

A MODEL OF FORWARD CURRENCY MARKET PARTICIPATION WITH RISK OF
EXCHANGE CONTROLS

by

Martin Gallagher

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APPROVAL

Name: Martin Gallagher

Degree: Doctor of Philosophy

Title of Thesis: A Model of Forward Currency Market Participation
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Examining Committee:

Chairperson: Kenji Okuda

Pao L. Cheng
Senior Supervisor

W. John Heaney

S. T. Easton

Edward Tower
External Examiner
Professor
Department of Economics
Duke University

Date Approved: Aug 7/82

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Author: _____

(signature)

Martin Gallagher

(name)

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ABSTRACT

There exists a body of literature concerned with the estimation of models of the "Modern Theory" of forward currency exchange. This thesis is ultimately concerned with the validity of the estimation procedures of this literature.

To constitute a theoretical alternative to its precursor, the "Interest Parity Theory", the "Modern Theory" requires that there does not exist a domestic asset and a "covered" foreign asset, which are perfect substitutes. A popular rationale within this literature for the absence of such assets, is the risk that within the period between the acquisition of a forward contract to repatriate the return on a nominally riskless foreign asset, and the maturity of that asset, the foreign authorities may impose regulations which prevent eventual repatriation at the contracted exchange rate. The modern theory models incorporate this risk in a simple manner at the aggregate level. As is recognized in the literature, if this manner is inappropriate, the estimation procedures employed in all of this literature are rendered invalid. This prompts the question; are the modern theory model specifications of aggregate behavior in the event of exchange control risk, consistent with the aggregate behavior which emerges from a model of optimal individual behavior with such risk?

Beginning with an expected utility model of optimal individual international portfolio selection, which treats forward currency market participation as only one of many ways in which the individual may tailor his portfolio to his tastes, expressions are derived which are analogous to the "arbitrage", "speculative" and "trade" functions of the modern theory models. These expressions are then compared and contrasted with the modern theory specifications. It emerges that the two models are not consistent. Had the equilibrium equation from which all of the modern theory model estimation embarks, been derived from the model within this thesis, it would have transpired that the 'parameters' which this literature attempts to estimate, are not parameters, but are variables. Furthermore they are random variables. Our model thus serves to engender misgivings about the soundness of the modern theory estimation literature.

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CHAPTER I: INTRODUCTION

Section 1: The Modern Theory empirical literature.

There exists a body of literature, documented below, concerned with the estimation of models of the 'Modern Theory' of forward exchange. Whilst these models are not identical they are, as we shall see, essentially similar. This thesis is ultimately concerned with the validity of the estimation procedures employed in this body of literature.

The 'Modern Theory' of forward exchange emerged as an attempt to explain the apparent existence of riskless profitable international arbitrage opportunities. Such opportunities would clearly be contrary to a fundamental tenet of economic theory, but more particularly they are inconsistent with the precursor of the 'Modern Theory', the 'Interest Parity Theory' of forward exchange.

The Interest Parity Theory ⁽¹⁾ may be summarized by equation (1) plus its concomitant definitions and assumptions

$$r_j^f = r_i^f \frac{f_{ji}}{s_{ji}} \quad (1)$$

where,

r_j^f is the risk free rate of interest plus one, in country j.

f_{ji} is the currency j price of a unit of currency i in a forward contract with the same common maturity date as the risk free assets of countries j and i.

s_{ji} is the spot currency j price of one unit of currency i .

It is assumed that there are no transaction costs or other market imperfections, so that f_{ji} and s_{ji} are both buying and selling rates, and each interest rate is both the borrowing and lending rate within the country with which it is associated.

The remainder the the Interest Parity Theory is in the form of a famous corollary introduced below.

The violation of (1) is viewed within the Interest Parity Theory as stimulating short term capital movements to take advantage of profitable covered arbitrage opportunities.

If

$$r_j^f > r_i^f \frac{f_{ji}}{s_{ji}} \quad (2)$$

capital is expected to flow from country i to country j . If

$$r_j^f < r_i^f \frac{f_{ji}}{s_{ji}} \quad (3)$$

the direction of flow is reversed. As a convenient indicator of the favourable investment location, the covered arbitrage margin is defined;

$$CAM_{ji} = \frac{r_j^f - r_i^f}{r_i^f} - \frac{f_{ji} - s_{ji}}{s_{ji}} \quad (4)$$

∞ (interest differential) - (forward premium)

(4) is the traditional equation for CAM_{ji} , it may be rewritten as

$$CAM_{ji} = \begin{bmatrix} r_j^f - \frac{f_{ji}}{s_{ji}} \\ r_i^f \end{bmatrix} \quad CAM_{ji} = \frac{1}{r_i^f} \begin{bmatrix} r_j^f - r_i^f \frac{f_{ji}}{s_{ji}} \end{bmatrix} \quad (5)$$

If (2) holds, then the term within parentheses in (5) is positive, CAM_{ji} is positive and short term capital move from country i to country j . If (3) holds CAM_{ji} is negative and such capital flows from country j to country i . In both cases, capital flows are expected to ensue and thus, within the confines of the theory, a non zero CAM_{ji} is viewed as a disequilibrium phenomenon. A corollary of the Interest Parity Theory therefore, is a tendency for CAM_{ji} to be zero.

Contrary to the Interest Parity Theorem there exists a wealth of data indicating a persistent tendency for covered arbitrage margins to be non zero (2). The theoretical resolution of this somewhat paradoxical phenomenon has taken two distinct avenues (3). One has retained the interest parity framework but has introduced market imperfections in the form of transactions costs and non parametric interest rates. There has also been consideration of the difficulties of selecting comparable assets for the calculation of covered arbitrage margins (4). Pursuit of the other avenue was prompted by the realization "that the traditional Keynesian emphasis on covered interest arbitrage was insufficient to fully explain the equilibrium rate of forward exchange and that other operations in the forward market were equally important in determining the equilibrium rate. These additional operations, which along with covered interest arbitrage, constitute the source of demand for and supply of forward exchange, can be broadly defined as commercial hedging and speculation. The addition of these components to the Keynesian theory has resulted in the challenging of the major conclusion of the prewar [interest parity]

theory, that is the new theory contends that the equilibrium forward premium rate can be considerably different from the interest-differential rate." (5) This "new" theory is now referred to as the "Modern Theory" of forward exchange.

Estimation of modern theory inspired models in the literature, has tended to embark from what is essentially a single common model. This model, which shall be referred to as the modern theory empirical model, is evident in the contributions of J. Stein (1965), H. Stoll (1968), J. Kesselman (1971), R. Haas (1974), B. McCallum (1977), W. Gaab (1978) and E. Tower & G. Pederson (1979). That Stein, Tower and Pederson may be included in this group is perhaps not clear. Following the presentation of the essentially common model of the remaining authors, justification for the inclusion of these three will be provided.

B. McCallum's derivation of a modern theory equilibrium condition, identical to that of Gaab and differing from the other works cited, only by the explicit inclusion of trading behaviour, is representative. It proceeds in the following manner.

Solving (1) for f_{ji} , provides the forward rate associated with an absence of profitable arbitrage opportunities. This rate, denoted F^*_{ji} , is universally referred to as the 'interest parity' forward rate.

Thus

$$F^*_{ji} = s_{ji} \frac{r_j^f}{r_i^f} \quad (5a)$$

The Interest Parity Theorem can be presented as

$$f_{ji} = F^*_{ji} \quad (5b)$$

$$\text{If } f_{ji} > F_{ji}^* = s_{ji} \frac{r_i^f}{r_j^f} \quad (6)$$

then

$$r_i^f \frac{f_{ji}}{s_{ji}} > r_j^f \text{ and from (3) we have that arbitrage will cause capital}$$

to flow from country j to country i . Associated with this flow are forward sales of currency i for the repatriation of return, and the repayment of debt. Hence (6) is associated with a negative excess demand for forward i arising from arbitrage.

If $f_{ji} < F_{ji}^*$ then (2) is satisfied, capital flows from country i to country j , and there is a positive excess demand for forward i arising from arbitrage.

Adopting a linear form, the excess demand for forward i arising from arbitrage between countries j and i , X_i^{jAi} , is written as

$$X_i^{jAi} = \alpha^{jAi} [F_{ji}^* - f_{ji}] \quad 0 < \alpha^{jAi} < \infty \quad (7)$$

The excess demand for forward currency i arising from speculation between currencies j and i , X_i^{jSi} , is expressed by,

$$X_i^{jSi} = \alpha^{jSi} [E[\tilde{s}_{ji}] - f_{ji}] \quad 0 < \alpha^{jSi} < \infty \quad (8)$$

where s_{ji} is the future spot currency j price of a unit of currency i , governing at the maturity date of the forward contract associated with f_{ji} , and E is an expectation operator.

The absolute value of $[E[\tilde{s}_{ji}] - f_{ji}]$ is the expected currency j profit per unit of currency i traded forward speculatively. For example,

if one unit of i is bought forward at price f_{ji} , and is eventually sold for \tilde{s}_{ji} units of j , the profit is $[\tilde{s}_{ji} - f_{ji}]$, the expected profit $[E[\tilde{s}_{ji}] - f_{ji}]$. If one unit of i is sold forward speculatively the expected profit is $[f_{ji} - E[\tilde{s}_{ji}]]$. Thus the implication of (8) is that speculative behaviour is a function ONLY of expected return.

The excess demand for forward i arising from trade between countries j and i , X_i^{jTi} , is expressed by

$$X_i^{jTi} = \alpha^T_o - \alpha^{jTi} f_{ji} \quad \alpha^T_o > 0 \quad \alpha^{jTi} > 0 \quad (9)$$

This equation does not appear in the works by Kesselman, Stoll or Haas, where trade inspired participation is viewed as implicitly included in the 'arbitrage' and 'speculative' functions. Whilst this distinction is not of great significance for our eventual purpose, it is of sufficient interest to warrant a somewhat detailed discussion below.

Using the market clearing condition

$$X_i^{jAi} + X_i^{jSi} + X_i^{jTi} = 0$$

provides the modern theory equilibrium equation

$$f_{ji} = \left[\frac{\alpha^T_o}{\alpha^{jAi} + \alpha^{jSi} + \alpha^{jTi}} \right] + \left[\frac{\alpha^{jSi}}{\alpha^{jAi} + \alpha^{jSi} + \alpha^{jTi}} \right] E[\tilde{s}_{ji}] + \left[\frac{\alpha^{jAi}}{\alpha^{jAi} + \alpha^{jSi} + \alpha^{jTi}} \right] F^*_{ji} \quad (10)$$

With the absence of an explicit trade excess demand function, this equilibrium equation has the form

$$f_{ji} = \left[\frac{\alpha^{jSi}}{\alpha^{jAi} + \alpha^{jSi}} \right] E[\tilde{s}_{ji}] + \left[\frac{\alpha^{jAi}}{\alpha^{jAi} + \alpha^{jSi}} \right] F^*_{ji} \quad (11)$$

McCallum and Gaab proceed with estimation from (10); Kesselman, Haas and Stoll proceed from (11). Other than this distinction, the principal difference between these works arrives in the specification of an operational measure of $E[\tilde{s}_{ji}]$.

In the specification of his structural equations, Stein (1965) allows for the possibility that forward market participation which is a function of $[F^*_{ji} - f_{ji}]$, (he terms this "hedging"), adjusts from actual positions to optimal positions, with a lag. As a consequence he arrives at a somewhat more complex equilibrium equation. Using his notation for parameters, but our own for variables, this "reduced form equation" as he terms it, has the form,

$$\begin{aligned} \left[\frac{\tilde{s}_{ji}(t) - f_{ji}(t)}{s_{ji}(t)} \right] &= \frac{(1 - e^{-t})}{1 + (g_1/h_1)} \left[\frac{r_i^f - r_i^f}{r_i^f} \right] \\ &+ \left[\frac{h_o - g_o}{g_1 + h_1} \right] + \left[\frac{H_o - h_o + h_i y_o}{g_1 + h_1} \right] e^{-At} \\ &- \left[\frac{g_1}{g_1 + h_1} \right] \left[\frac{E[\tilde{s}_{ji}] - s_{ji}(t)}{s_{ji}(t)} \right] \end{aligned} \quad (12)$$

[This is Stein's (1965) equation (5) p. 115.]

If there is no lag, the speed of adjustment coefficient λ tends to infinity, and the term $e^{-\lambda t}$ tends to zero. In this case equation (12) becomes

$$\begin{aligned} \left[\frac{s_{ji}(t) - f_{ji}(t)}{s_{ji}(t)} \right] &= \frac{1}{1 + g_1/h_1} \left[\frac{r_i^f - r_j^f}{r_i^f} \right] \\ &+ \left[\frac{h_0 - g_0}{g_1 + h_1} \right] - \left[\frac{g_1}{g_1 + h_1} \right] \left[\frac{E[\tilde{s}_{ji}] - s_{ji}(t)}{s_{ji}(t)} \right] \end{aligned} \quad (13)$$

After some preliminary estimation Stein (1965 p. 119) concludes that "Whatever lag may exist in the hedging function is negligible". He then proceeds to interpret the estimated coefficient of the interest differential as an estimate of $1/(1 + g_1/h_1)$. Thus he is using not (12), but (13), as the basis for his estimation. To justify the inclusion of Stein in this modern theory empirical model group, it remains to show that (13) is essentially a rearrangement of (11), the Kesselman, Stoll, Hass formulation.

(13) may be rewritten as

$$\begin{aligned} f_{ji} &= \left[\frac{g_0 - h_0}{g_1 + h_1} \right] s_{ji} + \left[\frac{h_1}{h_1 + g_1} \right] \left[\frac{r_j^f}{r_i^f} s_{ji} \right] \\ &+ \left[\frac{g_1}{g_1 + h_1} \right] E[\tilde{s}_{ji}] \end{aligned}$$

Or

$$f_{ji} = \delta_0 + \delta_1 F^*_{ji} + \delta_2 E[\tilde{s}_{ji}]$$

where $\delta_1 + \delta_2 = 1$ (14)

The only difference between (14) and (11) is the appearance of the δ_0 term in the latter. (This has arisen because of Stein's inclusion of intercepts to reflect transactions costs, in the specification of the hedging and speculative equations.)

Stein's estimation procedure is novel, in that he subsumes the expectations variable into the disturbance term, and then analyses the consequent estimation bias, to show its existence serves only to reinforce his conclusions ensuing from the estimation of δ_1 . In view of (14) however, it seems not unreasonable to claim that his estimation proceeds from the modern theory empirical model, and thus what we shall have to say about that model does pertain to his contribution.

Pederson and Tower (1979), employ Stein's (1965) model with its arbitrage response lag, to examine the sensitivity of the forward discount, to interest differentials in the long run. Using Stein's structural equations they derive a "reduced form", which expresses the forward discount as a linear function of current and past interest differentials.

Using our notation,

$$\left[\frac{s_{ji}(t) - f_{ji}(t)}{s_{ji}(t)} \right] = \sum_{k=0}^{\infty} w_k \left[\frac{r_i^f(t-k) - r_j^f(t-k)}{r_i^f(t-k)} \right] + c$$

The initial weight w_0 , is a measure of short run responsiveness, $w_L = \sum_k w_k$ the long run responsiveness. What is of interest from our perspective is that the w_k and c , are functions of the parameters of the structural equations specifying hedging and speculative behaviour. They are functions of some of the same parameters which compose the coefficients of the modern theory empirical model equations (10) and (11). Thus, whilst the Pederson Tower reduced form, differs from those of the modern theory empirical models, what we shall have to say of the latter has significance for the former.

Having delineated the literature which is the focus of our interest, we now return to an outline of the theme of our enquiry.

Section 2: The role of risk in the modern theory empirical model.

To constitute a theoretical alternative to the Interest Parity Theory, the modern theory empirical model requires that α^{jAi} in (7) above, be finite. If it were not, then equilibrium would require

$$f_{ji} = F^*_{ji}$$

which, recalling (5b) is the Interest Parity Theorem. Officer and Willet (1970) in a comprehensive survey of explanations of deviations from interest parity, provide a number of reasons for positive and finite α^{jAi} . In turn, in his most penetrating critical survey, Kohlhagen (1978) classifies the Officer and Willet explanations into four issues⁽⁶⁾, " (1) non monetary

returns, (2) default risk, (3) non unitary correlation of returns and (4) premature repatriation".

The first explanation "non monetary returns", first posited by S. Tsiang (1959), has not been addressed in the modern theory empirical literature. As Kohlhagen points out, the third explanation depends upon the existence of either (2) or (4). In their absence, the returns on the domestic bonds and covered foreign bonds are certain, and thus at an equilibrium these assets should be perfect substitutes. Stoll (1972) has shown that explanation (4) also cannot stand alone. He shows that in the absence of some form of default risk, premature liquidation cannot lead to a less than perfectly elastic arbitrage schedule (i.e. finite α^{jAi} .)

Explanation (2), whilst not posited exclusively, appears regularly in the modern theory empirical literature. Whilst in his 1968 paper, Haas does not address the question, in his associated 1971 paper he comments: (7)

"The riskiness of [a domestic security] and [a covered foreign security] may, however, differ if different degrees of default risk are associated with the promised payments ... [For securities of the same risk class] if there is uncertainty that the contractual rate F [i.e. the forward rate,] will be held to, the [covered foreign security] will be riskier than [the domestic security] ... In periods of international crisis [the covered foreign security] may suffer a loss relative to [the domestic security] because the forward exchange contract cannot be carried out, or can be carried out only at a loss due to a moratorium on payments or to other government restrictions on capital flows."

Hans Stoll makes the following comments: (8)

"Now the crucial question concerns the size of κ^{jAi} and the factors determining κ^{jAi} . We would argue that the value of κ^{jAi} depends on the amount of risk attaching to arbitrage operations and the degree of risk aversion on the part of arbitragers ... But what would be the nature of such risk? The exchange rate risk is already covered in the forward market. The remaining risk, which cannot be hedged by the arbitrageur, we claim is in the nature of a default risk. Not only does the arbitrageur face the possibility that the individual with whom he has a contract will fail to keep his bargain to buy or sell forward exchange, but, more important, he also faces a general danger that the governments concerned will freeze foreign balances or in some way put a halt to all international transactions, at least temporarily."

Robert Aliber comments in a similar vein: (9)

"The risk that ... arbitragers encounter is that of exchange controls. These controls applied within the home country of the investor may limit the ability of arbitragers to take advantage of a spread between the interest agio and the exchange agio. Even without these controls, that willingness to take advantage of this spread may be constrained by the concern that the monetary authorities may apply these controls."

It is of interest to note that Keynes, an early proponent of the Interest Parity Theory, was cognizant of the risks of "covered" bond arbitrage. The possibility of persistent deviation from interest parity was by no means paradoxical to him: (10)

"If questions of credit did not enter in, the factor of the rate of interest on short loans would be the dominating one [in the determination of the forward premium/discount]. But ... the various uncertainties of financial and political risk ... introduce a further element which quite transcends the factor of relative interest. The possibility of financial trouble or political disturbance, and the quite appreciable probability of a moratorium in the event of any difficulties arising, or of the sudden introduction of exchange regulations which would interfere with the movements of balances out of the country ... all of these

factors deter bankers, even when the exchange risk proper is eliminated, from maintaining large floating balances at certain foreign centres. Such risks prevent the business from being based ... on a mathematical calculation of interest rates."

Stein also quotes from Keynes⁽¹¹⁾

"Finally, in turbulent periods the fear of a "sudden introduction of exchange regulations which would interfere with the movement of balances out of the country ... deter bankers, even when the exchange risk is eliminated, from maintaining large floating balances at certain foreign centres⁽¹²⁾" For these reasons the potential investor may not invest more funds abroad although the forward rate is less than the measured interest parity."

In view of these quotations, it seems reasonable to assert that a popular foundation for the less than perfectly elastic arbitrage schedule in the modern theory empirical model, lies in the possibility of forward contract default, arising from the imposition of currency exchange regulation. In short, the slope of the arbitrage function is widely regarded as depending upon risk considerations.

The slope of the speculative function is universally regarded as depending upon risk considerations, Hans Stoll is explicit on this point:⁽¹³⁾

"... it is assumed that B [i.e. the slope of the speculative function] depends upon risk and the degree of risk aversion."

Section 3: The theme of the thesis: Can the modern theory model linear equilibrium equation be derived from a model of individual optimization?

That the slopes of the arbitrage and speculative schedules are

widely regarded as depending upon risk considerations is central to our theme. As Kesselman (1971, p. 238) emphasises, the modern theory empirical model "reduced form " "... requires the assumption that the slopes of the schedules remain constant over the period of study. Otherwise the specification will be underidentified for estimation." This assumption is reflected in the fact that the coefficients of the reduced forms (10) and (11), appear without a time index. If it is unwarranted, then to proceed with estimation as if the coefficients are constants, as all of the cited works do, is to fall prey to the identification problem. In short, such estimation would involve combining observations from differing underlying structures, and producing estimated coefficients which are invalid estimates of the parameters of any one of these structures. It is the question of whether or not these coefficients can reasonably be assumed to be unvarying in view of their dependence upon risk considerations, which is the major unifying theme of this thesis.

The structural equations of the modern theory empirical model include specifications of aggregate participation in a forward market. It is notable that these aggregate functions are immediately specified, they are not the consequence of aggregation from some form of individual behaviour which has been derived via an optimization process. It is perhaps, as a result that the empirical model specifications are somewhat lacking in their intuitive appeal. They purport to capture behaviour which is intimately related to the selection of risky financial positions, and yet

fail to display any hint of diversification. In short, in much of the theoretical underpinning of the modern theory empirical model, the return from holdings of covered foreign bonds and from a speculative position, are both risky. However, in the discussion of acquisition of these holdings, and in the specification of the excess demand for forward currency which ensues from these holdings, the diversification motive is nowhere to be seen. This prompts the question: "Are the modern theory model specifications of aggregate behaviour, in their constant coefficient entirety, consistent with the aggregate behaviour which emerges from a model of optimal individual behaviour?" This is the question we address.

Beginning with a model of optimal international portfolio selection, which treats forward market participation as only one of many ways in which the individual may tailor his portfolio to his tastes, we derive expressions which are analogous to the "arbitrage", "speculative" and "trade" functions of the modern theory empirical model. These expressions are then compared to those of the empirical model, and the critical question of the constancy of the coefficients of the empirical model equilibrium equation is addressed.

CHAPTER I INTRODUCTION FOOTNOTES

- (1) The interest parity theory is commonly ascribed to John Maynard Keynes. The following most succinct rendition from Keynes (1923), is perhaps the first statement of this theory.

"That is to say, forward quotations for the purchase of the currency of the dearer money market [i.e. higher rate of interest] tend to be cheaper than spot quotations by a percentage per month equal to the excess of the interest which can be earned in a month in the dearer market over what can be earned in the cheaper."

(parentheses added)

- (2) See Grubel (1966), Gaab (1978), Callier (1979).
- (3) For a comprehensive discussion and documentation of this literature see S. Kohlhagen (1978).
- (4) Grubel (1966) Ch. 8 and 13.
- (5) Glahe p. 8 - 9 (1967).
- (6) Kohlhagen (1978), p. 5.
- (7) Haas p. 115 (1971).
- (8) Stoll p. 58 (1968).
- (9) Aliber (1973).
- (10) Keynes Ch. 3. p. 105 (1923).
- (11) Stein (1965) p. 114.
- (12) Keynes (1923) p. 126.
- (13) Stoll p. 61 (1968).

