COMPARISON OF ALTERNATIVE MODELS OF THE SHORT-TERM INTEREST RATE

by

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ABSTRACT

The paper proposes a procedure for testing the alternative continuous time models of short term riskless interest rates. Parameters estimation and models comparison are presented using the Generalized Method of Moments. An empirical research to LIBOR in US dollar is given and found that the volatility of interest rate changes is to be less sensitive to the interest rate levels in contrast to previous findings. In addition the Brennan-Schwartz model is suggested to be superior to the others in term of data fit under daily observations, and CIR SR model cannot be rejected.

Keywords: continuous time models, short-term interest rate, CKLS, GMM, stochastic differential equation, conditional volatility

DEDICATION

I dedicate this paper to my families. Thank you for your patience, love and support throughout last one year. I am grateful too for the understanding and advice from my parents and my brother. You make it all worthwhile.

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1 INTRODUCTION

The stochastic dynamics of short term riskless interest rates is a crucial element in financial market and in pricing various interest rate related securities. Interest rate modeling has developed intensively but confused model selection choice still exists for academic researchers and practitioners. An enormous amount of work has been directed towards modeling and estimation of the short-term interest rate dynamics. The more popular models currently used in a wide range specify the single factor model in continuous time setting, which provides riskless instantaneous spot rate in short term interval (r_t) as the variable. Examples of models that have been proposed in the literature are: Merton(1973), Brennan and Schwartz (1977, 1979, 1980), Vasicek (1977), Dothan (1978), Cox, Ingersoll and Ross (1980, 1985), Constantinides and Ingersoll (1984), Shaefer and Shwartz (1984), Sundaresan (1984), Feldman (1989), Longstaff (1989), Hull and White (1990), Black and Karasinski (1991), and Longstaff and Shwartz (1992).

One of the key points in this area is how these models compare in terms of their ability to capture the actual behavior of the short-term riskless rate. Initiated with the paper of Chan et al. (1992), hereafter CKLS examined various one-factor continuous time stochastic models via the generalized method of moments (GMM).

This paper also takes the CKLS models as a starting point for analyzing US LIBOR as the short-term interest rate. The paper aims to illustrate how various models operate and to show the difference between US LIBOR and Treasury bill resulted in CKLS. We propose a consistent econometrics methodology for testing the models and conclude the justification in twofold: on the one hand, the US LIBOR is not mean reversion, which is consistent with CKLS results; on the

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other hand, the empirical results can not support the opinion that the volatility of the interest rate changes is highly sensitive to the interest rate level.

The paper is organized as follows. Section 2 presents the frameworks for the interest rate models and the relevant features of general interest rate models. Section 3 provides the literature review. Section 4 explains the estimation methodology and describes the interest rate data. Section 5 addresses the relative performance of the models under different parametric constraints and gives the comparisons. Section 6 concludes.

2 CONTINUOUS TIME INTEREST RATE MODELS

This section deals with term structure interest rate models with one single variable, which are concluded in CKLS and the 8 nested models. Alternative assumptions, parameters restriction and characteristics are given.

Stochastic differential equations (SDE) have been used to define the dynamics of interest rate processes. The general format of a one-factor diffusion model is as below

$$dr_t = (\alpha + \beta r_t)dt + \sigma r_t^\gamma dW_t \qquad (2.1)$$

where r_t represents instantaneous short term risk-free interest rate, dt is very short term interval and dW_t ($dW_t = \varepsilon\sqrt{dt}, \varepsilon \sim N(0,1)$) is a standard Brownian motion. Parameters α and β specify the drift (or instantaneous mean) and mean reversion that controls the speed at which the interest rate converges to its unconditional mean, respectively. σ is the instantaneous volatility parameter, while the constant γ measures the sensitivity of volatility (or the elasticity of volatility) with respect to the interest rate level r_t. In this model, the volatility of the interest rate depends upon the previous interest rate level through the sensitivity parameter γ.

By imposing a set of restrictions on ($\alpha, \beta, \sigma, \gamma$), a range of different term structure models for the short term interest rate can be obtained. These models are listed in Table 1 with the corresponding parameter restriction.

Nesting from unrestricted model of CKLS (1992), the 8 restricted models summarized in Table 1 have following specifications:

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Model 1: Merton

Merton model defines that the stochastic process for the risk free rate is simply a Brownian motion with drift.

Model 2: Vansicek

Vansicek model add the measure of mean reversion based on Merton and is the Ornstein-Ulenbeck process used for deriving an equilibrium model of discount bond prices.[1]

Model 3: CIR SR

CIR SR model which is the square root (SR) process appeared in Cox, Ingersoll, and Ross (CIR) term structure model, implied that the conditional volatility of the changes in interest rate is proportional to interest rate level.

Model 4: Dothan

Dothan model is a Brownian motion with conditional volatility depending upon previous r_t which can be nested within Brennan-Schwartz model by the parameter $\alpha = 0$.

