ON THE STATIONARY EINSTEIN-MAXWELL-KLEIN-GORDON EQUATIONS

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ABSTRACT

The stationary Einstein-Maxwell-Klein-Gordon (EMKG) equations for interacting gravitational, electromagnetic, and meson fields are examined. The theory is cast into the formalism of principal fiber bundles with a connection, wherein its relationship to current trends in theoretical physics is made manifest. The EMKG equations are shown to admit a "Higgs - like mechanism" for giving mass to the gauge field. A theorem specifying sufficient conditions for the stationarity of the spacetime metric to imply stationarity of the other fields is proved. By imposing additional constraints and symmetries, the EMKG equations are considerably simplified. An attempt is made to apply a solution-generating technique, and this meets with only partial success. Finally, a stationary, but non-static, solution is found, and the geometric and physical properties are discussed.

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1. INTRODUCTION

A. Motivation

In theoretical physics, a new paradigm is emerging[1]. It is difficult, at this point, to completely describe or even name, the emerging paradigm. Broadly speaking, what we have is a growing consensus among theoretical physicists that nature at its most fundamental level can be described by "quantum gauge theories". Furthermore, it is increasingly believed that the topological and metrical properties of spacetime itself play a highly non-trivial role at the level of elementary particles. The formerly complementary paradigms of (classical) general relativity theory and quantum field theory are shedding their separate identities, and are evolving into the theory of quantum gauge fields. In a sense the new paradigm is the fulfilment of the hopes of Einstein, Weyl, Schrödinger, and others that the fundamental forces of interaction could be unified in a geometrical setting.

At the heart of the new paradigm is the triad of gravity, gauge fields, and matter fields (also dubbed source fields, or generic "Higgs fields"). In Chapter 2, it will be outlined how this triad is consistent with the mathematical theory of principal fiber bundles with a connection. The simplest example of such a triad is the system consisting of a scalar (i.e., spin-zero) meson interacting with gravity and the electromagnetic field. The electromagnetic field is the simplest gauge field because the corresponding gauge group, U(1), is Abelian, and hence the gauge fields are not directly self-interacting. The Einstein-Maxwell-Klein-Gordon (EMKG) equations are an obvious choice for the field equations describing the dynamical behaviour of such a system.

Exact solutions of classical field equations are important in the new paradigm. There are two (not unrelated) reasons for this. First, in theories such as quantum chromodynamics[2] (the gauge theory of the strong interaction), in contrast to quantum electrodynamics, perturbative techniques have not born fruit. Second, certain exact solutions of classical field equations have revealed a richness and subtlety of the corresponding quantum field theories that could not have been revealed by perturbative techniques[3]. Examples of this are the t'Hooft monopoles, which are soliton solutions of the Yang-Mills-Higgs equations[3,4], the various instanton solutions of the vacuum Yang-Mills equations (in Euclidean spacetime)[5], and the gravitational instanton solutions of the vacuum Einstein equations with a strictly Riemannian metric[6,7,8].

In this thesis, the EMKG equations will be examined in some depth. By imposing certain symmetries (stationarity and axial symmetry of the fields) and other types of restrictions (isometric motion of the meson field and the Weyl-Majumdar-Papapetrou (WMP) condition), the field equations are simplified to the point where exact solutions can be found. A class of such solutions is displayed and some of its properties are discussed. The project undertaken here is to be regarded within the framework of the new paradigm. Hence the formal nature of Chapters 2 and 3 and the Appendices A and B, which cast the theory into the elegant geometric formalism which has proven so useful for understanding the structure and concepts of gauge theories.

B. A Brief History of the EMKG Equations

The history of the Klein-Gordon equation goes back to 1926 with the attempts of Schrödinger[9], Gordon[10], and Klein[11] to construct relativistic versions of quantum-mechanical wave-equations. In 1935, Yukawa[12] proposed a model for the strong nuclear interaction in which the force between nucleons was mediated by massive charged scalar mesons (pi mesons, or pions) which obeyed the Klein-Gordon equation in regions external to the nucleonic sources.

In 1947, Utiyama[13] examined the Einstein equations with a scalar meson source which obeyed a curved spacetime version of the Klein-Gordon equation. Subsequent investigations tended to deal with the subcase of massless scalar mesons. Prominent examples of this are the papers of Szekeres[14], Bergmann and Leipnik[15], and Yilmaz[16]. More recently, Eris and Gürses[17] displayed a technique for generating classes of solutions of the Einstein-Maxwell-massless-Klein-Gordon equations from stationary axially-symmetric solutions of the Einstein-Maxwell equations. In this context, it should be mentioned that the massless Klein-Gordon equation in curved spacetime is often modified by the addition of the term $\frac{1}{6}\hat{R}\psi$ where $\hat{R}$ is the scalar curvature of spacetime and ψ is the meson field. Such theories are often dubbed "conformal scalar" since the correspondingly modified Klein-Gordon equation is invariant under conformal transformations of the metric tensor[18].

Recently, there has been considerable interest in the Klein-Gordon and other wave-equations in a given semi-Riemannian spacetime. This is usually done in the context of black-hole physics or quantum field theory in a curved classical "background" spacetime. The solutions of the Klein-Gordon equation so found are never exact solutions of the combined Einstein-Maxwell-Klein-Gordon equations. For more details, see the book by Friedlander[19] and the review by Gibbons[20] and the references cited therein.

In the early 1960's, Das[21,22], Stephenson[23], De[24], and Das and Coffman[25] found exact, static, spherically symmetric solutions of the EMKG equations with the scalar meson having non-zero rest mass and charge. The solutions found by Das and Coffman are particularly interesting. First, they bear some resemblance to solitons[3], i.e., regular, finite energy, localized solutions of a wave-equation. Second, the solutions exist only if the fine-structure constant attains certain numerical values determined by a non-linear eigenvalue problem. Finally, the corresponding spacetimes are topologically nontrivial. In quantum gauge theories solitons are important, as mentioned above, and coupling constants are often energy-dependent[2].

C. Summary

In Chapter 2, the properties of a principal fiber bundle over a spacetime M_4 with structure group U(1) are developed. The bundle space, a five-dimensional smooth manifold, is called the world-bundle and denoted W. If W is endowed with a connection, determined by a 1-form field A over W, then W is the geometric setting for an Abelian gauge theory, and the local expressions A_i for A can be identified with the electromagnetic potential. The semi-Riemannian structure on M_4 determines the gravitational field in the usual manner. Scalar meson fields ψ are associated with smooth sections of the vector bundle associated with W by the fundamental representation of U(1). In any region of M_4 where $\psi \neq 0$ (strictly speaking, arg (ψ) is C^2), one may use the real gauge, wherein ψ is real.

The EMKG equations are introduced via an action principle. It is shown that in regions of M_4 where $\psi \neq 0$, certain configurations of the fields mimic solutions of the Einstein-Proca or Einstein-Maxwell-Lorentz equations.

The metric $\hat{g}$ of spacetime is assumed to be stationary in Chapter 3. The field equations then require that $\underset{\xi}{\pounds} T_{ij} = 0$, where T_{ij} is the sum of the energy-momentum tensors of the electromagnetic field, E_{ij}, and of the meson field, M_{ij}. It is shown that if one further demands that either $\underset{\xi}{\pounds} E_{ij} = 0$ or $\underset{\xi}{\pounds} M_{ij} = 0$ in a region of spacetime where $\psi \neq 0$, then $\underset{\xi}{\pounds} \sqrt{\psi^*\psi} = \underset{\xi}{\pounds} A_i = 0$, i.e., the meson and electromagnetic fields are stationary.

The theory is then decomposed into "3+1 form", so that all fields are over $M_3 \simeq M_4/T_1$ where T_1 is the one-parameter group of translations generated by the timelike Killing vector field ξ^i. Tensor analysis over M_3 is developed and the static case, including the Das-Coffman solutions, is reviewed.

In Chapter 4, we demand that the meson current J^i is parallel to ξ^i. This condition is called <u>isometric</u> <u>motion</u>, and has been used previously to simplify the Einstein-Maxwell-Lorentz equations. Used in the present context, it enables one to write the EMKG equations in terms of the metric $g_{\alpha\beta}$ of M_3, two complex potentials ϕ and Γ, and the real meson field η. Two significant results are then demonstrated. The first is that the magnetic field is parallel to the so-called twist vector (determined by the $\hat{g}_{4\alpha}$). The second is that the Weyl-Majumdar-Papapetrou (WMP) condition, which is a functional relationship between $\hat{g}_{44}$ and the real part of the electromagnetic potential ϕ, implies that either $e^2 = 16\pi m^2$ or that the real and imaginary parts of ϕ are functionally related. In the above, e and m are the charge and rest-mass of the scalar meson.

Chapter 5 constitutes an attempt to apply the Kramer, Neugebauer, and Stephani (KNS) solution-generating ansatz[26] to the EMKG equations. The stationary Einstein-Maxwell or Einstein-Maxwell-Lorentz equations with isometric motion for the fluid are derivable from an action principle on M_3 with a Lagrangian density which is invariant under a group of transformations isomorphic to SU(2,1) acting on ϕ, Γ, and, in the case where a fluid source is present, on the pressure, density, and fluid velocity fields. This group transforms among solutions of the field equations in a nontrivial way. However, if we apply the KNS technique to the stationary EMKG equations with isometric motion for the meson, then the transformations so obtained are, in general, trivial. However, along the way, one discovers a group isomorphic to SU(2,1) x T_1 which transforms among solutions of the stationary EMKG equations with the meson massless and electrically neutral. Mass breaks the SU(2,1) x T_1 symmetry much as it breaks conformal symmetry in electrodynamics.

In Chapter 6, the stationary EMKG equations with isometric motion are further simplified by imposing axial symmetry on the metric and the electromagnetic and meson fields. Interestingly enough, it is demonstrated that the metric can be put in the so-called Weyl-Lewis-Papapetrou form if and only if the WMP condition is satisfied. Finally, a class of exact solutions is found. These solutions are stationary (but non-static), axially symmetric, and satisfy the WMP and isometric motion conditions.

The final Chapter is a discussion of the properties of the solutions found in the previous Chapter. In particular it is shown that the metric has causal pathologies similar to those of the Gödel metric. The metric is not asymptotically flat, but is homogeneous. The sources are shown to be well-behaved in that T_{ij} satisfies the strong energy condition of Hawking and Ellis.

There are four appendices. The first summarizes the properties of principal fiber bundles with a connection. The second shows that a stationary spacetime M_4 has the structure of a principal fiber bundle over a strictly Riemannian $M_3 \simeq M_4/T_1$. The components $\hat{g}_{4\alpha}$ determine a connection on the bundle space M_4. The remaining two appendices are technical.

D. Suggestions for Future Research

In terms of the subject matter of the thesis itself, there are two projects which are worth pursuing. The first is the question of whether Theorem 3.1 could be strengthened so that the hypothesis that either $\mathcal{L}_\xi M_{ij} = 0$ or $\mathcal{L}_\xi E_{ij} = 0$ could be dropped. The second is to find other solutions of the EMKG equations, in particular solutions where $\mathrm{Re}(\phi) \neq$ a constant and/or $\eta \neq$ a constant. It would be gratifying if such a solution required the parameter e to satisfy an eigenvalue problem a la the Das-Coffman solution. The hope is that such a solution would be a more physically realistic classical model of an elementary particle than either the Das-Coffman solution or the solution found in this thesis.

The rather dramatic simplification in the stationary EMKG equations that results when isometric motion is imposed suggests the use of that condition in other equations for interacting systems of gravity, gauge fields, and sources. Under current investigation are the Einstein-Maxwell Dirac[27] and Einstein-Yang-Mills-Higgs[28,29] equations.

Finally, Appendix B suggests treating the stationary gravitational vacuum as a gauge theory. Under investigation by the author are questions of topology and quantization in this formalism.

2. THE WORLD-BUNDLE

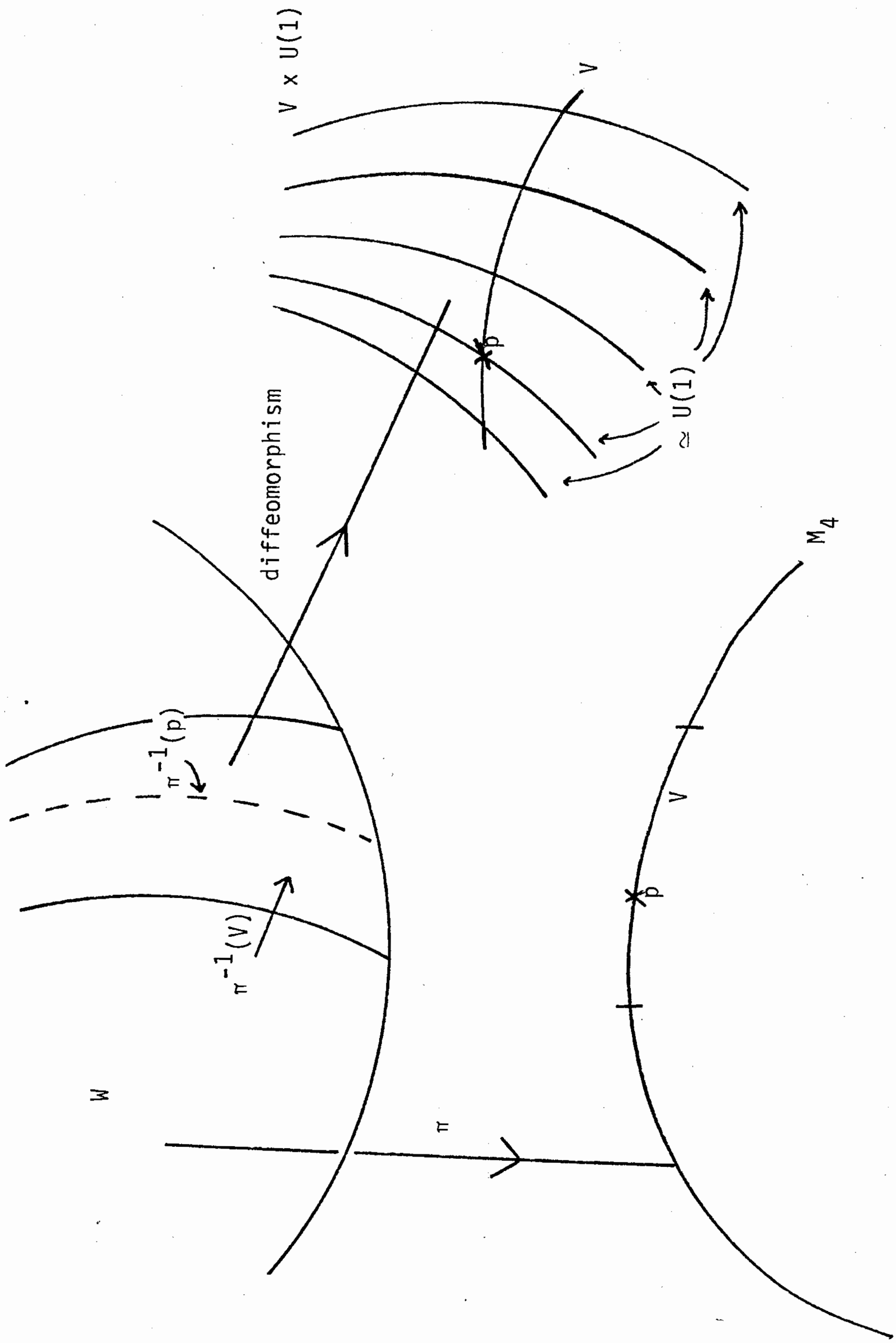
A. The World-Bundle

A natural geometrical setting for the physical theory described in this thesis is the world-bundle W. The latter is a principal fiber bundle with a connection. The pertinent properties of principal fiber bundles are summarized in Appendix A. More detailed treatments can be found in Kobayashi and Nomizu[30], Drechsler and Mayer[31], and Daniel and Viallet[32]. In spelling out the properties of W, most of the notation and conventions used subsequently will be displayed.

The principal fiber bundle structure of W is denoted by $W(M_4, U(1), \pi)$. Here W is a C^∞ manifold (the <u>bundle space</u>), M_4 (the <u>base manifold</u>) is a four-dimensional connected Hausdorff paracompact C^∞ manifold, U(1) (the <u>structure group</u>) is the one-parameter Abelian group of unitary transformations, and $\pi : W \to M_4$ is a surjective C^∞ map, called the <u>canonical projection</u>. Given $p \varepsilon M_4$, there is a neighborhood U_p of p such that $\pi^{-1}(U_p)$ is diffeomorphic to $U_p \times U(1)$. It is <u>not</u> assumed that W is trivial, i.e. that W is diffeomorphic to $M_4 \times U(1)$. See Figure 2.1.

The manifold M_4 is provided with an affine structure. In particular, it is assumed that M_4 has a semi-Riemannian structure. This means that there is a rank-two non-degenerate symmetric tensor field $\hat{g}$ on M_4 with signature -2. In the usual way[33] the metric $\hat{g}$ determines a torsion-free affine connection on M_4. On a coordinate patch (i.e. locally), where the metric has components $\hat{g}_{ij}$, the connection is determined by the <u>Christoffel symbols</u>

Figure 2.1 : The World-Bundle.



$$\Gamma^i_{jk} \equiv \frac{1}{2} \hat{g}^{im} (\hat{g}_{jm,k} + \hat{g}_{km,j} - \hat{g}_{jk,m}). \tag{2.1}$$

Lower case latin indices $i,j,\ldots,z \varepsilon \{1,2,3,4\}$ and the summation convention holds for repeated indices in a term. A comma denotes partial differentiation with respect to the coordinates, e.g. $\hat{g}_{ij,k} \equiv \frac{\partial}{\partial x^k} \hat{g}_{ij}$.

Covariant derivatives with respect to $\hat{g}$ are defined as usual, and locally are denoted by ∇_i. Thus, for example, if B_j is a covariant vector field on M_4, then

$$\nabla_i B_j = B_{j,i} - \Gamma^k_{ij} B_k.$$

The Riemann curvature tensor is defined so that its components in a chart, $\hat{R}^i{}_{jkm}$, are given by

$$\hat{R}^i{}_{jkm} \equiv \Gamma^i_{jm,k} - \Gamma^i_{jk,m} + \Gamma^r_{jm}\Gamma^i_{rk} - \Gamma^r_{jk}\Gamma^i_{rm}. \tag{2.2}$$

The Ricci tensor and curvature scalar are, respectively,

$$\hat{R}_{jk} \equiv \hat{R}^i{}_{jki}, \text{ and} \tag{2.3a}$$

$$\hat{R} \equiv \hat{g}^{jk} \hat{R}_{jk}. \tag{2.3b}$$

These tensors satisfy the usual algebraic identities:

$$\hat{R}_{ijkm} + \hat{R}_{imjk} + \hat{R}_{ikmj} = 0,$$

$$\hat{R}_{ijkm} = -\hat{R}_{ijmk} = -\hat{R}_{jikm} = \hat{R}_{kmij}, \tag{2.4}$$

$$\hat{R}_{ij} = \hat{R}_{ji} .$$

A number of differential identities are also satisfied by the Riemann tensor and its contractions, the most important of which are the Bianchi identities:

$$\nabla_l \hat{R}_{ijkm} + \nabla_m \hat{R}_{ijlk} + \nabla_k \hat{R}_{ijml} = 0. \tag{2.5}$$

This in turn implies the contracted Bianchi identities:

$$\nabla^k \hat{G}_{jk} = 0, \tag{2.6}$$

where $\hat{G}_{jk} \equiv \hat{R}_{jk} - \frac{1}{2}\hat{g}_{jk}\hat{R}$ is the Einstein tensor.

The semi-Riemannian affine structure on M_4 can be described in a coordinate-independent way in terms of differential forms[33]. This is accomplished by introducing the orthonormal family of 1-form fields $\omega^a = \omega^a_i\, dx^i$, such that

$$\hat{g}^{ij}\omega^a_i\omega^b_j = \eta^{ab} = \eta_{ab} \equiv \text{diagonal } (-1, -1, -1, 1). \tag{2.7}$$

Lower case latin indices $a,b,\ldots,h\varepsilon\ \{1,2,3,4\}$ and denote the tetrad (or vierbein or invariant) components. The affine connection on M_4 is determined by the connection 1-forms $\omega^a{}_b \equiv \omega_i{}^a{}_b\, dx^i$. These quantities are related to the Γ^i_{jk} by

$$\omega^a{}_b = \Gamma^r_{sk}\, \omega^a{}_r\, e^s{}_b\, dx^k, \tag{2.8}$$

where $e^s{}_b = \hat{g}^{st} \eta_{bc} \omega^c{}_t$ are the components of the orthonormal basis of tangent vectors dual to the $\omega^a{}_i$.

The curvature 2-forms $\Omega^a{}_b$ are defined as

$$\Omega^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b, \tag{2.9}$$

and are related to the curvature tensor by

$$\Omega^a{}_b = \frac{1}{2} \hat{R}^a{}_{bkm} dx^k \wedge dx^m, \tag{2.10}$$

where $\hat{R}^a{}_{bkm} \equiv \hat{R}^i{}_{jkm}\, \omega^a{}_i e^j{}_b$. The torsion-free property of the connection and the Bianchi identities are equivalent to, respectively:

$$d\omega^a + \omega^a{}_b \wedge \omega^b = 0, \text{ and} \tag{2.11}$$

$$d\Omega^a{}_b + \omega^a{}_c \wedge \Omega^c{}_b - \Omega^a{}_c \wedge \omega^c{}_b = 0 . \tag{2.12}$$

Equations (2.9), (2.11), and (2.12) are the Cartan structural equations for a connection on the bundle of orthonormal frames $O(M_4)$ over M_4. This is a principal fiber bundle over M_4 with structure group L_+^+, the proper orthochronous Lorentz group. The bundle space of $O(M_4)$ consists of all bases (frames) of the tangent spaces T_pM_4 at all $p \in M_4$[34]. The covariant derivative with respect to $\hat{g}$ acts upon all smooth (i.e., C^∞) sections of the vector bundles associated with $O(M_4)$ under the various representations of the Lie group L_+^+, i.e. smooth tensor fields. The $\omega^a{}_b$ and the $\Omega^a{}_b$ may be interpreted as the matrix elements of the local expressions for the connection 1-form and curvature 2-form fields, respectively, of a connection on the principal fiber bundle $O(M_4)$.

The world-bundle W is also endowed with a connection, denoted here by A. Thus (see Appendix A) A is a 1-form field over W, and it determines a unique smooth decomposition of T_wW into horizontal and vertical subspaces for each $w \in W$. If $U \subset M_4$ is the domain of a chart on M_4 and $S: U \to W$ is a local smooth section, then the local expressions for A are

$$\tilde{A}_j \equiv A(S_* \partial_j) = -ieA_j \mu , \tag{2.13}$$

where μ is a basis of $u(1)$, the Lie algebra of U(1). The factor i is introduced in order to make A_j real-valued ($u(1)$ is anti-hermitian). The constant factor (-e) is chosen for later convenience.

The curvature 2-form field F is

$$F = DA, \tag{2.14}$$

where D is the exterior covariant derivative. Since U(1) is Abelian, it follows that

$$\mathcal{F} = dA \tag{2.15}$$

The local expressions for $\mathcal{F}$ are of the form

$$\tilde{F}_{ij} \equiv -ieF_{ij}\mu = -ie(A_{i,j}-A_{j,i})\,\mu\,, \tag{2.16}$$

i.e.

$$F_{ij} = A_{i,j} - A_{j,i}\ . \tag{2.17}$$

The curvature 2-form $\mathcal{F}$ satisfies the Bianchi identity

$$D\mathcal{F} = d\mathcal{F} = 0\ , \tag{2.18}$$

which, locally, is equivalent to

$$F_{ij,k} + F_{ki,j} + F_{jk,i} = 0. \tag{2.19}$$

A cautionary note is in order here. A common practice is to essentially drop the distinction between the globally-defined forms A and $\mathcal{F}$ on W and the forms $A \equiv A_i dx^i$ and $F \equiv -\frac{1}{2}F_{ij}dx^i \wedge dx^j$. To do so implies that there is a global smooth section on M_4, and hence that W is trivial, i.e. isomorphic to $M_4 \times U(1)$. In order to keep open the possibility of the non-triviality of W, the objects A and F above will be assumed to be defined with respect to a local smooth section.