CHAPTER II. SPECIFICATION OF INDIVIDUAL FINAL WEALTH

As was suggested in the introduction, the modern theory empirical model incorporates the risks of international arbitrage and speculation, in a somewhat ad hoc manner at the aggregate level. Since the validity of the estimation of this model depends critically upon the constancy of coefficients which are recognized to be functions of these risks, we attempt to construct a model of optimal forward market participation which will provide these functional relationships in explicit form.

The model involves the specification of final wealth arising from an extensive opportunity set, including those opportunities for trade, speculation and arbitrage with which the modern theory model is exclusively concerned. Assuming that each transactor maximizes the expected utility of final wealth, we derive the individual's optimal risky portfolio. Within this portfolio there are elements which give rise to forward market participation, and which corresponds to the speculative, arbitrage and trade motivated participation of the modern theory model. Assuming that all transactors are "mean/variance" optimizers, we aggregate these elements over all transactors within a country, to provide aggregate market participation functions. From these functions, "trade", "arbitrage", and "speculative" functions analogous to those of the modern theory model, are then derived.

In this chapter the form of final wealth is presented. Before doing

so, however, some discussion of the modelling of international "covered" arbitrage risk, is warranted.

Section 1: Risk in covered arbitrage

This risk which we model is that of the imposition of currency exchange controls in the period between the acquisition of a forward contract and its maturity. The following assumptions are made with respect to controls.

1. The only form of control is one which a country imposes to restrict sale of its currency.
2. Controls are imposed identically on all sales of the domestic currency for a particular foreign currency, except those sales which are a consequence of "trade".
3. The imposition of controls by country i or country j , on the exchange of i for j , renders void all forward contracts for exchange between these currencies.
4. Beliefs about the future spot rate between currencies i and j , are contingent upon control configurations. That is, an individual is modelled as having a subjective conditional distribution for the future spot rate in the absence of controls, in the event of controls by either country alone, and in the event of simultaneous controls by both countries. It is further assumed that beliefs about the future spot rate between currencies i and j , are independent of the possibility

of imposition of controls on the exchange of currencies other than i or j.

5. Exchange is impeded by controls; only as a special case is it prevented. For example, if country j prohibits the sale of its currency for currency i, an exchange of j for i may be secured by purchasing commodities in country j, and selling them in country i. For such a transaction, an effective 'de facto' exchange rate may be calculated. Such de facto rates might also be suggested by the individual owning the 'blocked' currency j, having the belief that at some time in the future the restrictions may be removed. Any costs associated with the postponed exchange could be included in the calculation of a de facto exchange rate. Uncertainty in the timing of the liberalization of exchange and in the costs of postponed exchange, contribute to the randomness of such a rate. It is assumed that de facto exchange rates are the same for all individuals, and that their distributions are the same in the event of simultaneous imposition of controls, as in the event of unilateral control.

The following assumptions relate to other aspects of the opportunity set.

1. No transactions costs.
2. Unlimited borrowing and lending at an exogenous riskless rate in each country.

3. Unrestricted, marginless short positions in forward currencies and risky assets.
4. The risky opportunities within a country may be represented by a single index of returns.
5. All opportunities relate to a single common period. That is, all contracts are assumed to have the same duration. Thus, anticipating our theme, forward currency commitments arising from trade, speculation and arbitrage, all mature at the same time and have the dimension of flows per common period.

Section 2: The structure of final wealth.

Rather than immediately present the form of final wealth, we shall first provide a taxonomy of its components. In what follows, all choice variables relate to an individual 'k' of country j. With this understood, we shall omit these subscripts and, with respect to choice variables, use a single subscript i, to denote the currency in which the asset or debt associated with the choice, is established.

The individual is envisaged as having the opportunity to invest in the risky and/or riskless assets of 'n' countries, to borrow in these countries and/or to go short in their risky assets, and to engage in trade with these countries. For purposes of exposition only, the choices associated with these opportunities will be referred to as primary choices.

Concurrent with the making of primary choices, the individual is faced with the selection of exchange strategies for the repatriation of income, and the repayment of debt arising from these choices. We shall classify our taxonomy by the primary choice variables, and in each classification outline the influence of the primary choice and its associated exchange strategies on final wealth.

1. Lending.

Let A_i denote the number of units of currency j allocated to the risk free asset of country i ; r_i^f denote the risk free rate of country i plus one; and s_i the number of units of currency j paid spot currently for one unit of i . Then

$$A_i \frac{r_i^f}{s_i}$$

is a known future income, denominated in currency i . How may this income be repatriated? Let u_i denote the number of units (nb. currency i) of this income, which are left exposed to be repatriated at the end of period spot rate \tilde{s}_i . Let f_i denote the currency j forward price of one unit of currency i . Then, leaving some of the income exposed, and covering some in the forward market, provides, in the absence of any exchange control considerations, the following contribution to final wealth:

$$\left[A_i \frac{r_i^f}{s_i} - u_i \right] f_i + u_i \tilde{s}_i \quad (15)$$

With the possibility of exchange controls, it may be the case that neither f_i nor \tilde{s}_i turn out to be the rates which govern the eventual exchange. That they may not, follows from the assumptions that any controls serve to void forward contracts, and that the end of period spot rate, may be contingent upon the configuration of controls.

The influence of controls is reflected by use of the following binomially distributed random variable .

Let

$$P_{ji} = 0 \quad \text{if controls are imposed by country } j \text{ and/or country } i.$$

$$P_{ji} = 1 \quad \text{otherwise (i.e. in the absence of any controls).}$$

The obsolescence of f_i and \tilde{s}_i in the event of any controls is then reflected by changing (15) to

$$\left[A_i \frac{r_i^f}{s_i} - u_i \right] f_i P_{ji} + u_i \tilde{s}_i P_{ji} \quad (16)$$

We have now to represent the repatriation of the currency i income, given the imposition of controls. If country i controls the sale of its currency, the income is repatriated at a de facto rate \tilde{c}_i^i (i.e. A currency j price of another currency, in the event of control. The subscript

denotes the 'other' currency, the superscript denotes which country is controlling the sale of its currency.) If country j controls the sale of its currency this does not preclude access to the end of period spot market, however, the exchange rate in this market is contingent upon these controls. Hence repatriation is at rate \tilde{s}_i^j (i.e. A currency j future spot market price of another currency in the event of controls. The subscript denotes the 'other' currency, the superscript denotes which country is controlling the sale of its currency.)

These considerations can be reflected by the introduction of the following binomially distributed random variables.

Let

$$\begin{aligned} D_j &= 1 && \text{if } j \text{ (or if } j \text{ and } i) \text{ imposes controls.} \\ &= 0 && \text{otherwise} \end{aligned}$$

$$\begin{aligned} D_i &= 1 && \text{if } i \text{ (or if } i \text{ and } j) \text{ imposes controls.} \\ &= 0 && \text{otherwise} \end{aligned}$$

then

$$\begin{aligned} D_i(1 - D_j) &= 1 && \text{if ONLY } i \text{ imposes controls} \\ &= 0 && \text{otherwise} \end{aligned}$$

$$\begin{aligned} D_j(1 - D_i) &= 1 && \text{if ONLY } j \text{ imposes controls} \\ &= 0 && \text{otherwise} \end{aligned}$$

The repatriated income is then represented by the term:

$$A_i \frac{r^f}{s_i} (\tilde{D}_i \tilde{c}_i^i + \tilde{D}_j [1 - \tilde{D}_i] \tilde{s}_i^j) \quad (17)$$

Adding (16) and (17), provides the gross contribution of lending in country i to final wealth (i.e. 'gross' in the sense that we have yet to consider the financing of this lending),

$$\left[\left[A_i \frac{r^f}{s_i} - u_i \right] f_i + u_i \tilde{s}_i \right] \tilde{P}_{ji} + A_i \frac{r^f}{s_i} (\tilde{D}_i \tilde{c}_i^i + \tilde{D}_j [1 - \tilde{D}_i] \tilde{s}_i^j) \quad (18)$$

As an example of the responsiveness of this formulation, consider a simultaneous imposition of controls by countries i and j . Then:

$$P_{ji} = 0 \quad D_i = 1 \quad \text{and} \quad D_j (1 - D_i) = 0$$

(18) reduces to

$$A_i \frac{r^f}{s_i} \tilde{c}_i^i$$

which indicates that the currency i income is repatriated at the de facto rate \tilde{c}_i^i .

2. Borrowing.

Let O_i denote the number of units of j raised by borrowing in country

i. Then

$$O_i \frac{r_i^f}{s_i}$$

is a known future debt denominated in currency i . Let p_i denote the number of units (nb. currency i) of this debt, which are left exposed to be repaid via the end of period spot market. Then leaving some of the debt exposed, and covering some in the forward market, provides, in the absence of any exchange control considerations, the following debit terms in final wealth,

$$- \left[O_i \frac{r_i^f}{s_i} - p_i \right] f_i - p_i \tilde{s}_i \quad (19)$$

With the possibility of exchange controls, it may be that neither f_i nor \tilde{s}_i govern the eventual exchange. Following the discussion in the preceding section, this obsolescence of f_i and \tilde{s}_i in the event of controls, is reflected by changing (19) to,

$$- \left[O_i \frac{r_i^f}{s_i} - p_i \right] f_i \tilde{p}_{ji} - p_i \tilde{s}_i \tilde{p}_{ji} \quad (20)$$

We have now to represent the repayment of the debt, given the imposition of controls. If country j controls the sale of its currency, this debt

is repaid at the de facto exchange rate \tilde{c}_i^j . (i.e. A currency j price of currency i in the event of control by country j. The superscript denotes the 'controlling' country, the subscript the 'other' currency in the exchange.) If country i controls the sale of its currency, repayment of currency i debt by purchase of i in the end of period spot market is not precluded. However, the exchange rate in this market is contingent upon these controls. Hence the repayment is governed by the exchange rate \tilde{s}_i^i . (i.e. A currency j future spot market price of another currency in the event of control. The subscript denotes the 'other' currency, the superscript denotes which country is controlling the sale of its currency.) Once again, using the binomially distributed random variables introduced in the preceding section, this debt in the event of controls may be represented by the term,

$$- O_i \frac{r^f}{s_i} (\tilde{D}_j \tilde{c}_i^j + \tilde{D}_i [1 - \tilde{D}_j] \tilde{s}_i^i) \quad (21)$$

Adding (20) and (21) provides the currency j debt which arises from borrowing in country i,

$$- \left[\left[O_i \frac{r^f}{s_i} - p_i \right] f_i + p_i \tilde{s}_i^i \right] \tilde{p}_{ji}$$

$$- O_i \frac{r^f}{s_i} (\tilde{D}_j \tilde{c}_i^j + \tilde{D}_i [1 - \tilde{D}_j] \tilde{s}_i^i) \quad (22)$$

Consider a simultaneous imposition of controls by countries i and j .

Then,

$$P_{ji} = 0 \quad D_j = 1 \quad D_i = 1$$

and (22) becomes

$$- 0_i \frac{r_i^f}{s_i} \tilde{c}_i^j$$

This term indicates that the debt is repaid via the de facto rate \tilde{c}_i^j .

If there are no controls

$$P_{ji} = 1 \quad D_j = 0 \quad D_i = 0$$

and (22) becomes

$$- \left[0_i \frac{r_i^f}{s_i} - p_i \right] f_i - p_i \tilde{s}_i$$

This term indicates that in the absence of any controls, the debt is repaid employing both the forward market, and the future spot market.

A notable distinction between the form of foreign lending and foreign borrowing in final wealth, is that they are influenced differently by controls. For example, if country j controls sale of currency j , the repayment of borrowing in country i involves a de facto rate \tilde{c}_i^j , the repatriation of currency i income from lending in country i , however, is repatriated at the conditional spot rate \tilde{s}_i^j . In short, in the event of controls, the same exchange rate does not govern exchange associated

with borrowing and lending. Lending thus cannot be viewed as the opposite transaction to borrowing. For this reason, borrowing and lending cannot be modelled with a single choice variable, the sign of which would indicate whether at the optimum, the individual engages in borrowing or lending. This comment also pertains to the treatment of long and short positions in foreign risky assets.

3. Risky investment.

Let B_i denote the number of units of j allocated to the risky asset of country i , and \tilde{r}_i denote the risky rate of return plus one, from this asset. Then

$$B_i \frac{\tilde{r}_i}{s_i}$$

represents uncertain currency i income. Because of its uncertainty, even in the absence of control risk, the exchange risk associated with this income cannot be completely avoided. However, it may be "hedged" against

Hedging of a financial position, involves the acquisition of other positions such that because of the covariances of returns (or payments), the expected utility of total final wealth is enhanced.

One such financial position which readily presents itself in this context, involves the forward sale of currency i .

The value in currency i of the future income $B_i \frac{\tilde{r}_i}{s_i}$ fluctuates

directly with the future spot rate at which it is repatriated. Ignoring for the moment the possibility of controls, this rate is \tilde{s}_i . The sale of w_i units of currency i forward, anticipating their purchase at future spot \tilde{s}_i , creates a risky position the currency j return on which is $(f_i - \tilde{s}_i)$ per unit of i sold forward. This return fluctuates inversely with \tilde{s}_i , and thus the forward sale may be regarded as a hedge for $B_i \frac{\tilde{r}_i}{s_i}$. The net position, without considering controls is,

$$+ B_i \frac{\tilde{r}_i}{s_i} \tilde{s}_i + w_i (f_i - \tilde{s}_i)$$

or

$$\left[B_i \frac{\tilde{r}_i}{s_i} - w_i \right] \tilde{s}_i + w_i f_i \quad (23)$$

If controls are introduced either by i or j , or by both countries, the exchange rates \tilde{s}_i and f_i no longer apply. Multiplying (23) by the binomially distributed variable \tilde{P}_{ji} , ensures that in the event of controls, the terms in (23) disappear from final wealth. Thus final wealth includes the term

$$\left[\left[B_i \frac{\tilde{r}_i}{s_i} - w_i \right] \tilde{s}_i + w_i f_i \right] \tilde{P}_{ji} \quad (24)$$

In the event of restriction on the sale of currency i , repatriation of the income $B_i (\tilde{r}_i/s_i)$ is all effected at the de facto rate \tilde{c}_i^1 . In

the event of restriction on the sale of currency j but not currency i, this income may be repatriated via the future spot market, but at a rate \tilde{s}_i^j , contingent upon this restriction, These considerations are represented in final wealth by the term,

$$B_i \frac{\tilde{r}_i}{s_i} [\tilde{D}_i \tilde{c}_i^i + \tilde{D}_j (1 - \tilde{D}_i) \tilde{s}_i^j] \quad (25)$$

Combining (24) and (25) provides the gross contribution to final wealth of investment in the risky asset of country i,

$$\left[\left[B_i \frac{\tilde{r}_i}{s_i} - w_i \right] \tilde{s}_i + w_i f_i \right] \tilde{P}_{ji} + B_i \frac{\tilde{r}_i}{s_i} [\tilde{D}_i \tilde{c}_i^i + \tilde{D}_j (1 - \tilde{D}_i) \tilde{s}_i^j] \quad (26)$$

By the way of example of the responsiveness of this term, consider the case when the sale of currency j but not currency i, is controlled. Then,

$$P_{ji} = 0 \quad D_i = 0 \quad D_j = 1$$

and (26) becomes

$$B_i \frac{\tilde{r}_i}{s_i} \tilde{s}_i^j$$

This term indicates that all of this currency i income is repatriated via the end of period spot market, at the control contingent spot rate \tilde{s}_i^j .

4. Short positions in stock.

Let M_i denote the number of units of currency j acquired by going short in the risky asset of country i . Then $M_i \frac{\tilde{r}_i}{s_i}$ is an uncertain future debt denominated in currency i . This debt may be hedged by a forward purchase of k_i units of currency i . The net position, without considering controls is,

$$- M_i \frac{\tilde{r}_i \tilde{s}_i}{s_i} - k_i (f - \tilde{s}_i)$$

or

$$- \left[M_i \frac{\tilde{r}_i}{s_i} - k_i \right] \tilde{s}_i - k_i f_i \quad (27)$$

(The latter form indicates that hedging may be equivalent to some exchange via the future spot market, and some via the forward market.)

Multiplying (27) by \tilde{P}_{ji} ensures its disappearance from final wealth in the event of controls. Hence there is a term in final wealth:

$$- \left[\left[M_i \frac{\tilde{r}_i}{s_i} - k_i \right] \tilde{s}_i + k_i f_i \right] \tilde{P}_{ji} \quad (28)$$

In the event of restriction on the sale of currency j , repayment of the debt $M_i (\tilde{r}_i/s_i)$ is all effected at the de facto rate \tilde{c}_i^j . If the sale of currency i , but not that of j , is restricted, repayment is via the future spot market, at the control contingent rate \tilde{s}_i^i . These considerations

are embodied in the term,

$$- M_i \frac{\tilde{r}_i}{s_i} [\tilde{D}_j \tilde{c}_i^j + \tilde{D}_i (1 - \tilde{D}_j) \tilde{s}_i^i] \quad (29)$$

The gross influence of a short position in the risky asset of country i , on final wealth is represented by (28) and (29),

$$- \left[M_i \frac{\tilde{r}_i}{s_i} - k_i \right] \tilde{s}_i + k_i f_i \tilde{P}_{ji}$$

$$- M_i \frac{\tilde{r}_i}{s_i} [\tilde{D}_j \tilde{c}_i^j + \tilde{D}_i (1 - \tilde{D}_j) \tilde{s}_i^i] \quad (30)$$

Once more, by way of example, consider if the sale of i is controlled, but that of j is not. Then

$$P_{ji} = 0 \quad D_i = 1 \quad D_j = 0$$

and (30) becomes

$$- M_i \frac{\tilde{r}_i}{s_i} \tilde{s}_i^i$$

This term indicates that the entire debt is repaid via the future spot market at the control contingent rate \tilde{s}_i^i .

5. Exports.

The individual is treated as being in a position to contract to

supply a quantity of exports, for which the foreign currency payment at the end of the period is KNOWN. Let us denote this payment for exports to country i as E_i . The cost of generating a unit of this payment, is treated as being uncertain. It is denoted as \tilde{r}_i^E . This per unit cost \tilde{r}_i^E is assumed not to be a function of the magnitude of E_i . Thus it may be anticipated that it is the uncertainty of profit, rather than rising marginal cost, which will serve to ensure that the optimal choice of E_i is finite.

It is assumed that the cost of generating E_i is paid at the end of the period. Hence there is a term in final wealth,

$$- E_i \tilde{r}_i^E \quad (31)$$

The individual is faced with the problem of repatriating the known amount, E_i units of currency i . Without considering controls, there are three potential methods of repatriation. Some of E_i may be left exposed to be repatriated at future spot, let us denote this amount of currency i , e_i . Some of E_i , an amount denoted R_i , may be repatriated by borrowing in country i , exchanging the proceeds at current spot, and investing in the domestic riskless asset. The remainder of E_i , an amount $(E_i - e_i - R_i)$, may be sold forward. These opportunities may be summarized by the terms:

$$(E_i - e_i - R_i) f_i + R_i \frac{s_i}{r_i^f} r_j^f + e_i \tilde{s}_i \quad (32)$$

In the event of any form of controls terms involving f_i and \tilde{s}_i disappear from final wealth. This is ensured by multiplying such terms in (32) by \tilde{P}_{ji} . Hence there appears in final wealth, the terms

$$(E_i - e_i - R_i) f_i \tilde{P}_{ji} + e_i \tilde{s}_i \tilde{P}_{ji} \quad (33)$$

Note that the strategy associated with R_i is not influenced at all by intra period controls. Thus the contribution to final wealth

$$R_i \frac{s_i}{r_i} r_j^f \quad (34)$$

is unaltered by control considerations. With R_i uninfluenced by controls, in the event of controls, repatriation involves not the entire amount E_i , but the lesser amount $(E_i - R_i)$. We have now to model the repatriation of this amount, in the event of controls.

It has been assumed that controls are never imposed on trade inspired currency exchange. Thus in the event of controls, such exchange always takes place at some spot market control contingent rate. If country i alone imposes controls, repatriation is at the spot rate contingent on 'only i ' control, \tilde{s}_i^i . If country j alone imposes controls, the relevant exchange rate is \tilde{s}_i^j . If both countries impose controls simultaneously, then the relevant rate is that spot rate contingent on simultaneous controls, denoted \tilde{s}_i^{ij} . These considerations may all be represented by the following terms;

$$(E_i - R_i) \left[\tilde{D}_i (1 - \tilde{D}_j) \tilde{s}_i^i + \tilde{D}_j (1 - \tilde{D}_i) \tilde{s}_i^j + \tilde{D}_i \tilde{D}_j \tilde{s}_i^{ij} \right] \quad (35)$$

For example, in the event of simultaneous control

$$D_j = D_i = 1$$

and (35) reduces to

$$(E_i - R_i) \tilde{s}_i^{ij}.$$

Combining (31), (33), (34) and (35) provides the contribution of E_i to final wealth

$$\begin{aligned} & \left[(E_i - e_i - R_i) f_i + e_i \tilde{s}_i \right] \tilde{P}_{ji} + R_i \frac{s_i}{r_i} r_j^f \\ & + (E_i - R_i) \left[\tilde{D}_i (1 - \tilde{D}_j) \tilde{s}_i^i + \tilde{D}_j (1 - \tilde{D}_i) \tilde{s}_i^j + \tilde{D}_i \tilde{D}_j \tilde{s}_i^{ij} \right] \\ & - E_i \tilde{r}_i^E \end{aligned} \quad (36)$$

In the absence of controls, $P_{ji} = 1$, $D_j = D_i = 0$ and (36) reduces to (32), the contribution of E_i to final wealth without controls.

6. Imports.

The individual is viewed as contracting to purchase an amount of imports for which the end of period currency i payment, denoted I_i , is known.

The (domestic) currency j revenue which each unit of I_i generates, denoted \tilde{r}_i^I , is assumed to be uncertain. Hence there is a term in final wealth,

$$+ I_i \tilde{r}_i^I \quad (37)$$

The exchange strategies for meeting the known currency i debt I_i , are exactly analogous to those of the export case. Some of I_i , an amount denoted b_i , may be repaid via the future spot market. Some of I_i , an amount denoted Q_i , may be paid by borrowing domestically, exchanging the proceeds at current spot for currency i ; and investing in the riskless asset of currency i . The remainder of I_i , an amount $(I_i - b_i - Q_i)$ may be paid by purchasing forward currency i . These opportunities may be represented in final wealth by the terms

$$- (I_i - b_i - Q_i) f_i - Q_i \frac{s_i^f}{r_i^f} r_j^f - b_i \tilde{s}_i \quad (38)$$

Analogous to the export case, the possibility of controls requires that (38) be modified,

$$- (I_i - b_i - Q_i) f_i \tilde{p}_{ji} - Q_i \frac{s_i^f}{r_i^f} r_j^f - b_i \tilde{s}_i \tilde{p}_{ji} \quad (39)$$

In the event of controls, the problem is one of how to meet the currency i debt unaccounted for, $(I_i - Q_i)$. Since it is trade inspired, the exchange is always via the future spot market. If country j imposes controls, the

pertinent exchange rate is \tilde{s}_i^j , if country i imposes controls it is \tilde{s}_i^i , and if there is simultaneous control, it is \tilde{s}_i^{ji} .

