pointed out in Mo, Theorem 2.8.

The term stable (also superstable and \aleph_0 -stable) actually comes from Shelah (Sh1, Sh2); indeed, the modern notion of stability, as well as a great deal of what is known about it, is due to Shelah, beginning about 1969. Shelah's notion of forking independence, though never mentioned by name, is implicit throughout the current paper. The same goes for the finite equivalence relation theorem (Sh2, III, 2.8; see also CHL, Lemma 1.6); for example, it is used in deriving Proposition 1.14 of the current paper from Theorem 3.1 of

of CHL. His concepts of strong type and almost definability are used quite explicitly in the current paper. Also due to Shelah are the notion of imaginary elements and a structure referred to as M^{eq} ; this is essentially our structure N . (All of the above is in Sh2.)

There are other notion of rank besides Morley's, several due to Shelah and one, the U-rank, due to Lascar (Las), which for the purposes of the current paper is identical with Morley's (as pointed out in Bu, a relatively readable account of some of the highlights of CHL.) The notation for forking independence used here comes from Ma.

Sets of Morley rank 0 are finite and hence of little interest in studying ideas related to categoricity. Given the Morley rank, then, the natural thing to consider is definable sets of rank and degree 1. These, called strongly minimal sets, were first investigated by Marsh (Mar) in 1966, and later more thoroughly by Baldwin and Lachlan (BL) where they were used to prove a conjecture of Vaught's that every \aleph_1 -categorical theory in a countable language has one or \aleph_0 nonisomorphic models. Of course, given imaginary elements, the study of strongly minimal sets is virtually equivalent to studying strictly minimal sets (CHL).

Much of the work regarding categoricity has been devoted to providing answers to the following (from MO, more or less):

(1) Under what conditions on a structure M can it be finitely axiomatizable?

(2) Under what conditions is the rank of M finite?

(A structure is finitely axiomatizable, etc., iff its complete theory is.)

Baldwin (B) provided a partial answer to (2) in 1973; if M is \aleph_1 -categorical it has finite rank. Lachlan, in 1974, attempted to prove that the rank is finite for \aleph_0 -categorical M . To do this, he invented the notion of pseudoplane and showed (La) that the nonexistence of \aleph_0 -categorical pseudoplanes implies not only the finiteness of rank for \aleph_0 -categorical M , but also that stable and \aleph_0 -categorical imply \aleph_0 -stable. He proved also without assuming nonexistence of pseudoplanes, that superstable and \aleph_0 -categorical imply stable (as he mentions, this was known to Shelah).

Makowsky, meanwhile, showed in Mak that a structure which is the algebraic closure of a strongly minimal set cannot be finitely axiomatizable (extending a result known to Vaught) and provided an example of a superstable finitely axiomatizable theory.

In Z2, Zil'ber proved that if H is a strictly minimal \aleph_0 -categorical structure, either H interprets a rank 2, degree 1 pseudoplane or the Classification Theorem is true for H ; the Classification Theorem says that either H has in effect no structure at all or is essentially an affine or projective space over a finite field. In Z3, he introduced the notion of envelope in an attempt to prove that no complete totally categorical theory T can be finitely axiomatizable. The idea of the proof was to show

that if M is a model of such a T and $H \subseteq M$ is strongly minimal, then an envelope of any sufficiently large subset of H is a finite model of any fixed finite subset of T . Z3 contains an error, which Zil'ber has since repaired (in a non-trivial way).

Cherlin noticed that the Classification Theorem (for all strictly minimal, \aleph_0 -categorical H) is a consequence of the Classification Theorem for finite simple groups. (See CHL for proof.)

Using the Classification Theorem, Cherlin, Harrington and Lachlan (in CHL) expanded and reorganized Zil'ber's work. In particular, they generalized most of Zil'ber's work to \aleph_0 -categorical, \aleph_0 -stable structures, proved the rank is finite in \aleph_0 -categorical, \aleph_0 -stable structures, and introduced the powerful Coordinatization Theorem (Theorem 3.1 of CHL, Proposition 1.14 of the present paper). Using a notion of envelope that is the same as Zil'ber's except in one particularly perverse case (and in all cases the same as in the present paper), they show that \aleph_0 -categorical, \aleph_0 -stable structures are not finitely axiomatizable, addressing (1). On the other hand, Peretyat'kin (P) has found an example of an \aleph_1 -categorical finitely axiomatizable structure.

Zil'ber in Z4 and Z5, found a quite different proof of the Classification Theorem without using any deep group theory.

As mentioned above, one of the main tools of CHL is Zil'ber's notion of envelope. They also prove that except in the

previously mentioned perverse case, envelopes are unique in the sense that any two H -envelopes of A are isomorphic when considered as structures in their own right. This is the result the present paper extends. The main result of the present paper is that in \aleph_0 -categorical, \aleph_0 -stable structures, envelopes are as unique as could reasonably be expected, except in the perverse case (where they are not at all unique for either our envelopes or Zil'ber's). Along the way we prove that, for any subset A of an \aleph_0 -categorical, \aleph_0 -stable M , M is in a natural sense weakly homogeneous over A .

The first chapter of this paper is devoted to preliminaries, the bulk of which are from CHL. The second proves the weak homogeneity just mentioned (Theorem 2.7). The third proves that any two H -envelopes of A are isomorphic via a map fixing $A \cup H$ pointwise except in the perverse case, (Corollary 3.3). The final chapter shows that the structure is atomic over the union of H and any H -envelope (Theorem 4.5) and so if the structure is countable, the map from Corollary 3.3 extends to an automorphism of the structure (Corollary 4.6).

CHAPTER 1PRELIMINARIES

This chapter is devoted to setting the stage for the rest of the paper. It begins with a description of a structure N constructed from the given structure M ; N is essentially a version of Shelah's M^{eq} . [For an alternate description, see Ma, pp. B5-B8.]

From this point on M is assumed to be \aleph_0 -categorical and \aleph_0 -stable. The chapter defines most of the notions studied in the paper; most importantly, strictly minimal sets and envelopes. The definitions come essentially from CHL although they are a little more general. The Classification Theorem of Zil'ber and Cherlin for strictly minimal sets (see CHL, Theorem 2.1) is not stated but those of its consequences which I use are, in Propositions 1.9 through 1.12. Particularly important are 1.9(2) and (3) and 1.11(1). 1.9(2) states basically that a modular strictly minimal set behaves nicely when any parameters from the structure are named; 1.9(3) that any strictly minimal set is closely tied to a modular one; and 1.11(1) that any two modular strictly minimal sets are either not related at all or tied in the closest possible manner.

The result from CHL which this paper generalizes is Proposition 1.18. Also from CHL come Propositions 1.13 and 1.14; the latter, which states how powerful knowledge about strictly minimal sets is, is used repeatedly throughout the paper. Its basic content

is that any degree 1 type has an associated strictly minimal type that induces structure on the given type.

The chapter closes with a few well-known consequences of superstability and a simple application of these useful elsewhere in the paper.

Throughout, M and N (also M_1, N' , etc.) will refer to structures in a relational language. This entails no loss of generality for the purposes of this paper. If two or more structures are mentioned together there is no assumption that they share the same language. $|M|$ denotes the universe of M .

If $A \subseteq |M|^n$ for some $n < \omega$, A is definable if it is definable using parameters from $|M|$; A is B-definable if it is definable using parameters from B ; A is 0-definable if it is definable without parameters. The distinction between sequences, singletons, and the ranges of sequences will frequently be dropped; for example \bar{a} -definable means $\text{rng}(\bar{a})$ -definable.

If $A \subseteq M$, (M, A) denotes the expansion of M obtained by adding for each $a \in A$ a predicate U_a with $U_a^M = \{a\}$. If $A \subseteq |M|$ is non-empty, $M|A$ denotes the structure with universe A in language $L(M|A)$ which has a predicate symbol R_A for every 0-definable relation R on M ; and if R is n -ary, $R_A^{M|A} = R \cap A^n$. $\text{rk}_M(B)$ and $\text{deg}_M(B)$ (or just $\text{rk}(B)$, $\text{deg}(B)$ if M is understood) denote the Morley rank and degree of B for any $B \subseteq |M|^n$. $(\text{rk}, \text{deg})_M(B) = (\text{rk}_M(B), \text{deg}_M(B))$.

Proposition 1.1. If $A \subseteq |M|$ is definable and M is stable, then for any $n < \omega$, $B \subseteq A^n$,

(1) B is definable in M iff it is definable in $M|A$.

(2) $(\text{rk}, \text{deg})_M(B) = (\text{rk}, \text{deg})_{M|A}(B)$.

This is Proposition 1.4 of CHL.