The <u>fundamental representation</u> of U(1) is that carried by C = the complex plane. If $\alpha \varepsilon C$, then the action (left or right) of U(1) on α is just of the form $e^{i\theta}\alpha$, where $\theta \varepsilon R$. Denote the vector bundle associated to W by the fundamental representation by $V(M_4, C, \pi_v)$. Thus the fibers of V are isomorphic to C.

Let σ: $U \to C$ be a local smooth section of V. The set $\{ \sigma(m) | m \varepsilon U\}$ is called a <u>moving frame</u> over U. If $|\sigma(m)| \equiv \sqrt{\sigma(m)^* \sigma(m)} = 1$ for every $m \varepsilon U (\sigma(m)^*$ is the complex conjugate of $\sigma(m))$, then the moving frame can also be interpreted as a local section of W. In this case σ is called a <u>local gauge</u>.

A <u>gauge transformation</u> on W is a smooth map $v: U \to U(1)$. Gauge transformations act upon moving frames of vector bundles associated with representations of U(1). Let σ be a local gauge, and let A_i be the components of the local expression for the connection A with respect to σ. By equation (A.10), the exterior covariant derivative of σ is

$$D\sigma = ieA_j dx^j \otimes \sigma \tag{2.20}$$

The fundamental representation of the gauge transformation v is of the form e^{ieu}, where $u: U \to R$ is smooth. Thus σ is transformed to a new gauge σ' by

$$\sigma' = e^{ieu}\,\sigma . \tag{2.21}$$

Since the exterior covariant derivative is independent of the gauge, it follows that

$$D\sigma' = ieA'_j\, dx^j \otimes \sigma'.$$

From the fact that D is a derivation (see equation A.12 of Appendix A), and from (2.21), it follows that A'_j, the local expressions for A in the transformed gauge σ', are related to A_j by the formula

$$A'_j = A_j + u_{,j}. \tag{2.22}$$

Let $\hat{\psi}$ be a smooth section of the vector bundle associated with the fundamental representation of U(1), and denote its component with respect to the gauge σ by ψ, so that $\hat{\psi} = \psi\sigma$. The component of $\hat{\psi}$ with respect to σ' is then given by

$$\psi' = e^{-ieu}\psi. \tag{2.23}$$

The geometrical setting just sketched is now given the following physical interpretation. The base manifold M_4 is spacetime. The semi-Riemannian structure on M_4 determines the gravitational field. The connection on W determines the electromagnetic field. In particular, the real-valued tensor field F_{ij} is the electromagnetic field strength tensor field and A_j is the four-potential. Matter fields on spacetime are smooth sections of vector bundles associated with representations of U(1). In particular, smooth sections of the vector bundle associated with the fundamental representation of U(1) are the wave-functions of mesons, i.e. charged massive spin-zero bosons. Physically measureable quantities are required to be independent of the choice of gauge and spacetime coordinates.

For example, the field strength F_{ij} and the probability density of a meson, namely, $|\psi|^2 = \psi^*\psi$, are gauge-invariant.

Alternatively the world-bundle W has the structure of a five-dimensional semi-Riemannian manifold with a one-parameter isometry. The group which generates the isometry is U(1). This viewpoint is, of course, that of the Kaluza-Klein "unified" field theory[35,36], as updated and generalized by Kerner[37], Trautman[38], and Cho[39].

B. The Field Equations

If the world-bundle is to be a model for classical, i.e. non-second-quantized, interacting gravitational, electromagnetic, and mesonic fields, then the so far unprescribed fields $\hat{g}_{ij}$, A_j, and ψ must be governed by appropriate dynamical laws. It is postulated here that these dynamical laws are derivable from an action principle. Hence, it is assumed that there is a Lagrangian density $L(\psi; A_j; \hat{g}_{ij})$ depending on the fields and their derivatives such that when the action functional $I(\psi; A_j; \hat{g}_{ij}) \equiv \int_U L(\psi; A_j; \hat{g}_{ij})\ (-\hat{g})^{1/2} dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4$ is extremized, the consequent Euler-Lagrange equations are the dynamical laws. In the above expression, U is an open connected compact region of M_4, $\hat{g}$ is the determinant of the $\hat{g}_{ij}$, and the variations of the fields on the boundary ∂U are prescribed.

The criteria for choosing the form of L are the following:

(i) L is gauge and coordinate transformation invariant;

(ii) If the connections on $O(M_4)$ and W are both <u>flat</u>, i.e., if the respective curvature 2-forms $\Omega^a{}_b$ and F vanish, then L reduces to the usual free Lagrangian density $L_0(\psi) \equiv \eta^{ij}\partial_i\psi^*\partial_j\psi + m^2\psi^*\psi$ for a meson with rest mass m in Minkowski spacetime;

(iii) L should be the "simplest" Lagrangian density consistent with the previous criteria.

The first two criteria can be justified on physical grounds, while the third is invoked for primarily aesthetic reasons. The following is a short list of physically not untenable possibilities that are ruled out by the above:

1. Higher order self-interactions of the meson fields, e.g., terms in the Lagrangian density of the form $b(\psi^*\psi)^2$ violate (ii). Such terms may be useful in accounting at least phenomenologically for the interaction of the meson with the strong and/or weak forces.
2. The likelihood that existing mesons, e.g. pions, are not "fundamental", but rather are composite particles[2] cannot be dealt with in the formalism being used here.
3. The intimate connection between the electromagnetic and weak interactions as postulated in the Weinberg-Salaam theory[40] and which is on the verge on experimental verification cannot be accounted for within the model being developed here.
4. The possibility that the affine structure of spacetime has torsion[41] has been explicitely ruled out here.
5. If there is supersymmetry in the universe[42], then certain fermion fields such as the spin 3/2 "gravitino" field[43] would have to be included.
6. One interpretation of the Bohm-Aharonov effect is that the interaction of matter fields with the electromagnetic field may be gauge-dependent[44].

All of the above caveats suggest the limitations of the model which is being developed here. However all of the above, with the possible exception of 6, could be embraced within geometrical settings similar in principle to the world-bundle W. For example, possibility 3 could be developed in terms of a world-bundle with structure group SU(2) x U(1)[40].

We shall begin by considering the Lagrangian density L_0 for a free scalar field in Minkowski spacetime. L_0 can be made gauge and coordinate transformation invariant by making the following substitutions:

$$\eta^{ij} \to \hat{g}^{ij},$$

$$\frac{\partial}{\partial x^j} \to D_j \equiv \nabla_j + ieA_j,$$

so that the Lagrangian density becomes

$$L_{KG} \equiv \hat{g}^{ij} D_i^* \psi^* D_j \psi - m^2 \psi^* \psi. \qquad (2.24)$$

The constant e is to be interpreted as the charge of the meson.

The "kinetic" or "free field" terms in the Lagrangian density are now assumed to be

$$L_{EM} \equiv \hat{R} - \frac{1}{2} k F^{ij} F_{ij}, \qquad (2.25)$$

where $\hat{R}$ is the scalar curvature and k is a coupling constant. The complete Lagrangian density is

$$L = L_{EM} + 2k \, L_{KG}. \qquad (2.26)$$

This choice for L is governed by the criterion of simplicity, as L_{EM} is the simplest non-trivial scalar which can be constructed from the partial derivatives of $\hat{g}_{ij}$ and A_j. The units used here are "natural gaussian" if $k = 8\pi$. Hence $c = \hbar = G$ (the Newtonian gravitational constant) = 1.

The field equations are obtained by extremizing the action I (defined on p. 18) under independent variations of the fields ψ, ψ^*, A_j, and $\hat{g}^{ij}$, subject to the "boundary conditions"

$$\delta\psi = \delta\psi^* = \delta A_j = 0,$$

$$\hat{g}^{ms}(\delta\Gamma^r_{mr}) - \hat{g}^{mn}(\delta\Gamma^s_{mn}) = 0,$$

on ∂U. Hence the field equations are:

$$K \equiv (D^j D_j + m^2)\psi = 0, \tag{2.27a}$$

$$K^* \equiv (D^{*j} D^*_j + m^2)\,\psi^* = 0, \tag{2.27b}$$

$$M^i \equiv \nabla_j F^{ij} - J^i = 0, \tag{2.27c}$$

$$S_{ij} \equiv \hat{G}_{ij} + k\, T_{ij} = 0, \tag{2.27d}$$

where

$$J^i \equiv -ie\left[(D^{*i}\psi^*)\psi - \psi^* D^i \psi\right], \tag{2.28}$$

and

$$T_{ij} \equiv M_{ij} + E_{ij}, \tag{2.29}$$

with

$$M_{ij} \equiv 2\, D^*_{(i}\psi^*\, D_{j)}\psi - \hat{g}_{ij} \left[(D^{*l}\psi^*)\,(D_l\psi) - m^2\psi^*\psi \right], \tag{2.30}$$

$$E_{ij} \equiv -F_{ik}\, F_j{}^k + \frac{1}{4}\, \hat{g}_{ij}\, F_{kl}\, F^{kl}. \tag{2.31}$$

The differential identities satisfied by the fields are

$$\nabla_j J^j = 0, \tag{2.32}$$

$$\nabla_j T^{ij} = 0. \tag{2.33}$$

The field equations (2.27) are known as the Einstein-Maxwell-Klein-Gordon (EMKG) equations.

The following counting argument shows that in general the EMKG equations are determinate. One may impose four coordinate conditions ($C^i = 0$) and one gauge condition ($\wedge = 0$) on the fields ψ, ψ^*, A_j, and $\hat{g}_{ij}$. Hence the number of independent fields is 11 = 10 ($\hat{g}_{ij}$) + 4 (A_j) + + 1 (ψ) + 1 (ψ^*) - 4 ($C^i = 0$) - 1 ($\wedge = 0$). The number of independent field equations is 11 = 10 ($S_{ij} = 0$) + 4 ($M^i = 0$) + 1 ($K = 0$) + + 1 ($K^* = 0$) - 1($\nabla_j J^j = 0$) - 4($\nabla_j T^{ij} = 0$). Hence, the EMKG equations are determinate.

C. The Real Gauge

In the following, a choice of gauge will be imposed on the EMKG equations. This gauge, called the real gauge, will simplify subsequent computations, and will facilitate the comparison of the EMKG equations with related field equations involving gravity.

Let ψ be a smooth meson field, which is non-vanishing on some $U \subset M_4$. Then by defining the smooth functions

$$\eta \equiv (\psi^* \psi)^{\frac{1}{2}}, \tag{2.34}$$

$$\alpha \equiv \frac{-1}{2ie} \ln (\psi^*/\psi), \tag{2.35}$$

so that

$$\psi = \eta e^{ie\alpha}, \tag{2.36}$$

it is seen that η can be interpreted as the real-valued component of the meson field with respect to the gauge $e^{ie\alpha}$. This choice of gauge is called, for obvious reasons, the real gauge. Clearly the constant function $\varepsilon = 1$ over C is also a gauge, and shall be dubbed the canonical gauge. The component of the meson field in the canonical gauge is ψ itself. Let the components of the connection 1-form on W be $\overline{A}_j$ in the canonical gauge. Then the transformation from the canonical to the real gauge $e^{ie\alpha} = e^{ie\alpha}\varepsilon$ has the effect

$$\eta = e^{-ie\alpha}\psi, \tag{2.37}$$

$$A_j = \overline{A}_j + \alpha,_j, \tag{2.38}$$

by equations (2.22) and (2.23). Note that A_j are the components of the connection in the real gauge. Now in the canonical gauge, and by use of (2.36), the quantities $\overline{J}_i$, $\overline{M}_{ij}$, and the Klein-Gordon equation (2.27a) become, respectively,

$$\overline{J}_i = -2e^2(\overline{A}_i + \alpha,_i)\eta^2, \tag{2.39}$$

$$\overline{M}_{ij} = 2\eta,_i\eta,_j + 2e^2(\overline{A}_i + \alpha,_i)(\overline{A}_j + \alpha,_j)\eta^2 - \hat{g}_{ij}\Big[\hat{g}^{rs}\eta,_r\eta,_s + + e^2\hat{g}^{rs}(\overline{A}_r + \alpha,_r)(\overline{A}_s + \alpha,_s)\eta^2 - m^2\eta^2\Big], \tag{2.40}$$

$$K \equiv \Big\{\nabla^r\nabla_r\eta - e^2\hat{g}^{rs}(\overline{A}_r + \alpha,_r)(\overline{A}_s + \alpha,_s) + m^2\eta + + ie\eta^{-1}\nabla^r\Big[(\overline{A}_r + \alpha,_r)\eta^2\Big]\Big\}e^{ie\alpha} = 0. \tag{2.41}$$

The Klein-Gordon equation for ψ^* is just the complex conjugate of the last equation above. If $\eta^{-1} \neq 0$, then (2.41) and its complex conjugate hold if and only if the real and imaginary parts of $Ke^{-ie\alpha}$ vanish separately. Hence, in the real gauge, the EMKG equations reduce to:

$$\sigma_{ij} \equiv \hat{G}_{ij} + kT_{ij} = 0, \tag{2.42a}$$

$$m^i \equiv \nabla_j F^{ij} - J^i = 0, \tag{2.42b}$$

$$K_r \equiv \nabla^j\nabla_j\eta + (m^2 - e^2A^jA_j)\eta = 0, \tag{2.42c}$$

$$K_i \equiv \nabla^j(A_j\eta^2) = 0, \tag{2.42d}$$

where $T_{ij} \equiv M_{ij} + E_{ij}$ with

$$M_{ij} \equiv 2\eta,_i\eta,_j + 2e^2A_iA_j\eta^2 - \hat{g}_{ij}\left[\hat{g}^{rs}\eta,_r\eta,_s - (m^2 - e^2A^rA_r)\eta^2\right], \tag{2.43}$$

and

$$J^i \equiv -2e^2A^i\eta^2 . \tag{2.44}$$

Note that the equation $K_i = 0$ is equivalent to the differential identity $\nabla_i J^i = 0$, so that when solving the EMKG equations in the real gauge only equations (2.42a) - (2.42c) need be considered directly.

This section will be concluded by showing that if certain constraints are imposed on the solutions of the EMKG equations, then the resulting equations formally resemble other well-known classical field equations involving gravity. These results will be stated as theorems.

Theorem 2.1. The class of solutions of (2.42a) - (2.42c) for which $\eta \equiv$ a constant $\not\equiv 0$ satisfy the field equations

$$^{(P)}\sigma_{ij} \equiv \hat{G}_{ij} + k(-F_{il}F_j{}^l + \frac{1}{4}\hat{g}_{ij}F^{rs}F_{rs} + M^2A_iA_j) = 0, \tag{2.45}$$

$$^{(P)}m^i \equiv \nabla_j F^{ij} + M^2A^i = 0, \tag{2.46}$$

$$^{(P)}Q \equiv m^2 - e^2A^jA_j = 0, \tag{2.47}$$

where $M^2 \equiv 2e^2\eta^2$ = a constant. The differential identity $\nabla_i J^i = 0$ becomes

$$\nabla_j A^j = 0. \tag{2.48}$$

Proof: The constraint $\eta \equiv$ a constant reduces (2.42c) to (2.47). Substitute the constraint and (2.47) into (2.42a), (2.42b), and (2.42d) to get the desired result.//

The equations (2.45), (2.46), and (2.48) are the Einstein-Proca equations[45] for a massive vector meson interacting with the gravitational field. Equation (2.47) is a subsidiary condition and implies that the vector meson wave-function A_i is a timelike vector field. The theorem is reminiscent of the so-called Higgs mechanism[46], since a certain configuration of a massive scalar field effectively "gives mass" to the photon field, but otherwise plays no dynamical role. In contrast to the Higgs mechanism, where the mass of the original scalar meson is imaginary and there is a quartic self-interaction, the scalar meson here has real mass and obeys the linear Klein-Gordon equation.

Theorem 2.2. Let the conditions of the previous theorem hold. Define the vector field $v_i \equiv (e/m)A_i$ and the constants $\rho \equiv 2m^2\eta^2$ and $\sigma \equiv -2em\eta^2$. Then the equations (2.42a) - (2.42d) reduce to

$$^{(D)}\sigma_{ij} \equiv \hat{G}_{ij} + k(E_{ij} + \rho v_i v_j) = 0,$$

$$^{(D)}m^i \equiv \nabla_j F^{ij} - \sigma v^i = 0,$$

$$^{(D)}\gamma \equiv \nabla_j(\rho v^j) = 0,$$

$$^{(D)}Q \equiv v^i v_i - 1 = 0.$$

Proof: The proof is as in the previous theorem. //

The above are the equations for a charged dust in general relativity[47]. The dust is characterized by constant mass and charge densities and the velocity vector field is parallel to the electromagnetic four-potential.

Theorem 2.3. If $m^2 - e^2A^jA_j = 0$ and if there is a real-valued scalar field θ such that $\eta,_i = \theta A_i$, then equations (2.42a) - (2.42d) can be written in the form:

$$^{(F)}\sigma_{ij} \equiv \hat{G}_{ij} + k\left[E_{ij} + (\rho + p)v_i v_j - \hat{g}_{ij}\, p\right] = 0,$$

$$^{(F)}m^i \equiv \nabla_j F^{ij} - \sigma v^i = 0,$$

$$^{(F)}Y \equiv \nabla_j(\rho v^j) = 0,$$

$$^{(F)}Q \equiv v^j v_j - 2 = 0,$$

where $\rho \equiv m^2\eta^2$, $p \equiv \left[(m/e)\ \theta\right]^2$, $\sigma \equiv -2^{\frac{1}{2}}\ (m/e)\eta^2$, and $v_i \equiv 2^{\frac{1}{2}}(e/m)A_i =$ $= 2^{\frac{1}{2}}(e/m)\ \theta^{-1}\eta,_i$.

Proof: Substitute the conditions $m^2 - e^2A^jA_j = 0$ and $\eta,_j = \theta A_j$ into (2.42a) - (2.42d) to get the desired result.//

The above are the equations for a perfect fluid in general relativity[47]. Here the ratio of charge density σ to mass density ρ is a constant = $-2^{\frac{1}{2}}(em)^{-1}$. The fluid velocity is parallel to the electromagnetic four-potential. Also the pressure p and the mass density satisfy an equation of state of the form

$$\hat{g}^{ij}\rho,_{i}\rho,_{j} = 4\rho p.$$

If $\theta,_{i}$ is parallel to $\eta,_{i}$ (which includes the case θ = constant), then $F_{ij} = 0$.

3. THE STATIONARY CASE

In this section it will be assumed that the metric $\hat{g}$ is stationary. It will be demonstrated that under certain conditions the EMKG equations together with the stationarity of the metric imply the stationarity of the electromagnetic and meson fields. The metric will be put into the usual 3+1 form and tensor analysis on an associated 3-manifold M_3 will be developed. The components of the curvature tensor will be written in terms of tensor fields on M_3. Finally, the static case will be briefly reviewed.

A. Stationary Metrics

Let x^i be coordinates on $U \subset M_4$. Then the dx^i are a basis of 1-form fields dual to the coordinate basis $\partial_i \equiv \frac{\partial}{\partial x^i}$. The metric $\hat{g}$ can be written as

$$\hat{g} = \hat{g}_{ij} dx^i dx^j, \tag{3.1}$$

where $dx^i dx^j$ is shorthand for $1/2(dx^i \otimes dx^j + dx^j \otimes dx^i)$. The metric $\hat{g}$ is said to be stationary on M_4 if there is a smooth time-like vector field $\vec{\xi}$ over M_4 which generates a one-parameter group of isometries[48]. This means that $\vec{\xi}$ is a Killing vector field, i.e., that $\vec{\xi}$ satisfies the Killing equation

$$\underset{\xi}{\pounds}\, \hat{g} = 0, \tag{3.2}$$

where $\underset{\xi}{\pounds}$ is the Lie derivative[48] with respect to $\vec{\xi}$. The local form of (3.2) is

$$\underset{\xi}{\pounds}\, \hat{g}_{ij} \equiv \nabla_j \xi_i + \nabla_i \xi_j = 0. \tag{3.3}$$

Consider the class of observers whose world-lines are the trajectories (integral curves) of $\vec{\xi}$. Such observers will be called, following Trautman[48], Copernican. A Copernican chart (x^1, x^2, x^3, $x^4 \equiv t$) is characterized by $\partial_4 = \vec{\xi}$, i.e., the x^4-lines are the world-lines of a Copernican observer. Clearly, in such a chart, $\xi^i = \delta^i_4$.

The metric $\hat{g}$ is static if it is stationary and if the trajectories of $\vec{\xi}$ are orthogonal to a family of hypersurfaces. The latter condition is equivalent to[48]:

$$\xi_{[i} \nabla_j \xi_{k]} = 0. \tag{3.4}$$

The following results[48] are easily derived from (3.3) and (3.4): If $\hat{g}$ is stationary, then in a Copernican chart $\hat{g}_{ij,4} = 0$. If $\hat{g}$ is static as well, then in a Copernican chart, $\hat{g}_{\alpha 4} = 0$ for all $\alpha = 1,2,3$.

Trautman[48] has shown that neighboring Copernican observers in a stationary spacetime appear at rest in a Fermi-transported local frame if and only if the spacetime is static. In this sense, the physical distinction between stationary and static is that only in the latter neighboring Copernican observers will not rotate with respect to one another.

B. Stationarity of Electromagnetic and Meson Fields

It is usually assumed, for example in Das[22] or in Israel and Wilson[49], that non-gravitational fields in a stationary spacetime are also stationary. If Φ is an observable non-gravitational field, then Φ is stationary if

$\underset{\xi}{£}\Phi = 0$. If ψ is a wave-function, so not directly observable, then ψ is stationary if the Lie derivative of $|\psi|^2 \equiv \eta^2$, which is observable, is equal to zero. These latter conditions are not always necessary consequences of the field equations and the stationarity of $\hat{g}$, as has been demonstrated by Woolley[50] and Wainwright and Yaremovicz[51]. In this section it will be asked whether (3.3) and the EMKG equations imply

$$\underset{\xi}{£} F_{ij} = 0, \tag{3.5}$$

and

$$\underset{\xi}{£}|\psi| = \underset{\xi}{£}\eta = 0, \tag{3.6}$$

where F_{ij} is the electromagnetic field strength and ψ is the meson wave-function. Note that from (3.6) and from the fact that £ obeys the product rule, it follows that

$$\underset{\xi}{£}\psi = iE\psi, \tag{3.7}$$

where E is a real-valued function. The stationarity of ψ means that $|\psi|$ is constant along the trajectory of $\vec{\xi}$ through p, for each $p \varepsilon M_4$.

A necessary and sufficient condition for (3.5) is that there exists a smooth scalar field f such that

$$\underset{\xi}{£}A_i = f_{,i}. \tag{3.8}$$

This follows from (2.17) and the fact[51] that for any tensor field $T^{i\cdots}_{j\cdots}$:

$$\underset{\xi}{\pounds}\left(\nabla_k T^{i\cdots}_{j\cdots}\right) = \nabla_k\left(\underset{\xi}{\pounds} T^{i\cdots}_{j\cdots}\right). \qquad (3.9)$$

Now perform the gauge transformation

$$\overline{A}_i = A_i + \lambda,_i \ , \qquad (3.10a)$$

$$\overline{\psi} = e^{-ie\lambda}\ \psi \ , \qquad (3.10b)$$

where λ is chosen so that $\underset{\xi}{\pounds}\ \lambda = -f$. This choice for λ is possible because of the required smoothness of f. Now

$$\underset{\xi}{\pounds}\ \overline{A}_i = \underset{\xi}{\pounds}\ A_i + \underset{\xi}{\pounds}\ \lambda,_i$$

$$= 0,$$

and

$$\underset{\xi}{\pounds}\ (\overline{\psi}{*}\overline{\psi}) = \underset{\xi}{\pounds}\ (\psi{*}\psi) = 0.$$

Hence, without loss of generality, we may replace (3.5) by

$$\underset{\xi}{\pounds}\ A_i = 0. \qquad (3.5')$$

Wainwright and Yaremovicz[51] proved the elegant and general result that if $\hat{g}$ is homothetic, i.e., that $\underset{\xi}{\pounds}\, \hat{g}_{ij} = 2\gamma \hat{g}_{ij}$, where γ is a constant, and if F_{ij} is non-null ($F_{ij}F^{ij} \neq 0$), then the equations for a charged perfect fluid in general relativity imply that

$$\underset{\xi}{\pounds}\, v_i = \gamma\, v_i,\ \underset{\xi}{\pounds}\, \rho = -2\gamma\rho,\ \underset{\xi}{\pounds}\, p = -2\gamma p, \text{ and}$$

$$\underset{\xi}{\pounds}\, F_{ij} = \gamma F_{ij} + \tilde{\gamma}\, {}^*F_{ij},$$

where v_i, ρ, and p are as defined in section 2.C, ${}^*F_{ij}$ is the Hodge dual of F_{ij} (see section 4.A) and $\tilde{\gamma}$ is related to the complexion[52] α of F_{ij} by $\tilde{\gamma} \equiv \underset{\xi}{\pounds}\, \alpha$.

