PRICING THE VULNERABLE AMERICAN OPTIONS

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ABSTRACT

This thesis extends the models of Johnson and Stulz (1997), Klein (1996) and Klein and Inglis (2001) to price vulnerable American options. Most existing models mainly focus on the pricing of vulnerable European options, especially call options. This thesis focuses on vulnerable American options and especially put options. The model incorporates the default boundary at the time of maturity as in Klein and Inglis (2001), and allows the default barrier before maturity changes with the underlying asset price. The thesis compares the vulnerable American options with vanilla American options and studies some interesting properties of vulnerable American options under the assumption, which are quite different from those of vanilla American options.

DEDICATION

To my parents and my family.

(James) Jun Yang

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1 INTRODUCTION

Since in the over-the-counter (OTC) market derivative traders are not guaranteed by collaterals and daily mark-to-market that are managed by the exchange, counter-party credit risk does exist among these contracts. Considering the significantly increasing volume of OTC derivative trading in recent years, the study of the effects of credit risk on the prices of these so called vulnerable options is especially critical for the industry. A basic characteristic that needs to be taken into account by all participants is that an option that is subject to the credit risk would require higher expected yield, thus lowering the option price compared with a vanilla option with the same conditions. According to the Black-Scholes formula, investors tend to overvalue vulnerable options because the Black-Scholes PDE implies that the counterparty of options is default-free. Furthermore, it is not suitable for American options.

This paper constructs a pricing model for vulnerable American options whereas most prior researching papers focus on studying vulnerable European options. However, three papers — Johnson and Stulz (1987), Klein (1996), Klein and Inglis (2001) — play key role in the formation of the idea of this paper. Except the permission of early default and early exercise, the model of this paper preserves the sound feature of Klein and Inglis (2001) that the default boundary can be divided into two components: the fixed part represents the liabilities of the option issuer other than those arising from the vulnerable option; the variable part represents the potential payoff produced by the option itself. And it also assumes that the correlation between the total assets of the writer and the underlying asset of the option exists and the pay-out-ratio in the event of default is determined by the total assets as well as the total liabilities of the option writer, just as in Klein (1996) and Klein and Inglis (2001). The contribution of this paper is to provide a model to price vulnerable American options on which early default and early payoff is allowed and should be considered in pricing options. Therefore, the boundary condition of the model is generated not only at maturity but also at each time point prior to it, and the nominal claim of the option holder is the intrinsic value of the option at each point during contract period. The model extends Johnson and Stulz (1987), Klein (1996), and Klein and Inglis (2001) by allowing early default and early payoff, and employs the stochastic early default boundary where the variation of the boundary is determined by the potential payoff of the option. In addition, this paper primarily studies the pricing model and properties of put options while others mainly study call options.

This paper is organized as follows. Section 2 is literature review, which recommends the contributions and inadequacies of previous research in this field. Section 3 explains the model used to price vulnerable American options and the assumptions underlying the model, as well as the infinite difference method applied to value the option. In section 4, numerical examples are provided to study the properties of vulnerable American options while compared with vanilla American options, and exhibit the value of the approach in this paper. Section 5 follows as a conclusion.

2 LITERATURE REVIEW

As one of the three main kinds of risks (market risk, credit risk, operational risk), credit risk has long been recognized as a factor which affects the price of debt instruments, and the study of credit risk originates from its effects on bond yields. It can be traced back to Black and Scholes (1973) and Merton (1974), who first apply option pricing theory to price the risky corporate debt. The value of the risky debt of the firm is equal to the value of the identical risk-free bond minus a put option (expires at the maturity of the bond) on the firm's total assets. The call option can be valued by the option pricing theory. Since then, many models follow this approach. This approach is called structural model, compared with the reduced form approach, which takes the credit quality as the state variable. Klein outlines the comparison of some important models as Table 2.1 shows.

Table 2. 1 Review of risky debt models

Risky debt model	Default occurs at:	Time of valuation of nominal claim	Payout ratio linked to firm value	Fixed default boundary	Other liabilitie-s	Stocha-stic interest rates
Merton (1974)	Maturity	Maturity	Yes	Yes	No	No
Black and Cox (1976)	anytime	Default	Yes	Yes	Yes	No
Shimko et al (1993)	Maturity	Maturity	Yes	Yes	No	Yes
Longstaff and Schwartz (1995)	anytime	default	No	Yes	Yes	Yes

As an extension of Black and Scholes (1973), which presents a general approach in pricing European options and other liabilities as well as provides a closed-form solution that is a function of observable variables, Merton (1974) develops a theory of pricing bonds where default probability (credit risk) is significant and derives a closed-form pricing formula of bonds subject to credit risk. The contribution of the paper is notable since it was the first time that someone concentrated on the effects of credit risk when pricing bonds. The model assumes that there are only single, homogeneous debt class and equity claims for the debt issuer, and relates the default to the value of the total assets of the issuer. The relationship between debts and assets is clearly analyzed based on the relationship between shareholders and bondholders. Since the management is elected by shareholders, when having adequate assets, the bond issuer prefers to pay due debts in full, otherwise the equity would be valueless. Nonetheless, when the value of assets is less than the payment of due debts, the issuer would likely default since shareholders can not accept negative equity values. The default boundary therefore can be defined as the nominal claim of the debt at maturity. The model also assumes that the bondholder will receive the whole assets as recovered claim in the event of default, making the payout ratio endogenous. Accordingly, the payoff of the debt at maturity (defined as P) can be listed as

$$P = \mathrm{Min}\,[V, D]$$

Where V represents the value of the total assets of the debt issuer at the maturity of the debt, D represents the nominal claim of the debt at maturity. The equation can also be presented as

$$P = D - \mathrm{Max}\,[D - V, 0]$$

Where D represents the same nominal claim of a riskless debt, and the second term on the right side represents the potential loss due to credit risk.

The theory develop in Merton (1974) is extremely important since it works as a foundation for many posterior papers studying pricing models of vulnerable options, such as Klein (1996) and Klein and Inglis (1997).