The influence on final wealth, of the payment of $(I_i - Q_i)$ may be represented by,

$$- (I_i - Q_i) \left[\tilde{D}_i [1 - \tilde{D}_j] \tilde{s}_i^i + \tilde{D}_j [1 - \tilde{D}_i] \tilde{s}_i^j + \tilde{D}_i \tilde{D}_j \tilde{s}_i^{ij} \right] \quad (40)$$

Combining (37), (39) and (40) provides the contribution of I_i to final wealth,

$$+ I_i \tilde{r}_i^I - \left[(I_i - b_i - Q_i) f_i + b_i \tilde{s}_i \right] \tilde{P}_{ji} - Q_i \frac{s_i}{r_i} r_j^f - (I_i - Q_i) \left[\tilde{D}_i [1 - \tilde{D}_j] \tilde{s}_i^i + \tilde{D}_j [1 - \tilde{D}_i] \tilde{s}_i^j + \tilde{D}_i \tilde{D}_j \tilde{s}_i^{ij} \right] \quad (41)$$

This completes our taxonomy of primary choices and associated exchange strategies. In this taxonomy the influence on final wealth of borrowing, lending, going short and risky investment, were gross influences. (i.e. "gross" in the sense that the financing of investment, or the allocation of borrowing were not considered). Using W_0 to denote initial wealth, we ensure that all debt is allocated and all investment financed, in the formulation of final wealth, by introducing the term

$$W_0 + \sum_i [O_i + M_i - A_i - B_i] r_j^f \quad (42)$$

The individual is envisaged as creating a pool of currency j , composed of initial wealth, and the proceeds of borrowing ($\sum_i O_i$) and short positions ($\sum_i M_i$). This pool is then allocated to risky assets and foreign riskless assets. The remainder, the term multiplying r_j^f in (42), is allocated to the domestic (i.e. country j) bond.

Having discussed each major component of final wealth in turn, we now combine them and present final wealth in its entirety:

$$\begin{aligned}
 W_j &= \left[W_o + \sum_i [O_i + M_i - A_i - B_i] \right] r_j^f \\
 &+ \sum_i \left[\left[A_i \frac{r_i^f}{s_i} - u_i \right] f_i + u_i \tilde{s}_i \right] \tilde{P}_{ji} \\
 &+ \sum_i A_i \frac{r_i^f}{s_i} (\tilde{D}_i \tilde{c}_i^i + \tilde{D}_j [1 - \tilde{D}_i] \tilde{s}_i^j) \\
 &- \sum_i \left[\left[O_i \frac{r_i^f}{s_i} - p_i \right] f_i - p_i \tilde{s}_i \right] \tilde{P}_{ji} \\
 &- \sum_i O_i \frac{r_i^f}{s_i} (\tilde{D}_j \tilde{c}_i^j + \tilde{D}_i [1 - \tilde{D}_j] \tilde{s}_i^i) \\
 &+ \sum_i \left[\left[B_i \frac{r_i^f}{s_i} - w_i \right] \tilde{s}_i + w_i f_i \right] \tilde{P}_{ji}
 \end{aligned}$$

$$+ \sum_i B_i \frac{\tilde{r}_i}{s_i} (\tilde{D}_i \tilde{c}_i^i + \tilde{D}_j [1 - \tilde{D}_i] \tilde{s}_i^j)$$

$$- \sum_i \left[\left[M_i \frac{\tilde{r}_i}{s_i} - k_i \right] \tilde{s}_i + k_i f_i \right] \tilde{P}_{ji}$$

$$- \sum_i M_i \frac{\tilde{r}_i}{s_i} (\tilde{D}_j \tilde{c}_i^j + \tilde{D}_i [1 - \tilde{D}_j] \tilde{s}_i^i)$$

$$+ \sum_i R_i \frac{s_i}{r_i^f} r_j^f$$

$$+ \sum_i \left[(E_i - e_i - R_i) f_i + e_i \tilde{s}_i \right] \tilde{P}_{ji}$$

$$+ \sum_i (E_i - R_i) \left[\tilde{D}_i [1 - \tilde{D}_j] \tilde{s}_i^i + \tilde{D}_j [1 - \tilde{D}_i] \tilde{s}_i^j + \tilde{D}_i \tilde{D}_j \tilde{s}_i^{ij} \right]$$

$$- \sum_i E_i \tilde{r}_i^E$$

$$- \sum_i Q_i \frac{s_i}{r_i^f} r_j^f$$

$$\begin{aligned}
& - \sum_i \left[(I_i - b_i - Q_i) f_i + b_i \tilde{s}_i \right] \tilde{P}_{ji} \\
& - \sum_i (I_i - Q_i) \left[\tilde{D}_i [1 - \tilde{D}_j] \tilde{s}_i^i + \tilde{D}_j [1 - \tilde{D}_i] \tilde{s}_i^j + \tilde{D}_i \tilde{D}_j \tilde{s}_i^{ij} \right] \\
& + \sum_i I_i \tilde{r}_i^I
\end{aligned} \tag{43}$$

Final wealth as it appears above, does not exclude a priori, behaviour which, it might be readily suspected, is inconsistent with a conventionally optimal portfolio. As it stands it has a term for borrowing and covering in country i , and a term for lending and covering in country i . However, we emphasize that the above form is not final wealth at the individual's optimum, it is, instead, the argument in the individual's objective function. As will be shown below, some of the above choice variables cannot be simultaneously positive at an optimum which is defined by the maximization of expected utility.

Initially our aim has been to define final wealth over a comprehensive opportunity set which embodies the possibility of controls. This comprehensiveness, allied with the multiple configuration of controls, has forced upon us a most lengthy and unpalatable form of final wealth. Relief is in sight, however. Having attempted in this section to satisfy

the demands of demonstrable generality, in subsequent sections the expression for final wealth is simplified and collapsed to an equivalent, but more digestible, form.

Section 3: The simplification of the final wealth form.

Let

$$\tilde{G}_i \equiv \left[\tilde{D}_i \tilde{c}_i^i + \tilde{D}_j [1 - \tilde{D}_i] \tilde{s}_i^j \right] \quad (44)$$

$$\tilde{N}_i \equiv \left[\tilde{D}_j \tilde{c}_i^j + \tilde{D}_i [1 - \tilde{D}_j] \tilde{s}_i^i \right] \quad (45)$$

$$\tilde{S}_i^c \equiv \left[\tilde{D}_i [1 - \tilde{D}_j] \tilde{s}_i^i + \tilde{D}_j [1 - \tilde{D}_i] \tilde{s}_i^j + \tilde{D}_i \tilde{D}_j \tilde{s}_i^{ij} \right] \quad (46)$$

Note that \tilde{G}_i , \tilde{N}_i and \tilde{S}_i^c can be viewed as composites of control contingent exchange rates. \tilde{G}_i governs potentially controllable sales of i for j , \tilde{N}_i governs potentially controllable sales of j for i , and \tilde{S}_i^c governs those transactions which it is assumed, are never the direct target of controls.

Using the above definitions with (43), collecting terms and rearranging provides:

$$\begin{aligned}
W_j &= W_0 r_j^f + \sum_i A_i \left[\frac{r_i^f}{s_i} [f_i \tilde{P}_{ji} + \tilde{G}_i] - r_j^f \right] \\
&+ \sum_i B_i \left[\frac{\tilde{r}_i}{s_i} [\tilde{s}_i \tilde{P}_{ji} + \tilde{G}_i] - r_j^f \right] \\
&+ \sum_i O_i \left[r_j^f - \frac{r_i^f}{s_i} [f_i \tilde{P}_{ji} + \tilde{N}_i] \right] \\
&+ \sum_i M_i \left[r_j^f - \frac{\tilde{r}_i}{s_i} [\tilde{s}_i \tilde{P}_{ji} + \tilde{N}_i] \right] \\
&+ \sum_i E_i \left[f_i \tilde{P}_{ji} + \tilde{S}_i^c - \tilde{r}_i^E \right] \\
&+ \sum_i I_i \left[\tilde{r}_i^I - f_i \tilde{P}_{ji} - \tilde{S}_i^c \right] \\
&+ \sum_i R_i \frac{s_i}{r_i^f} \left[r_j^f - \frac{r_i^f}{s_i} [f_i \tilde{P}_{ji} + \tilde{S}_i^c] \right] \\
&- \sum_i Q_i \frac{s_i}{r_i^f} \left[r_j^f - \frac{r_i^f}{s_i} [f_i \tilde{P}_{ji} + \tilde{S}_i^c] \right] \\
&+ \sum_i (u_i + e_i + k_i - w_i - p_i - b_i) \left[[\tilde{s}_i - f_i] \tilde{P}_{ji} \right]
\end{aligned} \tag{47}$$

Letting

$$X_i \equiv (u_i + e_i + k_i - w_i - p_i - b_i)$$

and

$$Y_i \equiv (R_i - Q_i) \frac{s_i}{r_i^f}$$

$$\begin{aligned} \tilde{W}_j &= W_0 r_j^f + \sum_{\substack{i \\ i \neq j}} A_i \tilde{a}_i + \sum_i B_i \tilde{b}_i \\ &+ \sum_{\substack{i \\ i \neq j}} O_i \tilde{o}_i + \sum_i M_i \tilde{m}_i + \sum_i E_i \tilde{e}_i \\ &+ \sum_i I_i \tilde{i}_i + \sum_{\substack{i \\ i \neq j}} Y_i \tilde{y}_i + \sum_{\substack{i \\ i \neq j}} X_i \tilde{x}_i \end{aligned} \quad (48)$$

where the stochastic terms are defined by the corresponding terms within [] in (47). The exclusion of the $i=j$ terms is permitted by the fact that

$$\tilde{y}_{jj} = \tilde{a}_{jj} = \tilde{o}_{jj} = \tilde{x}_{jj} = 0.$$

Comparing \tilde{W}_j as it appears in (48) with its form in (43), the principle distinction is the collapse of what appear as separate choice

variables in the former, into single variables in the latter. In (48), Y_i and X_i are such 'net' variables. In our model, the mathematical legitimacy of this collapse serves to indicate that the 'net' variables and their components, are equivalent in that they have identical effect on \tilde{W}_j . Thus when final wealth was initially presented it might have simply been stated directly in the form of (48). Had we done so, all of the choice variables common to both equations, would of course retain their original definition, and supplementary definitions for the variables Y_i and X_i would be required. Associated with the original choice variables however, would be new repatriation strategies suggested by the stochastic return terms in (48). For example, A_i is the "number of units of currency j allocated to the risk free asset of country i ". Equation (43) is structured as if a part of the currency i income from A_i is sold forward, a part left exposed etc. In equation (48) the unit return on A_i is \tilde{a}_i , which, as examination of its form shows, involves complete forward coverage. Guided in this way by the form of the stochastic terms in (48), it is possible to provide a catalogue of transactions which may be considered AS IF they are the determinants of final wealth.

1. Allocate A_i to the covered bond of country i and finance this allocation with domestic borrowing. (nb. the unit return on this is \tilde{a}_i in (48)).
2. Allocate B_i , financed by domestic borrowing, to the risky asset of country i . Repatriate via the future spot market or, if this is

- precluded, by the most efficient alternative means (the unit return on this transaction is \tilde{b}_i in (48).)
3. Allocate O_i to the domestic bond, financed by covered borrowing in country i . (per unit return, \tilde{o}_i)
 4. Allocate M_i to the domestic bond, financed by a short position in the risky asset of country i . Repay the debt by means of the future spot market or, if precluded by controls, by the most efficient alternative means. (The per unit return on this transaction is \tilde{m}_i in (48).)
 5. Contract to export goods of the gross value of E_i units of currency i , to country i . Cover all of this currency i income in the forward market. (per unit return is \tilde{e}_i)
 6. Contract to import goods from country i at a total cost of I_i units of i . Cover this currency i debt completely in the forward market. (per unit return is \tilde{i}_i)
 7. The choice variable X_i may be positive or negative. For $X_i > 0$, engage in a speculative forward purchase of X_i units of currency i . If $X_i < 0$, engage in a speculative forward sale of X_i units of currency i . (It is shown below that the total return on either transaction may be modelled as it appears in (48), $+ X_i \tilde{x}_i$.)
 8. Y_i may be positive or negative. Guided by the form of \tilde{y}_i , Y_i positive may be interpreted as a number of units of currency j , allocated to the domestic (i.e. country j) bond, financed by borrowing

in country i and covering in the forward market.

Again guided by the form of \tilde{y}_i , Y_i negative may be interpreted as the number of units of j borrowed domestically for investment in the riskless asset of country i covered in the forward market.

Note the similarity between the transactions associated with Y_i positive, and O_i . Both involve covered borrowing in country i to finance domestic (country j) lending. However, they are NOT identical, the exchange channel employed for the repayment of the foreign debt, differs for the two transactions in the event of controls. Hence, their per unit return differs, a difference reflected in the forms of \tilde{y}_i and \tilde{o}_i .

Consider \tilde{y}_i and \tilde{o}_i ,

$$\tilde{y}_i \equiv \left[r_j^f - \frac{r_i^f}{s_i} [f_i \tilde{P}_{ji} + \tilde{S}_i^c] \right]$$

$$\tilde{o}_i \equiv \left[r_j^f - \frac{r_i^f}{s_i} [f_i \tilde{P}_{ji} + \tilde{N}_i] \right]$$

In the absence of controls $P_{ji} = 1$ and using (45) and (46),

$$y_i = o_i = \left[r_j^f - \frac{r_i^f}{s_i} f_i \right]$$

However, with any configuration of controls $P_{ji} \neq 1$ and we have:

$$(\tilde{y}_i | P_{ji} = 0) = \left[r_j^f - \frac{r_i^f}{s_i} \tilde{S}_i^c \right] \quad (49)$$

$$(\tilde{o}_i | P_{ji} = 0) = \left[r_j^f - \frac{r_i^f}{s_i} \tilde{N}_i \right] \quad (50)$$

Thus with controls the exchange transaction takes place at different rates.

From (49) it is apparent that with some controls the exchange associated with Y_i takes place at rate \tilde{S}_i^c . Explicitly, from (46) this rate is:

$$\tilde{S}_i^c \equiv \left[\tilde{D}_i [1 - \tilde{D}_j] \tilde{s}_i^1 + \tilde{D}_j [1 - \tilde{D}_i] \tilde{s}_j^1 + \tilde{D}_i \tilde{D}_j s_i^{ij} \right] \quad (46) \text{rptd.}$$

Note that all of these exchange rates are spot market rates, thus the exchange associated with Y_i is in the event of controls, always secured at a spot rate contingent upon the control configuration.¹ This is not the case for O_i .

From (50) with some controls the exchange takes place at rate \tilde{N}_i .

From (45) this rate is:

$$\tilde{N}_i \equiv \left[\tilde{D}_j \tilde{c}_i^j + \tilde{D}_i [1 - \tilde{D}_j] \tilde{s}_i^1 \right]$$

If country j imposes controls we have $D_j = 1$ and

$$(\tilde{N}_i | D_j = 1) = \tilde{c}_i^j$$

the exchange associated with O_i is excluded from the spot market and is instead secured at the de facto rate \tilde{c}_i^j .

In summary, the transactions associated with Y_i positive and O_i differ. With controls, the currency exchange involved in these transactions may take place at different rates. As a consequence the per unit return on the transactions may not be the same.

Y_i negative, interpreted above, is similar to A_i , however it also constitutes a distinct transaction. In the event of controls, currency exchange associated with A_i , may occur at a different exchange rate to that rate governing Y_i negative.

It has been shown that the transactions denoted Y_i do not duplicate those associated with either A_i or O_i . That this is the case arises from the fact that currency exchange involved in Y_i is never excluded from the spot market. Since in our model it is only trade exchange which is never the subject of control, this suggests that the choice variable Y_i and the trade choice variables E_i and I_i , are related. This relationship is explored below; before doing so, we conclude the discussion of the form of \tilde{W}_j as it appears in (48).

Within our model, the adoption of (48) and its associated transactions,

or (43) and its transactions, is a matter of assumption. It has been pointed out that the entire discussion might have ensued from the form of (48) from the outset; the only drawback being that without the explicitness of (43), there might be a suspicion of unreasonable restriction of the opportunity set. Because of its analytical convenience we choose to view (48) as representing "the way the individual behaves", a decision which has its precedents in the modern theory literature:

"The current flow of trade and payments will not be considered. ... In so far as traders operate in the forward market ...² they are considered either as arbitragers or speculators"

The intention here is that, in as far as traders behaviour may be duplicated by arbitrage and speculation, Stoll chooses not to model it explicitly but to view it as being contained within the "arbitrage" and "speculative" functions.

Similarly, as McCallum points out:

"Tsiang, for example, adopts the convention that traders hedge or cover completely, subsequently expressing choices not to hedge, as speculation."³

Section 4: Some comments on the modelling of speculation.

The term $+ X_i \tilde{x}_i$ in (48) is interpreted as the total return on forward currency i speculation. Writing \tilde{x}_i in full, the term is:

$$X_i \left[[\tilde{s}_i - f_i] \tilde{P}_{ji} \right]$$

Note that if any configuration of controls is imposed $P_{ji} = 0$ and this speculative component disappears from final wealth. This aspect of the formulation reflects a point made by Robert Aliber:

"But if controls are applied between the date when the contract is acquired and when it matures, the contract may be voided; both buyer and seller are relieved of their forward commitment without any direct loss. The speculator may lose the opportunity to secure a profit on his forward contract, but he has not incurred any loss nor any constraint on his use of owned funds comparable with the loss that the arbitragers might incur."⁴

X_i positive is construed as the number of units of i purchased forward speculatively. We are permitted to construe X_i negative, as the reverse transaction, because minus \tilde{x}_i constitutes the return on this reverse transaction. More explicitly, denote a speculative forward sale of i as SF_i , denote its unit return \tilde{sf}_i :

$$\tilde{sf}_i \equiv (f_i - \tilde{s}_i) \tilde{P}_{ji} = - (\tilde{s}_i - f_i) \tilde{P}_{ji} \equiv - \tilde{x}_i$$

The total return on the forward sale of i would appear in final wealth as:

$$+ SF_i \tilde{sf}_i$$

or

$$- SF_i \tilde{x}_i$$

or

$$+ X_i \tilde{x}_i$$

where X_i is negative. This elaboration on a standard method of using a single variable to denote a part of actions might seem unnecessary,

however it is of some importance. With the exception of X_i and Y_i , all of the choice variables in (48) are constrained to being positive. This is precisely because the negative of their associated unit returns is NOT the return on the "reverse" transaction. It is this factor which has forced the adoption of pairs of choice variables, rather than the more usual use of a single unconstrained variable to represent each pair.

Section 5: The implicit inclusion of spot speculation in final wealth.

Spot currency market speculation has not been explicitly included in final wealth, however, it may be shown that in as far as its return may be duplicated by forward speculation and arbitrage, it may be subsumed into these transactions.

Consider speculation on a rise in \tilde{s}_i . An individual in country j takes advantage of this belief by borrowing domestically, lending in country i and repatriating at the future favourable spot rate. In our model this future exchange may be impeded by controls, in which case exchange takes place at rate \tilde{G}_i . (From (44), \tilde{G}_i is a composite of the control contingent exchange rates which govern the sale of i).

The return on this spot speculation per unit of j borrowed denoted \tilde{R}_s , is:

$$\tilde{R}_s \equiv \frac{r^f}{s_i} [\tilde{s}_i \tilde{P}_{ji} + \tilde{G}_i] - r^f_j$$

$$\begin{aligned}
&= \frac{r_i^f}{s_i} [\tilde{s}_i \tilde{p}_{ji} + \tilde{G}_i] + \frac{r_i^f}{s_i} [f_i - f_i] \tilde{p}_{ji} - r_j^f \\
&= \frac{r_i^f}{s_i} [\tilde{s}_i - f_i] \tilde{p}_{ji} + \frac{r_i^f}{s_i} [f_i \tilde{p}_{ji} + \tilde{G}_i] - r_j^f \\
&\equiv \frac{r_i^f}{s_i} \tilde{x}_i + \tilde{a}_i
\end{aligned}$$

Thus, the return from borrowing an amount Z units of j for such spot speculation, may be duplicated by selling $Z \frac{r_i^f}{s_i}$ units of i forward speculatively, and simultaneously borrowing Z units of j for covered lending in country i .

In like manner it may also be shown that spot speculation on a fall in \tilde{s}_i can be duplicated by arbitrage and forward speculation.

Section 6: A digression on "triangular" speculation.

The opportunities structured explicitly in (48), reflect what may be termed "dual" transactions (i.e. transactions between country j and i , j and k , etc.) No explicit opportunity is included for a resident of country j to transact between countries i and k , and then back to his domestic currency, j . The following is an example of such a "triangular" transaction. A resident of country j sells currency i forward speculatively

for currency k, the repatriation of future currency k profit, or payment of loss, then involves a currency j exchange.

It is of interest to consider whether or not our formulation of final wealth implicitly includes such opportunities. Consider first a situation without the possibility of controls.

An individual with final wealth denominated in currency j, sells currency i forward for currency k. Per unit of i sold forward, the return in currency k is:

$$(f_{ki} - \tilde{s}_{ki})$$

where f_{ki} and \tilde{s}_{ki} are "k prices of i". This return is repatriated into currency j at the future spot rate \tilde{s}_{jk} . The currency j return on this triangular speculation may be written:

$$\tilde{R}_T = (f_{ki} - \tilde{s}_{ki}) \tilde{s}_{jk} \quad (51)$$

$$= (f_{ki} - \tilde{s}_{ki}) \tilde{s}_{jk} + (f_{ji} - f_{ji})$$

$$= (f_{ki} \tilde{s}_{jk} - f_{ji}) + (f_{ji} - \tilde{s}_{jk} \tilde{s}_{ki})$$

$$= f_{ki} (\tilde{s}_{jk} - f_{jk}) + (f_{ji} - \tilde{s}_{ji}) \quad (52)$$

where it has been assumed:

$$\tilde{s}_{jk} \tilde{s}_{ki} = \tilde{s}_{ji}$$

$$f_{jk} f_{ki} = f_{ji}$$

(52) indicates that \tilde{R}_T may be duplicated by buying speculatively f_{ki} units of currency k forward, and selling them at future spot for currency j , whilst simultaneously selling one unit of i forward for currency j speculatively.

\tilde{R}_T is equal to a linear combination of the returns on dual speculative transactions. Thus, without the consideration of controls, triangular speculation is implicitly included within the "dual" formulation of final wealth. With the consideration of controls however, this result is lost.

With the possibility of controls, (51) becomes:

$$\tilde{R}_T = [(f_{ki} - \tilde{s}_{ki}) \tilde{P}_{ki}] \tilde{u}_{jk}$$

where the appearance of \tilde{P}_{ki} reflects the property that speculative returns are zero if either or both currencies involved in the speculation become controlled; the replacement of \tilde{s}_{jk} with \tilde{u}_{jk} arises because the repatriation of the currency k income, or payment of currency k loss, from the initial speculation, may now be at some control contingent exchange rate. \tilde{u}_{jk} will not be specified precisely, it suffices to state that only in the special case of the prevention of currencies j and k exchange, is it zero.

Let \tilde{R}_D be the return on a linear combination of dual speculative transactions, all of which involve currency j .

$$\tilde{R}_D = \sum_i \alpha_i (f_{ji} - \tilde{s}_{ji}) \tilde{P}_{ji}$$

If some combination of dual transactions are to duplicate triangular speculation, we must have, for some \forall_i ,

$$\tilde{R}_T = \tilde{R}_D$$

for every configuration of outcomes for the random variables. However, one "outcome" may be readily stated where this equality does not hold.