- Definition 1.2. (1) A map $h: |M_1| \rightarrow |M_2|$ is a w.e. embedding of M_1 into M_2 if h is injective and for any $n < \omega$, $A \subseteq |M_1|^n$, A is 0-definable in M_1 iff $h(A)$ is 0-definable in M_2 .
- (2) $h: |M_1| \rightarrow |M_2|$ is an equivalence of M_1 and M_2 if h is a w.e. embedding of M_1 into M_2 and h^{-1} exists and is a w.e. embedding of M_2 into M_1 .
- (3) M_1 and M_2 are essentially identical if $|M_1| = |M_2|$ and the identity is an equivalence of M_1 and M_2 .
- (4) M_1 is a w.e. substructure of M_2 (M_2 is a w.e. extension of M_1) if $|M_1| \subseteq |M_2|$ and the identity is a w.e. embedding of M_1 into M_2 . We write $M_1 \subseteq_{\text{w.e.}} M_2$.
- (5) If $|M_0| \subseteq |M_1| \cap |M_2|$ and $|M_0|$ is 0-definable in M_1 , then $h: |M_1| \rightarrow |M_2|$ is a w.e. embedding of M_1 into M_2 (an equivalence of M_1 and M_2) over M_0 if $h|_{|M_0|}$ is the identity and h is a w.e. embedding of M_1 into M_2 (an equivalence of M_1 and M_2).

- Remarks. (i) All the above definitions are language-free.
- (ii) $M_1 \subseteq_{\text{w.e.}} M_2$ iff $|M_1|$ is a 0-definable subset of M_2 and M_1 is essentially identical to $M_2|_{|M_1|}$.
- (iii) If $M_1, M_2 \subseteq_{\text{w.e.}} N$ and $|M_1| \subseteq |M_2|$, then $M_1 \subseteq_{\text{w.e.}} M_2$.
- (iv) If $M_0 \subseteq_{\text{w.e.}} M_1$ and $M_1 \subseteq_{\text{w.e.}} M_2$, then $M_0 \subseteq_{\text{w.e.}} M_2$.

(v) If $A \subseteq |M_1|$ and $h: |M_1| \rightarrow |M_2|$ is a w.e. embedding of M_1 into M_2 , then h is a w.e. embedding of (M_1, A) into (M_2, A) .

Definition 1.3. (1) If $A, B \subseteq |M|$ and A and B are 0-definable, then A is B-small if there are $n < \omega$ and 0-definable C and F such that $C \subseteq B^n$ and F is a function from C onto A .

(2) If $B \subseteq |M|$ is 0-definable and $A \subseteq |M|$, A is B-small if there is 0-definable $C \subseteq |M|$ such that C is B-small and $A \subseteq C$.

(3) M_1 is a d-substructure of M_2 , and M_2 is a d-extension of M_1 , denoted $M_1 \subseteq_d M_2$, if $M_1 \subseteq_{w.e.} M_2$ and $|M_2|$ is $|M_1|$ -small (in M_2).

Remarks. (i) If $A, B, C \subseteq |M|$ are 0-definable, $A \subseteq C$ and C is B-small, then A is B-small according to definition 0.3(1); thus 0.3(1) and (2) agree on 0-definable subsets of $|M|$.

(ii) If $M \subseteq_{w.e.} N$, then for any 0-definable A with $|M| \subseteq A \subseteq |N|$, A is $|M|$ -small iff $N|A$ is a d-extension of M .

(iii) If $M_1 \subseteq_d M_2$ and $M_2 \subseteq_d M_3$, then $M_1 \subseteq_d M_3$.

(iv) If $M \subseteq_d M_2$, $M_1 \subseteq_{w.e.} M_2$ and $|M| \subseteq |M_1|$, then $M \subseteq_d M_1$.

(v) If $M \subseteq_d M_1$, $M \subseteq_d M_2$ and $M_1 \subseteq_{w.e.} M_2$, then $M_1 \subseteq_d M_2$.

Definition 1.4. N is a definable closure of M if :

(i) $M \subseteq_{w.e.} N$.

(ii) For all $k < \omega$ and all definable $R \subseteq |N|^k$ there is $|M|$ -small $B \subseteq |N|$ such that either $R \subseteq B^k$ or $R \cup B^k = |N|^k$.

(iii) If $M \subseteq_d M_1 \subseteq_{w.e.} N$ and $M_1 \subseteq_d M_2'$, then there is M_2 and $h: |M_2'| \rightarrow |M_2|$ such that $M_1 \subseteq_d M_2 \subseteq_{w.e.} N$ and h is an equivalence of M_2' and M_2 over M_1 .

(iv) $|N| = \bigcup \{A \subseteq |N| : A \text{ is } |M| \text{-small}\}$.

Proposition 1.5. (1) Suppose $M \subseteq_d M_1, M_2$ and

$|M_1| \cap |M_2| = |M|$. There is a structure unique up to essential

identity with universe $|M_1| \cup |M_2|$ that has M_1 and M_2 as

w.e. substructures. Denote this structure $M_1 \cup M_2$; we have further

that $M \subseteq_d M_1 \cup M_2$.

(2) For all M there exists N a definable closure of M . Further

if M has only a countable number of 0-definable relations, we can

choose N with only a countable number of 0-definable relations,

and this N is unique up to equivalence over M .

(3) If $M \subseteq_d M_1 \subseteq_{w.e.} N$ and N is a definable closure of M ,

then N is a definable closure of M_1 .

The proof is omitted. Here as elsewhere if neither a proof nor a reference is given, the reader should be able to supply his own proof if necessary.

For any $A \subseteq |N|$ where $M \subseteq_{w.e.} N$, (M, A) denotes

$(N, A) \upharpoonright |M|$.

Proposition 1.6. If $A \subseteq |N|$ and N is a definable closure of M , then (N, A) is a definable closure of (M, A) .

From now on, M will refer to an \aleph_0 -stable, \aleph_0 -categorical structure in a countable relational language and N to a (i.e., the) definable closure of M which has countably many 0-definable relations. Unless otherwise indicated, all sets considered will be $|M|$ -small subsets of $|N|$. The principal exception to this rule is that algebraic closures of small sets will not be small — $\text{acl}(A)$ always refers to the algebraic closure of A taken in N . Since any d -extension of M is \aleph_0 -categorical and \aleph_0 -stable, N retains much of the character of \aleph_0 -categorical, \aleph_0 -stable structures. For instance, although N is not \aleph_0 -categorical, any type (over \emptyset) realized in N is isolated. Also, if A is finite $\text{acl}(A)$ is not, but for any small B , $\text{acl}(A) \cap B$ is finite.

The notation $\text{tp}(\bar{a}|A)$ is used for the (complete) type of \bar{a} over A , and also for the solution set of this type. $\text{st}(\bar{a}|A)$ is the strong type of \bar{a} over A ; also its solution set. $\text{tp}(\bar{a}) = \text{tp}(\bar{a}|\emptyset)$ and $\text{st}(\bar{a}) = \text{st}(\bar{a}|\emptyset)$. Thus $\text{st}(\bar{a}_1|A) = \text{st}(\bar{a}_2|A)$ iff $\text{tp}(\bar{a}_1|A) = \text{tp}(\bar{a}_2|A)$ and for any A -definable $B \subseteq |N|^k$ with $\bar{a}_1, \bar{a}_2 \in B$ and any A -definable equivalence relation E on B with a finite number of classes, we have $\bar{a}_1 E \bar{a}_2$.

For any definable $B \subseteq |N|$, there is a point $[B] \in N$ which "names" B — that is, $(N, \{[B]\})$ is essentially identical to the expansion of N by a predicate U with $U^N = B$. Specifically, let $\varphi(x, \bar{a}_0)$ be a definition of B ; on the 0-definable set $\text{tp}(\bar{a}_0)$

define \sim by: $\bar{a} \sim \bar{a}'$ iff $\forall x[\varphi(x, \bar{a}) \leftrightarrow \varphi(x, \bar{a}')]]$. We may assume that $\text{tp}(\bar{a}_0)/\sim \subseteq |N|$, so let $[B] = \bar{a}_0/\sim$. Now for any b' with $\text{tp}(b') = \text{tp}([B])$, there is a unique B' such that

$[B'] = b' \text{ --- } x \in B' \text{ iff } \exists \bar{y} \in \text{tp}(\bar{a}_0) [b' = \bar{y}/\sim \wedge \varphi(x, \bar{y})]$. The

notation $[A]$ is unambiguous for definable A given a particular definition of $A \text{ ---}$ whenever it is used, a particular definition is assumed.

Also, if the definition used for A is $\varphi(x, \bar{a})$ and A' is a conjugate of A , then the definition used for A' is $\varphi(x, \bar{a}')$

for some $\bar{a}' \in \text{tp}(\bar{a})$ (it's irrelevant which $\bar{a}' \in \text{tp}(\bar{a})$). Note

that $\text{st}(\bar{a}_1|A) = \text{st}(\bar{a}_2|A)$ iff $\text{tp}(\bar{a}_1| \text{acl}(A)) = \text{tp}(\bar{a}_2| \text{acl}(A))$.

A set B is almost A-definable if $[B] \in \text{acl}(A)$; that is, there is C which is A -definable and an A -definable equivalence relation E on C with finitely many classes, one of which is B . If

$[B] \in \text{acl}(\phi)$, B is almost 0-definable.

For any definable B and any A , the closure of A in B , denoted $(A)_B$, is $B \cap \text{acl}(A \cup \{[B]\})$.