The analysis of bond pricing regarding credit risk is further extended in Black and Cox (1976). The paper studies the effects of three specific indenture provisions: safety covenant, subordinate arrangements and restrictions on the financing of interest and dividend payments. Besides default at maturity, early default is taken into account in Black and Cox (1976) when pricing bonds, which is more realistic in business. At the maturity of the debt, the default boundary and the expected payoff of the debt are the same as those in Merton (1974). But prior to expiry, the early default boundary is a threshold value K that could be a function of time until maturity τ and instantaneous interest rate r such that default can occur whenever the value of the issuer V reaches the boundary K. As in Merton (1974), the default payout ratio is still endogenous and linked to the assets of the issuer. Thus, the payoff of the debt in Black and Cox model can be presented as

$$B(V, T) = \operatorname{Min}(V, P)$$

$B(K, t) = K$, where P represents the nominal claim of the debt.

In addition, the paper provides the model accounting for multi-priority obligations where junior debt claims only can be paid off after senior claims have been paid in full. It offers alternative solutions for many practical business situations.

Longstaff and Schwartz (1995) make further extension based on Black and Cox (1976) with respect to the default payout ratio and interest rates. First, the model assumes an exogenous payout ratio (1 – w) in the event of financial distress so that the expected payoff under default equals 1 – w times the nominal claim of the debt. Here w represents the percentage written down of the claim. Second, in contrast with Merton (1974) and Black and Cox (1976), the model takes account of stochastic interest rates. The expected payoff of the debt is thus presented as

$$R^*(1-w^*I), (I=1 \mid V=K, I=0 \mid V>K)$$

However, the effects of credit risk on options were not systematically studied until Johnson and Stulz (1987), who first recognized the credit risk in the pricing of options. The reason is that options are traditionally traded in the exchange market, assuming no credit risk at all. But recently, the over-the-counter option market grew rapidly. There are considerably large credit risks in the so-called vulnerable options which are traded in the OTC market. Following Johnson and Stulz (1987), a number of models are developed to price the vulnerable options, some of which are outlined by Klein in Table 2.2

Table 2. 2 Review of risky options models

Risky option model	Default occurs at:	Time of valuation of nominal claim	Payout ratio linked to firm value	Fixed default boundary	Other liabilities	Independence assumption	Stochastic interest rates
Johnson and Stulz (1987)	Maturity	Maturity	Yes	No	No	Yes	No
Hull and White (1995)	Anytime	Default	Not directly	Yes	Yes	Yes	No
Jarrow and Turnbull (1995)	Anytime	Default	No	Yes	Yes	Yes	Yes/No
Klein (1996)	Anytime	Default	Yes	Yes	Yes	No	No
Klein and Inglis (1997)	Anytime	Default	Yes	Yes	Yes	No	Yes

Johnson and Stulz (1987) is the first important literature in the research of credit risk of options. They derive pricing formulas for vulnerable European options under the assumptions that default occurs when the value of the assets of the option writer is less than some boundary at the maturity of the option, and this boundary is the intrinsic value of the option at the maturity that is assumed to be the sole liability of the option writer. Following above considerations, in the event of default, the recovered nominal claim received by the option holder is linked to the assets of the counterparty and equals the value of the total assets. The payoff for the vulnerable European call option can be expressed as

$$S_T - X \mid S_T \geq X, V_T \geq S_T - X$$

$$V_T \mid S_T \geq X, V_T < S_T - X$$

0 otherwise

, where S_T is the value of the underlying asset of the option at maturity time T, X is the strike price, V_T is the value of the option writer's assets at time T.

The approach of Johnson and Stulz (1987) implies a variable default boundary (VDB) that is equal to the option value at maturity. The possibility of default, thus, is influenced not only by the assets of the option writer but also by the variation of the potential payoff caused by the option itself. This understanding is intuitive. When the value of the option equals zero, credit risk of the counterparty certainly does not exist. When the option has positive value, credit risk arises if the assets depreciate greatly such that less than the option value or can not catch up with the growth pace of the option value.

The pricing model for vulnerable European options in Johnson and Stulz (1987) includes endogenous payout ratio that is linked to the assets of the writer and the explicitly modelled correlation between the assets of the writer and the asset underlying the option, which have empirical meaning in the study.

The assumptions that the option value is the only liability of the writer and the option holder will receive all the assets from the writer as recovered claim are appropriate if the expected payoff of the option is quite large compared with the other debts of the issuer. In this case the effect of other small liabilities can be neglected for they could hardly trigger credit risk, and the claim from the option holder weighs most when default happens. However, in many business situations, we can not say the assumptions are proper since few traders in the market would "take positions of individual options as large as in the example in Johnson and Stulz (1987)"[1]. And there are usually others creditors who have the same priority of the claim as the option holder in the event of financial distress. Therefore, the model fails to price vulnerable options effectively given credit risk in many cases.

Klein (1996) maintains all the meaningful marks in Johnson Stulz (1987) but allows the presence of other liabilities of the option writer, which is pervasively adopted in the literature posterior to Johnson Stulz (1987), such as Hull and White (1995) and Jarrow and Turnbull (1995), and derives an analytical solution for the value of vulnerable European options based on these assumptions. The default boundary becomes a fixed default boundary (FDB) instead of VDB in the paper. This FDB could be lower than the value of the total liabilities (let the amount be D here) of the writer that are due at the maturity of the option since the writer could continue to operate with the assets less than D. It is presumed that default occurs only when the value of the assets of the option writer shift below the FDB at maturity. And the default payout ratio is still endogenous, but the proportional nominal claim is determined by the total assets and liabilities of the writer at the expiry of the option. Furthermore, a concept of deadweight costs is introduced in the model such that the payout ratio can be calculated as $(1-\alpha)*\frac{V_T}{D}$, where α represents deadweight cost, V_T represents the value of total assets at maturity T, and D is defined as above. The payoff for the vulnerable European call option can be expressed as

$$S_T - X \mid S_T \geq X, V_T \geq D^*$$

$$(1-\alpha)\frac{V_T}{D}(S_T - X) \mid S_T \geq X, V_T < D^*$$ [2]

[1] Klein (1996)
[2] In Klein and Inglis (2001), D in the model is showed as D* when Klein (1996) model is reviewed.