An analogous, but alas weaker, result for the EMKG equations will now be demonstrated:

Theorem 3.1. Let $\vec{\xi}$ (which is not necessarily timelike) generate an isometry of $\hat{g}$, so that $\underset{\xi}{\pounds}\, \hat{g}_{ij} = 0$. Further, suppose that the EMKG equations (2.42) hold, and that either $\underset{\xi}{\pounds}\, M_{ij} = 0$ or $\underset{\xi}{\pounds}\, E_{ij} = 0$. Then $\underset{\xi}{\pounds}\, \eta = 0$ and $\underset{\xi}{\pounds}\, A_i = 0$.

The proof of the theorem depends on the following two lemmas:

Lemma 3.1[51]. Let A_{ij} be a symmetric tensor field with $\underset{\xi}{\pounds}\, A_{ij} = 0$. If λ is an eigenvalue of A_{ij} with a non-null eigenvector, then $\underset{\xi}{\pounds}\, \lambda = 0$.

Proof: Let u^j be a non-null eigenvector of A_{ij} with eigenvalue λ. Take the Lie derivative of both sides of $A_{ij}u^j = \lambda \hat{g}_{ij}u^j$ to get the desired result. //

Lemma 3.2. In the real gauge, the eigenvalues of

$$M_{ij} \equiv 2(\eta,_i\, \eta,_j + e^2 A_i A_j \eta^2) + \hat{g}_{ij}\, [m^2\eta^2 - \hat{g}^{rs}(\eta,_r\, \eta,_s + e^2 A_r A_s \eta^2)]$$

$$\equiv 2Q_{ij} + \hat{g}_{ij}(m^2\eta^2 - Q),$$

where $Q_{ij} \equiv \eta,_i \eta,_j + e^2A_iA_j\eta^2$ and $Q \equiv \hat{g}^{ij} Q_{ij}$, are

$$\lambda_o = m^2\eta^2 - Q, \text{ and} \tag{3.11a}$$

$$\lambda_\pm = m^2\eta^2 \pm \sqrt{2Q_{ij}Q^{ij} + Q^2}\,. \tag{3.11b}$$

Furthermore, the eigenvectors at $p \varepsilon M_4$ corresponding to $\lambda_o(p)$ generate a subspace of T_pM_4 which contains at least one non-null vector.

Proof: Consider the secular equations for M_{ij}:

$$0 = \det(M_{ij} - \lambda\hat{g}_{ij}) = \det(2Q_{ij} - 2\Lambda\,\hat{g}_{ij}),$$

where $\Lambda \equiv Q - m^2\eta^2 + \lambda$. So we solve the secular equation $\det(Q_{ij} - \Lambda\hat{g}_{ij}) = 0$ for Λ and then find the eigenvalues λ of M_{ij} from $\lambda = \Lambda + m^2\eta^2 - Q$. A straightforward but tedious calculation yields the desired eigenvalues λ_o and $\lambda_\pm$.

Since $Q - m^2\eta^2 + \lambda_o = 0$, it follows that

$$(M_{ij} - \lambda_o\hat{g}_{ij}) = [2Q_{ij} - \hat{g}_{ij}(Q - m^2\eta^2 + \lambda_o)] = 2Q_{ij}.$$

Thus, if U^j is an eigenvector of M_{ij} with eigenvalue λ_o, then

$$Q_{ij}\, U^j = 0, \tag{3.12}$$

and hence $Q_{ij}U^iU^j = 0$, i.e., $(\eta,_iU^i)^2 + (e\eta AiU^i)^2 = 0$.

A necessary and sufficient condition, then, for (3.12) is

$$\eta,_i U^i = 0, \tag{3.13a}$$

$$A_i U^i = 0. \tag{3.13b}$$

If $\eta,_i$ and A_i are non-collinear, then the set of U^i satisfying the above equations generates a two-dimensional subspace of T_pM_4. Let r^i and s^i span this subspace. If one or both of r^i and s^i are non-null, then we are done. If they are both null, then the vector $t^i \equiv r^i + s^i$ is non-null since $\hat{g}_{ij}t^it^j = 2\hat{g}_{ij}r^is^j$ is non-zero by the fact that two null vectors are orthogonal if and only if they are collinear. If $\eta,_i$ and A_i are collinear, then the solutions of (3.13) span a three-dimensional subspace, and this subspace obviously contains a non-null vector. //

<u>Proof of Theorem 3.1</u>. From $\underset{\xi}{\mathcal{L}}\, \hat{g}_{ij} = 0$, it follows that $\underset{\xi}{\mathcal{L}}\, \hat{G}_{ij} = 0$[53]. Hence from $\hat{G}_{ij} + kT_{ij} = 0$ one has $\underset{\xi}{\mathcal{L}}\, T_{ij} = 0$. Thus, by hypothesis, $\underset{\xi}{\mathcal{L}}\, M_{ij} = 0$. Now, since M_{ij} can be written as

$$M_{ij} = 2Q_{ij} - \lambda_o \hat{g}_{ij} \quad ,$$

it follows that

$$0 = \underset{\xi}{\mathcal{L}}\, M_{ij} = 2\, \underset{\xi}{\mathcal{L}}\, Q_{ij} - \hat{g}_{ij}\, \underset{\xi}{\mathcal{L}}\, \lambda_o.$$

By Lemma 3.2, λ_o has a non-null eigenvector, and so by Lemma 3.1,

$\mathop{\pounds}_{\xi} \lambda_o = 0$. Thus

$$\mathop{\pounds}_{\xi} Q_{ij} = 0, \tag{3.14}$$

and since $\mathop{\pounds}_{\xi} Q = \hat{g}^{ij} \mathop{\pounds}_{\xi} Q_{ij}$,

it follows that $\mathop{\pounds}_{\xi} Q = 0$. But

$$0 = \mathop{\pounds}_{\xi} \lambda_o = \mathop{\pounds}_{\xi} (m^2\eta^2 - Q) = 2m^2\eta \mathop{\pounds}_{\xi} \eta - \mathop{\pounds}_{\xi} Q = 2m^2\eta \mathop{\pounds}_{\xi} \eta ,$$

and so

$$\mathop{\pounds}_{\xi} \eta = 0.$$

Finally, the last equation implies that $\mathop{\pounds}_{\xi} \eta_{,i} = (\mathop{\pounds}_{\xi} \eta)_{,i} = 0$, so that from (3.14) it follows that $\mathop{\pounds}_{\xi} (A_iA_j) = 0$. If each component $A_i \equiv 0$, then $\mathop{\pounds}_{\xi} A_i = 0$ follows trivially. If one component, say A_1, is non-zero, then $\mathop{\pounds}_{\xi}(A_1A_1) = 0$ implies $\mathop{\pounds}_{\xi} A_1 = 0$. Then from $\mathop{\pounds}_{\xi} (A_1A_j) = 0$, it follows that $\mathop{\pounds}_{\xi} A_j = 0$ for $j = 1,2,3,4$. //

C. The 3 + 1 Decomposition of the Metric

In a Copernican chart, the timelike Killing vector $\vec{\xi}$ satisfies

$$\xi^i = \delta^i_4 . \tag{3.15}$$

Hence, in a Copernican chart with coordinates (x^α, t), as we have seen above, the $\hat{g}_{ij}$ are functions of the x^α only. Henceforth, lower case greek indices are elements of {1,2,3}. Without loss of generality, the metric $\hat{g}$ can be written in a Copernican chart as follows[49]:

$$\hat{g} = -e^{-\omega(x)} g_{\alpha\beta}(x) dx^\alpha dx^\beta + e^{\omega(x)} \left[a_\alpha(x) dx^\alpha + dt \right]^2 . \tag{3.16}$$

The functions ω, $g_{\alpha\beta}$, and a_α depend on the x^α only, and the symbol $(x) \equiv (x^\alpha)$. The above shall be called the <u>Copernican form</u> of the metric and it is clearly preserved by the following coordinate transformations on $M_4{}^6$,

$$x'^\alpha = f^\alpha(x), \tag{3.17a}$$

$$t' = t + \lambda(x), \tag{3.17b}$$

where f^α and λ are arbitrary smooth functions of the x^α, if and only if the a_α transform as

$$a'_\alpha(x') = \frac{\partial x^\beta}{\partial x'^\alpha} \; a_\beta(x) - \lambda,_\alpha(x) . \tag{3.18}$$

Note the resemblance of the last equation to a gauge transformation of the electromagnetic four-potential A_i. In fact, the choice of a_α for a particular Copernican chart is only determined up to the addition of a gradient of an arbitrary smooth function.

The above considerations motivate the following reinterpretation of a stationary spacetime M_4: A stationary spacetime is a principal fiber bundle $M_4(M_3, \mathcal{T}_1, \Pi_s)$. The base manifold M_3, often called the associated space[21,22,25], is a connected Riemannian three-dimensional smooth manifold and the structure-group $\mathcal{T}_1$ is the one-parameter group of isometries generated by $\vec{\xi}$. In fact, $M_3 \simeq M_4/\mathcal{T}_1$. The bundle space M_4 is endowed with a connection. Given a moving frame, i.e., a smooth non-zero real-valued function on $U \subset M_3$, then this connection is locally determined by the smooth 1-form field $a_\alpha dx^\alpha$ over U. The curvature 2-form of the connection is given locally by

$$f_{\alpha\beta} \equiv a_{\alpha,\beta} - a_{\beta,\alpha} \; . \qquad (3.19)$$

In Appendix B it will be shown how the Riemannian structure on M_3 and the connection on the bundle space M_4 determine the semi-Riemannian structure on M_4.

The metric on M_3 is here chosen to be the $g_{\alpha\beta}$, as defined by (3.16). This choice is merely conventional, and others have used other conventions, e.g., Lichnerowicz[54] chose $-e^{-\omega} g_{\alpha\beta}$. The quantities $g_{\alpha\beta}$, a_α, and ω transform, respectively, as a rank two tensor, a covariant vector, and a scalar under coordinate transformations of M_3.

Henceforth, we shall work in M_3. Unless otherwise specified, greek indices are raised and lowered by $g_{\alpha\beta}$. In cases where confusion could arise, tensors in M_4 are "hatted", e.g.,

$$\hat{F}_{\cdot\beta}^{\alpha} = \hat{g}^{\alpha j} F_{\cdot\beta j},$$

but

$$F_{\cdot\beta}{}^{\alpha} = g^{\alpha\gamma} F_{\cdot\beta\gamma} \ .$$

Covariant derivatives of tensors in M_3 are denoted by a slash. Thus, for example, if B_α is a covariant vector field on M_3, then

$$B_{\alpha|\beta} = B_{\alpha,\beta} - \left\{{\gamma \atop \alpha\beta}\right\} B_\gamma,$$

where

$$\left\{{\gamma \atop \alpha\beta}\right\} \equiv \frac{1}{2} g^{\gamma\delta}(g_{\alpha\delta,\beta} + g_{\beta\delta,\alpha} - g_{\alpha\beta,\delta})$$

are the Christoffel symbols for the metric $g_{\alpha\beta}$ on M_3. The following formulae will be used repeatedly[49,54]:

$$\begin{aligned}
\hat{g}_{\alpha\beta} &= -e^{-\omega} g_{\alpha\beta} + e^{\omega} a_\alpha a_\beta \\
\hat{g}_{\alpha 4} &= e^{\omega} a_\alpha \\
\hat{g}_{44} &= e^{\omega} \\
\hat{g}^{\alpha\beta} &= -e^{\omega} g^{\alpha\beta} \qquad\qquad (3.20) \\
\hat{g}^{\alpha 4} &= e^{\omega} a^{\alpha} \\
\hat{g}^{44} &= e^{-\omega} - e^{\omega} a^{\alpha} a_{\alpha} \\
(-\hat{g})^{\frac{1}{2}} &= e^{-\omega} g^{\frac{1}{2}}
\end{aligned}$$

The alternating tensor on M_3 is defined as

$$\eta_{\alpha\beta\gamma} \equiv g^{\frac{1}{2}}\varepsilon_{\alpha\beta\gamma} ,$$

where

$$\varepsilon_{\alpha\beta\gamma} \equiv \begin{cases} +1, \text{ if } (\alpha\beta\gamma) \text{ is an even permutation of } (123), \\ -1, \text{ if } (\alpha\beta\gamma) \text{ is an odd permutation of } (123), \\ 0, \text{ otherwise.} \end{cases}$$

The $\eta_{\alpha\beta\gamma}$ are related to the components of the alternating tensor η_{ijkm} on M_4 by:

$$\eta_{\alpha\beta\gamma} = e^{\omega}\eta_{\alpha\beta\gamma 4} , \tag{3.21}$$

where

$$\eta_{ijkm} \equiv (-\hat{g})^{\frac{1}{2}} \varepsilon_{ijkm} , \tag{3.22}$$

with ε_{ijkm} defined analogously to $\varepsilon_{\alpha\beta\gamma}$.

Finally, the components of the curvature tensor $\hat{R}^{i}{}_{jkm}$ on a stationary M_4 in a Copernican chart can be expressed in terms of tensors on M_3[6]:

$$\hat{R}^{\mu}{}_{44\nu} = [\exp(2\omega)/4]\,[-2\omega_{|}{}^{\mu}{}_{\nu} - 3\omega_{,}{}^{\mu}\omega_{,\nu} + \delta^{\mu}_{\nu}\Delta_1\omega]$$

$$+ [\exp(4\omega)/4]\,[f^{\alpha}{}_{\nu}f^{\mu}{}_{\alpha}] \;,$$

$$\hat{R}^{4}{}_{\beta\gamma 4} = \frac{1}{4}\,[2\omega_{|\beta\gamma}+3\omega_{,\beta}\omega_{,\gamma}-g_{\beta\gamma}\Delta_1\omega] + [\exp(2\omega)/4]\,[-2a_{\beta}a^{\mu}\omega_{|\mu\gamma}$$

$$+ 2a^{\mu}f_{\mu\beta|\gamma}-3a_{\beta}a_{\gamma}a^{\mu}\omega_{,\mu}+a_{\beta}a_{\gamma}\Delta_1\omega+4a^{\mu}f_{\mu\beta}\omega_{,\gamma}$$

$$+ 2a^{\mu}f_{\mu\gamma}\omega_{,\beta}+a_{\gamma}f_{\beta\mu}\omega_{,}{}^{\mu}+g_{\beta\gamma}\omega_{,}{}^{\alpha}f_{\alpha\mu}a^{\mu} + 2f_{\gamma\beta}a^{\mu}\omega_{,\mu}$$

$$+ f_{\mu\gamma}f^{\mu}{}_{\beta}] + [\exp(4\omega/4]\,[a_{\beta}a^{\alpha}f_{\alpha\mu}f^{\mu}{}_{\gamma}] \;,$$

$$\hat{R}^{\mu}{}_{\alpha 4\nu} = [\exp(2\omega)/4]\,[-2a_{\alpha}\omega_{|}{}^{\mu}{}_{\nu}+2f^{\mu}{}_{\alpha|\nu}-3a_{\alpha}\omega_{,\nu}\omega_{,}{}^{\mu}$$

$$+ \delta^{\mu}_{\nu}a_{\alpha}\Delta_1\omega+4\omega_{,\nu}f^{\mu}{}_{\alpha}+2\omega_{,\alpha}f^{\mu}{}_{\nu}+2\omega_{,}{}^{\mu}f_{\nu\alpha}$$

$$+ g_{\alpha\nu}\omega_{,\beta}f^{\beta\mu}+\delta^{\mu}_{\nu}\omega_{,}{}^{\beta}f_{\alpha\beta}] + [\exp(4\omega)/4]\,[a_{\alpha}f^{\beta}{}_{\nu}f^{\mu}{}_{\beta}] \;,$$

$$\hat{R}^{\alpha}{}_{\beta\gamma\delta} = R^{\alpha}{}_{\beta\gamma\delta}+\frac{1}{2}\Big[2g_{\beta[\delta}\omega_{|\gamma]}{}^{\alpha} + 2\delta^{\alpha}_{[\gamma}\omega_{|\delta]\beta}+g_{\beta[\gamma}\delta^{\alpha}_{\delta]}\Delta_1\omega$$

$$+ \omega_{,}^{\alpha}g_{\beta[\delta}\omega_{,\gamma]}+\omega_{,\beta}\delta^{\alpha}_{[\gamma}\omega_{,\delta]}\Big] + [\exp(2\omega)/2]\,\Big[2a_{\beta}a_{[\delta}\omega_{|\gamma]}{}^{\alpha}$$

$$+ 2a_{\beta}f^{\alpha}{}_{[\gamma|\delta]}+2f^{\alpha}{}_{\beta|[\delta}a_{\gamma]} +\delta^{\alpha}_{[\delta}\,a_{\gamma]}a_{\beta}\Delta_1\omega$$

$$+ 3a_{\beta}\omega_{,}{}^{\alpha}a_{[\delta}\omega_{,\gamma]} + 4f_{\delta\gamma}a_{\beta}\omega_{,}{}^{\alpha} + 2a_{[\gamma}f_{\delta]\beta}\omega_{,}{}^{\alpha}$$

$$+ 2f^{\alpha}{}_{[\delta}a_{\gamma]}\omega_{,\beta} + 2a_{\beta}f^{\alpha}{}_{[\gamma}\omega_{,\delta]} + 4f^{\alpha}{}_{\beta}a_{[\gamma}\omega_{,\delta]}$$

$$+ g_{\beta[\gamma}a_{\delta]}\omega_{,}{}^{\mu}f^{\alpha}{}_{\mu} + a_{\beta}\omega_{,}{}^{\mu}f_{\mu[\delta}\delta^{\alpha}_{\gamma]} + \delta^{\alpha}_{[\gamma}a_{\delta]}\omega_{,}{}^{\mu}f_{\mu\beta}$$

$$+ f_{\beta[\gamma}f_{\delta]}{}^{\alpha} + f_{\gamma\delta}f^{\alpha}{}_{\beta}\Big]$$

$$+ [\exp(4\omega)/2]\,[a_{\beta}f^{\alpha}{}_{\mu}f^{\mu}{}_{[\delta}a_{\gamma]}] \;,$$

where $\Delta_1\omega \equiv g^{\alpha\beta}\omega_{,\alpha}\omega_{,\beta}$, $R^{\alpha}{}_{\beta\gamma\delta}$ are the components of the curvature tensor on M_3, and the $f_{\alpha\beta}$ are defined by (3.19).

D. The Static Case

The equations (3.4) are necessary and sufficient conditions that a stationary M_4 is also static. If we define a vector field $\vec{\tau}$ on M_4 by

$$\tau_i \equiv \eta_{ijkm}\, \xi^{[j}\nabla^{k}\xi^{m]} , \tag{3.24}$$

then clearly (3.4) is equivalent to $\tau_i = 0$. Now in a Copernican chart it turns out that

$$\tau_4 = 0 ,$$

and

$$\tau_\alpha = \frac{1}{2}\, e^{2\omega}\eta_{\alpha\beta\gamma} f^{\beta\gamma}. \tag{3.25}$$

The τ_α are the components of a vector field, called the <u>twist vector field</u>, on M_3. If M_4 is static, then it follows that $\tau_\alpha = 0$, and hence that

$$f_{\alpha\beta} = 0 .$$

Let us return for a moment to the principal fiber bundle interpretation of a stationary M_4. The $f_{\alpha\beta}$ are the components of the curvature 2-form of the connection defined locally by the a_α on the principle fiber bundle M_4 $(M_3, \mathcal{T}_1, \Pi_s)$. Thus the bundle space M_4 is static if and only if the connection if flat. In this interpretation it is manifest that the static property is local. A flat connection implies that the connection 1-form $a_\alpha dx^\alpha$ is closed, i.e., $d(a_\alpha dx^\alpha) = 0$. The converse of the Poincare' lemma ("Closed forms are locally exact")[33] implies that for each $p \varepsilon M_3$ there is a neighborhood of p in which there exists a function λ such that $a_\alpha dx^\alpha = d\lambda$, i.e., $a_\alpha = \lambda,_\alpha$. Thus the metric $\hat{g}$ of a static M_4 can be written as

$$\hat{g} = -e^{-\omega} g_{\alpha\beta} dx^\alpha dx^\beta + e^\omega d\lambda \ .$$

This is the standard form of a static metric[22,47,48], with timelike coordinate λ. The equations $\lambda(x^\alpha)$ = constant locally describe the spacelike surfaces orthogonal to the $\vec{\xi}$.

The properties of the (locally) static EMKG equations have been examined in detail, for example in Das[22] and Das and Coffman[25], in Bronnikov, et. al.[55], and in Stephenson[23]. Some interesting exact static solutions were found by Das and Coffman[25]. The metric and the electromagnetic and meson fields of these solutions are spherically symmetric. The electromagnetic field is, in addition, Coulomb-like and described by a single real-valued potential A. The so-called Weyl-Majumdar-Papapetrou (WMP) condition is satisfied:

$$\hat{g}_{44} = e^\omega = (1 + 4\pi A)^2 \ ,$$

and the "bare" charge e and mass m satisfy a balance condition*

$$e^2 = 4\pi m^2 \ .$$

* The units used in references 22 and 25 differ from those used here. In particular, one may transform the former units to the latter by writing $e^2 \equiv 4\pi\varepsilon^2$, where ε is the unit of charge in references 22 and 25, and e is the unit of charge used here.

In the following, (x, θ, ϕ, t) are modified spherical coordinates. In particular, x is the inverse of a radial coordinate, θ and ϕ are the usual angular coordinates on a sphere, and t, the timelike coordinate, is chosen so that $\xi^i = \delta^i_4$, i.e., the chart is Copernican. In these coordinates, the metric, electromagnetic, and meson fields are, respectively,

$$\hat{g} = - U^2(x)\ [\operatorname{csch}^4 x\ dx^2 + \operatorname{csch}^2 x\,(d\theta^2 + \sin^2\theta d\phi^2)] + U^{-2}(x)dt^2,$$

$$A(x) = (\tfrac{k}{2})^{-1}\ [U^{-1}(x) - 1]\ , \qquad (3.26)$$

$$\psi(x,t) = -k^{-1} x\ e^{\pm iet}.$$

The function $U(x) \equiv e^{-\omega/2}$ must satisfy the second-order nonlinear ordinary differential equation

$$\frac{d^2}{dx^2} U + (e x \operatorname{csch}^2 x)^2\ U^3 = 0. \qquad (3.27)$$

The equation (3.27) resembles the field equation for a static scalar field in one space dimension with a cubic self-interaction. The latter equation has soliton solutions[3], and so it is not too surprising that (3.27) has solutions which are soliton-like. One such class of solutions is found by choosing boundary conditions on U such that the total charge $\int_{M_3} J^i n_i \sqrt{g}\, d^3x$ is finite and is equal to e. The solution of this boundary-value problem entails solving a non-linear eigenvalue equation for e. The smallest value in the spectrum of e is about two orders of magnitude larger than the experimentally determined value of the fine-structure constant.

4. ISOMETRIC MOTION

The stationary Einstein-Maxwell equations can be written in a useful and elegant form in terms of two complex potentials[26,49,56,57]. If there are sources, then, in general, the above potentials do not exist. However, if one imposes the condition that the current four-vector J^i of the sources is parallel to the Killing vector field ξ^i, then the complex potentials can be defined[26]. This condition is called isometric motion since the motion of the sources is along the trajectories of ξ^i.

