CYCLES AND FACTORS IN CERTAIN GRAPHS

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Abstract

There are six chapters in this thesis.

In Chapter 1, we survey some important results and background in the history of the problems related to this thesis. Some frequently used definitions and notations are given.

In Chapter 2, we study the existence of cyclic 1-factorizations of circulants $C(n, S)$. A Γ-invariant 1-factorization of a Cayley graph $G = X(\Gamma, S)$ is a 1-factorization $\mathcal{F}$ of G such that Γ maps $\mathcal{F}$ to $\mathcal{F}$. In the case of circulants, that is, $\Gamma = Z_n$, a cyclic group of order n, we call a Γ-invariant 1-factorization a cyclic 1-factorization. Some necessary conditions and problems equivalent to the existence of cyclic 1-factorizations with a single 1-factor orbit are obtained. We also classify some special classes of graphs.

An isomorphic factorization of G is a partition of its edges into isomorphic subgraphs. In Chapter 3, we deal with another factorization problem — the isomorphic factorization of circulants. Some partial results are obtained.

In Chapter 4, we give a classification of 2-extendable Cayley graphs on dihedral groups. A graph G is said to be k-extendable if it contains a k-matching and any k-matching of G can be extended to a perfect matching of G.

In Chapter 5, we prove that the generalized Petersen graphs $GP(n, 3)$ and $GP(n, 2)$, where $gcd(2, n) = 1$, $n \not\equiv 5 \pmod 6$, are Hamilton-connected or Hamilton-laceable.

A Hamilton decomposition of $C(2p, S)$, where p is a prime, is given in Chapter 6.

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Dedication

To my family

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Chapter 1

Introduction

1.1 Definitions and notations

A *graph* G is a pair $(V(G), E(G))$, where $V(G)$ is a finite non-empty set of elements called *vertices* and $E(G)$ is a finite set of unordered pairs of elements of $V(G)$ called *edges*. We call $V(G)$ the vertex set of G and $E(G)$ the edge set of G. The number of vertices of G, denoted by $\nu(G)$, is called the *order* of G. The number of edges of G, denoted by $\varepsilon(G)$, is called the *size* of G.

An edge e with end vertices u and v is denoted by uv or (u, v); two edges $e_1 = uv$ and $e_2 = xy$ are *independent* if $\{u, v\} \bigcap \{x, y\} = \emptyset$.

A set of m independent edges of G is called an *m-matching* of G. If $m = \frac{1}{2}\nu(G)$, we call an m-matching M of G a *perfect matching* or a *1-factor* of G.

A *1-factorization* of G is a partition of $E(G)$ into 1-factors.

An *isomorphic factorization* of G is a partition of $E(G)$ into isomorphic subgraphs.

A *Hamilton cycle* of G is a cycle which contains every vertex of G. We call G hamiltonian if G has a Hamilton cycle. A *Hamilton decomposition* of G is a partition of $E(G)$ into Hamilton cycles.

An *automorphism* of G is a permutation φ of $V(G)$ such that both φ and φ^{-1} preserve adjacency. The automorphisms of G form a group $Aut(G)$ under composition, called the *automorphism group* of G.

A graph G is said to be *vertex-transitive* if $Aut(G)$ acts transitively on $V(G)$, that is, for any $u, v \in V(G)$, there is a $\varphi \in Aut(G)$ such that $\varphi(u) = v$.

Let Γ be a group with identity 1. Suppose that S is a subset of Γ with the properties that $1 \notin S$ and $S = S^{-1}$. The *Cayley graph* $G = X(\Gamma, S)$ is the simple graph whose vertex set $V(G) = \Gamma$, and edge set $E(G) = \{(g, h) : g^{-1}h \in S\}$.

Cayley graphs are a special class of vertex-transitive graphs. The main feature is that a graph G is a Cayley graph if and only if it admits a group Γ acting on G regularly. In fact, if $G = X(\Gamma, S)$ is a Cayley graph, then Γ acts on G regularly by left multiplication. If Γ is a cyclic group $(Z_n, +)$, we call the Cayley graph $X(Z_n, S)$ a *circulant*, and denote it by $C(n, S)$. In this case, S satisfies $0 \notin S$ and $S = -S$; the edge set $E(G) = \{(g, h) : h - g \in S\}$.

The dihedral group of order 2n, denoted D_n, is defined as follows:

$$D_n = \langle \rho, \tau : \rho^n = \tau^2 = 1, \tau\rho\tau = \rho^{-1} \rangle.$$

The *generalized Petersen graph* $GP(n, k)$, where $n \geq 2$ and $1 \leq k \leq n - 1$, is defined in the following way. It has vertices $u_0, u_1, \ldots, u_{n-1}, v_0, v_1, \ldots, v_{n-1}$ and edges u_iu_{i+1}, u_iv_i, v_iv_{i+k} for all $0 \leq i \leq n - 1$ with all subscripts reduced modulo n.

Note that $GP(5, 2)$ is just the Petersen graph.

For definitions and notations which do not appear here, we refer the reader to [18].

1.2 Background

The subjects of factors and cycles are fundamental to the study of graph theory. This thesis is mainly concerned with 1-factors and Hamilton cycles of graphs.

As early as 1859, M. Reiss [43] found that K_{2n} is 1-factorizable. Since then, there are many results about 1-factorizations of graphs. For example, D. König [32] showed that a bipartite graph G is 1-factorizable if and only if G is regular; B. Alspach [3] proved that the line graph $L(K_n)$ is 1-factorizable if and only if $n \equiv 0$ or $1 \pmod 4$; the famous Four-color Theorem [14] is also equivalent to the fact that every planar 2-connected cubic multigraph is 1-factorizable. In 1985, A. Hartman and A. Rosa [29] added some restrictions to 1-factorizations of K_n. A *cyclic 1-factorization* $\mathcal{F}$ of K_n is a 1-factorization which is invariant under a permutation which is an n-cycle. They studied the existence of cyclic 1-factorizations of K_n and proved that a cyclic 1-factorization of K_n exists if and only if n is even and $n \neq 2^t, t \geq 3$.

Some variations can be considered. R. Rees [42] studied cyclic k -matching decompositions of K_n for all n, where $k < n/2$. G. Korchmáros [33] considered cyclic 1-factorizations of K_n with an invariant 1-factor and applied the results to projective planes.

The existence of 1-factorizations of a large family of Cayley graphs was studied by R. Stong [46]. He obtained that a connected Cayley graph $G = X(\Gamma, S)$ has a 1-factorization if Γ is a cyclic group Z_{2n}, a dihedral group D_n, etc. However, we want an additional property. Since G has the group Γ acting on it regularly, it is natural to ask what the effect on a 1-factorization $\mathcal{F}$ of G is under the action of Γ on G ? If Γ sends $\mathcal{F}$ to $\mathcal{F}$, we call $\mathcal{F}$ a *Γ-invariant 1-factorization*, and G is called *Γ-invariant 1-factorizable*. We have the following problem.

Problem 1.1 Let $G = X(\Gamma, S)$ be a Cayley graph. Under what conditions does G admit a Γ-invariant 1-factorization?

This problem was first posed by D. Jungnickel [31]. In Chapter 2, we will consider the special case of circulants. We call a Z_n-invariant 1-factorization a *cyclic 1-factorization*. Note that the problems considered by Hartman, Rosa and Korchmáros are just some special cases of circulants.

The isomorphic factorization problem has attracted much attention. For example, M. N. Ellingham [22, 23, 24], F. Harary, W. Robinson, W.D. Wallis, N. Wormald [26, 27, 30] and S. Quinn [41] have all considered this problem. The graphs they have considered are complete graphs, complete multipartite graphs, and other regular graphs. Some of these families are still not completely understood. In 1984, Wormald [49] even obtained that almost all labeled r-regular graphs cannot be factorized into $t \geq 2$ isomorphic subgraphs, where $r > 2t$, but no examples of such non-factorizable graphs are known. Note that many of the graphs considered by the above authors are circulants. This led Alspach to ask the following problem.

Problem 1.2 Let G be a circulant. If $\varepsilon(G) \equiv 0 (\text{mod } t)$, does G admit a factorization into t isomorphic subgraphs?

It would be nice to give a positive answer or find a negative example because of Wormald's results on this problem. In Chapter 3, we give some partial results on this problem.

Isomorphic factorizations (1-factorizations) have relations to designs, latin squares, room squares, etc.

In studying graphs, it is often worth considering the extension of some small subgraphs with certain properties to a spanning subgraph with the same properties. One such interesting graph is a matching. In 1980, M. D. Plummer [38, 39, 40] considered extending an n-matching to a perfect matching (called an n-extension). He showed that every 2-extendable graph is either bipartite or a brick(which plays an important role in matching polyhedra). J. Liu and Q. Yu[35] generalized the concept of n-extension to (m, n)-extension and studied their properties. Recently, G. Schrag and L. Cammack [45] and Yu [50] classified the 2-extendable generalized Petersen graphs. O. Chan, C. C. Chen and Yu [20] classified all 2-extendable Cayley graphs on abelian groups. With Chen and Yu, we classify all 2-extendable Cayley graphs on dihedral groups. This will be given in Chaper 4.

The Petersen graph is probably the most important graph in graph theory. In 1969, M. E. Watkins [48] defined generalized Petersen graphs, which includes the Petersen graph as a member, and posed the question of whether or not every cubic $GP(n,k)$ other than $GP(5,2) \cong GP(5,3)$ has a 1-factorization. Meanwhile, G. N. Robertson [44] and J. A. Bondy [17] proved independently that $GP(n,2)$ is hamiltonian if and only if $n \not\equiv 5 (\text{mod } 6)$. In the latter paper, Bondy also proved that $GP(n,3)$ is hamiltonian whenever $n \neq 5$. Finally, F. Castagna and G. Prins provided an affirmative answer to Watkin's 1-factorization question in [19]. Then they conjectured that Robertson's examples were the only non-hamiltonian examples. This conjecture lasted for over 10 years. The first important contribution was made by K. Bannai [15] who showed that $GP(n,k)$ is hamiltonian when n and k are relatively prime and $GP(n,k)$ is not isomorphic to $G(n,2)$ with $n \equiv 5(\text{mod } 6)$. The second contribution was due to Alspach, P. J. Robinson and M. Rosenfold [11] who proved that $GP(n,k)$ is hamiltonian if $k \geq 3$ and n is sufficiently large. The conjecture was finally solved by Alspach [4]. The answer is that the generalized Petersen graph $GP(n,k)$ is hamiltonian if and only if it is neither $GP(n,2) \cong GP(n,n-2) \cong GP(n, \frac{n-1}{2}) \cong GP(n, \frac{n+1}{2})$, $n \equiv 5(\text{mod } 6)$ nor $GP(n, \frac{n}{2})$, $n \equiv 0(\text{mod } 4)$ and $n \geq 8$.

Even though $GP(n,2)$, where $n \equiv 5(\text{mod } 6)$, is not hamiltonian, it misses by very little in the following sense. Alspach has proved [9] that there is a Hamilton path between any two non-adjacent vertices of $GP(n,2)$.

He also made the following conjecture.

Conjecture 1 The generalized Petersen graph $GP(n,k)$, where $gcd(n,k) = 1$ and $GP(n,k)$ is not isomorphic to $GP(6m+5,2)$ for some integer m, is Hamilton-connected or Hamilton-laceable.

In Chapter 5, we will deal with the cases $k = 2, 3$. The results support the above conjecture.

The purpose of considering this problem is that while studying the existence of Hamilton cycles in metacirculant graphs, which is a large class of vertex transitive graphs including many known Cayley graphs. Alspach noticed that each pair of adjacent blocks contains a generalized Petersen graph as a subgraph (if each block is connected). If the above conjecture is true, then we will have that the metaciculant graphs with nonempty blocks are hamiltonian.

A Hamilton decomposition of $C(2p, S)$ is given in Chapter 6. For the case of vertex-transitive graph of order $2p$, $p \equiv 3 (\mathrm{mod}\ 4)$ and p is a prime, Alspach [2] already gave a Hamilton decomposition.

Chapter 2

The Cyclic 1-factorization of Circulants

2.1 Introduction

Let $G = C(n, S)$ be a circulant. Let $S = \{a_1, a_2, \ldots, a_k, n - a_1, n - a_2, \ldots, n - a_k\}$ and $S^+ = \{a_1, a_2, \ldots, a_k\}$, where $a_i \leq n/2$ for $1 \leq i \leq k$. Let $E_s = \{(a, b) : a - b = s$ or $a - b = n - s\}$. Then we have that E_s is a union of disjoint cycles of length $\frac{n}{gcd(s,n)}$ if $s \neq n/2$, and $E_{n/2}$ is a 1-factor of $C(n, S)$ if $n/2 \in S$. We call E_s an *even (odd) edge orbit* when s is even (odd), where $s \neq n/2$, and call $E_{n/2}$ the *diagonal orbit.*

Assume n is even and let $s = 2j + 1$ be an odd element in S. Then E_s can be partitioned into two 1-factors

$$\begin{aligned} F_1 &= \{(2i, 2i + 2j + 1) : 0 \leq i < n/2\} \text{ and} \\ F_2 &= \{(2i + 1, 2i + 2j + 2) : 0 \leq i < n/2\}. \end{aligned} \tag{2.1}$$

It is easy to see that $\{F_1, F_2\}$ is preserved under the action of Z_n. We also see that $E_{n/2}$ is invariant under the action of Z_n. Therefore, the difficulty in constructing a cyclic 1-factorization arises because of the even edge orbits.

Let $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$ be a Z_n-invariant 1-factorization of $C(n, S)$. Then for any $\mathbf{a} \in Z_n$,and $F_i \in \mathcal{F}$, we have $\mathbf{a}(F_i) = F_j$ for some $1 \leq j \leq m$, where $\mathbf{a}(u, v) = (u + a, v + a)$. In particular, for the element $\mathbf{1} \in Z_n, \mathbf{1}(u, v) = (u + 1, v + 1)$. If we define

$$\begin{aligned} \alpha : \ & Z_n \to Z_n \\ & x \mapsto x + 1, \end{aligned}$$

Then α is an automorphism and the action of the element $\mathbf{1}$ on Z_n is α. Therefore, $\mathcal{F}$ is α-invariant. Conversely, if $\mathcal{F}$ is α-invariant, then for any $\mathbf{m} \in Z_n$, $\mathbf{m}(\mathcal{F}) = \alpha^m(\mathcal{F})$.

Hence $\mathcal{F}$ is Z_n-invariant. This shows that a Z_n-invariant 1-factorization is equivalent to an α-invariant 1-factorization. It is convenient if we use α instead of Z_n. The main idea is to obtain the structure of cyclic 1-factorizations of $C(n, S)$ by studying the orbits of $< \alpha >$.

2.2 The structure

Definition 2.1 Let $\mathcal{F}$ be a cyclic 1-factorization of $C(n, S)$. Then $< \alpha >$ acting on $\mathcal{F}$ partitions the 1-factors into orbits, called *1-factor orbits.*

The number of 1-factors in a 1-factor orbit must be a divisor of n (Orbit-Stabilizer Theorem). If the number is m, then $< \alpha^m >$ fixes each 1-factor in such an orbit.

Lemma 2.1 *Let F be a 1-factor in a 1-factor orbit of length m. Then F contains $\lceil m/2 \rceil$ edges from disjoint edge orbits.*

Proof. Let $F, \alpha(F), \ldots, \alpha^{m-1}(F)$ be the 1-factor orbit, and let $\{a_1, a_2, \ldots a_h\} = \{s : E_s \bigcap F \neq \emptyset\}$. Then $F \subseteq \bigcup_{i=1}^{h} E_{a_i}$. Noticing that $\alpha^j(E_{a_i}) = E_{a_i}$ for any j, we have

$$F \bigcup \alpha(F) \bigcup \cdots \bigcup \alpha^{m-1}(F) \subseteq \bigcup_{i=1}^{h} E_{a_i}.$$

If there exists $(u, v) \in E_{a_i}$ for some $1 \leq i \leq h$, such that $(u, v) \notin F \bigcup \alpha(F) \bigcup \cdots \bigcup \alpha^{m-1}(F)$, then $(u, v), (u+1, v+1), \ldots, (u+n-1, v+n-1) \notin F$. This is a contradiction. Therefore, we have

$$F \bigcup \alpha(F) \bigcup \cdots \bigcup \alpha^{m-1}(F) = \bigcup_{i=1}^{h} E_{a_i}.$$

By counting the number of edges in both sides, we know that if m is even, then $h = m/2$, and if m is odd, then $h = \frac{m+1}{2}$ and one of E_{a_i} is $E_{n/2}$. ∎

Corollary 2.2 *A 1-factor orbit of odd length must contain $E_{n/2}$. In particular, a 1-factor orbit of length 1 is $E_{n/2}$.*

The following lemma was proved by Hartman and Rosa for complete graphs. It can be extended to circulants.

Lemma 2.3 *A 1-factor orbit of even length contains an even number of even edge orbits.*

Proof. Let F be a 1-factor in a 1-factor orbit of even length $2m$. Then F is stabilized by $< \alpha^{2m} >$ and contains m edges from distinct edge orbits, say, $(x_1, x_1+a_1), (x_2, x_2+a_2), \ldots, (x_m, x_m+a_m)$. Let $F_m = \{(x_1(\text{mod } 2m), (x_1+a_1)(\text{mod } 2m), \ldots, (x_m(\text{mod } 2m), (x_m+a_m)(\text{mod } 2m))\}$. Then F_m is a 1-factor of K_{2m}. Let l be the number of even edges in F_m. Then $m-l$ is the number of odd edges in F_m, and there are $m-l$ even vertices covered by these $m-l$ odd edges. Since the total number of even vertices in K_{2m} is m, then there are $m-(m-l)=l$ even vertices covered by even edges of F_m. But even edges cover two vertices of the same parity. Therefore, the number l of even edges per 1-factor must be even. ∎

From Lemma 2.3, the following two corollaries follow easily.

Corollary 2.4 *A 1-factor orbit of length 2 must be a partition of a non-diagonal odd edge orbit.*

Corollary 2.5 *Let $C(n,S)$ have a cyclic 1-factorization. If $n/2 \notin S$, then S^+ contains an even number of even elements.*

Proof. Since $n/2 \notin S$, then each 1-factor orbit has even cardinality by Corollary 2.2. Also, each 1-factor orbit contains an even number of even edge orbits by Lemma 2.3. Hence S^+ contains an even number of even elements. ∎

Lemma 2.6 *Let $C(n,S)$ have a 1-factor orbit of length m, and let F be a 1-factor in the orbit. Then $n \equiv 0(\text{mod } m)$ and $a_i \not\equiv 0(\text{mod } m)$ if $E_{a_i} \bigcap F \neq \emptyset$ and $a_i \neq n/2$.*

Proof. We have already seen that $n \equiv 0(\text{mod } m)$. If $E_{a_i} \bigcap F \neq \emptyset$, without loss of generality, say $(0,a_i) \in F$, then $\alpha^m(0,a_i) \in F$. Now if $a_i \equiv 0(\text{mod } m)$, then $\alpha^{a_i}(0,a_i) = (a_i, 2a_i) \in F$, implying that F contains two adjacent edges $(0,a_i)$ and $(a_i, 2a_i)$. This is a contradiction. ∎

We summarize as follows.

Lemma 2.7 *For a cyclic 1-factorization $\mathcal{F}$ of $C(n,S)$, let $m_1, m_2, \ldots, m_r$ be the lengths of 1-factor orbits. Then*

1. $\sum_{i=1}^{r} m_i = |S|$,

2. $n \equiv 0 (mod\ m_i)$ *for* $1 \leq i \leq r$, *and*

3. *if* $|S|$ *is odd, then only one of the* m_i*'s is odd; if* $|S|$ *is even, then all* m_i*'s are even.*

Definition 2.2 Let $m_1, m_2, \ldots, m_r$ satisfy the above conditions and $m_1 \geq m_2 \geq \ldots, \geq m_r$. We call $(m_1, m_2, \ldots, m_r)$ an *orbit vector.*

If we denote the 1-factor orbits by $\mathcal{F}_1, \ldots, \mathcal{F}_r$, and let $S_i{}^+ = \{a : F \bigcap E_a \neq \emptyset, F \in \mathcal{F}_i\}$, then $\bigcup_{F \in \mathcal{F}_i} F = \bigcup_{a \in S_i{}^+} E_a$, and $\mathcal{F}_i$ is a cyclic 1-factorization of $C(n, S_i)$ with a single 1-factor orbit. The problem now becomes:

1. Partition S^+ into $S_1{}^+, \ldots, S_r{}^+$, such that $C(n, S_i)$, $(i = 1, 2, \ldots, r)$, has a cyclic 1-factorization with a single 1-factor orbit.

2. Study the cyclic 1-factorization of circulants with a single 1-factor orbit.

In the next three sections, we will study the existence of cyclic 1-factorizations of circulants with a single 1-factor orbit. In fact, if one can give a characterization of cyclic 1-factorizations with a single 1-factor orbit, then one can characterize cyclic 1-factorizations with any orbit vector.

2.3 Equivalent conditions for existence of cyclic 1-factorizations with a single 1-factor orbit

Lemma 2.8 *If* $n \equiv 0 (mod\ m)$, $a \not\equiv 0 (mod\ m)$ *and* $a < n/2$, *then the edges of* E_a *can be partitioned into* n/m*-matchings which are* α*-invariant.*

Proof. Let $F = \{(i, i+a), (i+m, i+m+a), \ldots, (i+(n/m-1)m, i+(n/m-1)m+a)\}$ for any $i \in Z_n$. Then $F, \alpha(F), \ldots, \alpha^{m-1}(F)$ is a desired partition. ∎

Consider the following $m \times n/m$ array, denoted by $A(i, a, m)$.

$$\begin{pmatrix} (i,i+a) & (i+m,i+m+a) & \cdots & (i+(n/m-1)m,i+(n/m-1)m+a) \\ (i+1,i+1+a) & (i+1+m,i+1+m+a) & \cdots & (i+1+(n/m-1)m,i+1+(n/m-1)m+a) \\ & & \ddots & \\ (i+m-1,i+m-1+a) & (i+2m-1,i+2m-1+a) & \cdots & (i-1,i-1+a) \end{pmatrix}.$$

If we denote the first row by F and the first column by K, then we have

$$A(i, a, m) = \begin{pmatrix} F \\ \alpha(F) \\ \alpha^2(F) \\ \vdots \\ \alpha^{m-1}(F) \end{pmatrix},$$

and

$$A(i, a, m) = (K, \alpha^m(K), \alpha^{2m}(K), \ldots, \alpha^{(n/m-1)m}(K)).$$

For $a = n/2, n \equiv 0(\text{mod } m)$, and m odd, then $n \equiv 0(\text{mod } 2m)$, and we define in a similar way an $m \times \frac{n}{2m}$ array $A(i, n/2, m)$ for any $i \in Z_n$.

Let n be an even integer. We call a partition of Z_n into 2-subsets a *2-partition.*

Definition 2.3 Let $S^+ = \{a_1, a_2, \ldots, a_k\}$, where $a_i \leq n/2$ for $i = 1, 2, \ldots, k$. Let $m = 2k - 1$ or $2k$ according to $n/2 \in S^+$ or $n/2 \notin S^+$. If we can find $i_1, \ldots, i_k$ so that the elements in the first rows of $A(i_1, a_1, m), \ldots, A(i_k, a_k, m)$ form a 2-partition of Z_n, then we put $A(i_1, a_1, m), \ldots, A(i_k, a_k, m)$ together to obtain an $m \times n/2$ array

$$A = (A(i_1, a_1, m), A(i_2, a_2, m), \ldots, A(i_k, a_k, m)).$$

We call A a *1-factorization array* of $C(n, S)$.

The concept of starter plays an important roll in the study of 1-factorizations of K_n. We generalize it as follows.

Definition 2.4 Let $S^+ = \{a_1, a_2, \ldots, a_k\}$ or $S^+ = \{a_1, a_2, \ldots, a_k, n/2\}$, where $a_i < n/2$ for $1 \leq i \leq k$. Let $m = 2k+1$ or $2k$ according to $n/2 \in S^+$ or $n/2 \notin S^+$. Suppose $n \equiv 0(\text{mod } m)$ and $a_i \not\equiv 0(\text{mod } m)$ for $1 \leq i \leq k$.

An $(a_1, a_2, \ldots, a_k; m)$-*starter* of Z_n is a pair (U, P), where U is a $2k$-subset of Z_n such that for any $x, y \in U, x - y \not\equiv 0(\text{mod } m)$; and P is a 2-partition of U such that $\{\pm(x - y) : \{x, y\} \in P\} = \{\pm a_1, \ldots, \pm a_k\}(\text{mod } n)$.

Definition 2.5 Let $S = \{\pm b_1, \pm b_2, \ldots, \pm b_k\} \subseteq Z_n - \{0\}$ be a multiset. We define a multigraph $C^*(n, S)$ as follows: the vertex set is Z_n; for any $x, y \in Z_n$, the number of edges between x and y equals the multiplicity of $y - x$ in S. We call $C^*(n, S)$ a *circulant multigraph.*

Definition 2.6 Let $C(n, S)$ be a circulant. Let $S^+ = \{a_1, \ldots, a_k\}$ or $\{a_1, \ldots, a_k, n/2\}$ and $m = 2k + 1$ or $2k$ according to $n/2$ in S^+ or not. Suppose $n \equiv 0 (\text{mod } m)$ and $a_i \equiv b_i (\text{mod } m)$ for $1 \leq i \leq k$, where $0 < b_i < m$. We call $C^*(m, \{\pm b_1, \ldots, \pm b_k\})$ the *modulo m multigraph* of $C(n, S)$.

Note that we can assume that $b_i \leq k$ for $1 \leq i \leq k$.

Definition 2.7 Let G be a graph, and $\{E_1, \ldots, E_k\}$ be a partition of the edge set $E(G)$. Let F be a subset of $E(G)$. F is said to be *orthogonal* to $\{E_1, \ldots, E_k\}$, if $|F \bigcap E_i| = 1$, for $i = 1, 2, \ldots, k$.

If $X = \{x_1, x_2, \ldots, x_l\}$ is a collection of l-subsets of integers and m is an integer, we denote the set $\{x_i(\text{mod } m),\ x_2(\text{mod } m),\ \ldots,\ x_l(\text{mod } m)\}$ by $X(\text{mod } m)$.