0 otherwise

Since Klein (1996) does not discuss the relationship between FDB and the expected payoff of the option explicitly, the model indicates that the default barrier does not depend on the option value at maturity but just on the FDB, thus ignoring the situation in which the debt arising from the option weighs greatly in the total liabilities of the option writer and the fluctuation of the option value have to be taken account of when measuring credit risk. And in the situations mentioned above, the model turns out to be insensible in practice.

Klein and Inglis (2001) can be viewed as an integration of Johnson and Stulz (1987) and Klein (1996). They avoid the flaws of those two approaches by considering a stochastic default boundary that consists of two components: a fixed part that stands for the other liabilities of the option writer; a variable part that stands for the value of the option at expiry. Therefore, credit risk of vulnerable options is measured by the level of the assets, the amount of the debts other than the potential payoff of the option and the intrinsic value of the option at expiry. And they assume that only changes of the assets of the writer or changes of the asset underlying the option lead to defaults. Finally, the pricing equation of a vulnerable European call following above considerations turns to be as follows:

$$C = e^{-r\tau} * E^{*}[(\max(S_T - K) * \{(1 | V_T \geq D^{*} + S_T - K) + ((1-\alpha) * V_T * (S_T - K) / D^{*} + S_T - K | V_T < D^{*} + S_T - K)\}]$$

Where D^* represents the other liabilities of the option writer.

If D^* approximates zero, then the equation converges to Johnson and Stulz (1987) model. Reversely, if D^* approximates significantly large relative to the option value $S_T - K$, then the equation converges to Klein (1996) model.

As a whole, these papers all focus on the study of pricing vulnerable European options, for which the models can not be directly applied to vulnerable American options, thus leaving a blank for further research.

3 THE MODEL

In this section, the basic assumptions of the model for valuing vulnerable American put options are offered, part of which are standard assumptions following those of Merton (1974), Black and Cox (1976), Johnson and Stulz (1987), Klein (1996) and Klein and Inglis (2001).

Assumption 1. Let V denote the market value of the assets of the option writer. The dynamics of V are given by the diffusion process

$$\frac{dV}{V} = \mu_V dt + \sigma_V dZ_V$$

where μ_V is the instantaneous expected rate of return on the assets of the option writer, σ_V is the instantaneous standard deviation of the return (assumed to be constant) on the assets of the option writer. Z_V is a standard Wiener process.

Assumption 2. Let S represent the market value of the asset underlying the option. The dynamics of S are given by the diffusion process

$$\frac{dS}{S} = \mu_S dt + \sigma_S dZ_S$$

where μ_S is the instantaneous expected rate of return on the asset underlying the option, σ_S is the instantaneous standard deviation of the return (assumed to be constant) on the asset underlying the option. Z_S is a standard Wiener process. The instantaneous correlation between dZ_V and dZ_S is ρ.

Assumption 3. The markets are perfect and frictionless where there are no taxes transaction costs or information asymmetries. Securities can be traded in continuous time.

Assumption 4. The nominal claim of the option holder is the intrinsic value of the option.

Assumption 5. At the maturity of the option, t=T, default occurs only if the value of the option writer's assets at time T, V_T, is less than the threshold value $D^* + P_T$, where P_T=max(X- S_T,0) denotes the claim of the put option holder, D^* represent the value of the other liabilities of the option writer at maturity, X is the strike price of the put option and S_T represents the price of the underlying asset at maturity of the option.

Assumption 6. Before the maturity of the option, t<T, default occurs only if the value of the option writer's assets at time t, V_t, is less than the threshold value $D^* + P_t$, where P_t=max(X- S_t,0) denotes the claim of the put option holder and S_t represents the price of the underlying asset prior to the maturity of the option, t<T.

Assumption 7. At the time of default, t, the option holder receives (1- w_t) times the nominal claim, where w represents the percentage write-down of the nominal claim at time t.

Assumption 8. The percentage write-down on the nominal claim of the option holder upon default is w_t=1- (1-α) $V_t/(D^* + P_t)$ where α is the deadweight cost of the financial distress, which is represented as a percentage of the value of the assets of the option writer. The ratio $V_t/(D^* + P_t)$ represents the value of the option writer's assets available to pay the claim expressed as a proportion of total claims at time t.

Assumption 9. For simplicity, assume the underlying asset of the put option is stock and does not pay dividends. The other liabilities of the option writer D^* are zero-coupon debt.

Using the no-arbitrage approach, the price of a vulnerable put option P must satisfy the partial differential equation given by Johnson and Stulz (1987):

$$\frac{1}{2}\sigma_v^2 V^2 \frac{\partial^2 P}{\partial V^2} + rV\frac{\partial P}{\partial V} + \frac{1}{2}\sigma_s^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} + \rho\sigma_v\sigma_s VS\frac{\partial^2 P}{\partial V \partial S} - rP = P_t$$

The boundary conditions implied in the assumptions can be expressed as follows:

(1)At the maturity of the put option, t=T

$$X - S_T \mid S_T \leq X, V_T \geq D^* + X - S_T$$

$$(1-\alpha)\frac{V_T}{D^* + X - S_T}(X - S_T) \mid S_T \leq X, V_T < D^* + X - S_T$$

0 otherwise

(2) Prior to the maturity of the put option t<T

$$\lim_{S \to \infty} P(S, t) = 0$$

If default does not happen prior to maturity

$$P(S_f(t),t) = X - S_f(t) \mid S_f(t) \leq X, V_t > D^* + X - S_t$$

$$\frac{\partial P}{\partial S}\Big|_{(S_f(t),t)} = -1 \quad \mid S_f(t) \leq X, V_t > D^* + X - S_t$$

where $S_f(t)$ is the free boundary at t.