Model 5: GBM

GBM model is geometric Brownian motion process of Black and Scholes (1973), and adds the measure of mean reversion based on Dothan which can be nested within Brennan-Schwartz model by the parameter $\beta = 0$.

Model 6: Brennan-Schwartz

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[1] Ahangarani concluded that Vansicek model can specially derived by equilibrium approach. See, Pouyan Mashayekh Ahangarani, "An Empirical Estimation and Model Selection of the Short-Term Interest Rates," University of South Califonia, http://www.usc.edu/its/web/getting_started/ppages.html

Brennan-Schwartz implies that the conditional volatility proportionally depends upon the previous r_t. This process was used in deriving a numerical model for convertible bond prices and discount bond option prices.[2]

Model 7: CIR VR

CIR VR which is the variable-rate (VR) process appeared in Cox, Ingersoll, and Ross (CIR) term structure model.

Model 8: CEV

CEV is the constant elasticity of variance process.

Before presenting the estimation of one factor short term interest rate models, there are some features of general interest rate movement to provide us some intuitive form for an interest rate model.

1. Interest rates should not be allowed to become negative. Otherwise, there is opportunity of arbitrage.

2. The volatility of yields at different interest level varies. In particular, low level rates do not vary as much as high level rates.

3. Interest rates are mean-reverting. Interest rate increases tend to be followed by rate decrease; conversely, when rates drop, they tend to be followed by rate increases.

4. Based on the results reported in CKLS, the volatility of interest rates should be proportional to the level of the rate.

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[2] CKLS (1992) empirically concluded the high sensitivity of volatility elasticity. See, Chan, Karolyr, Longstaffl. Sanders, "An Empirical Comparison of Alternative Models of the Short-Term Interest Rate," *The Journal of Finance* 47, No.3 (1992): 1209-1227

Although no known model captures all of the characteristics mentioned above, models would be ranked based on the use of model and the number of listed conditions that are satisfied.

3 LITERATURE REVIEW

CKLS (1992) exploited a common econometric framework in which different well-know

models could be nested and evaluate their performance. They applied the generalized method of

moments (GMM) to estimate the parameters of the one-factor continuous-time models of the

short-term interest rate. Their most important finding is that the interest rate volatility is highly

sensitive to the level of the interest rate.

The observations they studies are US Treasury bill yield in monthly frequency and cover

the period from June 1964 to December 1989 (307 observations in total), which proved to include

the structural breaks in the interest rate process.[3] The more appropriate models that capture the

US short-term interest rate movements are those that allow the conditional volatility of short

interest rate changes to be highly dependent on the level of the short rate.

With regard to the parameter, the value of elasticity of volatility, γ is the most

important feature to differentiate interest rate models. They showed the unconstrained estimate of

γ is 1.5 with t-statistics (5.95) significant, the models allowed $\gamma \geq 1$ captured the dynamics of

the short-term interest rate better than those which required $\gamma < 1$. Accordingly, the Merton,

Vasicek and CIR SR models whose $\gamma < 1$ are found to be misspecified with χ^2 test failing to

reject the remaining models. The unrestricted model explains 2.59% of the total variation of the

actual yield changes and performs best. However, the non-rejected restricted models show better

perform in predicting volatility (squared yield changes) ranging from 13.29% to 20.49%.

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3 A structural break is identified during the period of October 1979 to the early 1980s when Federal Reserve monetary
policy was changing.

Subsequent researches either extended the basic CKLS formulation or considered alternative estimation procedures.

Nowman (1997) applied the Gaussian estimation techniques developed by Bergstrom (1990) for continuous time stochastic differential equations. The approach is based on the idea that any continuous time martingale can be written as Brownian motion after a suitable time change. For any $h \geq 0$, a discrete Gaussian model is:

$$r(t+h) = \frac{\alpha}{\beta}(e^{\beta h} - 1) + e^{\beta h} r(t) + M(h) \tag{3.1}$$

where M(h) construct a Brownian motion process as

$$M(h) = \int_0^h \sigma e^{\beta(h-\tau)} r^{\gamma}(t+\tau) dB(\tau). \tag{3.2}$$

The exact discrete model with Gaussian disturbances can be estimated directly by maximum likelihood which has the advantage that it produced an exact maximum likelihood estimator. Nowman estimated CKLS models for one-month sterling interbank middle rate from March 1975 to March 1995 monthly data containing 242 observations. The U.K. data set producing the value of γ is 0.2898 and t-statistic is 1.7620 for unrestricted model, and just reject Merton model at 95% confident level. The main conclusion was that British interest rate has a same feature with US interest rate that they both are not mean reversion, but remained significant difference in the value of γ. Broadly speaking, GMM and Gaussian both are methods to employ a discrete time approximation to the continuous system and conduct nonlinear regression or maximum likelihood.