Now we can state our theorem.

Theorem 2.9 *The following statements are equivalent.*

(1). *$C(n, S)$ has a cyclic 1-factorization with a single 1-factor orbit of length m.*

(2). *There exists an $m \times n/2$ 1-factorization array of $C(n, S)$.*

(3). *There exists an $(a_1, \ldots, a_k; m)$-starter of Z_n.*

(4). *(i) If $m = 2k$, there is a 2-partition of Z_{2k}, such that $\{\pm(x - y) : \{x, y\} \in P\} \equiv \{\pm a_1, \ldots, \pm a_k\}$ (mod $2k$).*

(ii) If $m = 2k + 1$, there is a 2-partition of $Z_{2k+1} - \{i\}$ for some i, such that $\{\pm(x - y) : \{x, y\} \in P\} \equiv \{\pm a_1, \ldots, \pm a_k\}$(mod $2k + 1$).

(5). *The system of equations*

$$x_i - y_i \equiv a_i (mod\ \ m) \qquad i = 1, 2, \ldots, \lfloor m/2 \rfloor, \tag{2.2}$$

has a solution covering $2\lfloor m/2 \rfloor$ elements of Z_m.

(6). *The modulo m multigraph $C^*(m, \{\pm b_1, \ldots, \pm b_k\})$ has a k-matching M which is orthogonal to $\{E_{b_1}, \ldots, E_{b_k}\}$, where $k = \lfloor m/2 \rfloor$.*

Proof. (1)$\Rightarrow$(2) Let $\mathcal{F} = \{F_1, \ldots, F_m\}$ be a cyclic 1-factorization with a single 1-factor orbit of length m. Without loss of generality, assume $F_i = \alpha^{i-1}(F_1)$ for $i = 2, 3, \ldots, m$. By Lemma 2.1, we have $F_1 \bigcap E_{a_i} \neq \emptyset$ for $i = 1, 2, \ldots, k$.

Let $e_1 = (i_1, i_1 + a_1), e_2 = (i_2, i_2 + a_2), \ldots, e_k = (i_k, i_k + a_k) \in F_1$. Then $\alpha^{hm}(\{e_1, \ldots, e_k\}) \subset F_1$, for $h = 0, 1, \ldots, n/m - 1$. In fact, we have

$$\bigcup_{h=0}^{n/m-1} \alpha^{hm}(\{e_1, \ldots, e_k\}) = F_1,$$

if $m = 2k$. Therefore, by counting both side, we have

$$\alpha^{hm}(\{e_1, \ldots, e_k\}) \bigcap \alpha^{pm}(\{e_1, \ldots, e_k\}) = \emptyset,$$

if $h \neq p, 0 \leq h, p \leq n/m - 1$. Thus

$$F_1 = \{e_1, \alpha^m(e_1), \ldots, \alpha^{(n/m-1)m}(e_1), \ldots, e_k, \alpha^m(e_k), \ldots, \alpha^{n/m-1}(e_k)\},$$

and we have

$$A = (A(i_1, a_1, m), \ldots, A(i_k, a_k, m))$$

is a 1-factorization array of $C(n, S)$.

If $m = 2k + 1$, we have

$$M = \{e_1, \alpha^m(e_1), \ldots, \alpha^{(n/m-1)m}(e_1), \ldots, e_k, \alpha^m(e_1), \ldots, \alpha^{(n/m-1)m}(e_k)\}$$

is a $(k\frac{n}{m} =)k\frac{n}{2k+1}$-matching. The remaining $\frac{n}{2m}$ edges of F_1 are diagonal edges, and say that $(i_{k+1}, i_{k+1} + \frac{n}{2})$ is one of them. Then

$$A = (A(i_1, a_1, m), \ldots, A(i_k, a_k, m), A(i_{k+1}, n/2, m))$$

is a 1-factorization array of $C(n, S)$.

(2)$\Rightarrow$(3) Let $A = (A(i_1, a_1, m), \ldots, A(i_k, a_k, m))$ or $A = (A(i_1, a_1, m), \ldots, A(i_k, a_k, m), A(i_{k+1}, n/2, m))$ be a 1-factorization array of $C(n, S)$.

Let $U = \{i_1, i_1 + a_1, \ldots, i_k, i_k + a_k\}$. Then for any $x, y \in U$, without loss of generality, we assume that $x = i_1$. If $y = i_1 + a_1$, then $y - x = a_1 \not\equiv 0 (\text{mod } m)$. If $y = i_j$ for some $1 < j \leq k$, we also have $y - x = i_j - i_1 \not\equiv 0(\text{mod } m)$, for otherwise, we will have $i_j = i_1 + mh$ for some h. This means $i_j = \alpha^{hm}(i_1)$, which is a contradiction, since A is a 1-factorization array. If $y = i_j + a_j$ for some j, then $y - x = i_j - i_1 + a_j$. As above, we also have $i_j + a_j = \alpha^{hm}(i_1)$ for some integer h, which is a contradiction. Hence we have proved that for any $x, y \in U$, $y - x \not\equiv 0 (\text{mod } m)$. Let $P = \{\{i_1, i_1 + a_1\}, \ldots, \{i_k, i_k + a_k\}\}$. Then (U, P) is an $(a_1, a_2, \ldots, a_k; m)$-starter.

(3)$\Rightarrow$(4) Let (U, P) be an $(a_1, a_2, \ldots, a_k; m)$-starter. Then $U(\text{mod } m) = Z_{2k}$ if $m = 2k$; and $U(\text{mod } m) = Z_{2k+1} - \{i\}$ for some $0 \le i \le 2k$ if $m = 2k+1$. In any case, $P(\text{mod } m)$ is a desired 2-partition.

(4)$\Rightarrow$(5) Let P be a 2-partition of Z_{2k} or $Z_{2k+1} - \{i\}$ in (4). Then we can assume $\pm(x_i - y_i) \equiv \pm a_i (\text{mod } m)$. By exchanging x_i and y_i if neccesary, we can assume $x_i - y_i \equiv a_i (\text{mod } m)$. Clearly, this solution $\{(x_i, y_i) : i = 1, 2, \ldots, \lfloor m/2 \rfloor\}$ covers $2\lfloor m/2 \rfloor$ elements of Z_m.

(5)$\Rightarrow$(6) Let $M = \{(x_i, y_i) : i = 1, 2, \ldots, \lfloor m/2 \rfloor\}$ be a solution of (2.2) satisfying (5). Then $x_i - y_i \equiv a_i (\text{mod } m) = b_i$, where $1 \le b_i < m$. Hence, $|M \bigcap E_{b_i}| = 1$. But $|M| = k$, and therefore, M is orthogonal to $\{E_{b_i}, \ldots, E_{b_k}\}$.

(6)$\Rightarrow$(1) Let M be a k-matching in (6) and $M \bigcap E_{b_i} = \{(x_i, y_i)\}$. Without loss of generality, we assume $y_i - x_i = b_i$ in Z_m, for $i = 1, 2, \ldots, k$.

Let $a_i = mm_i + b_i$, and let $x_i' = x_i, y_i' = mm_i + y_i$. Then $y_i' - x_i' = a_i$. There are two cases.

Case 1. $m = 2k$.

Let $M = \{(x_i', y_i') : i = 1, 2, \ldots k\}$, and $F = < \alpha^m > (M)$.

Claim 1. F is a 1-factor of $C(n, S)$.

By the definition of F, $F = \{(x_i' + mh, y_i' + mh) : h = 0, 1, \ldots, n/m - 1 \text{ and } i = 1, 2, \ldots, k\}$. If $y_i' + mh = y_j' + mh'$, then $y_i + mm_i + mh = y_j + mm_j + mh'$, which implies that $y_j - y_i \equiv 0(\text{mod m})$. Since $0 \le y_i, y_j \le m - 1$, we must have $i = j$ and hence $h = h'$.

By a similar argument, we can show that

$$x_i' + mh \ne y_j' + mh',$$

$$y_i' + mh \ne x_j' + mh',$$

and

$$x_i' + mh \ne x_j' + mh',$$

if $(i, h) \ne (j, h')$. Therefore, F is a matching. By counting the edges in F, we know that F is a 1-factor.

Claim 2. $F, \alpha(F), \ldots, \alpha^{m-1}(F)$ is a cyclic 1-factorization.

To prove Claim 2, we need only show that $F, \alpha(F), \ldots, \alpha^{m-1}(F)$ is a 1-factorization. This is equivalent to showing that

$$F \bigcup \alpha(F) \bigcup \cdots \bigcup \alpha^{m-1}(F) = E(C(n, S)).$$

Let $F_i = \{(x_i' + mh, y_i' + mh) : h = 0, 1, \ldots, n/m - 1\}$. Then $F = \bigcup_{i=1}^{k} F_i$. Note that $F_i \bigcup \alpha(F_i) \bigcup \cdots \bigcup \alpha^{m-1}(F_i)$ contains all the edges generated by a_i, that is, $E_{a_i} = F_i \bigcup \alpha(F_i) \bigcup \cdots \bigcup \alpha^{m-1}(F_i)$. Therefore,

$$F \bigcup \alpha(F) \bigcup \cdots \bigcup \alpha^{m-1}(F) =$$

$$\bigcup_{i=1}^{k} (F_i \bigcup \alpha(F_i) \bigcup \cdots \bigcup \alpha^{m-1}(F_i)) =$$

$$\bigcup_{i=1}^{k} E_{a_i} = E(C(n, S)).$$

Case 2. $m = 2k + 1$.

Let $\bigcup_{j=1}^{k} \{x_j, y_j\} = Z_m - \{i\}$ for some i. Let M be as in Case 1, and let

$$F = < \alpha^m > (M) \bigcup \{(i + mh, i + n/2 + mh) : h = 0, 1, \ldots, \frac{n}{2m} - 1\}.$$

Then $F, \alpha(F), \ldots, \alpha^{m-1}(F)$ is a cyclic 1-factorization of $C(n, S)$. The proof is similar to Case 1. ∎

Remark 1. In statement (4), let $a_i \equiv b_i (\mathrm{mod}\ m)$. Then we can assume that $b_i < m/2$, otherwise, we choose $b_i \equiv -a_i (\mathrm{mod}\ m)$. Also if we use *ordered 2-partition* instead of 2-partition, then we can drop the '$\pm$' sign. Hence we obtain that statement (4) is equivalent to the following.

There exists an ordered 2-partition $\vec{P} = \{(x_i, y_i) : i = 1, 2, \ldots, k\}$ *of* Z_m *(or* $Z_m - \{i\}$ *for some* i*, if* m *is odd) such that* $\{y_i - x_i : i = 1, 2, \ldots, k\} = \{b_1, b_2, \ldots, b_k\}$.

This observation will be very useful for finding cyclic 1-factorization of a circulant of small degree.

2. Note that $C^*(m, \{\pm b_1, \pm b_2, \ldots, \pm b_k\})$ is a multigraph in general, and if we delete the multiple edges, then we obtain a circulant $C(m, \{\pm d_1, \pm d_2, \ldots, \pm d_t\})$, where $\{d_1, d_2, \ldots, d_t\} = \{b_1, b_2, \ldots, b_k\}$. Let the multiplicity of d_i in $\{b_1, b_2, \ldots, b_k\}$ be r_i. Then we have an equivalent form of Theorem 2.9(6).

There exists an $\lfloor m/2 \rfloor$-matching of K_m such that $|M \bigcap E_{d_i}| = r_i$.

This observation allows us to work with the complete graph K_m.

3. It is interesting to note that the existence of a cyclic 1-factorization with a single 1-factor orbit m of the circulant $C(n, \{\pm a_1, \ldots, \pm a_k\})$ does not depend on n very much. It depends only on the congruence class of n modulo m, that is, regardless how large n is, we only need $n \equiv 0 (\text{mod } m)$.

4. Alspach has posed the following question:

If $F_1, \ldots, F_r$ is any 2-factorization of a $2r$-regular simple graph G, does there always exist an orthogonal r-matching ?

Statement (6) of Theorem 2.9 is similar to this question. M. Kouider and D. Sotteau [34] have given a positive answer to this question when the order of G is at least $3.23r$.

2.4 Necessary conditions

In this section, we apply Theorem 2.9 to obtain some necessary conditions for the existence of a cyclic 1-factorization of circulants.

Let $C^*(m, \{\pm b_1, \pm b_2, \ldots, \pm b_k\})$ be the modulo m multigraph of $C(n, S)$. Recall that $\{b_1, b_2, \ldots, b_k\}$ is a multiset in general, and that $\{b_1, b_2, \ldots, b_k\} = \{d_1, d_2, \ldots, d_t\}$. We can assume that $0 < d_1 < d_2 < \cdots < d_t \leq k$, and that r_j is the multiplicity of d_j in $\{b_1, \ldots, b_k\}$, for $1 \leq j \leq t$. Then we have

$$r_1 + r_2 + \cdots + r_t = k.$$

Let $c(j_1, \ldots, j_h)$ be the number of connected components of $C(m, \{\pm d_{j_1}, \ldots, \pm d_{j_h}\})$.

Lemma 2.10 *The number of connected components of $C(m, \{\pm d_{j_1}, \ldots, \pm d_{j_h}\})$ is given by*

$$c(j_1, \ldots, j_h) = gcd(m, d_{j_1}, \ldots, d_{j_h})..$$

Proof. Let $d = gcd(m, d_{j_1}, \ldots, d_{j_h})$. Then we can partition Z_m into a union of left cosets of $< d >$ as

$$Z_m = < d > \bigcup (1+ < d >) \bigcup \cdots \bigcup ((d-1)+ < d >).$$

We have $< d >= \{d, 2d, \ldots, (m/d-1)d\} \cong Z_{m/d}$. If we let $d'_{j_i} = \frac{d_{j_i}}{d}$, then the subgraph of $C(m, \{\pm d_{j_1}, \ldots, \pm d_{j_h}\})$ induced on $< d >$ is isomorphic to $C(m/d, \{\pm d'_{j_1}, \ldots, \pm d'_{j_h}\})$ which is a connected graph.

Furthermore, there are no edges between $i+ < d >$ and $j+ < d >$, for $i \neq j$. For if $(i + hd, j + ld)$ is an edge, then $i - j + (h - l)d \in S$ which implies $i - j \equiv 0(\text{mod d})$. But $0 \leq i, j < d$, and therefore, $i = j$. This is a contradiction.

We have proved that $C(m, \{\pm d_{j_1}, \ldots, \pm d_{j_h}\})$ is a disjoint union of d connected subgraphs. Thus $c(j_1, \ldots, j_h) = d = gcd(m, d_{j_1}, \ldots, d_{j_h})$. ∎

For convenience, we denote $\{d_1, d_2, \ldots, d_t\} - \{d_{i_1}, \ldots, d_{i_j}\}$ by $\{d_{i_1}, \ldots, d_{i_j}\}^c$ and let 2^t denote all nonempty subsets of $\{1, 2, \ldots, t\}$.

Theorem 2.11 *(Necessary conditions)*

1. *If $X = C(n, \{a_1, \ldots, a_k, n/2, n - a_1, \ldots, n - a_k\})$ has a cyclic 1-factorization with a single 1-factor orbit of length $2k + 1$, then*

(1) $n \equiv 0(\text{mod } 2k + 1)$,

(2) $a_i \not\equiv 0(\text{mod } 2k + 1)$, *for* $i = 1, 2, \ldots, k$,

(3) $gcd(2k + 1, a_1, \ldots, a_k) = 1$, *and*

(4) $r_{i_1} + r_{i_2} + \cdots + r_{i_j} \geq \frac{gcd(2k+1, \{d_{i_1}, \ldots, d_{i_j}\}^c) - 1}{2}$, *for all* $\{i_1, \ldots, i_j\} \in 2^t$.

2. *If $X = C(n, \{a_1, \ldots, a_k, n - a_1, \ldots, n - a_k\})$, where $a_i < n/2$, has a cyclic 1-factorization with a single 1-factor orbit of length $2k$, then*

(1) $n \equiv 0(\text{mod } 2k)$,

(2) $a_i \not\equiv 0(\text{mod } 2k)$, *for* $i = 1, 2, \ldots, k$,

(3) S^+ *contains an even number of even symbols,*

(4) $\frac{2k}{gcd(2k, a_1, \ldots, a_k)}$ *is even, and*

(5) *for any* $\{i_1, \ldots, i_j\} \in 2^t$,

$$r_{i_1} + \cdots + r_{i_j} \geq \frac{gcd(2k, \{d_{i_1}, \ldots, d_{i_j}\}^c)}{2}$$

if $\frac{2k}{gcd(2k, \{d_{i_1}, \ldots, d_{i_j}\}^c)}$ *is odd.*

Proof. The proof of 1. (1) and (2) were proved in Lemma 2.6.

(3) By Theorem 2.9, the modulo $2k+1$ multigraph $C^*(2k+1,\{\pm b_1,\ldots,\pm b_k\})$ has a k-matching M which is orthogonal to $\{E_{b_1},\ldots,E_{b_k}\}$. Then we must have that $C^*(2k+1,\{\pm b_1,\ldots,\pm b_k\})$ is connected, for otherwise, each connected component of $C^*(2k+1,\{\pm b_1,\ldots,\pm b_k\})$ is odd, and we cannot have a k-matching. By Lemma 2.10, we have $gcd(2k+1,b_1,\ldots,b_k)=1$, this is equivalent to $gcd(2k+1,a_1,\ldots,a_k)=1$.

(4) As used in (3), $C^*(2k+1,\{\pm b_1,\ldots,\pm b_k\})$ has a k-matching M which is orthogonal to $\{E_{b_1},\ldots,E_{b_k}\}$. This implies that M has r_j edges in E_{d_j} of $C(2k+1,\{\pm d_1,\ldots,\pm d_t\})$ for $j=1,2,\ldots,t$. Suppose that we have chosen r_{i_1} edges from $E_{d_{i_1}}$, r_{i_2} edges from $E_{d_{i_2}}$, ..., r_{i_j} edges from $E_{d_{i_j}}$. These edges are just between the connected components of $C(2k+1,\{\pm d_1,\ldots,\pm d_t\})-(E_{d_{i_1}}\bigcup\cdots\bigcup E_{d_{i_j}})=C(2k+1,\{\pm d_{i_1},\ldots,\pm d_{i_j}\}^c)$. But there are $gcd(2k+1,\{d_{i_1},\ldots,d_{i_j}\}^c)$ connected components, and each component has odd order, therefore, these edges $M\bigcap(E_{d_{i_1}}\bigcup\cdots\bigcup E_{d_{i_j}})$ must match those components, except one. Hence,

$$r_{i_1}+r_{i_2}+\cdots+r_{i_j}\geq\frac{gcd(2k+1,\{d_{i_1},\ldots,d_{i_j}\}^c)-1}{2}.$$

The proof of 2. (1) and (2) were proved in Lemma 2.6, and (3) was proved in Corollary 2.5.

(4) By Theorem 2.9 again, $C^*(2k,\{\pm b_1,\ldots,\pm b_k\})$ has a perfect matching M which is orthogonal to $\{E_{b_1},\ldots,E_{b_k}\}$. By Lemma 2.10, $C^*(2k,\{\pm b_1,\ldots,\pm b_k\})$ has $gcd(2k,b_1,b_2,\ldots,b_k)=gcd(2k,a_1,a_2,\ldots,a_k)$ isomorphic connected components. Hence, each component has order $\frac{2k}{gcd(2k,a_1,a_2,\ldots,a_k)}$, and this number must be even since $C^*(2k,\{\pm b_1,\ldots,\pm b_k\})$ has a perfect matching.

(5) The proof is similar to the proof of 1(4). The differences are that the order of $C(2k,\{\pm d_1,\ldots,\pm d_k\})$ is $2k$, so the k-matching M is a perfect matching, and the edges $M\bigcap(E_{d_{i_1}}\bigcup\cdots\bigcup E_{d_{i_k}})$ must match all components of $C(2k,\{\pm d_{i_1},\ldots,\pm d_{i_j}\}^c)$. Each components has order $\frac{2k}{gcd(2k,\{d_{i_1},\ldots,d_{i_j}\}^c)}$, and if this number is odd, then we will have

$$r_{i_1}+\cdots+r_{i_j}\geq\frac{gcd(2k,\{d_{i_1},\ldots,d_{i_j}\}^c)}{2}.$$

This finishes the proof. ∎

2.5 Some classes of circulants which have a cyclic 1-factorization

This section deals with another application of Theorem 2.9. By Remark 2 following Theorem 2.9, we need only find a desired k-matching of K_m in the following proofs.

Let $X = C(n, S)$ be a circulant, and let $C^*(m, \{\pm b_1, \dots, \pm b_k\})$ be the modulo m multigraph.

Theorem 2.12 *If all b_i's are distinct and m is odd, then X has a cyclic 1-factorization with a single 1-factor orbit.*

Proof. Since all b_i's are distinct, then $\{b_1, \dots, b_k\} = \{1, 2, \dots, k\}$. Now $(x_1, y_1) = (k, k+1), (x_2, y_2) = (k-1, k+2), \dots, (x_k, y_k) = (1, 2k)$, is a desired k-matching. ∎

Theorem 2.13 *If $b_1 = b_2 = \cdots = b_k = c$ for some $c \neq 0$, then*

(1) $X = C(n, \{\pm a_1, \dots, \pm a_k, n/2\})$ has a cyclic 1- factorization with a single 1-factor orbit if and only if $gcd(c, 2k+1) = 1$; and

(2) $X = C(n, \{\pm a_1, \dots, \pm a_k\})$ has a cyclic 1-factorization with a single 1-factor orbit if and only if $\frac{2k}{gcd(2k,c)}$ is even.

Proof. In both cases, the necessity follows from Theorem 2.11. To prove the sufficiency, first we let $gcd(2k+1, c) = 1$. Then $E_c = 0, c, 2c, 3c, \dots, (2k-2)c, (2k-1)c, 2kc$ is a Hamilton cycle in $C^*(2k+1, \{\pm b_1, \dots, \pm b_k\})$. It is now easy to check that $(x_1, y_1) = (0, c), (x_2, y_2) = (2c, 3c), \dots, (x_k, y_k) = ((2k-2)c, (2k-1)c)$ is a desired k-matching.

If $\frac{2k}{gcd(2k,c)} = h$ is even, then each $E_{b_i} = E_c$ in $C^*(2k, \{\pm b_1, \dots, \pm b_k\})$ is a union of $gcd(2k, c)$ cycles of even length h. Hence E_c gives a k-matching $\{(x_1, y_1), (x_2, y_2), \dots, (x_k, y_k)\}$, which satisfies Remark 2 following Theorem 2.9. ∎

Corollary 2.14 *Let $X = C(n, \{\pm a_1, \dots, \pm a_k\})$ be a $2k$-regular circulant. Let $0 < c < m$, and $a_i \equiv c (\mathrm{mod}\ m)$, for $i = 1, 2, \dots, k$. Let $c = 2^l p$ and $k = 2^h q$, where $l, h \geq 0$ and p, q are odd integers. Then X has a cyclic 1-factorization with a single 1-factor orbit if and only if $l \leq h$.*

Proof. If $l \leq h$, then $\frac{2k}{gcd(2k,c)} = \frac{2^{h+1}q}{gcd(2^{h+1}q, 2^l p)} = \frac{2^{h+1-l}q}{gcd(2^{h+1-l}q, p)}$ is even. If $l > h$, then $\frac{2k}{gcd(2k,c)}$ is odd. The corollary follows from Theorem 2.13. ∎

Theorem 2.15 *Let* $b_1 = b_2 = \cdots = b_h = c, b_{h+1} = \cdots = b_k = d$.

(1) *If* $gcd(2k+1, d) = 1$, *then* $X = C(n, \{\pm a_1, \ldots, \pm a_k, n/2\})$ *has a cyclic 1-factorization with a single 1-factor orbit.*

(2) *If both* c *and* h *are even, and* $gcd(2k, d) = 1$, *then* $X = C(n, \{\pm a_1, \ldots, \pm a_k\})$ *has a cyclic 1-factorization with a single 1-factor orbit.*

Proof. (1) If $gdc(2k+1, d) = 1$, then E_d lies along a Hamilton cycle in $C^*(2k+1, \{\pm b_1, \pm b_2, \ldots, \pm b_k\})$. Without loss of generality, we can assume that $d = 1$.

Case 1. Suppose that $h \leq c$. We take an h-matching $M = \{(0, c), (1, c+1), \ldots, (h-1, h+c-1)\}$ from E_c. After deleting the vertices of M, the subgraph remaining in E_1 is two disjoint paths: $P_1 = h, h+1, \ldots, c-1$, and $P_2 = h+c, h+c+1, \ldots, 2k$. But $|P_1| = c-h$ and $|P_2| = 2k-h-c+1$, where $|P|$ represents the number of vertices in the path P. If $c-h$ is odd, then $|P_2|$ is even; if $c-h$ is even, then $|P_1|$ is even. In any case, one of $|P_1|$ and $|P_2|$ is even. Hence, we can obtain a $(k-h)$-matching from $P_1 \bigcup P_2$. Together with M, we obtain a desired k-matching.

Case 2. Suppose that $h > c$. Let $h = pc + r$ where $p \geq 1$ and $0 \leq r < c$. One may choose p c-matchings $M_1, M_2, \ldots, M_p$ and an r-matching M_{p+1} as follows:

$$M_1 = \{(0, c), (1, c+1), \ldots, (c-1, 2c-1)\},$$

$$M_2 = \{(2c, 3c), (2c+1, 3c+1), \ldots, (3c-1, 4c-1)\},$$

$$\ddots$$

$$M_p = \{((2p-2)c, (2p-2)c+c), \ldots, ((2p-2)c+c-1, 2pc-1)\}, and$$
$$M_{p+1} = \{(2pc, 2pc+c), \ldots, (2pc+r-1, (2p+1)c+r-1)\}.$$

After deleting the vertices in $M_1 \bigcup \cdots \bigcup M_{p+1}$, the subgraph remaining in E_1 is two disjoint paths

$$P_1 = 2pc+r, 2pc+r+1, \ldots, 2pc+c-1$$

and

$$P_2 = (2p+1)c+r, (2p+1)c+r+1, \ldots, 2k.$$

Hence $|P_1| = c - r$ and $|P_2| = 2(k - pc) - c - r + 1$. A simple argument shows that $|P_1|$ and $|P_2|$ have different parity. Therefore, one may obtain a $(k-h)$-matching M_{p+2} from $P_1 \bigcup P_2$. Now $\bigcup_{i=1}^{p+2} M_i$ is a required k-matching of K_m.