(3) Prior to the maturity of the put option t<T

If default happens prior to maturity

$$(1-\alpha)\frac{V_t}{D^* + X - S_t}(X - S_t) \mid S_t \leq X, V_t \leq D^* + X - S_t$$

Where

P is the price of the put option,

V is the total market value of the asset of the put option writer,

$$dV = \mu_V V dt + \sigma_V V dZ_V$$

σ_v is the instantaneous standard deviation of the return on the assets of the option writer,

S is the market value of the asset underlying the put option,

$$dS = \mu_S S dt + \sigma_S S dZ_S$$

σ_S is Volatility of the stock price,

ρ is the instantaneous correlation between dZ_V and dZ_S,

r is the riskless rate of interest,

X is the strike price,

α is the deadweight costs of financial distress,

D* is the value of the other liabilities of the option writer,

T is the time to expire.

The boundary condition set (1) characterizes the payoffs of the put option at maturity. The boundary condition set (2) is the boundary conditions of the put option when the option writer does not default. The boundary condition set (3) expresses the amount which the put option holder will receive if the option writer's assets hit the variable default boundary $D^* + X - S_t$ prior to maturity. In (3), the amount available for distribution jumps down by α. We do not analyze directly whether this will sometimes make early exercise optimal when, if the absence of credit risk it would not be, but our valuation takes this into account.

4 NUMERICAL EXAMPLES

The above partial differential equation given the particular boundary conditions does not have an analytic solution and must be solved numerically. There are a number of approaches to solve the partial differential equations numerically, for example, the alternating direction implicit method (ADI) in Jones and Jocobs (1986).

This paper employs the three-dimensional binomial tree approach to solve the partial differential equation, which is suggested by Hull and White (1990), the main idea of which is briefly outlined here. After that, this section presents some numerical examples to examine some properties of the vulnerable American put options under the above assumptions. Some comparisons of vulnerable American put option with vanilla American put option are also illustrated.

The three-dimensional binomial tree approach is de facto the extension of the traditional two-dimensional binomial tree, which is widely used to price the vanilla options or other derivatives whose price is only depends on the price of one underlying asset. If the value of derivatives depends on two assets whose prices are correlated, the traditional two-dimensional binomial tree becomes three-dimensional. The ideas are as follows.

From assumption 1 and 2, V and S follow geometrical Brownian motion

$$\frac{dV}{V} = \mu_V dt + \sigma_V dZ_V$$

$$\frac{dS}{S} = \mu_S dt + \sigma_S dZ_S$$

Under risk neutral probability measure, V and S satisfy

$$d \ln V = (r - \frac{\sigma_V^2}{2})dt + \sigma_V dZ_1$$

$$d \ln S = (r - \frac{\sigma_S^2}{2})dt + \sigma_S dZ_2$$

Where the instantaneous correlation between dZ_1 and dZ_2 is ρ.

If define two new uncorrelated variables x_1 and x_2

$$x_1 = \sigma_S \ln V + \sigma_V \ln S$$

$$x_2 = \sigma_V \ln S - \sigma_S \ln V$$

Apply Ito's lemma, x_1 and x_2 follows

$$dx_1 = (\sigma_S (r - \frac{\sigma_V^2}{2}) + \sigma_V (r - \frac{\sigma_S^2}{2}))dt + \sigma_V \sigma_S \sqrt{2(1+\rho)} dZ_A$$

$$dx_2 = (\sigma_S (r - \frac{\sigma_V^2}{2}) - \sigma_V (r - \frac{\sigma_S^2}{2}))dt + \sigma_V \sigma_S \sqrt{2(1-\rho)} dZ_B$$

Where the instantaneous correlation between dZ_A and dZ_B is 0.

Because x_1 and x_2 are uncorrelated variables, traditional binomial tree for each of these two variables can be constructed. Then these two trees can be combined into a three-dimensional tree. Unlike the two-dimensional trees which have only two branches for each node, there are four branches for each node, whose probabilities are (assume that the probability of moving up is p for x_1 and q for x_2)

pq: when x_1 increases and x_2 increases;

p(1-q): when x_1 increases and x_2 decreases;

(1-p)q: when x_1 decreases and x_2 increases;

(1-p)(1-q): when x_1 decreases and x_2 decreases;

At each node, V and S can be calculated from x_1 and x_2

$$V=\exp(\frac{x_1+x_2}{2\sigma_S})$$

$$S=\exp(\frac{x_1-x_2}{2\sigma_V})$$

where exp stands for the exponential function.

After V and S are calculated, it is simple to work backward through the tree to determine the value of the option at each node, which is similar to the two-dimensional tree. Of course, to value the American option, the value of the option at each node should be compared with its intrinsic value and equal to whichever is larger, the same as valuing the vanilla American option if the option writer does not default. If the option writer defaults, the value of the option is only part of the payoff to the vanilla American put, which can be determined by the boundary condition. That is to say, the value of the option at each terminal node and the node at the boundary can be set according to the boundary conditions discussed in Section 3. The program is given in the appendices.

For more details about this method, please refer to Au (1997).

There are about ten factors that affect the price of vulnerable American options, i.e. the current stock price S, the Strike price X, the current total market value of the asset of the put option writer V, the value of liabilities of the option writer D*, the deadweight cost α, the volatility of the stock price σ_S, the risk-free interest rate r, the time to expire T, the volatility of the asset value σ_v, and the instantaneous correlation ρ. This paper uses the three-dimensional binomial tree method discussed above to examine the first 5 most important factors.

In the base case, the put option is at the money. The option writer is a highly leveraged firm (90% debt-asset ratio). The correlation between the value of the option writer's assets and the value of the asset underlying the option is zero. The exact parameters are S=100, X=100, V=200, σ_S=0.2, σ_v=0.2, D* =180, T=2, r=0.05, ρ=0.0, α=0.25, which are similar to Kou (1999).

Figure 4.1 shows the relationship between the put option value and the stock price S for vulnerable American put option and vanilla American put option. The parameters are the base case parameters except that the value of S is allowed to change. As the stock price S decreases, the put option will become more valuable. The value of vulnerable put is considerably lower than that of vanilla put, which means that default actually occurs. The reason is that V and D* are not large compared with S and X. When the put option is deep in-the-money, vulnerable put is exercised immediately as vanilla put does.