B is an atom over A if $\text{tp}(b|A)$ is the same for every $b \in B$. B is an atom (B is transitive) if B is an atom over ϕ .

Definition 1.7. (1) A set H is strongly minimal if it is definable and $(\text{rk}, \text{deg})H = (1, 1)$.

(2) If H is strongly minimal, there is no $[H]$ -definable equivalence relation on H with finite classes and H is an atom over $[H]$, we say H is strictly minimal.

The following version of the exchange principle is used

(Lemma 2, §1 of BL):

Proposition 1.8. If H is B -definable and strongly minimal, $a \in H$ and $c \in \text{acl}(B \cup \{a\}) - \text{acl}(B)$, then $a \in \text{acl}(B \cup \{c\})$.

If H is definable and $B \subseteq H$, B is independent over A if for any $b \in B$, $b \notin (A \cup (B - \{b\}))_H$. Otherwise B is dependent over A . From now on, if a definition is made "over A " and the A is omitted, it is understood to be ϕ . Thus, B is independent means B is independent over ϕ .

If H is strongly minimal and $A \subseteq H$, then $\dim_H(A)$ denotes the cardinality of a maximal independent subset of A . If $B \subseteq |N|$, $\dim_H(B) = \dim_H((B)_H)$; $\text{codim}_H(B)$ is the cardinality of a maximal subset of H independent over B . These notions are well-defined. Also, if $A \subseteq B \subseteq \text{acl}(A)$, $\dim_H(A) = \dim_H(B)$ and $\text{codim}_H(A) = \text{codim}_H(B)$.

If H is definable and $A \subseteq H$, then A is H -closed over B if $(A \cup B)_H = A$; the H is often omitted.

If H is strictly minimal, then H is modular if for any closed $A, B \subseteq H$, $\dim_H(A) + \dim_H(B) = \dim_H(A \cup B) + \dim_H(A \cap B)$.

If H is strictly minimal, $D[H, A]$ (the dependence relation of H over A) denotes $\{B \subseteq H : B \text{ finite and dependent over } A\}$; $D[H, \phi]$ is denoted $D[H]$.

Suppose H is strictly minimal and A is finite; then $H - (A)_H$ is a 0-definable strongly minimal atom in $(N, A \cup \{[H]\})$. Let M_1 be a small substructure of N containing $A \cup H \cup \{[H]\}$. In

M_1 there is a coarsest $A \cup \{[H]\}$ -definable equivalence relation E_A on $H - (A)_H$ with finite classes since M_1 is N_0 -categorical. But any relation on $H - (A)_H$ that's $A \cup \{[H]\}$ -definable in N is $A \cup \{[H]\}$ -definable in M_1 , so E_A is the coarsest $A \cup \{[H]\}$ -definable such relation in N . Let $H_A = [H - (A)_H] / E_A$ and H/A be a corresponding $A \cup \{[H]\}$ -definable subset of N as given by 0.6 and the definition of N .

Proposition 1.9. Let H be strictly minimal.

(1) If A is finite, H/A is strictly minimal in (N, A) ; if in addition either H is modular or $(A)_H \neq \emptyset$, then H/A is modular.

(2) If H is modular or $(A)_H \neq \emptyset$, $D[H, A] = D[H, (A)_H]$.

(3) Suppose H is not modular. Then there is a modular $H' \subseteq N$ such that:

(i) $\{[H]\}$ is $[H']$ -definable and $\{[H']\}$ is $[H]$ -definable.

(ii) If $\text{tp}([H_1]) = \text{tp}([H_2])$, then $\text{tp}([H_1], [H'_1]) = \text{tp}([H_2], [H'_2])$ for any non-modular strictly minimal H_1, H_2 and

(iii) For any $a \in H$, there is a unique $\{[H], a\}$ -definable bijection between H/a and H' .

(4) If H is not modular and $(A)_H = \emptyset$, then $D[H, A] = D[H, (A)_H]$.

(5) If H is not modular and $A \subseteq H'$ is finite, then H/A is not modular and there is a unique $A \cup \{[H]\}$ -definable bijection between $(H/A)'$ and H'/A .

This proposition contains Lemmas 2.3 - 2.7 of CHL.

If H is modular, the notation H' just refers to H .

If H_0 and H_1 are strictly minimal sets, they are orthogonal over A if for any $\bar{h}_i \in H_i$ independent over A , \bar{h}_i is independent over $A \cup H_{1-i}$ for $i = 0, 1$. We write $H_0 \perp_A H_1$.

Otherwise, H_0 and H_1 are nonorthogonal over A , written

$$H_0 \not\perp_A H_1.$$

Proposition 1.10. (1) For any strictly minimal H_0 and

H_1 with $([H_0])_{H_1} = ([H_1])_{H_0} = \phi$ and any A , $H_0 \perp H_1$ iff

$$H_0 \perp_A H_1.$$

(2) If $([H_0])_{H_1} = ([H_1])_{H_0} = \phi$, H_0 and H_1 are strictly minimal

sets, and $H_0 \perp H_1$, then for any A , $(A \cup H_1)_{H_0} = (A)_{H_0}$.

(3) The relation of being nonorthogonal is an equivalence relation on the set of strictly minimal sets.

(1) and (3) are parts of Lemma 1.5 of CHL; (2) follows easily from (1).

Proposition 1.11. Suppose H_0 and H_1 are 0-definable, nonorthogonal strictly minimal sets and for $i = 0, 1$, either H_i is modular or $(A)_{H_i} \neq \phi$.

(1) If H_0 and H_1 are both modular, there is a unique 0-definable bijection between them. In particular, $\dim_{H_0}(A) = \dim_{H_1}(A)$.

(2) If neither H_0 or H_1 is modular, then $\dim_{H_0}(A) = \dim_{H_1}(A)$.

(3) If H_0 is modular and H_1 is not, $\dim_{H_0}(A) + 1 = \dim_{H_1}(A)$.

This includes corollaries 2.8 and 2.9 of CHL.

Proposition 1.12. Suppose H_0 and H_1 are 0-definable non-orthogonal strictly minimal sets. There are three possibilities:

(1) H_1 is not modular, $(A)_{H_1} = \phi$, and either H_0 is modular or

$(A)_{H_0} \neq \phi$; then $\text{codim}_{H_0}(A) + 1 = \text{codim}_{H_1}(A)$.

(2) Interchange H_0 and H_1 in (1) .

(3) Otherwise, $\text{codim}_{H_0}(A) = \text{codim}_{H_1}(A)$.

Proposition 1.13. If M is an N_0 -categorical, N_0 -stable structure, then $\text{rk}(M)$ is finite.

This is Theorem 1.4 of CHL; it implies that for any A , $\text{rk}(A)$ ($= \text{rk}_N(A)$ by definition) is finite, since every A we consider is a small subset of N and so contained in an N_0 -categorical, N_0 -stable structure.

Proposition 1.14. If $H \subseteq N$ is transitive and definable, $\text{rk}(H) \geq 1$ and $\text{deg}(H) = 1$, there is a strictly minimal set J which is almost $[H]$ -definable, an atom over $[H]$ such that for any $a \in H$, $(a)_J \neq \phi$.

A set J such that for any $a \in H$, $(a)_J \neq \phi$ is said to coordinatize H . This proposition, the basic tool of this paper, is Theorem 3.1 of CHL.

Notation: $\text{Tp}(A|B) = \{\text{tp}(\bar{a}|B) : \bar{a} \in A\}$ for any $A, B \subseteq |N|$, and
 $\text{ST}(A|B) = \{\text{st}(\bar{a}|B) : \bar{a} \in A\}$. $\text{Tp}(A) = \text{Tp}(A|\emptyset)$ and $\text{ST}(A) = \text{ST}(A|\emptyset)$.

Definition 1.15. B is homogeneous over A if for any $\bar{a}_0, \bar{a}_1, b_0 \in B$ such that $\text{tp}(\bar{a}_0|A) = \text{tp}(\bar{a}_1|A)$, there is $b_1 \in B$ such that $\text{tp}(\bar{a}_0 \wedge \langle b_0 \rangle | A) = \text{tp}(\bar{a}_1 \wedge \langle b_1 \rangle | A)$. B is weakly homogeneous over A if for any $\bar{a}_0, \bar{a}_1, b_0 \in B$ such that $\text{st}(\bar{a}_0|A) = \text{st}(\bar{a}_1|A)$, there is $b_1 \in B$ with $\text{st}(\bar{a}_0 \wedge \langle b_0 \rangle | A) = \text{st}(\bar{a}_1 \wedge \langle b_1 \rangle | A)$.

Proposition 1.16. Let G be homogeneous and J a 0-definable atom with $(G)_J \subseteq G$. Then if $(G \cap M)_J \neq \emptyset$,

$$(G \cap M)_J = G \cap J.$$

Proof. Let $e \in (G \cap M)_J$ and $\bar{m} \in G \cap M$ be such that $e \in (\bar{m})_J$. If $f \in G \cap J$, $\text{tp}(e) = \text{tp}(f)$ and since G is homogeneous, there is $\bar{m}' \in G$ with $\text{tp}(\langle e \rangle \wedge \bar{m}) = \text{tp}(\langle f \rangle \wedge \bar{m}')$. So $f \in (\bar{m}')_J$ and $\bar{m}' \in G \cap M$. Thus $G \cap J \subseteq (G \cap M)_J$ and the reverse inclusion is immediate from $G \cap J = (G)_J$.