(2). The proof is similar to the proof in (1) except that both $|P_1|$ and $|P_2|$ are even in this case. Then we can obtain a $(k-h)$-matching from $P_1 \bigcup P_2$, and hence obtain a desired k-matching of K_m. ∎

Theorem 2.16 *Let* $(b_1, b_2, \ldots, b_k) = (a, a, \ldots, a, i, j)$, *where* $a, i, j < k$.

(1) *If* $gcd(a, 2k+1) = 1$, *then* $C(n, \{\pm a_1, \ldots, \pm a_k, n/2\})$ *has a cyclic 1-factorization with a single 1-factor orbit.*

(2) *If* $gcd(a, 2k) = 1$ *and* i, j *are even, then* $C(n, \{\pm a_1, \ldots, \pm a_k\})$ *has a cyclic 1-factorization with a single 1-factor orbit.*

Proof. (1) Without loss of generality, we assume that $a = 1$.

Case 1. i and j have the same parity.

Assuming that $i > j$, then $i > j + 1$. We choose edges $(0, i)$ and $(1, j + 1)$. The subgraph remaining in E_1 after deleting $\{0, 1, i, j + 1\}$ is the disjoint union of three paths: $P_1 = 2, 3, \ldots, j$; $P_2 = j + 2, j + 3, \ldots, i - 1$; and $P_3 = i + 1, i + 2, \ldots, 2k$. Then $|P_1| = j - 1, |P_2| = i - j - 2$, and $|P_3| = 2k - i$. If both i and j are even, then only $|P_1|$ is odd; if both i and j are odd, then only $|P_3|$ is odd. In any case, we can obtain a $(k-2)$-matching from $P_1 \bigcup P_2 \bigcup P_3$. This matching together with edges $(0, i)$ and $(1, j + 1)$ gives a desired k-matching.

Case 2. i is even and j is odd.

Subcase 1. $i < j$.

Choosing edges $(0, i)$ and $(1, j+1)$, then the remaining subgraph of E_1 - $\{0, 1, i, j\}$ is a disjoint union of three paths: $P_1 = 2, 3, \ldots, (i-1)$; $P_2 = (i+1), (i+2), \ldots, j$; and $P_3 = (j+2), (j+3), \ldots, 2k$. Then $|P_1| = i - 2$ is even, $|P_2| = j - i$ is odd, and $|P_3| = 2k - j - 1$ is even. Thus we can obtain a $(k-2)$-matching from $P_1 \bigcup P_2 \bigcup P_3$ which, together with $(0, i)$ and $(1, j + 1)$, gives a k-matching of K_m.

Subcase 2. $i > j$.

If $i > j + 1$, choosing edges $(0, i)$ and $(1, j + 1)$, the remaining subgraph of $E_1 - \{0, 1, i, j + 1\}$ is a disjoint union of three paths: $P_1 = 2, 3, \ldots, j$; $P_2 = (j + 2), (j +$

$3), \ldots, (i-1)$; and $P_3 = (i+1), (i+2), \ldots, 2k$. Then $|P_1| = j-1$ is even, $|P_2| = i-j-2$ is odd, and $|P_3| = 2k - i$ is even. Hence, from $P_1 \bigcup P_2 \bigcup P_3$, we can obtain a $(k-2)$-matching which, together with $(0, i)$ and $(1, j+1)$, gives a desired k-matching of K_m.

If $i = j + 1$, choosing edges $(0, i)$ and $(3, j+3)$, then the remaining subgraph of $E_1 - \{0, 3, i, j+3\}$ is a union of four disjoint paths: $P_1 = 1, 2; P_2 = 4, 5, \ldots, i-1; P_3 = \{i+1\}; P_4 = (j+4), (j+5), \ldots, 2k$. And $|P_1| = 2$, is even, $|P_2| = i - 4$ is even, and $P_4 = 2k - j - 3$ is even. Thus we can obtain a $(k-2)$-matching from $P_1 \bigcup P_2 \bigcup P_4$ which, together with $(0, i)$ and $(3, j+3)$, gives a desired k-matching of K_m.

(2) Again, we assume that $a = 1$. We choose edges $(0, i)$ and $(i-1, i-1+j)$. Then the subgraph $E_1 - \{0, i-1, i, i+j-1\}$ is a disjoint union of three paths: $P_1 = 1, 2, \ldots, i-2; P_2 = (i+1), (i+2), \ldots, (i+j-2)$; and $P_3 = (i+j), (i+j+1), \ldots, (2k-1)$. But each of the paths has odd length, and therefore, we can obtain a $(k-2)$-matching from $P_1 \bigcup P_2 \bigcup P_3$ which, together with $(0, i)$ and $(i-1, i-1+j)$, gives a desired k-matching of K_m. We have finished the proof. ∎

2.6 The classification of $C(2p, S)$, for prime p

The case with $n = 2p$, where p is a prime, can be solved completely. If $n = 4$, all the possible circulants of $C(4, S)$ are K_4, C_4, and $2K_2$. It is easy to see that $C(4, S)$ has a cyclic 1-factorization. So we assume that $p > 2$.

Theorem 2.17 *Let $n = 2p$, where $p > 2$ is a prime. Then $C(2p, S)$ has a cyclic 1-factorization if and only if one of the following conditions holds:*

(1) *S^+ does not contain an even symbol; or*

(2) *if S^+ contains an even symbol, then it has at most $\frac{p-1}{2}$ even symbols, at least $\frac{p+1}{2}$ symbols, and $p \in S^+$.*

Proof. Let $C(2p, S)$ have a cyclic 1-factorization. Suppose that S^+ contains an even symbol. Note that the only possible 1-factor orbit sizes are 2 and p. Thus, $C(2p, S)$ has a 1-factor orbit of length p since it contains an even symbol. But p is odd, so that $\frac{2p}{2} = p$ must belong to S^+, by Corollary 2.2.

Since we have only one 1-factor orbit of length p, each edge with even symbol must be in this 1-factor orbit and this 1-factor orbit contains at most $\frac{p-1}{2}$ non-diagonal edge orbits by Lemma 2.1. Therefore, the number of even symbols in S^+ is at most $\frac{p-1}{2}$.

Furthermore, if we have a 1-factor orbit of length p, then the degree of $C(2p, S)$ is at least p which implies that $|S^+| \geq \frac{p+1}{2}$. We have proved that either (1) or (2) holds if $C(2p, S)$ has a cyclic 1-factorization.

Conversely, if S^+ does not contain an even symbol, it is clear that $C(2p, S)$ has a cyclic 1-factorization such that each 1-factor orbit has length 2 or 1. If the even symbols in S^+ are $a_1, a_2, \ldots, a_l$, then $l \leq \frac{p-1}{2}$. But $|S^+| \geq \frac{p+1}{2}$, and $p \in S^+$, so that we can find $a_{l+1}, \ldots, a_{\frac{p-1}{2}}, p \in S^+$, where there are no additional a_i's when $l = \frac{p-1}{2}$.

Let $a_i \equiv b_i (\text{mod } p)$, for $i = 1, 2, \ldots, \frac{p-1}{2}$. Then $b_1, b_2, \ldots, b_{\frac{p-1}{2}}$ are all distinct (in fact, $a_i = b_i$). Hence $C(2p, \{\pm a_1, \pm a_2, \ldots, \pm a_{\frac{p-1}{2}}, p\})$ has a cyclic 1 - factorization with a single 1-factor orbit of length p by Theorem 2.12. Let $S_1 = S^+ - \{a_1, \ldots, a_{\frac{p-1}{2}}\}$. Then each element in S_1 is odd, implying that $C(2p, \pm S_1)$ has a cyclic 1-factorization with all 1-factor orbits of length 2.

Putting all these 1-factors together, we obtain a cyclic 1-factorization of $C(2p, S)$. ∎

2.7 Cyclic 1-factorizations of circulants with degree at most 11

In this section, by considering all possible orbit vectors, we can classify all cyclic 1-factorizable circulants of degree at most 11. For large degree circulants, the method works, but it is too complicated.

Like the proofs in section 2.5, we need only find an ordered partition of Z_{2k} or $Z_{2k+1} - \{i\}$ (for some i) for each $(b_1, b_2, \ldots, b_k)$, then the proof follows from Remark 1 following Theorem 2.9.

Theorem 2.18 1. *A 1-regular circulant has a cyclic 1-factorization.*

2. *A 2-regular circulant has a cyclic 1-factorization if and only if the only symbol in S^+ is odd.*

3. *A 3-regular circulant $C(n, \{\pm a_1, n/2\})$ has a cyclic 1-factorization if and only of one of the following conditions holds:*

(1) a_1 *is odd; or*

(2) a_1 *is even and* $a_1 \not\equiv 0(mod\ 3)$, $n \equiv 0(mod\ 3)$.

4. *A 4-regular circulant* $C(n, \{\pm a_1, \pm a_2\})$ *has a cyclic 1-factorization if and only if one of the following conditions holds:*

(1) *both* a_1 *and* a_2 *are odd; or*

(2) *both* a_1 *and* a_2 *are even,* $a_1, a_2 \not\equiv 0(mod\ 4)$, *and* $n \equiv 0(mod\ 4)$.

5. *A 5-regular circulant* $C(n, \{\pm a_1, \pm a_2, n/2\})$ *has a cyclic 1-factorization if and only if one of the following conditions holds:*

(1) *both* a_1 *and* a_2 *are odd;*

(2) *if, say,* a_1 *is even, and* a_2 *is odd, then* $a_1 \not\equiv 0(mod\ 3)$, *and* $n \equiv 0(mod\ 3)$;

(3) a_1 *and* a_2 *are even,* $a_1, a_2 \not\equiv 0(mod\ 4)$, *and* $n \equiv 0(mod\ 4)$; *or*

(4) $a_1, a_2 \not\equiv 0(mod\ 5)$, *and* $n \equiv 0(mod\ 5)$.

6. *A 6-regular circulant* $C(n, \{\pm a_1, \pm a_2, \pm a_3\})$ *has a cyclic 1-factorization if and only if one of the following conditions holds:*

(1) a_1, a_2 *and* a_3 *are odd;*

(2) *only one of the* a_i*'s is odd, say* a_3, *in which case* $a_1, a_2 \not\equiv 0(mod\ 4)$, $n \equiv 0(mod\ 4)$; *or*

(3) *only one of the* a_i*'s is odd, say* a_3, *in which case* $a_1, a_2, a_3 \not\equiv 0(mod\ 6)$, $n \equiv 0(mod\ 6)$.

7. *A 7-regular circulant* $C(n, \{\pm a_1, \pm a_2, \pm a_3, n/2\})$ *has a cyclic 1-factorization if and only if one of the following conditions holds:*

(1) a_1, a_2 *and* a_3 *are odd;*

(2) *only one of the* a_i*'s is even, say* a_1, *in which case* $a_1 \not\equiv 0(mod\ 3)$ *and* $n \equiv 0(mod\ 3)$;

(3) *only one of the* a_i*'s is odd, say* a_3, *in which case* $a_1, a_2 \not\equiv 0(mod\ 5)$, *and* $n \equiv 0(mod\ 5)$;

(4) *only one of the* a_i*'s is odd, say* a_3, *in which case* $a_1, a_2, \not\equiv 0(mod\ 4)$, *and*

$n \equiv 0(mod\ 4)$;

(5) *only one of the* a_i*'s is odd,* $a_1, a_2, a_3 \not\equiv 0(mod\ 6)$*, and* $n \equiv 0(mod\ 6)$*; or*

(6) $a_i \not\equiv 0(mod\ 7)$*, for* $i = 1, 2, 3$*, and* $n \equiv 0(mod\ 7)$*.*

Proof. 1. This is easy to see.

2. There is a 1-factor orbit of length 2 if and only if it is a partition of an odd edge orbit. Hence, the only symbol in S^+ is odd.

3. If a_1 is not odd, then the cyclic 1-factorization has a 1-factor orbit of length at least 3. But $C(n, \{\pm a_1, n/2\})$ is 3-regular, so that the cyclic 1-factorization must have a single 1-factor orbit of length 3. Hence $a_1 \not\equiv 0(\text{mod } 3)$, and $n \equiv 0$ (mod 3) by Theorem 2.11.

Conversely, if (1) holds, we have a cyclic 1-factorization with orbit vector $(2, 1)$. If (2) holds, then $a_1 \equiv 1$ or $2(\text{mod } 3)$. We can assume that $a_1 \equiv 1(\text{mod } 3)$ and then $\vec{P} = \{(0, 1)\}$ is the required ordered pair partition.

4. The possible lengths of 1-factor orbits are 2 and 4. If a_1 and a_2 have different parity, then the 1-factorization has a single 1-factor orbit of length 4. By Theorem 2.11, S^+ contains an even number of even elements. This is a contradiction. Therefore, a_1 and a_2 have same parity. If both a_1 and a_2 are even, then the cyclic 1-factorization must have a single 1-factor orbit of length 4. Hence $a_1, a_2 \not\equiv 0(\text{mod } 4)$, and $n \equiv 0(\text{mod } 4)$ by Theorem 2.11 again.

Conversely, if (1) holds, then $C(n, \{\pm a_1, \pm a_2\})$ has a cyclic 1-factorization with orbit vector $(2, 2)$ or orbit vector (4). If (2) holds, then we have $a_1 \equiv a_2 \equiv 2(\text{mod } 4)$, and $\frac{4}{gcd(4,2)} = 2$. By Theorem 2.13, $C(n, \{\pm a_1, \pm a_2\})$ has a cyclic 1-factorization with orbit vector (4).

5. Let $C(n, \{\pm a_1, \pm a_2, n/2\})$ have a cyclic 1-factorization. The possible 1-factor orbit vectors are $(2, 2, 1), (3, 2), (4, 1)$ and (5).

If the vector is $(2, 2, 1)$, we have that both a_1 and a_2 are odd. If the vector is $(3, 2)$, and at least one of a_1, a_2 is even, say a_1, then a_2 must be odd, and $a_1 \not\equiv 0(\text{mod } 3)$, $n \equiv 0(\text{mod } 3)$.

If the vector is $(4, 1)$, we have that both a_1 and a_2 are odd or even. In any case, we will have $a_1, a_2 \not\equiv 0(\text{mod } 4)$, and $n \equiv 0(\text{mod } 4)$.

If the vector is (5), then we will have $a_1, a_2 \not\equiv 0(\text{mod } 5)$ and $n \equiv 0(\text{mod } 5)$.

Conversely, if $C(n, \{\pm a_1, \pm a_2, n/2\})$ satisfies (1), (2) or (3), then, by the above proof, we know that $C(n, \{\pm a_1, \pm a_2, n/2\})$ has a cyclic 1-factorization with orbit vector $(2, 2, 1)$, $(3, 2)$ or $(4, 1)$, respectively. If (4) holds, let $a_i \equiv b_i (\text{mod } 5)$. We can assume that $b_i \leq 2$, for $i = 1, 2$, then $(b_1, b_2) = (1, 1)$ or $(2, 2)$ or $(1, 2)$. All cases are covered by Theorem 2.12 and Theorem 2.13.

6. The only possible orbit vectors are $(2, 2, 2), (4, 2)$ and (6).

If the vector is $(2, 2, 2)$, then (1) holds. If the vector is $(4, 2)$, we have (1) or (2) holds. If the vector is (6), that is, there is a single 1-factor orbit of length 6, then (3) holds by Theorem 2.11.

On the other hand, if (1) or (2) holds, it is easy to see that $C(n, \{\pm a_1, \pm a_2, \pm a_3\})$ has a cyclic 1-factorization. Let (3) hold, and let $a_i \equiv b_i (\text{mod } 6)$. As before, we can assume that $b_1, b_2, b_3 \leq 3$, then $(b_1, b_2, b_3) = (2, 2, 1)$ or $(2, 2, 3)$ since a_1, a_2 are even and a_3 is odd. The first case was covered by Theorem 2.15. For the second case, $\{(5, 1), (2, 4), (0, 3)\}$ is a desired ordered pair partition.

7. In fact, all cases except (6) are essentially proved above. The case (6) corresponds to the orbit vector (6), and the necessity follows from Theorem 2.11.

On the other hand, if (6) holds, let $a_i \equiv b_i (\text{mod } 7)$ for $i = 1, 2, 3$. Assuming $\{b_1, b_2, b_3\} \subseteq \{1, 2, 3\}$, all possible cases for (b_1, b_2, b_3) are: $(1, 1, 1), (2, 2, 2), (3, 3, 3)$, $(1, 2, 3), (1, 1, 2), (1, 1, 3), (2, 2, 3), (3, 3, 1)$ and $(3, 3, 2)$. All the cases are covered by Theorems 2.12, 2.13 and 2.15. ∎

Theorem 2.19 *An 8-regular circulant $C(n, \{\pm a_1, \pm a_2, \pm a_3, \pm a_4\})$ has a cyclic 1-factorization with a single 1-factor orbit if and only if*

1. $n \equiv 0 (\text{mod } 8)$ *and* $a_i \not\equiv 0 (\text{mod } 8)$ *for* $i = 1, 2, 3, 4$;

2. $\{a_1, a_2, a_3, a_4\}$ *contains an even number of even elements; and*

3. $\{a_1, a_2, a_3, a_4\} (\text{mod } 8) \neq \{\pm 1, \pm 2, 4, 4, \}, \{\pm 2, 4, 4, 4\}, \{\pm 2, \pm 2, \pm 2, 4\}$.

Proof. The necessity follows from Theorem 2.11 and some checking. For example, let $(a_1, a_2, a_3, a_4) = (1, 2, 4, 4)$. By the Remark 2 following Theorem 2.9, we need to show that a 4-matching M of K_8, such that M contains one edge with symbol 1, one edge with symbol 2 and two edges with symbol 4, does not exist. If M does exist, by symmetry, we can always choose $(0, 4)$ as an edge in M. The second edge with symbol 4 can be chosen as $(1, 5)$ or $(2, 6)$, and these are all possible choices up to symmetry.

But then we cannot find an edge with symbol 2 which is independent from $(0,4)$ and $(1,5)$, or an edge with symbol 1 which is independent from $(0,4)$ and $(2,6)$.

To prove the sufficiency, let $a_i \equiv b_i (\text{mod } 8)$, $i = 1,2,3,4$. We can assume that $b_i \leq 4$. We need consider only three cases: $(b_1, b_2, b_3, b_4) = (2,2,1,3), (2,4,1,3)$ and $(2,2,4,4)$, since all other cases are covered by Theorems 2.12, 2.13, 2.15, 2.16. For these three cases, $\{(2,4),(5,7),(0,1),(3,6)\}$, $\{(5,7),(0,4),(1,2),(3,6)\}$ and $\{(7,1),(3,5),(0,4),(2,6)\}$ are the respective required ordered pair partitions. ∎

Corollary 2.20 *An 8-regular circulant $C(n, \{\pm a_1, \pm a_2, \pm a_3, \pm a_4\})$ has a cyclic 1-factorization if and only if one of the following holds:*

1. *a_1, a_2, a_3 and a_4 are odd;*

2. *exactly two of a_i's are even, say a_1 and a_2, then $a_1, a_2 \not\equiv 0(\text{mod } 4)$, and $n \equiv 0(\text{mod } 4)$;*

3. *exactly two of a_i's are even, then $a_j, \not\equiv 0(\text{mod } 8)$ for $j = 1,2,3,4$, and $n \equiv 0(\text{mod } 8)$;*

4. *all a_i's are even, $a_i \not\equiv 0(\text{mod } 4)$, for $i = 1,2,3,4$, and $n \equiv 0(\text{mod } 4)$; or*

5. *all a_i's are even, $a_i \not\equiv 0(\text{mod } 8)$, for $i = 1,2,3,4$, and $n \equiv 0(\text{mod } 8)$, $\{a_1, a_2, a_3, a_4\}(\text{mod } 8) \neq \{\pm 2, 4, 4, 4\}, \{\pm 2, \pm 2, \pm 2, 4\}$.*

Theorem 2.21 *A 9-regular circulant $C(n, \{\pm a_1, \pm a_2, \pm a_3, \pm a_4, n/2\})$ has a cyclic 1-factorization with a single 1-factor orbit if and only if $0 \notin \{a_1, a_2, a_3, a_4\}(\text{mod } 9)$, $\{a_1, a_2, a_3, a_4\}(\text{mod } 9) \neq \{\pm 3, \pm 3, \pm 3, \pm 3\}$ and $n \equiv 0(\text{mod } 9)$.*

Proof. The proof is similar to the proof of Theorem 2.19. We need to check the following three cases: $(b_1, b_2, b_3, b_4) = (3,3,1,2), (3,3,1,4)$ and $(3,3,2,4)$. The corresponding ordered 2-partitions are: $\{(0,3),(2,5),(6,7),(8,1)\}$, $\{(5,8),(7,1),(2,3),(0,4)\}$ and $\{(2,5),(7,1),(6,8),(0,4)\}$. ∎

Theorem 2.22 *A 10-regular circulant $C(n, \{\pm a_1, \ldots, \pm a_5\})$ has a cyclic 1-factorization with a single 1-factor orbit if and only if $n \equiv 0(\text{mod } 10)$, $a_i \not\equiv 0(\text{mod } 10)$, $i = 1,2,3,4,5$, and $(a_1, a_2, a_3, a_4, a_5)(\text{mod } 10) \neq \pm(1,1,1,5,5), \pm(1,5,5,5,5), \pm(3,3,3,5,5), \pm(3,5,5,5,5), \pm(1,3,5,5,5), \pm(2,4,5,5,5), \pm(2,2,3,5,5)$ and $\pm(4,4,1,5,5)$.*

Proof. The necessity follows from Theorem 2.11 and some checking.

To prove the sufficiency, we give a table (see pages 29-30) providing ordered pair partitions of $(b_1, b_2, b_3, b_4, b_5)$ which are not covered by Theorems 2.12, 2.13, 2.15, 2.16. ∎

Theorem 2.23 *An 11-regular circulant $C(n, \{\pm a_1, \ldots, \pm a_5, n/2\})$ has a cyclic 1-factorization if and only if $n \equiv 0 (mod\ 11)$ and $a_i \not\equiv 0 (mod\ 11)$ for $i = 1, 2, 3, 4, 5$.*

Proof. The necessity follows from Theorem 2.11. To prove the sufficiency, we also give a table (see pages 31-32) as in the proof of Theorem 2.22. ∎

Remark. In fact, we can classify the cyclic 1-factorizable circulants with degrees 9, 10, and 11. The statements are too lengthy to give here.

2.8 Conclusion

Now we see that if we can characterize all cyclic 1-factorizable circulants with a single 1-factor orbit, then we can classify all cyclic 1-factorizable circulants by considering all the possible orbit vectors. We pose the following problem.

Problem 2.1 *Characterize all cyclic 1-factorizable circulants with a single 1-factor orbit.*

Remark. Even though there are many necessary conditions in Theorem 2.11, they are still not sufficient. For example, one can easily check that,if $n \equiv 0(\text{mod } 8)$ and $\{a_1, a_2, a_3, a_4\}(\text{mod } 8) = \{2, 2, 2, 4\}$ or $\{2, 4, 4, 4\}$, then $C(n, \{\pm a_1, \pm a_2, \pm a_3, \pm a_4\})$ has no cyclic 1-factorization with a single 1-factor orbit. So the first thing we need to do is find more necessary conditions.

Another thing we want to point out is that when $m = 2k + 1$ is a prime, then the necessary conditions in Theorem 2.11 become

(i) $n \equiv 0(\text{mod } m)$; and

(ii) $a_i \not\equiv 0(\text{mod } m)$ for $i = 1, 2, \ldots, k$.

The number of necessary conditions is much less than usual. Also we know that for $m = 2, 3, 5, 7$ and 11, the necessary conditions are sufficient. Perhaps this is true for all primes.

G. Korchmáros [33] has posed the following problem:

For which integer $n \equiv 2 (mod\, 4)$ does there exist a cyclic 1-factorization of K_n with an invariant 1-factor?

A cyclic 1-factorization of K_n with an invariant 1-factor is just a cyclic 1-factorization of $C(n, Z_n - \{0, n/2\})$.

By using the previous results, we can prove the following.

Theorem 2.24 1. *If K_n has a cyclic 1-factorization with an invariant 1-factor, and $n \equiv 2 (mod\, 4)$, then $n \equiv 2 (mod\, 8)$.*

2. *Let $n = 2(4m + 1)$ and $4m + 1 = p^t$, where p is a prime. Then K_n has no cyclic 1-factorization with an invariant 1-factor.*

Proof. 1. If K_n has a cyclic 1-factorization with an invariant 1-factor, then each 1-factor orbit must have even length, and each 1-factor orbit of even length contains even number of even symbols. So the total number of even symbols is even. For $n \equiv 2(\text{mod } 4)$, the total number of even symbols is $\frac{1}{2}(\frac{n}{2} - 1)$, which is even. Therefore, $n \equiv 2(\text{mod } 8)$.

