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CIRCUITS AND CYCLE DECOMPOSITIONS

by

Siming Zhan

M.Sc., Institute of System Sciences, Academia Sinica, 1986

A THESIS SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
in the Department
of
Mathematics & Statistics

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Circuits and Cycle Decompositions

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Abstract

This thesis contains two main chapters.

In Chapter 2, we study long circuits and long trails of graphs. We establish the existence of spanning circuits (or, trails) of $(3, 6)$ -edge-connected graphs. The results imply that every 7-connected line graph is Hamilton-connected which partially supports Thomassen's conjecture that every 4-connected line graph is hamiltonian. We also obtain an analogous result of Tutte's Bridge Lemma on circuits instead of on cycles. Therefore, Thomassen's conjecture is also true for planar graphs. Later, we find some applications of long circuits and long trails on the nowhere-zero integer flow problems, the cycle double cover problem and the vertex cycle cover problem by showing that every $(3, 6)$ -edge-connected graph has a nowhere-zero 4-flow and has a 3-circuit double cover, and by showing that every 2-edge-connected graph with minimal degree at least 3 and with order p has a vertex cycle cover with at most $2(p-1)$ edges. This is conjectured to be true for any 2-edge-connected graph by Bermond, Jackson and Jaeger.

In Chapter 3, we refine a result of de Werra on equitable edge-colourings of graphs. We use it to show that for an almost r -regular hamiltonian graph, there are at least $2\lfloor \frac{r}{2} \rfloor - 2$ edge-disjoint Hamilton cycles in its line graph. This result supports Bermond's conjecture that if a graph is Hamilton decomposable then its line graph is also Hamilton decomposable.

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Chapter 1

Introduction

The notation and terminology which are not specified in this thesis can be found in [16] or [15].

This chapter is devoted to introducing some basic concepts of graph theory and to surveying some results related to this thesis.

A *graph* $G = (V, E)$ consists of a non-empty set $V(G)$ of elements, called *vertices*, and a set $E(G)$ of elements called *edges*, together with an *incidence* relation that associates with each edge two vertices, called its *end vertices* or briefly *ends*. An edge with end vertices u and v is denoted by the unordered pair uv . The two end vertices of an edge are said to be *joined* by the edge and to be *adjacent* to one another. Adjacent vertices are also referred to as *neighbours*. The number of vertices of a graph G is denoted by $v(G)$ and called the *order* of G ; the number of edges is denoted by $e(G)$ and called the *size* of G . A graph G is *finite* if both $v(G)$ and $e(G)$ are finite, and *infinite* otherwise. We consider finite graphs only. For a given vertex u , the set of vertices adjacent to it is denoted by N_u and is called the *neighbourhood* of u in G and the set of edges incident with it is denoted by E_u and also denoted by $\nabla(\{u\})$ as a *trivial cut*. The cardinality of E_u is called the *degree* of u , denoted by $\text{deg}_G(u)$, or simply, $d_G(u)$ or $d(u)$. The *maximum degree* of a graph G is denoted by $\Delta(G)$ and the *minimum degree* of G is denoted by $\delta(G)$. We use $\delta(S)$ to denote the minimum degree among the vertices of $S \subseteq V$. A graph is *k-regular* if every vertex of G has

degree k . A graph is *almost k -regular* if its vertices have degree either k or $k + 1$.

An edge with identical end vertices is a *loop*; two or more edges with the same pair of end vertices are *multiple* edges. If there are loops in a graph, we can delete the loops from the graph. So we always assume that a graph has no loops. A *simple* graph is one with neither loops nor multiple edges.

A graph H is a *subgraph* of G , denoted by $H \leq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, and if every edge of H has the same pair of endvertices in H as it has in G . Meanwhile, G is also called a *supergraph* of H . H is a *spanning* subgraph of G if $V(H) = V(G)$. H is a *dominating* subgraph of G if every edge of G is incident with a vertex of $V(H)$. For any non-empty set $S \subseteq V(G)$, the *induced* subgraph on S , denoted by $G[S]$, is the subgraph of G whose vertex set is S , and whose edge set consists of those edges of G with both end vertices in S . The induced subgraph $G[V - S]$ is denoted by $G - S$; it is the subgraph obtained from G by deleting the vertices in S together with their incident edges. For a non-empty set $T \subseteq E(G)$, the *edge induced* subgraph on T , denoted by $G[T]$, is the subgraph of G whose vertex set is the set of end vertices of edges in T and whose edge set is T . The spanning subgraph with edge set $E - E'$ is written simply as $G - E'$. The graph obtained from G by adding a set of edges E' is denoted by $G + E'$. We write $G - u$, $G + e$ and $G - e$ instead of $G - \{u\}$, $G + \{e\}$ and $G - \{e\}$.

If f is a non-negative integer-valued function on $V(G)$ such that $f(u) \leq d_G(u)$, then an *f -factor* F of G is a spanning subgraph of G such that $f(u) = d_F(u)$. If f is a constant, say $f \equiv k$, then an *f -factor* F is also called a *k -factor*. If the edge set of G can be decomposed into edge-disjoint *f -factors*, then G is called *f -factorable*. The decomposition is also called an *f -factorization* of G .

A *walk* is finite sequence $W = v_0 e_1 v_1 e_2 v_2 \cdots e_k v_k$, whose terms are alternatively vertices and edges such that, for $1 \leq i \leq k$, the ends of e_i are v_{i-1} and v_i . We say that W is a walk from v_0 to v_k , or a (v_0, v_k) -walk. The vertices v_0 and v_k are called the *origin* and *terminus* of W and the remaining vertices of the walk are called *internal vertices*. The number k is the *length* of the walk. In a simple graph the walk

is determined by the vertices of the walk and hence we can simply use the vertex sequence $v_0v_1\cdots v_k$ to denote the walk. The reverse walk $v_k e_k v_{k-1} \cdots e_1 v_0$ of W is denoted by W^{-1} with origin v_k and terminus v_0 . If the edges of the walk are distinct, then W is called a *trail*. If, in addition, the vertices of the walk are distinct, W is called a *path*. A closed trail is a *circuit* and a circuit with distinct vertices is a *cycle*.

An *acyclic graph* is one that contains no cycles. It is also called a *forest*. A *tree* is a connected acyclic graph. Every non-trivial acyclic graph has at least two *leaves*, that is, the vertices of degree one. Every pair of vertices in a tree have a unique path connecting them. A *star* is a bipartite graph $K_{1,n}$ for some n . A *multistar* is a graph obtained from a star by replacing some edges by multiple edges.

A graph G is *connected* if for any pair of vertices u and v in G there is a (u, v) -path joining them; otherwise, the graph G is *disconnected*. Maximal connected subgraphs of a graph are connected *components* of the graph. The number of connected components in G is denoted by $\varpi(G)$.

A *vertex cut* in a connected graph is a subset of vertices whose removal will increase the number of connected components of the graph. A single vertex which forms a vertex cut is called a *cut vertex*. A graph G is *k-connected* if it has at least $k + 1$ vertices and it has no vertex cut with fewer than k -vertices. The *connectivity* of G is the maximum number k such that G is k -connected. A graph is 1-connected if and only if it is connected. A connected graph G without cut vertices is called a *block*. Every block of at least three vertices is 2-connected. A *block* of a graph is a subgraph which is a block and it is maximal with respect to this property. The *block-cut graph* of a graph G is a bipartite graph with the blocks and cut vertices of G as vertices and a block and a cut vertex are adjacent if and only if the cut vertex is in the block. A cut edge is regarded as a block here. The block-cut graph of a graph is acyclic. If G is 2-connected and e is an edge of G , then the block-cut graph of $G - e$ must be a trivial graph or a path with the end vertices of e in either end blocks.

For a proper vertex subset S of G let $\nabla(S)$ denote the set of edges with exactly one end vertex in S . We call $\nabla(S)$ an *edge cut* or, more briefly, a *cut*. A minimal

edge cut is a *bond*. We use $\rho(S)$ to denote the cardinality of $\nabla(S)$. The vertex set S and $V(G) - S$ are called *shores* of the cut $\nabla(S)$ (or $\nabla(V(G) - S)$). A cut is *trivial* if one of its shores is a single vertex. A cut is *cyclic* if the induced subgraphs on both shores contain cycles. A cut is *essential* if there is at least one induced subgraph on its shores containing no edges. A non-trivial graph G is λ -*edge-connected* if there is no cut with less than λ edges. A graph is *cyclically* λ_c -*edge-connected* if there is no cyclic cut with less than λ_c edges. A graph is *essentially* λ_e -*edge-connected* if there is no non-essential cut with less than λ_e edges.

One can obtain a new graph from graphs by binary graph operations, such as, *cartesian product*, *wreath product* (which is also called *lexicographical product*), *conjunction*, *join*, etc., of two graphs. One can also obtain a new graph from a graph by some monotone graph operation, such as taking the *square*, taking the *dual*, taking the *complement* or taking the *line graph*, etc.. In this thesis, we will mostly consider the line graph of a graph. The *line graph* of G is a graph with the edges of G as its vertices and two of them are adjacent if and only if they are adjacent in G as edges.

A *directed graph*, or briefly *digraph*, $D = (V, A)$ consists of a vertex set $V(D)$ and an arc set $A(D)$, where each arc is an ordered pair of vertices. The first vertex is the *origin* of the arc and the other is the *terminus* of the arc. We can obtain a digraph from an undirected graph G by assigning each edge an origin and a terminus. The resulting digraph, denoted by \vec{G} , is called an *orientation* of G . The graph G is called the *underlying graph* of D if D is an orientation of G . For a proper subset $S \subset V(D)$, we denote by $\nabla^+(S)$ the set of arcs with origin in S and terminus not in S and denote by $\nabla^-(S)$ the set of arcs with terminus in S and origin not in S . The cut $\nabla(S)$ is the union of $\nabla^+(S)$ and $\nabla^-(S)$. It is easy to see that $\nabla^+(S) = \nabla^-(V(D) - S)$.

For an abelian group Γ (with additive notation), a Γ -*flow* in a digraph D is a mapping ϕ from the arc set $A(D)$ to the group Γ such that for all $S \subseteq V(D)$,

$$\sum_{e \in \nabla^+(S)} \phi(e) - \sum_{e \in \nabla^-(S)} \phi(e) = 0. \quad (1.1)$$

A Γ -flow of D is a *nowhere-zero* Γ -flow if $\phi(e)$ takes a non-zero value for every arc

e of D . It is easy to see the following facts [67].

1. The mapping ϕ is a Γ -flow if and only if equation (1.1) is valid for all S consisting of a single vertex of G .
2. A Γ -flow takes zero value on each cut edge.
3. The equation (1.1) is still valid for all $S \subseteq V(D)$ if the orientation of some edges of D are reversed while their flows are simultaneously replaced by their (additive) inverses.

For an undirected graph G , there is a nowhere-zero Γ -flow for some orientation if and only if there is a nowhere-zero Γ -flow for every orientation. So the existence of a nowhere-zero Γ -flow is independent of the orientation. So we can simply say that G has a *nowhere-zero Γ -flow*. A *nowhere-zero k -flow* is a nowhere-zero Z -flow of G such that $0 < |\phi(e)| < k$ for each edge $e \in E(G)$. On the other hand, referring to Jaeger [63], Tutte's dichromatic polynomial theory and Tutte's unimodular matroid theory imply that there is a nowhere-zero Γ -flow of G for some abelian group Γ of order k if and only if there is a nowhere-zero k -flow of G . Hence the existence of a nowhere-zero flow is independent of the choice of the group as well.