Definition 1.17. If $M_1 \subseteq_{\text{w.e.}} N$, $H \subseteq |N|$ is strictly minimal, and $A \subseteq |N|$; then $E \subseteq |M_1|$ is an H-envelope of A in M_1 if E is a maximal subset of $|M_1|$ such that $(E \cup A)_H = (A)_H$. E is an H-envelope of A if E is an H-envelope of A in M .

Remarks: (1) If E is an H-envelope of A in M_1 , then

$$(A)_{M_1} \subseteq E \text{ and } (E)_{M_1} = E.$$

(2) If E is an H -envelope of A in M_1 , there is G an H -envelope of A in N with $E \subseteq G$. For any such G , $(G)_{M_1} = G \cap M_1 = E$. [Note: G is not small.] Any such G is called an extension of E to an H -envelope of A in N .

Proposition 1.18. If E is an H -envelope of A in $M_1 \subseteq$ w.e. N and either H is modular or $(A)_H \neq \emptyset$, then

(i) E is homogeneous

(ii) $ST(E) = \{st(\bar{a}) : \bar{a} \in M_1 \text{ and } (A \cup \bar{a})_H = (A)_H\}$.

This follows easily from Theorem 7.3 of CHL. The set A is assumed small; there is no need here to assume M_1 is small — of course, E will not then be small, but as long as A is, the result is true.

Morley rank and U -rank are identical in any \aleph_0 -categorical, \aleph_0 -stable structure and hence in N for the purposes of this paper (all types will be over small sets). The notation $\bar{a} \downarrow_B C$ means that $rktp(\bar{a}|B \cup C) = rktp(\bar{a}|B)$, and $A \downarrow_B C$ means that for every (finite)

$\bar{a} \in A$, $\bar{a} \downarrow_B C$. The following summarizes the basic facts about

forking used in this paper:

Proposition 1.19. (1) For any \bar{b}, A , there is finite

$\bar{a} \in A$ such that $\bar{b} \downarrow_{\bar{a}} A$.

(2) $A \downarrow_B C \Rightarrow C \downarrow_B A$. In particular, if $rktp(a) \geq 1$ and $a \in acl(\bar{b})$,

then $\text{rktp}(\bar{b}|a) < \text{rktp}(\bar{b})$.

(3) There do not exist $a, B, C_i (i < \omega)$ such that $C_i \downarrow_B \bigcup_{j \neq i} C_j$ and

$a \downarrow_B C_i$ for all $i < \omega$.

(4) If $C \subseteq D$, $\bar{a} \downarrow_B C$ and $\bar{a} \downarrow_C D$, then $\bar{a} \downarrow_B D$.

(5) If $A_i \subseteq A'_i \subseteq \text{acl}(A_i)$ for $i < 3$, then $A_0 \downarrow_{A_1} A_2$ iff

$A'_0 \downarrow_{A'_1} A'_2$.

These facts follow from the superstability of M and are part of the literature. See for example Ma, where (1) is A.10, (2) is A.5, (3) is D.2(i), (4) is A.4 and (5) is B.4.

Proposition 1.20. Suppose H is strictly minimal, either H is modular or $(A)_H \neq \emptyset$, $(A)_H$ is finite and $d \in N$. Then

$(A \cup \{d\})_H$ is finite.

Proof. Suppose not. Then choose $\{c_i : i < \omega\}$ in $(A \cup \{d\})_H$ independent over $(A)_H$. By 1.9(2), $\{c_i : i < \omega\}$ is independent over A . Thus $c_i \downarrow_A \{c_j : j \neq i\}$, but also $c_i \downarrow_A d$ for each $i < \omega$. 1.19(2) and (3) then give a contradiction.

CHAPTER 2WEAK HOMOGENEITY

The same conventions as in Chapter 1 are carried over here and elsewhere. In particular, M is an N_0 -categorical, N_0 -stable structure, N is its definable closure and any set mentioned is small unless otherwise indicated. The basic result of this chapter (Theorem 2.7) is that for any $\bar{a}_1, \bar{a}_2, b_1 \in N$ and $A \subseteq N$ which is small, if $\text{st}(\bar{a}_1|A) = \text{st}(\bar{a}_2|A)$ then there is $b_2 \in N$ so that $\text{st}(\bar{a}_1^{\wedge} \langle b_1 \rangle | A) = \text{st}(\bar{a}_2^{\wedge} \langle b_2 \rangle | A)$. That is, N is weakly homogeneous over any small $A \subseteq N$, and it follows that M is weakly homogeneous over any $A \subseteq M$. Also included is a technical lemma (2.1 and its corollary 2.2) useful in establishing 4.2 and 4.3 as well as 2.4. The chapter concludes with an example (2.8) showing that M need not be homogeneous over A .

Lemma 2.1. Suppose H and I are strictly minimal, H is almost 0-definable and I is almost b -definable, where $b \in H$. Also suppose there is $a \in N$ with $b \notin (a)_H$ and $(\langle a, b \rangle)_I \supsetneq (b)_I$. Then there is $J \subseteq N$ which is modular and almost 0-definable such that $J/b \not\perp I/b$.

Proof. If I is almost 0-definable, take $J = I'$. If not, but $H/b \not\perp I/b$, take $J = H'$. So assume I is not almost 0-definable and $H/b \perp I/b$; $[I] \in \text{acl}(b) - \text{acl}(\emptyset)$, so by exchange (1.8) $b \in \text{acl}([I])$ and I is an atom over b . Let $K = \text{st}([I])$; for each $[I_0] \in K$, I_0 is almost b_0 -definable, strictly minimal and an atom over b_0 for some $b_0 \in H$, so the same is true for I_0' . Let $J_0 = \cup\{I_0' : [I_0] \in K\}$ and define \sim on J_0 by $d_0 \sim d_1$ iff $d_1 \in (d_0)_{I_1'}$, for some $[I_1] \in K$. \sim is certainly reflexive. If $d_0 \sim d_1$, say $d_1 \in (d_0)_{I_1'}$ and $d_0 \in I_0'$; $d_1 \in \text{acl}(\langle [I_0], [I_1], d_0 \rangle)$ and $d_1 \notin \text{acl}(\langle [I_0], [I_1] \rangle)$ since $[I_0] \in \text{acl}(H)$, so $([I_0])_{I_1'} = \emptyset$. By exchange, $d_0 \in \text{acl}(\langle [I_0], [I_1], d_1 \rangle)$; again $(\langle [I_0], [I_1], d_1 \rangle)_{I_0'} = (d_1)_{I_0'}$, so $d_0 \in (d_1)_{I_0'}$, and so $d_1 \sim d_0$. Thus \sim is symmetric. Suppose $d_1 \in (d_0)_{I_1'}$ and $d_2 \in (d_1)_{I_2'}$; then $d_2 \in \text{acl}(\langle [I_1], [I_2], d_0 \rangle) -$ since $(\langle [I_1], [I_2], d_0 \rangle)_{I_2'} = (d_0)_{I_2'}$, $d_2 \in (d_0)_{I_2'}$. Thus \sim is an equivalence relation.

Let $J = J_0/\sim$ and define $f : I' \rightarrow J$ by $f(d) = d/\sim$. f is an almost b -definable injection (since I' is strictly minimal); if we show that f is onto it will demonstrate that J is strictly

minimal, modular and an atom over b , and $J/b = J \not\perp I' = I'/b \not\perp I/b = I$,
 so $J/b \not\perp I/b$. It suffices to show for any $[I_0], [I_1] \in K$ that

I_0 and I_1 are non-orthogonal. Suppose $[I_0], [I_1] \in K$ and

$I_0 \perp I_1$; choose $[I_2] \in K$ independent from both $[I_0]$ and $[I_1]$.

Either $I_0 \perp I_2$ or $I_1 \perp I_2$ and it follows from $(rk, deg) K = (1,1)$

for any independent $[I_0], [I_1] \in K$ that $I_0 \perp I_1$. Now choose

$b = b_0, b_1, \dots, b_k, \dots \in H$ independent over a and for each

$i < \omega$, $[I_i] \in K$ so that $st(\langle [I], b, a \rangle) = st(\langle [I_i], b_i, a \rangle)$.

$\langle a, b \rangle_I \neq \emptyset$ so $\langle a, b_i \rangle_{I_i} \neq \emptyset$; choose $d_i \in \langle a, b_i \rangle_{I_i}$ for

all $i < \omega$. $(b)_I = \emptyset$, so $d_i \notin (b)_I$ for $i < \omega$; since

$I_i \perp H/b_i$, if we let $B = \{b_i : i < \omega\}$, $d_i \notin (B)_{I_i}$ for $i < \omega$.