2. If K_n has a cyclic 1-factorization with an invariant 1-factor, then each 1-factor orbit has even length. The possible even lengths are $2, 2p, 2p^2, \ldots, 2p^{t-1}$. But $E_{2p^{t-1}}$ is not empty, so it must be in a 1-factor of orbit length $2p^l$ for some $l \leq t - 1$. By Lemma 2.6, we have $2p^{t-1} \not\equiv 0(\text{mod } 2p^l)$. This is a contradiction. ∎

$(b_1, b_2, b_3, b_4, b_5)$	*ordered pair partition*
(1 1 1 1 3)	(1, 2)(4, 5)(6, 7)(8, 9)(0, 3)
(1 1 1 3 3)	(1, 2)(5, 6)(8, 9)(0, 3)(4, 7)
(1 1 3 3 3)	(6, 7)(8, 9)(0, 3)(1, 4)(2, 5)
(1 3 3 3 3)	(7, 8)(0, 3)(1, 4)(2, 5)(6, 9)
(1 1 1 1 5)	(1, 2)(3, 4)(6, 7)(8, 9)(0, 5)
(1 1 5 5 5)	(3, 4)(8, 9)(0, 5)(1, 6)(2, 7)
(3 3 3 3 5)	(9, 2)(8, 1)(4, 7)(3, 6)(0, 5)
(3 3 5 5 5)	(6, 9)(1, 4)(0, 5)(2, 7)(3, 8)
(1 3 5 1 1)	(2, 3)(1, 4)(0, 5)(6, 7)(8, 9)
(1 3 5 3 3)	(2, 3)(4, 7)(0, 5)(8, 1)(6, 9)
(1 3 5 3 5)	(3, 4)(6, 9)(0, 5)(8, 1)(2, 7)
(1 3 5 1 3)	(2, 3)(6, 9)(0, 5)(7, 8)(1, 4)
(1 3 5 1 5)	(3, 4)(9, 2)(0, 5)(7, 8)(1, 6)
(2 2 1 1 3)	(6, 8)(7, 9)(1, 2)(4, 5)(0, 3)
(2 2 1 1 5)	(1, 3)(2, 4)(6, 7)(8, 9)(0, 5)
(2 2 1 3 3)	(9, 1)(4, 6)(7, 8)(0, 3)(2, 5)
(2 2 3 3 5)	(1, 3)(6, 8)(9, 2)(4, 7)(0, 5)
(2 2 1 5 5)	(1, 3)(4, 6)(8, 9)(0, 5)(2, 7)
(4 4 1 1 3)	(0, 4)(1, 5)(2, 3)(7, 8)(6, 9)
(4 4 1 1 5)	(7, 1)(2, 6)(3, 4)(8, 9)(0, 5)
(4 4 1 3 3)	(0, 4)(1, 5)(7, 8)(3, 6)(9, 2)
(4 4 3 3 5)	(7, 1)(4, 8)(3, 6)(9, 2)(0, 5)

$(b_1, b_2, b_3, b_4, b_5)$	*ordered pair partition*
(4 4 3 5 5)	(9, 3)(8, 2)(4, 7)(0, 5)(1, 6)
(2 2 1 3 5)	(6, 8)(7, 9)(2, 3)(1, 4)(0, 5)
(4 4 1 3 5)	(4, 8)(7, 1)(2, 3)(6, 9)(0, 5)
(2 4 1 3 5)	(6, 8)(7, 1)(3, 4)(9, 2)(0, 5)
(2 4 1 1 3)	(9, 1)(0, 4)(2, 3)(6, 7)(5, 8)
(2 4 1 1 5)	(9, 1)(8, 2)(3, 4)(6, 7)(0, 5)
(2 4 1 3 3)	(1, 3)(0, 4)(6, 7)(5, 8)(9, 2)
(2 4 3 3 5)	(9, 1)(8, 2)(3, 6)(4, 7)(0, 5)
(2 4 1 5 5)	(7, 9)(8, 2)(3, 4)(0, 5)(1, 6)
(2 4 3 5 5)	(9, 1)(4, 8)(3, 6)(0, 5)(2, 7)
(2 4 4 4 1)	(9, 1)(0, 4)(8, 2)(3, 7)(5, 6)
(2 2 4 4 1)	(9, 1)(3, 5)(0, 4)(8, 2)(6, 7)
(2 2 2 4 1)	(9, 1)(5, 7)(6, 8)(0, 4)(2, 3)
(2 4 4 4 3)	(7, 9)(0, 4)(1, 5)(8, 2)(3, 6)
(2 2 4 4 3)	(7, 9)(1, 3)(0, 4)(2, 6)(5, 8)
(2 2 2 4 3)	(1, 3)(5, 7)(6, 8)(0, 4)(9, 2)
(2 2 2 2 5)	(1, 3)(2, 4)(6, 8)(7, 9)(0, 5)
(4 4 4 4 5)	(7, 1)(2, 6)(4, 8)(9, 3)(0, 5)
(2 4 4 4 5)	(9, 1)(2, 6)(3, 7)(4, 8)(0, 5)
(2 2 4 4 5)	(9, 1)(4, 6)(8, 2)(3, 7)(0, 5)
(2 2 2 4 5)	(1, 3)(4, 6)(7, 9)(8, 2)(0, 5)

$(b_1, b_2, b_3, b_4, b_5)$	*ordered pair partition*
(1 1 2 3 4)	(1, 2)(7, 8)(3, 5)(6, 9)(0, 4)
(1 1 2 3 5)	(2, 3)(6, 7)(8, 10)(1, 4)(0, 5)
(1 1 2 4 5)	(1, 2)(3, 4)(7, 9)(6, 10)(0, 5)
(1 1 3 4 5)	(2, 3)(7, 8)(1, 4)(6, 10)(0, 5)
(2 2 1 3 4)	(1, 3)(6, 8)(9, 10)(2, 5)(0, 4)
(2 2 1 3 5)	(6, 8)(7, 9)(2, 3)(1, 4)(0, 5)
(2 2 1 4 5)	(7, 9)(8, 10)(3, 4)(2, 6)(0, 5)
(2 2 3 4 5)	(7, 9)(8, 10)(1, 4)(2, 6)(0, 5)
(2 4 1 5 5)	(7, 9)(8, 1)(2, 3)(0, 5)(10, 4)
(3 3 1 2 4)	(2, 5)(6, 9)(7, 8)(1, 3)(0, 4)
(3 3 1 2 5)	(3, 6)(4, 7)(1, 2)(8, 10)(0, 5)
(3 3 1 4 5)	(1, 4)(7, 10)(8, 9)(2, 6)(0, 5)
(3 3 2 4 5)	(10, 2)(1, 4)(6, 8)(3, 7)(0, 5)
(4 4 1 2 3)	(0, 4)(1, 5)(2, 3)(8, 10)(6, 9)
(4 4 1 2 5)	(2, 6)(4, 8)(9, 10)(1, 3)(0, 5)
(4 4 1 3 5)	(2, 6)(3, 7)(9, 10)(1, 4)(0, 5)
(4 4 2 3 5)	(2, 6)(3, 7)(8, 10)(1, 4)(0, 5)
(5 5 1 2 3)	(0, 5)(1, 6)(8, 9)(2, 4)(7, 10)
(5 5 1 2 4)	(0, 5)(1, 6)(8, 9)(2, 4)(3, 7)
(5 5 1 3 4)	(0, 5)(1, 6)(2, 3)(7, 10)(4, 8)
(5 5 2 3 4)	(0, 5)(3, 8)(7, 9)(1, 4)(2, 6)
(1 1 2 2 3)	(1, 2)(4, 5)(6, 8)(7, 9)(0, 3)
(1 1 2 2 4)	(1, 2)(9, 10)(3, 5)(6, 8)(0, 4)
(1 1 2 2 5)	(1, 2)(3, 4)(6, 8)(7, 9)(0, 5)
(1 1 3 3 2)	(1, 2)(5, 6)(4, 7)(0, 3)(8, 10)
(1 1 3 3 4)	(1, 2)(9, 10)(3, 6)(5, 8)(0, 4)

$(b_1, b_2, b_3, b_4, b_5)$	*ordered pair partition*
(1 1 3 3 5)	(1, 2)(8, 9)(3, 6)(4, 7)(0, 5)
(1 1 4 4 2)	(1, 2)(5, 6)(0, 4)(3, 7)(8, 10)
(1 1 4 4 3)	(1, 2)(9, 10)(3, 7)(0, 4)(5, 8)
(1 1 4 4 5)	(1, 2)(9, 10)(3, 7)(4, 8)(0, 5)
(1 1 5 5 2)	(7, 8)(9, 10)(0, 5)(1, 6)(2, 4)
(1 1 5 5 3)	(2, 3)(8, 9)(0, 5)(1, 6)(7, 10)
(1 1 5 5 4)	(2, 3)(9, 10)(0, 5)(1, 6)(4, 8)
(2 2 3 3 1)	(5, 7)(6, 8)(0, 3)(1, 4)(9, 10)
(2 2 3 3 4)	(1, 3)(8, 10)(2, 5)(6, 9)(0, 4)
(2 2 3 3 5)	(1, 3)(2, 4)(6, 9)(7, 10)(0, 5)
(3 3 4 4 1)	(6, 9)(7, 10)(0, 4)(1, 5)(2, 3)
(3 3 4 4 2)	(5, 8)(7, 10)(0, 4)(2, 6)(1, 3)
(3 3 4 4 5)	(3, 6)(4, 7)(8, 1)(9, 2)(0, 5)
(4 4 5 5 1)	(3, 7)(4, 8)(0, 5)(1, 6)(9, 10)
(4 4 5 5 2)	(3, 7)(9, 2)(0, 5)(1, 6)(8, 10)
(4 4 5 5 3)	(3, 7)(4, 8)(0, 5)(1, 6)(10, 2)
(2 2 4 4 1)	(7, 9)(8, 10)(0, 4)(1, 5)(2, 3)
(2 2 4 4 3)	(7, 9)(8, 10)(0, 4)(1, 5)(3, 6)
(2 2 4 4 5)	(1, 3)(7, 9)(2, 6)(4, 8)(0, 5)
(3 3 5 5 1)	(4, 7)(10, 2)(0, 5)(1, 6)(8, 9)
(3 3 5 5 2)	(3, 6)(1, 4)(0, 5)(2, 7)(8, 10)
(3 3 5 5 4)	(3, 6)(10, 2)(0, 5)(7, 1)(4, 8)
(2 2 5 5 1)	(7, 9)(8, 10)(0, 5)(1, 6)(2, 3)
(2 2 5 5 3)	(8, 10)(4, 6)(0, 5)(2, 7)(9, 1)
(2 2 5 5 4)	(4, 6)(10, 1)(0, 5)(3, 8)(9, 2)

Chapter 3

Isomorphic Factorizations of Circulants

3.1 Introduction

Definition 3.1 Let G and H be graphs. The *union* of G and H is defined by

$$G\bigcup H = (V(G)\bigcup V(H), E(G)\bigcup V(H)).$$

If $E(G)\bigcap E(H) = \emptyset$, we denoted $G\bigcup H$ by $G \oplus H$.

The union of k disjoint copies of G is denoted by kG.

Definition 3.2 Let G be a graph. G is said to be *t-divisible*, denoted by $t|G$, if $\varepsilon(G) \not\equiv 0 (\text{mod } t)$ or $E(G)$ can be partitioned into t isomorphic subgraphs $G_1, G_2, \ldots, G_t$. We call $G_1, G_2, \ldots, G_t$ a *t-isomorphic factorization* of G, or simply a *t-partition*, and write $G = G_1 \oplus G_2 \oplus \cdots \oplus G_t$.

If for all t such that $t|\varepsilon(G)$ we have $t|G$, we say that G is *divisible*.

Lemma 3.1 *If G is divisible, then nG is divisible for any integer n.*

Proof. We have that $\varepsilon(nG) = n\varepsilon(G)$. Let $t|\varepsilon(nG)$. Then $t|n\varepsilon(G)$.

Case 1. $t|n$.

Let $n = tm$. Then $nG = t(mG)$, and hence $t|nG$.

Case 2. $t|\varepsilon(G)$.

Since G is t-divisible, then $G = G_1 \oplus G_2 \oplus \cdots \oplus G_t$. Let $H_i = nG_i$, for $i = 1, 2, \ldots, t$. Then $nG = H_1 \oplus H_2 \oplus \cdots \oplus H_t$, and $H_i \cong H_j$ for $1 \leq i, j \leq t$. Therefore, $t|nG$.

Case 3. $n \not\equiv 0(\text{mod } t)$ and $\varepsilon(G) \not\equiv 0(\text{mod } t)$.

We will have that $t = t_1 t_2$ for some integers t_1 and t_2, such that $t_1|n$ and $t_2|\varepsilon(G)$. Since G is t_2-divisible, then $G = G_1 \oplus \cdots \oplus G_{t_2}$, where $G_i \cong G_j$ for $1 \leq i, j \leq t_2$. Let $H_i = \frac{n}{t_1} G_1$. Then $nG = H_1 \oplus \cdots \oplus H_t$, and $H_i \cong H_j$. Hence $t|H$. ∎

Lemma 3.2 *Any disconnected circulant graph is a disjoint union of isomorphic connected circulant graphs.*

Proof. Let $X = C(n, \{\pm a_1, \ldots, \pm a_k\})$ be a circulant which is disconnected. Then $gcd(a_1, a_2, \ldots, a_k, n) = d \neq 1$. Let $n = dm$ and $a_i = dc_i$ for $i = 1, 2, \ldots, k$.

Partition the vertex set Z_n into

$$< d >, 1+ < d >, \ldots, (d-1)+ < d > .$$

Then $X[i+ < d >]$, the induced subgraph of X on $i+ < d >$, is a circulant which is isomorphic to $C(m, \{\pm c_1, \pm c_2, \ldots, \pm c_k\})$ for each $1 \leq i \leq k$. Furthermore, there are no edges of X between any $i+ < d >$ and $j+ < d >$ for $i \neq j$. Therefore,

$$X = X[< d >] \oplus \cdots \oplus X[d - 1+ < d >] \cong dC(m, \{\pm c_1, \ldots, \pm c_k\}).$$

Moreover, $gcd(c_1, c_2, \ldots, c_k, m) = 1$, thus $C(m, \{\pm c_1, \pm c_2, \ldots, \pm c_k\})$ is connected. This completes the proof. ∎

From the proof above, we obtain a formula for circulant graphs, which we write as a corollary.

Corollary 3.3

$$C(dm, \{dc_1, \ldots, dc_k, d(m-c_1), \ldots, d(m-c_k)\}) = dC(m, \{c_1, \ldots, c_k, m-c_1, \ldots, m-c_k\}).$$

Remark. By Lemma 3.1 and Lemma 3.2, we need only consider connected circulants when we investigate the divisibility of circulants. From now on, we assume that all circulants are connected.

3.2 Main results

Theorem 3.4 *Let $X = C(n,S)$ be a d-regular circulant, and t be a positive integer. Then $t|X$ if $t|n$.*

Proof. Since $t|n$, then we can partition Z_n into the left cosets

$$< t >, 1+ < t >, \ldots, (t-1)+ < t > .$$

If d is odd, then $t|\frac{dn}{2}$ and $t|n$ imply that $\frac{n}{2} \equiv 0 (\mathrm{mod}\ t)$. If d is even, then $\frac{n}{2} \notin S$. Therefore, in any case, the diagonal edges (if they exist) must be in $X[i+ < t >]$ for $i = 0, 1, 2, \ldots, t-1$.

Let $S^+ = \{a_1, a_2, \ldots, a_k\}$. As defined in Chapter 2, $S^+(\mathrm{mod}\ t) = \{a_1(\mathrm{mod}\ t), \ldots, a_k(\mathrm{mod}\ t)\}$ is a multiset in general, but here we treat $S^+(\mathrm{mod}\ t)$ as a non-multiset. Let $S^+(\mathrm{mod}\ t) = \{j_1, \ldots, j_l\}$, where $j_i \neq j_h$ if $i \neq h$.

Case 1. $t \neq 2$ and $\frac{t}{2} \notin S^+(\mathrm{mod}\ t)$.

If $a_i \equiv j(\mathrm{mod}\ t)$, we can assume that $a_i = j + tm$ for some integer m. Between $< t >$ and $j+ < t >$, there is a 'parallel' n/t-matching $\{(0, mt+j), (t, (m+1)t + j), \ldots, ((\frac{n}{t}-1)t, (\frac{n}{t}+m-1)t+j)\}$. We call this the matching starting at $< t >$ generated by a_i, and denote it by $M(< t >, a_i)$.

Let $<< t >, j+ < t >>= \bigcup\{M(< t >, a_i) | a_i \equiv j(\mathrm{mod}\ t)\}$.

Let

$$X_0 = X[< t >] \bigcup << t >, j_1+ < t >> \bigcup \cdots \bigcup << t >, j_l+ < t >>$$

and

$$X_i = X[i+ < t >] \bigcup < i+ < t >, i+j_1+ < t >> \bigcup \cdots \bigcup < i+ < t >, i+j_l+ < t >>$$

for $i = 1, 2, \ldots, t-1$.

It is easy to see that $X_i \cong X_j$ and $E(X_i) \bigcap E(X_j) = \emptyset$ if $i \neq j$. Also, $E(X) = \bigcup_{i=0}^{t-1} E(X_i)$, therefore, $t|X$.

Case 2. $\frac{t}{2} \in S^+(\mathrm{mod}\ t)$.

Without loss of generality, let $\frac{t}{2} = j_l$, and $a_1 \equiv a_2 \equiv \cdots \equiv a_m \equiv \frac{t}{2}(\mathrm{mod}\ t)$. Then between $i+ < t >$ and $\frac{t}{2} + i+ < t >$, each edge orbit E_{a_h} contributes two perfect matchings, one with symbol a_h, denoted by F_h^+, another with symbol $n - a_h$, denoted by F_h^-, for $1 \leq h \leq m$.

Let $F_0 = \bigcup_{h=1}^m F_h{}^+$. Then $\frac{t}{2} + F_0 = \bigcup_{h=1}^m F_h{}^-$. Now, define

$$X_0 = X[< t >] \bigcup F_0 \bigcup << t >, j_1+ < t >> \bigcup \cdots \bigcup << t >, j_{l-1}+ < t >>$$

and

$$X_i = i + X_0 = X[i+ < t >] \bigcup < i+ < t >, i + j_1 + < t >> \bigcup \cdots \bigcup < i+ < t >, i + j_{l-1} + < t >> \bigcup (i + F_0)$$

for $i = 1, 2, \ldots, t-1$.

Then $X_0, X_1, \ldots, X_{t-1}$ gives a t-partition of X.

Case 3. $t = 2$.

In this case, Z_n is partitioned into $< t >$ and $1+ < t >$. Between $< t >$ and $1+ < t >$, all edges have odd symbols. Let $\{a_1, a_2, \ldots, a_l\}$ be the odd symbol set of X, and let $F_i = \{(2j, 2j + a_i) : 0 \leq j < \frac{n}{2}\}$. Then

$$X_0 = X[< t >] \bigcup F_1 \bigcup F_2 \bigcup \cdots \bigcup F_l$$

and

$$X_1 = X[1+ < t >] \bigcup (1 + F_1) \bigcup (1 + F_2) \bigcup \cdots \bigcup (1 + F_l)$$

is a 2-partition of G. This completes the proof. ∎

Corollary 3.5 *Let $X = C(n, S)$ be a d-regular circulant with $d > 2$. Then X is t-divisible if $t \geq d - 2$. In particular, 4-regular circulants are divisible.*

Proof. A result of Ellingham and Wormald [24] says that a d-regular graph is t-divisible if $t > d$. So we need only prove the corollary for $t \leq d$. But we give a proof for $d - 2 \leq t \leq d + 2$ here.

Let $t|\frac{dn}{2}$. If $t = d-1$, or $d+1$, then $gcd(t, d) = 1$. If d is odd, and $t = d-2$ or $d+2$, we still have $gcd(t, d) = 1$. Otherwise, say $gcd(t, d) = m \neq 1$, then $m|d$ and $m|(d-2)$ or $m|(d+2)$. Hence, $m = 2$ and d is odd. This is a contradiction. For any of the above cases, we have $t|n$. If d is even and $t = d-2$ or $d+2$, then $gcd(t, d) = 2$. Now $t|\frac{dn}{2}$ implies $\frac{t}{2}|\frac{n}{2}$. Again we have $t|n$. By Theorem 3.3, we know that X is t-divisible for $t = d-2, d-1, d+1$ and $d+2$. If $t = d$, and $d|n$, then we are done. So assume that $n \not\equiv 0 \pmod{d}$, in which case n is even. But it is well known that X has a 1-factorization, and this 1-factorization gives a d-isomorphic factorization. ∎

Corollary 3.6 *Let $X = C(n,S)$ be a d-regular circulant, and let d be a prime. Then X is divisible.*

Proof. Let t be any positive integer such that $t|\varepsilon(X)$. Then $t|\frac{dn}{2}$. We can assume that $2 \leq t < d$ from the above corollary. Then $gcd(t,d) = 1$, and hence $t|n$. By Theorem 3.3, X is t-divisible, and therefore X is divisible. ∎

Remark. Recall that in Chapter 2, we have that the necessary condition for $X = C(n,S)$ to have a cyclic 1-factorization with a single 1-factor orbit of length t is $t|n$. Unfortunately, it is not sufficient in general. But from the proof of Theorem 3.3, we know that we actually proved that $X = C(n,S)$ has a cyclic t-isomorphic factorization if $t|n$. Therefore, if we do not restrict our factor graph to be a 1-factor, we have proved:

Corollary 3.7 *There is a cyclic isomorphic factorization of $C(n,S)$ with single factor orbit of length t if and only if $t|n$.*

By the proof of Corollary 3.1, 3.2, we actually have proved:

Corollary 3.8 *A d-regular circulant $C(n,S)$ has a cyclic t-isomorphic factorization for $t = d-2, d-1, d+1, d+2$. If d is a prime, then $C(n,S)$ has a cyclic t-isomorphic factorization for $t \neq d$.*

Also notice that, by the proof of Theorem 3.3, it is not hard to determine the factor graph, which depends on the symbols of $C(n,S)$. For example, if $n/2 \notin S$ and $a_i \not\equiv a_j (\text{mod } t)$ for $a_i \neq a_j$, then the factor graph is a union of n/t k-stars and some isolated vertices. For the case $t = n$ (in this case, d is even), the factor graph is a union of a $d/2$-stars and $(n - d/2 - 1)$ isolated vertices.

Some special t's are more interesting, among them are $t = 2, d/2$.

If $t = 2$, we have the following result as a corollary of Theorem 3.3.

Corollary 3.9 *Let $X = C(n,S)$ be a d-regular circulant. Then X is 2-divisible if n is even or $d \not\equiv 0 (\text{mod } 4)$.*

The case $t = d/2$ is a weak form of the following problem:

Every connected circulant with even degree has a Hamilton decomposition.

Since a Hamilton decomposition is a special $d/2$-isomorphic factorization, it would be interesting to solve the weak form.

To end this chapter, we pose some problems.

Problem 3.1 *Prove that $C(n, S)$ is 2-divisible.*

Problem 3.2 *Prove that the $2k$-regular circulant $C(n, S)$ is k-divisible.*

Problem 3.3 *Let $C(2n, S)$ be a d-regular circulant. Show that there is a cyclic d-isomorphic factorization of $C(2n, S)$.*

Chapter 4

On 2-extendable Dihedral Cayley Graphs

4.1 Introduction

Recall that the dihedral group D_n is a group which is generated by two elements ρ and τ, where $\rho^n = \tau^2 = 1$ and $\tau\rho\tau = \rho^{-1}$. We denote $\{x\tau | x \in <\rho>\}$ by $<\rho>\tau$. From the relations $\rho^n = \tau^2 = 1$ and $\tau\rho\tau = \rho^{-1}$, we can easily obtain $(\rho^i\tau)^2 = 1$ and $\rho^i\tau\rho^{-j} = \tau\rho^{-(i+j)} = \rho^{i+j}\tau$, which are useful later. It is easy to see that D_n has a cyclic subgroup $<\rho>$ of index 2 which is isomorphic to Z_n. Moreover, $D_n = <\rho> \bigcup <\rho>\tau$.

Let X be a graph. If M is a k-matching of X and M^* is a perfect matching of X such that $M \subseteq M^*$, we call M^* *a matching extension of* M, or say M *can be extended to* M^*. A graph X is said to be *k-extendable* if it has k-matchings and any k-matching of X can be extended to a perfect matching of X.

Recently, O. Chan, C. C. Chen and Q. L. Yu classified the 2-extendable Cayley graphs on abelian groups. Their classification, as stated below, will be used in the proof later.

Theorem 4.1 *Let $X = X(\Gamma, S)$ be a Cayley graph on an abelian group Γ of even order. Then X is 2-extendable if and only if it is not isomorphic to any of the following graphs:*

(I) $C(2n, \{1, 2n-1\}), n \geq 3$;

(II) $C(2n, \{1, 2, 2n-1, 2n-2\}), n \geq 3$;

(III) $C(4n, \{1, 4n-1, 2n\}), n \geq 2$;

(IV) $C(4n+2, \{2, 4n, 2n+1\}), n \geq 1$; *and*

(V) $C(4n+2, \{1, 4n+1, 2n, 2n+2\}), n \geq 1$.

Stong [46] has proved that any Cayley graph on a dihedral group is 1-factorizable. His result implies that $X(D_n, S)$ is 1-extendable. In this chapter, we shall give a classification for 2-extendable Cayley graphs on dihedral groups by showing that, except for the five classes of graphs in Theorem 4.1, $X(D_n, S)$ is 2-extendable.

From now on, we shall assume that $X = X(D_n, S)$ is connected, that is, S is a generating set of D_n, or $< S >= D_n$. For convenience, we let $S' = S \bigcap < \rho >$ and $S'' = S \bigcap < \rho > \tau$. Then clearly, $S'' \neq \emptyset$ as $X(D_n, S)$ is connected. E_s, the set of edges with symbol s, is a perfect matching of $X(D_n, S)$ for $s \in S''$. Also, without loss of generality, we may always assume $\tau \in S''$.

We introduce a class of graphs, denoted by $C[2q, s, t]$ (where $s + t \equiv 0 (\text{mod } 2)$), which are defined as follows. The vertex set is $\{(i, j) | 0 \leq i \leq 2q-1; 0 \leq j \leq s-1\}$, which is the cartesian product of Z_{2q} and Z_s. The edge set consists of three types of pairs as given below:

(1) $(i, j)(i+1, j)$ and $(2q-1, j)(0, j)$, where $i = 0, 1, 2, \ldots, 2q-2$ and $j = 0, 1, 2, \ldots, s-1$;

(2) $(i, j)(i, j+1)$, where $i + j \equiv 0 (\text{mod } 2)$, $i = 0, 1, 2, \ldots, 2q-1$ and $j = 0, 1, 2, \ldots, s-1$; and

(3) $(2i+1, 0)(2i+1+t, s-1)$, where $i = 0, 1, \ldots, q-1$ and the first coordinate is computed modulo $2q$.

Clearly, $C[2q, s, t]$ is a 3-regular graph. Alspach and C. Q. Zhang [12] introduced the brick product of C_{2q} with P_s which is a $C[2q, s, t]$ without edges of type (3). As an example, the graph $C[6, 5, 1]$ is given in Figure 4.1a.

To conclude this section, we make the following observation which sketches the structure of Cayley graphs on dihedral groups.