If country j imposes controls on the sale of j for every i , then:

$$D_{ji} = 0 \text{ for all } i$$

and

$$R_D = 0$$

If countries k and i do not impose controls we have:

$$R_T = (f_{ki} - s_{ki}^a) u_{jk}^a$$

where s_{ki}^a and u_{jk}^a are actual end of period values. Since in general $R^T \neq 0$ we have $R^T \neq R^D$. Thus a series of dual speculative transactions cannot duplicate the triangular speculative transaction. Put simply, universal controls by country j , causes all speculative returns on dual transactions involving currency j to be zero. Return on speculation between two other currencies however, will not be zero, only the rate for repatriation of profit or payment of loss, will be influenced. Hence dual transactions cannot duplicate triangular speculative transactions.

Given this result, we are confronted with a demonstrably restrictive formulation of final wealth. In view of the theme of the overall analysis however, an examination of the consistency of the modern theory model,

with a model of optimal individual forward currency market participation, this formulation may prove sufficient for our purposes. Subsequent analysis suggests that consistency, or the lack of it, is most unlikely to hinge on the availability of "triangular" opportunities. If consistency cannot be demonstrated for the restricted opportunity set, there is no cause for optimism that it will emerge for a more comprehensive set. The soundness of this assertion is left to be judged in the light of the remaining analysis.

CHAPTER II. FOOTNOTES

1. Recall the "AS IF" qualification of page 44. The effect of Y_i on final wealth is identical to the net effect of its components R_i and Q_i . Its net effect in the forward market is also identical. Hence we have assumed the individual acts AS IF he is selecting Y_i . In fact the opportunities associated with Y_i do not exist. Since controls are assumed to apply to all capital movements, there are no such movements which can always be affected at spot market rates.
2. Hans Stoll (1968), p. 61.
3. B. McCallum (1977), p. 146.
4. R. Aliber (1973), p. 1458.

CHAPTER III. THE STRUCTURE OF AN INDIVIDUAL'S OPTIMAL PORTFOLIO

Section 1: The individual's optimization problem.

Individuals are assumed to maximize the expected utility of final wealth, all held in the form of a single currency. Individuals "of country j " are thus more precisely defined as those who intend final wealth to be in the form of currency j . The choice variables of the maximization problem are assumed to be those of equation (48) above. As was explained in the latter part of the section "Some comments on the modelling of speculation," all of the choice variables except Y_i and X_i are constrained to being positive. It is perhaps also clear from that section, that the X_i are unconstrained. It remains to formulate the constraints for the Y_i .

Y_i (recalling that the discussion is from country j 's perspective) is a number of units of j either allocated to the domestic bond and financed by covered borrowing in country i , or is a number of units of j borrowed domestically for covered riskless investment in country i . Which of these is the case at the individual's optimum, is indicated by the sign of Y_i . A positive sign denotes foreign borrowing, negative domestic borrowing. Recall that Y_i is distinguished from the somewhat similar choice variable A_i and O_i , by the property that exchange associated with Y_i is, in the

event of controls, always secured at a currency exchange market spot rate. Since in our model, it is only trade income and debt which is never excluded from the spot market, this suggests that the Y_i are constrained by the magnitude of the trade choice variables I_i and E_i . The form of these constraints may be derived by recalling the definitions of the constituents of Y_i .

R_i is defined (page 33) as the number of units of E_i repatriated by borrowing in country i , exchanging at current spot and lending domestically. Thus we have:

$$0 \leq R_i \leq E_i \quad (53)$$

Similarly, Q_i is defined (page 36) as the number of units of I_i paid by borrowing domestically, exchanging at current spot and lending in country i . Thus we have:

$$0 \leq Q_i \leq I_i \quad (54)$$

Using (53) and (54) we can write:

$$[R_i - Q_i] \leq E_i \quad (55)$$

$$[Q_i - R_i] \leq I_i \quad (56)$$

Combining (55) and (56) provides:

$$-I_i \leq [R_i - Q_i] \leq E_i$$

or, noting $Y_i \equiv (R_i - Q_i) s_i / r_i^f$:

$$- I_i \leq Y_i \frac{r_i^f}{s_i} \leq E_i \quad (57)$$

As long as the choice of Y_i satisfies (57), the opportunity set associated with \tilde{W}_j in (48) is identical to that associated with \tilde{W}_j in (43) and its concomitant constraints on R_i and Q_i . Thus in assuming that final wealth may be written as if it is generated by the choice variables enumerated in (48) we must not lose sight of these constraints on Y_i .

We are now in a position to present the individuals optimization problem. The problem facing individual k of country j is to:

$$\text{Max } E \left[U_{\cdot j}^k (\tilde{W}_j^k) \right]$$

w.r.t.

$$\begin{array}{cccc} A_{ji}^k & B_{ji}^k & O_{ji}^k & M_{ji}^k \\ E_{ji}^k & I_{ji}^k & Y_{ji}^k & X_{ji}^k \end{array}$$

s.t.

$$A_{ji}^k, B_{ji}^k, O_{ji}^k, M_{ji}^k, E_{ji}^k, I_{ji}^k \geq 0 \quad (58)$$

$$- I_{ji}^k \leq Y_{ji}^k \frac{r_i^f}{s_i} \leq E_{ji}^k$$

for all i

and where \tilde{W}_j^k is defined by (48).

Before deriving a solution to (58) we return to the assertion made earlier (page 40) that "some of the above choice variables cannot be simultaneously positive at an optimum which is defined by the maximization of expected utility."

Section 2: Some characteristics of any expected utility optimal portfolio.

The initial comprehensiveness of final wealth, has led us to an array of choice variables in (58), some of which, intuition suggests, must be zero at any optimum. For example, A_i is associated with covered lending in country i , O_i is associated with covered borrowing in country i . Similarly, B_i is associated with investment in the risky asset of country i , M_i is associated with a short position in the risky asset of country i . For both of these pairs of choice variables, intuition suggests that their components being simultaneously positive, is inconsistent with optimal choice. We shall demonstrate that such simultaneity is indeed inconsistent with expected utility maximization.

We shall show that there always exists a portfolio with one of A_i and O_i zero, which has greater expected utility than any portfolio with both positive. Similarly it will be shown that there always exists a portfolio with one of B_i and M_i zero, which has greater expected utility than any portfolio with both positive.

From (47) and (48), the per unit return on A_i is:

$$\tilde{a}_i \equiv \frac{r_i^f}{s_i} [f_i \tilde{P}_{ji} + \tilde{G}_i] - r_j^f$$

the per unit return on O_i is:

$$\tilde{o}_i \equiv r_j^f - \frac{r_i^f}{s_i} [f_i \tilde{P}_{ji} + \tilde{N}_i]$$

Thus

$$\tilde{a}_i = \frac{r_i^f}{s_i} [\tilde{G}_i - \tilde{N}_i] - \tilde{o}_i$$

and

$$\tilde{a}_i + \tilde{o}_i = \frac{r_i^f}{s_i} [\tilde{G}_i - \tilde{N}_i] \quad (59)$$

Using (44) and (45):

$$\begin{aligned} [\tilde{G}_i - \tilde{N}_i] &\equiv (\tilde{D}_i \tilde{c}_i^i + \tilde{D}_j [1 - \tilde{D}_i] \tilde{s}_i^j) \\ &\quad - (\tilde{D}_j \tilde{c}_i^j + \tilde{D}_i [1 - \tilde{D}_j] \tilde{s}_i^i) \end{aligned}$$

Depending upon the configuration of controls, the random variable $[\tilde{G}_i - \tilde{N}_i]$ is equal to one of four combinations of its underlying components.

1. With $D_i = 1, D_j = 0$

$$[\tilde{G}_i - \tilde{N}_i]_{D_i=1, D_j=0} = [\tilde{c}_i^i - \tilde{s}_i^i] \quad (60)$$

2. With $D_i = 0, D_j = 1$

$$[\tilde{G}_i - \tilde{N}_i]_{D_i=0, D_j=1} = [\tilde{s}_i^j - \tilde{c}_i^j] \quad (61)$$

3. With $D_i = 1, D_j = 1$

$$[\tilde{G}_i - \tilde{N}_i]_{D_i=1, D_j=1} = [\tilde{c}_i^i - \tilde{c}_i^j] \quad (62)$$

4. With $D_i = 0, D_j = 0$

$$[\tilde{G}_i - \tilde{N}_i]_{D_i=0, D_j=0} = 0 \quad (63)$$

Now it may plausibly be assumed that the terms in (60), (61) and (62) though stochastic, are always negative. Consider what is being assumed when it is assumed of (60) that:

$$[\tilde{c}_i^i - \tilde{s}_i^i] < 0. \quad (64)$$

Recall that all exchange rates are expressed as a currency j price of a unit of currency i . (64) is simply the assumption that the number of

units of j received by a controlled seller of one unit of currency i (i.e. \tilde{c}_i^i) is less than the number of units of j received by an uncontrolled seller (i.e. \tilde{s}_i^i). Thus (64) is simply a corollary of the broader assumption that controls work to dissuade exchange. Similarly, with respect to (61), assuming:

$$[\tilde{s}_i^j - \tilde{c}_i^j] < 0$$

involves assuming that the number of units of j paid for one unit of i by an uncontrolled seller of j (i.e. \tilde{s}_i^j), is less than the number paid by a controlled seller of j . Again this assumption would be covered by the broader assumption that controls "work". Finally, with respect to (62), assuming:

$$[\tilde{c}_i^i - \tilde{c}_i^j] < 0$$

involves assuming that the number of units of j received for a unit of i by a controlled seller of i (i.e. \tilde{c}_i^i), is less than the number of units of j paid by a controlled seller of j .

Given the above assumptions we may write:

$$[\tilde{G}_i - \tilde{N}_i] \leq 0 \tag{65}$$

and from (59):

$$\tilde{a}_i + \tilde{o}_i \leq 0 \tag{66}$$

(Note that it is (63) which prompts $(\tilde{a}_i + \tilde{o}_i) = 0$. This corresponds to our intuition. In the absence of controls, the transactions associated with A_i and O_i cancel each other out. Thus the return on one unit of A_i plus one unit of O_i , is zero.)

Using (66), it is possible to show that we can always find a portfolio with either A_i zero or O_i zero, which for every set of outcomes, generates greater final wealth than any portfolio with A_i and O_i both positive.

Let P_i denote the number of units devoted to transaction i . Let \tilde{p}_i denote the per unit return on that transaction. Then, writing A_i and O_i explicitly, the final wealth generated by any portfolio with A_i and O_i both positive is:

$$\tilde{W} = \sum_i P_i \tilde{p}_i + A_i \tilde{a}_i + O_i \tilde{o}_i$$

or

$$\tilde{W} = \sum_i P_i \tilde{p}_i + [A_i - O_i] \tilde{a}_i + O_i [\tilde{a}_i + \tilde{o}_i] \quad (67)$$

Now consider final wealth generated by a portfolio with the same values of P_i but with $[A_i - O_i]$ devoted to the transactions associated with \tilde{a}_i , and nothing devoted to those associated with \tilde{o}_i . In this case:

$$\tilde{W}^* = \sum_i P_i \tilde{p}_i + (A_i - O_i) \tilde{a}_i \quad (68)$$

Note that in our model the only constraint on the choice variable multiplying \tilde{a}_i , is that it is non negative. (The situation when $(A_i - O_i)$ is negative is addressed below.) This choice variable denotes the extent to which the individual engages in a series of self financing TRANSACTIONS. The associated return \tilde{a}_i is the net return per unit of the transaction

engaged in; specifically, it is the currency j net return on borrowing a unit of currency j domestically, and allocating it to the covered bond of country i .

Using (67) and (68):

$$\tilde{W}^* - \tilde{W} = -O_i(\tilde{a}_i + \tilde{o}_i)$$

With $O_i > 0$ and using (66), we have:

$$(\tilde{W}^* - \tilde{W}) \geq 0 \tag{69}$$

Using "State of the world" notation, (69) may be written equivalently:

$$W^*(\theta) - W(\theta) \geq 0 \tag{70}$$

for all states θ

Since its domain is such that the utility function is increasing, (70)

implies:

$$u(W^*(\theta)) \geq u(W(\theta))$$

for all θ

Thus,

$$f(\theta)u(W^*(\theta)) \geq f(\theta)u(W(\theta))$$

where $f(\theta)$ is the p.d.f. of θ , and:

$$\int_{\theta} f(\theta)u(W^*(\theta))d\theta > \int_{\theta} f(\theta)u(W(\theta))d\theta$$

(noting that there is some θ for which (70) is a strict inequality)

OR

$$E[u(W^*(\theta))] > E[u(W(\theta))] \quad (71)$$

The portfolio associated with \tilde{W} can be any portfolio with a positive allocation to the transaction associated with both \tilde{a}_i and \tilde{o}_i . The portfolio associated with \tilde{W}^* does not have both allocations positive. Thus (71) indicates that if \tilde{W}^* is feasible, then a portfolio W with positive allocations to both \tilde{a}_i and \tilde{o}_i , cannot be an expected utility optimal portfolio.

Recall that the feasibility of the portfolio associated with \tilde{W}^* required $[A_i - O_i] \geq 0$. If this is not the case, (71) may still be arrived at by specifying a slightly different, necessarily feasible, alternative prospect. We could have written (67) as:

$$\tilde{W} = \sum_i P_i \tilde{p}_i + [O_i - A_i] \tilde{o}_i + A_i [\tilde{o}_i + \tilde{a}_i]$$

and considered an alternative prospect:

$$\tilde{W}' = \sum_i P_i \tilde{p}_i + [O_i - A_i] \tilde{o}_i$$

(note if \tilde{W}^* is not feasible because $[A_i - O_i] < 0$, then \tilde{W}' is necessarily feasible.)

$$\tilde{W}' - \tilde{W} = -A_i [\tilde{o}_i + \tilde{a}_i]$$

With $A_i' > 0$ and using (66) we have:

$$(\tilde{W}' - \tilde{W}) \geq 0$$

which implies:

$$E[u(W'(\theta))] > E[u(W(\theta))] \quad (72)$$

Thus with \tilde{W}^* or \tilde{W}' , we can always define a prospect with a zero allocation to one of \tilde{a}_i or \tilde{o}_i , which has greater expected utility than any prospect with both allocations positive. Hence a portfolio with positive A_i and O_i cannot be optimal.

The exclusion of positive A_i and O_i from an optimum, proceeded from:

$$\tilde{a}_i + \tilde{o}_i \leq 0$$

By showing:

$$\tilde{b}_i + \tilde{m}_i \leq 0$$

we may immediately also exclude B_i and M_i positive.

From (47) and (48):

$$\tilde{b}_i \equiv \frac{\tilde{r}_i}{s_i} [\tilde{s}_i \tilde{P}_{ji} + \tilde{G}_i] - r_j^f$$

$$\tilde{m}_i \equiv r_j^f - \frac{\tilde{r}_i}{s_i} [\tilde{s}_i \tilde{P}_{ji} + \tilde{N}_i]$$

Thus

$$\tilde{b}_i = \frac{\tilde{r}_i}{s_i} [\tilde{G}_i - \tilde{N}_i] - \tilde{m}_i$$

$$\tilde{b}_i + \tilde{m}_i = \frac{\tilde{r}_i}{s_i} [\tilde{G}_i - \tilde{N}_i]$$

Since \tilde{r}_i (a rate of return plus unity) is non negative, and:

$$[\tilde{G}_i - \tilde{N}_i] \leq 0 \quad (65) \text{ rptd.}$$

we have:

$$\tilde{b}_i + \tilde{m}_i \leq 0$$

Thus B_i and M_i cannot both be positive in an optimum portfolio.

We turn next to the derivation of a result familiar from the text book theory of forward exchange.

In the contract default free models of many textbooks, it is a standard result that if conditions are not profitable for repatriating foreign trade income by borrowing in the "foreign" country and exchanging at current spot, then they are also not profitable for engaging in "arbitrage" involving borrowing in the "foreign" country.¹ To show this, consider a country j exporter anticipating one unit of currency i in the future. In the absence of contract default, he can sell the unit of i forward for a certain number of units of j , f_i . Alternatively, he may

borrow currently in country i $1/r_i^f$ units of i , exchange at spot for currency j , and invest in the currency j riskless asset. At the end of the period he receives with certainty:

$$\frac{r_j^f}{r_i^f} s_i \quad \text{units of currency } j.$$

If

$$f_i > \frac{r_j^f}{r_i^f} s_i$$

the exporter would use the forward market to repatriate his foreign income, he would not employ the "borrowing" strategy. This condition may be rewritten:

$$\frac{f_i}{s_i} r_i^f > r_j^f$$

which is equation (2) the condition for "arbitragers" to borrow in country j for investment in country i . Thus when it is not profitable for "traders" to borrow in country i , it is also not profitable for arbitragers to do so. This familiar result may also be derived from our model.

In our model, the return from investing in the domestic bond financed by covered borrowing in country i , is \tilde{o}_i . The number of units of country i export income repatriated by the "borrowing" method is R_i .

Since:

$$Y_i \equiv (R_i - Q_i) \frac{s_i}{r_i^f} \quad (73)$$

it is apparent from (48) (where Q_i and R_i appear only within Y_i) that it is only the difference between R_i and Q_i which influences final wealth. Thus we can assume that R_i and Q_i will not both be positive at an optimum. Since:

$$R_i, Q_i \geq 0$$

we have from (73):

$$Y_i > 0 \rightarrow R_i > 0, Q_i = 0 \quad (74)$$

From (48) the return per unit of positive Y_i is \tilde{y}_i . Using (47) and (48):

$$\tilde{o}_i \equiv \tilde{y}_i + \frac{r_i^f}{s_i} [\tilde{S}_i^c - \tilde{N}_i] \quad (75)$$

From (45) and (46):

$$[\tilde{S}_i^c - \tilde{N}_i] \equiv \tilde{D}_j (1 - \tilde{D}_i) \tilde{s}_i^j + \tilde{D}_i \tilde{D}_j \tilde{s}_i^{ij} - \tilde{D}_j \tilde{c}_i^j$$

Consider this term for each possible configuration of controls.

$$[\tilde{S}_i^c - \tilde{N}_i]_{D_j=D_i=0} = 0$$

$$[\tilde{S}_i^c - \tilde{N}_i]_{D_j=1, D_i=0} = [\tilde{s}_i^j - \tilde{c}_i^j] < 0$$

Negative, because we assume the number of units of j paid per unit of i

by a controlled seller of j (i.e. \tilde{c}_i^j), exceeds the number paid by an uncontrolled seller (\tilde{s}_i^j).

$$[\tilde{S}_i^c - \tilde{N}_i]_{D_j=0, D_i=1} = 0$$

$$[\tilde{S}_i^c - \tilde{N}_i]_{D_j=D_i=1} = [\tilde{s}_i^{ij} - \tilde{c}_i^j] < 0$$

Negative because we assume \tilde{c}_i^j (defined immediately above) exceeds the price for uncontrolled transactions \tilde{s}_i^{ij} .

The preceding, permits us to write:

$$[\tilde{S}_i^c - \tilde{N}_i] \leq 0 \quad (76)$$

From (75) and (76):

$$\tilde{y}_i \geq \tilde{o}_i \quad (77)$$

Let \tilde{W} be any feasible prospect with $O_i > 0$. Let $\tilde{W}'' = \tilde{W} + (1)\tilde{y}_i - (1)\tilde{o}_i$:

$$\tilde{W}'' - \tilde{W} = \tilde{y}_i - \tilde{o}_i \geq 0 \quad \text{by (77)}$$

Invoking the procedure following (69) above, \tilde{W}'' is preferred to \tilde{W} . Thus, as long as it is feasible (recall that the transactions generating \tilde{y}_i , denoted Y_i , are constrained), it is preferable to engage in Y_i rather than O_i . It follows that O_i will only be positive if Y_i is constrained by its positive boundary, it also follows that if at an optimum the choice of Y_i

is unconstrained then O_i is zero. In summary,

$$O_i > 0 \longrightarrow Y_i \text{ constrained at positive boundary} \quad (78)$$

or, using (57):

$$O_i > 0 \longrightarrow Y_i \frac{r_i^f}{s_i} = E_i$$

and using $Y_i \equiv (R_i - Q_i) \frac{s_i}{r_i^f}$ with (74) provides:

$$O_i > 0 \longrightarrow R_i = E_i \quad (79)$$

Recalling (53) (i.e. $0 \leq R_i \leq E_i$) provides:

$$0 \leq R_i < E_i \longrightarrow O_i = 0 \quad (80)$$

(80) is in part the familiar result that if zero currency i export income is repatriated by "borrowing" (i.e. $R_i = 0$), then covered borrowing in country i for riskless investment domestically, is also zero (i.e. $O_i = 0$).

(79) embodies the result that, only if it is optimal to repatriate all of the currency i export income by "borrowing", will it be optimal to engage in any currency i borrowing for arbitrage purposes.

The preceeding has focused on borrowing in country i . It is similarly possible to derive the familiar result pertaining to domestic borrowing.

It may be shown that:

$$0 \leq Q_i < I_i \longrightarrow A_i = 0 \quad (81)$$

That is, if it is not optimal to repay any currency i import debt by domestic "borrowing" (i.e. $Q_i = 0$), it is not optimal to engage in any domestic borrowing for covered investment in the riskless asset of country i (i.e. $A_i = 0$). (81) also conveys that if it is optimal to repay some, but not all, of this import debt by the domestic borrowing strategy (i.e. $0 < Q_i < I_i$), it is still not optimal to engage in any country i "arbitrage".

The intent of this section has been two-fold. Firstly, to demonstrate that in the plethora of substitution and rearrangement we have not lost sight of behaviour. Some a priori implausible choices have been excluded from the optimal portfolio, and some consistency between behaviour in this model and in the traditional model has been demonstrated. Secondly, as will eventually emerge, our major theme, the derivation of aggregate forward currency market excess demand equations from an optimization framework, is facilitated, by some of the results of this section.

Section 3. The Derivation of the structure of the individual's optimal portfolio.

The individual's optimization problem is one of maximization subject to inequality constraints. Such problems do not permit the derivation of a reduced form solution like that of a classical equality constrained problem. However, any actual solution can be viewed as if it had emerged

from a classical problem. In short, if it was known a priori which constraints were binding to the solution, that solution could be arrived at from a classical problem involving only these active equality constraints. This assertion will first be justified and then its significance will be explored.

Let X denote a vector of choice variables, c a vector of constants and $g(X)$ a vector of functions of X . Consider the following general maximization problem:

$$\text{Max } y = f(X)$$

w.r.t. X

s.t.

$$g(X) \leq c \quad (\text{N.B. These constraints may include non negativity constraints}) \quad (82)$$

Define the Lagrangian

$$Z = f(X) - \lambda^T [g(X) - c]$$

where λ^T is a row vector of multipliers.

At the solution the following is required:

$$\frac{\partial Z}{\partial X} = 0 \quad [g(X) - c] \leq 0$$

$$\lambda \geq 0 \quad \lambda^T [g(X) - c] = 0$$

The configuration of binding and non binding constraints at any solution may be represented by reordering and partitioning the above vectors.

At any solution the conditions for the optimum could be written as:

$$\frac{\partial Z}{\partial X} = 0 \quad [g(X) - c] = \begin{bmatrix} g(X)_B - c_B \\ g(X)_{NB} - c_{NB} \end{bmatrix} \begin{matrix} = \phi_B \\ < \phi_{NB} \end{matrix}$$

$$\lambda = \begin{bmatrix} \lambda_B \\ \lambda_{NB} \end{bmatrix} \begin{matrix} \geq 0 \\ = 0 \end{matrix}$$

where B indicates association with a binding constraint, NB a slack constraint.