Since $I_i \perp I_j$ for $i \neq j$, $i, j < \omega$, if we let $D = \{d_i : i < \omega\}$,

$d_i \notin (B \cup D - \{d_i\})_{I_i}$, so $d_i \not\downarrow_B D - \{d_i\}$ for $i < \omega$.

But also $d_i \not\downarrow_B a$, contradicting 1.19(3).

Corollary 2.2. Suppose H and I are strictly minimal,

H is almost \bar{c} -definable and I is almost $\bar{c}^\wedge \langle d \rangle$ -definable where

$d \in H$. Also suppose there is $a \in N$ with $d \notin (\bar{c}^\wedge \langle a \rangle)_H$ and

$(\bar{c}^\wedge \langle a, d \rangle)_I \supsetneq (\bar{c}^\wedge \langle d \rangle)_I$. Then there is $J \subseteq N$ which is modular, almost

\bar{c} -definable and an atom over \bar{c} such that $J/d \not\perp I/(\bar{c}^\wedge \langle d \rangle)$.

Proof. Apply 2.1 in (N, \bar{c}) to H/\bar{c} and I/\bar{c} with d/\bar{c}

taking the part of b .

Definition 2.3. (1) $\langle H_1, H_2 \rangle$ is A-suitable for $\langle \bar{c}_1, \bar{c}_2 \rangle$ if H_1 is strictly minimal, almost $\bar{c}_1 \cup A$ -definable and $\text{st}(\bar{c}_1^\wedge \langle H_1 \rangle | A) = \text{st}(\bar{c}_2^\wedge \langle H_2 \rangle | A)$. (2) $\langle \bar{c}_1, \bar{c}_2 \rangle$ is A-great if $\text{st}(\bar{c}_1 | A) = \text{st}(\bar{c}_2 | A)$ and for any $\langle H_1, H_2 \rangle$ A-suitable for $\langle \bar{c}_1, \bar{c}_2 \rangle$, either $\text{codim}_{H_1}(A \cup \bar{c}_1) = \text{codim}_{H_2}(A \cup \bar{c}_2)$ or both are infinite.

Remarks. (i) If $\langle \bar{c}_1, \bar{c}_2 \rangle$ is A-great, $d_1 \in \text{acl}(A \cup \bar{c}_1)$ and $\text{st}(\bar{c}_1^\wedge \langle d_1 \rangle | A) = \text{st}(\bar{c}_2^\wedge \langle d_2 \rangle | A)$, then $\langle \bar{c}_1^\wedge \langle d_1 \rangle, \bar{c}_2^\wedge \langle d_2 \rangle \rangle$ is A-great.

(ii) For any A , $\langle \phi, \phi \rangle$ is A-great.

(iii) For any B , $A \subseteq B \subseteq \text{acl}(A)$, $\langle \bar{c}_1, \bar{c}_2 \rangle$ is A-great iff $\langle \bar{c}_1, \bar{c}_2 \rangle$ is B-great.

Lemma 2.4. If $\langle \bar{c}_1, \bar{c}_2 \rangle$ is A-great, $\langle H_1, H_2 \rangle$ is A-suitable for $\langle \bar{c}_1, \bar{c}_2 \rangle$, $d_i \in H_i$ ($i = 1, 2$) and $\text{st}(\bar{c}_1^\wedge \langle d_1 \rangle | A) = \text{st}(\bar{c}_2^\wedge \langle d_2 \rangle | A)$, then $\langle \bar{c}_1^\wedge \langle d_1 \rangle, \bar{c}_2^\wedge \langle d_2 \rangle \rangle$ is A-great.

Proof. We may assume $d_1 \notin \text{acl}(A \cup \bar{c}_1)$. Let $\langle I_1, I_2 \rangle$ be A-suitable for $\langle \bar{c}_1^\wedge \langle d_1 \rangle, \bar{c}_2^\wedge \langle d_2 \rangle \rangle$; if I_1 (hence I_2) is not modular, $\langle I_1', I_2' \rangle$ is also A-suitable for $\langle \bar{c}_1^\wedge \langle d_1 \rangle, \bar{c}_2^\wedge \langle d_2 \rangle \rangle$. Since $(A \cup \bar{c}_1^\wedge \langle d_1 \rangle)_{I_1} = \phi$ iff $(A \cup \bar{c}_2^\wedge \langle d_2 \rangle)_{I_2} = \phi$, by 1.11 we may assume I_1 and I_2 are modular. Choose $\bar{a} \in A$ so that H_i is almost $\bar{c}_i^\wedge \bar{a}$ -definable and I_i is almost $\bar{c}_i^\wedge \langle d_i \rangle^\wedge \bar{a}$ -definable for $i = 1, 2$. $(A \cup \bar{c}_1^\wedge \langle d_1 \rangle)_{I_1} = (\bar{a}^\wedge \bar{c}_1^\wedge \langle d_1 \rangle)_{I_1}$ iff $(A \cup \bar{c}_2^\wedge \langle d_2 \rangle)_{I_2} = (\bar{a}^\wedge \bar{c}_2^\wedge \langle d_2 \rangle)_{I_2}$ and if both of these are true, since I_i is modular

we have $\text{codim}_{I_i} (A \cup \bar{c}_i^{\wedge} \langle d_i \rangle)$ infinite for $i = 1, 2$. So assume

there is $\bar{a}' \in A$, $\bar{a}' \supseteq \bar{a}$, with $(\bar{a}'^{\wedge} \bar{c}_1^{\wedge} \langle d_1 \rangle)_{I_1} \not\supseteq (\bar{a}^{\wedge} \bar{c}_1^{\wedge} \langle d_1 \rangle)_{I_1}$.

Since $d_1 \notin (\bar{a}'^{\wedge} \bar{c}_1)_{H_1}$, we apply 2.2 to find J_1 modular,

almost $\bar{a}^{\wedge} \bar{c}_1$ -definable, an atom over $\bar{a}^{\wedge} \bar{c}_1$ such that

$J_1/d_1 \perp I_1/(\bar{a}^{\wedge} \bar{c}_1^{\wedge} \langle d_1 \rangle)$. Since $\text{st}(\bar{c}_1^{\wedge} \langle d_1, [I_1] \rangle | A) = \text{st}(\bar{c}_2^{\wedge} \langle d_2, [I_2] \rangle | A)$

we have $\text{st}(\bar{c}_1^{\wedge} \langle d_1, [I_1], [H_1] \rangle | A) = \text{st}(\bar{c}_2^{\wedge} \langle d_2, [I_2], [H_2] \rangle | A)$

so choose J_2 so that

$\text{st}(\bar{c}_1^{\wedge} \langle d_1, [I_1], [H_1], [J_1] \rangle | A) = \text{st}(\bar{c}_2^{\wedge} \langle d_2, [I_2], [H_2], [J_2] \rangle | A)$.

$\langle J_1, J_2 \rangle$ is A -suitable for $\langle \bar{c}_1, \bar{c}_2 \rangle$ and since $\langle \bar{c}_1, \bar{c}_2 \rangle$ is A -great,

either $\text{codim}_{J_1} (A \cup \bar{c}_1) = \text{codim}_{J_2} (A \cup \bar{c}_2)$ or both are infinite.

$\text{codim}_{I_i} (A \cup \bar{c}_i^{\wedge} \langle d_i \rangle) = \text{codim}_{I_i/(\bar{a}^{\wedge} \bar{c}_i^{\wedge} \langle d_i \rangle)} (A \cup \bar{c}_i^{\wedge} \langle d_i \rangle)$

= (by 1.11(1)) $\text{codim}_{J_i/d_i} (A \cup \bar{c}_i^{\wedge} \langle d_i \rangle) = \text{codim}_{J_i} (A \cup \bar{c}_i^{\wedge} \langle d_i \rangle)$

$$= \begin{cases} \text{codim}_{J_i} (A \cup \bar{c}_i) & \text{if } J_i \perp H_i/(\bar{c}_i^{\wedge} \bar{a}) . \\ \text{codim}_{J_i} (A \cup \bar{c}_i) - 1 & \text{if } J_i \not\perp H_i/(\bar{c}_i^{\wedge} \bar{a}) . \end{cases}$$

Since $J_1 \perp H_1/(\bar{c}_1^{\wedge} \bar{a})$ iff $J_2 \perp H_2/(\bar{c}_2^{\wedge} \bar{a})$, either

$\text{codim}_{I_1} (A \cup \bar{c}_1^{\wedge} \langle d_1 \rangle) = \text{codim}_{I_2} (A \cup \bar{c}_2^{\wedge} \langle d_2 \rangle)$ or both are infinite.

Lemma 2.5. If $\langle \bar{a}_1, \bar{a}_2 \rangle$ is A -great and $b_1 \in \mathbb{N}$, then

there is $b_2 \in \mathbb{N}$ such that $\text{st}(\bar{a}_1^{\wedge} \langle b_1 \rangle | A) = \text{st}(\bar{a}_2^{\wedge} \langle b_2 \rangle | A)$.