In this chapter, after some preliminaries, the condition of isometric motion will be imposed on the current of the meson field. This will enable us to define complex potentials and write the stationary EMKG equations in a revealing form. Two significant results are demonstrated. The first is that the magnetic field is parallel to the twist vector defined in the previous chapter. The second is that the Weyl-Majumdar-Papapetrou (WMP) condition[58,59] implies that either $e^2 = 2km^2$, or that the electric and magnetic potentials are functionally related.

A. The Electric and Magnetic Potentials

In a stationary spacetime, the electric and magnetic fields can be covariantly defined. In particular,

$$E_i \equiv \xi^j F_{ij} , \tag{4.1}$$

$$H_i \equiv \xi^j \, {}^*F_{ij} , \tag{4.2}$$

where ξ^i is the timelike Killing vector field and F_{ij} is the electromagnetic field strength. The Hodge dual ${}^*F_{ij}$ is defined in the usual way:

$$*F_{ij} \equiv \frac{1}{2} \eta_{ijkm} F^{km} , \tag{4.3}$$

where η_{ijkm} was defined earlier (cf. equation (3.22)). Equations (4.1) and (4.2) generalize the usual coordinate-dependent definitions

$$E_\alpha \equiv F_{\alpha 4} , \tag{4.4}$$

$$H_\alpha \equiv *F_{\alpha 4} . \tag{4.5}$$

In a Copernican chart, (4.1) and (4.2) reduce to (4.4) and (4.5) since in this case $\xi^i = \delta^i_4$. Henceforth we shall work in the real gauge (cf. section 2.C) and we shall assume that the spacetime metric $\hat{g}$ is stationary and that either $\pounds_\xi M_{ij} = 0$ or $\pounds_\xi E_{ij} = 0$, so that, by Theorem 3.1, $\pounds_\xi \eta = 0$ and $\pounds_\xi A_i = 0$. The latter implies

$$\pounds_\xi F_{ij} = 0 . \tag{4.6}$$

From the above and from the fact[51] that $\pounds_\xi \eta_{ijkm} = 0$, it follows as well that

$$\pounds_\xi *F_{ij} = 0 . \tag{4.7}$$

In what follows, the EMKG equations and the stationarity of the various fields will be used to derive expressions for the "curl", i.e., the anti-symmetrized covariant derivative, of various vector fields. The work involved is greatly simplified if the abstract (differential geometric) viewpoint is adopted[60]. In particular, one needs the following[33]:

Let $\vec{X}$ be a smooth vector field over M_4, and let $\beta \in F^p(M_4)$, i.e., a smooth p-form field over M_4. Then the <u>contraction</u> of β by $\vec{X}$ is a (p-1) form field defined by

$$(\underset{X}{C}\ \beta)\ (\vec{Y}_1,\ldots,\vec{Y}_{P-1}) \equiv \beta\ (\vec{X},\ \vec{Y}_1,\ldots,\vec{Y}_{P-1})\ , \tag{4.8}$$

where $\vec{Y}_1, \ldots, \vec{Y}_{P-1}$ are an arbitrary set of contravariant vectors. Thus the electric and magnetic fields can be abstractly defined as the 1-form fields E and H which satisfy

$$E \equiv -2\ \underset{\xi}{C}\ F, \tag{4.9}$$

$$H \equiv -2\ \underset{\xi}{C}\ {*F}\ . \tag{4.10}$$

where F is a 2-form field related to the curvature 2-form field $\mathcal{F}$ on W by

$$F \equiv -\frac{i}{e}\mathcal{F} = \frac{1}{2}\ F_{ij}\ dx^i \wedge dx^j\ .$$

The Maxwell equations can then be expressed in the form

$$dF = 0 \tag{4.11}$$

$$d\ {*F} = {*J} \tag{4.12}$$

where $*J$ is a 3-form field dual to the current 1-form field, i.e.,

$$*J \equiv (*J_{ijk})\, dx^i \wedge dx^j \wedge dx^k \equiv \frac{1}{3}\, \eta_{rijk}\, J^r\, dx^i \wedge dx^j \wedge dx^k \;. \qquad (4.13)$$

We will also need the definition of the Lie derivative of a p-form. Given that the Lie derivative of a scalar field (0-form) f is $\underset{X}{\pounds} f = df(\vec{X})$, and the Lie derivative of a vector field $\vec{Y}$ is $\underset{X}{\pounds}\vec{Y} = [\vec{X},\vec{Y}] \equiv \vec{X}\vec{Y} - \vec{Y}\vec{X}$, then the Lie derivative of a p-form field θ is defined by[33]

$$(\underset{X}{\pounds}\theta)\,(\vec{Y}_1,\ldots,\vec{Y}_p) = \underset{X}{\pounds}\,[\theta(\vec{Y}_1,\ldots,\vec{Y}_p)] - \theta\,(\underset{X}{\pounds}\vec{Y}_1,\vec{Y}_2,\ldots,\vec{Y}_p) - \ldots$$

$$- \theta(\vec{Y}_1,\ldots,\vec{Y}_{p-1}, \underset{X}{\pounds}\vec{Y}_p) \;, \qquad (4.14)$$

for any set of p vector fields $\vec{Y}_1,\ldots,\vec{Y}_p$.

The following lemma, which is proved by Hicks[33], will be used repeatedly:

Lemma 4.1. Let $\theta \in F^p(M_4)$ and $\vec{X}$ a smooth vector field. Then

$$d(\underset{X}{C}\,\theta) = \underset{X}{\pounds}\,\theta - \underset{X}{C}\,(d\theta) \;. \qquad (4.15)$$

The lemma allows us to quickly compute the exterior derivatives of E and H. We also use the facts that $\underset{\xi}{\pounds}\, F = \underset{\xi}{\pounds}\, *F = 0$, which follow from (4.6) and (4.7), the definitions (4.9) and (4.10) of E and H, and the Maxwell equations. The upshot is

$$dE = -2d(\underset{\xi}{C}F) = -2[\underset{\xi}{\pounds}F - \underset{\xi}{C}\,(dF)] = 0 \;, \qquad (4.16)$$

$$dH = -2d(\underset{\xi}{C}{*F}) = -2[\underset{\xi}{\pounds}{*F} - \underset{\xi}{C}(d{*F})] = 2\underset{\xi}{C}{*J} \ . \tag{4.17}$$

The following theorem is thus proved:

Theorem 4.1. The magnetic field 1-form is closed if and only if $\underset{\xi}{C}{*J} = 0$. The latter is true if and only if $J^{[i}\xi^{j]} = 0$, i.e., $\vec{J}$ is parallel to $\vec{\xi}$.

We shall henceforth require $\vec{J}$ and $\vec{\xi}$ to be parallel. This condition is called isometric motion[26] since the "motion" of the source is along the trajectories of the timelike Killing vector field $\vec{\xi}$. A Copernican observer would observe the meson field at rest in this case. It will be shown later that there are solutions of the stationary EMKG equations with the charged meson field in isometric motion but such that Copernican observers would detect a magnetic field and a magnetic-like gravitational field, but would not detect an electric field or an electric-like ("Newtonian") gravitational field. The properties of these solutions will be discussed in detail in Chapter 7.

Henceforth, we shall work in a Copernican chart and in the real gauge. The following three theorems are consequences of the assumption of isometric motion.

Theorem 4.2. In a Copernican chart in the real gauge, the condition of isometric motion is equivalent to

$$\hat{A}^{\alpha} \equiv \hat{g}^{\alpha i}A_i = e^{\omega}g^{\alpha\beta}(a_{\beta}A_4 - A_{\beta}) = 0 \ . \tag{4.18}$$

Proof: In a Copernican chart $\hat{\xi}^{\alpha} \equiv \hat{g}^{\alpha i}\xi_i = 0$, and thus isometric motion implies $\hat{J}^{\alpha} \equiv \hat{g}^{i\alpha}J_i = 0$. From (2.44) and (3.20) we then obtain the desired result, since $\eta \neq 0$. //

The vanishing of the Lie derivative of the 4-potential A_i implies that in a Copernican chart A_α and A_4 are independent of the time coordinate $t \equiv x^4$. Hence $A_\alpha(x)$ and $A_4(x)$ may be regarded, respectively, as a vector field and a scalar field on M_3, the Riemannian base space of the principal fiber bundle M_4. From (4.18) it follows that these quantities are related by

$$A_\beta = A_4 a_\beta \ . \tag{4.19}$$

Theorem 4.3. In a Copernican chart, the x^4-components of E_i and H_i both vanish. The remaining components, E_α and H_α, are the components of locally exact 1-form fields on M_3, i.e., there is a neighborhood of each $p \in M_3$ on which there exists scalar fields A and B such that

$$E_\alpha = A_{,\alpha} \ , \tag{4.20}$$

and

$$H_\alpha = -B_{,\alpha} \ . \tag{4.21}$$

Proof: In a Copernican chart, $E_4 = \xi^j F_{4j} = F_{44} = 0$, and similarly for H_4. Since $\pounds_\xi E_i = 0$ and $\pounds_\xi H_i = 0$, it follows that E_α and H_α are independent of $t = x^4$. Hence E_α and H_α are the components of 1-form fields on M_3. Now by Theorem 4.1, $E_\alpha dx^\alpha$ and $H_\alpha dx^\alpha$ are closed 1-form fields on M_3, and hence, by the converse of the Poincare lemma[61], they are locally exact.//

The minus sign in (4.21) is chosen for later convenience. Since

$$E_\alpha = \xi^j F_{\alpha j} = F_{\alpha 4} = A_{\alpha,4} - A_{4,\alpha} = -A_{4,\alpha} \,,$$

without loss of generality we may choose $A = -A_4$. Thus (4.19) becomes

$$A_\beta = -Aa_\beta \,. \tag{4.22}$$

Theorem 4.4. The assumption of isometric motion implies that the magnetic field H_α is parallel to the twist vector τ_α (defined by equation (3.25)). In particular,

$$B_{,\alpha} = -H_\alpha = e^{-\omega} A \tau_\alpha \,. \tag{4.23}$$

Proof: Take the exterior derivative (in M_3) of the 1-form field $A_\beta dx^\beta$. From (4.22), the definitions of $F_{\alpha\beta}$ and $f_{\alpha\beta}$, and from $F_{\alpha 4} = A_{,\alpha}$, we obtain

$$-f_{\alpha\beta} A = F_{\alpha\beta} + a_\alpha F_{\beta 4} - a_\beta F_{\alpha 4} \,. \tag{4.24}$$

Hence, using (3.20) and (3.21),

$$\begin{aligned} B_{,\alpha} = -H_\alpha \equiv -{}^*F_{\alpha 4} &= -\frac{1}{2}\, \eta_{\alpha 4 ij} F^{ij} \\ &= -\frac{1}{2}\, e^{-\omega} \eta_{\alpha\beta\gamma}\, g^{\beta\delta} g^{\gamma\varepsilon} (F_{\delta\varepsilon} + a_\delta F_{\varepsilon 4} - a_\varepsilon F_{\delta 4}) \\ &= \frac{1}{2}\, e^{-\omega} \eta_{\alpha\beta\gamma}\, g^{\beta\delta} g^{\gamma\varepsilon} f_{\delta\varepsilon} A \\ &= e^{-\omega} A \tau_\alpha \,, \end{aligned}$$

the last equality following from (3.25). //

A Copernican observer will thus find that the magnetic field and the magnetic-like part of the gravitational field are parallel.

B. The Complex Potentials

The Maxwell equations (4.11) and (4.12) can be written as a single complex equation:

$$d\tilde{F} = \tilde{J} \,, \tag{4.25}$$

where

$$\tilde{F} \equiv F - i{*}F \text{ and } \tilde{J} \equiv -i{*}J \,. \tag{4.26}$$

This prompts the introduction of a smooth complex potential over M_4,

$$\phi \equiv A + iB \,, \tag{4.27}$$

so that

$$-2C_{\xi}\tilde{F} = d\phi, \text{ or} \tag{4.28}$$

$$\xi^j \tilde{F}_{ij} = \phi_{,i} \,,$$

where $\tilde{F}_{ij} \equiv F_{ij} - i{*}F_{ij}$. In a Copernican chart, ϕ can be considered as a complex-valued scalar field over M_3.

In the work of Kramer, Neugebauer, and Stephani[26], it was shown that for the Einstein-Maxwell-Lorentz equations with a charged perfect fluid source in isometric motion, one may define another complex potential in terms of ϕ and the "vertical part" of the metric, e^{ω} and a_{α}. It shall now be demonstrated that an analogous result holds for the EMKG equations with the meson field in isometric motion.

We begin by defining a complex 2-form field $\tilde{Z} \equiv \frac{1}{2} \tilde{Z}_{ij} dx^i \wedge dx^j$, where

$$\tilde{Z}_{ij} \equiv 2\left[\nabla_j \xi_i - i{*}(\nabla_j \xi_i)\right] + k\phi^* \tilde{F}_{ij} . \tag{4.29}$$

In the above $k \equiv 8\pi$, ϕ^* is the complex conjugate of ϕ, and $\tilde{Z}_{ij}$ is anti-symmetric since, by Killing's equation, $\nabla_j \xi_i = -\nabla_i \xi_j$. For convenience, we shall write

$$S_{ij} \equiv 2\nabla_j \xi_i = -S_{ji} , \tag{4.30}$$

$$\tilde{S}_{ij} \equiv S_{ij} - i\, {*}S_{ij} , \tag{4.31}$$

so that

$$\tilde{Z}_{ij} = \tilde{S}_{ij} + k\phi^* \tilde{F}_{ij} , \tag{4.32}$$

or, in terms of differential forms,

$$\tilde{Z} = \tilde{S} + k\phi^* \tilde{F} \tag{4.32a}$$

where $\tilde{S}$ has the obvious definition. Finally, we define a complex 1-form field $\zeta \equiv \zeta_i dx^i$ by contracting with $\vec{\xi}$:

$$\zeta \equiv -2C_{\xi} \tilde{Z} . \tag{4.33}$$

By Lemma 4.1,

$$d\zeta = -2\left[\pounds_{\xi} Z - C_{\xi} (d\tilde{Z})\right] . \tag{3.34}$$

In Appendix C it is shown that $d\zeta = 0$. Hence, locally there exists a smooth complex scalar field Γ such that

$$\zeta = -d\Gamma , \tag{3.35}$$

or,

$$\tilde{S}_i + k\phi^* \phi_{,i} = -\Gamma_{,i} , \tag{4.36}$$

where the $\tilde{S}_i$ are the components of the 1-form field $C_{\xi}\tilde{S}$, i.e.,

$$\tilde{S}_i \equiv \xi^j \tilde{S}_{ij} = 2\left[\nabla_j \xi_i - i\ {*}(\nabla_j \xi_i)\right]\xi^j . \tag{4.37}$$

In the manner of Kramer, Neugebauer, and Stephani[26], we now write Γ explicitely in terms e^{ω}, ϕ, ϕ^*, and the "twist potential" Ω, which is defined below. From Killing's equation, it follows that

$$\tilde{S}_i + \tilde{S}_i{}^* = -2(\xi^j \xi_j)_{,i} .$$

We can write $\xi^j\xi_j \equiv e^{\omega}$, because this invariant relation holds in a Copernican chart. The real part of (4.36) is then

$$\left[-2e^{\omega} + k\phi^*\phi + \Gamma + \Gamma^*\right]_{,i} = 0 .$$

Hence,

$$-2e^{\omega} + k\phi^*\phi + (\Gamma + \Gamma^*) = \text{a constant}.$$

If we choose the constant above to be zero, and write $\Omega \equiv \mathrm{Im}(\Gamma)$, we have, finally,

$$\Gamma = e^{\omega} - \frac{k}{2}\phi^*\phi + i\Omega . \tag{4.38}$$

It can be shown[49,62] that Ω satisfies

$$\Omega_{,i} = \tau_i + \frac{1}{2} ik(\phi^*\phi_{,i} - \phi^*_{,i}\phi) , \tag{4.39}$$

where τ_i is the twist vector on M_4, defined by equation (3.24).

In a Copernican chart,

$$\tilde{S}_4 = 2\left[\nabla_4\xi_4 - i^*(\nabla_4\xi_4)\right] = 0 ,$$

since $\nabla_i\xi_j = -\nabla_j\xi_i$. Thus, by (4.36), $\Gamma_{,4} = 0$. The complex potential Γ, like ϕ, can be thought of as a scalar field over M_3. The same considerations obviously apply to Ω.

It will now be shown that ϕ and Γ satisfy formally similar field equations. An analogous result was shown for perfect fluid sources by Kramer, Neugebauer, and Stephani[26]. These field equations are equivalent to the Maxwell equations and the Einstein equations $\hat{G}_{i4} = - kT_{i4}$.

First consider the Maxwell equations in the form

$$\nabla_j \tilde{F}^{ij} = J^i \ . \tag{4.40}$$

After contracting with respect to ξ_i and using (4.30), one obtains

$$\Box\phi + \frac{1}{2} S_{ij} \tilde{F}^{ij} = -\xi^j J_j \ ,$$

where $\Box \equiv \nabla^i \nabla_j$. The so-called dual-product identity[52] is needed. This states that any two anti-symmetric tensors A_{ij} and B_{ij} satisfy the following identity

$$*A_{jk} \, *B^{ik} = A^{ik} B_{jk} - \frac{1}{2} \delta^i_j A_{rs} B^{rs} \ . \tag{4.41}$$

This enables one to easily establish that

$$\begin{aligned} \frac{1}{2} S_{ij} \tilde{F}^{ij} &= (\xi^j \xi_j)^{-1} \phi_{,i} \tilde{S}^i \\ &= -e^{-\omega} \phi_{,i} (\Gamma^{,i} + k\phi * \phi^{,i}) \ , \end{aligned} \tag{4.42}$$

where $\Gamma^{,i} \equiv \hat{g}^{ij} \Gamma_{,j}$. Hence, the Maxwell equations for the case of stationary gravitational and electromagnetic fields with stationary mesonic source in isometric motion are equivalent to

$$\Box\phi - e^{-\omega}(\Gamma^{,i} + k\phi^*\phi^{,i})\phi_{,i} = -\xi^i J_i \ . \tag{4.43}$$

In order to obtain an analogous equation for Γ, we begin by considering the complex-valued anti-symmetric tensor field $\tilde{Z}_{ij}$ defined by equation (4.32). We want an equation analogous to the Maxwell equation (4.40), i.e., an equation of the form

$$\nabla_j \tilde{Z}^{ij} = T^i \tag{4.44}$$

where the "current" T^i depends on the sources of the gravitational field. From (4.32),

$$\nabla_j \tilde{Z}^{ij} = \nabla_j \tilde{S}^{ij} + k\left[\phi^*_{,j}F^{ij} + \phi^* \nabla_j \tilde{F}^{ij}\right] . \tag{4.45}$$

Now

$$\begin{aligned}\nabla_j \tilde{S}^{ij} &= 2\nabla_j\left[\nabla^j\xi^i - i*(\nabla^j\xi^i)\right] \\ &= 2\hat{R}^i{}_j\xi^j \\ &= -2k(T^i{}_j - \tfrac{1}{2}\delta^i_j T)\,\xi^j \ ,\end{aligned} \tag{4.46}$$

by the Einstein equations and by use of the following facts[47]:

$$\nabla_j \nabla^j \xi^i = \hat{R}^i{}_j \xi^j \quad ,$$

$$\nabla_j \ {*}(\nabla^j \xi^i) = 0.$$

Also, from (4.40), the last term in (4.45) becomes

$$k\phi^* \nabla_j \ \tilde{F}^{ij} = k\phi^* J^i \quad . \tag{4.47}$$

Hence (4.45) can be written as

$$\nabla_j \ \tilde{Z}^{ij} = k\left[\phi^*_{,j} \ \tilde{F}^{ij} - 2(T^i{}_j - \frac{1}{2} \ \delta^i{}_j T) \ \xi^j + \phi^* J^i\right] \ . \tag{4.48}$$

The right hand side of (4.48) satisfies the criterion mentioned above for the "current" T^i in (4.44). In fact, since the dual-product identity enables one to show that

$$\phi^*_{,j} \ \tilde{F}^{ij} = \xi^k(\tilde{F}^*_{jk}) \ \tilde{F}^{ij} = 2\xi^j E^i{}_j \ ,$$

where $E^i{}_j$ is the electromagnetic energy-momentum tensor, and since $E^i{}_i = 0$, (4.48) reduces to

$$\nabla_j \tilde{Z}^{ij} = k\left[-2(M^i{}_j - \frac{1}{2} \ \delta^i{}_j M) \ \xi^j + \phi^* \ J^i\right] \ . \tag{4.49}$$

If one now contracts (4.49) by ξ_i and uses the dual-product identity, one obtains

$$\Box\Gamma - e^{-\omega}(\Gamma^{,i} + k\phi^*\phi^{,i})\,\Gamma_{,i} = k\left[-2(M_{ij} - \tfrac{1}{2}\hat{g}_{ij}M)\xi^i\xi^j + \phi^*\xi^i J_i\right]. \quad (4.50)$$

This equation is of the same form as (4.43).

So far, we have found that under the assumptions of stationarity and isometric motion the Maxwell equations, the Einstein equations $\hat{G}_{i4} = -kT_{i4}$, and the Klein-Gordon equations are equivalent, respectively, to (4.43), (4.50), and (2.42c). The remaining field equations are the Einstein equations $\hat{G}_{\alpha\beta} = -kT_{\alpha\beta}$. The latter can be cast into covariant form using the projection operator[34] $h^i{}_j$ defined by

$$h^i{}_j \equiv \delta^i{}_j - e^{-\omega}\xi^i\xi_j \quad (4.51)$$

The results of acting on the Ricci tensor $\hat{R}_{ij}$ and the energy-momentum tensor T_{ij} with $h^i{}_j$ are the tensors $\overline{R}_{ij}$ and $\overline{T}_{ij}$:

$$\overline{R}_{ij} \equiv h^r{}_i h^s{}_j \hat{R}_{rs} ,$$

$$\overline{T}_{ij} \equiv h^r{}_i h^s{}_j T_{rs} .$$

In a Copernican chart, the components $\overline{R}_{\alpha\beta}$ are

$$\overline{R}_{\alpha\beta} = \hat{R}_{\alpha\beta} - a_\alpha\hat{R}_{\beta 4} - a_\beta\hat{R}_{\alpha 4} + a_\alpha a_\beta\hat{R}_{44} \quad . \quad (4.52)$$

It turns out that, for our purposes, the most convenient form of the $\hat{G}_{\alpha\beta} = -kT_{\alpha\beta}$ equations is the following*:

$$\sigma_{ij} \equiv \bar{R}_{ij} + e^{-\omega} h_{ij} \hat{R}_{rs} \xi^r \xi^s = -k(\bar{T}_{ij} - h_{ij} T + e^{-\omega} T_{rs} \xi^r \xi^s) \ .$$

For easy reference, the EMKG equations in the form in which they will be henceforth used are collected here:

$$\bar{R}_{ij} + e^{-\omega} h_{ij} \hat{R}_{rs} \xi^r \xi^s = -k(\bar{T}_{ij} - h_{ij} T + e^{-\omega} h_{ij} T_{rs} \xi^r \xi^s) \ , \tag{4.53}$$

$$\Box \phi - e^{-\omega}(\Gamma^{,i} + k\phi^* \phi^{,i}) \ \phi_{,i} = -\xi^i J_i \ , \tag{4.54}$$

$$\Box \Gamma - e^{-\omega}(\Gamma^{,i} + k\phi^* \phi^{,i}) \Gamma_{,i} = k \left[-2(M_{ij} - \tfrac{1}{2} \hat{g}_{ij} M) \xi^i \xi^i + \phi^* \xi^i J_i \right] , \tag{4.55}$$

$$\Box \eta + e^{-\omega}(m^2 e^{\omega} - e^2 A^2) \ \eta = 0 \ . \tag{4.56}$$

C. The Field Equations as Tensor Equations on M_3

In this section we will work in a Copernican chart and in the real gauge. The components of the electromagnetic energy-momentum tensor E_{ij} can be expressed in terms of tensors on M_3 as follows:

---

* The $\sigma_{\alpha\beta}$ used here are not quite the same as the quantities of the same name appearing in equation (F_1) of Kloster, Som, and Das[6]. Denoting the latter by $\sigma^{(K)}_{\alpha\beta}$ it turns out that $\sigma_{\alpha\beta} = \sigma^{(K)}_{\alpha\beta} - g_{\alpha\beta} g^{\mu\nu} \sigma^{(K)}_{\mu\nu}$.

$$E_{\alpha\beta} = \frac{1}{2}\ \{(e^{-\omega}g_{\alpha\beta} + e^{\omega}a_{\alpha}a_{\beta})\Delta_1(\phi,\phi^*) - e^{-\omega}(\phi_{,\alpha}\phi^*_{,\beta} + \phi^*_{,\alpha}\phi_{,\beta})$$

$$- i(\eta^{\mu}{}_{\beta}{}^{\nu}a_{\alpha} + \eta^{\mu}{}_{\alpha}{}^{\nu}a_{\beta})\ \phi^*_{,\nu}\phi_{,\mu}\}\ ,$$

$$E_{44} = \frac{1}{2}\ e^{\omega}\Delta_1(\phi,\phi^*)\ , \qquad (4.57)$$

$$E_{\beta 4} = \frac{1}{2}\ \{-i\eta^{\mu}{}_{\beta}{}^{\nu}\phi^*_{,\nu}\phi_{,\mu} + e^{\omega}a_{\beta}\Delta_1(\phi,\phi^*)\}\ ,$$

where $\Delta_1(\phi;\phi^*) \equiv g^{\alpha\beta}\phi_{,\alpha}\phi^*_{,\beta}$. This result was obtained by Israel and Wilson using slightly different notation and different units[49].