In principle, the equations:

$$\frac{\partial Z}{\partial X} = 0 \quad g(X)_B = c_B \quad \lambda_{NB} = 0 \quad (83)$$

provide solutions for optimal X and λ_B .

Let us consider the classical problem:

$$\text{Max } f(X)$$

$$\text{w.r.t. } X$$

s.t.

$$g_B(X) = c_B \quad (84)$$

$$\text{Define } Z^* = f(X) - \lambda_B^T [g_B - c_B]$$

At the optimum:

$$\frac{\partial Z^*}{\partial X} = 0 \quad \frac{\partial Z^*}{\partial \lambda_B} = [g_B - c_B] = 0 \quad (85)$$

In principle these equations provide solutions for optimal X and λ_B .

Now with $\lambda_{NB} = 0$ we have:

$$\frac{\partial Z}{\partial X} = \frac{\partial Z^*}{\partial X}$$

and (83) is seen to be the same set of equations as (85). Thus the same solutions for X and λ_B emerge from both the inequality constrained problem (82), and the classical problem, (84).

The significance of this result lies in the following considerations. If it was known a priori which constraints were binding, then a reduced form could be arrived at. With the assumption that for small changes in the parameters of the problem, the set of binding constraints does not change, it would then be possible to arrive at the functions of interest, and to engage in the usual exercises in comparative statics. However, since it is not possible to know a priori the binding constraints, the unequivocal solution cannot be determined. Instead, "contingent solutions" can be determined; solutions which are contingent upon the a priori specification of a set of binding constraints.

To engage in the contingent analysis associated with contingent solutions would be a massive, repetitive task. Fortunately, for our purpose, the examination of each possible contingent solution is unnecessary. Rather than engage in a taxonomy of contingent solutions, we can derive the structure of any such solution, Since our concern is with the

structure of the Modern Theory Model, this will prove to be sufficient for our purposes.

The structure of any contingent solution, may be arrived at by specifying in a very general way, the constraints which are binding. To facilitate this specification the choice variables are consolidated into vectors.

Define the vectors:

$$\begin{bmatrix} A_{j1}^K \\ \cdot \\ \cdot \\ \cdot \\ A_{jn}^K \end{bmatrix} \equiv A$$

$$\begin{bmatrix} B_{j1}^K \\ \cdot \\ \cdot \\ \cdot \\ B_{jn}^K \end{bmatrix} \equiv B$$

and similarly for all the remaining choice variables of the optimization problem (58).

(N.B. All analysis is from the perspective of person K of country j. At a subsequent stage the indices, person K and country "of domicile" j, will be applied to the vector symbols A, B etc ... Until then it is convenient to omit them.)

Let us collect those choice variables which are involved only in non negativity constraints, in a single vector:

$$\begin{bmatrix} A \\ B \\ O \\ M \end{bmatrix} \equiv G$$

We attempt next to specify a contingent solution to the optimization problem, which is sufficiently general not to contradict ANY possible actual solution. We proceed to specify this very general solution.

The vector of choice variable denoted X is unconstrained. Thus we can simply state that at the optimum, X takes on some unspecified value. Using * to denote the optimal choice $X = X^*$. Now with respect to Y, recall:

$$-I_{ji}^K \leq Y_{ji}^K \frac{r_i^f}{s_i} \leq E_{ji}^K$$

or in matrix notation $-I \leq [r/s]Y \leq E$ where $[r/s]$ is a diagonal matrix

$$\begin{bmatrix} r_1^f/s_1 & & & \\ & \ddots & & \\ & & r_n^f/s_n & \\ & & & \ddots \end{bmatrix}$$

Thus we may have a subset of the vector Y, let us denote it Y^I , constrained by a subset of positive elements of the vector I, denoted I^C . Hence

$$-I^{C*} = [r/s]^I Y^{I*} < \emptyset \text{ where } [r/s]^I \text{ is a diagonal matrix compatible}$$

with Y^I .

Similarly we may have a subset of the vector Y , denoted Y^E constrained by a subset of positive elements of the vector E , denoted E^C . Hence:

$$E^{C*} = [r/s]^E Y^{E*} > \emptyset \text{ where } [r/s]^E \text{ is compatible with } Y^E.$$

There may be a subset of Y which is non zero at the optimum and is not at a constraint boundary. This may be denoted

$$Y^{U*} \neq \emptyset$$

There may be a subset of positive elements of I which do not serve to constrain elements of Y . This vector is denoted I^P

$$I^{P*} > \emptyset$$

Similarly there may be a positive non constraining subset of E denoted E^P

$$E^{P*} > \emptyset$$

Finally, the vector of choice variables G , which is only constrained to being non negative, may have some positive elements and some zero elements.

With the elements of G appropriately rearranged, its form at any optimum may be represented by defining two new vectors Z and θ , and writing:

$$G = \begin{bmatrix} Z^* > \emptyset \\ \theta^* = \emptyset \end{bmatrix}$$

Collecting these vectors, any solution is consistent with the following:

$$\begin{aligned}
 X &= X^* \\
 -I^{C^*} &= [r/s]^I Y^{I^*} < \emptyset \\
 E^{C^*} &= [r/s]^E Y^{E^*} > \emptyset \\
 &Y^{U^*} \neq \emptyset \\
 &I^{P^*} > \emptyset \\
 &E^{P^*} > \emptyset \\
 &Z^* > \emptyset
 \end{aligned} \tag{86}$$

with any remaining choice variables not embraced by any of these vectors, all equal to zero. Denote the vector of these zero value variables N .

The above general contingent 'solution' may be viewed as if it had emerged from a classical optimization problem. In particular, the relationship between the above optimal choices, and the parameters of the optimization problem would be provided by the solution to the following classically constrained problem:

$$\text{Max } E[U_j^K (w_j^K)]$$

w.r.t.

A B O M
E I Y X

s.t.

$$\begin{aligned}
 -I^C &= [r/s]^I Y^I \\
 E^C &= [r/s]^E Y^E \\
 N &= \emptyset
 \end{aligned} \tag{87}$$

where \tilde{W}_j^K is defined by (48). Using vector notation it has the form:

$$\tilde{W}_j^K = W^o r_j^f + A^T \tilde{a} + B^T \tilde{b} + O^T \tilde{o} + M^T \tilde{m} + E^T \tilde{e} + I^T \tilde{i} + Y^T \tilde{y} + X^T \tilde{x} \quad (88)$$

where $\tilde{a}^T = [\tilde{a}_1 \dots \tilde{a}_n]$, and similarly for the other stochastic terms.

Note \tilde{W}_j^K includes the vector of zero choice variable N . Setting these choice variables equal to zero in (88), and employing the choice variable vectors of the general contingent solution with their appropriately defined associated stochastic return vectors, provides:

$$\begin{aligned} \tilde{Y}_j^K &= W^o r_j^f + Z^T \tilde{z} + E^c{}^T \tilde{e}^c + E^p{}^T \tilde{e}^p + I^c{}^T \tilde{i}^c + I^p{}^T \tilde{i}^p + Y^I{}^T \tilde{y}^I + Y^E{}^T \tilde{y}^E \\ &\quad + Y^u{}^T \tilde{y}^u + X^T \tilde{x} \end{aligned} \quad (89)$$

The optimization problem (87) is then equivalent to:

$$\text{Max } E[U_j^K(\tilde{Y}_j^K)]$$

w.r.t.

$$Z, E^c, E^p, I^c, I^p$$

$$Y^I, Y^E, Y^u, X$$

s.t.

$$\begin{aligned} -I^c &= [r/s] Y^I \\ E^c &= [r/s] Y^E \end{aligned} \quad (90)$$

Rather than employ the Lagrangian technique to solve (90), we can use the constraints to eliminate Y^I and Y^E from the objective function, and maximize the resulting function without constraint.

Substituting the constraints into (89) provides:

$$\begin{aligned} \tilde{Y}_j^K = & W^O r_j^f + Z^T \tilde{z} + E^c{}^T (\tilde{e}^c + [s/r]^E \tilde{y}^E) + E^P{}^T \tilde{e}^P + I^c{}^T (\tilde{i}^c - [s/r]^I \tilde{y}^I) \\ & + I^P{}^T \tilde{i}^P + Y^u{}^T \tilde{y}^j + X^T \tilde{x} \end{aligned} \quad (91)$$

Let us consolidate the choice vectors and return vectors of (91) into two vectors:

$$\begin{bmatrix} \tilde{z} \\ (\tilde{e}^c + [s/r]^E \tilde{y}^E) \\ \tilde{e}^P \\ (\tilde{i}^c - [s/r]^I \tilde{y}^I) \\ \tilde{i}^P \\ \tilde{y}^u \\ \tilde{x} \end{bmatrix} \equiv \tilde{M}_j^K \quad \begin{bmatrix} z \\ E^c \\ E^P \\ I^c \\ I^P \\ Y^u \\ X \end{bmatrix} \equiv C_j^K \quad (92)$$

Note that both the choice vector and return vector are indexed for person K of country j. Whilst the return on any single transaction is the same for all persons of country j, the choice variables included within C_j are associated with a single person K, and thus the vector of returns on these

choice variables is also associated with person K. This point is of some importance and will be returned to below when the question of aggregation arises.

Using the vectors \check{M}_j^K and C_j^K results in:

$$\check{Y}_j^K = W_j^{of} + C_j^K T_j^K \check{M}_j^K \quad (93)$$

and the optimization problem may be written in its most concise form:

$$\text{Max } E[U_j^K (\check{Y}_j^K)] \quad (94)$$

w.r.t. C_j^K

It is assumed that U_j^K is quadratic for all individuals K and countries j. Using (93) U_j^K is expanded in an exact Taylor series around W_j^{of} to provide:

$$E[U_j^K] = E[U_j^K + U_j^K' C_j^K T_j^K \check{M}_j^K + \frac{U_j^K''}{2!} C_j^K T_j^K \check{M}_j^K \check{M}_j^K T_j^K C_j^K]$$

At the optimum:

$$\frac{\partial E[U_j^K]}{\partial C_j^K} = U_j^K' E[\check{M}_j^K] + U_j^K'' E[\check{M}_j^K \check{M}_j^K T_j^K] C_j^K^* = 0$$

and:

$$C_j^K^* = - \frac{U_j^K'}{U_j^K''} [E[\check{M}_j^K \check{M}_j^K T_j^K]]^{-1} E[\check{M}_j^K]$$

$$C_j^{K*} = T_j^K \left[E[\tilde{M}_j^K \tilde{M}_j^{KT}] \right]^{-1} E[\tilde{M}_j^K] \quad (95)$$

where

$$T_j^K \equiv - \frac{U_j^{K'}}{U_j^K} > 0 \quad (96)$$

is the individual's risk tolerance function evaluated at $W_j^o r_j^f$.

Thus the solution to (94) is provided by (95); allied with the constraints

$$\begin{aligned} -I^{cK*} &= [r/s] I_Y^{IK*} \\ E^{cK*} &= [r/s] E_Y^{EK*} \end{aligned} \quad \text{it is the solution to the constrained problem (90).}$$

(95) is the relationship between the optimal choices and the parameters of the choice problem, which emerge from a very general specification of the contingent solution. It is perhaps apparent that the specification of any precise contingent solution, rather than the rather vague general specification in (86), would serve only to alter the composition of the vectors C_j^K and \tilde{M}_j^K in (95). Most pertinent from our perspective, the specification of any particular contingent solution would not alter the structure of (95). Thus it is contended that (95) conveys the structure of the relationship between the optimal choices and the parameters of the choice problem for any contingent solution to the classical optimization

problem (90). Finally, since the solution to the actual choice problem (58), is necessarily the same as some contingent solution , we have that (95) conveys the structure of the above relationship for our actual choice problem.

CHAPTER III. FOOTNOTES

1. For an extended discussion see for example "International Monetary Theory and Policy," by Miltiades Chacholiades, McGraw-Hill, 1978, New York, pp. 21-31.

CHAPTER IV. THE STRUCTURE OF AN OPTIMAL AGGREGATE PORTFOLIO

In Chapter III the structure of the solution to an individual's choice problem was derived. That structure was arrived at by specifying sets of active and slack constraints, discarding the latter, and solving the resulting classical, equality constrained problem. It was noted in the derivation, that the active constraint specification, from which the solution ensued, pertained to an individual. This specification determined which choice variables were zero at this individual's optimum, and hence which stochastic return terms did not appear in this individual's equivalent classical contingent choice problem (90). Consequently when those stochastic returns which did appear in (90) were collected in a single vector \tilde{M} , that vector pertained to this individual K and hence required a superscript K. Now that the question of aggregation is to be addressed this point is of some significance.

Section 1: Aggregation using an implication of Pareto Optimality for constrained individuals portfolios.

The structure of the individual K's optimal portfolio is expressed by:

$$c_j^{K*} = T_j^K [E[\tilde{M}_j^K \tilde{M}_j^{K T}]]^{-1} E[\tilde{M}_j^K]$$

$$-I^{cK*} = [r/s] I_Y^{IK*} \quad E^{cK*} = [r/s] E_Y^{EK*} \quad (95) \text{ rptd.}$$

Recall (from (96)) that T_j^K is a scalar. Hence the aggregation of C_j^{K*} over K would be straightforward if it was reasonable to assume that \tilde{M}_j^K was the same for all individuals K of country j . We proceed to argue that such an assumption is indeed reasonable.

It may be shown that with Pareto Optimality in the distribution of the risky opportunities of this model, at any optimum the SAME constraints are binding for the choices of ALL individuals of the SAME country. (Please see appendix for proof). Invoking such Pareto-optimality, then, given this result, the specification of a set of binding constraints for individual K 's optimum, implies that for the overall optimum, the same constraints are binding for all other individuals of country j . Recalling that the specification of binding constraints for K , determined which choice variables appeared in C_j^K , it then follows immediately that the choice vector C_j^K contains the same variables for all K , and hence the associated return vector \tilde{M}_j^K is the same for all K . (N.B. In making this argument it has been assumed that when a non negativity constraint is not binding, the choice variable is positive. Thus, when for all K such a constraint is slack, the associated choice variable appears in all C_j^K .)

Given \tilde{M}_j^K is the same for all K , the structure of individual K 's optimal portfolio is:

$$C_j^{K*} = T_j^K [E[\tilde{M}_j \tilde{M}_j^T]]^{-1} E[\tilde{M}_j]$$

$$- I^{CK*} = [r/s] L_Y^{KI*} \quad E^{CK*} = [r/s] L_Y^{KE*}$$

Summing over K provides the structure of country j's aggregate portfolio:

$$C_j^* = T_j [E[\tilde{M}_j \tilde{M}_j^T]]^{-1} E[\tilde{M}_j] \quad (97)$$

$$\text{where } T_j \equiv \sum_K T_j^K$$

$$- I^{c*} = [r/s] I_Y I^{I*}$$

$$E^{c*} = [r/s] E_Y E^{E*}$$

where $I^{c*} = \sum_K I^{cK*}$ and similarly for Y^{I*} , E^{c*} and Y^{E*} .

It is emphasized that (97) is not a reduced form in the usual sense; since we don't know which constraints are binding at the optimum, we don't know which variables and parameters of the model, compose (97). However, if we posit some solution as optimal, (97) provides the relationship between that solution, and the parameters which are pertinent to it. If we then assume that for small changes in these parameters, the composition of the solution does not change (i.e. the configuration of binding and non binding constraints does not change, so that which variables compose C_j , and which parameters compose $E[\tilde{M}_j \tilde{M}_j^T]$ and $E[\tilde{M}_j]$, also do not change) then (97) may be treated as a reduced form. In particular, it may be viewed as a set of demand functions for participation in risky transactions. Since these transactions involve the exchange of securities and currencies, (97)

is the source of contingent demand functions for securities and currencies. Thus we can derive from (97), contingent demand functions for forward currencies. Again we choose not to derive a series of such functions; but instead to derive a general contingent demand function for a forward currency which displays the structure of any such demand function.

Recall that in the Modern Theory Model, the ultimate focus of our interest, there is a functional classification of the sources of demand for forward exchange. This classification demarks an "arbitrage" demand a "speculative" demand and a "trade" demand. To permit a comparison of the excess demand functions specified in the Modern Theory, and those which emerge from our model of optimal behaviour, we shall derive a "general contingent excess demand function" for each of the classifications adopted in the Modern Theory Model.

CHAPTER V: THE DERIVATION OF "ARBITRAGE," "SPECULATIVE" AND "TRADE" EXCESS
DEMAND FUNCTIONS FOR FORWARD CURRENCY.

In the conclusion of chapter IV, it was emphasized that whilst the model does not permit the derivation of an unequivocal reduced form, we have in (97), the structure of any contingent reduced form. This "reduced form" now permits the derivation of the structure of the aggregate demand function for each risky opportunity in the model. In particular, it provides the structure of the aggregate optimal demand for participation in "arbitrage", forward currency speculation and "trade".

At the outset, it is perhaps reassuring to demonstrate that the structure of our demand equations (97), is identical to that of the demand equations of the well-known "Capital Asset Pricing Model" when that model involves quadratic utility functions. Jan Mossin¹ for example, in a model involving a single country, a riskfree rate of interest, and risky opportunities represented by shares in companies, derives the vector of the optimal number of shares of each company, for an individual. He employs the following notation:

Z_j the number of shares of company j bought by the individual.

Z a vector of the Z_j .

p_j the beginning of the period price of a share of company j .

j the end of period expected value of a share of company j .

$m_j \equiv (\mu_j - r_p)$ the "risk margin". This is the expected net return on a share of company j ("net" may be either net of explicit borrowing costs incurred to finance the purchase of the share; or net of the opportunity cost of the share).

m a vector of the m_j .

C the variance covariance matrix of the end of period values of shares.

W the individuals initial wealth.

c a parameter of the individuals utility function $U_j(\tilde{W}) = \tilde{W} - c\tilde{W}^2$.

The vector of the optimal number of shares of each company for the individual to hold, is then:

$$Z = \begin{bmatrix} \frac{1}{2c} & -rW \end{bmatrix} [C + mm^T]^{-1} m$$

Our optimal portfolio for the individual has the form:

$$C_j^K = - \frac{U'(W_0 r_j^f)}{U''(W_0 r_j^f)} \left[E[\tilde{M}_j \tilde{M}_j^T] \right]^{-1} E[\tilde{M}_j]$$

In our model $E[\tilde{M}_j]$ is a vector of the expected net returns on risky transactions and is thus analogous to Mossin's vector of "risk margins" m .

Let us define Σ_j as the variance covariance matrix of the elements of \tilde{M}_j . Then:

$$\begin{aligned}\Sigma_j &\equiv E[\tilde{M}_j - E[\tilde{M}_j]] [\tilde{M}_j - E[\tilde{M}_j]]^T \\ &= E[\tilde{M}_j \tilde{M}_j^T - \tilde{M}_j E[\tilde{M}_j]^T - E[\tilde{M}_j] \tilde{M}_j^T + E[\tilde{M}_j] E[\tilde{M}_j]^T] \\ &= E[\tilde{M}_j \tilde{M}_j^T] - E[\tilde{M}_j] E[\tilde{M}_j]^T\end{aligned}$$

Thus:

$$E[\tilde{M}_j \tilde{M}_j^T]^{-1} = [\Sigma_j + E[\tilde{M}_j] E[\tilde{M}_j]^T]^{-1}$$

Finally, the utility functions in our model are quadratic:

$$U(\tilde{W}) = \tilde{W} - c\tilde{W}^2$$

Hence: $-\frac{U'}{U}$ evaluated at $W_o r_j^f$ is:

$$-\left[\frac{1 - 2cW_o r_j^f}{-2c} \right] = \frac{1}{2c} - W_o r_j^f$$

Substituting into the expression for C_j^K provides:

$$C_j^K = \left[\frac{1}{2c} - W_o r_j^f \right] [\Sigma_j + E[\tilde{M}_j] E[\tilde{M}_j]^T]^{-1} E[\tilde{M}_j]$$

which is identical in structure, to Mossin's solution. With homogeneous beliefs assumed both here and by Mossin, this similarity of structure is maintained at the aggregate level.

Having demonstrated a consistency with the literature, we turn now to

the derivation of the demand functions of interest.

Let us rewrite (97) as:

$$C^*_j = T_j [\Sigma_j + E[\tilde{M}_j] E[\tilde{M}_j]^T]^{-1} E[\tilde{M}_j]$$

$$= \frac{T_j \det. (\Sigma_j)}{\det. (\Sigma_j + E[\tilde{M}_j] E[\tilde{M}_j]^T)} \Sigma_j^{-1} E[\tilde{M}_j]$$

where "det." indicates "determinant of".²

Or:

$$C^*_j = \theta_j \Sigma_j^{-1} E[\tilde{M}_j] \quad (98)$$

$$\text{where } \theta \equiv \frac{T_j \det. (\Sigma_j)}{\det. (\Sigma_j + E[\tilde{M}_j] E[\tilde{M}_j]^T)} \quad (99)$$

Recalling the definition of \tilde{M}_j note that in general:

$$\theta = \theta_j (r_i^f, f_i, s_i, E[\tilde{s}_i], \dots, \dots) \quad (100)$$

$$i = 1, \dots, n.$$

Let us denote the ki^{th} element of Σ_j^{-1} as η^J_{ki} . Then, in general the t^{th} element of C^*_j is, from (98):

$$C^*_{jt} = \theta_j \eta^J_{tt} E[\tilde{M}_{jt}] + \theta_j \sum_{i \neq t} \eta^J_{ti} E[\tilde{M}_{ji}] \quad (101)$$

Section 1: An "arbitrage" excess demand function.

In our model, the choice variables A_i , O_i and Y_i are associated with transactions which are traditionally referred to as covered arbitrage. (For elaboration on these choice variables please see p. 44). Using (101) we can provide expressions for the optimal aggregate participation in these transactions.

Assume that we are at an optimum where the amount allocated to the covered bond of country p, A_{jp} , is positive, and thus the choice variable A_{jp} is contained in C_j^* of (98). Let A_{jp} be the t^{th} element of C_j and then by implication we have $\tilde{a}_{jp} \equiv \tilde{M}_{jt}$. Using (101):

$$A_{jp} = \theta_j \eta_{tt}^J E[\tilde{a}_{jp}] + \theta_j \sum_{i \neq t} \eta_{ti}^J E[\tilde{M}_{ji}] \quad (102)$$

Now from (47) and (48):

$$E[\tilde{a}_{jp}] \equiv E \left[\frac{r_p^f}{s_p} [f_p \tilde{P}_{jp} + \tilde{G}_p] - r_j^f \right] \quad (103)$$

$$= \frac{r_p^f}{s_p} [f_p E[\tilde{P}_{jp}] + E[\tilde{G}_p]] - r_j^f \quad (104)$$

Recall $P_{ji} = 0$ if controls are imposed by country j and/or country i.
 $= 1$ in the absence of controls.

*Henceforth choice variables pertain to the aggregate value for a country.

Then $E[\tilde{P}_{jp}^c] = (1 - P_{jp}^c)$ where P_{jp}^c is the probability of having some configuration of controls between currencies j and p . Hence (104) may be rewritten as:

$$E[\tilde{a}_{jp}^f] \equiv \frac{r_p^f}{s_p} [f_p - F_p^*] - \frac{r_p^f}{s_p} [P_{jp}^c f_p - E[\tilde{G}_p]]$$

where $F_p^* \equiv \frac{r_j^f}{r_p^f} s_p$ is the "interest parity rate."