Proof. Choose $\langle \bar{c}_1, \bar{c}_2 \rangle$ so that $\bar{a}_1 \subseteq \bar{c}_1$, $\langle \bar{c}_1, \bar{c}_2 \rangle$ is A-great and $\text{rktp}(b_1 | \bar{c}_1)$ is minimal. If $\text{rktp}(b_1 | \bar{c}_1) = 0$ choose b_2 so that $\text{st}(\bar{c}_1 \wedge \langle b_1 \rangle | A) = \text{st}(\bar{c}_2 \wedge \langle b_2 \rangle | A)$ and we are finished. So suppose $\text{rktp}(b_1 | \bar{c}_1) \geq 1$; let H_1 be a strictly minimal almost \bar{c}_1 -definable atom over \bar{c}_1 that coordinates $\text{st}(b_1 | \bar{c}_1)$ — such H_1 exists by 1.14. Choose H_2 so that $\text{st}(\bar{c}_1 \wedge \langle [H_1] \rangle | A) = \text{st}(\bar{c}_2 \wedge \langle [H_2] \rangle | A)$ — then $\langle H_1, H_2 \rangle$ is A-suitable for $\langle \bar{c}_1, \bar{c}_2 \rangle$. For $d_1 \in (\bar{c}_1 \wedge \langle b_1 \rangle)_{H_1}$, there is $d_2 \in (\bar{c}_2 \wedge \langle b_2 \rangle)_{H_2}$ so that $\text{st}(\bar{c}_1 \wedge \langle d_1 \rangle | A) = \text{st}(\bar{c}_2 \wedge \langle d_2 \rangle | A)$; if $d_1 \in (\bar{c}_1 \cup A)_{H_1}$ this is clear, and if $d_1 \notin (\bar{c}_1 \cup A)_{H_1}$, then $\text{codim}_{H_1}(\bar{c}_1 \cup A) \geq 1$. Since $\langle \bar{c}_1, \bar{c}_2 \rangle$ is A-great, $\text{codim}_{H_2}(\bar{c}_2 \cup A) \geq 1$, so choose any $d_2 \in H_2$, $d_2 \notin \text{acl}(\bar{c}_2 \cup A)$. By 2.4, $\langle \bar{c}_1 \wedge \langle d_1 \rangle, \bar{c}_2 \wedge \langle d_2 \rangle \rangle$ is A-great; also $\text{rktp}(b_1 | \bar{c}_1 \wedge \langle d_1 \rangle) < \text{rktp}(b_1 | \bar{c}_1)$ by 1.19(2). This contradiction finishes the proof.

Lemma 2.6. If $\text{st}(\bar{a}_1 | A) = \text{st}(\bar{a}_2 | A)$ there are \bar{b}_1 and \bar{b}_2 such that $\langle \bar{a}_1 \wedge \bar{b}_1, \bar{a}_2 \wedge \bar{b}_2 \rangle$ is A-great.

Proof. Choose \bar{b}_1, \bar{b}_2 so that $\text{st}(\bar{a}_1 \wedge \bar{b}_1 | A) = \text{st}(\bar{a}_2 \wedge \bar{b}_2 | A)$, $\langle \bar{b}_1, \bar{b}_2 \rangle$ is A-great and $\text{rktp}(\bar{a}_1 | \bar{b}_1)$ is minimal; this can be done since $\langle \phi, \phi \rangle$ is A-great. If $\text{rktp}(\bar{a}_1 | \bar{b}_1) = 0$ we're done, so assume $\text{rktp}(\bar{a}_1 | \bar{b}_1) \geq 1$. Let H_1 be an almost \bar{b}_1 -definable, strictly minimal atom over \bar{b}_1 that coordinatizes $\text{st}(\bar{a}_1 | \bar{b}_1)$. Choose H_2 so that $\text{st}(\bar{a}_1 \wedge \bar{b}_1 \wedge \langle [H_1] \rangle | A) = \text{st}(\bar{a}_2 \wedge \bar{b}_2 \wedge \langle [H_2] \rangle | A)$;

$\langle H_1, H_2 \rangle$ is A -suitable for $\langle \bar{b}_1, \bar{b}_2 \rangle$. Let $c_1 \in (\bar{a}_1 \wedge \bar{b}_1)_{H_1}$ and choose $c_2 \in (\bar{a}_2 \wedge \bar{b}_2)_{H_2}$ so that $\text{st}(\bar{a}_1 \wedge \bar{b}_1 \wedge \langle c_1 \rangle | A) = \text{st}(\bar{a}_2 \wedge \bar{b}_2 \wedge \langle c_2 \rangle | A)$; by 2.4 $\langle \bar{b}_1 \wedge \langle c_1 \rangle, \bar{b}_2 \wedge \langle c_2 \rangle \rangle$ is A -great. By 1.19(2) $\text{rktp}(\bar{a}_1 | \bar{b}_1 \wedge \langle c_1 \rangle) < \text{rktp}(\bar{a}_1 | \bar{b}_1)$, a contradiction.

From the two preceding lemmas the following is immediate.

Theorem 2.7. If $\text{st}(\bar{a}_1 | A) = \text{st}(\bar{a}_2 | A)$ and $b_1 \in N$ there is $b_2 \in N$ such that $\text{st}(\bar{a}_1 \wedge \langle b_1 \rangle | A) = \text{st}(\bar{a}_2 \wedge \langle b_2 \rangle | A)$. In particular, M is weakly homogeneous over any $A \subseteq M$.

Corollary 2.8. If $\text{st}(\bar{a}_1 | A) = \text{st}(\bar{a}_2 | A)$ then $\langle \bar{a}_1, \bar{a}_2 \rangle$ is A -great.

Proof. If we had a counterexample $\langle \bar{a}_1, \bar{a}_2 \rangle$, $A \subseteq N$, $\langle H_1, H_2 \rangle$ A -suitable for $\langle \bar{a}_1, \bar{a}_2 \rangle$ with $\text{codim}_{H_1}(A \cup \bar{a}_1) \neq \text{codim}_{H_2}(A \cup \bar{a}_2) < \aleph_0$, then by taking an elementary submodel prime over $\bar{a}_1, \bar{a}_2, [H_1], [H_2]$ and a suitable countable subset of $A \cup H_1 \cup H_2$, we would get a counterexample in countable N . So suppose N is countable, and $\langle H_1, H_2 \rangle$ is A -suitable for $\langle \bar{a}_1, \bar{a}_2 \rangle$.

We have $\text{st}(\bar{a}_1 \wedge \langle [H_1] \rangle | A) = \text{st}(\bar{a}_2 \wedge \langle [H_2] \rangle | A)$ and using 2.6 and a back-and-forth argument we get an automorphism of N fixing A pointwise that takes \bar{a}_1 to \bar{a}_2 and $[H_1]$ to $[H_2]$. The conclusion is immediate.

The following shows that there is an \aleph_0 -categorical \aleph_0 -stable M and (algebraically closed in M) $A \subseteq M$ such that M is not homogeneous over A .

Example 2.8. Let $L(M)$ have one unary predicate symbol V and three binary predicate symbols R, \sim_1 and \sim_2 . Let

$|M| = \aleph_0$, where $|B| = |C| = |D| = \aleph_0$, and :

$$(1) \quad V^M = C \cup D.$$

(2) \sim_1^M, \sim_2^M are both equivalence relations on V^M .

(3) \sim_1^M has two classes C and D .

(4) Every class of \sim_2^M is infinite; \sim_2^M has \aleph_0 classes that are subsets of C , \aleph_0 classes that are subsets of D , and none that intersect both C and D .

(5) Let $C/\sim_2^M = \{C_0, C_1, \dots\}$, $D/\sim_2^M = \{D_0, D_1, \dots\}$, and

$$B = \{\{C_i, D_j\} : i, j < \omega\}.$$

(6) $R^M(a, b)$ iff $a \in B$, $b \in C \cup D$ and $b/\sim_2^M \in a$.

Let $A = \{\{C_i, D_j\} : i \geq 2, j \geq 1\}$. Choose $a_0 \in C_0$, $a_1 \in C_1$ and $b_0 \in D_0$. We have $tp(a_0|A) = tp(b_0|A)$, but there is no $b_1 \in M$ with $tp(\langle a_0, a_1 \rangle|A) = tp(\langle b_0, b_1 \rangle|A)$.

CHAPTER 3ST(E|A ∪ H), THE SET OF STRONG TYPES

The result of this brief chapter is that given H strictly minimal in M and any $A \subseteq M$ with either H modular or $(A)_H \neq \emptyset$, the set of strong types over $A \cup H$ realized in any H -envelope of A doesn't depend on the choice of envelope. This is Lemma 3.1. From this and 2.7 it follows easily (Corollary 3.3) that in countable M the envelope is unique up to an isomorphism fixing $A \cup H$ pointwise.

Lemma 3.1. If $A \subseteq N$ is small, H is almost A -definable and strictly minimal with either H modular or $(A)_H \neq \emptyset$, and $E \subseteq M$ is an H -envelope of A , then $\text{ST}(E|A \cup H) = \{\text{st}(\bar{b}|A \cup H) : \bar{b} \in M$ and $(A \cup \bar{b})_H = (A)_H\}$.