The components of the meson energy-momentum tensor M_{ij} are:

$$M_{\alpha\beta} = 2(\eta_{,\alpha}\eta_{,\beta} + e^2a_{\alpha}a_{\beta}A^2\eta^2) + (-e^{-\omega}g_{\alpha\beta} + e^{\omega}a_{\alpha}a_{\beta})\mu\ ,$$

$$M_{44} = 2e^2A^2\eta^2 + e^{\omega}\mu\ ,$$

$$M_{\alpha 4} = a_{\alpha}M_{44}\ ,$$

$$M \equiv \hat{g}^{ij}M_{ij} = 2(e^{-\omega}A^2\eta^2 - e^{\omega}\Delta_1\eta + 2\mu)\ ,$$

where

$$\mu \equiv e^{\omega}\Delta_1\eta + e^{-\omega}(m^2e^{\omega} - e^2A^2)\eta^2\ .$$

Finally, the components of the meson current J_i are

$$J_4 = 2e^2A\eta^2\ ,$$

$$(4.59)$$

$$J_{\alpha} = a_{\alpha}J_4\ .$$

We shall first write the field equations equivalent to $\hat{G}_{\alpha\beta} = -kT_{\alpha\beta}$, i.e., equations (4.53). In a Copernican chart, the latter are:

$$\sigma_{\alpha\beta} \equiv \hat{R}_{\alpha\beta} - a_\alpha \hat{R}_{\beta 4} - a_\beta \hat{R}_{\alpha 4} + (a_\alpha a_\beta - e^{-2\omega} g_{\alpha\beta}) \hat{R}_{44}$$

$$= - k \Big[T_{\alpha\beta} - a_\alpha T_{\beta 4} - a_\beta T_{\alpha 4} + (a_\alpha a_\beta - e^{-2\omega} g_{\alpha\beta}) T_{44}$$

$$+ e^{-\omega} g_{\alpha\beta} T \Big] .$$

From equations (3.20) and (3.23), the left hand side, in a Copernican chart, is

$$\sigma_{\alpha\beta} \equiv R_{\alpha\beta} + \frac{1}{2} (\omega_{,\alpha} \omega_{,\beta} + e^{-2\omega} \tau_\alpha \tau_\beta) \quad ,$$

where $R_{\alpha\beta}$ is the Ricci tensor constructed from the matric $g_{\alpha\beta}$ on M_3 and τ_α is the twist vector. Since (4.38) and (4.39) imply that

$$\Gamma_{,\alpha} + k\phi^* \phi_{,\alpha} = (e^\omega)_{,\alpha} + i\tau_\alpha \quad , \tag{4.60}$$

the $\sigma_{\alpha\beta}$ become:

$$\sigma_{\alpha\beta} \equiv R_{\alpha\beta} + \frac{1}{2} e^{-2\omega} \mathrm{Re} \Big[(\Gamma_{,\alpha} + k\phi^* \phi_{,\alpha}) (\Gamma^*_{,\beta} + k\phi\phi^*_{,\beta}) \Big] .$$

Hence, from (4.57) and (4.58), the field equations (4.53) become

$$R_{\alpha\beta} + \frac{1}{2} e^{-2\omega} \mathrm{Re}\left[(\Gamma_{,\alpha} + k\phi^*\phi_{,\alpha}) (\Gamma^*_{;\beta} + k\phi\phi^*_{,\beta})\right]$$

$$+ k\left[-e^{-\omega}\mathrm{Re}\phi_{,\alpha}\phi^*_{;\beta} + 2\eta_{,\alpha}\eta_{,\beta} + 2e^{-2\omega}g_{\alpha\beta}(m^2e^{\omega}-e^2A^2)\eta^2\right] \qquad (4.61)$$

$$= 0 \quad .$$

It is straightforward to show that the remaining EMKG equations become:

$$\Delta_2\phi - e^{-\omega} (\Gamma^{,\alpha}+k\phi^*\phi^{,\alpha})\phi_{,\alpha} = 2e^2e^{-\omega}\mathrm{Re}(\phi)\eta^2 \quad , \qquad (4.62)$$

$$\Delta_2\Gamma - e^{-\omega} (\Gamma^{,\alpha} + k\phi^*\phi^{,\alpha}) \Gamma_{,\alpha} = 2ke^{-\omega}\left[-(m^2e^{\omega} - e^2A^2) + \frac{1}{2} ie^2\mathrm{Im}(\phi^2)\right] \eta^2 \quad , \qquad (4.63)$$

$$\Delta_2\eta - e^{-2\omega}(m^2e^{\omega} - e^2A^2)\,\eta = 0 \quad , \qquad (4.64)$$

where Δ_2 is the Laplacian operator on M_3, i.e.,

$$\Delta_2\phi \equiv \phi^{|\alpha}{}_{|\alpha} = g^{-\frac{1}{2}} (g^{\frac{1}{2}}g^{\alpha\beta}\phi_{,\alpha})_{,\beta} \;.$$

The equations (4.61) - (4.64) are the desired tensor equations on M_3. They are equivalent to the EMKG equations for stationary gravitational and electromagnetic fields whose stationary mesonic source is in isometric motion.

D. The WMP Condition

The Weyl-Majumdar-Papapetrou (WMP) condition, that $\hat{g}_{44}$ is functionally dependent on A_4 (the x^4-component of the electromagnetic 4-potential A_i), has been used by various authors[47,21,58,59] in order to simplify the static Einstein-Maxwell equations to the point that exact solutions can be readily found. If sources are present, then, characteristically, a "balance condition" on the charge and mass densities of the form (charge density)/(mass density) = constant often results. If one imposes the WMP condition on the sourceless static Einstein-Maxwell equations, then one finds[47] a solution which may be interpreted as arising from an arbitrary number of charged point-masses mutually at rest.

In the next theorem it will be demonstrated that an analogous situation arises for the stationary EMKG equations with isometric motion. The form of the WMP condition to be used is

$$e^{\omega} = \frac{e^2}{m^2} A^2 \quad . \tag{4.65}$$

It is worth noting that the above can be written in an invariant manner, namely,

$$m^2 = e^2 A^i A_i \quad . \tag{4.65a}$$

A solution of the stationary EMKG equations which satisfies (4.65) or (4.65a) will be said to be of the <u>WMP class</u>.

<u>Theorem 4.5</u>. Let $(g_{\alpha\beta}, e^{\omega}, a_{\alpha}, A, B, \eta)$ be of the WMP class (in the real gauge and in a Copernican chart) with $A \neq 0$. Then one or both of the following must hold:

(i) $e^2 = 2km^2$;

(ii) The potentials A and B are functionally related.

Proof: The proof uses the fact that a solution of the EMKG equations in a Copernican chart must satisfy the contracted Bianchi identities on M_3, namely

$$R_{\alpha}{}^{\beta}{}_{|\beta} - \tfrac{1}{2} R_{,\alpha} = 0 \; . \tag{4.66}$$

The WMP condition (4.65) is now imposed on the field equations (4.61) - (4.64). The result is:

$$R_{\alpha\beta} + \alpha A^{-2} A_{,\alpha} A_{,\beta} + \beta A^{-2} B_{,\alpha} B_{,\beta} + 2k\eta_{,\alpha}\eta_{,\beta} = 0 \quad , \tag{4.67}$$

$$\Delta_2 A - A^{-1}(2\Delta_1 A - \Delta_1 B + 2m^2\eta^2) = 0 \quad , \tag{4.68}$$

$$\Delta_2 B - 3A^{-1}\Delta_1(A,B) = 0 \quad , \tag{4.69}$$

$$\alpha\Delta_1 A - \left(1 + \frac{km^2}{e^2}\right)\Delta_1 B + 2\alpha m^2\eta^2 = 0 \; , \tag{4.70}$$

$$\Delta_2 \eta = 0 \; , \tag{4.71}$$

where

$$\alpha \equiv 2 - \frac{km^2}{e^2}, \text{ and } \beta \equiv \tfrac{1}{2} - \frac{km^2}{e^2} \; .$$

The relation (4.23) has been used to eliminate τ_α from the field equations. From the above, one finds that

$$R_{\alpha}{}^{\beta}{}_{|\beta} - \tfrac{1}{2}R_{,\alpha} = \beta A^{-3}(A_{,\alpha}B_{,\beta} - B_{,\alpha}A_{,\beta})\, B^{,\beta} \quad .$$

From the fact that $g_{\alpha\beta}$ is Riemannian and from the definition of β, it follows that (4.66) is satisfied only if either $e^2 = 2km^2$, or if A and B are functionally related, or if B = a constant. Since the third case is clearly a special case of the second, the theorem is proved. //

If $e^2 = 2km^2$, so that $\beta = 0$ and $\alpha = 3/2$, then the field equations (4.67) - (4.71) reduce to:

$$R_{\alpha\beta} + \frac{3}{2}\, A^{-2}A_{,\alpha}A_{,\beta} + 2k\,\eta_{,\alpha}\eta_{,\beta} = 0 \ , \qquad (4.72)$$

$$\Delta_2 A - A^{-1}\Delta_1 A = 0 \ , \qquad (4.73)$$

$$\Delta_2 B - 3A^{-1}\Delta_1(A,B) = 0 \ , \qquad (4.74)$$

$$\Delta_1 B - \Delta_1 A - 2m^2\eta^2 = 0 \ , \qquad (4.75)$$

$$\Delta_2 \eta = 0 \ . \qquad (4.76)$$

Since these equations identically satisfy (4.66), there are seven independent equations for the six independent unknowns. (Three of the $g_{\alpha\beta}$ can be chosen arbitrarily, leaving three components of $g_{\alpha\beta}$ plus the three scalars A, B, and η as independent fields.) This makes the task of solving these equations a difficult one.

If B = B(A), then (4.67) - (4.71) become:

$$R_{\alpha\beta} + A^{-2}\left[\alpha + \beta\,(B')^2\right] A_{,\alpha}A_{,\beta} + 2k\eta_{,\alpha}\eta_{,\beta} = 0\ , \tag{4.77}$$

$$\Delta_2\, A - A^{-1}\ \{\left[2 - (B')^2\right]\Delta_1 A + 2m^2\eta^2\ \} = 0\quad , \tag{4.78}$$

$$B'\Delta_2 A + (B'' - 3A^{-1}B')\Delta_1 A = 0\ , \tag{4.79}$$

$$[\alpha - (1 + \frac{km^2}{e^2})\ (B')^2]\ \Delta_1 A + 2\alpha m^2\eta^2 = 0\ , \tag{4.80}$$

$$\Delta_2\eta = 0\ , \tag{4.81}$$

where $B' \equiv \frac{d}{dA}\,B$. The consistency of equations (4.78) - (4.80) requires that B satisfy the following ordinary non-linear differential equation:

$$\alpha B'' - 2\alpha A^{-1}B' - 2\beta A^{-1}(B')^3 = 0.$$

This equation has two first integrals:

$$B'_{(\pm)} \equiv \frac{1 \pm \sqrt{1 - 4\alpha\beta B_0^2\ A^4}}{2\beta B_0 A^2}\quad , \tag{4.82}$$

where B_0 is a constant of integration. If $(\alpha\beta) \leq 0$, then $B'_{(\pm)}$ is real. But if $(\alpha\beta) > 0$, then, to insure the reality of B', it is necessary that $|A| \leq (4\alpha\beta B_0^2)^{-1/4}$.

There are six independent equations for the five remainings unknowns in this case. In general, the task of finding solutions would be formidable because of the non-polynomial character of (4.82).

We conclude this chapter with a theorem giving a necessary and sufficient condition for a stationary WMP solution to be static.

Theorem 4.6. A WMP solution with $A \neq 0$, $\eta \neq 0$, and $m \neq 0$ is static if and only if $\alpha = 0$, i.e., $e^2 = \frac{1}{2}km^2$.

Proof: If $\alpha = 0$, then (4.70) implies that $\Delta_1 B = 0$, i.e., $B =$ a constant. But then, by (4.23), $\tau_\alpha = 0$, so $\hat{g}$ is static. On the other hand, if $\hat{g}$ is static, then $\Delta_1 B = 0$, so (4.70) becomes $\alpha(\Delta_1 A + 2m^2\eta^2) = 0$. Since $m \neq 0$ and $\eta \neq 0$, it follows that $\alpha = 0$. //

This theorem is consistent with results obtained earlier by Das[22]. It also shows that, in a sence, non-trivial stationary non-static and static WMP solutions are "disjoint" if B and A are not functionally related. This is so because in the former case $e^2 = 2km^2$, while in the latter $e^2 = \frac{1}{2}km^2$.

5. GENERATING SOLUTIONS OF THE EMKG EQUATIONS

One of the more promising recent developments in general relativity is the discovery of techniques for generating new solutions of the stationary Einstein-Maxwell equations from known solutions. The origins of this project go back to the work of Bonnor[63], Buchdahl[64], Ehlers[65], Ernst[66], Harrison[67], Matzner and Misner[68], Geroch[69], and Kloster, Som and Das[6]. In the early 1970's it was discovered independently by Kramers, Neugebauer and Stephani[26], and Kinnersley[70], that the group SU(2,1) plays a fundamental role in transforming stationary electrovac solutions into one another. At the present time, Kinnersley and Chitre[71] and Ernst and Hauser[60] have developed the solution-generating technique to the point that some sort of "general solution" of the stationary axially symmetric Enstein-Maxwell equations may be at hand.

In this chapter, the method of Kramer, Neugebauer, and Stephani[26] (KNS) will be applied to the stationary EMKG equations with the meson source in isometric motion, with the hope of discovering a non-trivial group of transformations of the fields which generate "new solutions from old". To accomplish this, the Lagrangian density must be of the form $R + L_0$, where

$$L_0 \equiv L_{AB}\,(\Theta) g^{\alpha\beta}\, \Theta^A{}_{,\alpha}\, \Theta^B{}_{,\beta} \quad ,$$

R is the curvature scalar on M_3, the Θ^A are the fields $\phi, \phi^*, \Gamma, \Gamma^*, \eta$, the indices A,B,... have the range {1,2,3,4,5} and the L_{AB} are functions of the Θ^A. This, unfortunately, is not the case, since it turns out that the Lagrangian density is of the form $R + L_0 + H$, where

$$H \equiv 4kf^{-2}\left[m^2 f - \frac{e^2}{4} (\phi + \phi^*)^2 \right] \eta^2 \ ,$$

with

$$f \equiv e^{\omega} = \tfrac{1}{2} (\Gamma + \Gamma^* + k\phi^*\phi).$$

The KNS ansatz relies on the fact that a Lagrangian density of the form of L_0 above can be interpreted as a semi-Riemannian metric with components L_{AB} in a chart on some manifold ("potential space") with coordinates Θ^A. The "solution-generating group" is simply the group of isometries of the metric L_{AB}. The procedure that will be followed here is to find the group of transformations which preserves L_0 and then find the subset which also preserves H. In general, as shall be shown below, this subset is trivial. However, along the way, one discovers the isometries of L_0, and this is a group of transformations among solutions of the EMKG equations with a massless neutral meson source. It seems that the meson's mass breaks the KNS symmetry much as it breaks the conformal group symmetry of relativstic wave-equations in flat spacetime[72].

A. The KNS form of the Lagrangian Density

The Einstein-Hilbert Lagrangian density for the EMKG equations is given by equations (2.24) - (2.26), namely

$$\hat{L} = \hat{R} + 2k\left[-\tfrac{1}{4} F_{ij}F^{ij} + D^{*i}\psi^* D_i\psi - m^2\psi^*\psi \right] . \qquad (5.1)$$

If one simply imposes the conditions of stationarity and isometric motion on $\hat{L}$ (in a chart where one may use the real gauge), then the Euler-Lagrange equations of the "reduced" Lagrangian density so obtained are *not* the correct field equations (4.53) - (4.56) or (4.61) - (4.64), since some of the field equations have already been used. However, by examining the field equations, it is possible to "guess" an appropriate Lagrangian density. The following does the job:

$$L = R + L_0 + H, \tag{5.2}$$

where

$$\begin{aligned} L_0 \equiv & -\tfrac{1}{2}kf^{-2}(\Gamma + \Gamma^*)\phi^{*,\alpha}\phi_{,\alpha} + \tfrac{1}{2}f^{-2}\Gamma^{*,\alpha}\Gamma_{,\alpha} + \tfrac{1}{2}kf^{-2}\phi\phi^{*,\alpha}\Gamma_{,\alpha} \\ & + \tfrac{1}{2}kf^{-2}\phi^*\Gamma^{*,\alpha}\phi_{,\alpha} + 2k\eta^{,\alpha}\eta_{,\alpha} , \end{aligned} \tag{5.3}$$

$$H \equiv 2kf^{-2}\left[m^2 f - \frac{e^2}{4}(\phi + \phi^*)^2\right]\eta^2 . \tag{5.4}$$

The quantity f is

$$f \equiv e^{\omega} = \tfrac{1}{2}(\Gamma + \Gamma^* + k\phi^*\phi) , \tag{5.5}$$

where the equation just preceeding (4.38) on page 55 has been used. The terms $(R + L_0)$ are the Lagrangian density for the stationary EMKG equations with a massless neutral meson source.

The quantity L_0 is a quadratic form in the gradients of the five fields ϕ, ϕ^*, Γ, Γ^*, η. Following KNS, we interpret these fields as coordinates in a five-dimensional manifold K_5, called the "potential space". Capital latin indices range over {1,2,3,4,5} and denote components of geometric objects on K_5. In particular, we write

$$(\Theta^A) \equiv (\phi,\phi^*,\Gamma,\Gamma^*,\eta) \ , \tag{5.6}$$

$$(L_{AB}) \equiv \frac{1}{4} k f^{-2} \begin{bmatrix} 0 & -(\Gamma+\Gamma^*) & 0 & \phi^* & 0 \\ -(\Gamma+\Gamma^*) & 0 & \phi & 0 & 0 \\ 0 & \phi & 0 & k^{-1} & 0 \\ \phi^* & 0 & k^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 8f^2 \end{bmatrix} \tag{5.7}$$

Hence, the $L_{AB} = L_{BA}$ are the components of the metric $\tilde{L}_0$ defined by:

$$\tilde{L}_0 \equiv L_{AB}\, d\Theta^A\, d\Theta^B \ , \tag{5.8}$$

where "d" here denotes the exterior derivative on K_5. The matrix (L_{AB}) is block diagonal, i.e.,

$$(L_{AB}) = \left[\begin{array}{c|c} L_{ab} & 0 \\ \hline 0 & 2k \end{array}\right] \ , \tag{5.9}$$

where $a,b,\ldots \in \{1,2,3,4\}$.

Let G be the group of isometries of $\tilde{L}_0$. Clearly, G leaves the Lagrangian density L_0 invariant. In the next section, we shall see how the infinitesimal generators of G are obtained, and we will display the corresponding finite transformations.

B. Transformations which Preserve L_0

The infinitesimal generators of G are the Killing vector fields of $\tilde{L}_0$ in K_5. To obtain them, we solve Killing's equation in K_5:

$$
\begin{aligned}
\underset{V}{\pounds} L_{AB} &= V_{A||B} + V_{B||A} \\
&= V^C{}_{,B} L_{AC} + V^C{}_{,A} L_{CB} + V^C L_{AB,C} ,
\end{aligned}
\tag{5.10}
$$

where $||$ denotes the covariant derivative on K_5 with respect to the metric L_{AB}. We shall consider the three cases (i) $A = B = 5$, (ii) $A = 5$, $B = b$, and (iii) $A = a$, $B = b$.

(i) Equation (5.10) becomes

$$\underset{V}{\pounds} L_{55} = 2\, L_{5C}\, V^C{}_{,5} + L_{55,C} V^C = 0 .$$

But $L_{55} = 2K$ = a constant, and $L_{5C} = \delta_{5C} 2k$, so we have $V^5{}_{,5} = 0$, i.e.,

$$V^5 = \varepsilon\ (\Theta^a) \quad . \tag{5.11}$$

(ii) In this case,

$$\underset{V}{\pounds} L_{5b} = L_{5C} V^C{}_{,b} + L_{Cb} V^C{}_{,5} + L_{5b,C} V^C = 0 .$$

By (5.11), and since $L_{5b} = 0$, we have

$$2k\, \varepsilon_{,b} + L_{ab} V^a{}_{,5} = 0 .$$

But the first term is independent of $\Theta^5 = \eta$ and since $L_{ab,5} = 0$, it follows that

$$(L_{ab}V^a)_{,5} = -2k\varepsilon_{,b} .$$

Thus,

$$V_b = L_{ab}V^a = -2k\varepsilon_{,b}\eta + \chi_b \quad , \tag{5.12}$$

where χ_b are arbitrary functions of the Θ^a. Thus the Killing vector fields V^A of L_{AB} are of the form

$$(V_A) = (V_a, 2k\varepsilon) = (\chi_b - 2k\varepsilon_{,b}\eta, 2k\varepsilon). \tag{5.13}$$

(iii) Finally, we consider

$$\underset{V}{\pounds} L_{ab} = V_{a||b} + V_{b||a} = 0 .$$

By (5.13), the above is equivalent to the two equations

$$\underset{\chi}{\pounds} L_{ab} = \chi_{a||b} + \chi_{b||a} = 0 \quad , \tag{5.14}$$

$$\underset{\nabla\varepsilon}{\pounds} L_{ab} = 2\varepsilon_{||ab} = 0 \quad . \tag{5.15}$$

In Appendix D it is shown that (5.15) implies that ε = a constant. Hence, (5.13) reduces to

$$(V_A) = (\chi_a, 2k\varepsilon) \ . \tag{5.16}$$

The equations (5.14) are precisely the equations for the generators of the eight parameter group of KNS transformations of stationary Einstein-Maxwell solutions into one another[26]. This group is isomorphic to SU(2,1), and it was shown by Das and Kloster[72] that the group is generated by three continuous transformations and one discrete transformation. The action of these transformations on the fields ϕ, Γ and η are given below:

$$\begin{aligned}
T_1^{(\alpha)}\ (\phi,\Gamma,\eta) &= (\alpha\phi,\ |\alpha|^2\Gamma,\ \eta) \\
T_2^{(\gamma)}\ (\phi,\Gamma,\eta) &= (\phi,\ \Gamma + i\gamma,\ \eta) \\
T_3^{(\delta)}\ (\phi,\Gamma,\eta) &= (\phi + 2\delta^*,\ \Gamma - 2k\delta\phi - 2k|\delta|^2,\ \eta) \\
T\ (\phi,\Gamma,\eta) &= (\phi\Gamma^{-1},\ \Gamma^{-1},\ \eta)
\end{aligned} \tag{5.17}$$

The parameters α and δ are complex, while γ is real. The fields ϕ^* and Γ^* transform as the complex conjugates of ϕ and Γ.