Irrespective of the estimation methodology one employs, another significant issue relates to what data is used to proxy the instantaneous interest rate. Episcopos (2000) subsequently

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examined the parameters of the CKLS for the 1-month interbank rate in ten countries applying Gaussian methodology, providing the same time interval and frequency with CKLS. It was interesting that the parameters estimation results of Episcopos are generally lower than CKLS. The main result is that interest rate volatility is not as sensitive to the interest rate level as stated in previous literatures. In particular, parameter γ in the general specification is less than unity in most countries of the sample. In addition, comparison of the various one-factor models leads to the conclusion that the CEV model is superior in terms of data fit. These results are in contrast to previous existing studies in the literature.

4 THE METHODOLOGY AND DATA

4.1 The Econometric Approach

In this section Generalized Method of Moments (GMM) is described to estimate the equation 2.1. To ensure comparability with previous studies, the methodology is based on the setting in CKLS to estimate the parameters of the interest rate models and examine their explanatory power for the dynamic behavior of short-term interest rates.

The approach focus on the unrestricted process is explained here, and the same approach can then be used for the nested models after imposing different parameters restriction. Following CKLS, a discretization of Equation 2.1 yields the discrete-time econometric specification to estimate the parameters of the continuous-time model.

$$r_{t+1} - r_t = \alpha + \beta r_t + \varepsilon_{t+1} \tag{4.1}$$

$$E[\varepsilon_{t+1}] = 0, \qquad E[\varepsilon_{t+1}^2] = \sigma^2 r_t^{2\gamma} \tag{4.2}$$

Define θ is the parameter vector with elements of $\alpha, \beta, \sigma^2, \gamma$. Given the vector of proposed moment conditions $f_t(\theta)$ is

$$f_t(\theta) = \begin{bmatrix} \varepsilon_{t+1} \\ \varepsilon_{t+1} r_t \\ \varepsilon_{t+1}^2 - \sigma^2 r_t^{2\gamma} \\ (\varepsilon_{t+1}^2 - \sigma^2 r_t^{2\gamma}) r_t \end{bmatrix} \tag{4.3}$$

The population orthogonality conditions to approve the null hypothesis implied in Equation 4.1 and 4.2 are

$$E \begin{bmatrix} \varepsilon_{t+1} \\ \varepsilon_{t+1} r_t \\ \varepsilon_{t+1}^2 - \sigma^2 r_t^{2\gamma} \\ (\varepsilon_{t+1}^2 - \sigma^2 r_t^{2\gamma}) r_t \end{bmatrix} = 0 \qquad (4.4)$$

and the sample counterparty for T observations is

$$m_T(\theta) = \frac{1}{T} \sum_{t=1}^{T} f_t(\theta) \qquad (4.5)$$

then choose the parameters which minimize the quadratic

$$J_T(\theta) = m_T'(\theta) W_T(\theta) m_T(\theta) \qquad (4.6)$$

where W is a 4 by 4 vector positive definite weighting matrix as

$$W = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (4.7)$$

If the model is "just-identified" there is one parameter for each restriction to satisfy the restriction, otherwise, it is impossible to set every moment to zero if model is over-identified. The test statistic is distributed χ^2 under the null, which measures the goodness-of-fit for the model. In addition, volatility plays an important role in application of term structure models and is examined as a direct measure to differentiate the ability of models.

GMM is applied extensively in a wide range due to several advantages of its properties. Firstly, weak assumption is required for hypothesis testing over other estimation. It has no requirement of normal distribution for the samples and supplies consistent estimates even without normality. Therefore, "nests" almost all other common estimators. Secondly, compared with

Gaussian estimation and other methods, it has no assumption that the variance of the stochastic variables remains constant between discrete observations. Thirdly, GMM provides asymptotic properties for estimators to encompass many of the estimators usefully and one doesn't have to derive each estimator property separately. Besides, GMM provides an alternative when MLE doesn't work. However, GMM only works quite well for large sample but does not provide an answer when the sample size is small. The nonlinear GMM estimation in this paper is implemented by Cliff's GMM Library (2003).

4.2 The Data

The data are the 1 month US LIBOR which is collected from British Bankers' Associate. The LIBOR is officially a daily reference rate based on the interest rates at which banks offer to lend funds to other banks changing throughout the day and used as a benchmark to price derivative or capital market transaction. The data for empirical research are in daily form and cover the period from January 2, 1986 to June 29, 2006 in daily frequency, providing 5200 observations.[4]

In addition, the monthly data, which are obtained through choosing the first daily rate from daily data, are provided to double check the parameters estimation and prove the power of explanation. The monthly data cover the period from January 1986 to June 2006, providing 246 observations.

Table 2 shows the means, standard deviations, and first six autocorrelations of the daily rate of return and their changes for the 1-month LIBOR. The unconditional average level of the spot rate is 5.12% with a standard deviation of 2.22% during January 1986 to June 2006. The first six autocorrelations of the interest rate levels decay slowly, while those of the day-to-day changes

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[4] Although we didn't obtain accurate monthly LIBOR during 1964 to 1989 to ensure comparability to CKLS estimation results, the data in this paper avoid the structural break that can be used as a main reason to explain the differences.

are generally small and do not follow consistently positive or negative pattern. This offers some evidence that interest rates are stationary.