Obeservation 4.2 *Any connected Cayley graph $X = X(D_n, S)$ can be decomposed into two subgraphs on $< \rho >$ and $< \rho > \tau$ together with a class of perfect matchings joining them. Furthermore, the two subgraphs on $< \rho >$ and $< \rho > \tau$ are isomorphic to the same circulant on Z_n.*

Proof. Let $X[<\rho>]$ and $X[<\rho>\tau]$ be the induced subgraphs on $<\rho>$ and $<\rho>\tau$, respectively. Then $X[<\rho>] = X(<\rho>, S') \cong C(n, S^*)$, where $S^* = \{i|\rho^i \in S'\}$, which is a circulant and $\phi : X[<\rho>] \to X[<\rho>\tau]$ defined by $\phi(\rho^i) = \rho^i\tau$ is an isomorphism (note that $X[<\rho>]$ may be edgeless).

The class of perfect matchings is $\{E_s|s \in S''\}$. ∎

We set $E_1 = E(X[<\rho>]), E_2 = E(X[<\rho>\tau])$ and $E_3 = E(X(D_n, S''))$. Then $E_i \bigcap E_j = \emptyset$ if $i \neq j$, and $E(X) = E_1 \bigcup E_2 \bigcup E_3$.

4.2 Basic Lemmas

We need the following lemmas in the proof of the main theorem.

Lemma 4.3 *If n is odd, then $C(n, S) \times K_2 \cong C(2n, 2S \bigcup \{n\})$.*

Proof. $C(n, S) \times K_2$ has two subgraphs X_1 and X_2 each of which is isomorphic to $C(n, S)$, and there is an isomorphism $\theta : X_1 \to X_2$, such that the set of edges $\{v(\theta(v))|v \in V(X_1)\}$ is a perfect matching between X_1 and X_2.

We can label the vertices of X_1 by $0, 2, 4, \ldots, 2(n-1)$, and then $2S$ is the symbol set of X_1. Similarly, labelling the vertices of X_2 by $\{n, n+2, n+4, \ldots, 3n-2\}(\text{mod } 2n) = \{1, 3, 5, \ldots, 2n-1\}$ (as n is odd) will turn X_2 into a circulant with $2S$ as the symbol set.

Note that the mapping $\phi : X_1 \to X_2$ defined by $\phi(v) = (n+v)(\text{mod } 2n)$ is an isomorphism. So if we add n to the symbol set $2S$, then we obtain the desired perfect matching between X_1 and X_2. Therefore, $C(n, S) \times K_2 \cong C(2n, 2S \bigcup \{n\})$. ∎

Lemma 4.4 *Let $X = X(D_n, \{\rho^i\tau, \rho^j\tau, \rho^{\pm k}\})$ be connected.*

(1) If $X(D_n, \{\rho^i\tau, \rho^j\tau\})$ is connected, then X is a 3-regular or 4-regular circulant.

(2) If $X(D_n, \{\rho^i\tau, \rho^j\tau\})$ is disconnected, then X has $C_{2m} \times P_h$ as a spanning subgraph for some $m \geq 2$ and $h \geq 2$.

Proof. (1) Let $X_1 = X(D_n, \{\rho^i\tau, \rho^j\tau\})$. Since $\rho^i\tau$ and $\rho^j\tau$ are of order 2, X_1 is a 2 - regular graph. If it is connected, then it is a $2n$ - cycle

$$1(\rho^i\tau)(\rho^{i-j})(\rho^{2i-j}\tau)(\rho^{2(i-j)})\ldots(\rho^{(n-1)(i-j)})(\rho^{ni-(n-1)j}\tau)1.$$

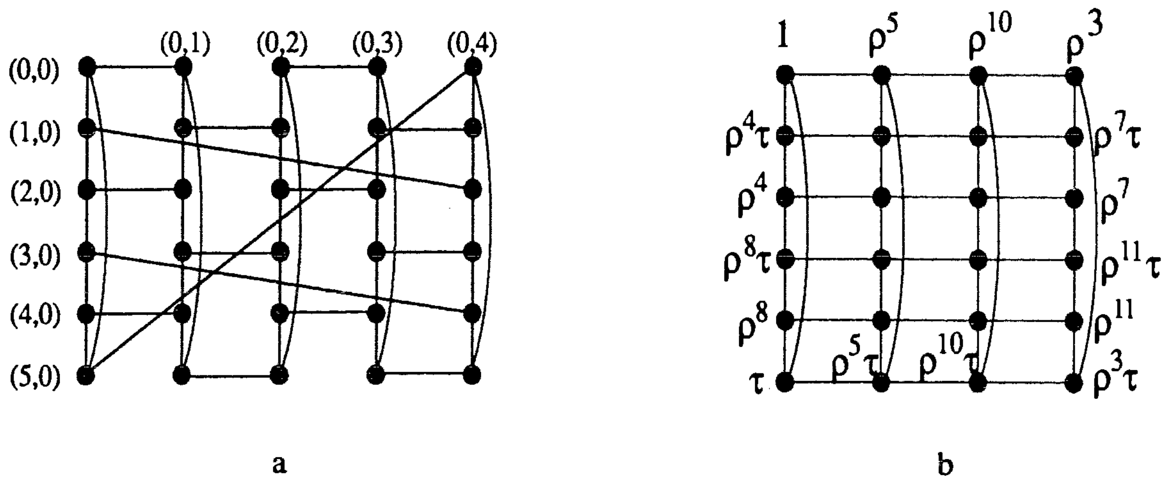


Figure 4.1:

We use $\{0, 1, 2, \ldots, 2n-1\}$ to relabel this cycle so that $\rho^{ti-(t-1)j}\tau \leftrightarrow 2t-1$ and $\rho^{t(i-j)} \leftrightarrow 2t$. Then the cycle becomes 0 1 2 ...(2n-1) 0 after the relabelling.

Let $\rho^k = \rho^{h(i-j)}$. Then edges of X with symbol ρ^k and ρ^{-k} become edges with symbol $2h$ and $-2h$, respectively, after relabelling. Therefore, $X = X(D_n, \{\rho^i\tau, \rho^j\tau, \rho^{\pm k}\})$ $\cong C(2n, \{1, 2n-1, \pm 2h\})$. If $h = n/2$, then X is 3-regular. Otherwise, it is 4-regular.

(2) If $X_1 = X(D_n, \{\rho^i\tau, \rho^j\tau\})$ is disconnected, then it is a union of h disjoint even cycles C_{2m}, for some $m > 1, h > 1$. We can arrange the vertices of each cycle in a column such that the first column begins with 1, the second column begins with ρ^k (note that ρ^k does not belong to the first column, for otherwise X will be disconnected), the third column begins with ρ^{2k}, and so on. We thus obtain a $2m \times h$ array in which each row forms an h-path whose edges have the same symbol ρ^k or ρ^{-k} (an example with $X = X(D_{12}, \{\tau, \rho^4\tau, \rho^{\pm 5}\})$ is illustrated in Figure 4.1b). Therefore, X has a spanning subgraph $C_{2m} \times P_h$. ∎

We quote the following result which is implied in the proof of Theorem 3.1 of [12].

Lemma 4.5 *Let* $X = X(D_n, \{\rho^i\tau, \rho^j\tau, \rho^k\tau\})$ *be connected. If* $X(D_n, \{\rho^i\tau, \rho^j\tau\})$ *is disconnected, then* X *is isomorphic to* $C[2q, s, t]$ *for some* $q \geq 2, s \geq 2$ *and* $t \geq 1$.

We also need the following result from [51].

Lemma 4.6 $C_{2m} \times P_h$ *($m \geq 2, h \geq 2$) is 2-extendable.*

4.3 The Main Theorem

In this section, we shall prove the following result which is a characterization of 2-extendable Cayley graphs on dihedral groups.

Theorem 4.7 *Let $X = X(D_n, S)$ be connected. Then X is 2-extendable if and only if it is not isomorphic to any of the following graphs:*

(I) $C(2n, \{1, 2n-1\}), n \geq 3$;

(II) $C(2n, \{1, 2, 2n-1, 2n-2\}), n \geq 3$;

(III) $C(4n, \{1, 4n-1, 2n\}), n \geq 2$;

(IV) $C(4n+2, \{2, 4n, 2n+1\}), n \geq 1$; *and*

(V) $C(4n+2, \{1, 4n+1, 2n, 2n+2\}), n \geq 1$.

Proof. It is not hard to see that each class of graphs in (I) - (V) can be realized by Cayley graphs on dihedral groups. If X is isomorphic to any graph in these classes, then X is not 2-extendable by Theorem 4.1.

Let $X = X(D_n, S)$. We shall show that if X is not isomorphic to any of the graphs in the five classes, then X is 2-extendable. If $n = 2$, then $X = X(D_2, S)$ is either C_4 or K_4. In any case, X is 2-extendable. So we may assume that $n \geq 3$. Recall that $E_1 = E(X[<\rho>]), E_2 = E(X[<\rho>\tau])$, $E_3 = E(X(D_n, S''))$ and $\tau \in S$. Let $M = \{e_1, e_2\}$, where e_1 and e_2 are any two independent edges of X.

Case 1. $M = \{e_1, e_2\} \subseteq E_1$ or $M \subseteq E_2$.

Since $X[<\rho>] \cong X[<\rho>\tau]$, we may assume that $M \subseteq E_1$. Suppose $e_1 = (\rho^i)(\rho^j)$ and $e_2 = (\rho^k)(\rho^h)$. Then i, j, k and h are all distinct. Let

$$M^* = (E_\tau \bigcup \{e_1, e_2, (\rho^i\tau)(\rho^j\tau), (\rho^k\tau)(\rho^h\tau)\} - \{(\rho^i)(\rho^i\tau), (\rho^j)(\rho^j\tau), (\rho^k)(\rho^k\tau), (\rho^h)(\rho^h\tau)\}.$$

Then M^* is a perfect matching containing M.

Case 2. $M \bigcap E_3 \neq \emptyset, M \bigcap (E_1 \bigcup E_2) \neq \emptyset$.

Without loss of generality, assume $e_1 = (\rho^i)(\rho^j) \in E_1$ and $e_2 = (\rho^k)(\rho^{k+h}\tau) \in E_3$, where k, i and j are all distinct and $\rho^h\tau \in S''$. Then

$$(E_{\rho^h\tau} \bigcup \{e_1, (\rho^{i+h}\tau)(\rho^{j+h}\tau)\}) - \{(\rho^i)(\rho^{i+h}\tau), (\rho^j)(\rho^{j+h}\tau)\}$$

is a perfect matching of X which contains M.

Case 3. $e_1 \in E_1, e_2 \in E_2$.

Let $G_1, G_2, \ldots, G_r$ be the components of $X[<\rho>]$. Then $G_i \cong G_j$ for $1 \leq i, j \leq r$. Let G'_i be the subgraph of $X[<\rho>\tau]$ induced by $\{x\tau | x \in V(G_i)\}$. Then $G'_i \cong G_i$ $(1 \leq i \leq r)$.

In this case, we have three subcases to consider.

Case 3.1. e_1 and e_2 lie in G_i and G'_j, respectively, and $i \neq j$.

Let $e_1 = (\rho^i)(\rho^j)$ and $e_2 = (\rho^k\tau)(\rho^h\tau)$. Then

$$E_\tau \bigcup \{e_1, e_2, (\rho^i\tau)(\rho^j\tau), (\rho^k)(\rho^h)\} - \{(\rho^i)(\rho^i\tau), (\rho^j)(\rho^j\tau), (\rho^k)(\rho^k\tau), (\rho^h)(\rho^h\tau)\}$$

is a perfect matching containing M.

Case 3.2. e_1 and e_2 lie in G_i and G'_i, respectively, and $|V(G_i)| = |V(G'_i)|$ is even.

It is easy to see that every connected circulant of even order is 1-factorizable and each component of $X[<\rho>]$ is a circulant. Hence e_1 can be extended to a perfect matching M_1 in $X[<\rho>]$ and e_2 can be extended to a perfect matching M_2 in $X[<\rho>\tau]$. Then $M_1 \bigcup M_2$ is a perfect matching of X as required.

Case 3.3. e_1 and e_2 lie in G_i and G'_i, respectively, and $|V(G_i)| = |V(G'_i)|$ is odd.

Let $e_1 = (\rho^i)(\rho^j)$ and $e_2 = (\rho^k\tau)(\rho^h\tau)$.

(a) If $X[<\rho>]$ is disconnected, then so is $X(D_n, S' \bigcup \{\tau\})$. Since X is connected, there exists $\rho^m\tau \in S''$ so that $\rho^i \cdot (\rho^m\tau) = \rho^{i+m}\tau \notin V(G'_i)$. Therefore, $\{x \cdot (\rho^m\tau) | x \in V(G_i)\} \bigcap V(G'_i) = \emptyset$. In this case,

$$M^* = E_{\rho^m\tau} \bigcup \{e_1, e_2, (\rho^{i+m}\tau)(\rho^{j+m}\tau), (\rho^{k-m})(\rho^{h-m})\} - \{(\rho^i)(\rho^{i+m}\tau), (\rho^j)(\rho^{j+m}\tau), (\rho^{k-m})(\rho^k\tau), (\rho^{h-m})(\rho^h\tau)\}$$

is a perfect matching containing e_1 and e_2.

(b) If $X[<\rho>]$ is connected, then $e_1 \in E(X[<\rho>]), e_2 \in E(X[<\rho>\tau])$ and n is odd. Let $n = 2k + 1$.

If $|S'| \geq 4$, then $X' = X(D_n, S' \bigcup \{\tau\}) \cong C(n, S^*) \times K_2 \cong C(2n, 2S^* \bigcup \{n\})$, where $S^* = \{i | \rho^i \in S'\}$ (by Lemma 4.3). Hence X' is a circulant of degree at least 5 and is

2-extendable by Theorem 4.1. But X' is a spanning subgraph of X which contains e_1 and e_2. Hence $\{e_1, e_2\}$ can be extended to a perfect matching of X.

Suppose now $S' = \{\rho^{\pm i}\}$. Then e_1 and e_2 have the same symbol. If $S'' = \{\tau\}$, then X is 3-regular and $X \cong C(4k+2, \{2k+1, 2, 4k\})$, which is a graph belonging to class (IV). Hence we must have $|S''| \geq 2$.

When $|S''| = 2$ and $X(D_n, S'')$ is disconnected, $X(D_n, S' \bigcup S'')$ has $C_{2m} \times P_h$ as a spanning subgraph by Lemma 2.2, where $h \geq 2$. But $2mh = 2n = 2(2k+1)$, and we must have that h is odd. Hence $h \geq 3$. Therefore, we can rearrange the columns in the proof of Lemma 4.4, such that $e_1, e_2 \in E(C_{2m} \times P_h)$. But $C_{2m} \times P_h$ is 2-extendable (by Lemma 4.6). Hence e_1 and e_2 can be extended to a perfect matching of X.

When $|S''| = 2$ and $X(D_n, S'')$ is connected, $X(D_n, S' \bigcup S'')$ is a 4-regular circulant by Lemma 4.4 again. If $X = X(D_n, S) = X(D_{2k+1}, S) \cong C(4k+2, \{1, 4k+1, 2k, 2k+2\})$, then X is a graph of class (V), which is not 2-extendable. (For instance, $X(D_5, \{\tau, \rho\tau, \rho^2, \rho^3\}) \cong C(10, \{1, 4, 6, 9\})$ is such a graph.) In all other cases, $X(D_n, S)$ is 2-extendable by Theorem 4.1.

Now assume $|S''| > 2$. We shall show that e_1 and e_2 can be extended to a perfect matching of X. Note again that e_1 and e_2 have the same symbol. Without loss of generality, we assume that $e_1 = 1(\rho^i), e_2 = (\rho^i\tau)(\rho^{2i}\tau)$. If $\rho^i\tau \in S''$, then $(E_{\rho^i\tau} \bigcup \{e_1, e_2\}) - \{1(\rho^i\tau), (\rho^i)(\rho^{2i}\tau)\}$ is a perfect matching containing e_1 and e_2. If $\rho^i\tau \notin S''$, then there is a $\rho^j\tau \in S''$ such that $j \neq 0, j \neq 2i$ as $|S''| \geq 3$. Let

$$M^* = E_{\rho^j\tau} \bigcup \{e_1, e_2, (\rho^j\tau)(\rho^{i+j}\tau), (\rho^{i-j}\tau)(\rho^{2i-j}\tau)\} - \{1(\rho^j)\tau), (\rho^i)(\rho^{i+j}\tau), (\rho^{i-j})(\rho^i\tau), (\rho^{2i-j})(\rho^{2i}\tau)\}.$$

Then M^* is a perfect matching of X which extends e_1 and e_2.

Case 4. $\{e_1, e_2\} \subseteq E_3$.

If e_1 and e_2 have the same symbol $\rho^i\tau$, then $E_{\rho^i\tau}$ is a perfect matching of X which contains e_1 and e_2. So we assume that e_1 has symbol $\rho^i\tau$ and e_2 has symbol $\rho^j\tau, i > j$.

Case 4.1. If $X_1 = X(D_n, \{\rho^i\tau, \rho^j\tau\})$ is disconnected, then X_1 is a disjoint union of some even cycles. If e_1, e_2 belong to different cycles, then we can easily extend e_1 and e_2 to a perfect matching of X. So suppose that e_1 and e_2 belong to the same cycle and no perfect matching of this cycle contains both e_1 and e_2. Let $G_1, G_2, \ldots, G_h$ be the disjoint cycles of X_1, where $G_i \cong C_{2m} (1 \leq i \leq h)$ and $e_1, e_2 \in E(G_1)$. Since X is vertex-transitive, we may assume $e_1 = 1(\rho^i\tau)$. Thus G_1 is a $2m$-cycle:

$$1(\rho^i\tau)(\rho^{i-j})(\rho^{2i-j}\tau)(\rho^{2(i-j)}) \cdots (\rho^{(m-1)(i-j)})(\rho^{mi-(m-1)j}\tau)1$$

where $m(i-j) \equiv 0 (\bmod n)$.

(a) Suppose $S' - \{\rho^{(i-j)}, \rho^{2(i-j)}, \ldots, \rho^{(m-1)(i-j)}\}$ is not empty, containing some ρ^k. Since $\rho^k \notin V(G_1)$, we may assume that $\rho^k \in V(G_2)$. Then the subgraph X' of $X(D_n, \{\rho^i\tau, \rho^j\tau, \rho^{\pm k}\})$ induced by $V(G_1) \bigcup V(G_2)$ is isomorphic to $C_{2m} \times K_2$. By Lemma 4.6, $C_{2m} \times K_2$ is 2-extendable. Thus there is a perfect matching M' of X' containing e_1 and e_2. For other $G_i, i \geq 3$, simply choose a perfect matching M_i of G_i. Then $M' \bigcup (\bigcup_{j=3}^{h} M_j)$ is a perfect matching of X containing e_1 and e_2.

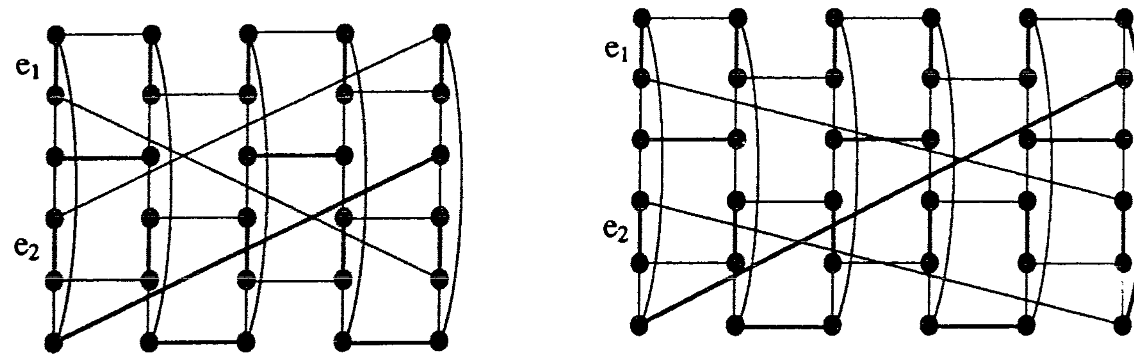
(b) If $S' - \{\rho^{(i-j)}, \rho^{2(i-j)}, \ldots, \rho^{(m-1)(i-j)}\} = \emptyset$, then $X(D_n, S' \bigcup \{\rho^i\tau, \rho^j\tau\})$ is disconnected. Since X is connected, there is a $\rho^r\tau \in S''$ such that the edges with symbol $\rho^r\tau$ join G_1 and another G_k, say. Let $X'' = X(D_n, \{\rho^i\tau, \rho^j\tau, \rho^r\tau\})$. Then each component of X'' is also a Cayley graph on a dihedral group D_b for some b. So, without loss of generality, we assume that X'' is connected. By Lemma 4.5, X'' is isomorphic to $C[2q, s, t]$ for some $q \geq 2, s \geq 2$ and $t \geq 1$. For convenience, we assume that $X'' = C[2q, s, t]$, and we can assume that $e_1 = (0,0)(1,0)$ and $e_2 = (2p+1, 0)(2p+2, 0)$ for some p.

If s is odd, let
$M^* = \{(0,j)(1,j) | \, j = 0,1,2,\ldots, s-2\} \bigcup \{(2,i)(2,i+1) | \, i = 0,2,4,\ldots, s-3\} \bigcup$
$\{(2q-1,0)(2q-1+t, s-1), (2q-1,1)(2q-1,2), \ldots, (2q-1, s-2)(2q-1, s-1)\}$
$\bigcup \{(i,j)(i+1,j) | \, i = 3,5,\ldots, 2q-3; j = 0,1,2,\ldots, s-2\} \bigcup B$,
where B is a perfect matching of $(C_{2q} \times \{s-1\}) - \{(2q-1, s-1), (2q-1+t, s-1)\}$ which is a union of paths of odd length (since $2q-1+t-(2q-1) = t$ is odd). Then M^* is a perfect matching of X which contains e_1 and e_2.

If s is even, let
$M^* = \{(0,j)(1,j) | \, j = 0,1,2,\ldots, s-2\} \bigcup \{(2,i)(2,i+1) | \, i = 0,2,4,\ldots, s-2\} \bigcup$
$\{(2q-1,0)(2q-1+t, s-1), (2q-1,1)(2q-1,2), \ldots, (2q-1, s-3)(2q-1, s-2)\}$
$\bigcup \{(i,j)(i+1,j) | \, i = 3,5,\ldots, 2q-3; j = 0,1,2,\ldots, s-2\} \bigcup B$,
where B is a perfect matching of $(C_{2q} \times \{s-1\}) - \{(2, s-1), (2q-1+t, s-1)\}$ which is a union of paths of odd length (since $2q-1+t-2 = 2q-3+t$ is odd). Then M^* is a perfect matching of X which contains e_1 and e_2. (We illustrate the above patterns wit2 $C[6,5,3]$ and $C[6,6,2]$ in Figure 4.2a and Figure 4.2b, respectively.)

Case 4.2. $X_1 = X(D_n, \{\rho^i\tau, \rho^j\tau\})$ is connected. Then $X_1 \cong C_{2n}$. We assume that no perfect matching of C_{2n} contains both e_1 and e_2.

(a) If $S = \{\rho^i\tau, \rho^j\tau\}$, then $X \cong C_{2n} = C(2n, \{1, 2n-1\}), (n > 3)$, which is in class (I).



a b

Figure 4.2:

(b) If $S = \{\rho^i\tau, \rho^j\tau, \rho^{n/2}\}$, then n is even, say $n = 2m$. By Lemma 4.4, we have $X(D_n, S)$ is a 3-regular circulant and $X(D_n, S) \cong C(2n, \{1, 2n-1, n\}) = C(4m, \{1, 4m-1, 2m\})$. This is a graph of class (III).

(c) If $S = \{\rho^i\tau, \rho^j\tau, \rho^{\pm k}\}$, $(k \neq n/2)$, then $X(D_n, S)$ is a 4-regular circulant by Lemma 4.4. By Theorem 4.1, $X(D_n, S)$ is 2-extendable if it is either not isomorphic to $C(4k+2, \{1, 4k+1, 2k, 2k+2\})$, (which belongs to class (V)), or to $C(2n, \{1, 2, 2n-1, 2n-2\})$, (which is a graph in class (II)).

(d) If $|S'| \geq 3$, then $X(D_n, S' \bigcup \{\rho^i\tau, \rho^j\tau\})$ is a circulant of degree at least 5, by the proof of Lemma 4.4. By Theorem 4.1, $X(D_n, S' \bigcup \{\rho^i\tau, \rho^j\tau\})$ is 2-extendable. Hence $\{e_1, e_2\}$ can be extended to a perfect matching of X.

(e) If $|S'| = 0$, then $|S''| \geq 3$. We have $\rho^k\tau \in S''$ for some k distinct from i and j. We shall show that, for some $\rho^k\tau \in S'', X' = X(D_n, S^*)$ has a perfect matching containing $\{e_1, e_2\}$, where $S^* = \{\rho^i\tau, \rho^j\tau, \rho^k\tau\}$. This is also a perfect matching of X.

For convenience, we can assume that $\rho^i\tau = \tau$. Then

$$X(D_n, \{\tau, \rho^j\tau\} = 1(\tau)(\rho^{-j})(\rho^{-j}\tau)(\rho^{-2j)})(\rho^{-2j}\tau)\dots(\rho^j\tau)1.$$

Also assume that $e_1 = 1(\tau)$, $e_2 = (\rho^{-qj}\tau)(\rho^{-(q+1)j})$. Let $\rho^k = \rho^{-mj}$. We can assume that $m > q+1$, (or else consider ρ^{-k}). Now let

$$\begin{aligned} M^* = \{&e_1, (\rho^{-j})(\rho^{-(m+1)j}\tau), (\rho^{-2j}\tau)(\rho^{-3j}), \dots, (\rho^{-(q-1)j}\tau)(\rho^{-qj}), \\ &e_2, (\rho^{-(q+1)j}\tau)(\rho^{-(q+2)j}), \dots, (\rho^{-mj}\tau)(\rho^{-(m+1)j}), (\rho^{-(m+2)j})(\rho^{-(m+2)j}\tau), \dots, (\rho^j)(\rho^j\tau)\}. \end{aligned}$$

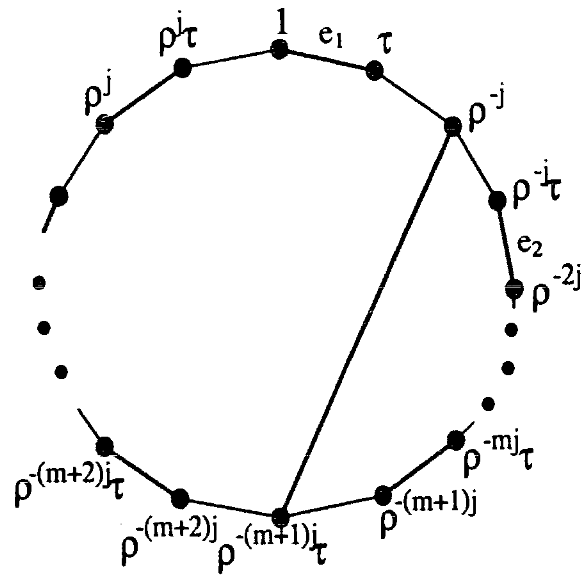


Figure 4.3:

Then M^* is a perfect matching of X which contains e_1 and e_2 (see Figure 4.3). ∎

Chapter 5

Hamilton Connectivity of $GP(n,k)$

5.1 Introduction

In this chapter, all subscripts are taken modulo n.