Chapter 2

Long Paths and Cycles

2.1 Introduction.

Constant large connectivity (or minimal degree) cannot always guarantee the graph to be hamiltonian in the following sense: for any given positive integer n , there exists a non-hamiltonian graph G of connectivity (or minimal degree) at least n . For instance, $K_{n,n+1}$ is an n -connected (or minimal degree n) non-hamiltonian graph. Furthermore the longest cycle in an n -connected graph (or minimal degree n) might be fairly short with respect to the order of the graph.

Theorem 2.1.1 (Jackson and Parsons [60]) *For a given integer $r \geq 3$ and any real $\epsilon > 0$, there exists an integer $N(r, \epsilon)$ such that if r is even and $p \geq N(r, \epsilon)$, or if r is odd and p is even and $p \geq N(r, \epsilon)$, then there exists an r -regular r -connected graph of order p such that the length of a longest cycle in the graph is less than ϵp .*

A simple, but important, necessary condition for a graph to be hamiltonian (see [31, 32]) is the following.

Theorem 2.1.2 *Let G be a hamiltonian graph, let S be a non-empty proper subset of the vertex set $V(G)$, and let $\varpi(G - S)$ be the number of components of the graph*

$G - S$. Then

$$t(G) =: \min_{S \in \mathcal{V}(G)} \frac{|S|}{\varpi(G - S)} \geq 1. \quad (2.1)$$

The parameter $t(G)$ is the so-called *toughness* of the graph G . The graph G is also called *t-tough* if $t \leq t(G)$. It was also conjectured by Chvátal (later modified) that every 2-tough graph is hamiltonian.

Conjecture 2.1.3 (Chvátal [31]) *Every 2-tough graph is hamiltonian.*

There are many sufficient conditions for graphs to be hamiltonian. The following results are typical examples. Here $\kappa(G)$ is the connectivity of G , $d(u, v)$ is the distance between vertices u and v , and $\alpha(G)$ is the (vertex) *independence number*, that is, the cardinality of the largest independent vertex subset.

Theorem 2.1.4 *If any one of the following conditions is true for a graph G of order $p \geq 3$, then the graph is hamiltonian.*

1. $\delta(G) \geq \frac{p}{2}$ (Dirac[36]);
2. $\deg_G(u) + \deg_G(v) \geq p$ for any $uv \notin E(G)$ (Ore[79]);
3. $\kappa(G) \geq \alpha(G)$ (Chvátal and Erdős [33]);
4. $d_i + d_{p-i} \geq p$, for degree sequence $d_1 \leq d_2 \leq \dots \leq d_p$, and $i < \frac{p}{2}$ (Chvátal[30]);
5. $\delta(G) \geq \frac{p+\kappa(G)}{3}$, when G is 2-connected graph (Häggkvist and Nicoghossian [52]);
6. $\min\{\max\{\deg_G(u), \deg_G(v)\} : d(u, v) = 2\} \geq \frac{p}{2}$ (Fan [37]);
7. $\max_{d(u,v)=2} |N(u) \cup N(v)| \geq \frac{2p-1}{3}$, when G is 2-connected (Lindquester[72]);
8. $d(U) + d(V) \geq \frac{p+2}{2}$, for an equal bipartite graph $G = (U, V; E)$ (Jackson[57]).

There are many improvements on these results, and there are many other sufficient conditions as well (see survey articles Bermond[11], Berr and Thomassen [14], Bondy[15], Gould [50], etc.). Most sufficient conditions require the graph to be of ‘high density’, have high average degree, or involve a ‘local (forbidden) structure’. All results except (3) in Theorem 2.1.4 require as many as $O(p^2)$ edges. One can show that the Chvátal and Erdős Theorem (3) requires $O(p^{\frac{3}{2}})$ edges by an argument using the complementary version of Turán’s theorem. Our main focus in this chapter is on ‘low density’ graphs, which only require as many as $O(2p)$ edges.

2.2 Circuits and Trails

A long circuit (a long trail) can be measured in two different ways: one way is in terms of the number of edges in the circuit (the trail) and another is in terms of the number of vertices in the circuit (the trail). A *longest circuit* (a *longest trail*) is a circuit (a trail) which contains the maximum number of edges. A *largest circuit* (a *largest trail*) is a circuit (a trail) which contains the maximum number of vertices. A circuit (a trail) is a *dominating circuit* (a *dominating trail*) if every edge of the graph has one end in the circuit (in the internal vertices of the trail) and a circuit (a trail) is a *spanning circuit* (a *spanning trail*) if, in addition, it contains all vertices of the graph. A graph with a spanning circuit is also called *supereulerian*. A graph is *dominating trailable* if for every pair of edges there is a dominating trail joining them. A graph is *spanning trailable* if for every pair of edges there is a spanning trail joining them.

Unlike the *long cycle (path) problem*, which has a vast literature, the *long circuit (long trail) problem* has received attention only in recent years, even though Euler's paper on circuits appeared 250 years ago. Being a useful tool in graph theory, long circuits play an important role just as long cycles do. Circuits are the natural objects in matroid theory. They have many applications in flow problems and in cycle covers. They are also closely related to the Chinese Postman Problem.

Large fixed connectivity can guarantee the graph to be supereulerian. To see this we need a result of Tutte[95] and Nash-Williams [77] on packing edge-disjoint spanning trees (or, equivalently, connected factors). Here $\varpi(G - S)$ is the number of connected components of $G - S$.

Theorem 2.2.1 (Tutte [95] and Nash-Williams [77]) *In order that a graph G is decomposable into n connected factors it is necessary and sufficient that for any edge subset S ,*

$$|S| \geq n(\varpi(G - S) - 1). \quad (2.2)$$

Every 4-edge-connected graph satisfies the inequality (2.2) and hence can be decomposed into two connected factors. Actually Kundu showed more.

Theorem 2.2.2 (Kundu [70]) *Every λ -edge-connected graph has at least $\lceil \frac{\lambda-1}{2} \rceil$ -edge-disjoint spanning trees.*

Jaeger[62, 63] observed that if G has two edge-disjoint spanning trees, then G is supereulerian.

Theorem 2.2.3 (Jaeger [62, 63]) *Every 4-edge-connected graph is supereulerian.*

A 3-edge-connected graph need not be supereulerian. In Theorem 2.1.1 of Jackson and Parsons [60] (when $\epsilon < 1$ and $r = 3$), infinitely many examples of 3-connected non-supereulerian cubic graphs are given since cycle and circuit coincide in cubic graphs. Actually the longest circuit might be quite short by taking a small value for ϵ . One can also obtain infinitely many examples of 3-connected $(2r + 1)$ -regular graphs whose longest and largest circuits are fairly short.

Proposition 2.2.4 *For any given integer $r \geq 1$ and any real $\epsilon > 0$, there exists an integer $M(r, \epsilon)$ such that for all even $p \geq M(r, \epsilon)$, there exists a $(2r + 1)$ -regular 3-connected graph of order p such that the length of a longest circuit and the order of a largest circuit in the graph are less than ϵp .*

Proof. We construct 3-connected graphs H_{2r+3} and H_{2r+5} of orders $2r + 3$ and $2r + 5$, respectively, with exactly three vertices of degree $2r$ and the remaining vertices of degree $2r + 1$ for $r \geq 2$ as follows: H_{2r+3} is the graph obtained from $K_{2r+2} - F$ by adding a new vertex x and edges from x to $2r$ vertices of $K_{2r+2} - F$, where F is a 1-factor of K_{2r+2} . The graph H_{2r+5} is obtained from $K_{2r+4} - (C \cup F)$ by adding a new vertex x and edges from x to $2r + 1$ vertices of $K_{2r+4} - (C \cup F)$, where C is a Hamilton cycle and F is a 1-factor of K_{2r+4} such that $K_{2r+4} - (C \cup F)$ is 3-connected.

The vertices of H_{2r+3} and H_{2r+5} have degree $2r+1$ except three vertices in each graph which have degree $2r$.

Let $\epsilon' = \frac{\epsilon}{(2r+5)(2r+6)}$ and $N(3, \epsilon')$ be the integer of Theorem 2.1.1. We can take $N(3, \epsilon')$ to be even. Let p be an even number and $p \geq (2r+5)N(3, \epsilon')$. Let $p' = \lfloor \frac{p}{2r+3} \rfloor$ and $q = \frac{p - (2r+3)p'}{2}$. Let $G_{p'}$ be a cubic graph whose longest cycle has length less than $\epsilon'p'$. Construct a $(2r+1)$ -regular graph Q of order p from $G_{p'}$ by replacing q vertices of $G_{p'}$ with copies of H_{2r+5} and replacing the remaining vertices of $G_{p'}$ with copies of H_{2r+3} . For each vertex u of $G_{p'}$ let the three edges incident with u be incident with the three vertices of degree $2r$ in the copy H_{2r+3} or the copy H_{2r+5} . Since H_{2r+3} , H_{2r+5} and $G_{p'}$ are 3-connected, Q must be 3-connected of order p .

Let T be a circuit of Q and let C be the corresponding cycle in $G_{p'}$ obtained from Q by shrinking all copies of H_{2r+3} and H_{2r+5} to single vertices. Since $|C| \leq \epsilon'p'$, $|V(T)| \leq \epsilon'p'(2r+5) < \epsilon p$ and $|E(T)| \leq \epsilon'p'(1 + \frac{(2r+5)(2r+1)}{2}) < \epsilon p$. \square

Hence, a 3-connected $(2r+1)$ -regular graph might not have a dominating circuit. On the other hand, the class of graphs which are 3-edge-connected but not 4-edge-connected is very important in graph theory. Most of the problems in nowhere-zero integer flows and cycle covers are involved with this class (see next two sections).

It is well known that the decision problem to determine whether a 3-connected cubic planar graph without faces of length less than 5 is hamiltonian is NP-complete. Therefore the decision problem to determine whether a 3-edge-connected graph is supereulerian is also NP-complete since a circuit is a cycle in cubic graphs. But from Theorem 2.2.3, we know that every 4-edge-connected graph is supereulerian. It is of interest to discuss graphs whose edge-connectivity is between 3 and 4. We introduce a 'fractional' edge-connectivity: A graph G is (n, m) -edge-connected for positive integers m and n if G is n -edge-connected and any n -cut is edge-disjoint from any bond of size $l < m$ except the n -cut itself.

Proposition 2.2.5 *Let l , m and n be positive integers.*

1. Every (n, m) -edge-connected graph is (n', m') -edge-connected for integers $n' < n$ or $n' = n$ but $m' \leq m$.
2. Every $(n+1)$ -edge-connected graph is (n, m) -edge-connected for any integer $m \geq 1$.
3. Every n -edge-connected graph is (n, m) -edge-connected for $m \leq n$.
4. Every 3-edge-connected essentially 7-edge-connected graph with at least 4 vertices other than a multistar is $(3, 6)$ -edge-connected, and in general, every n -edge-connected essentially $(m+n-2)$ -edge-connected graph with $p \geq 4$ other than a multistar is (n, m) -edge-connected for $n \geq 2$.

Proof. In (2), we suppose that G is (n, m) -edge-connected and suppose that $n' < n$. Then G is n' -edge-connected and has no n' -cut. Hence G is (n', m') -edge-connected. We can use the same argument to prove (3).