Substituting this expression for $E[\tilde{a}_{jp}^f]$, into (102) provides:

$$A_{jp} = \theta_j \eta_{tt}^J \frac{r_p^f}{s_p} [f_p - F_p^*] + \theta_j \left[\sum_{i \neq t} \eta_{ti}^J E[\tilde{M}_{ji}^f] - \eta_{tt}^J \frac{r_p^f}{s_p} [P_{jp}^c f_p - E[\tilde{G}_p]] \right]$$

(105)

Recall from section 2, Chapter III, "Some characteristics of any expected utility optimal portfolio," our optimum cannot involve both positive A_{jp} and O_{jp} . Since by assumption A_{jp} is positive, we have O_{jp} is zero. Also in that section it was shown that:

$$O_i > 0 \longrightarrow Y_i = E_i \frac{s_i}{r_i^f} \quad (78) \text{ rptd.}$$

Whilst it was not done so, it could also have been shown by the same procedure that:

$$A_i > 0 \longrightarrow Y_i = -I_i \frac{s_i}{r_i^f} \quad (106)$$

Thus with $A_{jp} > 0$ we have $Y_{jp} = -I_{jp} \frac{s_p}{r_p^f}$. Following the procedure

immediately preceding (91), Y_{jp} could be eliminated from final wealth, and the only two terms involving Y_{jp} and I_{jp} , replaced by a single term:

$$+ I_{jp} \left(\tilde{i}_{jp}^c - \frac{s_p}{r_p^f} \tilde{y}_{jp}^I \right)$$

Using the definitions of the stochastic terms, this term may be rewritten as:³

$$+ I_{jp} \left[\tilde{r}_p^I - s_p \frac{r_j^f}{r_p^f} \right] \quad (107)$$

This expression indicates that the forward market activity associated with I_{jp} and Y_{jp} separately, nets out to zero. That is, the net result is as if the foreign currency debt associated with I_{jp} is repatriated entirely by the "discount" method, without any resort to the forward market. (Note that the term within parentheses in (107) is the per unit j net return on imports, the foreign currency cost of which is paid by this discounting method.)

Given these results, we have that if at the optimum A_{jp} is positive, then the transactions denoted A_{jp} are the only source from country j , of forward market activity associated with arbitrage.⁴ Since A_{jp} denotes the number of units of j allocated to the covered bond of country p , we have that the NET supply of forward p from country j , arising from arbitrage is:

$$S_{jp}^j (A_{jp}) \equiv A_{jp} \frac{r_p^f}{s_p} \quad (108)$$

where A_{jp} is provided by (105).

Note that (108) has been derived from the assumption of positive A_{jp} . For any optimum exhibiting positive A_{jp} , this net supply from arbitrage function, will have the structure of (108).

For any such optimum the terms which compose (108) will depend upon the configuration of the \tilde{M}_{ji} in (105). What is significant from our perspective is that independently of this configuration of the \tilde{M}_{ji} , having substituted for A_{jp} (108) may be rewritten generally as:

$$S_{jp}^j (A_{jp}) = \theta_j \left[\eta_{tt}^j \left[\frac{r_p^f}{s_p} \right]^2 [f_p - F_p^*] + \sum_{jp}^A (M_j) \right] \quad (109)$$

where $\sum_{jp}^A (M_j)$ is meant to symbolize a summation of terms which depend upon the parameters of the vector M_j , but which for our purposes need not be considered explicitly.

(109) can be viewed as a family of functions giving country j 's participation in the jp forward currency market arising from arbitrage. The family is characterized by positive A_{jp} , each member corresponding to a particular configuration of \tilde{M}_j . If instead of positing an optimum with positive A_{jp} , we assume O_{jp} is positive, a different family of market participation functions emerges. However, as we shall see, this family

has a general form akin to (109).

Assume that we are at an optimum where O_{jp} is positive, and thus the choice variable O_{jp} is contained in the C_j^{*} vector of (110).*

$$C_j^{*} = \theta_j' \Sigma_j^{-1} E[\tilde{M}_j'] \quad (110)$$

Let O_{jp} be the t^{th} element of C_j^{*} , and we have by implication $\tilde{O}_{jp} \equiv \tilde{M}_{jt}'$.

Using an equation which corresponds to (101) we have:

$$O_{jp} = \theta_j' \eta_{tt}^{J'} E[\tilde{O}_{jp}] + \theta_j' \sum_{i \neq t} \eta_{ti}^{J'} E[\tilde{M}_{ji}'] \quad (111)$$

From (47) and (48):

$$E[\tilde{O}_{jp}] \equiv E \left[r_j^f - \frac{r_p^f}{s_p} [f_p \tilde{P}_{jp} + \tilde{N}_p] \right]$$

or using $E[P_{jp}^c] = [1 - P_{jp}^c]$:

$$E[\tilde{O}_{jp}] \equiv \frac{r_p^f}{s_p} [P_{jp}^c f_p - E[\tilde{N}_p]] - \frac{r_p^f}{s_p} [f_p - F_p^*]$$

* When positive A_{jp} was assumed and derivation proceeded from (98), the notation \tilde{M}_j and \tilde{M}_{ji}^{jp} was used. Since simultaneously positive A_{jp} and O_{jp} has been ruled out, the assumption of positive O_{jp} necessarily means that O_{jp} has the vector \tilde{M}_j differs from the original. Hence the "new solution" (110).

Substituting into (111) provides:

$$\begin{aligned}
 O_{jp} = & -\theta_j' \eta_{tt}^{J'} \frac{r_p^f}{s_p} [f_p - F_p^*] + \theta_j' \eta_{tt}^{J'} \frac{r_p^f}{s_p} [P_{jp}^c f_p - E[\tilde{N}_p]] \\
 & + \theta_j' \sum_{i \neq t} \eta_{ti}^{J'} E[\tilde{M}_{ji}] \quad (112)
 \end{aligned}$$

O_{jp} denotes the optimal number of units of j allocated to the domestic bond, financed by covered borrowing in country p , hence it generates a demand for forward currency p .

$$D_p^j(O_{jp}) \equiv O_{jp} \frac{r_p^f}{s_p} \quad (113)$$

Again recall from the section "Characteristics of any optimal portfolio,"

A_{jp} and O_{jp} cannot both be positive at an optimum. Thus in positing

O_{jp} positive, we have A_{jp} is zero. Also from that section, (78) provides:

$$O_{jp} > 0 \rightarrow Y_{jp} = E_{jp} \frac{s_p}{r_p^f}$$

Once more following the procedure preceding (91), Y_{jp} could be eliminated

from final wealth and the only two terms involving Y_{jp} and E_{jp} , replaced

by a single term:

$$+ E_{jp} \left(\tilde{a}_{jp}^c + \frac{s_p}{r_p^f} \tilde{y}_{jp}^E \right)$$

Using the definitions of the stochastic terms, this term may be rewritten:⁵

$$+ E_{jp} \left[\frac{s_p}{r_f} r_j^f - \tilde{r}_p^E \right] \quad (114)$$

This expression indicates that the forward market activity associated with E_{jp} and Y_{jp} separately, nets out to zero. That is, the net result is as if the foreign return associated with E_{jp} is repatriated entirely by the "discounting" method, without any resort to the forward market. (Note that the term within parentheses in (114) is the per unit j net return of exports, the foreign proceeds of which are repatriated in this way).

Given these results, we have that if at the optimum O_{jp} is positive, then the transactions denoted O_{jp} are the only source, from country j , of forward market activity associated with arbitrage.⁶ Thus with O_{jp} positive the NET demand from country j for forward p arising from arbitrage is provided by (113). Substituting (112) into (113) and rearranging, provides:

$$D_p^j(O_{jp}) \equiv \sum_{jp}^o (M_j) - \theta_j' \eta_{tt}^{J'} \left[\frac{r_p^f}{s_p} \right]^2 [f_p - F_p^*] \quad (115)$$

where once again $\sum_{jp}^o (M_j)$ is a summation of terms which is a function of the parameters of \tilde{M}_j .

As anticipated, the "family" of forward market participation functions (115) (in this case net demand functions) has a structure akin

to (109).

To complete this taxonomy of country j forward market participation arising from arbitrage with country p , we posit an optimum with Y_{jp} NOT being constrained by the value of I_{jp} or E_{jp} . Recalling the footnote to (110), in this case we write the optimal vector as:

$$c_j^{*''} = \theta_j'' \sum_{M_j}^{-1} E[\tilde{M}_j''] \quad (116)$$

Let Y_{jp} be the t^{th} element of $c_j^{*''}$ and by implication the unit return on Y_{jp} is $\tilde{y}_{jp} \equiv \tilde{M}_{jt}''$.

Using the equation which corresponds to (101) provides:

$$Y_{jp} = \theta_j'' \eta_{tt}^{J''} E[\tilde{y}_{jp}] + \theta_j'' \sum_{\substack{i \\ i \neq t}} \eta_{ti}^{J''} E[\tilde{M}_{ji}''] \quad (117)$$

From (47) and (48):

$$E[\tilde{y}_{jp}] \equiv E \left[r_j^f - \frac{r_p^f}{s_p} [f_p \tilde{P}_{jp} + \tilde{S}_p^c] \right]$$

or using $E[P_{jp}^c] = 1 - P_{jp}^c$

$$E[\tilde{y}_{jp}] \equiv \frac{r_p^f}{s_p} [P_{jp}^c f_p - E[\tilde{S}_p^c]] - \frac{r_p^f}{s_p} [f_p - F_p^*]$$

Substituting into (117) provides:

$$\begin{aligned}
 Y_{jp} = & -\theta_j'' \eta_{tt}^{J''} \frac{r_p^f}{s_p} [f_p - F_p^*] + \theta_j'' \eta_{tt}^{J''} \frac{r_p^f}{s_p} [P_{jp}^c f_p - E[\tilde{S}_p^c]] \\
 & + \theta_j'' \sum_{i \neq t} \eta_{ti}^{J''} E[\tilde{M}_{ji}'']
 \end{aligned} \tag{118}$$

In the "characteristics" section it was shown that, from (78):

$$Y_{ji} < E_{ji} \frac{s_i}{r_i^f} \rightarrow O_{ji} = 0$$

from (106):

$$-I_{ji} \frac{s_i}{r_i^f} < Y_{ji} \rightarrow A_{ji} = 0$$

Combining these:

$$\left[-I_{ji} \frac{s_i}{r_i^f} < Y_{ji} < E_{ji} \frac{s_i}{r_i^f} \right] \rightarrow A_{ji} = O_{ji} = 0 \tag{119}$$

Thus at this posited optimum, the transactions denoted Y_{jp} , are the only source from country j of participation in the currencies j and p forward market, arising from arbitrage. Recalling the definition of Y_{ji} (see p. 45), Y_{jp} positive generates a demand for forward p , Y_{jp} negative a supply. Regarding a negative demand as a supply, we can write the demand function for forward p arising from arbitrage as:

$$D_p^j(Y_{jp}) \equiv Y_{jp} \frac{r_p^f}{s_p} \tag{120}$$

Substituting (118) into (120) and rearranging provides:

$$D_{jp}^j (Y_{jp}) = \sum_{jp} Y_{jp} (M_j) - \theta_j^* \eta_{tt}^{J*} \left[\frac{r_p^f}{s_{jp}} \right]^2 [f_p - F_p^*] \quad (121)$$

We have now, in equations (109), (115) and (121), a complete taxonomy of jp forward market, net participation functions, arising from arbitrage by "residents" of country j . That is to say, some net participation requires either that A_{jp} or O_{jp} be positive; or that Y_{jp} be non-zero and not at a constraint boundary. We have considered each of these cases in turn, and, it is to be emphasized, we have shown that at most only one of these sources is non-zero at any optimum. For convenience, let us multiply (109) by minus unity, and view it as a demand function. It may then be stated that in general, the net demand for forward p in the pj market arising from arbitrage by "residents" at country j , whatever its source, has the form:

$$D_{jp}^j (\text{Arb.}) = \sum_{jp}^* (M_j^*) - \theta_j^* \eta_{tt}^{J*} \left[\frac{r_p^f}{s_{jp}} \right]^2 [f_{jp} - F_{jp}^*] \quad (122)$$

where at this optimum the sub vector of returns entering our "reduced form" (98), is denoted \tilde{M}_j^* ; and consequently:

$$\theta_j^* \equiv \frac{T_j \det(M_j^*)}{\det(M_j^* + E[M_j^*] E[M_j^*]^T)} \quad (123)$$

$$\eta^{J*} \equiv \sum_j^{-1} \quad (124)$$

$$\sum_{jp}^* = g(\text{parameters of } M_j^*) \quad (125)$$

(Note that thus far, the currency j forward price of p has been denoted f_p ; with the consideration of country p 's perspective next, this price is denoted f_{jp} in (122). Similarly $F_p^* \equiv F_{jp}^*$, $s_p \equiv s_{jp}$).

Now all of the preceding analysis could have been conducted from the perspective of country p . By replacing the index p with the index j , and vice versa, in (122), we can arrive at the net demand for forward j in the pj market, arising from arbitrage by "residents" of country p .

$$D_p^p(\text{Arb.}) = \sum_{pj}^* (M_p^*) - \theta_p^* \eta_{tt}^{P*} \left[\frac{r_j^f}{s_{pj}} \right]^2 [f_{pj} - F_{pj}^*] \quad (126)$$

Multiplying this expression by f_{pj} the currency p forward price of currency j , provides the net supply of forward p in the pj market, arising from arbitrage by country p .

$$S_p^p(\text{Arb.}) = f_{pj} \sum_{pj}^* (M_p^*) - \theta_p^* \eta_{tt}^{P*} f_{pj} \left[\frac{r_j^f}{s_{pj}} \right]^2 [f_{pj} - F_{pj}^*] \quad (127)$$

Assume $f_{pj} = \frac{1}{f_{jp}}$ and $s_{pj} = \frac{1}{s_{jp}}$

Then $F_{pj}^* \equiv \frac{r_p^f}{r_j^f} s_{pj} = \frac{r_p^f}{r_j^f s_{jp}} = \frac{1}{F_{jp}^*}$

Hence the second term in (127) may be rewritten as:

$$\begin{aligned} & \theta_p^* \eta_{tt}^{p*} f_{pj} \left[\frac{r_j^f}{s_{pj}} \right]^2 \left[\frac{F_{jp}^* - f_{jp}}{F_{jp}^* f_{jp}} \right] \\ &= \theta_p^* \eta_{tt}^{p*} \frac{r_j^f r_p^f}{f_{jp} s_{pj}} [F_{jp}^* - f_{jp}] \end{aligned}$$

and (122) becomes:

$$S_p^P (\text{Arb.}) = f_{pj} \sum_{pj}^* (M_p^*) - \theta_p^* \eta_{tt}^{p*} \frac{r_j^f r_p^f}{f_{jp} s_{pj}} [F_{jp}^* - f_{jp}] \quad (128)$$

Using (122) and (128), we can derive an expression for the excess demand for currency p in the jp forward market arising from arbitrage, X_p^{jAp} :

$$\begin{aligned} X_p^{jAp} &\equiv D_p^j (\text{Arb.}) - S_p^P (\text{Arb.}) \\ &= \left[\theta_J^* \eta_{tt}^{j*} \left[\frac{r_p^f}{s_{jp}} \right]^2 + \theta_p^* \eta_{tt}^{p*} \frac{r_j^f r_p^f}{f_{jp} s_{pj}} \right] [F_{jp}^* - f_{jp}] \\ &\quad + \left[\sum_{jp}^* - f_{pj} \sum_{pj}^* \right] \end{aligned}$$

or:

$$X_p^{jAp} = \pi^{jAp} + \lambda^{jAp} [F_{jp}^* - f_{jp}] \quad (129)$$

where

$$\lambda^{jAp} \equiv \theta_J^* \eta_{tt}^{J*} \left[\frac{r_p^f}{s_{jp}} \right]^2 + \theta_p^* \eta_{tt}^{p*} \frac{r_j^f r_p^f}{f_{jp}^2 s_{pj}} \quad (130)$$

and

$$\pi^{jAp} \equiv \sum_{jp}^* - f_{pj} \sum_{pj}^* \quad (131)$$

Equation (129) is what we have previously referred to as the "general contingent excess demand functions" for forward currency p in the pj market, arising from arbitrage. Henceforth, it will be referred to as an "arbitrage" function. Before comparing the form of this function, with that of the arbitrage function of the 'Modern Theory Model', we derive excess demand functions arising from speculation and trade.

Section 2: A speculative excess demand function.

The choice variable X_{jp} denotes country j 's net speculative position with respect to forward currency p . X_{jp} positive, indicates the number of units of p purchased forward for speculative purposes, X_{jp} negative, indicates the number of units of p sold forward.

Assume that X_{jp} is the s^{th} element of C_j^* in (98). Then:

$$\tilde{x}_{jp} \equiv \tilde{M}_{js}^*$$

and using (101), we have:

$$x_{jp} = \theta_j^* \eta_{ss}^{J*} E[\tilde{x}_{jp}] + \theta_j^* \sum_{i \neq s} \eta_{si}^{J*} E[\tilde{M}_{ji}^*] \quad (132)$$

From (47) and (48):

$$\begin{aligned} E[\tilde{x}_{jp}] &\equiv E \left[[\tilde{s}_{jp} - f_{jp}] \tilde{P}_{jp} \right] \\ &= E[\tilde{s}_{jp} \tilde{P}_{jp}] - f_{jp} P_{jp}^{NC} \end{aligned} \quad (133)$$

where P_{jp}^{NC} is the probability of 'no controls' between j and p .

Now $E[\tilde{s}_{jp} \tilde{P}_{jp}] = E[\tilde{s}_{jp}] P_{jp}^{NC} + \text{Cov}(\tilde{s}_{jp}, \tilde{P}_{jp})$. It seems not unreasonable to assume $\text{Cov}(\tilde{s}_{jp}, \tilde{P}_{jp})$ is zero.

Recall $P_{jp} = 1$ with 'no controls', $P_{jp} = 0$ with controls. \tilde{s}_{jp} is the spot rate which it is currently believed will, in the absence of controls, govern at the end of the period. If \tilde{s}_{jp} is 'high' we might expect country j to impose controls and hence $P_{jp} = 1$, if \tilde{s}_{jp} is 'low', then $1/\tilde{s}_{jp} = \tilde{s}_{pj}$ is 'high' and we might expect country p to impose controls, again $P_{jp} = 1$. Thus we might expect the values of \tilde{s}_{jp} and \tilde{P}_{jp} NOT to be related and hence their covariance is zero, with this assumption, (133) becomes:

$$E[\tilde{x}_{jp}] = [E[\tilde{s}_{jp}] - f_{jp}] P_{jp}^{NC} \quad (134)$$

Substituting (134) into (132) provides:

$$X_{jp} = \theta_j^* \eta_{ss}^{j*} P_{jp}^{NC} [E[\tilde{s}_{jp}] - f_{jp}] + \theta_j^* \sum_{i \neq s} \eta_{si}^{j*} E[\tilde{M}_{ji}^*] \quad (135)$$

Viewing negative demand as supply, (135) may be considered as country j 's speculative demand for forward p .

Thus:

$$D_p^j(\text{spec}) \equiv X_{jp}$$

Once again the corresponding function for country p with respect to currency j may be arrived at by transposing superscript and subscripts. This provides a demand for forward j , multiplying by f_{pj} , the currency p forward price of j , provides country p 's net supply of forward p ;

$$S_p^D(\text{spec}) = \theta_p^* \eta_{ss}^{p*} P_{jp}^{NC} f_{pj} [E[\tilde{s}_{pj}] - f_{pj}] + f_{pj} \theta_p^* \sum_{i \neq s} \eta_{si}^{p*} E[\tilde{M}_{pi}^*] \quad (136)$$

If we assume $E[\tilde{s}_{jp}^*] = \frac{1}{E[\tilde{s}_{pj}]}$

then:

$$[E[\tilde{s}_{pj}] - f_{pj}] = \frac{[f_{jp} - E[\tilde{s}_{jp}]]}{f_{jp} E[\tilde{s}_{jp}]} \quad (137)$$

Substituting (137) into (136) we arrive at the excess demand for currency p arising from speculation $X_p^{j\text{sp}}$:

$$X_p^{j\text{sp}} \equiv D_p^j (\text{spec}) - S_p^p (\text{spec})$$

$$X_p^{j\text{sp}} = \left[\theta_j^* \eta_{ss}^{J*} + \theta_p^* \eta_{ss}^{P*} \frac{f_{pj}^2}{E[s_{jp}]} \right] P_{jp}^{\text{NC}} \left[E[\tilde{s}_{jp}] - f_{jp} \right] \\ + \theta_j^* \sum_{i \neq s} \eta_{si}^{J*} E[\tilde{M}_{ji}] - \theta_p^* f_{pj} \sum_{i \neq s} \eta_{si}^{P*} E[\tilde{M}_{pi}^*]$$

or:

$$X_p^{j\text{sp}} = \pi^{j\text{sp}} + \lambda^{j\text{sp}} \left[E[\tilde{s}_{jp}] - f_{jp} \right] \quad (138)$$

where $\pi^{j\text{sp}}$ and $\lambda^{j\text{sp}}$ are defined by the corresponding terms in the preceding equation .

Equation (138) is the "general contingent excess demand function" for forward p in the p_j market, arising from speculation. Henceforth it is referred to as a "speculative" function.

Section 3: A "trade" excess demand function.

The aggregated variable E_{ji} , denoted the number of units of currency i accruing to residents of country j , from exports to country i . Similarly, the aggregated variable I_{ji} , denotes the number of units of currency i owed to residents of country i , by residents of country j , arising from imports. We have chosen to view final wealth as it appears in (47) and

(48), as representing "the way an individual behaves" (see discussion p. 49). Now the returns on E_{ji} and I_{ji} in (48) arise from covering ALL of the associated currency i income or debt, in the forward market. Hence, associated with E_{ji} is a supply of forward currency i , associated with I_{ji} a demand for forward i .

Assume E_{jp} is positive and non-constraining (recall footnote 6); assume that it is the q^{th} element of C_j^* in (98). Then:

$$\tilde{M}_{jq}^* \equiv \tilde{e}_{jp}$$

From (101) we have:

$$E_{jp} = \theta_j^* \eta_{qq}^{J*} E[\tilde{e}_{jp}] + \theta_j^* \sum_{i \neq q} \eta_{qi}^{J*} E[\tilde{M}_{ji}^*] \quad (139)$$

From (47) and (48):

$$E[\tilde{e}_{jp}] \equiv E[f_{jp} P_{jp} + S_p^c - \tilde{r}_p^E]$$

$$E[\tilde{e}_{jp}] \equiv f_{jp} P_{jp}^{NC} + E[S_p^c - \tilde{r}_p^E]$$

Substituting into (139):

$$E_{jp} = \theta_j^* \eta_{qq}^{J*} f_{jp} P_{jp}^{NC} + \theta_j^* \eta_{qq}^{J*} E[S_p^c - \tilde{r}_p^E] + \sum_{i \neq q} \eta_{qi}^{J*} E[\tilde{M}_{ji}^*] \quad (140)$$

This is the optimal supply of forward currency p by country j arising from exports. We now derive the optimal demand of forward p arising from imports.