Figure 4.2 illustrates the relationship between the put option value and the strike price X for vulnerable American put option and vanilla American put option. The parameters are the base case parameters except that the value of X is allowed to change. As the stock price X increases, the value of the put option increases. Like last paragraph, the value of vulnerable put is considerably lower than that of vanilla put, which means that default actually occurs.

Figure 4.3 sketches the relationship between the put option value P and the value of the option writer's assets V for vulnerable American put option and vanilla American put option. The parameters are the base case parameters except that the value of V is allowed to change. The higher the initial value of V, the less likely the value of V will hit the variable default boundary in the future. Thus if the value of V large enough, the value of vulnerable put should be equal to that of vanilla put, as Figure 3 shows.

Figure 4.4 presents the effect of the option writer's debt D* on the value of vulnerable American put option. The parameters are the base case parameters except that the price of stock S and option writer's debt D* are allowed to change. As the value of D* increases, the possibility of default increases and the value of V will be more likely to hit the variable default boundary. Therefore, as D* increases, P decreases. At extremes, when D*=0, the price of vulnerable American option is equal to that of vanilla American option. When D is large, the price of vulnerable American option will approach zero.

Figure 4.5 demonstrates the effect of deadweight costs α on the value of the vulnerable American put option. The parameters are the base case parameters except that the price of stock S and the deadweight costs α are allowed to change. As the value of α increases, the payoff to the put option holder when the option writer defaults decreases. Therefore, as α increases, P decreases. At extremes, when α=1, the payoffs to the put is zero when default occurs, leading to the zero put price when the put option is deep in-the-money.

Table 4.3 A summary of the comparison between vulnerable and vanilla American put option

parameter	Vulnerable	Vanilla
S	-	-
X	+	+
V	+	N/A
D	-	N/A
α	-	N/A

Figure 4.1 Vulnerable American put values as a function of stock price S

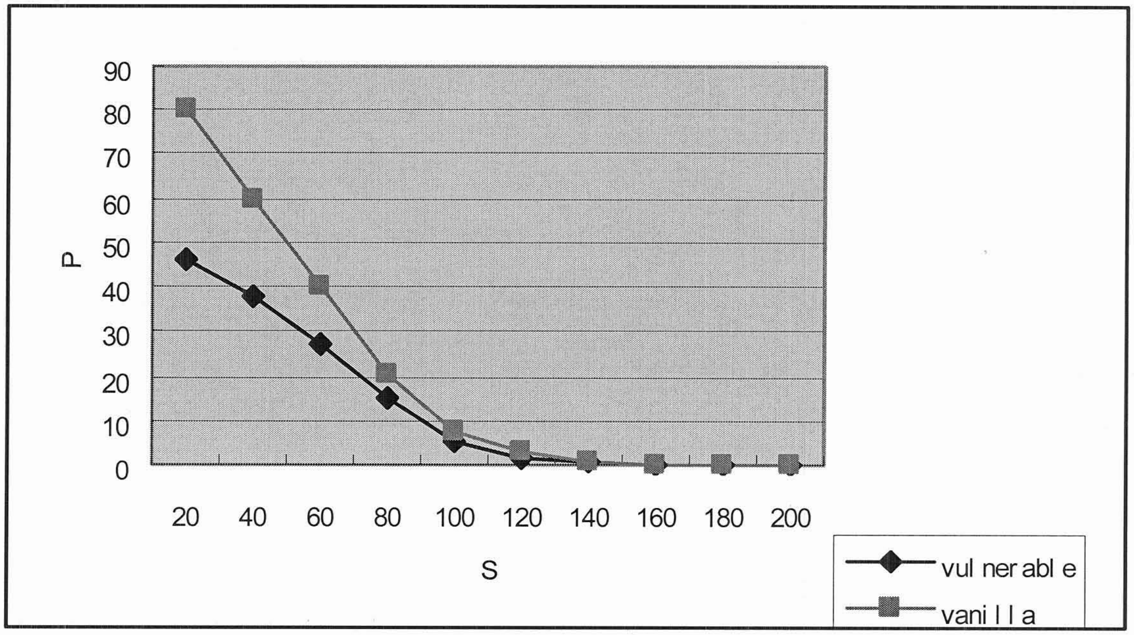

Figure 4.1 Vulnerable American put values as a function of stock price S: comparison between vulnerable American put and vanilla American put. Calculations of vulnerable put option prices are based on the base case parameters. The numerical solution is based on a three-dimensional binomial tree using 50 steps. Calculations of vanilla put option prices are based on the base case parameters for vanilla put and the solution is based on binomial tree using 100 steps.