In (4), let G be n -edge-connected and essentially $(n+m-2)$ -edge-connected. It is trivially true if $m \leq n$. So suppose $m > n$. Then any n -cut $\nabla(X)$ of G must be a trivial cut, i.e., one of its shores is a single vertex. Let $\nabla(Y)$ be a bond different from $\nabla(X)$ and having non-empty intersection with $\nabla(X)$. If both shores of $\nabla(Y)$ have an edge, then by the essential edge-connectivity of G , $\rho(Y) \geq m+n-2 \geq m$. If some shore of $\nabla(Y)$ has no edge, then $\nabla(Y)$ must be a trivial cut. So we can assume $Y = \{y\}$ and $X = \{x\}$. We can also choose y such that $\deg(y)$ is minimum among all possible choice of y . Since the two cuts have a non-empty intersection, the two vertices x and y must be adjacent in G . If $\nabla(\{x, y\})$ is a non-essential cut, then $\rho(\{x, y\}) \geq n+m-2$ which implies $\rho(Y) \geq m$. If $\nabla(\{x, y\})$ is an essential cut, then there is no edge in the subgraph induced by $V(G) - \{x, y\}$ (we will show that this is impossible). Since G is not a multistar and $p \geq 4$, then there is a vertex u other than y adjacent to x . If there is a vertex v adjacent to y but not adjacent to x , then $\nabla(\{y, v\})$ is a proper subset of $\nabla(Y)$. This contradicts to the minimality of $\nabla(Y)$. Hence every vertex in $V(G) - \{x, y\}$ must be adjacent to x . By the edge-connectivity of G and by the minimality of

y , we have $d(u) \geq \max\{d(x), d(y)\}$ for every $u \in V(G) - \{x, y\}$. So we have $\sum\{d(u) : u \in V(G) - \{x, y\}\} \geq (|V(G)| - 2) \max\{d(x), d(y)\} \geq 2 \max\{d(x), d(y)\}$. But $\sum\{d(u) : u \in V(G) - \{x, y\}\} = d(x) + d(y) - 2e(x, y)$. This is also a contradiction. \square

A possible reason why a 3-edge-connected graph might not be supereulerian is that there are too many 3-cuts. In the following result, certain number of 3-cuts are allowed.

Theorem 2.2.6 *Every (3, 6)-edge-connected graph is supereulerian. Every 3-edge-connected essentially 7-edge-connected graph is supereulerian.*

Actually, we will prove a stronger result (Theorem 2.2.9). We split the proof into the following lemmas.

Lemma 2.2.7 *If graph G is a (3, 6)-edge-connected, then for any subset $S \subseteq E(G)$ which is not a 3-cut,*

$$|S| \geq 2\varpi(G - S). \quad (2.3)$$

Proof Let G_1, \dots, G_r be the components of $G - S$ such that $\nabla(V(G_i))$ are 3-cuts in G ; let G_{r+1}, \dots, G_s be the components such that $\nabla(V(G_j))$ are the cuts disjoint from all 3-cuts $\nabla(V(G_i))$ and let G_{s+1}, \dots, G_ϖ be the remaining components of $G - S$ (any of the above three classes of components might be empty). Then for $1 \leq i \leq r$, $r + 1 \leq j \leq s$ and $s + 1 \leq k \leq \varpi$, we have

$$\varrho(V(G_i)) = 3 \quad (2.4)$$

$$\varrho(V(G_j)) \geq 4 \quad (2.5)$$

$$\varrho(V(G_k)) \geq 6 \quad (2.6)$$

$$\bigcup_{k=s+1}^{\varpi} \nabla(V(G_k)) \supseteq \bigcup_{i=1}^r \nabla(V(G_i)) \quad (2.7)$$

$$\sum_{k=s+1}^{\varpi} \varrho(V(G_k)) \geq \sum_{i=1}^r \varrho(V(G_i)) \quad (2.8)$$

$$S \supseteq \bigcup_{t=1}^{\varpi} \nabla(V(G_t)). \quad (2.9)$$

The above equalities or inequalities (2.4), (2.5), (2.6) and (2.9) are direct consequences from the definition and (2.7) and (2.8) are true because every edge in $\nabla(V(G_i))$ must have another end in some $V(G_k)$. Therefore,

$$\begin{aligned} |S| &\geq |\bigcup_{t=1}^{\varpi} \nabla(V(G_t))| \\ &\geq \frac{1}{2} \sum_{t=1}^{\varpi} \varrho(V(G_t)) \\ &= \frac{1}{2} \sum_{i=1}^r \varrho(V(G_i)) + \frac{1}{2} \sum_{j=r+1}^s \varrho(V(G_j)) + \frac{1}{2} \sum_{k=s+1}^{\varpi} \varrho(V(G_k)) \\ &\geq \frac{1}{2} 3r + \frac{1}{2} 4(s-r) + \frac{1}{2} \sum_{k=s+1}^{\varpi} \varrho(V(G_k)). \end{aligned}$$

If $r \leq 2(\varpi - s)$, then, by 2.6, we have

$$\begin{aligned} |S| &\geq \frac{1}{2} 3r + \frac{1}{2} 4(s-r) + \frac{1}{2} 6(\varpi - s) \\ &= 2\varpi + (\varpi - s) - \frac{1}{2} r \\ &\geq 2\varpi. \end{aligned}$$

If $r > 2(\varpi - s)$, then, by 2.8, we have

$$\begin{aligned} |S| &\geq \frac{1}{2} 3r + \frac{1}{2} 4(s-r) + \frac{1}{2} 3r \\ &= 2\varpi + r - 2(\varpi - s) \\ &> 2\varpi. \end{aligned}$$

□

Lemma 2.2.8 *If G is $(3, 6)$ -edge-connected, then for any pair of edges e_1 and e_2 of G , the subgraph $G - \{e_1, e_2\}$, or $G - e_1$, if e_1 and e_2 are in a 3-cut, can be decomposed into two connected factors F_1 and F_2 .*

Proof. *Case 1.* If the edges e_1 and e_2 are in a 3-cut, then let S be an edge subset of $G - e_1$. If S is a 2-cut containing e_2 in $G - e_1$, then

$$|S| = 2 = 2(\varpi((G - e_1) - S) - 1). \quad (2.10)$$

If S is a 3-cut in $G - e_1$, then

$$|S| = 3 > 2(\varpi((G - e_1) - S) - 1). \quad (2.11)$$

If S is none of above, then $S \cup \{e_1\}$ is not a 3-cut and hence by Lemma 2.2.7,

$$|S \cup \{e_1\}| \geq 2\varpi(G - (S \cup \{e_1\})). \quad (2.12)$$

Therefore,

$$|S| \geq 2\varpi((G - e_1) - S) - 1 > 2(\varpi((G - e_1) - S) - 1). \quad (2.13)$$

Case 2. If the edges e_1 and e_2 are not in a 3-cut, then for an edge subset S of $G - \{e_1, e_2\}$, either

$$|S| = 2 = 2(\varpi((G - \{e_1, e_2\}) - S) - 1) \quad (2.14)$$

or

$$|S \cup \{e_1, e_2\}| \geq 2\varpi(G - (S \cup \{e_1, e_2\})).$$

That is,

$$|S| \geq 2(\varpi((G - \{e_1, e_2\}) - S) - 1). \quad (2.15)$$

Applying Theorem 2.2.1 to either case, we can derive two edge-disjoint connected factors from inequalities 2.10, 2.11, 2.12, 2.13, 2.14 and 2.15. \square

Theorem 2.2.9 *If G is $(3, 6)$ -edge-connected, then for any three edges e_1, e_2 and e_3 of G , there are spanning circuits T_1, T_2 and T_3 such that*

$$\{e_1, e_2, e_3\} \subseteq T_1 \quad (2.16)$$

$$\{e_2, e_3\} \subseteq T_2, e_1 \notin T_2 \quad (2.17)$$

$$e_3 \in T_3, e_1 \notin T_3, e_2 \notin T_3 \quad (2.18)$$

unless e_1, e_2 and e_3 form a 3-cut of G in the cases (2.16) and (2.18).

Proof Let F_1 and F_2 be the two connected factors of $G - \{e_1, e_2\}$ or $G - e_1$ from Lemma 2.2.8. Without loss of generality, we assume that $e_3 \in F_1$. Let $F_1^1 = F_1 \cup \{e_1, e_2\}$, $F_1^2 = F_1 \cup \{e_2\}$ and $F_1^3 = F_1$. Let $B^i = O(F_1^i)$, $1 \leq i \leq 3$, be the sets of odd vertices of F_1^i , respectively. Then $|B^i| = 2k_i$ is even, and hence their vertices can be paired off. Let $P_1^i, P_2^i, \dots, P_{k_i}^i$ be paths of F_2 joining the two vertices of each pair. Then by adding the binary sum (or symmetric differences) $P_1^i \Delta P_2^i \Delta \dots \Delta P_{k_i}^i$ to F_1^i , we have

$$T_i = F_1^i \cup (P_1^i \Delta P_2^i \Delta \dots \Delta P_{k_i}^i) \quad (2.19)$$

to be the spanning circuits as desired. \square

Proposition 2.2.10 *If G is $(3, 6)$ -edge-connected, then G is spanning trailable.*

Proof Let $x = uv$ and $y = st$ be any two edges of G . If x and y are independent edges, and if one of the edges $\{us, ut, vs, vt\}$ is not an edge in G , say, $z = us$ is not an edge in G , then by Proposition 2.2.9 there is a spanning circuit T containing x, y and z in $G \cup \{z\}$ (since $G \cup \{z\}$ still satisfies the condition of Proposition 2.2.9). $T - z$ is a spanning trail with end edges x and y in G . If all edges $\{us, ut, vs, vt\}$ are in G , then by Proposition 2.2.9 there is a spanning circuit T containing edge $z = us$ but not x and y . Then $T \Delta \{x, y, z\}$ is a spanning trail with end edges x and y . By Proposition 2.2.9, if x and y are incident with a common vertex (say, $u = s$), then there is a spanning circuit T containing y but not x . Then $T \cup \{x\}$ is a spanning trail

with end edges x and y . □

Consequently, we can obtain the following result.

Theorem 2.2.11 *If G is either 4-edge-connected or 3-edge-connected and essentially 7-edge-connected, then for any three edges e_1 , e_2 and e_3 of G , there are spanning circuits T_1 , T_2 and T_3 such that*

$$\{e_1, e_2, e_3\} \subseteq T_1 \tag{2.20}$$

$$\{e_2, e_3\} \subseteq T_2, e_1 \notin T_2 \tag{2.21}$$

$$e_3 \in T_3, e_1 \notin T_3, e_2 \notin T_3 \tag{2.22}$$

unless e_1 , e_2 and e_3 form a 3-cut of G in the cases (2.20) and (2.22). Furthermore, G is spanning trailable. □

Let G be a $(3, 6)$ -edge-connected graph or 3-edge-connected and essentially 7-edge-connected graph. To see the proof of Theorem 2.2.6, we take T to be the spanning circuit T_1 in G in Theorem 2.2.9 and Theorem 2.2.11 respectively, while e_1 , e_2 and e_3 are any three edges which do not form a 3-cut. Then T is the required circuit in Theorem 2.2.6.

If G is essentially λ_e -edge-connected and G is not a star $K_{1,p-1}$, then we can obtain a 2-edge-connected essentially λ_e -edge-connected graph G' by deleting all vertices of degree 1 and their incident edges. If $\lambda_e \geq 3$, we can obtain a 3-edge-connected essentially λ_e -edge-connected graph G'' by replacing suspended paths (paths whose internal vertices are of degree 2) by edges. G'' is unique up to isomorphism since the above graph operation will not reduce the essential edge-connectivity of the graph. Furthermore, $V(G'')$ is a dominating set of G .

Lemma 2.2.12 *If G is an essentially λ_e -edge-connected graph with $\lambda_e \geq 3$, then G has a dominating circuit if G'' has a spanning circuit. Moreover, G is dominating trailable if G'' is spanning trailable.*

Proof Since $V(G'')$ is a dominating set, a spanning circuit of G'' induces a dominating circuit in G .