From the two continuous transformations T_2 and T_3 and the discrete transformation T, one may generate the two remaining KNS transformations as follows:

$$T_4^{(\lambda)}\,(\phi,\Gamma,\eta) \equiv (T\circ T_2^{(\lambda)}\circ T)\,(\phi,\Gamma,\eta)$$

$$= (\phi(1+i\lambda\Gamma)^{-1},\ \Gamma(1+i\lambda\Gamma)^{-1},\eta) \tag{5.18}$$

$$T_5^{(\beta)}\,(\phi,\Gamma,\eta) \equiv (T\circ T_3^{(\beta)}\circ T)\,(\phi,\Gamma,\eta)$$

$$=\left(\frac{\phi+2\beta^*\Gamma}{1-2k\beta\phi-2k|\beta|^2\Gamma}\ ,\ \frac{\Gamma}{1-2k\beta\phi-2k|\beta|^2\Gamma},\eta\right)$$

The parameter λ is real and β is complex.

The infinitesimal generator $(0,0,0,0,2k\varepsilon)$ generates the finite transformation

$$P(\phi,\Gamma,\eta) = (\phi,\Gamma,\eta + 2k\varepsilon) \quad . \tag{5.19}$$

The transformations (5.17) and (5.19) form a group G isomorphic to $SU(2,1) \times \mathcal{T}_1$ where $\mathcal{T}_1$ is the one-parameter group of translations.

C. The Triviality of G

The group $G \simeq SU(2,1) \times \mathcal{T}_1$ preserves the term L_0, defined by equation (5.3), in the Lagrangian density $L = R + L_0 + H$. Since, the scalar curvature R on M_3 does not depend on any of the fields Θ^A, it is trivially preserved under G. It remains to examine the effect of G on the term H in L. Unfortunately, it turns out that the subset of G which preserves H is trivial.

First we consider the effect of the KNS transformations, which are isomorphic to SU(2,1), on H. To this end, we consider the effect of the transformations T_1, T_2, T_3, and T on H. Under each of these transformations, the quantity H will become

$$H' = 2k(f')^{-2}\left[(m')^2 f' - \frac{1}{4}(e')^2 (\phi' + \phi'^*)^2\right](\eta')^2 \quad , \tag{5.20}$$

where

$$f' = \frac{1}{2}(\Gamma' + \Gamma'^* + k\phi'^*\phi') \quad . \tag{5.21}$$

(i) Let $(\phi', \Gamma', \eta') = T_1^{(\alpha)}(\phi, \Gamma, \eta)$. Thus,

$$f' = |\alpha|^2 f \tag{5.22}$$

and H' is

$$H' = 2k|\alpha|^{-2} f^{-2}\left[(m')^2 f - \frac{1}{4}(e')^2 |\alpha|^{-2}(\alpha\phi + \alpha^*\phi^*)^2\right]\eta^2 \quad . \tag{5.23}$$

If the parameters e and m are non-zero, then the subset of T_1 which preserves H consists of transformations of the form:

$$\begin{aligned} \phi' &= a\phi \quad , \\ \Gamma' &= a^2\Gamma \quad , \\ m' &= \pm\, am \quad , \\ e' &= \pm\, ae \quad , \\ \eta' &= \eta \quad , \end{aligned} \tag{5.24}$$

where a is <u>real</u> parameter.

(ii) The transformations T_2 preserve H with no restriction on the real parameter γ, i.e.,

$$\phi' = \phi ,$$

$$\Gamma' = \Gamma + i\gamma ,$$

$$m' = m , \qquad (5.25)$$

$$e' = e ,$$

$$\eta' = \eta ,$$

preserve H.

(iii) Under T_3, $f' = f$, and H' becomes

$$H' = 2kf^{-2}\left[(m')^2 f - \tfrac{1}{4}(e')^2\,(\phi + \phi^* + 2(\delta + \delta^*))^2\right]\eta^2 . \qquad (5.26)$$

Hence, the subset of T_3 which preserves H is given by

$$\phi' = \phi - 2id ,$$

$$\Gamma' = \Gamma - 2ikd\phi - 2kd^2 ,$$

$$m' = m , \qquad (5.27)$$

$$e' = e ,$$

$$\eta' = \eta ,$$

where the parameter d is real.

(iv) Under the discrete transformation T, one obtains

$$f' = |\Gamma|^{-2} f, \tag{5.28}$$

and

$$H' = 2k|\Gamma|^2 f^{-2} \left[m^2 f - \frac{1}{4} e^2 |\Gamma|^{-2} (\Gamma^* \phi + \Gamma \phi^*)^2 \right] \eta^2 \ . \tag{5.29}$$

(The constants m and e must transform as $m' = m$ and $e' = e$ under T if they are to remain constant.) Hence, in general, H is not invariant under T.

Under the transformations $P^{(\varepsilon)}(\phi,\Gamma,\eta) \equiv (\phi,\Gamma,\eta + 2k\varepsilon)$, H becomes

$$H' = 2kf^{-2} \left[(m')^2 f - \frac{1}{4}(e')^2 (\phi + \phi^*)^2 \right] (\eta^2 + 4k\varepsilon\eta + 4k^2\varepsilon^2) \ .$$

Hence, the subset of P which preserves H is the identify element ($\varepsilon = 0$).

Thus the subset of G which preserves L is generated by $T_1^{(a)}$, $T_2^{(\gamma)}$, and $T_3^{(d)}$, with the parameters a, γ, and d all real. The full group G does transform solutions of the EMKG equations with massless neutral meson fields into one another.

The transformations $T_1^{(a)}$ and $T_2^{(\gamma)}$, given by (5.24) and (5.25). respectively, are clearly trivial. To show that the remaining transformation, $T_3^{(d)}$, given by (5.27), is trivial, it suffices to show that e^{ω} and the twist vector τ_α get mapped into themselves.

Let ϕ' and Γ' denote the result of the action of the transformation (5.27) on ϕ and Γ. Then since ϕ' and Γ' are solutions of the EMKG equations, there are quantities ω' and Ω' such that, by equation (4.38)

$$\Gamma' = e^{\omega'} - \frac{k}{2}\phi'^*\phi' + i\Omega' \quad . \tag{5.30}$$

Now, by (5.27), we have,

$$\Gamma - 2ikd\phi - 2kd^2 = \Gamma' = e^{\omega'} - \frac{1}{2}k\phi^*\phi + ikd(\phi^* - \phi) - 2kd^2 + i\Omega' \quad . \tag{5.31}$$

Hence, by using (4.38) for Γ on the left hand side,

$$e^{\omega} + i\Omega = e^{\omega'} + i\left[kd(\phi + \phi^*) + \Omega' \right] \quad .$$

Since d is real, it follows that

$$\omega' = \omega \quad , \tag{5.31}$$

$$\Omega' = \Omega - kd(\phi + \phi^*) \quad . \tag{5.32}$$

If we take the gradient (in M_3, of course) of both sides of the latter equation, we get

$$\Omega'_{,\alpha} = \Omega_{,\alpha} - kd(\phi + \phi^*)_{,\alpha}$$

The transformed twist potential Ω' must satisfy (4.39), and so from (5.33),

$$\tau'_\alpha - \frac{1}{2}\, ik(\phi'^*\phi'_{,\alpha} - \phi'^*_{,\alpha}\phi') = \tau_\alpha - \frac{1}{2}\,(\phi^*\phi_{,\alpha} - \phi^*_{,\alpha}\phi) - kd(\phi + \phi^*)_{,\alpha} \quad ,$$

and hence, using (5.27) again, we obtain

$$\tau_\alpha{}' = \tau_\alpha \quad . \tag{5.34}$$

The results (5.31) and (5.34) show that the transformation $T_3{}^{(d)}$ is trivial.

6. AXIAL SYMMETRY

A uniformly rotating source produces stationary fields. Thus it seems natural to introduce the additional symmetry that the metric is invariant under rotations about some given line (the "polar axis"). This will be accomplished by assuming that there exists another Killing vector $\vec{\rho}$ whose trajectories are closed curves in M_4 and which commutes with the timelike Killing vector $\vec{\xi}$, i.e., $\mathop{£}_{\xi}\vec{\rho} = [\vec{\xi},\vec{\rho}] = -\mathop{£}_{\rho}\vec{\xi} = 0$. A subset of the Copernican charts, with coordinates (r,z,θ,t), has an angular coordinate θ whose curves are the trajectories of $\vec{\rho}$. It will be further assumed that $\mathop{£}_{\rho}\eta = \mathop{£}_{\rho} A_i = 0$, so that, it terms of the above charts, the components on the metric $g_{\alpha\beta}$ on M_3, the complex potentials ϕ and Γ, and the meson field η, depend only on the radial coordinate r and the axial coordinate z. With these assumptions, the stationary EMKG equations with isometric motion and with the WMP condition holding, will be written as partial differential equations on an auxilliary Euclidean space E^3. Finally, a class of exact solutions of these reduced equations with A = a constant will be displayed.

A. The Periodic Killing Vector Field

An axially symmetric metric is characterized by the existence of a Killing vector field $\vec{\rho}$ whose trajectories are closed curves, i.e., $\vec{\rho}$ is "periodic". It is further assumed that $\vec{\rho}$ is compatible with the timelike Killing vector field $\vec{\xi}$[73]:

$$\mathop{£}_{\xi}\vec{\rho} = \left[\vec{\xi},\vec{\rho}\right] = -\left[\vec{\rho},\vec{\xi}\right] = -\mathop{£}_{\rho}\vec{\xi} = 0 \quad . \tag{6.1}$$

This condition implies that one may choose charts on M_4 such that the trajectories of both $\vec{\rho}$ and $\vec{\xi}$ are coordinate curves[33]. Such charts form a subset of the Copernican charts. If one denotes by x^3 and $x^4 \equiv t$ the coordinates whose curves are, respectively, the trajectories of $\vec{\rho}$ and $\vec{\xi}$, then clearly in such a chart, $\rho^i = \delta^i_3$ and $\xi^i = \delta^i_4$. Furthermore, in such a Copernican chart $g_{\alpha\beta}$, a_α, and ω are independent of both x^3 and t. In general, it is <u>not</u> possible to guarantee in advance that a solution of the appropriate field equations will be such as to make $\vec{\rho}$ spacelike everywhere[73]. Spacetime will, in general, contain smooth closed non-spacelike curves.

It is well known (cf., for example, Krasinski[73], and references therein) that there exists a subset of charts of the above type, with coordinates denoted (r, z, θ, t), in which the metric has the form:

$$\hat{g} = -e^{-\omega}\left[e^{\nu}(dr^2 + dz^2) + e^{2\lambda}d\theta^2\right] + e^{\omega}(a\,d\theta + dt)^2. \qquad (6.2)$$

In the above, θ is an angular coordinate, whose curves are the trajectories of $\vec{\rho}$, and (r,z) are rectangular Cartesian coordinates in the half-plane θ = a constant, t = a constant. The domain of these coordinates is the open set in M_4 whose image in $\mathbb{R}^4$ under the coordinate map is $0 < r < \infty$, $-\infty < z < \infty$, $0 < \theta < 2\pi$, $-\infty < t < \infty$. The functions ω, ν, λ, and a depend on (r,z) only. The polar axis $r = 0$ is <u>at least</u> a coordinate singularity. The coordinates (r,z,θ,t) are called <u>cylindrical Copernican</u> coordinates. The considerations of Chapter 3 suggest that the additional requirement that either $\pounds_\rho M_{ij} = 0$ or $\pounds_\rho E_{ij} = 0$, suffices for $\pounds_\rho \eta = \pounds_\rho A_i = 0$. In any case, we shall simply assume that the latter hold. The result is that the complex electromagnetic potential ϕ, the gravitational potential Γ, and the meson field η depend on (r,z) only.

B. The Cylindrically Symmetric EMKG Equations

The Riemannian metric g on M_3 in cylindrical Copernican coordinates is

$$g = e^{\nu(r,z)}\,(dr^2 + dz^2) + e^{2\lambda(r,z)}\,d\theta^2 \quad . \tag{6.3}$$

The components of the Ricci tensor $R_{\alpha\beta}$ of g in these coordinates are easily computed:

$$\begin{aligned} R_{11} &= \frac{1}{2}\Delta\nu + \lambda_{,11} + \lambda^2_{,1} - \frac{1}{2}(\lambda_{,1}\nu_{,1} - \lambda_{,2}\nu_{,2}) \quad , \\ R_{22} &= \frac{1}{2}\Delta\nu + \lambda_{,22} + \lambda^2_{,2} + \frac{1}{2}(\lambda_{,1}\nu_{,1} - \lambda_{,2}\nu_{,2}) \quad , \\ R_{12} &= \lambda_{,12} + \lambda_{,1}\lambda_{,2} - \frac{1}{2}(\lambda_{,1}\nu_{,2} + \nu_{,1}\lambda_{,2}) \quad , \\ R_{33} &= e^{2\lambda-\nu}(\Delta\lambda + |\nabla\lambda|^2) = e^{\lambda-\nu}\Delta(e^{\lambda}) \quad , \end{aligned} \tag{6.4}$$

where $x^1 = r$, $x^2 = z$, $\Delta\nu \equiv \nu_{,11} + \nu_{,22}$, and $|\nabla\lambda|^2 \equiv \lambda^2_{,1} + \lambda^2_{,2}$.

The following theorem is further motivation for imposing the WMP condition on the EMKG equations:

Theorem 6.1. Assume the stationary EMKG equations with isometric motion hold. Furthermore, assume that all the fields are cylindrically symmetric. Then the metric $\hat{g}$ on M_4 can be put in the Weyl-Lewis form[74,75], namely, in cylindrical Copernican coordinates,

$$\hat{g} = -e^{-\omega}\left[e^{\nu}(dr^2 + dz^2) + r^2 d\theta^2\right] + e^{\omega}\,(a\,d\theta + dt)^2 \quad , \tag{6.5}$$

if and only if the WMP condition (4.65) holds.

Proof: Consider the field equation (4.61), with $\alpha = \beta = 3$. By use of (6.4) and the facts $\phi_{,3} = \Gamma_{,3} = \eta_{,3} = 0$, the field equation becomes

$$e^{\lambda-\nu}\Delta(e^{\lambda}) + 2ke^{-2\omega+2\lambda}\,(m^2e^{\omega} - e^2A^2)\,\eta^2 = 0.$$

If the metric is of the form (6.5), i.e., $e^{\lambda} = r$, then the above equation implies that (4.65) holds. On the other hand, if (4.65) holds, then e^{λ} must be harmonic in (r,z). Following Synge[47], we reason as follows: Write $\bar{r} \equiv e^{\lambda(r,z)}$. Then $\bar{r}$ harmonic implies that there exists a conjugate harmonic function $\bar{z}(r,z)$ such that

$$\bar{r} + i\bar{z} = f(r + iz),$$

where f is analytic. The transformation $(r,z) \to (\bar{r},\bar{z})$ is then conformal[76], so it preserves the form $e^{\nu(r,z)}(dr^2 + dz^2)$, i.e.,

$$e^{\nu(r,z)}\,(dr^2 + dz^2) \to e^{\bar{\nu}(\bar{r},\bar{z})}\,(d\bar{r}^2 + d\bar{z}^2)\quad .$$

Thus we have succeeded in putting $\hat{g}$ in the Weyl-Lewis form (6.5).//

All the field equations (4.72) - (4.76) can now be written as partial differential equations on an auxilliary Euclidean $\mathbb{R}^3$. The equations below are the EMKG equations with stationarity, isometric motion, the WMP condition, $e^2 = 16\pi m^2$, and axial symmetry imposed:

$$\Delta\nu = -\frac{3}{2}A^{-2}\,|\nabla A|^2 - 2k\,|\nabla\eta|^2\quad , \tag{6.6}$$

$$\nu_{,r} = r\,\{\frac{3}{2}A^{-2}\left[A^2_{,r} - A^2_{,z}\right] + 2k\left[\eta^2_{,r} - \eta^2_{,z}\right]\}\quad , \tag{6.7}$$

$$\nu_{,z} = r\,\{3A^{-2}\,A_{,r}A_{,z} + 4k\eta_{,r}\eta_{,z}\}\quad , \tag{6.8}$$

$$\nabla^2 B - 3A^{-1}\, \nabla A \cdot \nabla B = 0 \quad , \tag{6.9}$$

$$\nabla^2 A - A^{-1} |\nabla A|^2 = 0 \quad , \tag{6.10}$$

$$|\nabla B|^2 - |\nabla A|^2 = 2\, m^2\, e^{\nu}\, \eta^2 \quad , \tag{6.11}$$

$$\nabla^2 \eta = 0 \quad , \tag{6.12}$$

where $A_{,r} \equiv \frac{\partial A}{\partial r}$, $A_{,z} \equiv \frac{\partial A}{\partial z}$, ∇ is the usual gradient operator in cylindrical polar coordinates on E^3 ($\mathbb{R}^3$ with a Euclidean metric), ∇^2 is the Laplacian, $\nabla A \cdot \nabla B$ is the scalar product, and $|\nabla A| \equiv (\nabla A \cdot \nabla A)^{\frac{1}{2}}$.

Some comments on these equations are in order before more restrictions are imposed on them.

(i) The equations (6.7) and (6.8) are integrable. This is actually a consequence of the contracted Bianchi identities on M_3 (equations 4.66), and the field equations (6.10) and (6.12). Hence, given harmonic functions η and $\ln A$ on E^3, ν can be computed from the following path-independent contour integral in E^3:

$$\nu(r,z) = \int_C \vec{v} \cdot d\vec{s} \quad , \tag{6.13}$$

where

$$\vec{v} \equiv \left(r\left[\frac{3}{2}\, A^{-2}(A^2_{,r} - A^2_{,z}) + 2k\, (\eta^2_{,r} - \eta^2_{,z})\right],\; r\left[3A^{-2}A_{,r}A_{,z} + 4k\eta_{,r}\eta_{,z}\right], 0\right),$$

$$d\vec{s} \equiv (dr,\, dz,\, r d\theta) \quad ,$$

and C is a contour from a given point to the point (r,z,θ). Also it should be noted that equation (6.6) is not independent, but is a consequence of the rest of the field equations. The situation here is not without precedent in the quest for exact axially symmetric solutions. See Synge[47], page 311- for the case of static vacuum solutions, and Eris and Gürses[17] for the case of a neutral massless meson field source of the Einstein-Maxwell equations.

(ii) There are no non-trivial static solutions of (6.6) - (6.12). This follows immediately from (6.11) since $\tau_\alpha = 0$ imples that $B_{,\alpha} = 0$, which, in turn, holds if and only if $A_{,\alpha} = 0$ and either $m = 0$ or $\eta = 0$.

If $m = 0$, then there is a family of solutions of the form:

$$\hat{g} = -e^{\nu(r,z)} (dr^2 + dz^2) - r^2 d\theta^2 + dt^2 \quad , \tag{6.14}$$

where $\nu(r,z)$ is given by

$$\nu(r,z) = 2k \int_C \left[r(\eta^2_{,r} - \eta^2_{,z})\, dr + 2r\eta_{,r}\eta_{,z} dz \right] \quad , \tag{6.15}$$

for each harmonic function $\eta(r,z)$. These are static solutions of the Einstein equations with an electrically neutral massless meson field source, and are a subclass of the solutions found for that case by Eris and Gürses[17]. If $\eta \equiv 0$, then from (6.7) and (6.8), ν = a constant and hence $\hat{g}$ is flat and the electromagnetic and meson fields vanish.

C. The Case A = a Constant

The Lorentz force on the meson is given by $F_{ij}J^j$ which, by the assumption of isometric motion, is proportional to F_{i4}. Since $F_{i4} = A_{,i}$, the condition A = a constant is equivalent to the vanishing of the Lorentz force. Thus the motion of the meson is geodesic[77] (at least at the classical level). This constitutes a physical interpretation (or at least motivation) for the condition A = a constant. The result of the latter is that the field equations (6.6) - (6.12) become:

$$\Delta\nu = -2k|\nabla\eta|^2 \quad , \tag{6.16}$$

$$\nu_{,r} = 2kr\,(\eta_{,r}^2 - \eta_{,z}^2) \quad , \tag{6.17}$$

$$\nu_{,z} = 4kr\,\eta_{,r}\eta_{,z} \quad , \tag{6.18}$$

$$\nabla^2 B = 0 \quad , \tag{6.19}$$

$$|\nabla B|^2 = 2m^2 e^{\nu}\eta^2 \quad , \tag{6.20}$$

$$\nabla^2\eta = 0 \quad . \tag{6.21}$$

The existence and uniqueness of solutions of the above field equations are shown by the following theorem:

Theorem 6.2. If ν, B, and η are C^2 functions in the auxilliary space E^3, then:

(i) Equations (6.16) - (6.21) have solutions only if η = a constant;

(ii) The following is a solution, unique up to the choice of parameters a_0, m, ν, A, B_0, and η:

$$\hat{g} = -(2kA^2)^{-1}\left[e^{\nu}(dr^2 + dz^2) + r^2 d\theta^2\right] + (2kA^2)\left[a(r)d\theta + dt\right]^2 , \qquad (6.22)$$

$$a(r) = \pm\left[(2^{3/2}kA^3)^{-1}e^{\nu/2}\eta m\right] r^2 + a_o , \qquad (6.23)$$

A = a constant,

$$B = \pm\,(2^{\frac{1}{2}}e^{\nu/2}\eta m)\; z + B_o , \qquad (6.24)$$

η = a constant.

Proof:

(i) If $\eta(r,z)$ is a solution of (6.21) then there is a solution $\nu(r,z)$ unique up to a constant of integration, of (6.17) and (6.18). Write $f(r,z) \equiv 2^{\frac{1}{2}}me^{\nu/2}\eta$. If $B(r,z)$ is C^2 and satisfies (6.19) and (6.20), then there exists a function $\alpha(r,z)$ such that

$$B_{,r} = f\cos\alpha , \qquad (6.25)$$

$$B_{,z} = f\sin\alpha , \qquad (6.26)$$

if and only if

$$f^{-1} f_{,r} + \alpha_{,z} = -r^{-1}\cos^2\alpha , \qquad (6.27)$$

$$f^{-1} f_{,z} - \alpha_{,r} = -r^{-1}\cos\alpha\,\sin\alpha . \qquad (6.28)$$

Now define $\lambda \equiv \ln|f|$, differentiate (6.27) with respect to r and (6.28) with respect to z, and add the resulting equations to get

$$\Delta\lambda - r^{-1}(\cos 2\alpha\,\lambda_{,r} + \sin 2\alpha\,\lambda_{,z}) - 2r^{-2}\cos^2\alpha = 0 \quad . \tag{6.29}$$

Use (6.17), (6.18) and (6.21) to express $\Delta\lambda$, $\lambda_{,r}$ and $\lambda_{,z}$ in terms of η, $\eta_{,r}$ and $\eta_{,z}$ only. Then (6.29) is a quadratic equation in r^{-1}:

$$C_1 r^{-2} + C_2 r^{-1} + C_3 = 0 \quad , \tag{6.30}$$

where

$$C_1 \equiv 2\cos^2\alpha \quad ,$$

$$C_2 \equiv 2\eta^{-1}\cos\alpha\,(\cos\alpha\,\eta_{,r} + \sin\alpha\,\eta_{,z}) \quad ,$$

$$C_3 \equiv \eta^{-2}\left[16\pi(\cos\alpha\,\eta_{,r} + \sin\alpha\,\eta_{,z})^2 + |\nabla\eta|^2\right] \quad .$$

Thus $C_2^2 - 4C_1C_3 = 4\,\eta^{-2}\left[(\vec{n}\cdot\nabla\eta)^2 - 2|\nabla\eta|^2 - 4k\,\eta^2\,(\vec{n}\cdot\nabla\eta)^2\right]$, where $\vec{n} \equiv (\cos\alpha, \sin\alpha)$. Since $|\vec{n}|^2 = 1$, it follows that $(\vec{n}\cdot\nabla\eta)^2 - 2|\nabla\eta|^2 < 0$ for $\nabla\eta \neq 0$. Thus if $|\nabla\eta| \neq 0$, $C_2^2 - 4C_1C_3 < 0$, so r^{-1} must be complex. Hence, in order to get solutions depending on real values of r, it must be that $|\nabla\eta| = 0$, i.e., η = constant.