Figure 4.2 Vulnerable American put values as a function of strike price X

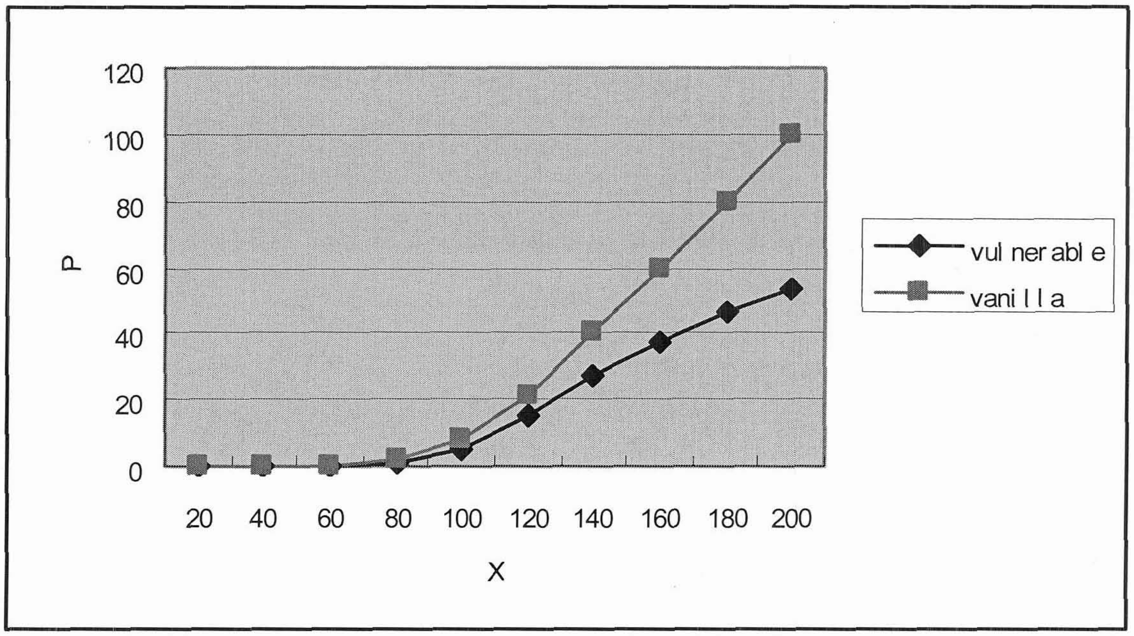

Figure 4.2 Vulnerable American put values as a function of strike price X: comparison between vulnerable American put and vanilla American put. Calculations of vulnerable put option prices are based on the base case parameters. The numerical solution is based on a three-dimensional binomial tree using 50 steps. Calculations of vanilla put option prices are based on the base case parameters for vanilla put and the solution is based on binomial tree using 100 steps.

Figure 4.3 Vulnerable American put values as a function of the value of the option writer's assets V

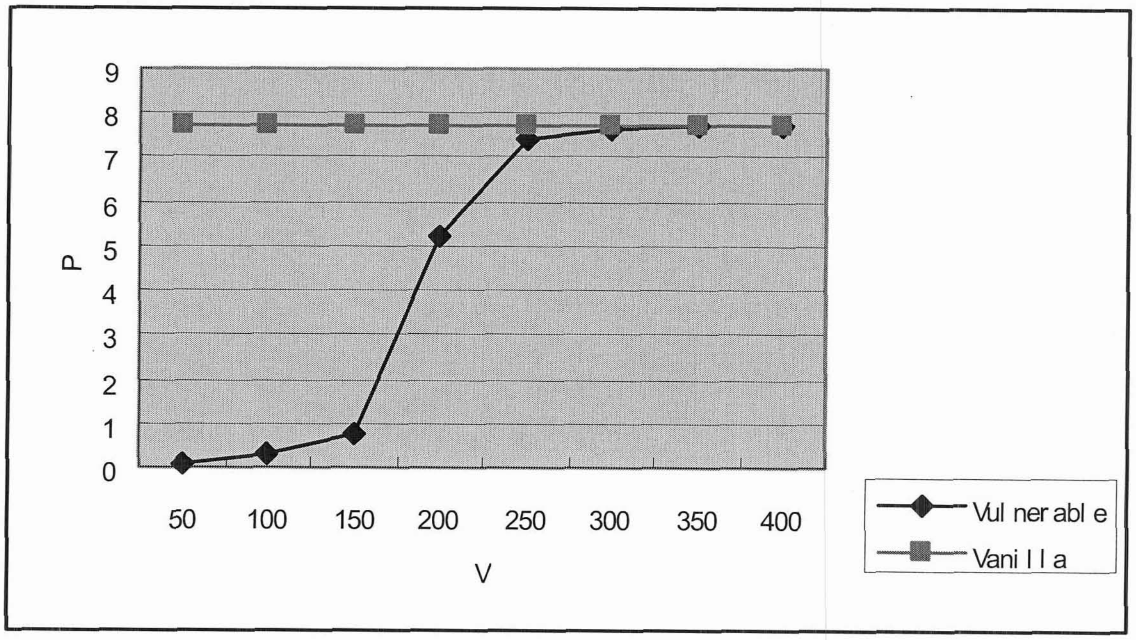

Figure 4.3 Vulnerable American put values as a function of the value of the option writer's assets V: comparison between vulnerable American put and vanilla American put. Calculations of vulnerable put option prices are based on the base case parameters. The numerical solution is based on a three-dimensional binomial tree using 50 steps. Calculations of vanilla put option prices are based on the base case parameters for vanilla put and the solution is based on binomial tree using 100 steps.

Figure 4.4 The effect of the option writer's debt D* (i.e. D in the figure) on the value of vulnerable American put option

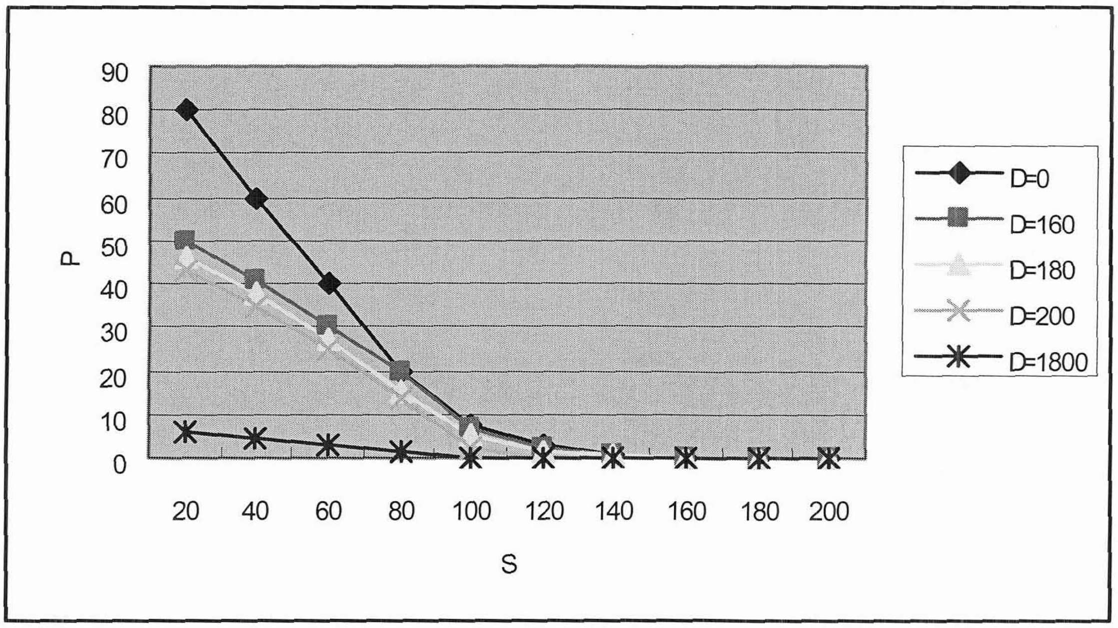

Figure 4.4 The effect of the option writer's debt D* (i.e. D in the figure) on the value of vulnerable American put option. Calculations of vulnerable put option prices are based on the base case parameters. The numerical solution is based on a three-dimensional binomial tree using 50 steps.