Comparing to CKLS summary results, daily LIBOR supplies lower values in means and standard deviations than T-bill whose yield summarized 6.715% and 2.675% separately. Since GMM has better estimation for large samples, the LIBOR in daily frequency has much more observation than that of T-bill in CKLS (307 observations). The higher autocorrelation coefficient can give the evidence as well, sequentially, the parameters estimated in the next section will be adjust to be monthly for comparing with CKLS in union unit.

5 THE EMPIRICAL RESULTS

In this section, two issues are now addressed: Parameter estimation and model selection based on a statistical criterion. Through estimating the unrestricted and the eight restricted interest rate models, the explanatory power of the nested model and that of the unrestricted model are compared.

5.1 Daily LIBOR Estimation

Table 4 gives the report of the parameter estimates, asymptotic t-statistics, and GMM minimized the criterion (χ^2) values for the unrestricted model and for the other eight models during January 1986 to June 2006. This χ^2 measure provides a goodness-of-fit test for the models, which express that the model is misspecified under a high value of χ^2. As shown from the t-statistics for each parameter, the models provide varied explanation power for interest rate changes. At 95% confident level, Vasicek and Merton models can be rejected, which are followed by the CEV, CIR SR, GBM, CIR VR, Brennan-Schwarz and Dothan models. At 90% confident level, Vasicek and Merton models can be rejected as well, which are followed by CEV, CIR SR, GBM, CIR VR, Brennan-Schwarz and Dothan models. The 8 nesting models produce the same ranking in their identification level for both 95% and 90% confident levels, in which Vasicek and Merton models are over identified and are followed closely by CEV model that has quite close value of χ^2 to the critical values.

Another important property of this ranking is observed that the models can be basically classified by γ values, that is, generally γ of those models been rejected have values equal to

zero ($\gamma = 0$). Moreover, differences in the minimized GMM criterion value (χ^2) between models with the same value of γ are generally smaller than differences in models with different values of γ. However, the t-statistics values of γ for both unrestricted and restricted models are 1.4297 and 1.3306 respectively and are weakly different to zero. The results appear that the relation between the volatility and the elasticity level of interest rate is another feature to classify the models but not strongly significant for any dynamic model of the short term riskless rate.

More estimation of the models provides a number of insights about the dynamics of the interest rate. First, in all unrestricted models and restricted models the mean reversion parameter β has the correct negative sign (except Merton has zero value) but it is not clearly significant. This means that the interest rates display weak tendency to return to the average trend level (unconditional mean). Similarly, the drift parameters are statistically weak for most models. Second, the conditional volatility coefficient of the models with $\gamma > 1$ provides much higher t-statistics than those with $\gamma < 1$ and statistically significant. Thirdly, the unconstrained estimate of γ is 0.9376 which is close to CEV model (0.9897) but lower than all the non-rejected restricted models. The t-statistics for γ is 1.4297. In particular, four of the nested models imply $1 \leq \gamma \leq 1.5$ and t-statistics is 1.3306. and the γ of unrestricted model is to close to zero, which express the conditional volatility increase equally with the level of interest rate, so called "level effect" and the fairly sensitivity. The results indicate that the conditional volatility of the process is not sensitive to the level of the short-term yield.

Further relative performance of the alternative models is to test their power for forecasting the interest rate changes. R^2 values in Table 5.1 provide the information about how well each model is able to forecast further level and volatility of the short-term rate. The first R^2 computed as the proportion of the total variation of the actual yield changes and describes the fit

of the various models for the actual yield changes. Except for Merton, Dothan, GBM, CIR VR and CEV, which have no explanatory power for interest rate changes, the models appear similar weak forecast ability. Mostly, the models can explain only 0.03% to 0.05% which is fairly poor forecast of the total variation in their rate changes. The second R^2 computed as the proportion of the total variation of the volatility. Except Merton, all other nested models provide the explanatory power that is higher than first R^2, which could explain 0.53% at highest but still appear poor ability for forecast. The results suggest that the dynamic of actual interest rates is difficult to forecast and keep to the stochastic process.

Compared with T-bill in CKLS, there are several differences in the parameters' estimate: (1) most of the parameters of LIBOR provide lower values than T-bill, (2) with respect of χ^2 values; different models are rejected for LIBOR and T-bill in all constricted models. (3) LIBOR supplied the value of γ that is clearly different with T-bill at unrestricted model, which address that the volatility is not highly sensitive to the interest rate level. (3) LIBOR and T-bill have consistent property of no mean reversion. Combining the statistics in χ^2 and R^2, Brennan-Schwartz is considered to the comparatively better model. However, one possible reason for CKLS higher estimation results is attribute to the structural break, which cause the great shift in interest rate and fail to provide accurate parameters.

5.2 Monthly LIBOR Estimation

Summary statistics and parameters estimates for monthly data is expressed in Table 3 and Table 5 in order to compare with the results from daily form estimation and check the results.