Proof. Given $\bar{b} \in M$ and applying 1.19(1), (4) and (5) we can find $\bar{a}_0 \in A$, $\bar{h} \in H$ such that \bar{h} is independent from A , $(\bar{a}_0 \wedge \bar{b})_H \neq \emptyset$ if H is not modular, H is almost \bar{a}_0 -definable, and $\bar{b} \downarrow_{\bar{a}_0 \wedge \bar{h}} A \cup H$. If in addition $(A \cup \bar{b})_H = (A)_H$, then $\bar{h} \downarrow_{\bar{a}_0} \bar{a}_0 \wedge \bar{b}$ by 1.9(2). So by 1.19(2) and (5), $\bar{b} \downarrow_{\bar{a}_0} \bar{a}_0 \wedge \bar{h}$, so by (4), $\bar{b} \downarrow_{\bar{a}_0} A \cup H$. That is, $\text{rktp}(\bar{b}|A \cup H) = \text{rktp}(\bar{b}|\bar{a}_0)$. Thus if $(A \cup \bar{b})_H = (A)_H$ and $\text{rktp}(\bar{b}|A \cup H) = r$ there is $\bar{a} \in \text{acl}(A)$ such that $(\text{rk}, \text{deg}) \text{tp}(\bar{b}|\bar{a}) = (r, 1)$, and H is \bar{a} -definable.

We now show by induction on r that:

For all A, \bar{b} with $\bar{b} \in M$, $(A \cup \bar{b})_H = (A)_H$ and E an H -envelope of A , if $\text{rktp}(\bar{b}|A \cup H) = r$ there is $\bar{e} \in E$ such that $\text{st}(\bar{e}|A \cup H) = \text{st}(\bar{b}|A \cup H)$.

If $r = 0$ and the conditions apply, there is $\bar{a} \in \text{acl}(A)$ with $\text{rktp}(\bar{b}|\bar{a}) = 0$, $\bar{b} \in (A)_M \subseteq E$ and we're finished. Suppose $\text{rktp}(\bar{b}|A \cup H) = k \geq 1$, \bar{b} and A fulfill the conditions, and the induction hypothesis holds for all $r < k$. Choose $\bar{a} \in \text{acl}(A)$ so that $(\text{rk}, \text{deg}) \text{tp}(\bar{b}|\bar{a}) = (k, 1)$, H is \bar{a} -definable and let I be

an almost \bar{a} -definable strictly minimal atom over \bar{a} that coordinatizes $\text{tp}(\bar{b}|\bar{a})$. Choose $a_1 \in (\bar{a} \wedge \bar{b})_I$.

Case 1: $H/\bar{a} \perp I$. Then for any $B \subseteq N$, $(B \cup \{a_1\})_{H/\bar{a}} = (B)_{H/\bar{a}}$.

Thus $(\bar{b} \cup A \cup \{a_1\})_{H/\bar{a}} = (\bar{b} \cup A)_{H/\bar{a}} = (A)_{H/\bar{a}} = (A \cup \{a_1\})_{H/\bar{a}}$; thus

$(\bar{b} \cup A \cup \{a_1\})_H = (A \cup \{a_1\})_H$ and also E is an H -envelope of

$A \cup \{a_1\}$. We also have $\text{rktp}(\bar{b}|A \cup \{a_1\} \cup H) < k$, so by the

induction hypothesis there is $\bar{e} \in E$ with $\text{st}(\bar{b}|A \cup \{a_1\} \cup H)$

$= \text{st}(\bar{e}|A \cup \{a_1\} \cup H)$.

Case 2: $H/\bar{a} \not\perp I$. Then if $(A \cup H)_I \neq \emptyset$, $(A \cup H/\bar{a})_I \neq \emptyset$, so

$(A \cup H/\bar{a})_I = I$, so $(A \cup H)_I = I$. Then $a_1 \in (A \cup H)_I$; but then

$\text{rktp}(\bar{b}|A \cup H) \leq \text{rktp}(\bar{b}|\bar{a} \wedge \langle a_1 \rangle) < \text{rktp}(\bar{b}|\bar{a}) = \text{rktp}(\bar{b}|A \cup H)$, a

contradiction. So $(A \cup H)_I = \emptyset$ and so every point of I

the same strong type over $A \cup H$. Applying 1.18(2) in (N, \bar{a}) , since

E is also an H/\bar{a} -envelope of A , we get $\bar{e}' \in E$ with $\text{st}(\bar{e}' | \bar{a})$

$= \text{st}(\bar{b}|\bar{a})$ and $a_2 \in (\bar{e}' \wedge \bar{a})_I$. Since $\text{st}(a_1|A \cup H) = \text{st}(a_2|A \cup H)$,

by 2.7 there is $\bar{c} \in M$ with $\text{st}(\langle a_1 \rangle \wedge \bar{b} | A \cup H) = \text{st}(\langle a_2 \rangle \wedge \bar{c} | A \cup H)$.

Since $a_1 \in \text{acl}(\bar{a} \wedge \bar{b})$, $(A \cup \{a_1\} \cup \bar{b})_H = (A \cup \bar{b})_H = (A)_H = (A \cup \{a_1\})_H$

and so $(A \cup \{a_2\} \cup \bar{c})_H = (A \cup \{a_2\})_H$. $a_2 \in \text{acl}(E)$, so E is an

H -envelope of $A \cup \{a_2\}$. By the induction hypothesis, since

$\text{rktp}(\bar{c}|A \cup \{a_2\} \cup H) < \text{rktp}(\bar{c}|A \cup H) = \text{rktp}(\bar{b}|A \cup H)$, there is $\bar{c} \in E$ with $\text{st}(\bar{e}|A \cup \{a_2\} \cup H) = \text{st}(\bar{c}|A \cup \{a_2\} \cup H)$, so $\text{st}(\bar{e}|A \cup H) = \text{st}(\bar{b}|A \cup H)$.

Thus $\text{ST}(E|A \cup H) \supseteq \{\text{st}(\bar{b}|A \cup H) : \bar{b} \in M \text{ and } (A \cup \bar{b})_H = (A)_H\}$ and the opposite inclusion is clear.

Corollary 3.2. Given A, H and E as in Lemma 3.1, E is weakly homogeneous over $A \cup H$.

Proof. Suppose $\bar{e}_1, \bar{e}_2, f_1 \in E$ and $\text{st}(\bar{e}_1|A \cup H) = \text{st}(\bar{e}_2|A \cup H)$. By 2.5 there is $f_2' \in M$ with $\text{st}(\bar{e}_1^{\wedge} \langle f_1 \rangle | A \cup H) = \text{st}(\bar{e}_2^{\wedge} \langle f_2' \rangle | A \cup H)$; thus $(\bar{e}_2^{\wedge} \langle f_2' \rangle \cup A)_H = (A)_H$ and since E is an H -envelope of $A \cup \bar{e}_2$, there is $f_2 \in E$ by 3.1 such that $\text{st}(f_2|A \cup \bar{e}_2 \cup H) = \text{st}(f_2'|A \cup \bar{e}_2 \cup H)$. So $\text{st}(\bar{e}_1^{\wedge} \langle f_1 \rangle | A \cup H) = \text{st}(\bar{e}_2^{\wedge} \langle f_2 \rangle | A \cup H)$.

Corollary 3.3. Suppose A and H are as in Lemma 3.1, E_1 and E_2 are both H -envelopes of A (in M), and M is countable. Then there is an isomorphism of $N|(E_1 \cup H)$ and $N|(E_2 \cup H)$ that fixes $A \cup H$ pointwise.

Proof. A simple back-and-forth argument based on 3.1 and 3.2.

CHAPTER 4M IS ATOMIC OVER E U H.

In this final chapter, it is shown that under the usual assumptions that E is an H -envelope of A for $H, A \subseteq M$ and H strictly minimal with either H modular or $(A)_H \neq \phi$, that the structure M is atomic over $E \cup H$. This result (Theorem 4.5), along with Corollary 3.3, easily gives the final result (Corollary 4.6) that if M is countable, an H -envelope of A is unique up to an automorphism of M fixing $A \cup H$ pointwise.

Definition 4.1. \bar{c} is A-good if $\text{tp}(\bar{c}|A)$ is isolated and if for any H which is strictly minimal and almost $\bar{c} \cup A$ -definable, $(A \cup \bar{c})_H$ is either finite or all of H .

Remarks: (i) If $d \in \text{acl}(A \cup \bar{c})$, \bar{c} is A-good iff $\bar{c}^{\wedge} \langle d \rangle$ is A-good.

(ii) If \bar{c} is A-good, H is strictly minimal and almost $\bar{c} \cup A$ -definable and $d \in H$, then $\text{tp}(d|A \cup \bar{c})$ is isolated. Hence $\text{tp}(\bar{c}^{\wedge} \langle d \rangle | A)$ is isolated.

Lemma 4.2. If \bar{c} is A-good and H is strictly minimal and almost $A \cup \bar{c}$ -definable, then for any $d \in H$, $\bar{c}^{\wedge} \langle d \rangle$ is A-good.