Let $G = GP(n,k)$ be the generalized Petersen graph with vertex set $V(G) = \{u_i, v_i : i = 0, 1, \ldots, n-1\}$ and edge set $E(G) = \{u_iu_{i+1}, v_iv_{i+k}, u_iv_i : i = 0, 1, \ldots, n-1\}$. We call $E_1 = \{u_iu_{i+1} : i = 0, 1, \ldots, n-1\}, E_2 = \{v_iv_{i+k}, u_iv_i : i = 0, 1, \ldots, n-1\}$ and $E_3 = \{u_iv_i : i = 0, 1, \ldots, n-1\}$ *Type I, Type II* and *Type III* edges of $GP(n,k)$, respectively.

The classification of hamiltonian generalized Pertersen graphs was carried out by many people. Their results, stated below, will be used later.

Theorem 5.1 *The generalized Petersen graph $GP(n,k)$ is hamiltonian if and only if neither*

(i) $GP(n,k) \cong GP(n,2) \cong GP(n,n-2) \cong GP(n,\frac{n-1}{2}) \cong GP(n,\frac{n+1}{2}), n \equiv 5 (mod\ 6)$, *nor*

(ii) $GP(n,k) \cong GP(n,n/2), n \equiv 0 (mod\ 4)$ *and* $n \geq 8$.

The exceptional graphs (i) may not have a Hamilton cycle but, they come so close. Alspach [6] has pointed out the following.

Theorem 5.2 *Any two non-adjacent vertices of* $GP(6m+5,2), m \geq 0$, *are joined by a Hamilton path.*

In this chapter, we will study Conjecture 1 of Chapter 1. The first thing we need to do is to distinguish bipartite generalized Petersen graphs. We have the following theorem.

Theorem 5.3 *$G = GP(n,k)$ is bipartite if and only if n is even and k is odd.*

Proof Let $G = GP(n,k)$ be a bipartite graph. We have that $u_0u_1 \ldots u_{n-1}u_0$ and $u_0v_0v_ku_ku_{k-1}\ldots u_1u_0$ are cycles of length n and $k+3$, respectively, implying that n is even and k is odd.

Conversely, suppose that n is even and k is odd. Let

$$X = \{u_0, u_2, \ldots, u_{2i}, \ldots, u_{n-2}, v_1, v_3, \ldots, v_{2j+1}, \ldots, v_{n-1}\},$$

and

$$Y = \{u_1, u_3, \ldots, u_{2j+1}, \ldots, u_{n-1}, v_0, v_2, \ldots, v_{2i}, \ldots, v_{n-2}\}$$

be a 2-partition of $V(G)$. If $E(G[X]) \neq \emptyset$, then there exists an edge $v_iv_j \in E(G)$. This implies that $i-j$ (or $j-i$) $= k$ which is odd, and hence i and j have different parity. This is a contradiction.

Similarly, $E(G[Y]) = \emptyset$. Therefore, $G = GP(n,k)$ is a bipartite graph with bipartition X and Y. ∎

The following lemmas simplify many cases in later proofs.

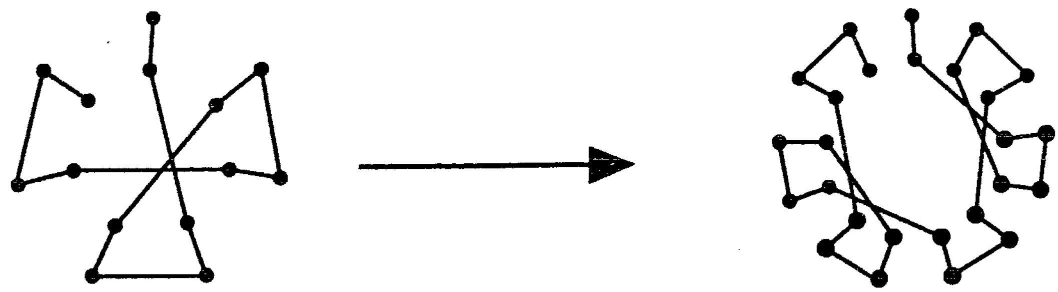
Lemma 5.4 *If $gcd(n,k) = 1$, then there exist Hamilton paths from u_0 to v_{k-1} and v_{k+1} in $GP(n,k)$, respectively.*

Proof. The path $u_0u_1u_2\ldots u_{n-1}v_{n-1}v_{n-1-k}v_{n-1-2k}\cdots v_{n-1-(n-1)k}$ is a Hamilton path in $GP(n,k)$, but $n-1-(n-1)k = n-nk+k-1 \equiv k-1 (\bmod n)$, so $v_{n-1-(n-1)k} = v_{k-1}$.

The path $u_0u_{n-1}u_{n-2}\ldots u_1v_1v_{1-k}v_{1-2k}\cdots v_{1-(n-1)k}$ is a Hamilton path in $GP(n,k)$ and $v_{1-(n-1)k} = v_{1-nk+k} = v_{k+1}$. Therefore, there exist Hamilton paths from u_0 to v_{k-1} and v_{k+1} in $GP(n,k)$, respectively. ∎

Lemma 5.5 *If $GP(n,k)$ is a hamiltonian generalized Petersen graph, then it is edge-hamiltonian.*

Proof. It is easy to see that if C is a Hamilton cycle of $GP(n,k)$, it must contain edges of each type. But $Aut(GP(n,k))$ acts transitively on each edge type, and therefore, $GP(n,k)$ is edge-hamiltonian. ∎



(a)

Figure 5.1: A Type A insertion

5.2 $GP(n,3)$

One of our main results is next.

Theorem 5.6 *1. $GP(n,3)$ is Hamilton-connected if and only if n is odd and $n \neq 5$.*

2. $GP(n,3)$ is Hamilton-laceable if and only if n is even and $n \neq 6$.

Proof. Proof of 1. Let $GP(n,3)$ be Hamilton-connected. Then $GP(n,3)$ is not bipartite, and hence n is odd by Theorem 5.3. Also, we know that $n \neq 5$ since $GP(5,3)$ is the Petersen graph, which is not hamiltonian.

Conversely, let n be odd, and $n \neq 5$. To prove $GP(n,3)$ is Hamilton-connected, it suffices to prove that there are Hamilton paths from u_0 to v_m, from u_0 to u_m, and from v_0 to v_m, for $m = 2,4,\ldots,n-1$ since both $(u_0u_1\ldots u_{n-1})(v_0v_1\ldots v_{n-1})$ and $(u_0)(v_0)(u_1u_{n-1})(u_2u_{n-2})\ldots(v_{\frac{n-1}{2}}v_{\frac{n+1}{2}})$ are automorphisms of $GP(n,3)$.

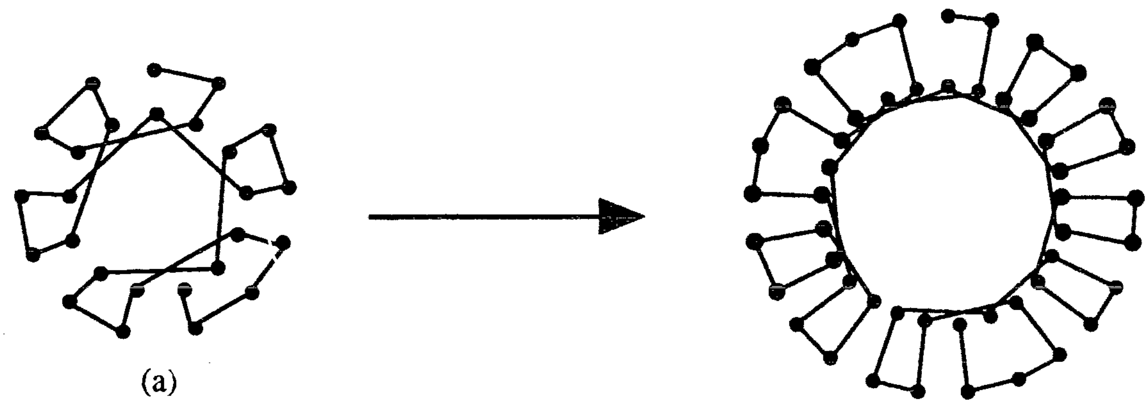
Case 1. There is a Hamilton path from u_0 to v_m for $m = 2,4,\ldots,n-1$.

Note that if $n \not\equiv 3(\text{mod } 6)$, then $gcd(n,3) = 1$. We know that there are Hamilton paths from u_0 to v_2 and v_4, respectively by Lemma 5.4.

Subcase 1.1. $n \equiv 1(\text{mod } 6)$.

(i) $m \equiv 0(\text{mod } 6)$.

In this case, $n - m \geq 1$ and $m \geq 6$. For $m = 6, n = 7$, a Hamilton path from u_0 to v_6 in $GP(7,3)$ is given in Figure 5.1a. For $m = n-1$, successive Type A insertions



(a)

Figure 5.2: A Type B and a Type C insertions

in Figure 5.1a starting at $u_i = u_2$ give a Hamilton path from u_0 to v_{n-1} in $GP(n,3)$. In fact, we need $\frac{n-7}{6}$ Type A insertions. Figure 5.1 shows one Type A insertion in $GP(7,3)$.

Let $n-m \geq 7$. Then $n \geq 13$. A Hamilton path from u_0 to v_6 in $GP(13,3)$ is given in Figure 5.2a.

Let $m = 6h$ and $m < n-1$. Then $n \geq 13$. Successive $h-1$ Type B insertions starting at $u_i = u_1$ in Figure 5.2a give a Hamilton path P from u_0 to v_m in $GP(6(h-1)+13,3)$. Followed by successive $\frac{n-6(h-1)-13}{6}$ Type C insertions based on P starting at $i = m+4$ in $GP(6(h-1)+13,3)$ give a Hamilton path from u_0 to v_m in $GP(n,3)$, for all $n \equiv 1 (\text{mod } 6)$. Figure 5.2 shows one Type B insertion and one Type C insertion in $GP(13,3)$.

Remark. The proof in (i) indicates the general strategy we follow.

1. Find a small graph for which it is easy to perform the insertions.

2. Find two types of insertions and the inserting points such that the insertion preserves the local property of the inserting points, that is, we can perform the next insertion after each insertion.

In order to simplify the proof, we indicate only the above two steps. We do not give the figures as in (i) since it is easy to draw a figure following the description.

(ii) $m \equiv 2 (\text{mod } 6)$.

In this case, $n-m \geq 5$ and $m \geq 2$. If $m = 2$, then we are done by Lemma 5.4. So assume that $m \geq 8$. Then $n \geq 13$. A Hamilton path from u_0 to v_8 in $GP(13,3)$ is given in Figure 5.3a.

One type of insertion is a Type D starting at $i = 4$. The other is a Type E starting at $i = m+3$.

(iii) $m \equiv 4 (\mathrm{mod}\ 6)$.

In this case, $n-m \geq 3$ and $m \geq 4$. If $m = 4$, we are done by Lemma 5.4. So assume that $m \geq 10$ and hence $n \geq 13$. A Hamilton path from u_0 to v_{10} in $GP(13,3)$ is given in Figure 5.3b.

One type of insertion is a Type F starting at $i = 3$. The other is a Type G starting at $i = m+2$.

Subcase 1.2. $n \equiv 3 (\mathrm{mod}\ 6)$.

(i) $m \equiv 0 (\mathrm{mod}\ 6)$.

We have that $m \geq 6$ and $n-m \geq 3$. A Hamilton path from u_0 to v_6 in $GP(9,3)$ is given in Figure 5.3c. A sequence of Type B insertions starting at $i = 3$ gives a Hamilton path from u_0 to v_{n-3} in $GP(n,3)$ for all $n \equiv 3 (\mathrm{mod}\ 6)$.

We may assume $n-m \geq 9$. A Hamilton path from u_0 to v_6 in $GP(15,3)$ is given in Figure 5.3d.

One type of insertion is a Type B starting at $i = 3$. The other is a Type H starting at $i = m+2$.

(ii) $m \equiv 2 (\mathrm{mod}\ 6)$.

In this case, $n-m \geq 1$ and $m \geq 2$. We have $n \geq 9$. A Hamilton path from u_0 to v_8 in $GP(9,3)$ is shown in Figure 5.3e. A sequence of Type B insertions starting at $i = 1$ gives a Hamilton path from u_0 to v_{n-1} in $GP(n,3)$ for all $n \equiv 3 (\mathrm{mod}\ 6)$.

We may assume that $n-m \geq 7$. A Hamilton path from u_0 to v_2 in $GP(9,3)$ is given in Figure 5.3f. A sequence of Type I insertions starting at $i = 5$ gives a Hamilton path from u_0 to v_2 in $GP(n,3)$ for all $n \equiv 3 (\mathrm{mod}\ 6)$.

Let $m \geq 8$. A Hamilton path from u_0 to v_8 in $GP(15,3)$ is given in Figure 5.3g.

One type of insertion is a Type A starting at $i = 2$. The other is a Type I starting at $i = m+3$.

(iii) $m \equiv 4 (\text{mod } 6)$.

In this case, $m \geq 4$ and $n - m \geq 5$. A Hamilton path from u_0 to v_4 in $GP(9,3)$ is given in Figure 5.3h. A sequence of Type E insertions starting at $i = 6$ gives a Hamilton path from u_0 to v_4 in $GP(n,3)$ for all $n \equiv 3 (\text{mod } 6)$.

We may assume $m \geq 10$. A Hamilton path from u_0 to v_{10} in $GP(15,3)$ is given in Figure 5.3i.

One type of insertion is a Type A starting at $i = 6$. The other is a Type E starting at $i = m + 2$.

Subcase 1.3. $n \equiv 5 (\text{mod } 6)$.

(i) $m \equiv 0 (\text{mod } 6)$.

In this case, $m \geq 6$ and $n - m \geq 5$. A Hamilton path from u_0 to v_6 in $GP(11,3)$ is given in Figure 5.3j.

One type of insertion is a Type B starting from $i = 1$. The other is a Type I starting from $i = m + 2$.

(ii) $m \equiv 2 (\text{mod } 6)$.

In this case, we have that $m \geq 2$ and $n - m \geq 3$. By Lemma 5.4, we can assume that $m \geq 8$. Then $n \geq 13$. A Hamilton path from u_0 to v_8 in $GP(11,3)$ is given in Figure 5.3k.

One type of insertion is a Type B starting at $i = 1$. The other is a Type G starting at $i = m + 2$.

(iii) $m \equiv 4 (\text{mod } 6)$.

We have that $m \geq 4$ and $n - m \geq 1$. A Hamilton path from u_0 to v_{10} in $GP(11,3)$ is given in Figure 5.3l. A sequence of Type B insertions starting at $i = 1$ gives a Hamilton path from u_0 to v_{n-1} in $GP(n,3)$ for all $n \equiv 5 (\text{mod } 6)$.

Let $m < n - 1$. Then $m \leq n - 7$. A Hamilton path from u_0 to v_4 in $GP(11,3)$ is given in Figure 5.4a.

One type of insertion is a Type J insertion followed by a sequence of Type B insertions starting at $i = 1$. The other is a Type G starting at $i = m + 6$.

Case 2. There is a Hamilton path from u_0 to u_m for $m = 2, 4, \ldots, n - 1$.

Subcase 2.1. $n \equiv 1 (\text{mod } 6)$.

(i) $m \equiv 0 (\text{mod } 6)$.

We have that $m \geq 6$ and $n - m \geq 1$. By Lemma 5.5, we can assume $m < n - 1$. Then $n - m \geq 7$, and $n \geq 13$. A Hamilton path from u_0 to u_6 in $GP(13,3)$ is given in Figure 5.4b.

One type of insertion is a Type E starting at $i = 2$. The other is a Type D starting at $i = m + 2$

(ii) $m \equiv 2 (\text{mod } 6)$.

We have that $m \geq 2$ and $n - m \geq 5$. A Hamilton path from u_0 to u_2 in $GP(7,3)$ is given in Figure 5.4c.

One type of insertion is a Type G starting at $i = 1$. The other is a Type J insertion followed by a sequence of Type B insertions starting at $i = m + 1$.

(iii) $m \equiv 4 (\text{mod } 6)$.

In this case, $m \geq 4$ and $n - m \geq 3$. A Hamilton path from u_0 to u_4 in $GP(7,3)$ is shown in Figure 5.4d. A sequence of Type K insertions starting at $i = 1$ gives a Hamilton path from u_0 to u_{n-3} in $GP(n,3)$ for all $n \equiv 1 (\text{mod } 6)$.

We may assume $n - m \geq 9$. A Hamilton path from u_0 to u_4 in $GP(13,3)$ is given in Figure 5.4e.

One type of insertion is a Type K starting at $i = 1$. The other is a Type A starting at $i = m + 2$.

Subcase 2.2. $n \equiv 3 (\text{mod } 6)$.

(i) $m \equiv 0 (\text{mod } 6)$.

We have that $m \geq 6$ and $n - m \geq 3$. The smallest graph is $GP(9,3)$. A Hamilton path from u_0 to u_6 in $GP(9,3)$ is given in Figure 5.4f. For $m = n - 3$, successive Type I insertions starting at $i = 3$ give a Hamilton path from u_0 to u_{n-3} in $GP(n,3)$ for all $n \equiv 3 (\text{mod } 6)$.

Now let $m \leq n - 9$. Then $n \geq 15$. A Hamilton path from u_0 to u_6 in $GP(15,3)$ is given in Figure 5.4g.

One type of insertion is a Type I starting at $i = 3$. The other is a Type J insertion followed by a sequence of Type B insertions starting at $i = m + 5$.

(ii) $m \equiv 2 (\text{mod } 6)$.

In this case, $m \geq 2$ and $n - m \geq 1$. By Lemma 5.5, we can assume $m < n - 1$. Then $n - m \geq 7$. A Hamilton path from u_0 to u_2 in $GP(9,3)$ is given in Figure 5.4h.

One type of insertion is a Type G starting at $i = 1$. The other is a Type B starting at $i = m + 1$.

(iii) $m \equiv 4 (\text{mod } 6)$.

In this case, we have $m \geq 4$ and $n - m \geq 5$. A Hamilton path from u_0 to u_4 in $GP(9,3)$ is given in Figure 5.4i. A sequence of Type I insertions starting at $i = 1$ gives a Hamilton path from u_0 to u_{n-5} in $GP(n,3)$ for all $n \equiv 3 (\text{mod } 6)$.

Let $n - m \geq 11$. A Hamilton path from u_0 to u_4 in $GP(15,3)$ is given in Figure 5.4j.

One type of insertion is a Type I starting at $i = 1$. The other is a Type D starting at $i = m + 6$.

Subcase 2.3. $n \equiv 5 (\text{mod } 6)$.

In this case, we assume $n \geq 11$ because if $n = 5$, it is the Petersen graph!

(i) $m \equiv 0 (\text{mod } 6)$.

We have that $m \geq 6$ and $n - m \geq 5$. A Hamilton path from u_0 to u_6 in $GP(11,3)$ is given in Figure 5.4k.

One type of insertion is a Type C starting at $i = 2$. The other is a Type A starting at $i = m + 2$.

(ii) $m \equiv 2 (\text{mod } 6)$.

We have that $m \geq 2$ and $n - m \geq 3$. A Hamilton path from u_0 to u_2 in $GP(11,3)$ is given in Figure 5.4l. One Type J insertion followed by a sequence of Type B insertions starting at $i = 7$ give a Hamilton path from u_0 to u_2 in $GP(n,3)$ for all $n \equiv 5 (\text{mod } 6)$.

We may assume $m \geq 8$. A Hamilton path from u_0 to u_8 in $GP(17,3)$ is given in Figure 5.5a.

One type of insertion is a Type H starting at $i = 4$. The other is a Type J insertion followed by a sequence of Type B insertions starting at $i = m + 5$.

(iii) $m \equiv 4 (\text{mod } 6)$.

We have that $m \geq 4$ and $n - m \geq 1$. We can assume $m < n - 1$ by Lemma 5.4. Then $n - m \geq 7$ and $n \geq 11$. A Hamilton path from u_0 to u_4 in $GP(11, 3)$ is given in Figure 5.5b.

One type of insertion is a Type I starting at $i = 1$. The other is a Type J insertion followed by a sequence of Type B insertions starting at $i = m + 1$.

Case 3. There is a Hamilton path from v_0 to v_m for $m = 2, 4, \ldots, n - 1$.

Subcase 3.1. $n \equiv 1 (\text{mod } 6)$.

(i) $m \equiv 0 (\text{mod } 6)$.

We have $m \geq 6$ and $n - m \geq 1$. A Hamilton path from v_0 to v_6 in $GP(7, 3)$ is given in Figure 5.5c. For $m = n - 1$, successive Type E insertions starting at $i = 3$ give a Hamilton path from v_0 to v_{n-1} in $GP(n, 3)$ for all $n \equiv 1 (\text{mod } 6)$.

Let $m \leq n - 7$. Then $n \geq 13$. A Hamilton path from v_0 to v_6 in $GP(13, 3)$ is given in Figure 5.5d.

One type of insertion is a Type E starting at $i = 3$. The other is a Type B starting at $i = m + 2$.

(ii) $m \equiv 2 (\text{mod } 6)$.

In this case, $m \geq 2$ and $n - m \geq 5$. A Hamilton path from v_0 to v_2 in $GP(7, 3)$ is $v_0u_0u_6v_6v_3u_3u_2u_1v_1v_4u_4u_5v_5v_2$. A Hamilton path from v_0 to v_2 in $GP(13, 3)$ is given in Figure 5.5e. For $m = 2$, successive Type D insertions starting at $i = 7$ give a Hamilton path from v_0 to v_2 in $GP(n, 3)$ for all $n \equiv 1 (\text{mod } 6)$.

We may assume $m \geq 8$. A Hamilton path from v_0 to v_8 in $GP(13, 3)$ is given in Figure 5.5f.

One type of insertion is a Type C starting at $i = 3$. The other is a Type F starting at $i = m + 2$.

(iii) $m \equiv 4 (\text{mod } 6)$.

We have $m \geq 4$ and $n - m \geq 3$. By Lemma 5.5, we may assume $n - m \geq 9$. A Hamilton path from v_0 to v_4 in $GP(13,3)$ is given in Figure 5.5g. For $m = 4$, successive Type F insertions starting at $i = 6$ give a Hamilton path from v_0 to v_4 in $GP(n,3)$ for all $n \equiv 1 (\text{mod } 6)$.

Let $m \geq 10$. A Hamilton path from v_0 to v_{10} in $GP(19,3)$ is given in Figure 5.5h.

One type of insertion is a Type H starting at $i = 3$. The other is a Type F starting at $i = m + 2$.

Subcase 3.2. $n \equiv 3(\text{mod } 6)$.

(i) $m \equiv 0(\text{mod } 6)$.

We have $m \geq 6$ and $n - m \geq 3$. By Lemma 5.5, we may assume $n - m \geq 9$. A Hamilton path from v_0 to v_6 in $GP(15,3)$ is given in Figure 5.5i.

One type of insertion is a Type E starting at $i = 2$. The other is a Type D starting at $i = m + 3$.

(ii) $m \equiv 2(\text{mod } 6)$.

We have that $m \geq 2$ and $n - m \geq 1$. A Hamilton path from v_0 to v_8 in $GP(9,3)$ is given in Figure 5.5j. Successive Type C insertions starting at $i = 3$ give a Hamilton path from v_0 to v_{n-1} in $GP(n,3)$ for all $n \equiv 3(\text{mod } 6)$.

Let $n - m \geq 7$. A Hamilton path from v_0 to v_2 in $GP(9,3)$ is given in Figure 5.5k. Successive Type L insertions starting at $i = 5$ give a Hamilton path from v_0 to v_2 in $GP(n,3)$ for all $n \equiv 3(\text{mod } 6)$.

Let $m \geq 8$. A Hamilton path from v_0 to v_8 in $GP(15,3)$ is given in Figure 5.5l

One type of insertion is a Type C starting at $i = 3$. The other is a Type F starting at $i = m + 2$.

(iii) $m \equiv 4(\text{mod } 6)$.

We have that $m \geq 4$ and $n - m \geq 5$. A Hamilton path from v_0 to v_4 in $GP(9,3)$ is given in Figure 5.6a. Successive Type B insertions starting at $i = 6$ give a Hamilton path from v_0 to v_4 in $GP(n,3)$ for all $n \equiv 3(\text{mod } 6)$.

We may assume $m \geq 10$. A Hamilton path from v_0 to v_{10} in $GP(15,3)$ is given in Figure 5.6b.

One type of insertion is a Type E starting at $i = 3$. The other is a Type B starting at $i = m + 2$.

Subcase 3.3. $n \equiv 5 (\text{mod } 6)$.

In this case, we assume $n \geq 11$ since $GP(5,3)$ is the Petersen graph.

(i) $m \equiv 0 (\text{mod } 6)$.

We have $m \geq 6$ and $n - m \geq 5$. A Hamilton path from v_0 to v_6 in $GP(11,3)$ is given in Figure 5.6c. For $m = n - 5$, one Type M insertion starting at $i = 1$ followed by a sequence of Type H insertions starting at $i = 4$ give a Hamilton path from v_0 to v_{n-5} in $GP(n,3)$ for all $n \equiv 5 (\text{mod } 6)$.

Let $m < n - 5$. A Hamilton path from v_0 to v_6 in $GP(17,3)$ is given in Figure 5.6d.