Figure 4.5 The effect of the deadweight costs α (i.e. alpha in the figure) on the value of vulnerable American put option

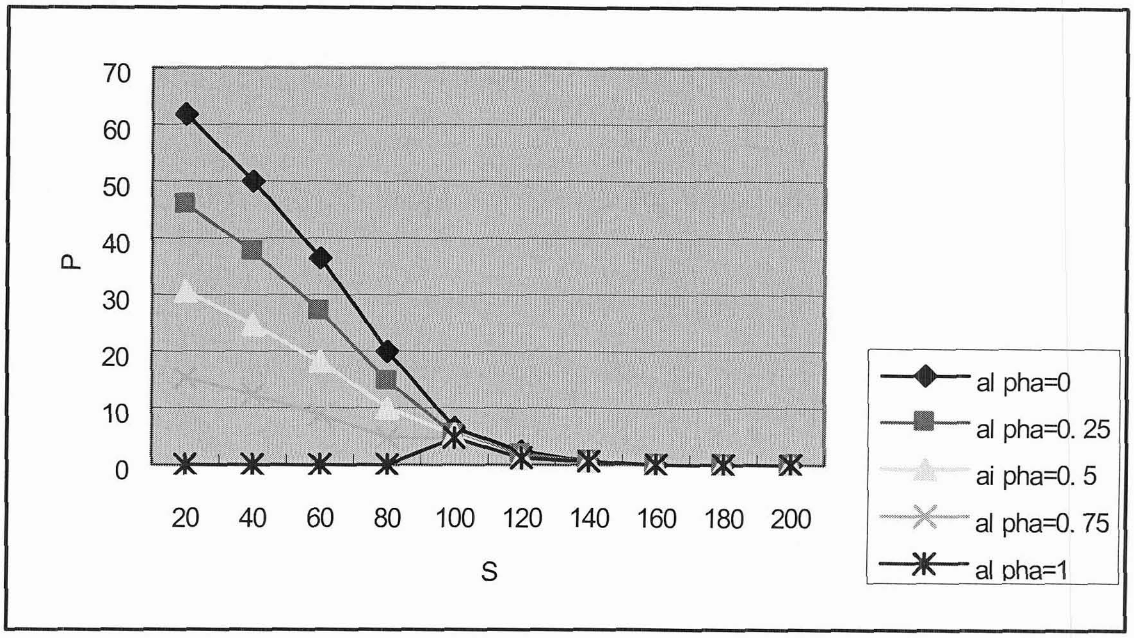

Figure 4.5 The effect of the deadweight costs α (i.e. alpha in the figure) on the value of vulnerable American put option. Calculations of vulnerable put option prices are based on the base case parameters. The numerical solution is based on a three-dimensional binomial tree using 50 steps.

5 CONCLUSION

The main purpose of this paper is to extend the models developed by Johnson and Stulz (1987), Klein (1996), Klein and Inglis (2001), to price vulnerable American options, which are usually subject to the risk of financial distress happening to the option writer and can be exercised before the expiry. We retain the assumptions hold by Klein and Inglis (2001) that the default boundary depends not only on the debt produced by the option itself but also on the other liabilities owned by the option issuer, and the correlation between the total assets of the writer and the underlying asset of the option is allowed to derive the pricing model, which are suitable for most business situations. Also, the pay out ratio in the event of default is endogenous and linked to the assets of the option writer. On the basis of these assumptions, the paper extends Klein and Inglis (2001) by releasing the restriction that default can occur only at maturity, and adopts a variable early default boundary. Moreover, this paper focuses on put options instead of call options and studies properties of vulnerable American put options. We provide numerical examples that compare the results of the model with those of vanilla American options.

According to the results from the numerical examples in this paper, the value of an in-the-money vulnerable American put has negative correlation with the asset underlying the option and positive correlation with the strike price. We also learn that there is a divergence between the prices of vulnerable and vanilla American put, and this divergence that represents the credit risk of vulnerable options will expand as the option price increases. Meanwhile, the price of a vulnerable American put will go up accompanying the growth of the value of the total assets of the option writer, and the price will keep stable after the assets exceed a certain level. This is logical since the higher the value of the assets, the lower the possibility of default, that is, the lower the credit risk, and the credit risk converges to zero when the assets of the writer are adequate enough such that they will not influence the price of the option any longer. Finally, the price of a vulnerable American put is negatively correlated to the total debts of the option writer because the higher the liabilities of the writer, the higher the credit risk that the counterparty takes.

The results of numerical examples further confirm the effectiveness of the approach in this paper, and the pricing model can be employed to general vulnerable American options after simple adjustments.

APPENDICES

Appendix A Program: pricing the vulnerable American options (Excel/VBA program)