(ii) Clearly if η = constant, then ν = constant. Thus the solution of equations (6.16) - (6.21) reduces to finding a C^2 function $B(r,z)$ such that $\nabla^2 B = 0$ and $|\nabla B|^2 = p^2 \equiv 2m^2 e^{\nu}\eta^2$ = constant. Now the equations (6.27) and (6.28) become, respectively,

$$\alpha_{,z} = -r^{-1}\cos^2\alpha \quad ,$$

$$\alpha_{,r} = r^{-1}\cos\alpha\,\sin\alpha \quad .$$

The only C^1 functions $\alpha(r,z)$ satisfying these last two equations are constant functions $\alpha = \pm \pi/2, \pm 3\pi/2, \ldots$. Hence, by (6.25) and (6.26) $B_{,r} = 0$ and $B_{,z} = \pm p$. In conclusion, $B = \pm pz + B_0$, where B_0 is an arbitrary constant of integration, η = constant, and ν = constant is the unique class of solutions of equations (6.16) - (6.21).//

In the next chapter, the geometrical and physical properties of these solutions will be elucidated.

7. PROPERTIES OF THE SOLUTION

In this chapter, some of the properties of the solution of the EMKG equations obtained in the last chapter will be displayed. It will be shown, in particular, that the metric $\hat{g}$ of the solution is static if and only if it is flat. The latter will be shown to hold only if the mass m of the meson is zero or if the constant wave-function $\eta = 0$. The causality properties of $\hat{g}$ will be examined, and it will be shown that the trajectories of the periodic Killing vector field $\vec{\rho}$ are timelike in a region of M_4 exterior to the cylinder $0 < r \leq r_0 \equiv (k^{\frac{1}{2}}M)^{-1}$. This causal pathology is similar to that found in the Gödel solution[47]. The "sources" will be shown to obey the strong energy condition of Hawking and Ellis[34]. Finally, the physical relevance of the solution will be commented upon.

A. Geometric Properties of the Metric

We will consider here some of the properties of the metric

$$\hat{g} = -(2kA^2)^{-1}\left[e^{\nu}(dr^2 + dz^2) + r^2 d\theta^2\right] + (2kA^2)\left[a(r)d\theta + dt\right]^2 \quad , \qquad (7.1)$$

where

$$a(r) = \pm (2^{3/2}k\,A^3)^{-1}\, e^{\nu/2}\,\eta m\, r^2 + a_0 \quad . \qquad (7.2)$$

In the above, η and m are arbitrary positive constants, A is a non-zero constant, and ν and a_0 are arbitrary constants.

(1) It is easy to see that the metric g on M_3, given by

$$g = e^{\nu}\,(dr^2 + dz^2) + r^2 d\theta^2 \quad , \qquad (7.3)$$

is flat. Hence with no loss of generality, one may choose $\nu = 0$, so that (7.1) and (7.2) become:

$$\hat{g} = -(2kA^2)^{-1}\left[dr^2 + dz^2 + r^2 d\theta^2\right] + (2kA^2)\left[a(r)d\theta + dt\right]^2 \quad , \tag{7.4}$$

$$a(r) = \pm\, (2^{3/2}\, kA^3)^{-1}\, \eta m\, r^2 + a_o \quad . \tag{7.5}$$

Now make the following coordinate transformation:

$$r' = C^{-1} r \quad ,$$

$$z' = C^{-1} z \quad ,$$

$$\theta' = \theta \quad ,$$

$$t' = Ct \quad ,$$

where $C^2 \equiv 2kA^2$. After dropping the primes, (7.5) and (7.6) can be written in the form:

$$\hat{g} = -\left[dr^2 + dz^2 + r^2 d\theta^2\right] + \left[a(r)d\theta + dt\right]^2 \quad , \tag{7.5'}$$

$$a(r) = \pm\, k^{\frac{1}{2}} M r^2 + a_o \quad , \tag{7.6'}$$

where $M \equiv \eta m$. In these coordinates, the WMP condition $e^{\omega} = 1 = (e^2/m^2)\, A^2$ implies that A is given by

$$A = \pm\, (2k)^{-\frac{1}{2}} \quad . \tag{7.7}$$

The magnetic potential B is then of the form

$$B = \pm\, 2^{\frac{1}{2}} Mz + B_0 \quad . \tag{7.8}$$

The meson field is, of course, still given by η = a constant.

(2) The components of the twist vector τ_α are most easily computed by substituting (7.8) into (4.23). The result is

$$\begin{aligned} \tau_1 &= \tau_3 = 0 \quad , \\ \tau_2 &= \pm\, 2k^{\frac{1}{2}} M \quad . \end{aligned} \tag{7.9}$$

Hence, $\hat{g}$ is static only if $M = 0$, i.e., either $m = 0$ or $\eta = 0$.

(3) The invariant components of $\hat{R}_{ijkm}$ are now computed. The orthonormal tetrad used is ω defined by:

$$\begin{aligned} \omega^1 &= dr \quad , \\ \omega^2 &= dz \quad , \\ \omega^3 &= r d\theta \quad , \\ \omega^4 &= a(r) d\theta + dt \quad . \end{aligned} \tag{7.10}$$

The connection 1-form $\omega^a{}_b$ and the curvature 2-form $\Omega^a{}_b$ are easily computed from the structure equations (2.11) and (2.9), respectively. The result is

$$\omega^1{}_3 = \pm k^{\frac{1}{2}} M \omega^4 - r^{-1}\omega^3 \;,$$

$$\omega^1{}_4 = \pm k^{\frac{1}{2}} M \omega^3 \;, \tag{7.11}$$

$$\omega^3{}_4 = \mp k^{\frac{1}{2}} M \omega^1 \;,$$

$$\omega^1{}_2 = \omega^2{}_3 = \omega^2{}_4 = 0 \;,$$

$$\Omega^1{}_3 = 3kM^2 \omega^1 \wedge \omega^3 \;,$$

$$\Omega^1{}_4 = kM^2 \omega^1 \wedge \omega^4 \;, \tag{7.12}$$

$$\Omega^3{}_4 = kM^2 \omega^3 \wedge \omega^4 \;,$$

$$\Omega^1{}_2 = \Omega^2{}_3 = \Omega^2{}_4 = 0 \;.$$

Finally, the invariant components $\hat{R}_{(abcd)}$ of the curvature tensor can be obtained from $\Omega^a{}_b = \frac{1}{2}\hat{R}^{(a)}{}_{(bcd)}\omega^c \wedge \omega^d$, and the non-vanishing components are:

$$\hat{R}_{(1331)} = 3kM^2 \;,$$

$$\hat{R}_{(1441)} = kM^2 \;, \tag{7.13}$$

$$\hat{R}_{(3443)} = kM^2 \;.$$

Thus $\hat{g}$ is flat only if $\eta = 0$ or $m = 0$. Note that the $\hat{R}_{(abcd)}$ are constants.

(4) Since $\hat{g} \equiv \det \hat{g}_{ij} = -r^2$, there is at least a coordinate singularity on the polar axis $r = 0$. In order that the polar axis be part of the smooth manifold M_4, the condition of elementary flatness[47] must be satisfied for regions of M_4 which contain the surface $r = 0$. Thus we must check that

$$\lim_{r\to o} \frac{C_r}{R_r} = 2\pi \quad , \tag{7.14}$$

where C_r is the circumference of a circle centered on the polar axis and R_r is its radius, both computed with respect to the metric $\hat{g}$. We need only consider circles for which z and t are constant. Hence

$$C_r = 2\pi|(a(r))^2 - r^2|^{\frac{1}{2}} = 2\pi|kM^2r^4 + (\pm 2k^{\frac{1}{2}}Ma_o - 1)r^2 + a_o^2|^{\frac{1}{2}} \quad ,$$

and $R_r = r$. Thus

$$\lim_{r\to o} \frac{C_r}{R_r} = 2\pi|kM^2r^2 \pm 2kMa_o - 1 + a_o^2 r^{-2}|^{\frac{1}{2}} \quad .$$

We see that the elementary flatness condition (7.14) is satisfied only if $a_o = 0$. Henceforth, it is assumed the above holds, so that

$$a(r) = \pm k^{\frac{1}{2}}Mr^2 \quad . \tag{7.15}$$

(5) The magnitude of the periodic Killing vector field $\vec{\rho}$ is

$$\hat{g}(\vec{\rho},\vec{\rho}) = \hat{g}_{33} = (a(r))^2 - r^2 = (kM^2r^2-1)r^2 \quad . \tag{7.16}$$

Write $r_0 \equiv (k^{\frac{1}{2}}M)^{-1}$. Thus $\vec{p}$ is spacelike for $0 < r < r_0$, null for $r = r_0$, and timelike for $r > r_0$.

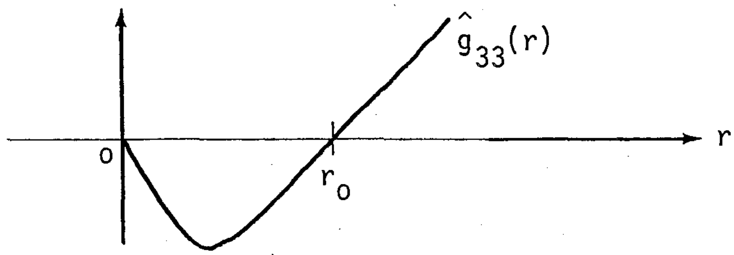


Figure 7.1: Graph of $\hat{g}_{33}(r)$.

There is a type of horizon at r_0. A Copernican observer at r_0 would be rotating at the speed of light, as seen by another Copernican observer at some $r \neq r_0$. For $r > r_0$, the trajectories of $\vec{p}$ are smooth closed timelike curves. Such causal pathologies are observed in the Gödel solution[34,47].

(6) The surface defined by $r = 0$ consists of the locus of fixed points of the Killing vector field $\vec{p}$. However, with $a_0 = 0$, the manifold M_4 can be extended to include this surface. This can be easily seen by transforming to rectangular Cartesian coordinates (x, y, z, t). The metric is then of the form:

$$\hat{g} = -\left[dx^2 + dy^2 + dz^2\right] + \left[\pm r_0^{-1}(xdy - ydx) + dt\right]^2 .$$

Clearly the metric is regular at $x = y = 0$.

Define the "twist charge"[78] (or, in the parlance of Gibbons and Hawking[79], the "nut charge") enclosed by a closed surface S in M_3 as

$$N_S \equiv k^{-1} \oint_S \tau_\alpha \, dS^\alpha$$

Since τ_α is a constant vector field on all of M_3, including the polar axis, it follows that $N_S = 0$ for any S. This means that, unlike the Taub-NUT metric[80], the metric considered here cannot be interpreted as a "gravitational dyon"[81].

B. The Strong Energy Condition

The non-vanishing components of T_{ij}, the energy-momentum tensor field of the electromagnetic and mesonic "sources", are:

$$\begin{aligned} T_{11} &= -T_{22} = M^2 \quad , \\ T_{33} &= M^2 \left[r^2 + 3(a(r))^2 \right] \quad , \\ T_{34} &= 3M^2 \, a(r) \quad , \\ T_{44} &= 3M^2 \quad . \end{aligned} \tag{7.17}$$

The non-vanishing components of the current four-vector field J_i are:

$$\begin{aligned} J_3 &= \pm (8k)^{\frac{1}{2}} M^2 a(r) \quad , \\ J_4 &= \pm (8k)^{\frac{1}{2}} M^2 \quad . \end{aligned} \tag{7.18}$$

It will be shown here that the T_{ij} satisfy the strong energy condition of Hawking and Ellis[34]. This condition means that

$$T_{ij}W^iW^j \geq \frac{1}{2} W^iW_iT^j_j \quad , \tag{7.19}$$

for any timelike vector W^i.

We will show that (7.17) satisfy (7.19) by considering the eigenvalues λ_a and the corresponding set of linearly independent eigenvectors $V^i_{(a)}$ of T_{ij}. Thus $T_{ij} V^j_{(a)} = \lambda_a V_{(a)i}$. The eigenvalues are easily computed from $\det(T_{ij} - \lambda g_{ij}) = 0$. We have

$$\begin{aligned} \lambda_1 &= \lambda_2 = -M^2 \quad , \\ \lambda_3 &= M^2 \quad , \\ \lambda_4 &= 3M^2 \quad . \end{aligned} \tag{7.20}$$

The desired set of eigenvectors are*

$$\begin{aligned} V^j_{(1)} &= 2^{-\frac{1}{2}} (1,0,r^{-1}, -r^{-1}a(r)) \quad , \\ V^j_{(2)} &= 2^{-\frac{1}{2}} (1,0,-r^{-1},r^{-1}a(r)) \quad , \\ V^j_{(3)} &= (0,1,0,0) \quad , \\ V^j_{(4)} &= (0,0,0,1) \quad . \end{aligned} \tag{7.21}$$

The eigenvectors $V^j_{(1)}$, $V^j_{(2)}$, and $V^j_{(3)}$ are spacelike, while the eigenvector $V^j_{(4)}$ is timelike. The $V^j_{(a)}$ form an orthonormal tetrad.

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* The author thanks Mr. Ted Biech for doing the computation correctly.

Let W^i be an arbitrary timelike vector. Without loss of generality, W^i can be taken to be a unit vector. Denote the invariant components of W^i with respect to the orthonormal tetrad $V^j_{(a)}$ by W^a, i.e., $W^i = W^a V^i_{(a)}$. Hence,

$$T_{ij}W^iW^j = \sum_{a,b=1}^{4} W^aW^b\lambda_a\eta_{ab} = M^2\left[(W^1)^2 + (W^2)^2 - (W^3)^2 + 3(W^4)^2\right]$$

$$= M^2\left[\,4(W^1)^2 + 4(W^2)^2 + 2(W^3)^2 + 3\,\right] .$$

Furthermore, $\frac{1}{2}\, W^iW_iT^j{}_j = M^2$. So we finally have

$$T_{ij}W^iW^j - \frac{1}{2}\, W^iW_iT^j{}_j = M^2\left[4(W^1)^2 + 4(W^2)^2 + 2(W^3)^2 + 2\right] \geq 0 ,$$

demonstrating that the strong energy condition holds for the solution being considered here.

C. The Physical Relevance of the Solution

Som and Raychaudhuri[82] discovered a solution of the Einstein-Maxwell-Lorentz equations with a charged dust source in which the metric is formally identical to the metric (7.5') with $a_0 = 0$, and with the electromagnetic field of the same form as that determined by the potentials (7.7) and (7.8). This is not surprising in view of Theorem 2.2, since the meson field in our solution is a constant.

Theorem 2.1 provides a third interpretation of our solution. The potentials A and B determine the field strength of a stationary Proca field with mass M.

The physical relevance of the solution is problematic. On the one hand the electromagnetic and mesonic sources are not obviously unphysical, given the fact that the Hawking and Ellis strong energy condition is satisfied. On the other hand, the metric (7.5') is not asymptotically flat and has causal pathologies of the sort which violate our intuition, and, in any case, have never been observed.

If our metric were, like the Taub-NUT metric, asymptotically flat, then, like the latter, which also has causal pathologies, our (Euclideanized?) solution could play a role in quantum gravity[83]. From the quantum point of view, the lack of asymptotic flatness is more serious than the existence of closed smooth timelike curves.

Non-asymptotically flat metrics with physically well-behaved sources are often dubbed "cosmological". In this context, the solutions given by (7.5'), (7.6') with $a_o = 0$, (7.7) and (7.8), could be interpreted as a model for a universe consisting of a gas of protons, diffuse enough so that spin can be ignored, and with a constant magnetic field $H_\alpha = -B_{,\alpha} = \pm 2^{\frac{1}{2}} M \delta_{\alpha 2}$ due to the proton charge.

APPENDIX A - Principal Fiber Bundles with a Connection

In this Appendix the basic definitions and properties of principal fiber bundles with a connection are summarized. The purpose of this is primarily to standardize notation. For proofs, examples, and detailed discussion, the reader is referred to Kobayashi and Nomizu[30], Drechsler and Mayer[31], Daniel and Viallet[32], Trautman[38], and Cho[39].

A fiber bundle is a manifold which is locally isomorphic to the Cartesian product of two other manifolds. More precisely we have:

Def.A.1: A smooth fiber bundle is a smooth manifold P together with another smooth manifold M, called the base space, and a smooth surjection $\pi: P \to M$, called the projection, such that the following property, called local triviality, is satisfied:

There exists a smooth manifold F, called the fiber space, such that for each $m \in M$ there is an open neighborhood U of m such that $\pi^{-1}(U)$ is diffeomorphic to U x F.

A smooth fiber bundle shall be denoted by $P(M,F,\pi)$.

Def.A.2: A smooth fiber bundle $P(M,F,\pi)$ is trivial if P is diffeomorphic to M x F.

Def.A.3: A smooth map $s:U \to P$, $U \subset M$, such that $\pi \circ S = id_M$ is called a smooth section of P over U. A smooth section whose domain is M is called a global section.

All trivial fiber bundles admit global sections, but the converse is generally not true.

The two types of smooth fiber bundles we shall be concerned with here are vector bundles and principal fiber bundles. (Henceforth, the smoothness of the various manifolds and maps shall be taken for granted and the adjective "smooth" shall be dropped.)

Def.A.4: A fiber bundle $P(M,V,\pi)$ is a vector bundle if the fibers V are vector spaces.

Def.A.5: A fiber bundle $P(M,G,\pi)$ is a principal fiber bundle if G is a Lie group and if in addition the following hold:

(i) G acts smoothly to the right on P without fixed points.

(ii) The base space M is the quotient space of P by the equivalence relation of right multiplication, i.e., $M = P/G$.

The right (left) action of $\alpha\varepsilon G$ on $p\varepsilon P$ is denoted $R_\alpha p \equiv p\cdot\alpha (L_\alpha p \equiv \alpha\cdot p)$, and satisfies, for each $\alpha,\beta\varepsilon G, p\cdot(\alpha\beta) = (p\cdot\alpha)\cdot\beta$ and analogously for the left action. That G acts on P without fixed points means that if $p\cdot\alpha = p$ for some $p\varepsilon P$, then α is the identity element e of G.

Def.A.6: Let μ be a vector field over a Lie group G. (Recall that G has a smooth manifold structure, so $\mu:G \to TG$.) Then μ is right (left) invariant if for every $\alpha,\beta\varepsilon G$, $R_{\alpha*}\mu(\beta) = \mu(\alpha\beta)$ $(L_{\alpha*}\mu(\beta) = \mu(\beta\alpha))$, where $R_{\alpha*}$ is the Jacobian of the map $R_\alpha:G \to G$.

All vector bundles, trivial or otherwise, admit the global section $\tilde{o}:M \to P$ by $m \mapsto (m,\vec{o})$, where $\vec{o}$ is the zero-vector of V. However, a principal fiber bundle is trivializable (i.e., isomorphic to M x G) if and only if it admits a global section[31].

A principal fiber bundle $P(M,G,\pi)$ is naturally associated with a family of vector bundles $E_r(M,V,\pi_r)$ as follows:

Let $r:G \to GL(V)$ be a representation of G, i.e., V is a vector space on which GL(V) acts to the left. Thus r induces a map $G \times V \to V$ by $(\alpha,v) \mapsto r(\alpha)v$, for any $\alpha\varepsilon G$ and $v\varepsilon V$. Write

$E_r \equiv (P \times V)/G,$

and define $\pi_r : E_r \to M$ by $\pi_r(z) \equiv \pi(p)$ for each $z = [(p,v)] \equiv$ the set of all points in $P \times V$ equivalent to (p,v) under right action by G. Drechsler and Mayer[31] show that E_r (M,V,π_r) satisfies the axioms for a vector bundle. Hence we have:

Def.A.7: The vector bundle $E_r(M,V,\pi_r)$ is the vector bundle associated to P by the representation r of G, with E_r and π_r as defined above.

Let $\{\mu_A\}$, $A = 1, \ldots, n = \dim. G$ be a left invariant basis field on G. The μ_A are thus a basis of the Lie algebra G' of G[38]. The μ_A induce, in a natural way, a linearly independent set of vector fields $\{\mu_A^*\}$ over the bundle space P, i.e., each $\mu_A^*(p) \varepsilon T_pP$. The $\mu_A{}^*$ are called fundamental vector fields over P corresponding to the μ_A. The subspace V_p of T_pP spanned by the $\mu_A^*(p)$ is called the vertical subspace of T_pP.

Def.A.8: A connection on a principal fiber bundle $P(M,G,\pi)$ is a smooth assignment of a horizontal subspace $H_p \subset T_pP$ such that

(i) $T_pP = H_p \oplus V_p$.

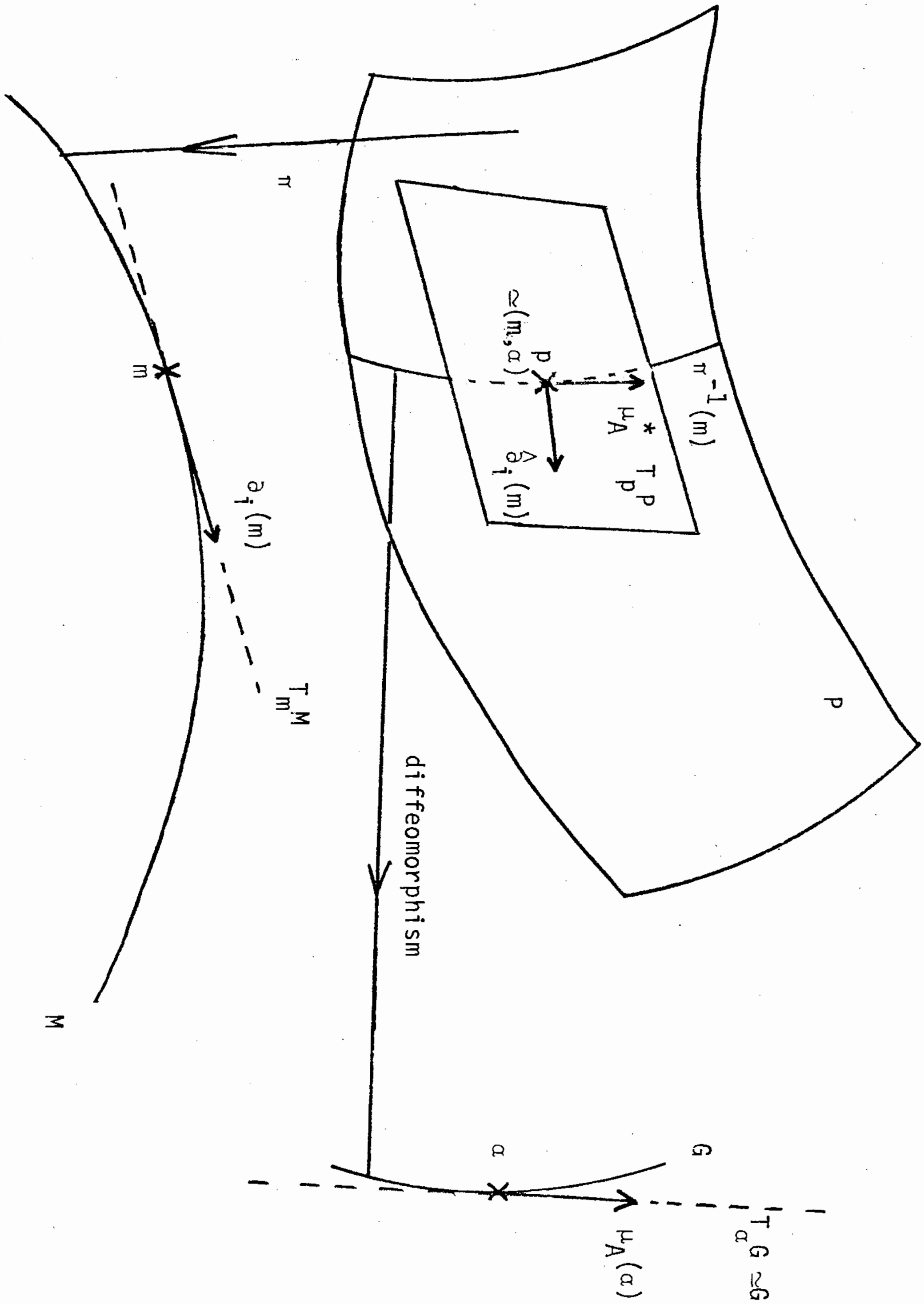
(ii) For every $\alpha \varepsilon G$, $p \varepsilon P$, $H_{p \cdot \alpha} = R_{\alpha *} H_p$, i.e, H_p is right invariant.