Use one Type M insertion starting at $i = 1$ followed by a sequence of Type H insertions starting at $i = 4$. The other is a Type D starting at $i = m + 3$.

(ii) $m \equiv 2 (\text{mod } 6)$.

We have $m \geq 2$ and $n - m \geq 3$. A Hamilton path from v_0 to v_2 in $GP(11,3)$ is given in Figure 5.6e. A sequence of Type B insertions starting at $i = 8$ gives a Hamilton path from v_0 to v_2 in $GP(n,3)$ for all $n \equiv 5 (\text{mod } 6)$.

Let $m \geq 8$. By Lemma 5.5, we may assume $n - m \geq 9$. A Hamilton path from v_0 to v_8 in $GP(17,3)$ is shown in Figure 5.6f.

One type of insertion is a Type C starting at $i = 4$. The other is a Type A starting at $i = m + 3$.

(iii) $m \equiv 4 (\text{mod } 6)$.

We have $m \geq 4$ and $n - m \geq 1$. A Hamilton path from v_0 to v_{10} in $GP(11,3)$ is given in Figure 5.6g. Successive Type H insertions starting at $i = 4$ give a Hamilton path from v_0 to v_{n-1} in $GP(n,3)$ for all $n \equiv 5 (\text{mod } 6)$.

We may assume $n - m \geq 7$. A Hamilton path from v_0 to v_4 in $GP(11,3)$ is given in Figure 5.6h. Successive Type F insertions starting at $i = 6$ give a Hamilton path from v_0 to v_4 in $GP(n,3)$ for all $n \equiv 5 (\text{mod } 6)$.

Let $m \geq 10$. A Hamilton path from v_0 to v_{10} in $GP(17,3)$ is given in Figure 5.6i.

One type of insertion is a Type H starting at $i = 3$. The other is a Type F starting at $i = m+2$.

This completes the proof of 1.

Proof of 2. The neccessity is easy to see from Theorem 5.1 and Theorem 5.3.

To prove the sufficiency, recall that a bipartition of $GP(n,3)$ is

$$X = \{u_0, u_2, \ldots, u_{2i}, \ldots, u_{n-2}, v_1, v_3, \ldots, v_{2j+1}, \ldots, v_{n-1}\},$$

and

$$Y = \{u_1, u_3, \ldots, u_{2j+1}, \ldots, u_{n-1}, v_0, v_2, \ldots, v_{2i}, \ldots, v_{n-2}\}.$$

By the same reason as in the proof of 1 and by Lemma 5.5, we need only prove that there are Hamilton paths in $GP(n,3)$ from u_0 to v_m for $m = 2,4,\ldots,n-2$, and from u_0 to u_m for $m = 3,5,\ldots,n-3$, and from v_0 to v_m for $m = 1,3,\ldots,n-1$, respectively.

Case 1 There are Hamilton paths in $GP(n,3)$ from u_0 to v_m for $m = 2,4,\ldots,n-2$.

Subcase 1.1. $n \equiv 0 (\text{mod } 6)$.

In this case, we can assume $n \geq 12$ since $GP(6,3)$ is not 3-regular.

(i) $m \equiv 0(\text{mod } 6)$.

We have $m \geq 6$ and $n - m \geq 6$. A Hamilton path from u_0 to v_6 in $GP(12,3)$ is given in Figure 5.6j.

One type of insertion is a Type K starting at $i = 1$. The other is a Type D starting at $i = m+3$.

(ii) $m \equiv 2(\text{mod } 6)$.

We have $m \geq 2$ and $n - m \geq 4$. A Hamilton path from u_0 to v_2 in $GP(12,3)$ is given in Figure 5.6k. Successive Type C insertions starting at $i = 9$ give a Hamilton path from u_0 to v_2 in $GP(n,3)$ for all $n \equiv 0(\text{mod } 6)$.

We may assume $m \geq 8$. A Hamilton path from u_0 to v_8 in $GP(12,3)$ is given in Figure 5.6l. Successive Type I insertions starting at $i = 3$ give a Hamilton path from u_0 to v_{n-4} in $GP(n,3)$ for all $n \equiv 0(\text{mod } 6)$.

Let $m < n - 4$. A Hamilton path from u_0 to v_8 in $GP(18,3)$ is given in Figure 5.7a.

One type of insertion is a Type I starting at $i = 3$. The other is a Type F starting at $i = m + 2$.

(iii) $m \equiv 4 \pmod 6$.

We have $m \geq 4$ and $n - m \geq 2$. A Hamilton path from u_0 to v_{10} in $GP(12,3)$ is given in Figure 5.7b. Successive Type F insertions starting at $i = 2$ give a Hamilton path from u_0 to v_{n-2} in $GP(n,3)$ for all $n \equiv 0 \pmod 6$.

Let $m < n - 2$. Then $n - m \geq 8$. A Hamilton path from u_0 to v_4 in $GP(12,3)$ is given in Figure 5.7c. Successive Type C insertions starting at $i = 9$ give a Hamilton path from u_0 to v_4 in $GP(n,3)$ for all $n \equiv 0 \pmod 6$.

Let $m \geq 10$. A Hamilton path from u_0 to v_{10} in $GP(18,3)$ is given in Figure 5.7d.

One type of insertion is a Type D starting at $i = 2$. The other is a Type C starting at $i = m + 5$.

Subcase 1.2. $n \equiv 2 \pmod 6$.

(i) $m \equiv 0 \pmod 6$.

We have $m \geq 6$ and $n - m \geq 2$. A Hamilton path from u_0 to v_6 in $GP(8,3)$ is given in Figure 5.7e. Successive Type L insertions starting at $i = 1$ give a Hamilton path from u_0 to v_{n-2} in $GP(n,3)$ for all $n \equiv 2 \pmod 6$.

Let $n - m \geq 8$. Then $n \geq 14$. A Hamilton path from u_0 to v_6 in $GP(14,3)$ is given in Figure 5.7f.

One type of insertion is a Type K starting at $i = 2$. The other is a Type D starting at $i = m + 5$.

(ii) $m \equiv 2 \pmod 6$.

We have $m \geq 2$ and $n - m \geq 6$. A Hamilton path from u_0 to v_2 in $GP(8,3)$ is given in Figure 5.7g. A sequence of Type C insertions starting at $i = 5$ gives a Hamilton path from u_0 to v_2 in $GP(n,3)$ for all $n \equiv 2 \pmod 6$.

Let $m \geq 8$. Then $n \geq 14$. A Hamilton path from u_0 to v_8 in $GP(14,3)$ is given in Figure 5.7h.

One type of insertion is a Type B starting at $i = 1$. The other is a Type C starting at $i = m + 3$.

(iii) $m \equiv 4 (\mathrm{mod}\ 6)$.

We have $m \geq 4$ and $n - m \geq 4$. Since $(u_0)(v_0)(u_1 u_{n-1})(v_1 v_{n-1}) \cdots (u_{\frac{n}{2}-1} u_{\frac{n}{2}+1})$ $(v_{\frac{n}{2}-1} v_{\frac{n}{2}+1})(u_{\frac{n}{2}})(v_{\frac{n}{2}})$ is an automorphism interchanging v_{n-4} and v_4 and there is a Hamilton path joining u_0 to v_4 (by Lemma 5.4), we may assume $m < n - 4$. Then $n \geq 14$.

A Hamilton path from u_0 to v_4 in $GP(14, 3)$ is given in Figure 5.7i.

One type of insertion is a Type K starting at $i = 1$. The other is a Type A starting at $i = m + 3$.

Subcase 1.3. $n \equiv 4 (\mathrm{mod}\ 6)$.

(i) $m \equiv 0 (\mathrm{mod}\ 6)$.

We have that $m \geq 6$ and $n - m \geq 4$. Since $(u_0)(v_0)(u_1 u_{n-1})(v_1 v_{n-1}) \cdots (u_{\frac{n}{2}-1} u_{\frac{n}{2}+1})$ $(v_{\frac{n}{2}-1} v_{\frac{n}{2}+1})(u_{\frac{n}{2}})(v_{\frac{n}{2}})$ is an isomorphism interchanging v_{n-4} and v_4 and there is a Hamilton path joining u_0 to v_4 (by Lemma 5.4), we may assume $m < n - 4$. Then $n \geq 16$.

A Hamilton path from u_0 to v_6 in $GP(16, 3)$ is shown in Figure 5.7j. Successive Type D insertions starting at $i = 9$ give a Hamilton path from u_0 to v_6 in $GP(n, 3)$ for all $n \equiv 4 (\mathrm{mod}\ 6)$.

We may assume $m \geq 12$. A Hamilton path from u_0 to v_{12} in $GP(22, 3)$ is given in Figure 5.7k.

One type of insertion is a Type H starting at $i = 5$. The other is a Type D starting at $i = m + 3$.

(ii) $m \equiv 2 (\mathrm{mod}\ 6)$.

In this case, $n - m \geq 2$. Since there is an isomorphism interchanging v_{n-2} and v_2 and there is a Hamilton path joining u_0 to v_2 (by Lemma 5.4), we may assume $m < n - 2$ and $m \geq 8$. Then $n \geq 16$. A Hamilton path from u_0 to v_8 in $GP(16, 3)$ is given in Figure 5.7l.

One type of insertion is a Type A starting at $i = 2$. The other is a Type C starting at $i = m + 4$.

(iii) $m \equiv 4 (\mathrm{mod}\ 6)$.

We have $m \geq 4$ and $n - m \geq 6$. A Hamilton path from u_0 to v_4 in $GP(10, 3)$ is given in Figure 5.8a.

One type of insertion is a Type B starting at $i = 1$. The other is a Type C starting at $i = m + 3$.

Case 2 There is a Hamilton path from u_0 to u_m for $m = 3, 5, \ldots, n-3$.

Subcase 2.1. $n \equiv 0 (\text{mod } 6)$.

In this case, $n \geq 12$.

(i) $m \equiv 1 (\text{mod } 6)$.

We have $m \geq 7$ and $n - m \geq 5$. A Hamilton path from u_0 to u_7 in $GP(12,3)$ is given in Figure 5.8b.

One type of insertion is a Type D starting from $i = 2$. The other is a Type C starting at $i = m + 2$.

(ii) $m \equiv 3 (\text{mod } 6)$.

We have $m \geq 3$ and $n - m \geq 3$. A Hamilton path from u_0 to u_3 in $GP(12,3)$ is given in Figure 5.8c. Successive Type I insertions starting at $i = 4$ give a Hamilton path from u_0 to u_3 in $GP(n,3)$ for all $n \equiv 0 (\text{mod } 6)$. But $(u_0)(v_0)(u_1 u_{n-1})(v_1 v_{n-1}) \cdots (u_{\frac{n}{2}-1} u_{\frac{n}{2}+1})$ $(v_{\frac{n}{2}-1} v_{\frac{n}{2}+1})(u_{\frac{n}{2}})(v_{\frac{n}{2}})$ is an isomorphism interchanging u_{n-3} and u_3, there is a Hamilton path from u_0 to u_{n-3} in $GP(n,3)$ for all $n \equiv 0 (\text{mod } 6)$.

Let $m \geq 9$ and $m < n - 3$. A Hamilton path from u_0 to u_9 in $GP(18,3)$ is given in Figure 5.8d.

One type of insertion is a Type D starting at $i = 4$. The other is a Type I starting at $i = m + 1$.

(iii) $m \equiv 5 (\text{mod } 6)$.

We have $m \geq 5$ and $n - m \geq 1$. By Lemma 5.5, we can assume $m < n - 1$, and hence $m \leq n - 7$. A Hamilton path from u_0 to u_5 in $GP(12,3)$ is given in Figure 5.8e.

One type of insertion is a Type J insertion followed by a sequence of Type B insertions starting at $i = 1$. The other is a Type C starting at $i = m + 4$.

Subcase 2.2. $n \equiv 2 (\text{mod } 6)$.

(i) $m \equiv 1 (\text{mod } 6)$.

In this case, $m \geq 1, n - m \geq 1$. We may assume $m \geq 7, n - m \geq 7$ by Lemma 5.5. Then $n \geq 14$. A Hamilton path from u_0 to u_7 in $GP(14,3)$ is given in Figure 5.8f.

One type of insertion is a Type K starting at $i = 1$. The other is a Type D starting at $i = m + 4$.

(ii) $m \equiv 3 \pmod 6$.

We have $m \geq 3$ and $n - m \geq 5$. A Hamilton path from u_0 to u_3 in $GP(8, 3)$ is given in Figure 5.8g.

One type of insertion is a Type G starting at $i = 2$. The other is a Type J insertion followed by a sequence of Type B insertions starting at $i = m + 1$.

(iii) $m \equiv 5 \pmod 6$.

We have $m \geq 5$ and $n - m \geq 3$. A Hamilton path from u_0 to u_5 in $GP(8, 3)$ is given in Figure 5.8h.

One type of insertion is a Type J insertion followed by a sequence of Type I insertions starting at $i = 1$. The other is a Type G starting at $i = m + 2$.

Subcase 2.3. $n \equiv 4 \pmod 6$.

(i) $m \equiv 1 \pmod 6$.

We have that $m \geq 7$ and $n - m \geq 3$. A Hamilton path from u_0 to u_7 in $GP(10, 3)$ is given in Figure 5.8i. Successive Type I insertions starting at $i = 1$ give a Hamilton path from u_0 to u_{n-3} in $GP(n, 3)$ for all $n \equiv 4 \pmod 6$.

Let $m < n - 3$. A Hamilton path from u_0 to u_7 in $GP(16, 3)$ is given in Figure 5.8j.

One type of insertion is a Type I starting at $i = 1$. The other is a Type D starting at $i = m + 4$.

(ii) $m \equiv 3 \pmod 6$.

We have that $m \geq 3$ and $n - m \geq 1$. By Lemma 5.5, we may assume $n - m \geq 7$. A Hamilton path from u_0 to u_3 in $GP(10, 3)$ is given in Figure 5.8k. Successive Type I insertions starting at $i = 4$ give a Hamilton path from u_0 to u_3 in $GP(n, 3)$ for all $n \equiv 4 \pmod 6$.

Let $m \geq 9$. A Hamilton path from u_0 to u_9 in $GP(16, 3)$ is given in Figure 5.8l.

One type of insertion is a Type D starting at $i = 4$. The other is a Type I starting at $i = m + 1$.

(iii) $m \equiv 5 (\text{mod } 6)$.

We have $m \geq 5$ and $n - m \geq 5$. A Hamilton path from u_0 to u_5 in $GP(10,3)$ is given in Figure 5.9a.

One type of insertion is a Type L starting at $i = 2$. The other is a Type H starting at $i = m + 1$.

Case 3 There is a Hamilton path from v_0 to v_m for $m = 1, 3, \ldots, n-1$.

Subcase 3.1. $n \equiv 0 (\text{mod } 6)$.

In this case, $n \geq 12$.

(i) $m \equiv 1 (\text{mod } 6)$.

We have $m \geq 1$ and $n - m \geq 5$. A Hamilton path from v_0 to v_1 in $GP(12,3)$ is given in Figure 5.9b. Successive Type E insertions starting at $i = 4$ give a Hamilton path from v_0 to v_1 in $GP(n,3)$ for all $n \equiv 0 (\text{mod } 6)$.

Let $m > 1$. Then $m \geq 7$. A Hamilton path from v_0 to v_7 in $GP(12,3)$ is given in Figure 5.9c. One Type M insertion at $i = 8$ followed by successive Type H insertions starting at $i = 11$ give a Hamilton path from v_0 to v_7 in $GP(n,3)$ for all $n \equiv 0 (\text{mod } 6)$.

Let $m \geq 13$. A Hamilton path from v_0 to v_{13} in $GP(18,3)$ is given in Figure 5.9d.

One type of insertion is a Type F starting at $i = 2$. The other is a Type M insertion at $i = m + 1$ followed by a sequence of Type H insertions starting at $i = m + 4$.

(ii) $m \equiv 3 (\text{mod } 6)$.

We have $m \geq 3$ and $n - m \geq 3$. If $m = 3$, or $n - 3$, then v_0 and v_m are adjacent in $GP(n,3)$ and hence there is a Hamilton path joining them in $GP(n,3)$ by Lemma 5.5. Let $n - m \geq 9$ and $m \geq 9$. A Hamilton path from v_0 to v_9 in $GP(18,3)$ is given in Figure 5.9e.

One type of insertion is a Type H starting at $i = 2$. The other is a Type B starting at $i = m + 6$.

(iii) $m \equiv 5 (\text{mod } 6)$.

We have $m \geq 5$ and $n - m \geq 1$. A Hamilton path from v_0 to v_{11} in $GP(12,3)$ is given in Figure 5.9f. One Type M insertion at $i = 1$ followed by succesive Type H insertions starting at $i = 4$ give a Hamilton path from v_0 to v_{n-1} in $GP(n,3)$ for all $n \equiv 0 \pmod 6$.

Let $m < n-1$. Then $n - m \geq 7$. A Hamilton path from v_0 to v_5 in $GP(12,3)$ is given in Figure 5.9g. A sequence of Type B insertions starting at $i = 7$ gives a Hamilton path from v_0 to v_5 in $GP(n,3)$.

Let $m \geq 11$. A Hamilton path from v_0 to v_{11} in $GP(18,3)$ is given in Figure 5.9h.

One type of insertion is a Type H starting at $i = 3$. The other is a Type B starting at $i = m+2$.

Subcase 3.2. $n \equiv 2 \pmod 6$.

(i) $m \equiv 1 \pmod 6$.

We have $m \geq 1$ and $n-m \geq 1$. A Hamilton path from v_0 to v_1 in $GP(8,3)$ is given in Figure 5.9i. A sequence of Type A insertions starting at $i = 4$ gives a Hamilton path from v_0 to v_1 in $GP(n,3)$ for all $n \equiv 2 \pmod 6$. Since $(u_0)(v_0)(u_1u_{n-1})(v_1v_{n-1})\cdots(u_{\frac{n}{2}-1}u_{\frac{n}{2}+1})(v_{\frac{n}{2}-1}v_{\frac{n}{2}+1})(u_{\frac{n}{2}})(v_{\frac{n}{2}})$ is an automorphism interchanging v_{n-1} and v_1, we also have that there is a Hamilton path from v_0 to v_{n-1} in $GP(n,3)$ for $n \equiv 2 \pmod 6$.

Let $m > 1$ and $n - m \geq 7$. Then $m \geq 7$. A Hamilton path from v_0 to v_7 in $GP(14,3)$ is given in Figure 5.9j.

One type of insertion is a Type A starting at $i = 3$. The other is a Type K starting at $i = m+4$.

(ii) $m \equiv 3 \pmod 6$.

We have $m \geq 3$ and $n - m \geq 5$. Since v_0 and v_3 are adjacent, we may assume $m \geq 9$ by Lemma 5.5. A Hamilton path from v_0 to v_9 in $GP(14,3)$ is given in Figure 5.9k.

One type of insertion is a Type H starting at $i = 2$. The other is a Type F starting at $i = m+2$.

(iii) $m \equiv 5 \pmod 6$.

We have $m \geq 5$ and $n - m \geq 3$. Since v_0 is adjacent to v_{n-3} in $GP(n,3)$, by Lemma 5.5, we can assume that $n - m \geq 9$. Then $n \geq 14$. A Hamilton path from v_0 to v_5 in $GP(14,3)$ is given in Figure 5.9l.

One type of insertion is a Type B starting at $i = 2$. The other is a Type C starting at $i = m + 3$.

Subcase 3.3. $n \equiv 4 \pmod 6$.

(i) $m \equiv 1 \pmod 6$.

We have $m \geq 1$ and $n - m \geq 3$. A Hamilton path from v_0 to v_1 in $GP(10,3)$ is given in Figure 5.10a. Successive Type L insertions starting at $i = 5$ give a Hamilton path from v_0 to v_1 in $GP(n,3)$ for all $n \equiv 4 \pmod 6$.

Let $m \geq 7$. Since v_{n-3} is adjacent to v_0, by Lemma 5.5, we may assume $m \leq n-9$. A Hamilton path from v_0 to v_7 in $GP(16,3)$ is given in Figure 5.10b.

One type of insertion is a Type K starting at $i = 3$. The other is a Type F starting at $i = m + 2$.

(ii) $m \equiv 3 \pmod 6$.

We have $m \geq 3$ and $n - m \geq 1$. A Hamilton path from v_0 to v_9 in $GP(10,3)$ is given in Figure 5.10c. Successive Type L insertions starting at $i = 5$ give a Hamilton path from v_0 to v_{n-1} in $GP(n,3)$ for all $n \equiv 4 \pmod 6$.

Let $m < n - 1$. Then $n - m \geq 7$. By Lemma 5.5, we may assume $m \geq 9$. A Hamilton path from v_0 to v_9 in $GP(16,3)$ is given in Figure 5.10d.

One type of insertion is a Type D starting at $i = 3$. The other is a Type E starting at $i = m + 2$.

(iii) $m \equiv 5 \pmod 6$.

We have $m \geq 5$ and $n - m \geq 5$. A Hamilton path from v_0 to v_5 in $GP(10,3)$ is given in Figure 5.10e. One Type M insertion at $i = 6$ followed by a sequence of Type H insertions starting at $i = 9$ give a Hamilton path from v_0 to v_5 in $GP(n,3)$ for all $n \equiv 4 \pmod 6$.

Let $m \geq 11$. Then $n \geq 16$. A Hamilton path from v_0 to v_{11} in $GP(16,3)$ is given in Figure 5.10f.

One type of insertion is a Type I starting at $i = 2$. The other is a Type B starting at $i = m + 2$.

Combining all the cases, we have finished the proof of the theorem. ∎

5.3 $GP(n, 2)$

Theorem 5.7 *If $gcd(n,2) = 1$, then $GP(n,2)$ is Hamilton-connected except for $n \equiv 5 (mod\ 6)$.*

Proof. We divide the proof into several cases.

Case 1 There are Hamilton paths from u_0 to $v_1, v_2, \ldots, v_{n-1}$, respectively. As shown in the begining of Theorem 5.6, we need only prove that there is a Hamilton path from u_0 to v_m for $m = 1, 3, \ldots, n-2$.

By Lemma 5.4, there is a Hamilton path from u_0 to each of v_1, v_3. So some of the time we can assume $m > 3$.

Subcase 1.1. $n \equiv 1(\text{mod } 6)$.

(i) $m \equiv 1(\text{mod } 6)$.

We have $n - m \geq 6$. We can assume $m \geq 7$. A Hamilton path from u_0 to v_7 in $GP(13,2)$ is given in Figure 5.10g.

One type of insertion is a Type N starting at $i = 1$. The other is a Type O insertion starting at $i = m + 5$ followed by a sequence of Type R insertions starting at $i = m + 8$.

(ii) $m \equiv 3(\text{mod } 6)$.

We have $n - m \geq 4$. We may assume $m \geq 9$ by Lemma 5.4. A Hamilton path from u_0 to v_9 in $GP(13,2)$ is given in Figure 5.10h.

One type of insertion is a Type N starting at $i = 2$. The other is a Type P insertion starting at $i = m + 3$ followed by a sequence of Type R insertions starting at $i = m + 6$.

(iii) $m \equiv 5(\text{mod } 6)$.

We have $m \geq 5$ and $n - m \geq 2$. A Hamilton path from u_0 to v_5 in $GP(7,2)$ is given in Figure 5.10i. Successive Type Q insertions starting at $i = 3$ give a Hamilton path from u_0 to v_{n-2} in $GP(n,2)$ for all $n \equiv 1(\text{mod } 6)$.

Let $m < n - 2$. A Hamilton path from u_0 to v_5 in $GP(13,2)$ is given in Figure 5.10j.

One type of insertion is a Type Q starting at $i = 3$. The other is a Type R starting at $i = m + 4$.

Subcase 1.2. $n \equiv 3 \pmod 6$

(i) $m \equiv 1 \pmod 6$.

We may assume $m \geq 7$ by Lemma 5.4. A Hamilton path in $GP(9,2)$ from u_0 to v_7 is given in Figure 5.10k. Successive Type N insertions starting at $i = 1$ give a Hamilton path from u_0 to v_{n-2} in $GP(n,2)$ for all $n \equiv 3 \pmod 6$.

We may assume $m < n - 2$. A Hamilton path from u_0 to v_7 in $GP(15,2)$ is given in Figure 5.10l.

One type of insertion is a Type S starting at $i = 3$. The other is a Type R starting at $i = m + 4$.

(ii) $m \equiv 3 \pmod 6$.

We have $m \geq 3$ and $n - m \geq 6$. A Hamilton path from u_0 to v_3 in $GP(9,2)$ is given in Figure 5.11a.

One type of insertion is a Type Q starting at $i = 1$. The other is a Type O insertion starting at $i = m + 5$ followed by a sequence of Type R insertions starting at $i = m + 8$.

(iii) $m \equiv 5 \pmod 6$.

We have $m \geq 5$ and $n - m \geq 4$. A Hamilton path from u_0 to v_5 in $GP(9,2)$ is given in Figure 5.11b.

One type of insertion is a Type N starting at $i = 2$. The other is a Type P insertion starting at $i = m + 3$ followed by a sequence of Type R insertions starting at $i = m + 6$.

Case 2 There are Hamilton paths from u_0 to $u_1, u_2, \ldots, u_{n-1}$, respectively.

By the same reason as given in the begining of Theorem 5.6, we need only show that u_0 is joined by a Hamilton path to u_m for $m = 2, 4, \ldots, n-1$.

Subcase 2.1. $n \equiv 1 \pmod 6$.

(i) $m \equiv 0 \pmod 6$.

In this case, $m \geq 6$, $n - m \geq 1$. By Lemma 5.5, we can assume that $n - m \geq 7$. Then $n \geq 13$. A Hamilton path from u_0 to u_6 in $GP(13, 2)$ is given in Figure 5.11c.

One type of insertion is a Type T starting at $i = 3$. The other is Type N starting at $i = m + 4$.

(ii) $m \equiv 2(\text{mod } 6)$.

A Hamilton path from u_0 to u_2 in $GP(7, 2)$ is given in Figure 5.11d. A sequence of Type S insertions starting at $i = 4$ gives a Hamilton path from u_0 to u_2 in $GP(n, 2)$ for all $n \equiv 1(\text{mod } 6)$.

Assume $m \geq 8$. Then $n \geq 13$. A Hamilton path from u_0 to u_8 in $GP(13, 2)$ is given in Figure 5.11e.