Firstly, from the results of χ^2, only Vasicek model can absolutely be rejected under 95% confident level, however, Brennan-Schwarz, Merton and CIR VR models address χ^2 values that

are quite close to the critical value. But under 90% confident level, Vasicek, CIR VR, Brennan-Schwarz, Merton and GBM models orderly provide χ^2 values that are excess to critical value and can be rejected weakly. The results refer to the instability under different data options, however, CIR SR model cannot be rejected which is contrast to the results in CKLS. Secondly, value of γ for unrestricted model is 0.3789 keeps being much lower than CKLS estimates. The t-statistics for both unrestrained model and nested models are not significant either, which prove that the conditional volatility is not sensitive to the level of interest rate level. Moreover, other parameters including α, β, σ^2 are consistently lower than CKLS estimates and t-statistics of β are not significant to express the property of non mean reversion.

On the other hand, the lower R^2 values appear that the forecast performance is poor which provide consistent results with the daily data estimates. Vasicek model provides best forecast of total variation relatively, and CIR VR has the highest value of R_2^2 and that of Merton and Vasicek models maintain the zero forecast.

Generally, the estimates in monthly frequency can support the empirical result from daily data even if they show some differences in rejecting models. Especially, the much lower value of γ supplies the conclusion that conditional volatility in not sensitive to the level of interest rate. On the other hand, the difference in monthly and daily estimates suggests the fluctuation of the model under different period.

6 CONCLUSION

Taking as a starting point the stochastic differential equation for the instantaneous spot rate of interest used by CKLS, in this paper, the eight alternative short term interest rate model are estimated and compared in order to determine which model is the best model to fit the actual British interest rate data. The estimation procedure proposed here is GMM.

The results of the test show that the popular models: Merton (1973) and Vasicek (1977) can be rejected and perform poorly relative to less well-known models under daily observations. It is found that there is no significant evidence to support the previous finding that the volatility of interest rate changes is highly sensitive to the interest rate level.

These results also provide that some features between US LIBOR and Treasury bill. First, they both are not mean reversion. Second, LIBOR supplies much lower estimates, especially; γ of unrestricted value is 0.9376 which is much lower than T-bill. And the statistics result cannot reject CIR SR model compared with CKLS. However, the nested models generally perform poor for LIBOR data than that of T-bill. Due to the structural break, it assumes that the estimation in CKLS is too high and actual movement of short term interest rate is difficult to forecast.

Table 1   Alternative one-factor short-term interest rate models and parameter relationship

	Model	α	β	γ
Merton(1973)	$dr_t = \alpha dt + \sigma dW_t$		0	0
Vasicek(1977)	$dr_t = (\alpha + \beta r_t)dt + \sigma dW_t$			0
CIR SR(1985)	$dr_t = (\alpha + \beta r_t)dt + \sigma r_t^{1/2} dW_t$			1/2
Dothan(1978)	$dr_t = \sigma r_t dW_t$	0	0	1
GBM	$dr_t = \beta r_t dt + \sigma r_t dW_t$	0		1
Brennan-Schwartz(1980)	$dr_t = (\alpha + \beta r_t)dt + \sigma r_t dW_t$			1
CIR VR(1980)	$dr_t = \sigma r_t^{3/2} dW_t$	0	0	3/2
CEV	$dr_t = \beta r_t dt + \sigma r_t^{\gamma} dW_t$	0		
CKLS(1992)	$dr_t = (\alpha + \beta r_t)dt + \sigma r_t^{\gamma} dW_t$			

Table 2   Summary Statistics (Daily Data)

Variables	N	Mean	Standard Deviation	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6
r_t	5200	0.0512	0.0222	0.9994	0.9987	0.9982	0.9976	0.9970	0.9964
$r_{t+1} - r_t$	5199	0.0000	0.0008	0.0588	0.0618	0.0025	0.0194	0.0147	0.0312

Table 3   Summary Statistics (Monthly Data)

Variables	N	Mean	Standard Deviation	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6
r_t	246	0.0514	0.0223	0.9887	0.9795	0.9684	0.9517	0.9341	0.9145
$r_{t+1} - r_t$	245	-0.0001	0.0033	-0.0876	0.0901	0.2390	0.0476	0.1008	0.1724

Table 4 Estimates of Alternative Models for the Short-Term Interest Rate (Daily Data)