Assume I_{jp} is positive and non-constraining, (recall footnote 4), and is the g^{th} element of C_j^* in (98). Then:

$$\tilde{M}_{jg} \equiv \tilde{i}_{jp}$$

From (101):

$$I_{jp} = \theta_j^* \eta_{gg}^{J*} E[\tilde{i}_{jp}] + \theta_j^* \sum_{i \neq g} \eta_{gi}^{J*} E[\tilde{M}_{ji}^*] \quad (141)$$

From (47) and (48):

$$\begin{aligned} E[\tilde{i}_{jp}] &\equiv E[\tilde{r}_p^I - f_{jp} \tilde{p}_{jp} - \tilde{s}_p^c] \\ &= E[\tilde{r}_p^I - \tilde{s}_p^c] - f_{jp} P_{jp}^{NC} \end{aligned}$$

Thus:

$$\begin{aligned} I_{jp} &= \theta_j^* \left[\eta_{gg}^{J*} E[\tilde{r}_p^I - \tilde{s}_p^c] + \sum_{i \neq g} \eta_{gi}^{J*} E[\tilde{M}_{ji}^*] \right] \\ &\quad - \theta_j^* \eta_{gg}^J P_{jp}^{NC} f_{jp} \quad (142) \end{aligned}$$

This is the optimal demand for forward currency p by country j arising from imports. The net demand for forward p by country j arising from trade, $D_p^j(\text{trade})$, is:

$$D_p^j(\text{trade}) \equiv I_{jp} - E_{jp}$$

Using (140) and (142)

$$D_p^j(\text{trade}) = \sum_{jp}^T (M_j^*) - \theta_j^* P_{jp}^{NC} [\eta_{gg}^{J*} + \eta_{qq}^{J*}] f_{jp} \quad (143)$$

where $\sum_{jp}^T (M_j^*)$ simply denotes the terms not written explicitly.

The net demand for forward j by country p arising from trade $D_j^P(\text{trade})$, has the form:

$$D_j^P(\text{trade}) = \sum_{jp}^T (M_j^*) - \theta_p^* P_{jp}^{\text{NC}} [\eta_{gg}^{p*} + \eta_{qq}^{p*}] f_{jp} \quad (144)$$

Multiplying by the forward p price of j , f_{pj} , provides the net supply of forward p by country p , arising from trade:

$$S_p^P(\text{trade}) \equiv [D_j^P(\text{trade})] f_{pj} \quad (145)$$

The excess demand for forward p in the pj market arising from trade, X_p^{jTp} , is then:

$$X_p^{jTp} \equiv D_p^j(\text{trade}) - S_p^P(\text{trade})$$

Using (143), (144) and (145):

$$X_p^{jTp} = \sum_{jp}^T - \frac{\sum_{pj}^T}{f_{jp}} - P_{jp}^{\text{NC}} \left[\theta_j^* [\eta_{gg}^{j*} + \eta_{qq}^{j*}] - \theta_p^* \frac{[\eta_{gg}^{p*} + \eta_{qq}^{p*}]}{f_{jp}^3} \right] f_{jp} \quad (146)$$

or:

$$X_p^{jTp} = \pi^{jTp} + \lambda^{jTp} f_{jp} \quad (147)$$

where π^{jTp} and λ^{jTp} are defined by the corresponding terms in (146).

Equation (147) is the "general contingent excess demand function" for forward p in the p_j market, arising from trade. In the terminology of the modern theory model this is a "trade function."

CHAPTER V. FOOTNOTES

1. & 2. Jan Mossin (1973), p. 55.

3. $\tilde{i}_{jp}^c \equiv \tilde{i}_{jp}$ the superscript 'c' was introduced in (89) to denote that this is a return associated with a choice variable which at this optimum acts as a binding constraint.

$\tilde{y}_{jp}^I \equiv \tilde{y}_{jp}$ the superscript 'I' was introduced in (89) to denote that the associated choice variable Y_{jp} is constrained by I_{jp} at this optimum. Thus:

$$\left(\tilde{i}_{jp}^c - \frac{s_p}{r_p^f} \tilde{y}_{jp}^I \right) \equiv \left(\tilde{i}_{jp} - \frac{s_p}{r_p^f} \tilde{y}_{jp} \right)$$

Using (47) and (48) to provide \tilde{i}_{jp} and \tilde{y}_{jp} we have:

$$\begin{aligned} \left(\tilde{i}_p^c - \frac{s_p}{r_p^f} \tilde{y}_{jp}^I \right) &\equiv \left[\tilde{r}_p^I - f_p \tilde{P}_{jp} - \tilde{S}_p^c \right] - \frac{s_p}{r_p^f} \left(r_j^f - \frac{r_p^f}{s_p} [f_p \tilde{P}_{jp} + \tilde{S}_p^c] \right) \\ &= \tilde{r}_p^I - s_p \frac{r_p^f}{r_p^f} \tilde{y}_{jp} \end{aligned}$$

4. In stating this, we are netting out the offsetting forward market activity associated with Y_{jp} and I_{jp} . That is, the forward market activity associated with the CONSTRAINED Y_{jp} is excluded from the arbitrage function. For consistency, to reflect this "netting out," when we get to the trade function the forward market activity arising from the CONSTRAINING I_{jp} is excluded from that function.

5. See footnote #3, then in the same vein we have:

$$\tilde{e}_{jp}^c \equiv \tilde{e}_{jp} \quad \tilde{y}_{jp}^E \equiv \tilde{y}_{jp} \quad \text{Thus:} \quad \tilde{e}_{jp}^c + \frac{s_p}{r_p^f} \tilde{y}_{jp}^E \equiv \tilde{e}_{jp} + \frac{s_p}{r_p^f} \tilde{y}_{jp}$$

Using (47) and (48) for \tilde{e}_{jp} and \tilde{y}_{jp} provides:

$$\tilde{e}_{jp}^c + \frac{s_p}{r_p} \frac{\tilde{y}_{jp}^E}{r_p} = \frac{s_p}{r_p} r_j^f - r_p^E$$

6. We are netting out the offsetting forward market activity associated with constrained Y_{jp} and constraining E_{jp} . Here we are excluding the activity arising from Y_{jp} from the arbitrage function, the offsetting activity of the constraining E_{jp} is excluded from the trade function.

CHAPTER VI: THE EXCESS DEMAND FUNCTIONS OF THE "MODERN THEORY MODEL" COMPARED AND CONTRASTED WITH THOSE OF OUR MODEL.

Section 1. The arbitrage function.

Using our notation, the arbitrage function of the Modern Theory Model has the following form:

$$X_p^{jAp} = \alpha^{jAp} [F_{jp}^* - f_{jp}] \quad 0 < \alpha^{jAp} < \infty \quad (7) \text{ rptd.}$$

where X_p^{jAp} is the excess demand for currency p arising from arbitrage between countries j and p; AND, of particular interest from our perspective, α^{jAp} is a positive, finite, CONSTANT.

This 'modern theory' specification is to be compared and contrasted with the 'arbitrage function' which emerges from our model:

$$X_p^{jAp} = \pi^{jAp} + \lambda^{jAp} [F_{jp}^* - f_{jp}] \quad (129) \text{ rptd.}$$

1. In the modern theory specification

$$0 < \alpha^{jAp} < \infty$$

Writing λ^{jAp} , the term in (129) which corresponds to α^{jAp} , explicitly,

we have:

$$\lambda^{jAp} \equiv \theta_j^* \eta_{tt}^{j*} \frac{r_p^f}{s_{jp}} + \theta_p^* \eta_{tt}^{p*} \frac{r_p^f}{f_{jp}^2} \frac{r_j^f}{s_{pj}} \quad (130) \text{ rptd.}$$

It may be shown that this term is also positive and finite.

Repeating (99):

$$\theta_j \equiv \frac{T_j \det(\Sigma_j)}{\det(\Sigma_j + E[\tilde{M}_j]E[\tilde{M}_j]^T)}$$

$$\text{Now } T_j \equiv \sum_K T_j^K$$

From (96) $T_j^K > 0$ for all K and j , thus $T_j > 0$ for all j .

Σ_j is a variance covariance matrix. It is thus positive definite and:

$$\det(\Sigma_j) > 0 \quad (\text{it is assumed that there is no exact linear dependence in the } \tilde{M}_{ji})$$

Since:

$$X^T [\Sigma_j + E[\tilde{M}_j] E[\tilde{M}_j]^T] X$$

$$= X^T \Sigma_j X + X^T E[\tilde{M}_j] E[\tilde{M}_j]^T X$$

$$= \text{Var}(X^T \tilde{M}_j) + (X^T E[\tilde{M}_j])^2 > 0 \quad \text{Thus: } \det(\Sigma_j + E[\tilde{M}_j]E[\tilde{M}_j]^T) > 0$$

Using these results we have:

$$\theta_j > 0 \quad \text{for all } j. \quad (148)$$

Note η_{tt}^{J*} is the tt^{th} element of Σ_{j*}^{-1} . Thus $\eta_{tt}^{J*} = \frac{1}{\det(\Sigma_{j*}^S)} \det(\Sigma_{j*}^S)$

where Σ_{j*}^S is Σ_{j*} with the t^{th} column and t^{th} row removed. It is also a variance/covariance matrix, is positive definite and thus has a positive determinant. Hence:

$$\eta_{tt}^{J*} > 0 \text{ for all } J \text{ and } t \quad (149)$$

In view of (148) and (149), we have, from (130):

$$\lambda^{jAp} > 0$$

We may rewrite θ_j as:

$$\theta_j = \rho_j \sum_K T_j^K$$

$$\text{where } \rho_j \equiv \frac{\det(\Sigma_j)}{\det(\Sigma_j + E[\tilde{M}_j] E[\tilde{M}_j]^T)}$$

is positive.

Using (96):

$$\theta_j = \rho_j \sum_K \begin{bmatrix} U_j^K' \\ -\frac{j}{U_j^K''} \end{bmatrix}$$

$$\lim_{U_j^K'' \rightarrow 0^-} \theta_j \rightarrow \infty \quad \text{for all } j$$

Looking at (130) repeated above, this implies that the existence of a single investor in either country j or p , who is approximately risk neutral would cause λ^{jAp} to approach infinity. Our assumption that all investors are risk averse however, ensures that λ^{jAp} is finite.

Hence, like its analogue in the modern theory model α^{jAp} , we have:

$$0 < \lambda^{jAp} < \infty$$

2. The modern theory model specification (7), implies that x_p^{jAp} is zero if, and only if, the forward rate f_{jp} is equal to the "interest parity" rate F_{jp}^* . (Or, equivalently, if and only if, the "covered arbitrage margin" is zero.) In view of the term π^{jAp} in (129), the arbitrage function which emerges from our model, does not have this implication. The condition that the "covered arbitrage margin" be zero is neither necessary nor sufficient for x_p^{jAp} to be zero.

3. The arbitrage function which emerges from our model, embodies a great many more terms, embedded in the sum π^{jAp} , than does the arbitrage function of the modern theory model.

4. In the modern theory model, α^{jAp} is regarded as parametric. The model recognizes that the value of α^{jAp} "... depends on the amount of risk attaching to arbitrage operations and the degree of risk aversion on the part of arbitragers."¹

λ^{jAp} , as is immediately apparent from (130), depends upon the degree of risk aversion of market participants. It also is a function of the moments of the joint density functions of the \tilde{M}_{ji} and \tilde{M}_{pi} , embedded in the θ and η terms. Hence it depends upon the "amount of risk." However, as warrants emphasis, λ^{jAp} cannot be regarded as parametric. It is a function of f_{jp} , and thus cannot be a constant in a model where f_{jp} is variable.

5. α^{jAp} is regarded as a constant. Its analogue λ^{jAp} , as a function of f_{jp} , is necessarily a variable. Furthermore, if, as is the case in the

estimation of the modern theory model, f_{jp} is regarded as a random variable, consistency requires that λ^{jAp} also be regarded as a random variable.

Section 2. The speculative function.

Using our notation, the speculative function of the Modern Theory Model has the form:

$$X_p^{jSp} = \alpha^{jSp} [E[\tilde{s}_{jp}] - f_{jp}] \quad (8) \text{ rptd.}$$

where X_p^{jSp} is the excess demand for currency p arising from speculation between countries j and p ; AND α^{jSp} is a positive, finite, CONSTANT.

This specification is to be compared and contrasted with the "speculative function" for currencies j and p , which emerges from our model:

$$X_p^{jSp} = \pi^{jSp} + \lambda^{jSp} [E[\tilde{s}_{jp}] - f_{jp}] \quad (138) \text{ rptd.}$$

1. In the modern theory specification:

$$\alpha^{jSp} > 0$$

its analogue λ^{jSp} has the form:

$$\lambda^{jSp} \equiv \left[\theta_j^* \eta_{ss}^{j*} + \theta_p^* \eta_{ss}^{p*} \frac{f_{pj}^2}{E[\tilde{s}_{jp}]} \right] P_{jp}^{NC}$$

Using the arguments of point 1 made with respect to the arbitrage function, we have:

$$\lambda^{jSp} > 0$$

2. In the modern theory model theoretical exposition, λ^{jSp} is typically regarded as being finite. However, that there is a theoretical rationale for its being non finite is recognized,² and in at least one paper is given the prominence of constituting a distinct model.³

Once again utilizing an argument from point 1 of the arbitrage schedule discussion; in our model, risk aversion is necessary and sufficient for the finiteness of λ^{jSp} . A single risk neutral market participant in country j or p would cause λ^{jSp} to be non finite.

3. The modern theory specification implies that x_p^{jSp} is zero if, and only if, $E[\tilde{s}_{jp}]$ is equal to f_{jp} . In view of the term π^{jSp} in (138), our model does not have this implication. In short, in our model it may be desirable to open a forward speculative position when the expected return on that position is zero or negative, because covariance properties influence the variance of total final wealth so as to enhance its expected utility. Conversely, it may not be desirable to open a speculative position even though the expected return is positive, because doing so may increase the variance of total final wealth to such a degree, that the expected utility of final wealth is lowered.

4. The specification which emerges from our model clearly involves a great many more terms, embodied in π^{jSp} than does that of the modern theory. Confined as it is to two currencies and to two income paying assets, the modern theory model suppresses the extensive interdependency between risky positions which emerge from any general model of portfolio selection.

Recalling that both π^{jSp} and λ^{jSp} are functions of elements of the inverses of the variance/covariance matrices of all return pertinent to countries j and p , suggests that X_p^{jSp} depends not only on exchange rates between j and p , but also on EVERY exchange rate in the model involving either currency j or p .⁴ Furthermore, it also depends upon the return distributions of ALL of the "risky assets" (i.e. stocks) of the model. (Thus the model would support the contention, that a forward dollar price of pounds depends in part upon belief about the performance of the Dow Jones Industrial Average.)

5. In the modern theory model κ^{jSp} is a parameter. We have seen that its analogue λ^{jSp} , is a function both of $E[\tilde{s}_{jp}]$ and f_{jp} . In a model with these variable, λ^{jSp} is also necessarily, variable. Furthermore, in an estimation model with f_{jp} a random variable, then λ^{jSp} is also, necessarily a random variable.

Section 3. The trade function.

In all of the papers which compose, what has been characterized here as, the modern theory model empirical literature, the specification of the arbitrage and speculative functions is uniform. There is some diversity however, in the treatment of the excess demand for forward exchange arising from trade.

In the work of Stoll, Kesselman and Haas, "traders are considered

either as arbitragers or speculators"⁵ Thus their behaviour is regarded as being implicitly captured by the arbitrage and speculative functions, (7) and (8).

McCallum, whilst adopting the same arbitrage and speculative functions as those of Stoll, Haas and Kesselman, chooses to specify a "trade" function explicitly. His specification has the form:

$$X_p^{jTp} = \kappa_o^T - \kappa^{jTp} f_{jp} \quad \kappa_o > 0 \quad \kappa^{jTp} > 0 \quad (9) \text{ rptd.}$$

where X_p^{jTp} is the excess demand for forward p arising from trade between countries j and p .

Reflecting the two treatments of trade in the modern theory literature, we have two avenues to pursue here; to consider whether or not the collapse of trade excess demand into either speculative or arbitrage excess demand is in our model legitimate, and to compare and contrast McCallum's explicit specification (9), with the trade function which emerges from our model:

$$X_p^{jTp} = \pi^{jTp} - \lambda^{jTp} f_{jp} \quad (147) \text{ rptd.}$$

In the literature, the rationale for the collapse of trade induced participation in a forward currency market, into the arbitrage and speculative functions, is provided verbally. For example:

"Suppose a home country exporter is to be paid in foreign exchange in three months time, i.e. he has extended trade credit.

If he protects himself against the danger that the exchange rate will move against him, by selling forward foreign exchange, he is acting like an arbitrager. If he does nothing, he has a net asset denominated in foreign exchange, which is speculation."⁶

Perhaps this argument may be structured in the following way. A country j exporter has 'E' units of currency p accruing in the future. The future number of units of currency j he will receive in exchange for E can be written as:

$$\tilde{\omega} = (E - e - R) f_{jp} + R \frac{r_j^f}{r_p^f} s_{jp} + e \tilde{s}_{jp} \quad (150)$$

where R is the number of units of p "against which" he borrows in country p (i.e. he borrows an amount R/r_p^f), for exchange at current spot, and investment in the domestic riskless asset; and e is the number of units of E left uncovered, to be repatriated at future spot.

This return $\tilde{\omega}$ may be written as:

$$= E \frac{r_j^f}{r_p^f} s_{jp} + (E - R) \left(f_{jp} - \frac{r_j^f}{r_p^f} s_{jp} \right) + e (\tilde{s}_{jp} - f_{jp})$$

or

$$= E F_{jp}^* + (E - R) (f_{jp} - F_{jp}^*) + e (\tilde{s}_{jp} - f_{jp}) \quad (151)$$

(151) indicates that the exporter's return from selling some of E forward, leaving some exposed etc., may be duplicated by: (a) repatriating all of E by borrowing against it in country p etc., (b) engaging in arbitrage by borrowing $(E - R) \frac{s_{jp}}{r_p^f}$ units of currency j for investment in the

riskless asset of country p, (c) engaging in speculation by buying e units of currency p forward.

Since the repatriation of E in this scenario does not utilize the forward market, this trader's participation in that market, may be viewed as being governed entirely by the motives of arbitrage and speculation. In short, since the trader's return can be duplicated by repatriation not involving the forward market, plus arbitrage and speculation which do involve the forward market, his behaviour in the forward market may be subsumed under that of "specialist" arbitragers and speculators. This perspective suggests that if in our model the trader's return cannot be duplicated by complete repatriation not involving the forward market, plus 'arbitrage' and 'speculation', then it would be illegitimate in our model to subsume trade behaviour under that of arbitrage and speculation.

From our initial definition of final wealth (43), we have as the analogue to (150):

$$\tilde{\omega}^* = (E - e - R) f_{jp} \tilde{P}_{jp} + (E - R) \tilde{S}_p^c + R \frac{r_j^f}{r_p^f} s_{jp} + e \tilde{s}_{jp} \tilde{P}_{jp} \quad (152)$$

This is the gross return to an exporter (i.e. ignoring the cost of generating the export income), using the same array of repatriation opportunities as those of (150) but allowing for the risk of exchange control. Using the definition:

$$F_{jp}^* \equiv \frac{r_j^f}{r_p^f} s_{jp}$$

and adding and subtracting $E F_{jp}^*$ to (152) provides:

$$\begin{aligned} \Omega^* &= E F_{jp}^* + (E - R) \left[(f_{jp} \tilde{P}_{jp} + \tilde{S}_p^c) - F_{jp}^* \right] \\ &\quad + e(\tilde{s}_{jp} - f_{jp}) P_{jp} \end{aligned} \quad (153)$$

This expression indicates that the return from exports arising from selling some of E forward, leaving some of it exposed etc., (as embodied in (152)), can indeed be duplicated by repatriating ALL of E by a method not involving the forward market, and simultaneously engaging in arbitrage and speculation.

The first term in (153) corresponds to borrowing E/r_p^f units of p , exchanging the proceeds at current spot and investing in the domestic riskless asset. Thus all of E is repatriated without resort to the forward market.

The second term in (153) corresponds to an arbitrage transaction. It involves borrowing $(E - R) \frac{s_{jp}}{r_p^f}$ units of the domestic currency j , exchanging at spot for currency p and investing in the riskless asset of currency p . The proceeds, $(E - R)$ units of currency p , are then sold forward. The appearance of the terms \tilde{P}_{jp} and \tilde{S}_p^c reflect the possibility that in this model despite the forward sale, f_{jp} may not be the actual exchange rate, it may be that with the imposition of controls, a different rate, \tilde{S}_p^c , governs the exchange.

The third term in (153) corresponds to a speculative purchase of e units of forward p .

It may be concluded that the suppression of trade induced participation in a forward currency market, can in our model be legitimate.* It should be emphasized however, that in our machinations, this suppression of trade induced behaviour in a forward market, into speculative or arbitrage induced behaviour in that market, has not been engineered. In going from (43) to (48) above, we did not use the manipulation involved in going from (152) to (153). Hence, as discussed at length in the section above "A trade excess demand function," there remains explicitly in our model, trade induced behaviour in forward currency markets. As a consequence it is of interest to compare and contrast a 'trade excess demand function' which emerges from our model, with that explicit trade function which appears in a section of the Modern Theory Model literature.

Using our notation, the trade function of the Modern Theory Model has the form:

$$x_p^{jTp} = \kappa_o^T - \kappa_{jp}^{jTp} \quad \kappa_o^T > 0 \quad \kappa_{jp}^{jTp} > 0 \quad (9) \text{ rptd.}$$

where x_p^{jTp} is the excess demand for forward currency p arising from trade between countries j and p . This is to be compared with the function which emerges from our model:

* We have not considered the return on import income, however, it could be treated in parallel manner.

$$X_p^{jTp} = \pi^{jTp} - \lambda^{jTp} f_{jp} \quad (147) \text{ rptd.}$$

1. In the modern theory specification:

$$\frac{d X_p^{jTp}}{d f_{jp}} = - \alpha^{jTp} < 0$$

This property stems essentially from the following considerations: (a) As f_{jp} rises, country j exports and a consequent supply of forward p are encouraged, country j imports and a consequent demand for forward p , are discouraged. Thus as f_{jp} rises, the country j net demand for forward p declines. In our model, this net demand is provided by (143):

$$D_p^j(\text{trade}) = \sum_{jp}^T - \theta_j^{*PNC} [\eta_{gg}^{J*} + \eta_{qq}^{J*}] f_{jp} = \sum_{jp}^T - \beta_j f_{jp} \quad (143) \text{ rptd.}$$

$$\text{where } \beta_j \equiv \theta_j^{*PNC} [\eta_{gg}^{J*} + \eta_{qq}^{J*}]$$

Using the approach involved in the discussion of the arbitrage function, it may be shown that β_j is positive. This reflects the sense of the preceding paragraph. In a partial equilibrium framework, increasing f_{jp} encourages country j exports to p and discourages imports. (143) however, shows that these forces may not be sufficient to ensure:

$$\frac{\partial D_p^j}{\partial f_{jp}} < 0$$

Since:

$$\frac{\partial D_{jp}^j}{\partial f_{jp}} = - \left[\beta_j + \frac{\partial \beta_j}{\partial f_{jp}} f_{jp} \right] + \frac{\partial \Sigma_{jp}^T}{\partial f_{jp}}$$

$$\frac{\partial D_{jp}^j}{\partial f_{jp}} > 0 \text{ cannot be precluded.}$$

(b) As f_{jp} rises, $f_{pj} = 1/f_{jp}$ declines. This encourages a supply of currency p from country p importers, and discourages the demand for currency p from country p exporters to country j. Hence the net supply of forward currency p from country p traders is positively related to f_{jp} . In our model this net supply is provided by (145). Using (144) and $f_{pj} = 1/f_{jp}$ it has the form:

$$S_p^P(\text{Trade}) = \sum_{pj}^T / f_{jp} - \theta_p^* P_{jp}^{NC} [\eta_{gg}^{P*} + \eta_{qq}^{P*}] / f_{jp}^2$$

or:

$$S_p^P = \sum_{pj}^T / f_{jp} - \beta_p / f_{jp}^2$$

Now β_p is positive, which again reflects the sense of the preceding comments. However, since the signs of \sum_{pj}^T and $\partial \beta_p / \partial f_{jp}$ are unrestricted it cannot be concluded that

$$\partial S_p^P / \partial f_{jp} > 0.$$

(c) If as f_{jp} rises, the country j net demand for forward p declines, and country p net supply increases, then the excess demand for forward p is

negatively related to f_{jp} . Hence the modern theory specification. In our model however, the relationship between the excess demand from trade and the forward rate, is complex and is not necessarily inverse.