Let $x = uv$ and $y = st$ be two edges of G . We choose two edges x' and y' of G'' as follows: if $x \in E(G'')$, choose $x' = x$; if x is incident with a vertex v of degree 2 and if w is the other vertex adjacent to v in G , choose $x' = vw$ in G'' ; and if x is incident with a vertex u of degree 1 in G , choose x' to be any edge of G'' incident with the vertex v . Choose y' the same way, but different from x' if possible. If T is a spanning trail with end edges x' and y' , by naturally restoring the trail back to G , we can easily get a dominating trail with end edges x and y . \square

Proposition 2.2.13 *Every essentially 7-edge-connected graph is dominating trailable.*

Connectivity can not guarantee that a graph has a Hamilton cycle. But if a graph is planar, then Tutte showed that *every 4-connected planar graph is hamiltonian*. That result is a consequence of *Tutte's Bridge Lemma* [97, 101] (see also Ore [80]).

A *plane graph* G is a plane embedding of a planar graph G . A *simple face* of a plane graph is a face whose edges form a cycle. A *simple face* of a planar graph G is a simple face of some plane graph of G . So every face in a 2-connected plane graph is simple.

The theory of *bridges* was developed in Tutte's papers [97, 101], and Tutte's books *Introduction to the Theory of Matroids* [100], *Connectivity in Graph* [98] and *Graph Theory* [102]. Let H be a subgraph of G . A *vertex-attachment*, or briefly, an *attachment*, in G is a vertex of H that is incident in G with some edge not belonging to H . A *bridge* B of H is a subgraph of G that satisfies the following conditions.

1. Each vertex-attachment of B is a vertex of H .
2. B is not a subgraph of H .
3. No proper subgraph of B satisfies conditions (1) and (2).

B is also called an H -bridge of G .

There are two kinds of bridges. A single edge not in H but with both ends in H is a bridge of H . Such a bridge is called a *trivial bridge*. Let B^* be a connected component of $G - V(H)$ with some vertices adjacent to some vertices of $V(H)$. Let B be a subgraph of G consisting of the connected component B^* and the edges with one end in H and the other end in B^* and all end vertices of such edges. Evidently B is an H -bridge, and is called a *proper H -bridge* of G .

We define an *edge-attachment* of H to be an edge of G not in H but with at least one end vertex in H . All edge-attachments of a proper bridge form a cut $\nabla(B^*)$. Therefore, an edge-attachment is also a very natural object in graph theory.

Theorem 2.2.14 (Tutte's Bridge Lemma [97, 101]) *Let G be any planar graph. Let x be an edge lying on simple faces F and K while y is another edge on F . Then there exists a cycle J passing through x and y such that none of its bridges have more than three attachments while the special bridges having edges in common with F and K have at most two attachments.*

Tutte's Bridge Lemma can not be directly generalized to arbitrary graphs since connectivity cannot guarantee the existence of a Hamilton cycle. We propose the following possible generalizations of Tutte's Bridge Lemma.

Conjecture 2.2.15 *Every graph has a circuit C such that every C -bridge has at most three edge-attachments in C .*

Conjecture 2.2.16 *Let G be any graph. Let x be an edge lying on the minimal cycles F and K while y is another edge on F . Then there exists a circuit J passing*

through x and y such that none of its bridges have more than three edge-attachments while the special bridges having edges in common with F and K have at most two edge-attachments.

Theorem 2.2.11 implies that every essentially 7-edge-connected graph has a circuit C such that every C -bridge has at most two edge-attachments in G . Every (3, 6)-edge-connected graph, and hence every 4-edge-connected graph and every 3-edge-connected essentially 7-edge-connected graph, has a circuit such that every C -bridge has only trivial bridges. Moreover, Conjecture 2.2.15 is true for planar graphs.

Proposition 2.2.17 *Let G be any planar graph. Let x be an edge lying on the simple faces F and K while y is another edge on F . Then there exists a circuit J passing through x and y such that none of its bridges have more than three edge-attachments while the special bridges having edges in common with $F \cup K$ have at most two edge-attachments.*

Proof. Proceed by induction on the number of edges of the planar graph. It is easy to check the planar graphs having less than five edges. Let G be any planar graph with n edges.

If G has a vertex cut u , which is also regarded as a subgraph of G , then we can split G into u -bridges G_1, G_2, \dots , and G_ϖ by splitting the vertex u , where ϖ is the number of bridges. The simple faces F and K must be in one of the G_i . Without loss of generality, we assume F and K are in G_1 . By the induction hypothesis, G_1 has a circuit C_1 passing through x and y such that none of the C_1 -bridges have more than three edge-attachments while the special C_1 -bridges having edges in common with $F \cup K$ have at most two edge-attachments. If u is not in C_1 , then C_1 is a circuit of G as required since all C_1 -bridges in G_1 are also C_1 -bridges in G except one bridge, say B , of G_1 which contains the vertex u while the corresponding C_1 -bridge in G consists of B and $G_2, G_3, \dots, G_\varpi$. If u is in C_1 , then let G_2, G_3, \dots, G_t be these C_1 -bridges which have at least four edge-attachments. These bridges must share a common vertex (attachment) u in G . All edge-attachments of G_i must

be in a common block of G_i ; and hence we can find a simple face F_i containing two edge-attachments, say x_i and y_i . By the induction hypothesis, there exists a circuit C_i in G_i passing through x_i and y_i such that none of its bridges have more than three edge-attachments while the special bridges having edges in common with F_i have at most two edge-attachments. The circuit C_i contains both edges and hence contains the vertex u . The circuit $C = C_1 \cup (\cup_{i=2}^t C_i)$ is the required circuit in G since every C -bridge of G is a C_i -bridge for some $i = 1, 2, \dots, t$.

So we may assume that G is a 2-connected plane graph such that F and K are faces of the plane graph. Let $V_{\geq k}$ be the set of vertices of degree at least k . Let G' denote the *pseudo-cubic* plane graph derived from the plane graph G by replacing each vertex $u \in V_{\geq 4}$ by a cycle C_u , which also forms a simple face of G' , of length equal to $\deg_G(u)$ and by distributing the incident edges of u to the vertices of the cycle in a cyclic fashion in the plane. Let F' and K' be the (unique) faces expanded from F and K by this process and let x' and y' be the edges in G' corresponding to x and y . F' and K' are simple faces since F and K are simple faces. By Tutte's Bridge Lemma, there exists a cycle J passing through x' and y' such that none of J -bridges have more than three vertex-attachments while the special J -bridges having edges in common with $F' \cup K'$ have at most two vertex-attachments. Conversely, G can be obtained from G' by shrinking all cycles C_u for $u \in V_{\geq 4}$ (and by deleting loops). Let C be the corresponding circuit in G obtained from J by this shrinking process. If C is the required circuit, then the proof is completed. Otherwise, there exist C -bridges of G such that they have at least four edge-attachments or there exist C -bridges which have edges in common with $F \cup K$ and have at least three edge-attachments. After shrinking all C_u for $u \in V_{\geq 4}$, the subgraph of G corresponding to a J -bridge B' of G' must be a union of some C -bridges of G . On the other hand, for every C -bridge B of G there must be a J -bridge, denoted by $\pi(B)$, of G' such that the corresponding subgraph in G contains B . Since G is 2-connected, every C -bridge in G must have at least two vertex-attachments. First we show that if a C -bridge B has at least four edge-attachments, then it has exactly two vertex-attachments in which one is incident with one of the edge-attachment of B and the other is incident with the remaining

edge-attachments of B , and show that if a C -bridge B has some edges in common with $F \cup K$, then B has at most two edge-attachments.

Let u be a vertex-attachment of B . Then it must be incident with some edge-attachments of B . If $u \notin V_{\geq 4}$, then u must be incident with one edge-attachment, which also corresponds to an edge-attachment of $\pi(B)$ in G' . If $u \in V_{\geq 4}$ and if u is incident with one edge-attachment e of B in G , then there must be at least one edge-attachment, which is either the edge corresponding to e or some edge in the cycle C_u of G' , of $\pi(B)$ in G' incident with vertices of C_u . If $u \in V_{\geq 4}$ and if u is incident with at least two edge-attachments of B in G , then there must be at least two edge-attachments, which are either the edges of G' corresponding to the edge-attachments of B or some edges appearing in pairs in the cycle C_u of G' , of $\pi(B)$ in G' incident with vertices of C_u . These edge-attachments corresponding to different vertices of $V_{\geq 4}$ are distinct. Hence, if B has at least four edge-attachments in G , then $\pi(B)$ has at least four edge-attachments in G' unless B has exactly two vertex-attachments in which one is incident with one of the edge-attachment of B and the other is incident with the remaining edge-attachments of B , and if B has some edges in common with $F \cup K$ and has at least three edge-attachments in G , then $\pi(B)$ must have some edges in common with $F' \cup K'$ and have at least three edge-attachments. The latter case is impossible since the number of vertex-attachments is the same as the number of edge-attachments of a bridge in a pseudo-cubic graph. So if B has at least four edge-attachments in G , then $\pi(B)$ has at least four edge-attachments in G' unless B has exactly two vertex-attachments in which one is incident with one of the edge-attachment of B and the other is incident with the remaining edge-attachments of B .

Let B_1, B_2, \dots, B_k be the C -bridges having at least four edge-attachments in G . Let v_i be the vertex-attachment of B_i which is incident with exactly one edge-attachment of B_i and also let w_i be the other end of the edge-attachment. Let u_i be the vertex-attachment of B_i which is incident with the remaining edge-attachments of B_i for $1 \leq i \leq k$. Let B_i^* be the graph obtained from B_i by deleting the vertex v_i and the edge $v_i w_i$. Since G is 2-connected, the graph obtained from B_i by identifying

the vertices u_i and v_i must be 2-connected. Hence the block-cut graph of B_i^* is a path while one end block contains the vertex u_i and another end block contains w_i . Let $T_i^1, T_i^2, \dots, T_i^{r_i}$ be the consecutive blocks containing at least two edges in B_i^* where T_i^1 is the block containing u_i . Let z_i^j be the vertex in T_i^j and T_i^{j+1} for $1 \leq j \leq r_i - 1$. Let $z_i^0 = u_i$ in T_i^1 and let $z_i^{r_i} = w_i$ in $T_i^{r_i}$ if $w_i \in T_i^{r_i}$. The vertices z_i^{j-1} and z_i^j are in the outer face of T_i^j . Take any two edges from the outer face incident with each of these vertices z_i^{j-1} and z_i^j . By the induction hypothesis, there exists a circuit C_i^j passing through the two selected edges such that none of the C_i^j -bridges of T_i^j have more than three edge-attachments while the special bridges having edges in common with the outer face of T_i^j have at most two edge-attachments. Finally, the extended circuit $C_1 \cup \cup_{i,j} C_i^j$ is the required circuit in G . \square

If either Conjecture 2.2.15 or Conjecture 2.2.16 is true, then it implies the following conjectures.

Conjecture 2.2.18 (Bondy [44]) *There exists a constant c , $0 < c < 1$, such that every cyclically 4-edge-connected cubic graph has a cycle of length at least cp .*

Conjecture 2.2.19 (Jaeger [44]) *Every cyclically 4-edge-connected graph G has a cycle C such that $G - V(C)$ is acyclic.*

Conjecture 2.2.20 (Jackson, Fleischner [44]) *Every cyclically 4-edge-connected cubic graph has a dominating cycle.*

Conjecture 2.2.21 (Thomassen [91]) *Every 4-edge-connected line graph is hamiltonian.*

Actually, the circuit C in Conjecture 2.2.15 and the circuit J in Conjecture 2.2.16 are dominating circuits when G is 3-edge-connected and cyclically 4-edge-connected. Hence both of them imply Conjecture 2.2.20 and Conjecture 2.2.21. Conjecture 2.2.18 and Conjecture 2.2.19 are weaker than Conjecture 2.2.20 and Conjecture 2.2.21. We may propose the following stronger conjectures.