	α [5] (t-statistics)	β (t-statistics)	σ^2 (t-statistics)	γ (t-statistics)	χ^2 Test (p-value)	d.f.	R_1^2	R_2^2
Unrestricted	0.0085 (1.6215)	-0.1945 (-1.5984)	0.0338 (0.5493)	0.9376 (1.4297)			0.0005	0.0033
Merton	0.0021 (1.0000)	0.0	0.0000[6]	0.0	11.6355 (0.0030)	2	0.0000	0.0000
Vasicek	0.0063 (1.5000)	-0.1374 (-1.1404)	0.0000[7]	0.0	10.4824 (0.0012)	1	0.0003	0.0000
CIR SR	0.0085 (1.3333)	-0.1797 (-1.4912)	0.0021	0.5	2.2224 (0.1360)	1	0.0004	0.0010
Dothan	0.0	0.0	0.0444 (5.2500)	1.0	2.7463 (0.4324)	3	0.0000	0.0030
GBM	0.0	-0.0169 (-0.3333)	0.0423 (5.0000)	1.0	2.6488 (0.2660)	2	0.0000	0.0030
Brennan-Schwarz	0.0085 (2.0000)	-0.1945 (-1.5862)	0.0465 (5.5000)	1.0000	0.0349 (0.8518)	1	0.0005	0.0036
CIR VR	0.0000	0.0000	0.6193 (5.0517)	1.5000	4.4832 (0.2138)	3	0.0000	0.0053
CEV	0.0000	-0.0148 (-0.2800)	0.0402 (0.4872)	0.9897 (1.3306)	2.6476 (0.1037)	1	0.0000	0.0029

[5] For consistent unit of time and comparability with CKLS parameters, α, β, σ^2 are adjusted into monthly by multiplying 21, which is (5200/(20*12+6)).
[6] Zero here is caused by digital limitation, so is not value of real "zero".
[7] Also see footnotes 6.

Table 5 Estimates of Alternative Models for the Short-Term Interest Rate (Monthly)

|	α (t-statistics)	β (t-statistics)	σ^2 (t-statistics)	γ (t-statistics)	χ^2 Test (p-value)	d.f.	R_1^2	R_2^2
Unrestricted	0.0077	-0.1778	0.0013	0.3789			0.0099	0.0171
	1.1324	-1.3729	1.0000	2.0219				
Merton	-0.0014	0.0	0.0001	0.0	5.5556	2	0.0	0.0
	-0.4516				0.0622			
Vasicek	0.0102	-0.23	0.0001	0.0	4.019	1	0.0165	0.0
	1.5	-1.7997			0.045			
CIR SR	0.0075	-0.1678	0.0024	0.5	0.3428	1	0.0088	0.0237
	1.1029	-1.3140			0.5582			
Dothan	0.0	0.0	0.0316	1	4.9737	3	0.0	0.0445
			5.0967742		0.1737			
GBM	0.0	-0.0268	0.0319	1	4.6942	2	0.0002	0.0454
		-0.4661	5.1452		0.0956			
Brennan-Schwarz	0.0074	-0.1532	0.0315	1	3.6572	1	0.0073	0.0443
	1.0882	-1.2025	5.0806		0.0558			
CIR VR	0.0	0.0	0.4146	1.5	7.2586	3	0.0	0.0665
			4.4153355		0.0641			
CEV	0.0	-0.0486	0.0013	0.3849	1.3166	1	0.0007	0.0176
		-0.8265	0.9286	2.0289	0.2512			