Declaration

```
Public Type xcell
x1 As Double
x2 As Double
End Type
Public Type optioncell
S As Double
V As Double
Value As Double
End Type

Function Americanputtest(S0 As Double, K As Double, r As Double, _
sigmaS As Double, V As Double, sigmaV As Double, rho As Double, T As Double, _
NoPeriods As Integer, alpha As Double, Dstar As Double) As Double
Dim xvalue() As xcell, putvalue() As optioncell
ReDim xvalue(1 To NoPeriods + 1, 1 To NoPeriods + 1)
ReDim putvalue(1 To NoPeriods + 1, 1 To NoPeriods + 1)
Dim q1 As Double, q2 As Double, x1 As Double, x2 As Double
Dim mux1 As Double, mux2 As Double, sigmax1 As Double, sigmax2 As Double
Dim u1 As Double, u2 As Double, d1 As Double, d2 As Double
Dim pup1 As Double, pup2 As Double, pdown1 As Double, pdown2 As Double
Dim xi As Integer, xj As Integer, period As Integer, dt As Double
Dim temp As Double, S As Double
''S0 = stock price
''K = strike price
''r = risk-free interest rate
''sigmaS = volatility of S
''V = value of the option writer's assets
''sigmaV = volatility of V
''rho = correlation between S and V
''T = time to maturity of the option
''NoPeriods = number of the steps of the tree
''alpha = deadweight costs
''Dstar = liabilities of the option writer
S = S0
''define two uncorrelated variables
q1 = 0
q2 = 0
```

```
x1 = sigmaV * Log(S) + sigmaS * Log(V)
x2 = sigmaV * Log(S) - sigmaS * Log(V)
mux1 = sigmaV * (r - q1 - sigmaS * sigmaS / 2) + sigmaS * (r - q2 - sigmaV * sigmaV / 2)
mux2 = sigmaV * (r - q1 - sigmaS * sigmaS / 2) - sigmaS * (r - q2 - sigmaV * sigmaV / 2)
sigmax1 = sigmaS * sigmaV * Sqr(2 * (1 + rho))
sigmax2 = sigmaS * sigmaV * Sqr(2 * (1 - rho))
dt = T / NoPeriods
u1 = sigmax1 * Sqr(dt)
d1 = -u1
u2 = sigmax2 * Sqr(dt)
d2 = -u2
pup1 = (mux1 * dt + u1) / (2 * u1)
pup2 = (mux2 * dt + u2) / (2 * u2)
pdown1 = 1 - pup1
pdown2 = 1 - pup2
"construct the three-dimensional binomial tree
For period = NoPeriods + 1 To 1 Step -1
xvalue(1, 1).x1 = x1 + d1 * (period - 1)
xvalue(1, 1).x2 = x2 + d2 * (period - 1)
putvalue(1, 1).S = Exp((xvalue(1, 1).x1 + xvalue(1, 1).x2) / (2 * sigmaV))
putvalue(1, 1).V = Exp((xvalue(1, 1).x1 - xvalue(1, 1).x2) / (2 * sigmaS))
For xj = 1 To period
For xi = 2 To period
xvalue(xj, xi).x1 = u1 * 2 + xvalue(xj, xi - 1).x1
xvalue(xj, xi).x2 = xvalue(xj, xi - 1).x2
putvalue(xj, xi).S = Exp((xvalue(xj, xi).x1 + xvalue(xj, xi).x2) / (2 * sigmaV))
putvalue(xj, xi).V = Exp((xvalue(xj, xi).x1 - xvalue(xj, xi).x2) / (2 * sigmaS))
xvalue(xi, xj).x1 = xvalue(xi - 1, xj).x1
xvalue(xi, xj).x2 = u2 * 2 + xvalue(xi - 1, xj).x2
putvalue(xi, xj).S = Exp((xvalue(xi, xj).x1 + xvalue(xi, xj).x2) / (2 * sigmaV))
putvalue(xi, xj).V = Exp((xvalue(xi, xj).x1 - xvalue(xi, xj).x2) / (2 * sigmaS))
Next xi
Next xj
"calculate the option value
For xi = 1 To period
For xj = 1 To period
If period = NoPeriods + 1 Then
If putvalue(xi, xj).V > (Dstar + Application.Max(K - putvalue(xi, xj).S, 0)) Then
 putvalue(xi, xj).Value = Application.Max(K - putvalue(xi, xj).S, 0)
 Else
  putvalue(xi, xj).Value = (1 - alpha) * (putvalue(xi, xj).V / _
  (Dstar + Application.Max(K - putvalue(xi, xj).S, 0))) * Application.Max(K - putvalue(xi, xj).S,
0)
End If
Else
If putvalue(xi, xj).V > (Dstar + Application.Max(K - putvalue(xi, xj).S, 0)) Then
 putvalue(xi, xj).Value = Application.Max(K - putvalue(xi, xj).S, Exp(-r * dt) * ((pup1 * pup2) *
putvalue(xi + 1, xj + 1).Value + _
(pdown1 * pup2) * putvalue(xi + 1, xj).Value + (pup1 * pdown2) * putvalue(xi, xj + 1).Value + _
(pdown1 * pdown2) * putvalue(xi, xj).Value))
 Else
```

```
 temp = Application.Max(K - putvalue(xi, xj).S, Exp(-r * dt) * ((pup1 * pup2) * putvalue(xi + 1,
xj + 1).Value + _
(pdown1 * pup2) * putvalue(xi + 1, xj).Value + (pup1 * pdown2) * putvalue(xi, xj + 1).Value + _
(pdown1 * pdown2) * putvalue(xi, xj).Value))
  putvalue(xi, xj).Value = (1 - alpha) * (putvalue(xi, xj).V / _
  (Dstar + temp)) * temp
End If
End If
Next xj
Next xi
Next period
Americanputtest = putvalue(1, 1).Value
End Function
```

Appendix B Program: pricing the vanilla American options

(matlab program)

```
function vanillaAmericanput=vanillaAmericanput(S0,K,r,sigmaS,T,steps)
% S0 = stock price
% K = strike price
% r = risk-free interest rate
% sigmaS = volatility of stock price
% T = time to maturity of the option
% steps = number of the steps of the binomial tree
dt=T/steps;
u=exp(sigmaS*sqrt(dt));
d=1/u;
p=(exp(r*dt)-d)/(u-d);
S=S0*d^steps;
% boundary condition at the time of maturity
put(1)=max(K-S,0);
for i =2:steps+1
  S=S*u^2;
  put(i)=max(K-S,0);
end
% compute the value of the option
for i=steps-1:-1:0
  S=S0*d^i;
  put(1)=max(K-S,exp(-r*dt)*((1-p)*put(1)+p*put(2)));
  for j=2:i+1
    S=S*u^2;
    put(j)=max(K-S,exp(-r*dt)*((1-p)*put(j)+p*put(j+1)));
  end
end
vanillaAmericanput=put(1)
```

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