- Conjecture 2.2.22** 1. *Every 3-edge-connected graph is supereulerian if and only if it is not contractable to a cubic non-hamiltonian graph.*
2. *Every 3-edge-connected essentially 4-edge-connected non-cubic graph is supereulerian.*
3. *Every cyclically 4-edge-connected cubic graph has a 2-factor consisting of at most two cycles.*
4. *Every 3-edge-connected essentially 4-edge-connected graph has a spanning trail.*
5. *Every cyclically 4-edge-connected cubic graph has a hamiltonian path.*
6. *Every 3-edge-connected essentially 5-edge-connected graph is supereulerian.*

One can also make similar conjectures on planar graphs.

- Conjecture 2.2.23** 1. *Every 3-edge-connected planar graph is supereulerian if and only if it is not contractable to a cubic non-hamiltonian planar graph.*
2. *Every 3-edge-connected essentially 4-edge-connected non-cubic planar graph is supereulerian.*
3. *Every cyclically 4-edge-connected cubic planar graph has a 2-factor consisting of at most two cycles.*
4. *Every 3-edge-connected essentially 4-edge-connected cubic graph has a spanning trail.*
5. *Every cyclically 4-edge-connected cubic planar graph has a Hamilton path.*
6. *Every 3-connected essentially λ_e -edge-connected planar graph is hamiltonian for some $\lambda_e \geq 5$.*
7. *Every cyclically 4-edge-connected cubic bipartite planar graph has a Hamilton path.*

It is easy to see that Conjecture 2.2.22 (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5), (2) \Rightarrow (6), and Conjecture 2.2.22 (i) \Rightarrow Conjecture 2.2.23 (i), for $1 \leq i \leq 6$. In the next section, we will see that spanning and dominating circuits (trails) are closely related to the *nowhere-zero 5-flow conjecture* and the *cycle double conjecture*. Actually, the above Conjecture 2.2.22 (1)-(5) on trails or Hamilton paths will imply both of them.

2.3 Hamilton Cycles of Line Graphs

Harary and Nash-Williams [53] derived the relation between the hamiltonicity of the line graph and the dominating cyclability of the graph itself.

Theorem 2.3.1 (Harary and Nash-Williams [53]) *If G is a graph with at least four vertices, then its line graph $L(G)$ is hamiltonian if and only if G has a dominating circuit.*

There are many results on hamiltonian line graph in recent years. All results in Theorem 2.2.19 also imply that the graph $L(G)$ is hamiltonian. Following are some selected results.

Theorem 2.3.2 *If G is a graph of order p , then each of the following conditions imply that $L(G)$ is hamiltonian.*

1. $\deg_G(u) + \deg_G(v) \geq p - 1$, for any $uv \notin E(G)$ (Lesniak and Williamson [71]).
2. $\deg_G(u) + \deg_G(v) \geq p$, for any $uv \in E(G)$ (Brualdi and Shanny [17]).
3. $\deg_G(u) + \deg_G(v) \geq 2\lfloor \frac{p-2}{2} \rfloor$, for any $uv \in E(G)$ (Clark [34]).
4. $\deg_G(u) + \deg_G(v) \geq \frac{2p+3}{3}$, for any $uv \notin E(G)$ in a 2-edge-connected graph G (Benhocine, Clark, Kohler and Veldman [9]).
5. $\deg_G(u) + \deg_G(v) + \deg_G(w) \geq p + 1$, for any three independent vertices in a 2-edge-connected graph G (Catlin [22]).
6. $\deg_G(u) + \deg_G(v) \geq \frac{n}{5} - 2$, for any $uv \in E(G)$ in a 3-edge-connected graph which is not contractable to the Petersen graph P_{10} (Chen and Lai [28],[29]).

In [76], Matthews and Sumner obtained a 3-connected non-hamiltonian claw-free graph, which is also a line graph of a graph. Again one can obtain infinitely many

3-connected non-hamiltonian line graphs $L(G)$ by setting $r = 3$ and $\epsilon < \frac{3}{4}$ for the graph G in Theorem 2.1.1 and by applying Theorem 2.3.1.

Thomassen observed that the line graph of a 4-edge-connected graph is hamiltonian. The proof can be obtained from Theorem 2.2.3 and Theorem 2.3.1.

Theorem 2.3.3 (Thomassen [93]) *The line graph of a 4-edge-connected graph is hamiltonian.*

Thomassen conjectured that every 4-connected line graph is hamiltonian.

Conjecture 2.3.4 (Thomassen [93, 14]) *Every 4-connected line graph is hamiltonian.*

Similar to Theorem 2.3.1, in [110] we proved the following result.

Lemma 2.3.5 *If G is a graph with at least four vertices, then its line graph $L(G)$ is Hamilton-connected if and only if G has a dominating trail.*

From Corollary 2.2.11, Corollary 2.2.13 and Lemma 2.3.5, we can obtain the following result.

Theorem 2.3.6 (Zhan [110, 111]) *Every 7-connected line graph is Hamilton-connected. Every line graph of a 4-edge-connected graph is Hamilton-connected.*

Proof. From Lemma 2.3.5, we only need to show that G is dominating trailable. If $L(G)$ is 7-connected, then G must be either essentially 7-edge-connected or G is a star. By Proposition 2.1.13, G is dominating trailable. If G is 4-edge-connected, by Theorem 2.2.11 G is spanning trailable. hence $L(G)$ is Hamilton-connected. \square

To support Thomassen's Conjecture, we can prove that Thomassen's conjecture is true for planar graphs.

Theorem 2.3.7 *Every 4-connected line graph of a planar graph is hamiltonian.*

Proof. Let G be a planar graph such that its line graph $L(G)$ is 4-connected. Then G is either a star or essentially 4-edge-connected. Let J be the circuit in Proposition 2.2.17 while x and y are any edges and F and K are any faces satisfying the assumption of Proposition 2.2.17. If G is a star, then $L(G)$ is a complete graph and hence is hamiltonian. If G is essentially 4-edge-connected, then G has a dominating circuit J . $L(G)$ is also hamiltonian. \square

2.4 Flows and Cycle Covers

In this section, we consider some applications of the longest (or the largest) circuits or trails. We will address the relationship of the longest (or the largest) circuits or trails to nowhere-zero integer flow problems and cycle cover problems.

2.4.1 Nowhere-zero Integer Flows

The concept of a *flow* in a graph is a useful model of *Operation Research*, and it is also essentially identical to the concept of a *current* in an electrical network. The concept of a flow can be regarded as the dual concept of *tension* (or *potential difference*) (see Jaeger [67]. Tutte observed that the whole theory of vertex-colourings of graphs can be formulated in terms of tension. In a plane graph, the *four-colour theorem* can be reformulated in terms of flows: *Every 2-edge-connected planar graph has a nowhere-zero 4-flow*. Tutte [99] introduced the concept of nowhere-zero integer flows and also made the following conjectures.

Conjecture 2.4.1 (Tutte [99])

1. *Every 2-edge-connected graph has a nowhere-zero 5-flow;*
2. *Every 2-edge-connected graph with no minor isomorphic to the Petersen graph P_{10} has a nowhere-zero 4-flow;*
3. *Every 2-edge-connected graph with no 3-cuts has a nowhere-zero 3-flow.*

Jaeger [63] showed that every 2-edge-connected graph has a nowhere-zero 8-flow and Seymour [86] improved Jaeger's result to a nowhere-zero 6-flow. We restate their results as follows.

Theorem 2.4.2 (Jaeger [63]) *Every 2-edge-connected graph has a nowhere-zero 8-flow.*

Theorem 2.4.3 (Seymour [86]) *Every 2-edge-connected graph has a nowhere-zero 6-flow.*

There are several other partial results related to Tutte's flow conjectures. Hoffman, Locke and Meyerowitz [55] showed that every Cayley graph of degree at least 2 has a cycle double cover.

Theorem 2.4.4 (Jaeger [64]) *Every 2-edge-connected graph without 3-cuts has a nowhere-zero 4-flow.*

Theorem 2.4.5 (Jaeger [67]) *If $G - e$ has a nowhere-zero 4-flow for some edge e of a 2-edge-connected graph G , then G has a nowhere-zero 5-flow.*

A dual version of Grötsch's well known 3-colour theorem on triangle-free planar graphs can be formulated as follows.

Theorem 2.4.6 (Grötsch [51]) *Every 2-edge-connected planar graph without 3-cuts has a nowhere-zero 3-flow.*

Theorem 2.4.7 (Steinberg and Younger [87, 88]) *Every 2-edge-connected graph embeddable in a real projective plane has a nowhere-zero 5-flow. Every 2-edge-connected graph with at most one 3-cut embeddable in a real projective plane has a nowhere-zero 3-flow.*

Lemma 2.4.8 *If a 2-edge-connected graph G has a spanning circuit, then G has a nowhere-zero 4-flow. If a 2-edge-connected graph G has a spanning trail, then G has a nowhere-zero 5-flow.*

Proof. Let T be a spanning circuit in G and let $H = G - E(T)$. Then $O(H)$, the set of odd vertices in H , can be paired off into $\frac{|O(H)|}{2}$ pairs. Let P_i be a path in T joining the i -th pair of vertices. Hence $J = H \cup (P_1 \Delta P_2 \Delta \cdots \Delta P_{\frac{|O(H)|}{2}})$ is an even

subgraph of G . Since $E(J) \cup E(T) = E(G)$ and J and T both have a Z_2 -flow, G has a $Z_2 \times Z_2$ -flow. If G has a spanning trail with ends u and v , then the graph $G + uv$ has a spanning circuit and hence has a nowhere-zero 4-flow. Applying Theorem 2.4.5 to $G + uv$, one can obtain a nowhere-zero 5-flow. \square

Therefore, the truth of Conjecture 2.2.22 (1)-(5) would imply the nowhere-zero 5-flow conjecture is true.

Theorem 2.4.9 *If a graph G is a $(3, 6)$ -edge-connected, then G has a nowhere-zero 4-flow.*

Proof. It is a direct consequence of Proposition 2.4.8 and Lemma 2.2.9. \square

Theorem 2.4.10 *Every 3-edge-connected essentially 7-edge-connected graph has a nowhere-zero 4-flow.*

We propose the following conjectures.

Conjecture 2.4.11 *If G is $(3,4)$ -edge-connected, then G has a nowhere-zero 4-flow. Every 3-edge-connected essentially 5-edge-connected graph has a nowhere-zero 4-flow.*

Conjecture 2.4.12 *If G is $(3,4)$ -edge-connected, then G has a nowhere-zero 3-flow. Every 3-edge-connected essentially 5-edge-connected graph has a nowhere-zero 3-flow.*

2.4.2 Cycle Covers.

A family of cycles of G is a *cycle double cover* of G if every edge of G is in precisely two of these cycles. A *k -circuit double cover* of G is a family of k even subgraphs such that every edge in precisely two of these even subgraphs. A *k -circuit double cover* is a

cycle double cover. Seymour [85] showed that every planar graph has a cycle double cover. A result of Alspach, Goddyn and Zhang [3] on cycle covers implies that every 2-edge-connected graph without a Petersen graph minor has a cycle double cover. Tarsi [90] (see also [49] for a short proof) showed that every 2-edge-connected graph with a Hamilton path has a cycle double cover. We restate the theorem as follows.

Theorem 2.4.13 (Tarsi [90]) *Every 2-edge-connected graph with a Hamilton path has a 6-circuit double cover.*

Therefore, the truth of Conjecture 2.2.22(1-5) would imply the *cycle double cover conjecture*. Catlin [23] eventually showed that if a 2-edge-connected graph has at most ten 3-cuts (improved to at most 13 3-cuts in [24]) and is not contractable to the Petersen graph, then the graph has a 3-circuit double cover.

Theorem 2.4.14 (Alspach [75]) *If a 2-edge-connected graph G has a cycle through all odd vertices of G , then G has a cycle double cover.*

Then M^cGuinness [75] generalized the preceding to circuits.