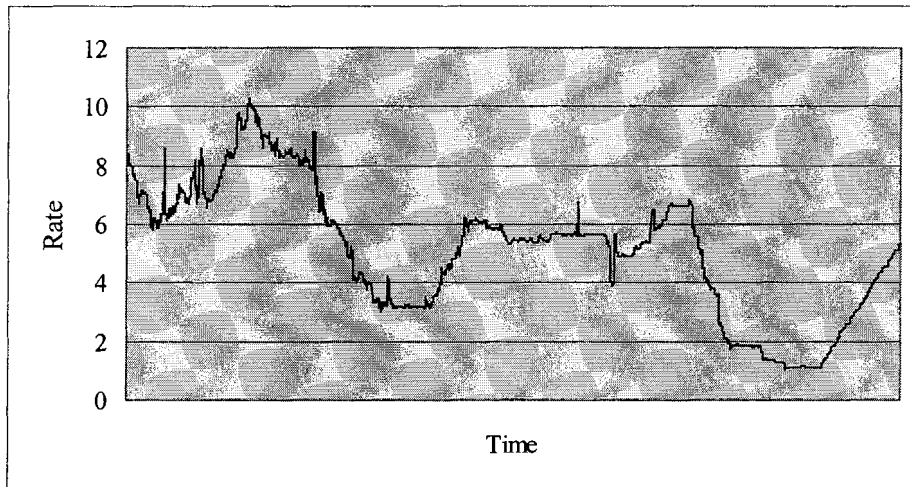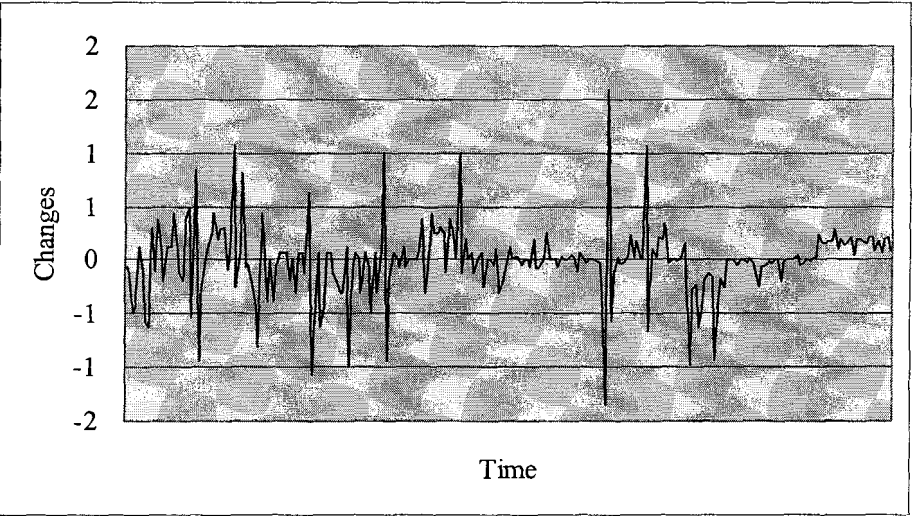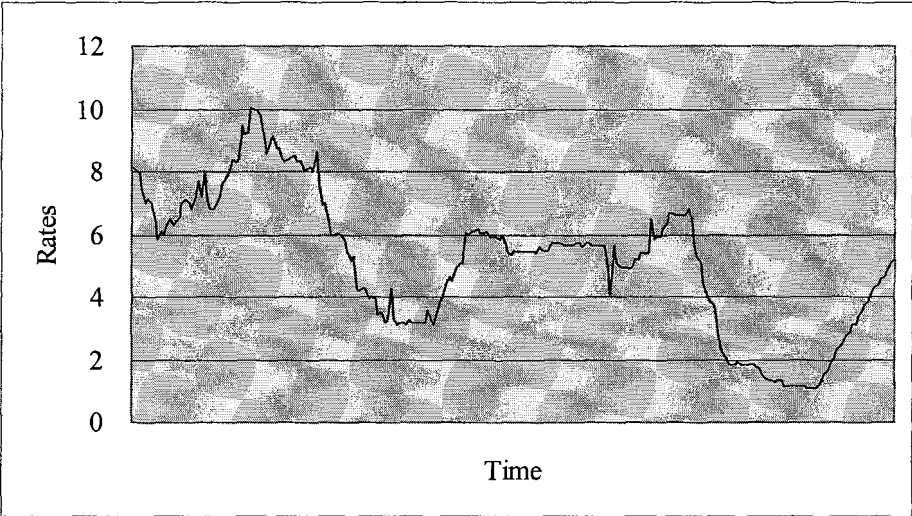**Figure 1   LIBOR Daily Rate and its Changes (Jan 86 to Jun 06)**
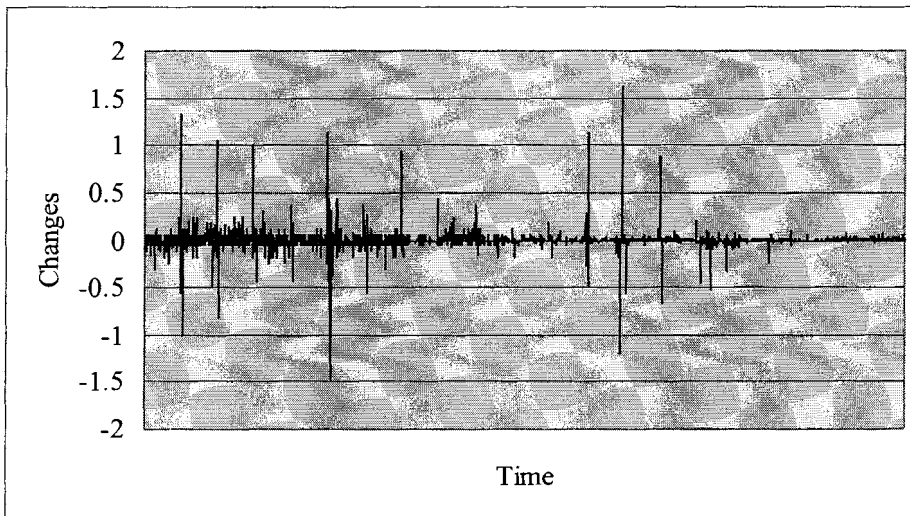
Figure 2   LIBOR Monthly Rate and its Changes (Jan 86 to Jun 06)

APPENDICES

Appendix A

Generalized Method of Moment Estimation (GMM)

Developed by Hansen (1982), GMM is an extension to the classical method of moments estimator. The basic idea is to choose parameters of the model so as to match the moments of the model to those of the data as closely as possible.

Starting from basic method of moments estimator, let x be a random variable, a moment of the distribution of x is $\gamma = E[g(x)]$, then a sample moment is $\hat{\gamma} = \frac{1}{N}\sum g(x)$.

For a continuous function g, typical moments are:

---First: $\gamma = E[x]$

---Second: $\gamma = E[x^2]$

---Third: $\gamma = E[x^3]$

---...