Proof. By the remarks, $\text{tp}(\bar{c}^{\wedge} \langle d \rangle | A)$ is isolated and we can assume $d \notin \text{acl}(A \cup \bar{c})$. Let I be strictly minimal and almost $A \cup \bar{c}^{\wedge} \langle d \rangle$ -definable. We may assume I is modular, since if

$(A \cup \bar{c}^{\wedge} \langle d \rangle)_I \neq \emptyset$ there is a direct connection between $(A \cup \bar{c}^{\wedge} \langle d \rangle)_I$ and $(A \cup \bar{c}^{\wedge} \langle d \rangle)_{I'}$, — i.e., if one is finite, the other is, and if

the second is I' , the first is I . Choose $\bar{a} \in A$ so that H is almost $\bar{a} \bar{c}$ -definable and I is almost $\bar{a} \bar{c}^{\wedge} \langle d \rangle$ -definable. If

$(A \cup \bar{c}^{\wedge} \langle d \rangle)_I = (\bar{a} \bar{c}^{\wedge} \langle d \rangle)_I$, we're finished; if not, find $\bar{a}' \in A$,

$\bar{a} \subseteq \bar{a}'$ such that $(\bar{a}' \bar{c}^{\wedge} \langle d \rangle)_I \supsetneq (\bar{a} \bar{c}^{\wedge} \langle d \rangle)_I$. Since also $d \notin (\bar{a}' \bar{c})_H$,

we apply 2.2 to find J which is almost $\bar{a}' \bar{c}$ -definable, modular and an atom over $\bar{a}' \bar{c}$ such that $J/d \not\subseteq I / (\bar{a}' \bar{c}^{\wedge} \langle d \rangle)$. Since \bar{c} is A-good,

$(A \cup \bar{c})_J$ is either finite or J ; thus by 1.20, $(A \cup \bar{c}^{\wedge} \langle d \rangle)_J$ is

finite or J and so $(A \cup \bar{c}^{\wedge} \langle d \rangle)_{J/d}$ is finite or J/d . Thus by

1.11(1) $(A \cup \bar{c}^{\wedge} \langle d \rangle)_{I/(\bar{a}^{\wedge} \bar{c}^{\wedge} \langle d \rangle)}$ is finite or all of $I/(\bar{a}^{\wedge} \bar{c}^{\wedge} \langle d \rangle)$ —

but then $(A \cup \bar{c}^{\wedge} \langle d \rangle)_I$ is finite or I .

Lemma 4.3. If H is strictly minimal, almost A -definable, either H is modular or $(A)_H \neq \emptyset$, and E is an H -envelope of A , then:

(1) The empty sequence is $E \cup H$ -good.

(2) If, in addition, $(A)_H$ is finite then the empty sequence is E -good.

Proof. Suppose I is strictly minimal and almost $\bar{e}^{\wedge} \bar{h}$ -definable, where $\bar{e} \in E$ and $\bar{h} \in H$ is independent over E . Also suppose H is almost \bar{e} -definable. We need to check that $(E \cup H)_I$ is finite or all of I . Also, if $(A)_H$ is finite and $\bar{h} = \emptyset$, we need to check that $(E)_I$ is finite or I .

Case 1. $I/(\bar{e}^{\wedge} \bar{h}) \not\perp H/(\bar{e}^{\wedge} \bar{h})$. If $(E \cup H)_I \supset (\bar{e}^{\wedge} \bar{h})_I$, then

$(E \cup H)_{I/(\bar{e}^{\wedge} \bar{h})} \neq \emptyset$, so from 1.11(1) and $(E \cup H)_{H/(\bar{e}^{\wedge} \bar{h})} = H/(\bar{e}^{\wedge} \bar{h})$ we

get $(E \cup H)_{I/(\bar{e}^{\wedge} \bar{h})} = I/(\bar{e}^{\wedge} \bar{h})$. Thus either $(E \cup H)_I = I$ or

$(E \cup H)_I = (\bar{e}^{\wedge} \bar{h})_I$. Also when $\bar{h} = \emptyset$, if $(A)_H$ is finite, so is

$(E)_{H/\bar{e}}$; by 1.11 $\dim_{I/\bar{e}}(E) \leq \dim_{H/\bar{e}}(E) + 1$ so $(E)_{I/\bar{e}}$ and $(E)_I$

are finite.

Case 2. $I/(\bar{e}^{\wedge} \bar{h}) \perp H/(\bar{e}^{\wedge} \bar{h})$. Proceed by induction on $lh(\bar{h})$.

Suppose $\bar{h} = \emptyset$; expand E to G an H/\bar{e} -envelope of A in N . By

1.10(2), $(G \cup I/\bar{e})_{H/\bar{e}} = (G)_{H/\bar{e}}$, so $I/\bar{e} \subseteq G$; by 1.16, if

$(E)_{I/\bar{e}} \neq \emptyset$ then $(E)_{I/\bar{e}} = I/\bar{e}$. Thus $(E)_I$ is finite or I .

Again by 1.10(2) $(E \cup H)_{I/\bar{e}} = (E)_{I/\bar{e}}$ so $(E \cup H)_I$ is finite or I .

Now suppose that for any $\bar{h}' \in H$ independent over E with $\ell h(\bar{h}') \leq k$

and any J strictly minimal, almost $\bar{e}^{\wedge} \bar{h}'$ -definable, we have that

$(E \cup H)_J$ is either finite or J ; also suppose that $\bar{h} = \bar{h}'^{\wedge} \langle h_0 \rangle$

where $\ell h(\bar{h}') = k \geq 0$.

If $(E \cup \bar{h}'^{\wedge} \langle h_0 \rangle)_I = (\bar{e}^{\wedge} \bar{h}'^{\wedge} \langle h_0 \rangle)_I$, we are done since

$(E \cup H)_I = (E \cup \bar{h}'^{\wedge} \langle h_0 \rangle)_I$; so assume there is $\bar{e}' \in E$, $\bar{e}' \supseteq \bar{e}$ with

$(\bar{e}'^{\wedge} \bar{h}'^{\wedge} \langle h_0 \rangle)_I \supsetneq (\bar{e}^{\wedge} \bar{h}'^{\wedge} \langle h_0 \rangle)_I$. Also $h_0 \notin (\bar{e}'^{\wedge} \bar{h}')_H$, so applying 2.2

gives J which is a modular, almost $\bar{e}'^{\wedge} \bar{h}'$ -definable atom over $\bar{e}'^{\wedge} \bar{h}'$

such that $J/h_0 \not\perp I/(\bar{e}'^{\wedge} \bar{h}'^{\wedge} \langle h_0 \rangle)$. By the induction hypothesis

$(E \cup H)_J$ is either finite or J , so $(E \cup H)_{J/h_0}$ is finite or J/h_0

by 1.20. By 1.11(1), $(E \cup H)_{I/(\bar{e}'^{\wedge} \bar{h}'^{\wedge} \langle h_0 \rangle)}$ is finite or

$I/(\bar{e}'^{\wedge} \bar{h}'^{\wedge} \langle h_0 \rangle)$, so $(E \cup H)_I$ is finite or I .

Lemma 4.4. Suppose $B \subseteq M$ and some sequence is B -good.

Then M is atomic over B .

Proof. Let $\bar{b} \in M$; choose $\bar{a} \in N$ so that \bar{a} is

B -good and $\text{rktp}(\bar{b}|\bar{a})$ is minimal. If $\text{rktp}(\bar{b}|\bar{a}) = 0$, $\text{tp}(\bar{b}|\bar{a} \cup B)$

is isolated; since $\text{tp}(\bar{a}|B)$ is isolated, $\text{tp}(\bar{b}|B)$ is isolated. Suppose

$\text{rktp}(\bar{b}|\bar{a}) \geq 1$ and let H be a strictly minimal almost \bar{a} -definable

atom over \bar{a} that coordinatizes $\text{st}(\bar{b}|\bar{a})$. Choose $a' \in (\bar{b}^{\wedge}\bar{a})_H$; by 4.2, $\bar{a}^{\wedge}\langle a' \rangle$ is B-good. Since $\text{rktp}(\bar{b}|\bar{a}^{\wedge}\langle a' \rangle) < \text{rktp}(\bar{b}|\bar{a})$, we have a contradiction.

Theorem 4.5. Suppose H is strictly minimal and almost A-definable, and either H is modular or $(A)_H \neq \emptyset$. Also suppose $E \subseteq M$ is an H-envelope of A . Then:

- (1) M is atomic over $E \cup H$.
- (2) If in addition $(A)_H$ is finite, M is atomic over E .

Proof. Immediate from 4.3 and 4.4.

Corollary 4.6. Let A, H and M be as in Theorem 4.5.

Then if E_1 and E_2 are both H-envelopes of A and M is countable, there is an automorphism of M mapping E_1 onto E_2 and fixing $A \cup H$ pointwise.

Proof. Immediate from 3.3, 4.5 and the uniqueness of countable prime models. (See, for example, ChK, Theorem 2.3.3, P. 95.)

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