Theorem 2.4.15 (M^cGuinness [75]) *If a 2-edge-connected graph G has a circuit through all odd vertices of G , then G has a cycle double cover.*

Bondy showed that if a graph has two edge disjoint spanning trees, then G has a cycle double cover (see M^cGuinness [75]). Therefore, every 4-edge-connected graph has a cycle double cover. By Lemma 2.2.9 and Theorem 2.4.15, we can conclude that the following result is true.

Theorem 2.4.16 *If a graph G is a $(3, 6)$ -edge-connected, then G has a 3-circuit double cover. And if a graph G is 2-edge-connected essentially 7-edge-connected, then G has a 3-circuit double cover.*

Proof. If G has a spanning circuit C , then $S = O(G) = O(G - E(C))$, the set of odd vertices, must be even and hence can be paired off. Let P_i be a path in C joining the i -th pair of such vertices and let $P = P_1 \Delta P_2 \cdots P_{\lfloor \frac{|S|}{2} \rfloor}$ be the binary sum of the paths. Then C , $P \cup (G - E(C))$ and $(C - E(P)) \cup (G - E(C))$ are the required three even subgraphs which cover G exactly twice. If G is $(3, 6)$ -edge-connected, then G has such a spanning circuit. If G is 2-edge-connected essentially 7-edge-connected, then G is a subdivision of some 3-edge-connected essentially 7-edge-connected graph H . We know that H has a spanning circuit and hence has a 3-circuit double cover. It is also easy to see that a graph has a k -circuit double cover if and only if a subdivision of G has a k -circuit double cover. Therefore, the graph G has a 3-circuit double cover. \square

It is also conjectured [75] that every 2-edge-connected graph with a dominating circuit has a cycle double cover.

An *edge cycle cover*, or briefly a *cycle cover*, of a graph is a collection of cycles such that each edge appears in at least one of the cycles. A *vertex cycle cover* is a collection of cycles such that each vertex appears in at least one of the cycles.

A problem related to nowhere-zero integer flows and cycle double covers is to minimize the number of edges (i.e., $\sum_{C \in \mathcal{C}} |C|$) in a cycle cover \mathcal{C} of a 2-edge-connected graph. Itai and Rodeh made the following conjecture.

Conjecture 2.4.17 (Itai and Rodeh [56]) *Every 2-edge-connected graph has a cycle cover using at most $|E(G)| + |V(G)| - 1$ edges.*

Bermond, Jackson and Jaeger [13] also made a similar conjecture on vertex cycle covers. They also commented that the bound is the best possible because of the graph $K_{n,2}$.

Conjecture 2.4.18 (Bermond, Jackson and Jaeger [13]) *Every 2-edge-connected graph has a vertex cycle cover using at most $2|V(G)| - 2$ edges.*

Itai and Rodeh [56] showed that Conjecture 2.4.17 is true for graphs with two edge-disjoint spanning trees. Bermond, Jackson and Jaeger [13] showed that Conjecture 2.4.17 is true for planar graphs. U. Jamshy, A. Raspaud and M. Tarsi [68], and C. Q. Zhang [113] proved that if a 2-edge-connected graph has a nowhere-zero 3-flow, then there is a cycle cover using at most $|E(G)| + |V(G)| - 3$ edges. G.-H. Fan [38] showed that if a 2-edge-connected graph has a nowhere-zero 4-flow, then there is a cycle cover using at most $|E(G)| + |V(G)| - 2$ edges. A. Raspaud [82] showed that if a 2-edge-connected graph has a nowhere-zero 4-flow, then there is a cycle cover using at most $|E(G)| + |V(G)| - 3$ edges. There are also several other partial results on Conjecture 2.4.17. Bermond, Jackson and Jaeger [13] proved that there is a cycle cover of a 2-edge-connected graph with at most $|E(G)| + \min\{\frac{2}{3}|E(G)|, \frac{7}{3}(|V(G)| - 1)\}$ edges by using Jaeger's 8-flow theorem. P. Fraisse [46] showed that there is a cycle cover of a 2-edge-connected graph with at most $|E(G)| + \frac{5}{4}(|V(G)| - 1)$ edges.

- Theorem 2.4.19** 1. *Every (3, 6)-edge-connected graph has a cycle cover using at most $|E(G)| + |V(G)| - 1$ edges.*
2. *Every 2-edge-connected essentially 7-edge-connected graph G has a cycle cover using at most $|E(G)| + |V(G)| - 1$ edges.*

Proof. If G is a (3,6)-edge-connected graph, then by Theorem 2.2.9, there is a spanning circuit T in G . Pair off the vertices of $O(G - E(T))$, the set of odd vertices of $G - T$. Let P_i be a path in T joining the vertices of the i -th pair of $O(G - T)$. Then the vertex degrees of the subgraph $H = P_1 \Delta P_2 \Delta \cdots \Delta P_{\frac{|O(G-E(T))|}{2}}$ have the same parity as in G and $G - E(T)$. Let K be the subgraph obtained by deleting edges of cycles of H . So K is a forest and has the same parity as H and hence has the same parity as $G - E(T)$. Therefore $(G - E(T)) \cup K$ is an even subgraph. The cycles of T and $(G - E(T)) \cup K$ form a cycle cover using at most $|E(G)| + |E(K)| \leq |E(G)| + |V(G)| - 1$ edges. Each edge of G is in at most two cycles of the cycle cover.

If G is 2-edge-connected essentially 7-edge-connected, then let G' be the graph obtained from G by replacing all suspended paths by edges joining the end vertices

of the suspended paths. Then G' must be $(3, 6)$ -edge-connected. By (1), there is a cycle cover of G' using at most $|E(G')| + |V(G')| - 1$ edges and each edge of G' is in at most two cycles of the cycle cover. Restoring the graph G back from G' , we can obtain a cycle cover of G , corresponding to the cycle cover of G' , using at most $|E(G)| + |V(G)| - 1$ edges. \square

For the vertex cycle cover conjecture (Conjecture 2.4.18), Bermond, Jackson and Jaeger [13] showed that there is a vertex cycle cover of a 2-edge-connected graph G using at most $\frac{10}{3}(|V(G)| - 1)$ edges. P. Fraisse [46] showed that there is a vertex cycle cover of a 2-edge-connected graph G with at most $\frac{50}{23}(|V(G)| - 1)$ edges. He also showed that Conjecture 2.4.18 is true for planar graphs. X. Yu [109] proved it is true for 2-edge-connected graphs with no Petersen minors by using a theorem of Alspach, Goddyn and Zhang [3]. By using the longest circuits, we show here that the conjecture is true for 2-edge-connected graphs with minimum degree at least 3.

Theorem 2.4.20 *If a 2-edge-connected graph G has minimal degree $\delta(G) \geq 3$, then there is a spanning even subgraph with at most $2|V(G)| - 2$ edges.*

Lemma 2.4.21 *If G is a connected graph of order p , then there is a connected spanning subgraph G' such that $O(G) = O(G')$ and $E(G') \leq 2(p - 1)$.*

Proof. If $|S| \geq 2(\varpi(G - S) - 1)$ for any edge subset $S \subseteq E(G)$, then by Theorem 2.2.1, there are two edge-disjoint spanning trees T_1 and T_2 in G . Let $U = O(G) \Delta O(T_1)$. Then $|U|$ must be even and hence the vertices U can be paired off. Let P_i be a path in T_2 joining the i -th pair of U . Therefore, $G' = T_1 \cup (P_1 \Delta P_2 \Delta \cdots \Delta P_{\frac{|U|}{2}})$ is a connected spanning subgraph of G with $O(G) = O(G')$ and $E(G') \leq 2(p - 1)$. If $|S| < 2(\varpi(G - S) - 1)$ for some edge subset $S \subseteq E(G)$, then let S_0 be such a subset such that $\varpi(G - S_0)$ is the largest. If $\varpi(G - S_0) = p$, then $|E(G)| \leq 2(p - 1)$ and hence $G' = G$ is desired. So we can assume that $\varpi(G - S_0) < p$, and that $G_1, G_2, \dots, G_\omega$ are the connected components of $G - S_0$ with at least one of them non-trivial. By

the maximality of $\varpi(G - S_0)$, we can conclude that for each non-trivial component G_k ,

$$|S| \geq 2(\varpi(G_k - S) - 1)$$

for any edge subset $S \subseteq E(G_k)$. Suppose that there is a subset $S_k \subseteq E(G_k)$ such that $|S_k| < 2(\varpi(G_k - S_k) - 1)$. Then $\varpi(S_0 \cup S_k) = \varpi(G - S_0) + \varpi(G - S_k) - 1$ and hence

$$2(\varpi(S_0 \cup S_k) - 1) = 2(\varpi(G - S_0) - 1) + 2(\varpi(G - S_k) - 1) > |S_0| + |S_k| \geq |S_0 \cup S_k|. \quad (2.23)$$

This contradicts the maximality of $\varpi(G - S_0)$. So for each non-trivial component G_k of $G - S_0$ and for any edge subset $S \subseteq E(G_k)$,

$$|S| \geq 2(\varpi(G_k - S) - 1).$$

Therefore, by the above arguments, there exist connected spanning subgraphs G'_k of G_k for each non-trivial component such that $O(G_k) = O(G'_k)$ and $E(G'_k) \leq 2(|V(G_k)| - 1)$. Let G' be the graph obtained from G by replacing each non-trivial component G_k of $G - S_0$ by G'_k . We have $O(G') = O(G)$ and $E(G') \leq 2(p - 1)$ as required. \square

Corollary 2.4.22 *Every eulerian graph has a connected spanning circuit with at most $2(p - 1)$ edges. Every even graph has a spanning even subgraph with at most $2(p - 1)$ edges.*

Corollary 2.4.23 *Every 4-edge-connected graph and every 3-edge-connected essentially 7-edge-connected graph have a spanning circuit containing less than $2(p - 1)$ edges. They also have a spanning circuit containing at least $q - p + 1$ edges.*

The following result is in Lovász's book "Combinatorial Problems and Exercises" 7.42(b), page 54.

Lemma 2.4.24 (Lovász [73]) *If G is a 2-edge-connected graph with minimal degree at least 3, then there is a spanning even subgraph.*

Proof of Theorem 2.4.20. Let G be a 2-edge-connected graph with minimal degree at least 3. Then by Lemma 2.4.24, there is a spanning even subgraph G' . Applying Corollary 2.4.22 to G' , We can obtain a spanning even subgraph with at most $2(p - 1)$ edges. \square

Theorem 2.4.20 can be improved slightly to 2-edge-connected graphs with at most two vertices of degree 2.

Chapter 3

Edge-Disjoint Hamilton Cycles

3.1 Introduction

A graph G has a *Hamilton cycle decomposition* if its edge set $E(G)$ can be decomposed into edge-disjoint Hamilton cycles and possibly one perfect matching. The graph G is also called *Hamilton (cycle) decomposable*. For convenience, K_1 and K_2 are regarded as Hamilton decomposable. Similarly, one can define a graph G to be *Hamilton path decomposable*.

A matching \mathbf{M} is *suborthogonal* to a Hamilton cycle decomposition if each Hamilton cycle in the decomposition contains at most one matching edge; a matching \mathbf{M} is *orthogonal* to a Hamilton cycle decomposition if each Hamilton cycle in the decomposition contains precisely one matching edge. A matching \mathbf{M} is *almost orthogonal* to a Hamilton cycle decomposition if each Hamilton cycle but one in the decomposition contains precisely one matching edge. One can also define a matching to be *suborthogonal* (or *orthogonal*, or *almost orthogonal*) to a Hamilton path decomposition.