In most cases, the first moment is set to be mean, the second moment is variance (σ^2), the third moment is skewness, and the forth is kurtosis.

Method of moments is to estimate a population moments using the corresponding sample moment, accordingly, is to estimate a function of the population moment using the corresponding

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function of sample moments. It is an approach to obtain consistent estimators for the multiple regression models.

Hansen (1982) GMM, assumes r_i is $T \times 1$ observations, θ is $a \times 1$ unknown parameters, and $f(\theta, r_i)$ is $r \times 1$ vector functions. Then

$$f(\cdot):(v^r \times v^a) \to v^n$$

where r_i and $f(\theta, r_i)$ are random, and θ satisfies that

$$E(f(\theta, r_i)) = 0$$

and f satisfies that

$$r_T = (r_1', \cdots, r_T')'$$

so for all the observations

$$m(\theta, r_T) = \frac{1}{T} \sum_{i=1}^{T} f(\theta, r_i)$$

is the sample mean for $f(\theta, r_i)$, $m(\cdot)$ is a r vector. The basic idea of GMM is to minimize the quadratic

$$Q(\theta, r_T) = m(\theta, r_r)' W_n m(\theta, r_T)$$

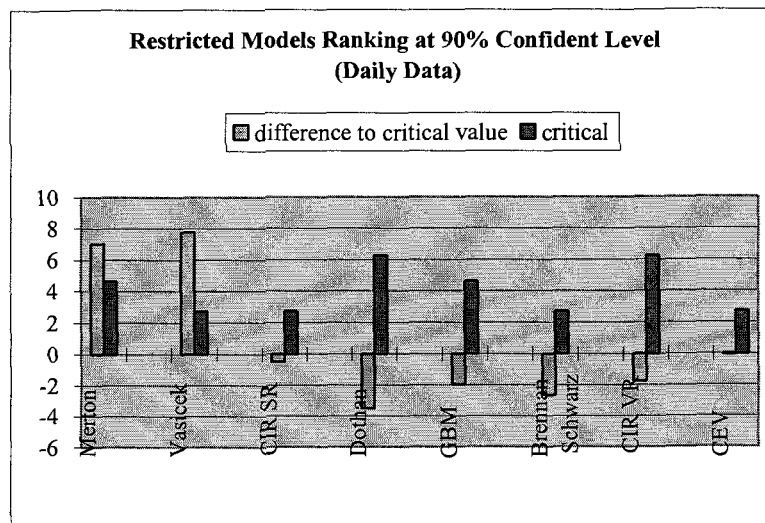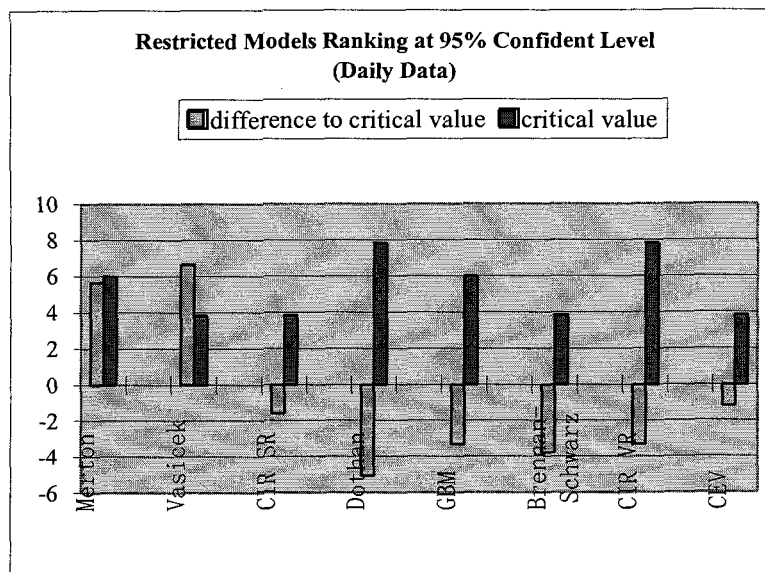where W_n is $r \times r$ weighted matrix (n=1,2,...)

Appendix B

Critical Value of Chi-Square and Ranking of the Models

From Robert S. Pindyck and Daniel L. Rubinfeld, Econometric Models and Economic

Forecasts (4th edition). Chi-square (χ^2) critical values are

(1) at 95% confident level: 3.84(d.f.=1); 5.99 (d.f.=2); 7.81(d.f.=3);

(2) at 90% confident level: 2.71(d.f.=1); 4.61 (d.f.=2); 6.25(d.f.=3);

Restricted Models Ranking at 95% Confident Level (Daily Data)

Restricted Models Ranking at 90% Confident Level (Daily Data)

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Stephen Bond, Anke Hoeffler, Jonathan Temple, "GMM Estimation of Empirical Growth Models," University of Oxford, http://ideas.repec.org/p/nuf/econwp/0121.html