2. The trade function which emerges from our model embodies a great many more terms, embodied in the sum π^{jTp} , than does the trade function of the modern theory.

3. In the modern theory specification, α_o^T and α^{jTp} are parametric. In (140) however, both π^{jTp} and λ^{jTp} are functions of f_{jp} and thus cannot be treated as constants in a model with f_{jp} variable.

4. Following from point #3; in an estimation model with f_{jp} a random variable π^{jTp} and λ^{jTp} would necessarily be random variables. In the modern theory estimation procedures, the terms α_o^T and α^{jTp} are regarded as constants.

Having derived excess demand functions from our model, and having compared their form to those of the modern theory model, we are now in a position to answer the question: is the assumed "constancy" of the coefficients of the modern theory empirical model "reduced" form consistent with our model?

Section 4. Conclusion.

The "modern theory model" in its most explicit form (i.e. inclusive of a "trade" function) uses the market clearing condition:

$$X_P^{jAp} + X_P^{jSp} + X_P^{jTp} = 0$$

to arrive at a "reduced form" for f_{jp} . Using the modern theory specifications (7), (8) and (9) this provides:

$$\begin{aligned} f_{jp} &= \frac{\alpha_o^T}{\alpha^{jAp} + \alpha^{jSp} + \alpha^{jTp}} \\ &+ \frac{\alpha^{jSp}}{\alpha^{jAp} + \alpha^{jSp} + \alpha^{jTp}} E[\tilde{s}_{jp}] \\ &+ \frac{\alpha^{jAp}}{\alpha^{jAp} + \alpha^{jSp} + \alpha^{jTp}} F_{jp}^* \end{aligned} \quad (10) \text{ rptd.}$$

This "reduced form" (or a similar form omitting the trade function) is the basic estimation equation from which the estimation of the modern theory literature ensues.

If our model were the "true" model, then the "reduced form (10)", would be an approximation to the expression which emerges from the market clearing condition, and the excess demand functions of our model. It would be an approximation of:

$$\begin{aligned} f_{jp} &= \left[\frac{\pi^{jAp} + \pi^{jSp} + \pi^{jTp}}{\lambda^{jAp} + \lambda^{jSp} + \lambda^{jTp}} \right] \\ &+ \left[\frac{\lambda^{jSp}}{\lambda^{jAp} + \lambda^{jSp} + \lambda^{jTp}} \right] E[\tilde{s}_{jp}] \\ &+ \left[\frac{\lambda^{jAp}}{\lambda^{jAp} + \lambda^{jSp} + \lambda^{jTp}} \right] F_{jp}^* \end{aligned} \quad (154)$$

In view of the points made above with respect to the excess demand functions of our model, the following observations may be made regarding (154):

1. None of the "coefficients" in (154) are constants. With f_{jp} , F_{jp}^* and $E[\tilde{s}_{jp}]$ variable, they are necessarily variable. With f_{jp} a random variable, these supposed coefficients are also random variables.
2. Since the "coefficients" of (154) are functions of f_{jp} , (154) is not a reduced form.
3. (10) as an approximation of (154) has omitted variables.

If our model is the "true" model then in view of these points, the estimation procedures of the modern theory literature, in any of their forms, do not have the desirable properties which are the justification for their employment.

CHAPTER VI. FOOTNOTES

1. H. Stoll, (1968) p. 58.
2. H. Stoll, (1968) p. 61: "If there is no ... risk aversion [$x^{jSp} \rightarrow \infty$]"
3. H. Stoll, (1968) p. 63.
4. In our model "triangular" transactions have been ignored. However, since we have shown that in this model, "dual" speculation, for example, cannot duplicate "triangular" speculation, a more general model would have X^{jSp} a function of exchange rates involving ALL currencies, not simply those involving currency j or currency p.
5. H. Stoll, (1968) p. 61.
6. H. Stoll, (1968) p. 61.

APPENDIX I

AN IMPLICATION OF PARETO OPTIMALITY FOR THE CONSTRAINED CHOICE PROBLEM

In this appendix we attempt to prove that with Pareto Optimality, in the distribution of the risky opportunities of this model, the SAME constraints in our optimization problem, must be binding for all individuals of the same country. We deal first with those choice variables which are constrained only in that they must be non negative. Associated with any such variable, A_i^K for example, there are the following necessary conditions for an optimum:

$$\frac{\partial E[U^K(w_i^K)]}{\partial A_i^K} \leq 0 \quad \text{for all individuals } K, \text{ and choice variables } i \quad (1)$$

$$A_i^K \geq 0 \quad \text{for all } K \text{ and } i \quad (2)$$

$$A_i^K \frac{\partial E[U^K(w_i^K)]}{\partial A_i^K} = 0 \quad \text{all } K \text{ and } i \quad (3)$$

where * denotes an optimal value, or "evaluated at the optimum."

As conditions for a Pareto Optimal risk allocation we have, using

state of the world notation ¹:

$$p^K U^K [W^K(\theta)] = U^1 [W^1(\theta)] \quad (4)$$

for all individuals K and states θ

where $U^K [W^K(\theta)] \equiv \frac{\partial U^K(\tilde{W}^K)}{\partial \tilde{W}^K}$ evaluated at $\tilde{W} = W(\theta)$

Again using state of the world notation:

$$E[U^K(W^K(\theta))] \equiv \sum_{\theta} U^K(W^K(\theta))p(\theta)$$

thus

$$\frac{\partial E[U^K(\theta)]}{\partial c_i^K} = \sum_{\theta} U^K c_i(\theta)p(\theta) \quad (5)$$

where c_i^K is any choice variable and:

$$\frac{\partial W^K(\theta)}{\partial c_i^K} = c_i(\theta)$$

Substituting the P-0 condition (4) into (5) provides:

$$p^K \frac{\partial E[U^K(\theta)]}{\partial c_i^K} = \sum_{\theta} U^1 c_i(\theta)p(\theta)$$

¹ see p. 107 "Theory of Financial Markets" by Jan Mossin, Prentice Hall, New Jersey, 1973.

or

$$p^K \frac{\partial E[U^K(\theta)]}{\partial c_i^K} = \frac{\partial E[U^1(\theta)]}{\partial c_i^1} \quad (6)$$

for all individuals K and choice variables c_i

Now if A_i^1 is positive (i.e. the constraint $A_i^1 \geq 0$ is not binding for person 1) we have, from (3):

$$\frac{\partial E[U^1(w_*^1)]}{\partial A_i^1} = 0$$

and from (6) (noting $p^K > 0$):

$$\frac{\partial E[U^K(w_*^K)]}{\partial A_i^K} = 0 \quad \text{for all K}$$

which implies that for no person K is the constraint on A_i binding. Since the selection of person 1 is arbitrary, it may be stated that if a non negativity constraint is not binding for one person, then it is not binding for anyone. It follows immediately that if a non negativity constraint is binding for one person, it is binding for all. We next attempt to prove, that this result also holds for the remaining constraints of our optimization problem.

Other than non negativity constraints, we have as constraints:

$$- I_i^K \leq Y_i^K \frac{r_i^f}{s_i} \leq E_i^K \quad \text{for all } i$$

These constraints may be rewritten:

$$- (I_i^K + \bar{Y}_i^K) \leq 0$$

$$(\bar{Y}_i^K - E_i^K) \leq 0 \quad \text{for all } i$$

where $\bar{Y}_i^K \equiv Y_i^K \frac{r_i^f}{s_i}$

For the variables within these constraints we also require:

$$I_i^K, E_i^K \geq 0 \quad \text{for all } i$$

which may be rewritten:

$$- I_i^K \leq 0$$

$$- E_i^K \leq 0 \quad \text{for all } i$$

The individual is assumed to maximize a Lagrangian which looks in part like the following:

$$Z^K = E[U^K(W^K)] + y_{1i}^K (I_i^K + \bar{Y}_i^K) - y_{2i}^K (\bar{Y}_i^K - E_i^K) + y_{3i}^K I_i^K$$

$$+ y_{4i}^K E_i^K + \dots$$

where the y_{ji}^K are multipliers.

For a maximum, we have amongst the necessary conditions:

$$y_i^K \geq 0 \quad \text{for all } K \text{ and } i \quad (7)$$

$$\frac{\partial Z^K}{\partial \bar{Y}_i^K} = \frac{\partial E[U^K(\bar{W}^K)]}{\partial \bar{Y}_i^K} + y_{1i}^K - y_{2i}^K = 0 \quad (8)$$

$$\frac{\partial Z^K}{\partial I_i^K} = \frac{\partial E[U^K(\bar{W}^K)]}{\partial I_i^K} + y_{1i}^K + y_{3i}^K = 0 \quad (9)$$

$$\frac{\partial Z^K}{\partial E_i^K} = \frac{\partial E[U^K(\bar{W}^K)]}{\partial E_i^K} + y_{2i}^K + y_{4i}^K = 0 \quad (10)$$

for all k and i

Invoking Pareto Optimality provides (6). Using (6), (8), (9) and (10) provides:

$$p^K [y_{1i}^K - y_{2i}^K] = y_{1i}^1 - y_{2i}^1 \quad (11)$$

$$p^K [y_{1i}^K + y_{3i}^K] = y_{1i}^1 + y_{3i}^1 \quad (12)$$

$$p^K [y_{2i}^K + y_{4i}^K] = y_{2i}^1 + y_{4i}^1 \quad (13)$$

We will first assume that there is at least one individual, whom

we shall index l , who is an exporter to country i . It will then be shown using this assumption that the same and only the same constraints on choices involving country i are binding for all individuals of country j . We shall then assume the existence of at least one individual who is an importer from country i . Using this assumption we will again prove the assertion. Without either variety of trader the proof is immediate:

If neither a country j exporter to country i , nor an importer from country i exists,

$$E_i^K = I_i^K = 0 \text{ for all } K$$

and using the constraint $-I_i^K \leq Y_i^K \leq E_i^K$ (14)

$$Y_i^K = 0 \text{ for all } K$$

Assume that there is at least one country j exporter to country i .

Index this individual, l . Then from the complementary slackness condition,

$$y_{4i}^K E_i^K = 0 \text{ for all } K$$

we have $y_{4i}^l = 0$ and (13) becomes:

$$p^K [y_{2i}^K + y_{4i}^K] = y_{2i}^l \quad (15)$$

There are also complementary slackness conditions:

$$y_{1i}^K (I_i^K + \bar{Y}_i^K) = 0$$

for all K

$$y_{2i}^K (\bar{Y}_i^K - E_i^K) = 0$$

We have assumed $E_i^1 > 0$. Either $E_i^1 \neq \bar{Y}_i^1$ in which case $y_{2i}^1 = 0$, or $E_i^1 = \bar{Y}_i^1$ and we have $y_{1i}^1 = 0$. Thus we have

$$y_{1i}^1 y_{2i}^1 = 0 \quad (16)$$

Multiplying (15) by y_{1i}^1 provides:

$$\rho^K [y_{2i}^K y_{1i}^1 + y_{4i}^K y_{1i}^1] = y_{2i}^1 y_{1i}^1 = 0 \quad \text{using (16)}$$

Since all y 's ≥ 0 and $\rho^K > 0$ we have:

$$y_{2i}^K y_{1i}^1 = 0 \quad (17)$$

Adding (11) and (15) gives:

$$\rho^K [y_{1i}^K + y_{4i}^K] = y_{1i}^1$$

Multiplying this equation by y_{2i}^K :

$$\rho^K [y_{1i}^K y_{2i}^K + y_{4i}^K y_{2i}^K] = y_{1i}^1 y_{2i}^K = 0 \quad \text{using (17)}$$

Since y 's ≥ 0 and $\rho^K > 0$ we have:

$$y_{1i}^K y_{2i}^K = 0 \quad (18)$$

Multiplying (11) by y_{1i}^K and using (18)

$$\rho^K y_{1i}^{K^2} = y_{1i}^K y_{1i}^1 - y_{1i}^K y_{2i}^1$$

$$\text{Thus } y_{1i}^K > 0 \rightarrow y_{1i}^1 > 0 \quad (19)$$

Multiplying (11) by y_{1i}^1 and using (16) and (17) provides:

$$\rho^K y_{1i}^K y_{1i}^1 = y_{1i}^1{}^2$$

$$\text{Thus } y_{1i}^1 > 0 \rightarrow y_{1i}^K > 0 \quad (20)$$

Combining (19) and (20):

$$y_{1i}^1 > 0 \leftrightarrow y_{1i}^K > 0 \quad \text{for all } K \quad (21)$$

This result implies that the constraint associated with y_{1i} is either binding for everyone, or binding for no-one.

Using (21) and (16):

$$y_{1i}^K > 0 \rightarrow y_{1i}^1 > 0 \rightarrow y_{2i}^1 = 0$$

and since y 's ≥ 0

$$y_{2i}^1 > 0 \rightarrow y_{1i}^K = 0$$

Thus:

$$y_{1i}^K y_{2i}^1 = 0 \quad (22)$$

Multiplying (11) by y_{2i}^1 and using (22) and (16) provides:

$$-\rho^K y_{2i}^K y_{2i}^1 = -y_{2i}^1{}^2$$

$$\text{Thus } y_{2i}^1 > 0 \rightarrow y_{2i}^K > 0 \quad (23)$$

Multiplying (11) by y_{2i}^K and using (18) and (17):

$$- y_{2i}^K{}^2 = -y_{2i}^K y_{2i}^1$$

$$\text{Thus } y_{2i}^K > 0 \rightarrow y_{2i}^1 > 0 \quad (24)$$

Combining (23) and (24):

$$y_{2i}^K > 0 \leftrightarrow y_{2i}^1 > 0 \quad (25)$$

This result implies that the constraint associated with y_{2i} , is either binding for everyone, or binding for no-one.

Using (11), (21), (25) and (22):

$$p^K [y_{1i}^K - y_{2i}^K] = y_{1i}^1 - y_{2i}^1 \quad (11) \text{ rptd.}$$

$$y_{1i}^1 > 0 \leftrightarrow y_{1i}^K > 0 \quad (21) \text{ rptd.}$$

$$y_{2i}^K > 0 \leftrightarrow y_{2i}^1 > 0 \quad (25) \text{ rptd.}$$

$$y_{1i}^K y_{2i}^1 = 0 \quad (22) \text{ rptd.}$$

We have:

$$p^K y_{1i}^K = y_{1i}^1 \quad (26) \text{ and } p^K y_{2i}^K = y_{2i}^1 \quad (27)$$

Then from (12) $p^K [y_{1i}^K + y_{3i}^K] = y_{1i}^1 + y_{3i}^1$ and from (26) we have:

$$p^K y_{3i}^K = y_{3i}^1 \quad \text{for all } K$$

Since $\rho^K > 0$

$$y_{3i}^K > 0 \iff y_{3i}^1 > 0 \quad (28)$$

Thus the constraint associated with y_{3i} is either binding for everyone or binding for no-one.

Using:

$$\rho^K [y_{2i}^K + y_{4i}^K] = y_{2i}^1 \quad (25) \text{ rptd.}$$

and (27) we have:

$$y_{4i}^K = 0 \quad \text{for all } K. \quad (29)$$

Recall that by assumption, the constraint associated with y_{4i} is not binding for individual 1, (29) indicates that it is not binding for anyone else either.

Collecting the results ensuing from the assumptions of Pareto Optimality and the existence of at least one country j exporter to country i , we have:

$$y_{1i}^1 > 0 \iff y_{1i}^K > 0 \quad (21) \text{ rptd.}$$

$$y_{2i}^1 > 0 \iff y_{2i}^K > 0 \quad (25) \text{ rptd.}$$

$$y_{3i}^1 > 0 \iff y_{3i}^K > 0 \quad (28) \text{ rptd.}$$

$$y_{4i}^K = 0 \quad (29) \text{ rptd.}$$

for all K .

Together, these results imply that the same constraints are binding for all individuals of country j , with respect to choices involving country i . In like manner, the same conclusion may be made from the assumption of at least one country j importer from country i .

Index the importer individual l . From complementary slackness:

$$y_{3i}^K I_i^K = 0 \quad \text{for all } K$$

Since by assumption $I_i^1 > 0$ we have:

$$y_{3i}^1 = 0$$

and (12) becomes:

$$p^K [y_{1i}^K + y_{3i}^K] = y_{1i}^1 \quad (30)$$

Again consider the complementary slackness conditions:

$$y_{1i}^K (I_i^K + \bar{Y}_i^K) = 0$$

$$y_{2i}^K (\bar{Y}_i^K - E_i^K) = 0$$

Either $-I_i^1 = \bar{Y}_i^1$ in which case $y_{2i}^1 = 0$, or $-I_i^1 \neq \bar{Y}_i^1$ and we have $y_{1i}^1 = 0$.

Thus:

$$y_{1i}^1 y_{2i}^1 = 0 \quad (31) \quad (26) \text{ rptd.}$$

Multiplying (30) by y_{2i}^1 provides:

$$p^K [y_{1i}^K y_{2i}^1 + y_{3i}^K y_{2i}^1] = y_{1i}^1 y_{2i}^1 = 0 \quad \text{using (31)}$$

$$\text{Thus } y_{1i}^K y_{2i}^1 = 0 \quad (32) \quad (22) \text{ rptd.}$$

Subtracting (11) from (30):

$$\rho^K [y_{3i}^K + y_{2i}^K] = y_{2i}^1 \quad (32)$$

and multiplying by y_{1i}^K

$$\rho^K [y_{3i}^K y_{1i}^K + y_{2i}^K y_{1i}^K] = y_{2i}^1 y_{1i}^K = 0 \text{ using (32)}$$

Thus

$$y_{2i}^K y_{1i}^K = 0 \quad (33) \quad (18) \text{ rptd.}$$

Multiplying (32) by y_{1i}^1

$$\rho^K [y_{3i}^K y_{1i}^1 + y_{2i}^K y_{1i}^1] = y_{2i}^1 y_{1i}^1 = 0 \text{ using (31)}$$

Thus

$$y_{2i}^K y_{1i}^1 = 0 \quad (34) \quad (17) \text{ rptd.}$$

Having rederived (16), (17), and (18), we may now invoke the procedure following the original appearance of equation (18), and state:

$$y_{1i}^1 > 0 \iff y_{1i}^K > 0 \quad (35) \quad (21) \text{ rptd.} \\ \text{for all } K$$

Similarly, we may invoke the procedure following (21) and state immediately:

$$y_{2i}^K > 0 \iff y_{2i}^1 > 0 \quad (36) \quad (25) \text{ rptd.}$$

and

$$\rho^K y_{2i}^K = y_{2i}^1 \quad (37) \quad (27) \text{ rptd.}$$

Using (32) $p^K [y_{3i}^K + y_{2i}^K] = y_{2i}^1$ and (37) provides:

$$y_{3i}^K = 0 \quad \text{for all } K \quad (38)$$

Using (13) $p^K [y_{2i}^K + y_{4i}^K] = y_{2i}^1 + y_{4i}^1$ and (37) provides:

$$p^K y_{4i}^K = y_{4i}^1$$

Thus

$$y_{4i}^K > 0 \iff y_{4i}^1 > 0 \quad (39)$$

Collecting results we have:

$$y_{1i}^1 > 0 \iff y_{1i}^K > 0 \quad (35) \text{ rptd.}$$

$$y_{2i}^1 > 0 \iff y_{2i}^K > 0 \quad (36) \text{ rptd.}$$

$$y_{3i}^K = 0 \quad (38) \text{ rptd.}$$

$$y_{4i}^1 > 0 \iff y_{4i}^K > 0 \quad (39) \text{ rptd.}$$

The interpretation of these results is as before, the same constraints are binding for all individuals of country j with respect to choices involving country i .

APPENDIX II

DEFINITIONS OF TERMS APPEARING IN FINAL WEALTH

All of the following pertain to "person K of country j" however, the j and k have been suppressed.

- W_j final wealth denominated in units of j.
- W_0 initial wealth.
- O_i #j raised by borrowing in country i.
- M_i #j raised by going short in the risky asset of country i.
- A_i #j allocated to the risk free asset of country i.
- B_i #j allocated to the risky asset of country i.
- E_i #i accruing from exports to country i.
- I_i #i for which the individual is liable arising from imports from country i.
- R_i number of units of E_i repatriated by borrowing in country i, exchanging at current spot and lending domestically.
- Q_i number of units of I_i paid by borrowing domestically, exchanging at current spot and lending in country i.
- u_i # i accruing from lending in country i which are left exposed for repatriation at future spot.

- w_i #i sold forward to hedge the uncertain income from B_i .
- p_i #i for which the individual is liable through borrowing in country i, which are left exposed to be repaid at future spot.
- k_i #i bought forward to hedge the uncertain liability arising from M_i .
- e_i #i accruing from exports to country i, which are left exposed for repatriation at future spot.
- b_i #i for which the individual is liable as a result of imports from country i, which are left exposed for repatriation at future spot.
- r_j^f risk free rate of country j, plus one.
- \tilde{r}_j risky rate of return of country j, plus one.
- f_i #j paid forward for one unit of i.
- s_i #j paid spot currently for one unit of i (the beginning of the period spot rate).
- \tilde{s}_i s_i governing at the end of the period in the absence of the imposition of controls.
- \tilde{s}_i^i s_i governing at the end of the period in the event that only country i imposes controls.
- \tilde{s}_i^j s_i governing at the end of the period in the event that only country j imposes controls.
- \tilde{s}_i^{ji} s_i governing at the end of the period in the event that BOTH countries i and j impose controls.
- \tilde{c}_i^j the end of period de facto currency j price of a unit of i, (an exchange rate), for those transactions where the sale of j is controlled.
- \tilde{c}_i^i the end of period de facto currency j price of a unit of i for those transactions where the sale of i is controlled.

r_i^E the uncertain cost, in units of j , of generating one unit of E_i .

r_i^I the uncertain return, in units of j , on one unit of I_i .

Binomially distributed random variables used to reflect the influence of controls.

$D_j = 1$ if j (or if j and i) impose controls.
 $= 0$ otherwise.

$D_i = 1$ if i (or if i and j) impose controls.
 $= 0$ otherwise.

$P_{ji} = 0$ if controls are imposed by country j and/or country i .
 $= 1$ otherwise (i.e. in the ABSENCE of any controls.)

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