If two graphs G and H are Hamilton decomposable, one may ask whether the graph obtained from G and H by a certain binary graph operation, e.g., cartesian product, lexicographic product, or conjunction, is Hamilton decomposable. There are

many results on these topics [5], [6], [11], [7], [89], and there are several good survey papers on this topic [11] and [4]. If a graph G is Hamilton decomposable, one may also ask whether the graph obtained from G by a monotone graph operation, such as taking the line graph, preserves the Hamilton decomposable property.

In 1957, Kotzig [69] first studied the Hamilton decomposition of line graphs of cubic graphs. Kotzig proved that a cubic graph has a Hamilton cycle if and only if its line graph has a Hamilton decomposition. Later in 1983, Jaeger [65] showed that if a 4-regular graph G is Hamilton decomposable, then its line graph $L(G)$ is also Hamilton decomposable. Recently, Jackson's results on compatible eulerian tours of 4-regular graphs imply that if a 4-regular graph G is 3-edge-connected, then its line graph is Hamilton decomposable. We restate these results as follows.

Theorem 3.1.1 (Kotzig, [69]) *A cubic graph G has a Hamilton cycle if and only if its line graph $L(G)$ can be decomposed into two Hamilton cycles.*

Theorem 3.1.2 (Jaeger, [65]) *If a 4-regular graph G can be decomposed into two Hamilton cycles, then its line graph can be decomposed into three Hamilton cycles.*

Theorem 3.1.3 (Jackson, [59]) *If a 4-regular graph is 3-edge-connected, then its line graph can be decomposed into three Hamilton cycles.*

Bermond made following general conjecture

Conjecture 3.1.4 (Bermond, [12]) *If an r -regular graph can be decomposed into $\lfloor \frac{r}{2} \rfloor$ Hamilton cycles and possibly a 1-factor, then its line graph can be decomposed into $r - 1$ Hamilton cycles.*

As far as I know, even for complete graphs K_n and for $n = 6$ or $n \geq 8$ the conjecture is still open, which was also conjectured by McKay (see, [4]).

Conjecture 3.1.5 (McKay) *$L(K_n)$ can be decomposed into $n - 2$ Hamilton cycles for every $n \geq 3$.*

In this chapter, our main result is the following theorem. In section 3.2.1 and section 3.3, we will provide some necessary tools used in the decompositions. We will show the main theorem in section 3.4.

Theorem 3.1.6 *Let G be a graph with a Hamilton cycle and minimal degree δ . If $|\deg(u) - \deg(v)| \leq 1$ when δ is odd, and $|\deg(u) - \deg(v)| \leq 2$ when δ is even for any two vertices u and v , then its line graph $L(G)$ has at least $2\lfloor \frac{\delta}{2} \rfloor - 2$ edge-disjoint Hamilton cycles.*

Corollary 3.1.7 *The line graph of an r -regular hamiltonian graph has at least $2\lfloor \frac{r}{2} \rfloor - 2$ edge-disjoint Hamilton cycles.*

As a consequence, we can get a partial result related to Bermond's conjecture.

Corollary 3.1.8 *If a graph G can be decomposed into r Hamilton cycles, then its line graph $L(G)$ can be decomposed into at least $2r - 2$ Hamilton cycles and a 2-factor.*

If a graph G can be decomposed into r Hamilton cycles and a 1-factor, then its line graph $L(G)$ can be decomposed into at least $2r - 2$ Hamilton cycles and a 4-factor.

Using Dirac's theorem, we can get the following corollary.

Corollary 3.1.9 *If a graph G has minimal degree $\delta \geq \frac{|V(G)|}{2}$ and for any two vertices u and v of G , $|\deg(u) - \deg(v)| \leq 1$ when δ is odd, and $|\deg(u) - \deg(v)| \leq 2$ when δ is even, then its line graph $L(G)$ has at least $2\lfloor \frac{\delta}{2} \rfloor - 2$ edge-disjoint Hamilton cycles.*

3.2 Almost Even Factorizations

3.2.1 Balanced Orientations

Recall that an *orientation* of an undirected graph G is digraph which can be obtained from G by assigning each edge an *origin* and a *terminus*. An orientation of G is denoted by \vec{G} . For a vertex u , we use $E_{\vec{G}}^+(u)$ to denote the set of edges of G which have u as an origin in \vec{G} ; and $E_{\vec{G}}^-(u)$ to denote the set of edges of G which have u as a terminus in \vec{G} .

Therefore,

$$od_{\vec{G}}(u) = |E_{\vec{G}}^+(u)|$$

and

$$id_{\vec{G}}(u) = |E_{\vec{G}}^-(u)|$$

are the outdegree and indegree of \vec{G} , respectively.

A digraph D is balanced if for each vertex $u \in V(D)$,

$$|id_D(u) - od_D(u)| \leq 1. \quad (3.1)$$

Lemma 3.2.1 *Every graph G has a balanced orientation.*

Proof If the graph G is an even graph, then the orientation \vec{G} of G along an eulerian tour of each connected component of G is balanced. Actually, $id_D(u) = od_D(u)$ for all $u \in V(G)$. If G is not a even graph, then the number of odd vertices of G must be even and non-zero. The supergraph obtained from G by adding a new vertex and adding new edges between the odd vertices of G and the new vertex must be an even supergraph and hence has a balanced orientation. This orientation will induce a balanced orientation of G by deleting the new vertex from the supergraph. \square

3.2.2 Equitable Edge-Colourings

Consider an edge-colouring of a graph G with colours $1, 2, \dots$, and k . For each vertex u of G , let $C_i(u)$ be the set of edges incident with u of colour i ; and for two vertices u and v of G , let $C_i(u, v)$ be the set of edges joining u and v of colour i . Denote

$$c_i(u) = |C_i(u)|$$

and

$$c_i(u, v) = |C_i(u, v)|.$$

Therefore,

$$C_i = \bigcup_{u \in V(G)} C_i(u), 1 \leq i \leq k \quad (3.2)$$

are the colour classes. In this chapter, we do not distinguish between the colour classes, the edge subsets and the edge spanned subgraphs of the same colour.

An edge-colouring of G is equitable if, for every vertex $u \in V(G)$ and $1 \leq i < j \leq k$,

$$|c_i(u) - c_j(u)| \leq 1,$$

and it is balanced if, in addition, for all $u, v \in V(G), u \neq v$ and $1 \leq i < j \leq k$,

$$|c_i(u, v) - c_j(u, v)| \leq 1.$$

Thus, an edge colouring is balanced if the colours occur as uniformly as possible at each vertex and if the colours are shared out as uniformly on multiple edges between two vertices.

Similarly one can define *equitable arc-colourings* on a digraph D given an arc-colouring of a digraph D with colours $1, 2, \dots$, and k . For each vertex u of D , let $C_i^+(u)$ and $C_i^-(u)$ be the set of arcs with origin u and terminus u of colour i , respectively; and let $C_i^+(u, v)$ ($C_i^-(u, v)$) be the set of arcs with origin u and terminus v (with origin v and terminus u of colour i , respectively). Let

$$c_i^+(u) = |C_i^+(u)|$$

$$\begin{aligned}c_i^+(u, v) &= |C_i^+(u, v)| \\c_i^-(u) &= |C_i^-(u)| \\c_i^-(u, v) &= |C_i^-(u, v)|.\end{aligned}$$

An arc-colouring of D is called *equitable* if, for every vertex $u \in V(D)$ and $1 \leq i < j \leq k$,

$$|c_i^+(u) - c_j^+(u)| \leq 1$$

and

$$|c_i^-(u) - c_j^-(u)| \leq 1,$$

and it is called *balanced* if, in addition, for all $u, v \in V(D)$, $u \neq v$ and $1 \leq i < j \leq k$,

$$|c_i^+(u, v) - c_j^+(u, v)| \leq 1,$$

and

$$|c_i^-(u, v) - c_j^-(u, v)| \leq 1.$$

Let $p : V(G) \rightarrow Z^+$ be a function assigning to each vertex u of a graph G a positive integer $p(u)$. A k -colouring (C_1, C_2, \dots, C_k) of G is an *equitable (k, p) -colouring* if for each vertex u and each colour i , $1 \leq i \leq k$,

$$p(u) \lfloor \frac{d_G(u)}{kp(u)} \rfloor \leq c_i(u) \leq p(u) \lceil \frac{d_G(u)}{kp(u)} \rceil.$$

The graph G is also called *equitable (k, p) -colourable*.

In other words, a graph G has an equitable (k, p) -colouring if and only if for every vertex $u \in V(G)$ and colour $1 \leq i \leq k$,

$$c_i(u) = p(u) \lfloor \frac{d_G(u)}{kp(u)} \rfloor + \epsilon_i(u)$$

for some $0 \leq \epsilon_i(u) \leq p(u)$.

If a graph G is k -regular and $p \equiv 1$, then the equitable $(k, 1)$ -colouring problem is equivalent to the edge-colouring problem. So it is not surprising that completely determining the equitable (k, p) -colourable graphs is very hard.

D. de Werra [103] proved the following theorem.

Theorem 3.2.2 (de Werra [103]) *For each integer $k \geq 1$, any bipartite graph has an equitable $(k, 1)$ -colouring.*

Theorem 3.2.3 (de Werra [104]) *Let k be a positive integer and $p : V(G) \rightarrow \mathbb{Z}^+$ a given function. If each odd cycle H of G with degrees $d_H(u) \equiv 0 \pmod{2p(u)}$, ($u \in V(H)$), meets either a vertex u with $d_G(u) \geq k(\frac{d_H(u)}{2} + 1) - 1$ or a vertex u with $d_G(u) \leq k\frac{d_H(u)}{2} - 1$, then G has an equitable (k, p) -colouring.*

3.2.3 Almost Even Equitable (k, p) -Colourings

An *even equitable (k, p) -colouring* is an equitable (k, p) -colouring such that $c_i(u)$ is even for $1 \leq i \leq k$ and $u \in V(G)$. An *almost even equitable (k, p) -colouring* is an equitable (k, p) -colouring such that for each vertex u , $u \in V(G)$, all $c_i(u)$ but possibly one are even for $1 \leq i \leq k$.

Proposition 3.2.4 *For every graph G , there is an equitable $(k, 2)$ -colouring of G .*

Proof Let G be a graph. By Lemma 3.2.1, G has a balanced orientation \vec{G} .