(iii) $\pi_* H_p = T_{\pi(p)}M$.

Denote by $\{\partial_i\}$ the coordinate basis fields over a coordinate patch U of M. The existence of a connection on P allows one to uniquely associate with any vector $v \varepsilon T_mM$ a vector $\hat{v} \varepsilon H_p$ where $p \varepsilon \pi^{-1}(m)$[31]. The vector $\hat{v}$ is called the horizontal lift of v. The horizontal lifts $\hat{\partial}_i(p)$ are a basis of H_p, but in general are not part of a coordinate basis[39].

The existence of a connection on $P(M,G,\pi)$ implies the existence of a 1-form field ω on P with values in the Lie algebra G'.

Figure A.1 : Principal Fiber Bundle.



<u>Def.A.9</u>: A 1-form field $\omega: P \rightarrow T^*P$ is called a <u>connection 1-form field</u> on P if it satisfies:

(i) For any fundamental vector field v^* corresponding to a vector field v over M, $\omega(v^*) = v$.

(ii) A vector field u is horizontal if and only if $\omega(u) = 0$.

Let $\{(U_{(\alpha)}, x^i_{(\alpha)})\}$ be an atlas of M such that on each $U_{(\alpha)} \subset M$ there is a local section of P, denoted $S_{(\alpha)}: U_{(\alpha)} \rightarrow P$. Given a connection 1-form ω, one can define a family of local 1-form fields $A_{(\alpha)}$ over $U_{(\alpha)}$ by

$$A_{(\alpha)i} = A_{(\alpha)}(\partial_i) \equiv A_{(\alpha)}{}^B{}_i \mu_B = \omega(S_{(\alpha)*}\partial_i) \quad , \tag{A.1}$$

where $S_{(\alpha)*}: T_m M \rightarrow T_{S_{(\alpha)}m} P$ is the Jacobian of $S_{(\alpha)}$ and the μ_B are a basis of the Lie algebra G'. In fact, the $A_{(\alpha)}$ can be expressed as

$$A_{(\alpha)} = A_{(\alpha)i}\, dx^i \equiv A_{(\alpha)}{}^B{}_i \mu_B \otimes dx^i \quad , \tag{A.2}$$

where $\{dx^i\}$ are the basis 1-forms dual to $\{\partial_i\}$. The $A_{(\alpha)}{}^B{}_i$ are smooth real-valued functions on $U_{(\alpha)}$ and depend on the choice of the local section $S_{(\alpha)}$

Transform to a new local section $S'_{(\alpha)} \equiv S_{(\alpha)} \cdot \gamma$, where $\gamma: U_{(\alpha)} \rightarrow G$. It can be shown that the expressions

$$A'_{(\alpha)} \equiv \omega(S'_{(\alpha)*}\partial_i)$$

are related to the $A_{(\alpha)i}$ by

$$A'_{(\alpha)i} \equiv \gamma^{-1} A_{(\alpha)i}\gamma + \gamma^{-1}\partial_i\gamma \quad . \tag{A.3}$$

A few definitions are necessary before the curvature 2-form can be properly defined.

Def.A.10: Let $r:G \rightarrow GL(V)$ be a representation of G. A V-valued k-form ν over P is said to be a type r if for every $\alpha \varepsilon G$, $\alpha^{*}\nu = r(\alpha^{-1})\nu$.

Def.A.11: The horizontal part, denoted hor ν, of a k-form ν of type r, is defined by

$$\text{hor}\ \nu(v_1, \ldots, v_k) \equiv \nu\ (\text{hor}\ v_1\ , \ldots , \text{hor}\ v_k) \quad ,$$

where $v_1, \ldots, v_k \ \varepsilon\ T_pP$, and hor v_j is the projection of v_j onto H_p.

Def.A.12: The covariant exterior derivative of a k-form ν of type r is a (k + 1)-form of type r defined by

$$D\nu \equiv \text{hor}\ d\nu \quad . \tag{A.4}$$

The connection 1-form ω is of the type "ad", i.e., it takes its values in the set of automorphisms of the Lie algebra G' induced by the adjoint representation of G in its Lie algebra[32].

Def.A.13: The Curvature 2-form Ω is of type ad and is given by $\Omega \equiv D\omega$.

From the definitions of D and ω it can be shown that[31]

$$\Omega = d\omega + \frac{1}{2}[\omega,\omega] \quad , \tag{A.5}$$

where the commutator of two G'-valued forms $\nu = \nu^B \mu_B$ and $\tau = \tau^B \mu_B$ is

$$[\nu,\tau] = \nu^B \wedge \tau^C\ [\mu_B,\mu_C] = \nu^B \wedge \tau^C\ C^E{}_{BC}\ \mu_E \quad . \tag{A.6}$$

The real numbers C^{E}_{BC} are the structure constants of G with respect to the basis $\{\mu_B\}$ of G'. The local expression for Ω corresponding to a section $S_{(\alpha)}:U_{(\alpha)} \rightarrow P$ is

$$F_{(\alpha)} \equiv \frac{1}{2} F_{(\alpha)}{}^{B}{}_{ij}\mu_B dx^i \wedge dx^j \quad . \tag{A.7}$$

The $F_{(\alpha)}{}^{B}{}_{ij}$ are given by

$$\begin{aligned} F_{(\alpha)ij} &\equiv F_{(\alpha)}{}^{B}{}_{ij}\, \mu_B = \Omega(S_{(\alpha)*}\partial_i, S_{(\alpha)*}\, \partial_j) \\ &= A_{(\alpha)j,i} - A_{(\alpha)i,j} + [A_{(\alpha)i}, A_{(\alpha)j}] \quad , \end{aligned} \tag{A.8}$$

where

$$[A_{(\alpha)i}, A_{(\alpha)j}] \equiv A_{(\alpha)i}A_{(\alpha)j} - A_{(\alpha)j}\, A_{(\alpha)i} \quad ,$$

the product of the $A_{(\alpha)}$'s being the product defined by the Lie algebra structure.

Under a change of local section $S'_{(\alpha)} = S_{(\alpha)}\cdot\gamma$, the $F_{(\alpha)}$ transforms as

$$F'_{(\alpha)} = \gamma^{-1}\, F_{(\alpha)}\gamma \quad . \tag{A.9}$$

The definition A.12 of the exterior covariant derivative $D\nu$ of a k-form of type r allows one to define the exterior covariant derivative of a smooth local section $\psi:U_{(\alpha)} \rightarrow E_r$ on a vector bundle $E_r(M,V,\pi_r)$ associated with $P(M,G,\pi)$. This, of course, amounts to defining a connection on $E_r(M,V,\pi_r)$.

The definition (A.4) for D is not very convenient for computations, so, following Trautman[38], we use the local expression corresponding to (A.4), namely:

$$D_i\psi^a = \frac{\partial}{\partial x^i}\psi^a + r^a_{bB}\, A_{(\alpha)}{}^B{}_i\psi^b \quad . \tag{A.10}$$

In the above, the indicies a, b are for the components in the vector space V, and r^a_{bB} is the a^{th} component of

$$\frac{d}{dt}\, r\left[\exp(t\mu_B)e_b\right]\Big|_{t=0} \quad , \tag{A.11}$$

where $t\varepsilon[0,1]$, and the e_b are the basis of V. Thus the matrices $r_B = (r^a_{bB})$ are a basis of the r-representation of G'. Hence $(r^a_{bB}A_{(\alpha)}{}^B{}_i)$ are the r-representation of the components of the connection 1-form ω. So (A.10) can be written in matrix form as:

$$D_i\psi = \frac{\partial}{\partial x^i}\psi + \mathcal{A}_{(\alpha)i}\psi \quad , \tag{A.12}$$

where ψ stands for the column vector (ψ^a) and $\mathcal{A}_{(\alpha)i}$ is the matrix with elements $r^a_{bB}A_{(\alpha)}{}^B{}_i$.

Now suppose that M is a semi-Riemannian manifold. Let $\psi^{i\ldots}_{j\ldots}$ be the components of a V-valued tensor field over M. Then (A.12) has the immediate generalization:

$$D_i\psi^{j\ldots}_{k\ldots} = \nabla_i\psi^{j\ldots}_{k\ldots} + \mathcal{A}_{(\alpha)i}\psi^{j\ldots}_{k\ldots} \tag{A.13}$$

where ∇_i is the covariant derivative associated with the semi-Riemannian structure on M.

APPENDIX B - The Principal Fiber Bundle Structure of a Stationary Spacetime

In this Appendix, we shall consider the principal fiber bundle $M_4(M_3, T_1, \pi)$, where M_4 is a smooth four-dimensional manifold, T_1 is a 1-parameter Lie group isomorphic to R (i.e., the "translation group"), M_3, is a smooth three-dimensional manifold defined by $M_3 \equiv M_4/T_1$, and, finally, $\pi: M_4 \rightarrow M_3$ is a smooth surjection. In addition, it is assumed that:

(i) M_3 is Riemannian with negative-definite metric h.

(ii) There is a connection a on the principal fiber bundle.

It will now be shown that the above is sufficient to determine a semi-Riemannian structure on M_4 with metric $\hat{g}$ having the signature -2 and such that $\hat{g}$ is stationary, i.e., there exists a Killing vector field $\vec{\xi}$ on M_4 such that $\hat{g}(\vec{\xi},\vec{\xi}) > 0$. It should be noted that the usual "3 + 1 decomposition" of a stationary spacetime amounts to establishing the converse of the above[69].

The existence of the metric $\hat{g}$ and the Killing vector field $\vec{\xi}$ on M_4 will be established locally. Thus we work in a chart (U, x^α) on M_3. It is also assumed that over U there is a smooth local section $\sigma: U \rightarrow T_1$. The coordinate basis over U is $\{\partial_\alpha\}$, and the dual basis is $\{dx^\alpha\}$. Hence, the metric h over U is

$$h = h_{\alpha\beta} dx^\alpha \otimes dx^\beta \tag{B.1}$$

where $h_{\alpha\beta} = h_{\beta\alpha}$ are smooth functions of the coordinates x^α. Let ξ be a left-invariant vector field over T_1. Hence, ξ is a basis of T_1', the Lie algebra of T_1[39].

Now we will horizontally lift the basis$\{\partial_\alpha\}$ of $T_U M_3 \equiv \{T_p M_3 : p\varepsilon U\}$ to $T_{\pi^{-1}(U)} M_4 \equiv \{T_{\pi^{-1}(p)} M_4 : p\varepsilon U\}$. The resulting linearly independent vector fields $\hat{\partial}_\alpha$ are uniquely determined by the ∂_α and the connection a. (See Appendix A.) The $\hat{\partial}_\alpha$ are horizontal and hence satisfy $a(\hat{\partial}_\alpha) = 0$. However, the $\hat{\partial}_\alpha$ are not, in general, part of a coordinate basis on M_4 because, as pointed out by Cho[39], $[\hat{\partial}_\alpha, \hat{\partial}_\beta] \neq 0$. To get a set of coordinate basis vectors on M_4, choose a trivial local section $\bar{\sigma}: U \to M_4$ such that $\bar{\sigma}(p) = (x^\alpha, t_0)$, where t_0 = a constant. Define the 1-form field a on U as the "pullback" $\mathrm{a} \equiv \bar{\sigma}^* a$. Note also that

$$a(\bar{\sigma}_* v) = (\bar{\sigma}^* a)(v) = \mathrm{a}(v) \quad , \tag{B.2}$$

for any $v \varepsilon T_U M_3$. Thus, since

$$a(\bar{\sigma}_* \partial_\alpha) = \mathrm{a}(\partial_\alpha) = \mathrm{a}_\alpha \xi = a(\mathrm{a}_\alpha \vec{\xi}) \quad ,$$

where $\vec{\xi}$ is the vector field over M_4 induced by ξ and $\mathrm{a} = \mathrm{a}_\alpha \xi \otimes dx^\alpha$ it follows that

$$a(\bar{\sigma}_* \partial_\alpha - \mathrm{a}_\alpha \vec{\xi}) = 0 \quad ,$$

i.e., $(\bar{\sigma}_* \partial_\alpha - \mathrm{a}_\alpha \vec{\xi}) \varepsilon T_{\pi^{-1}(U)} M_4$ is horizontal. But since $\vec{\xi}$ is vertical, $\pi_* \vec{\xi} = 0$, and so $\pi_*(\hat{\partial}_\alpha) = \partial_\alpha = \pi_*(\bar{\sigma}_* \partial_\alpha - \mathrm{a}_\alpha \vec{\xi})$.
Thus, by the uniqueness of the horizontal lift it follows that

$$\hat{\partial}_\alpha = \bar{\sigma}_* \partial_\alpha - \mathrm{a}_\alpha \vec{\xi} \quad . \tag{B.3}$$

To obtain the metric $\hat{g}$ on M_4 we use the prescription of Cho[39]:

$$\hat{g}_{ij}\hat{\partial}_\alpha{}^i\hat{\partial}_\beta{}^j = h_{\alpha\beta} \quad , \tag{B.4}$$

$$\hat{g}_{ij}\hat{\partial}_\alpha^i\xi^j = 0 \quad , \tag{B.5}$$

$$\hat{g}_{ij}\xi^i\xi^j = e^\omega \quad . \tag{B.6}$$

The quantities $\hat{\partial}_\alpha{}^i$ and ξ^i are the components with respect to the local basis $(\bar{\sigma}_*\partial\alpha,\vec{\xi})$ over M_4 of the vector fields $\hat{\partial}_\alpha$ and $\vec{\xi}$, respectively. The prescription (B.4) - (B.6) is natural in the sense that the metric $\hat{g}_{ij}$ makes horizontal and vertical vectors orthogonal. Since $(\bar{\sigma}_*\partial_\alpha,\vec{\xi})$ satisfies the relations[39]

$$[\bar{\sigma}_*\partial_\alpha,\bar{\sigma}_*\partial_\beta] = 0 \quad ,$$

$$[\bar{\sigma}_*\partial_\alpha,\vec{\xi}] = 0 \quad ,$$

it is a coordinate basis over $\pi^{-1}(U)$. Denote the corresponding coordinates by (x^α,x^4). In this basis $\xi^i = \delta^i_4$. Hence equations (B.4) - (B.6) become

$$\hat{g}_{\alpha\beta} - e^\omega a_\alpha a_\beta = h_{\alpha\beta} \quad , \tag{B.7}$$

$$\hat{g}_{\alpha 4} - e^\omega a_\alpha = 0 \quad , \tag{B.8}$$

$$\hat{g}_{44} = e^\omega \quad . \tag{B.9}$$

These equations may be solved for $\hat{g}_{\alpha\beta}$, $\hat{g}_{\alpha 4}$, and $\hat{g}_{44}$ uniquely in terms of $h_{\alpha\beta}$, a_α, and e^ω. In fact, by defining a new metric $g_{\alpha\beta}$ on M_3 by

$$h_{\alpha\beta} \equiv -e^{-\omega} g_{\alpha\beta} \quad , \tag{B.10}$$

it can be seen that the equations (B.7) - (B.9) formally resemble the equations which give the components of a stationary metric in a Copernican chart.

From equation (B.6), the vector field $\vec{\xi}$ is timelike with respect to the metric $\hat{g}_{ij}$. Furthermore, in the chart (x^α, x^4), the Lie derivative of the metric $\hat{g}_{ij}$ with respect to $\vec{\xi}$ is

$$\mathop{£}_{\xi} \hat{g}_{ij} = \hat{g}_{ij,k}\xi^k = \hat{g}_{ij,4} = 0 \quad , \tag{B.11}$$

since the $g_{\alpha\beta}$, a_α, and e^ω, from which the $\hat{g}_{ij}$ are constructed, are independent of x^4. Hence $\vec{\xi}$ is a timelike Killing vector field over M_4, $\hat{g}$ is stationary, and (x^α, x^4) are coordinates of a Copernican chart.

APPENDIX C - The Proof that the 1-form ζ is Closed

It will be shown here the $d\zeta = 0$, where ζ is the 1-form field defined by equation (4.33).

From the definition of $\tilde{\tilde{Z}}$, it is clear that $\underset{\xi}{£}\tilde{Z} = 0$. Hence

$$d\zeta = 2\underset{\xi}{C}\,(d\tilde{Z})$$

Using the properties of exterior derivatives of forms[33] and (4.28) and (4.25), it can be shown that

$$d\zeta = 2\underset{\xi}{C}d\tilde{S} - 4k\underset{\xi}{C}\left[(\underset{\xi}{C}\tilde{F}^*)\wedge\tilde{F}\right] + 2k\phi^*\,\underset{\xi}{C}\tilde{J} \quad .$$

The assumption of isometric motion causes the last term of the above to vanish. Furthermore, from the properties of the contraction operation (see Hicks[33], page 91), and from the facts[47] $\Box\xi^i = \hat{R}^i{}_j\xi^j$ and $\nabla_j\, {}^*(\nabla^j\xi^i) = 0$, it follows that

$$d\zeta = 2\,\{2i\eta_{rkji}\hat{R}^r{}_m\xi^m\xi^k dx^i\wedge dx^j - 2k\tilde{F}^*_{ij}\xi^j\tilde{F}_{km}\xi^m dx^i\wedge dx^k\} \quad ,$$

where $\hat{R}^i{}_j$ is the Ricci tensor on M_4. We now invoke the Einstein equations in the form $\hat{R}^r{}_m = -k(T^r{}_m - \frac{1}{2}\,\delta^r_m T)$, to arrive at

$$d\zeta = -2k\{2i\eta_{rkji}(T^r{}_m - \tfrac{1}{2}\,\delta^r{}_m T)\xi^m\xi^k + 2\tilde{F}^*_{ik}\tilde{F}_{jm}\xi^m\xi^k\}\, dx^i\wedge dx^j$$

$$= -2k\{2i\eta_{rkji}(E^r{}_m + M^r{}_m - \tfrac{1}{2}\,\delta^r{}_m M) + (\tilde{F}^*_{ik}\tilde{F}_{jm} - \tilde{F}^*_{jk}\tilde{F}_{im})\}\;\xi^k\xi^m dx^i\wedge dx^j \quad .$$

It is now claimed that

$$(2i\eta_{rkji}E^r{}_m + \tilde{F}^*_{ik}\tilde{F}_{jm} - \tilde{F}^*_{jk}\tilde{F}_{im})\xi^m\xi^k = 0 \quad , \qquad \text{(C.1)}$$

$$\eta_{rkij}(M^r{}_m - \frac{1}{2}\,\delta^r{}_m M)\xi^k\xi^m = 0 \quad . \tag{C.2}$$

These facts are most easily demonstrated in a Copernican chart. Equation (C.1) holds independently of the isometric motion condition, and it is established by explicit verification for the three cases $(ij) = (\alpha\beta)$, $(ij) = (\alpha 4)$, and $(ij) = (44)$. Equation (C.2) holds only if the condition of isometric motion is invoked, and follows almost immediately from equations (4.58).

Upon using (C.1) and (C.2) in the previous expression for $d\zeta$, the desired result is obtained.

APPENDIX D - Properties of the Potential Space Metric L_{AB}

In this appendix, the contravariant components L^{AB} and the Christoffel symbols K^C_{AB} of the potential space metric $\tilde{L}_o$ will be displayed. These will be used to show that $\varepsilon_{\|AB} = 0$ implies that ε = a constant.

From (5.9), it follows that

$$(L^{AB}) = \left[\begin{array}{c|c} (2k)^{-1}\,(L^{ab}) & 0 \\ \hline 0 & (2k)^{-1} \end{array}\right] \tag{D.1}$$

where

$$(L^{ab}) \equiv \text{cofactor }(L_{ab})|\det(L_{ab})$$

$$= 2f \begin{bmatrix} 0 & -k^{-1} & 0 & \phi \\ -k^{-1} & 0 & \phi^* & 0 \\ 0 & \phi^* & 0 & (\Gamma+\Gamma^*) \\ \phi & 0 & (\Gamma+\Gamma^*) & 0 \end{bmatrix} . \tag{D.2}$$

The Christoffel symbols of the metric L_{AB} on K_5 are

$$K^a_{11} = - \delta^a_1 k f^{-1} \phi^* \quad ,$$

$$K^a_{12} = 0 \quad ,$$

$$K^1_{31} = - \frac{1}{2} f^{-1} \quad ,$$

$$K^3_{31} = - \frac{1}{2} k f^{-1} \phi^* \quad ,$$

$$K^a_{14} = 0 \quad ,$$

$$K^a_{22} = - \delta^a_2 k f^{-1} \phi \quad ,$$

$$K^a_{23} = 0 \quad , \qquad \text{(D.3)}$$

$$K^2_{24} = - \frac{1}{2} f^{-1} \quad ,$$

$$K^4_{24} = - \frac{1}{2} k f^{-1} \phi \quad ,$$

$$K^1_{24} = \Gamma^3_{24} = 0 \quad ,$$

$$K^a_{33} = - \delta^a_3 f^{-1} \quad ,$$

$$K^a_{34} = 0 \quad ,$$

$$K^a_{44} = - \delta^a_4 f^{-1} \quad ,$$

$$K^5_{AB} = K^A_{5B} = K^A_{B5} = 0 \quad .$$

Now consider the equations

$$\varepsilon_{||ab} \equiv \varepsilon_{,ab} - K^C_{ab} \varepsilon_{,C} = 0 \quad . \qquad \text{(D.4)}$$

Since $K^a_{12} = K^a_{14} = K^a_{23} = K^a_{34} = 0$ for all a = 1,2,3,4, ε must be of the form,

$$\varepsilon = F(\phi,\Gamma) + G(\phi^*,\Gamma^*) \quad , \tag{D.5}$$

where F and G are arbitrary differentiable functions of their arguments.

The equation

$$\varepsilon_{\|11} \equiv \varepsilon_{,11} - K^a_{11}\, \varepsilon_{,a} = 0 \quad ,$$

reduces to

$$F_{,11} + k\, f^{-1}\, \phi^*\, F_{,1} = 0 \quad .$$

Assume that $F_{,1}$ is not identically zero. The first integral of the last equation is then

$$\ln(F_{,1}) = -2\ln(f) + \zeta(\phi^*,\Gamma,\Gamma^*) \quad ,$$

where ζ is an arbitrary differentiable function. But since F does not depend on ϕ^* it follows that

$$\zeta_{,2} = -k\, f^{-1}\, \phi \quad .$$

But this is impossible because $\zeta_{,21} = 0$, but $(-kf^{-1}\phi)_{,1} \neq 0$. Hence our assumption $F_{,1} \not\equiv 0$ must be wrong. The upshot is that ε is of the form

$$\varepsilon = F(\Gamma) + G(\phi^*,\Gamma^*) \quad .$$

Similarly, the $\varepsilon_{\|22} = 0$ equation yields

$$\varepsilon = F(\Gamma) + G(\Gamma^*) \quad .$$

Now $\varepsilon_{||33} = F_{,22} + f^{-1} F_{,2} = 0$. The first integral is, if $F_{,2} \neq 0$,

$$\ln(F_{,2}) = -2 \ln(f) + \rho(\phi,\phi^*,\Gamma^*) \quad ,$$

where ρ is an arbitrary differentiable function. If we differentiate both sides with respect to Γ^* we obtain

$$-f^{-1} + \rho_{,4} = 0 \quad .$$

Again, the last equation is impossible since ρ does not depend on Γ. Hence, $F_{,2} = 0$, and $\varepsilon = G(\Gamma^*)$.

Finally, by exactly similar reasoning, the $\varepsilon_{||44} = 0$ equation yields ε = a constant.

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