Let $B(V^+, V^-; E)$ be a bipartite graph with two disjoint parts $V^+ = \{u^+ : u \in V(G)\}$ and $V^- = \{u^- : u \in V(G)\}$ and with edge set $E(B) = \{u^+v^- : uv \in A(\vec{G})\}$. We use α to denote the natural bijection from $E(B)$ to $A(\vec{G})$ and β to denote the natural bijection from $A(\vec{G})$ to $E(G)$:

$$\alpha(u^+v^-) = uv$$

$$\beta(uv) = uv.$$

It is easy to see that $d_B(u^+) = od_{\vec{G}}(u)$ and $d_B(u^-) = id_{\vec{G}}(u)$. Since \vec{G} is balanced, then

$$d_B(u^+) = od_{\vec{G}}(u) \geq \lfloor \frac{d_G(u)}{2} \rfloor \geq k$$

and

$$d_B(u^-) = id_{\overrightarrow{G}}(u) \geq \lfloor \frac{d_G(u)}{2} \rfloor \geq k.$$

By de Werra's Theorem 3.2.2, the bipartite graph B has an equitable $(k, 1)$ -colouring (C_1, C_2, \dots, C_k) :

$$\begin{aligned} \lfloor \frac{d_B(u^+)}{k} \rfloor &\leq c_i(u^+) \leq \lceil \frac{d_B(u^+)}{k} \rceil \\ \lfloor \frac{d_B(u^-)}{k} \rfloor &\leq c_i(u^-) \leq \lceil \frac{d_B(u^-)}{k} \rceil. \end{aligned}$$

This edge colouring of B induces an edge colouring $(\alpha \circ \beta(C_1), \alpha \circ \beta(C_2), \dots, \alpha \circ \beta(C_k))$ of G :

$$c_i(u) = c_i(u^+) + c_i(u^-)$$

and hence

$$c_i(u) \geq \lfloor \frac{d_B(u^+)}{k} \rfloor + \lfloor \frac{d_B(u^-)}{k} \rfloor \geq 2 \lfloor \frac{d_G(u)}{2k} \rfloor$$

and

$$c_i(u) \leq \lceil \frac{d_B(u^+)}{k} \rceil + \lceil \frac{d_B(u^-)}{k} \rceil \leq 2 \lceil \frac{d_G(u)}{2k} \rceil.$$

Therefore, $c_i(u) = 2 \lfloor \frac{d_G(u)}{2k} \rfloor + \epsilon_i$, where $0 \leq \epsilon_i \leq 2$. In the following proposition, we try to control the ϵ 's. An edge-colouring is *even* if all $c_i(u)$ are even for $u \in V(G)$ and all colours $1 \leq i \leq k$. An edge-colouring is *almost even* if for every vertex $u \in V(G)$, there is at most one $c_i(u)$ which is odd, $1 \leq i \leq k$. \square

Proposition 3.2.5 *For all $k \leq \min\{\lfloor \frac{d_G(u)}{2} \rfloor : u \in V(G)\}$, there is an almost even equitable $(k, 2)$ -colouring.*

Proof If G is almost $2k$ -regular, i.e., $d_G(u) = 2k$ or $d_G(u) = 2k + 1$ for every vertex $u \in V(G)$, then any equitable $(k, 2)$ -colouring from Proposition 3.2.4 will be almost even.

If G is not almost $2k$ -regular, then we construct an almost $2k$ -regular graph G^* by splitting each vertex u of degree at least $2k + 2$ into vertices u_1, u_2, \dots, u_l , where

$l = \lceil \frac{d_G(u)}{2k} \rceil$, and distributing the edges of $E_G(u)$ to each u_i as follows: Each vertex u_i ($1 \leq i \leq k-1$) is incident with $2k$ edges of $E_G(u)$; vertex u_l is incident with the rest. If the degree of u_l is even, then add loops at the vertex u_l such that the degree of u_l is $2k$. If the degree of u_l is odd, then add a new vertex u^* and one edge between u_l and u^* and add loops at u_l and u^* such that the degree of u_l is $2k$ and the degree of u^* is $2k+1$.

By Proposition 3.2.4, G^* has an equitable $(k, 2)$ -colouring which also is almost even since G^* is almost even. Restoring G back from G^* , the edge-colouring of G^* induces an almost even equitable $(k, 2)$ -colouring of G . \square

Corollary 3.2.6 *If G is an even graph with minimal degree at least $2k$, then G can be decomposed into k even factors.*

Edge colourings of G can be regarded as a certain factorization of the graph. As a direct consequence, the above result implies the following classical result.

Theorem 3.2.7 (Petersen,[81]) *Every $2k$ -regular graph has a 2-factorization.*

3.3 Complementary Poles

3.3.1 Complementary Poles of K_C

Let $\mathcal{H} = \{H_1, H_2, \dots, H_r\}$ be a collection of Hamilton cycles of G . Two vertices of G are said to be *poles* of \mathcal{H} if they divide each cycle of \mathcal{H} into two paths of equal length, when the order is even, or almost equal length, when the order of G is odd. The paths are called *semicircles*. Furthermore, if G is a graph with vertex set $C = \{C_i^+, C_i^- : 0 \leq i \leq k\}$, then two vertices C_j^+ and C_j^- are *complementary poles* of G if each pair C_i^+ and C_i^- are never in the same semicircle for all $i \neq j$. Let K_C be the complete graph with vertex set C . Then we have the following lemma.

Lemma 3.3.1 K_C has a Hamilton decomposition with complementary poles C_0^+ and C_0^- .

Proof The following construction is classical. Let σ be the permutation

$$\sigma = (C_0^+)(C_0^-)(C_1^+C_2^+ \cdots C_k^+C_1^-C_2^- \cdots C_k^-) \quad (3.3)$$

acting on the set of vertices of K_C , and let H be the Hamilton cycle

$$H = \begin{cases} C_0^+C_1^+C_k^-C_2^+C_{k-1}^- \cdots C_{\frac{k+2}{2}}^+C_0^-C_{\frac{k+2}{2}}^- \cdots C_k^+C_2^-C_1^-C_0^+, & \text{for } k \text{ even} \\ C_0^+C_1^+C_k^-C_2^+C_{k-1}^- \cdots C_{\frac{k+3}{2}}^-C_0^-C_{\frac{k+3}{2}}^+ \cdots C_k^+C_2^-C_1^-C_0^+, & \text{for } k \text{ odd.} \end{cases} \quad (3.4)$$

Then

$$\mathcal{H} = (\sigma^0(H) = H, \sigma^1(H), \sigma^2(H), \dots, \sigma^{k-1}(H)) \quad (3.5)$$

is a Hamilton decomposition with complementary poles C_0^+ and C_0^- . \square

Remarks.

1. Each pair C_i^+ and C_i^- , $1 \leq i \leq k$, are also complementary poles of the Hamilton decomposition of $K_C - F$.
2. The Hamilton cycle decomposition is symmetric between C^+ 's and C^- 's.

3.3.2 Orthogonalities.

Lemma 3.3.2 *Let \mathcal{H} be the Hamilton decomposition in Proposition 3.3.1 with complementary poles C_0^+ and C_0^- . Then for each i , $1 \leq i \leq k$, there are edge-disjoint orthogonal matchings M_i^+ , M_i^- , such that each edge in M_i^+ is in the same semicircle as the vertex C_i^+ is, and each edge in M_i^- is in the same semicircle as the vertex C_i^- .*

Proof If k is even, then for $1 \leq i \leq k$,

$$M_i^+ = \{\sigma^j(C_i^+ C_{i+1}^-) : 0 \leq j \leq k-2\} \cup \{C_k^+ C_0^-\}$$

and

$$M_i^- = \sigma^k(M_i^+)$$

are edge-disjoint matchings both orthogonal to the Hamilton decomposition. If k is odd, then for $1 \leq i \leq k$,

$$M_i^+ = \{\sigma^j(C_i^+ C_{i+1}^-) : 2 \leq j \leq k\} \cup \{C_{i+1}^- C_0^-\}$$

and

$$M_i^- = \sigma^k(M_i^+)$$

are edge-disjoint matchings both orthogonal to the Hamilton decomposition, where σ is the permutation in Lemma 3.3.1. \square

3.4 Edge-Disjoint Hamilton Cycles

Theorem 3.4.1 *If a graph G has a Hamilton cycle and has minimal degree δ , and if $|\deg(u) - \deg(v)| \leq 1$ when δ is odd and $|\deg(u) - \deg(v)| \leq 2$ when δ is even for any two vertices u and v , then its line graph $L(G)$ has at least $2\lfloor \frac{\delta}{2} \rfloor - 2$ edge-disjoint Hamilton cycles.*

Proof Let C_0 be a Hamilton cycle in G and let $r = \lfloor \frac{\delta}{2} \rfloor - 1$. Then $\delta(G) \geq 2r + 2$ and $\delta(G - C_0) \geq 2r$. By Proposition 3.2.5, there is an almost even equitable $(r, 2)$ -colouring $\{C_1, C_2, \dots, C_r\}$ of $G - C_0$. Applying Lemma 3.2.1, we can obtain balanced orientations of C_i , $0 \leq i \leq r$. Since $|\deg(u) - \deg(v)| \leq 1$ when δ is odd and $|\deg(u) - \deg(v)| \leq 2$ when δ is even for any two vertices u and v , we have three possible cases: $\deg_{G-C_0}(u) = 2r$, $\deg_{G-C_0}(u) = 2r + 1$ and $\deg_{G-C_0}(u) = 2r + 2$. We denote

$$C = \{C_i^+, C_i^- : 0 \leq i \leq r\}$$

and

$$C(u) = \{C_i^+(u), C_i^-(u) : 0 \leq i \leq r\}.$$

Hence,

$$E_G(u) = \bigcup_{i=0}^r \{C_i^+(u) \cup C_i^-(u)\}.$$

By Lemma 3.3.1, the Hamilton decomposition (3.5)

$$\mathcal{H} = (\sigma^0(H) = H, \sigma^1(H), \sigma^2(H), \dots, \sigma^{r-1}(H)) \quad (3.6)$$

of K_C has complementary poles C_0^+ and C_0^- . Denote the semicircles by

$$L_{1,1}, L_{1,2}, L_{2,1}, L_{2,2}, \dots, L_{r,1}, L_{r,2}, \quad (3.7)$$

where $L_{i,1} \cup L_{i,2} = \sigma^{i-1}(H)$

Case 1. If $\deg_{G-C_0}(u) = 2r$, then $|C_i(u)| = 2$ and $|C_i^+(u)| = |C_i^-(u)| = 1$, for $1 \leq i \leq r$. Let

$$L_{1,1}(u), L_{1,2}(u), L_{2,1}(u), L_{2,2}(u), \dots, L_{r,1}(u), L_{r,2}(u) \quad (3.8)$$

be the corresponding semicircles of $K_{E_G(u)}$.

Case 2. If $\deg_{G-C_0}(u) = 2r + 1$, then $|C_i(u)|$ are all equal to 2 with one exception which is equal to 3, and hence $|C_i^+(u)|$ and $|C_i^-(u)|$ are equal to 1 with one exception, say, $|C_{i_0}^+(u)| = 2$. Regard $K_{E_G(u)}$ as a graph obtained from K_C by inserting a duplicated vertex $C_{i_0}^+$. By Lemma 3.3.1, the Hamilton decomposition (3.5) has an orthogonal matching $M_{i_0}^+$. We obtain semicircles

$$L_{1,1}(u), L_{1,2}(u), L_{2,1}(u), L_{2,2}(u), \dots, L_{r,1}(u), L_{r,2}(u) \quad (3.9)$$

of $K_{E_G(u)}$ by subdividing the matching edges of semicircles of K_C using the duplicated vertex.

Case 3. If $\deg_{G-C_0}(u) = 2r + 2$, then $|C_i(u)|$ are all equal to 2 except one which is equal to 4 and $|C_i^+(u)|$ and $|C_i^-(u)|$ are equal to 1 except for some i_0 satisfying $|C_{i_0}^+(u)| = |C_{i_0}^-(u)| = 2$. Regard $K_{E_G(u)}$ as a graph obtained from K_C by inserting a duplicated vertex of $C_{i_0}^+$ and a duplicated vertex of $C_{i_0}^-$. By Lemma 3.3.1, the Hamilton decomposition (3.5) has disjoint orthogonal matchings $M_{i_0}^+$ and $M_{i_0}^-$. We obtain semicircles

$$L_{1,1}(u), L_{1,2}(u), L_{2,1}(u), L_{2,2}(u), \dots, L_{r,1}(u), L_{r,2}(u), \quad (3.10)$$

of $K_{E_G(u)}$ by subdividing the matching edges of $M_{i_0}^+$ using the duplicated vertex $C_{i_0}^+(u)$ and subdividing the matching edges of $M_{i_0}^-$ using the duplicated vertex $C_{i_0}^-$ from the semicircles of K_C .

Note that the vertex and its duplicated vertex are in the same semicircles.

We construct $2r$ edge-disjoint Hamilton cycles $H_{i,j}$, $1 \leq i \leq r$, $j = 1, 2$, as follows:

$$H_{i,j} = \bigcup_{u \in V(G)} L_{i,j}(u). \quad (3.11)$$

□

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