One type of insertion is a Type S starting at $i = 3$. The other is a Type O insertion starting at $i = m+4$ followed by a sequence of Type R insertions starting at $i = m+7$.

(iii) $m \equiv 4(\text{mod } 6)$.

A Hamilton path from u_0 to u_4 in $GP(7, 2)$ is given in Figure 5.11f. A sequence of Type T insertions starting at $i = 1$ gives a Hamilton path from u_0 to u_{n-3} for all $n \equiv 1(\text{mod } 6)$.

We may assume $m < n - 3$. A Hamilton path from u_0 to u_4 in $GP(13, 2)$ is given in Figure 5.11g.

One type of insertion is a Type T starting at $i = 1$. The other is a Type S starting at $i = m + 6$.

Subcase 2.2. $n \equiv 3(\text{mod } 6)$.

(i) $m \equiv 0(\text{mod } 6)$.

We have $m \geq 6$ and $n - m \geq 3$. A Hamilton path from u_0 to u_6 in $GP(9, 2)$ is given in Figure 5.11h.

One type of insertion is a Type S starting at $i = 3$. The other is a Type O insertion starting at $i = m+2$ followed by a sequence of Type R insertions starting at $i = m+5$.

(ii) $m \equiv 2(\text{mod } 6)$.

We have $m \geq 2$ and $n - m \geq 1$. Since u_0 and u_{n-1} are adjacent, we may assume that $m < n - 1$. A Hamilton path from u_0 to u_2 in $GP(9, 2)$ is given in Figure 5.11i.

A sequence of Type S insertions starting at $i = 6$ gives a Hamilton path from u_0 to u_2 for all $n \equiv 3 \pmod 6$.

Let $m \geq 8$. A Hamilton path from u_0 to u_8 in $GP(15,2)$ is given in Figure 5.11j.

One type of insertion is a Type R starting at $i = 1$. The other is a Type S starting at $i = m + 4$.

(iii) $m \equiv 4 \pmod 6$.

We have $m \geq 4$ and $n - m \geq 5$. A Hamilton path from u_0 to u_4 in $GP(9,2)$ is given in Figure 5.11k.

One type of insertion is a Type N starting at $i = 1$. The other is a Type O insertion starting at $i = m + 4$ followed by a sequence of Type R insertions starting at $i = m + 7$.

Case 3 There are Hamilton paths from v_0 to each of $v_1, v_2, \ldots, v_{n-1}$. We need only prove that there is a Hamilton path from v_0 to v_m for $m = 2, 4, \ldots, n-1$.

Subcase 3.1. $n \equiv 1 \pmod 6$.

(i) $m \equiv 0 \pmod 6$.

We have $m \geq 6$ and $n - m \geq 1$. A Hamilton path from v_0 to v_6 in $GP(7,2)$ is given in Figure 5.11l. Successive Type S insertions starting at $i = 2$ give a Hamilton path from v_0 to v_{n-1} in $GP(n,2)$ for all $n \equiv 1 \pmod 6$.

A Hamilton path from v_0 to v_6 in $GP(13,2)$ is given in Figure 5.12a.

One type of insertion is a Type T starting at $i = 2$. The other is a Type S starting at $i = m + 2$.

(ii) $m \equiv 2 \pmod 6$.

We have $m \geq 2$ and $n - m \geq 5$. We may assume $m \geq 8$ by Lemma 5.5. Then $n \geq 13$. A Hamilton path from v_0 to v_8 in $GP(13,2)$ is given in Figure 5.12b. A sequence of Type S insertions starting at $i = 4$ gives a Hamilton path from v_0 to v_{n-5} for all $n \equiv 1 \pmod 6$.

Let $m < n - 5$. A Hamilton path from v_0 to v_8 in $GP(19,2)$ is given in Figure 5.12c.

One type of insertion is a Type S starting at $i = 4$. The other is a Type T starting at $i = m + 7$.

(iii) $m \equiv 4 (\mathrm{mod}\ 6)$.

We have $m \geq 4$ and $n - m \geq 3$. A Hamilton path from v_0 to v_4 in $GP(7,2)$ is $v_0v_2u_2u_3u_4u_5v_5v_3v_1u_1u_0u_6v_6v_4$. A Hamilton path from v_0 to v_{10} in $GP(13,2)$ is given in Figure 5.12d. A sequence of Type R insertion starting at $i = 4$ gives a Hamilton path from v_0 to v_{n-3} in $GP(n,2)$ for all $n \equiv 1 (\mathrm{mod}\ 6)$. A Hamilton path from v_0 to v_4 in $GP(13,2)$ is given in Figure 5.12e. A sequence of Type S insertion starting at $i = 9$ gives a Hamilton path from v_0 to v_4 in $GP(n,2)$ for all $n \equiv 1 (\mathrm{mod}\ 6)$.

We may assume $m \geq 10$ and $m < n - 3$. A Hamilton path from v_0 to v_{10} in $GP(19,2)$ is given in Figure 5.12f.

One type of insertion is a Type R starting at $i = 4$. The other is a Type Q starting at $i = m + 7$.

Subcase 3.2. $n \equiv 3 (\mathrm{mod}\ 6)$.

(i) $m \equiv 0 (\mathrm{mod}\ 6)$.

A Hamilton path from v_0 to v_6 in $GP(9,2)$ is given in Figure 5.12g. A sequence of Type T insertions starting at $i = 2$ gives a Hamilton path from v_0 to v_{n-3} in $GP(n,2)$ for all $n \equiv 3 (\mathrm{mod}\ 6)$.

Let $m < n - 3$. A Hamilton path from v_0 to v_6 in $GP(15,2)$ is given in Figure 5.12h.

One type of insertion is a Type T at $i = 2$. The other is a Type N starting at $i = m + 6$.

(ii) $m \equiv 2 (\mathrm{mod}\ 6)$.

We have $m \geq 2$ and $n - m \geq 1$. A Hamilton path from v_0 to v_8 in $GP(9,2)$ is given in Figure 5.12i. Successive Type S insertions starting at $i = 4$ give a Hamilton path from v_0 to v_{n-1} in $GP(n,2)$ for all $n \equiv 3 (\mathrm{mod}\ 6)$.

Since v_0 is adjacent to v_2, we may assume that $m \geq 8$ by Lemma 5.5. A Hamilton path from v_0 to v_8 in $GP(15,2)$ is given in Figure 5.12j.

One type of insertion is a Type N at $i = 5$. The other is a Type R starting at $i = m + 2$.

(iii) $m \equiv 4(\text{mod } 6)$.

A Hamilton path from v_0 to v_4 in $GP(9,2)$ is given in Figure 5.12k. Successive Type T insertions starting at $i = 6$ give a Hamilton path from v_0 to v_4 in $GP(n,2)$ for all $n \equiv 3(\text{mod } 6)$.

Let $m \geq 10$. A Hamilton path from v_0 to v_{10} in $GP(15,2)$ is given in Figure 5.12l.

One type of insertion is a Type N at $i = 5$. The other is a Type T starting at $i = m + 2$.

This completes the proof. ∎

Remark. The requirment of $gcd(n,k) = 1$ is important in Conjecture 1. For example, $GP(6,2)$ is hamiltonian, but it is not Hamilton-connected as there is no Hamilton path joining u_0 to u_2! But for $k = 3$, we have proved that except for $n = 5$, $GP(n,3)$ is Hamilton-connected or Hamilton-laceable.

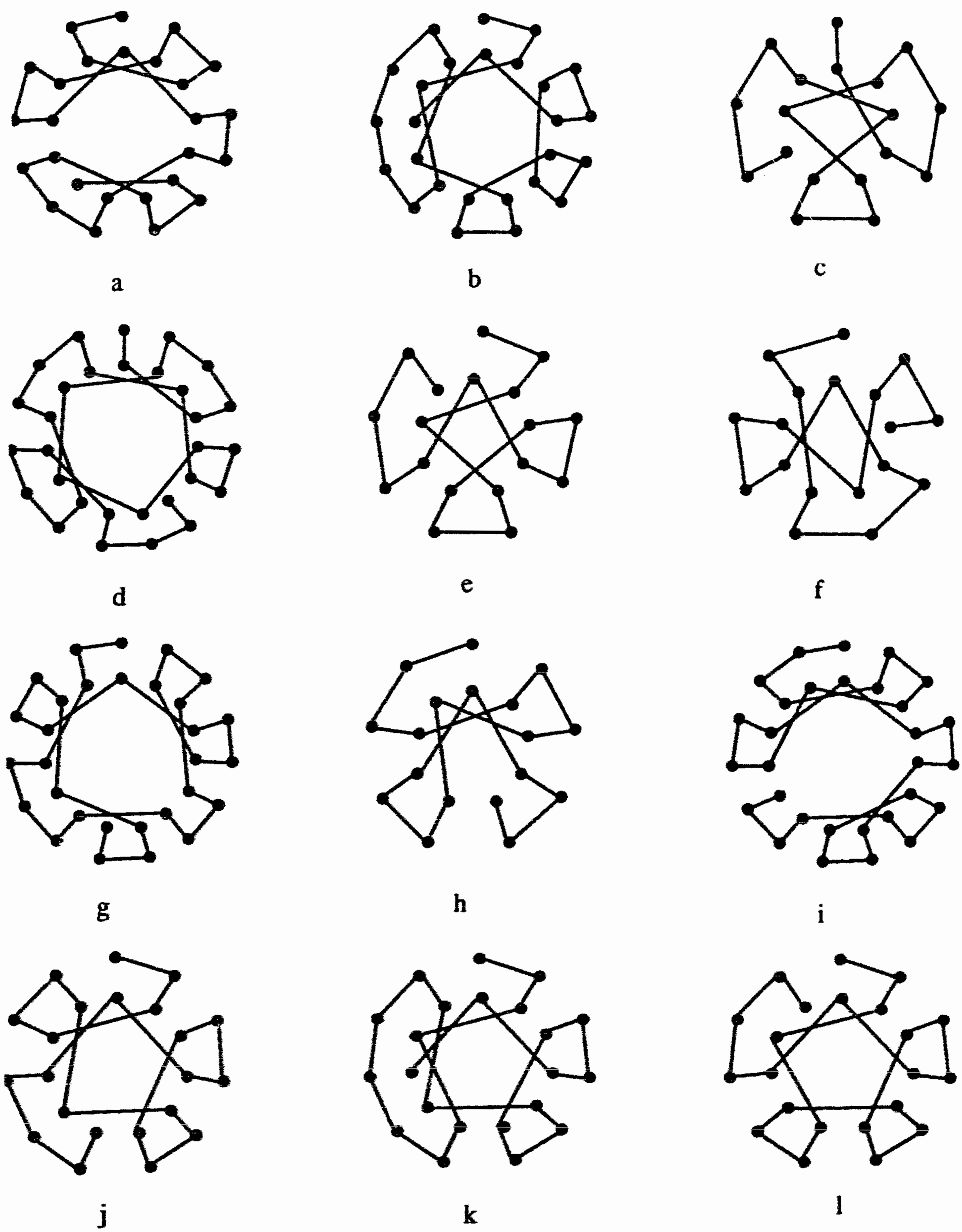


Figure 5.3: Figure 5.3a - Figure 5.3l

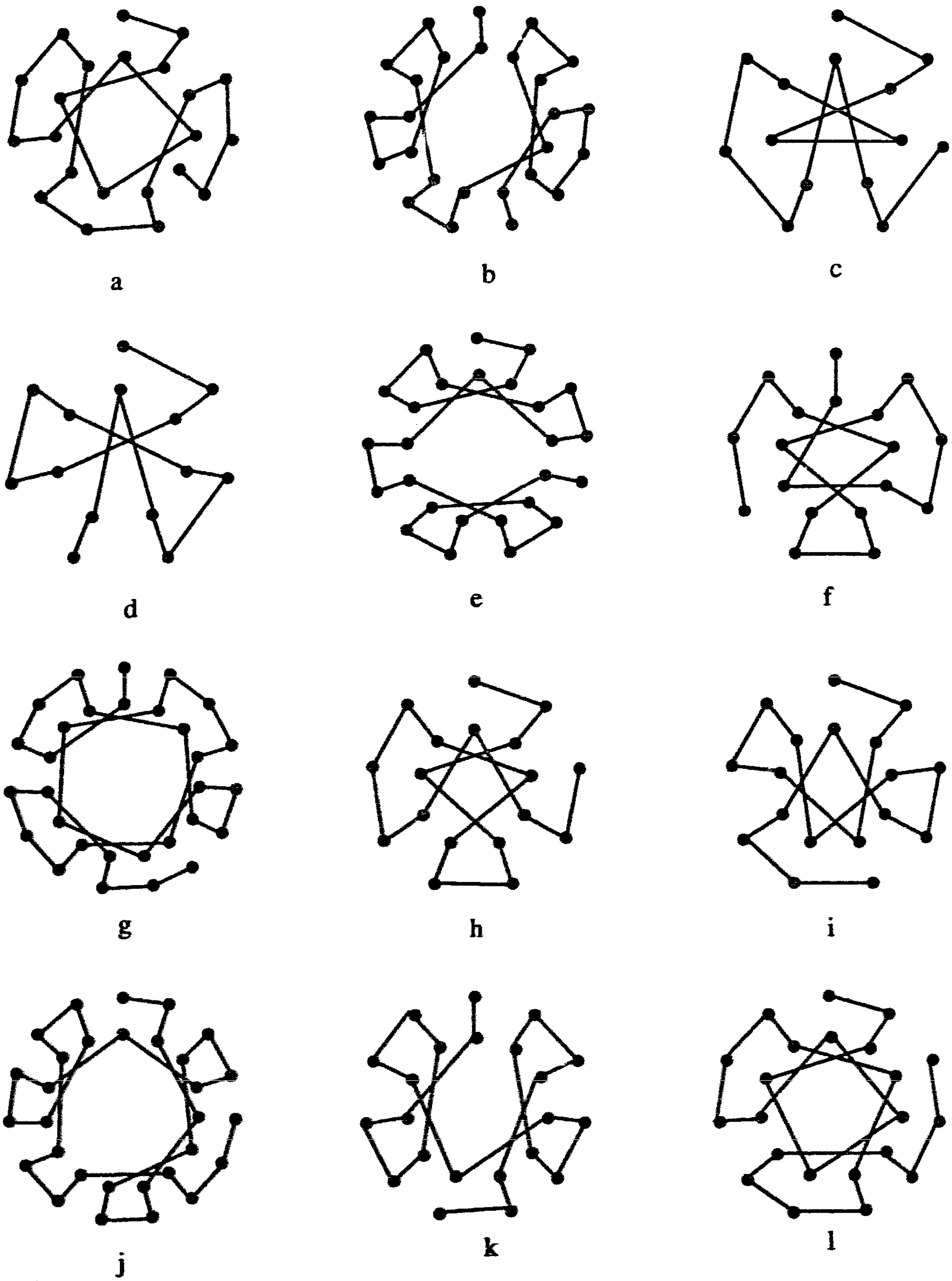


Figure 5.4: Figure 5.4a - Figure 5.4l

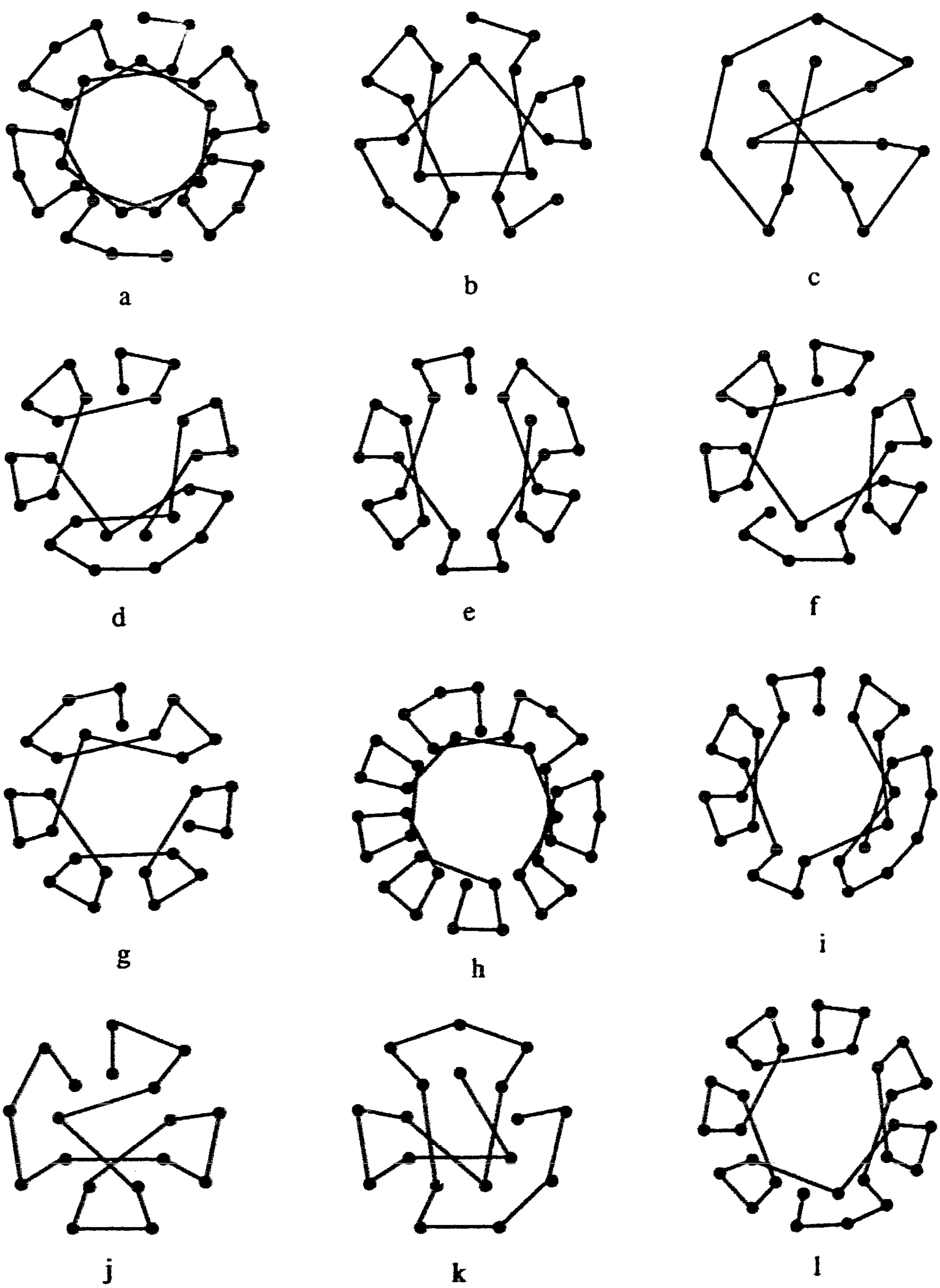


Figure 5.5: Figure 5.5a - Figure 5.5l

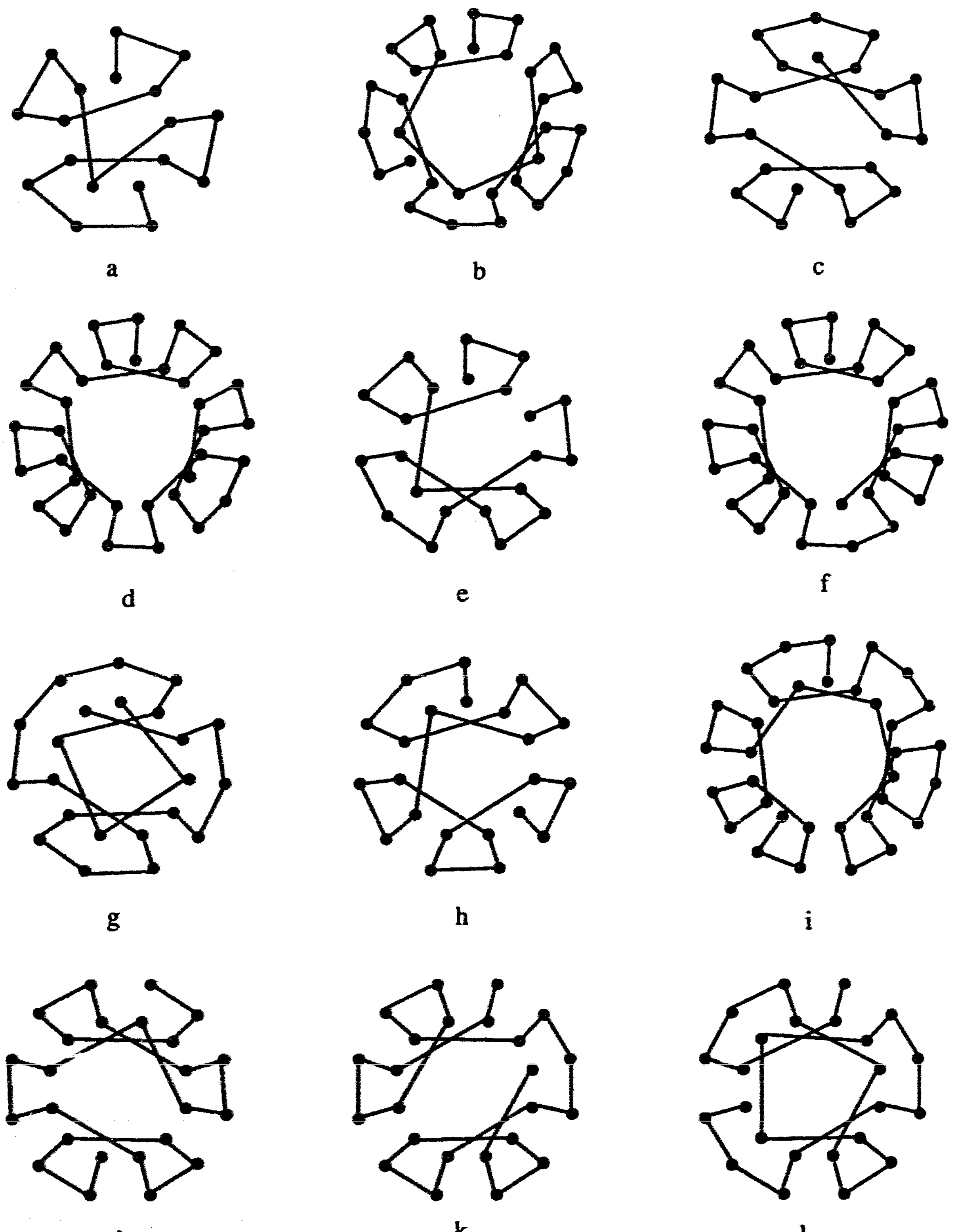


Figure 5.6: Figure 5.6a - Figure 5.6l

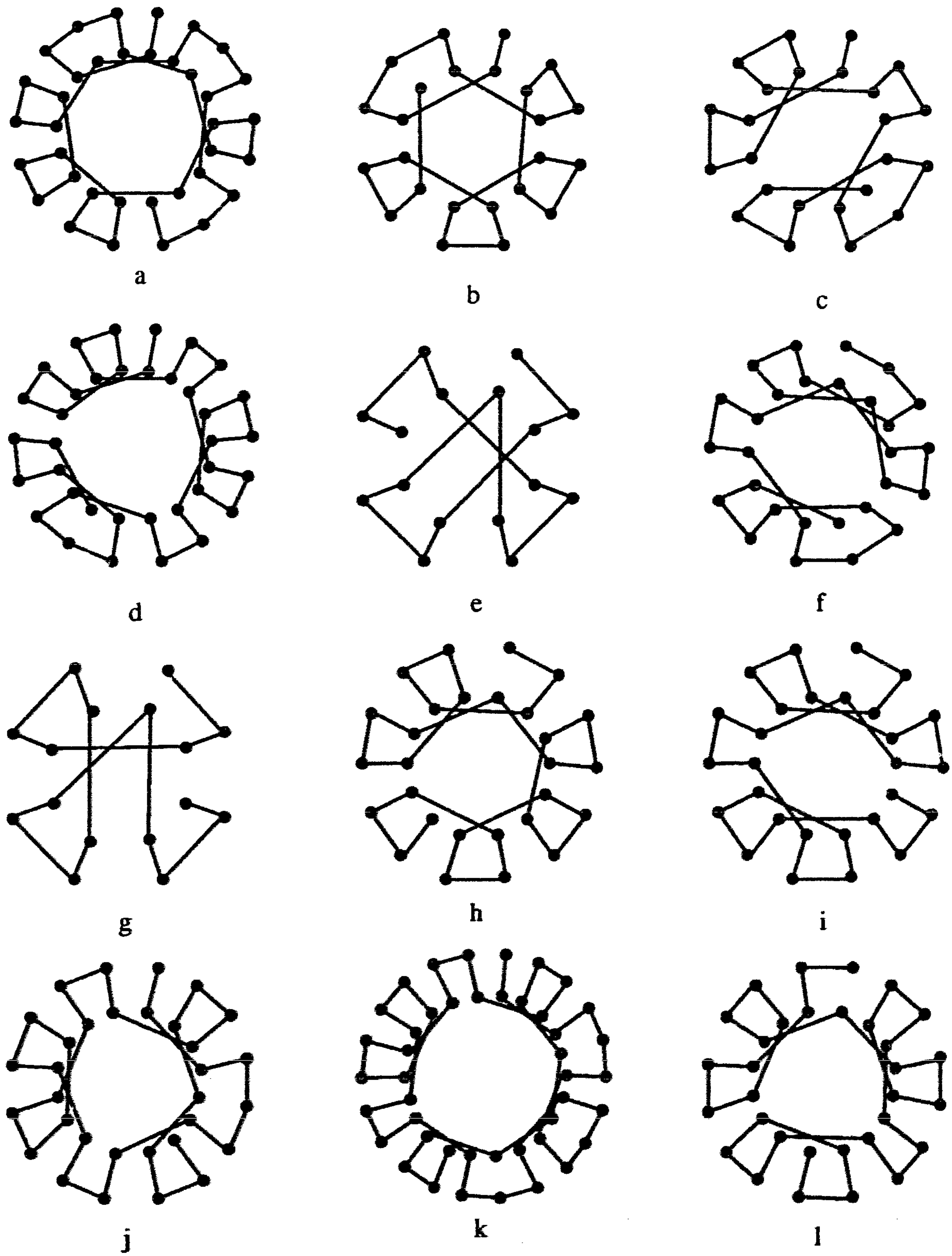


Figure 5.7: Figure 5.7a - Figure 5.7l

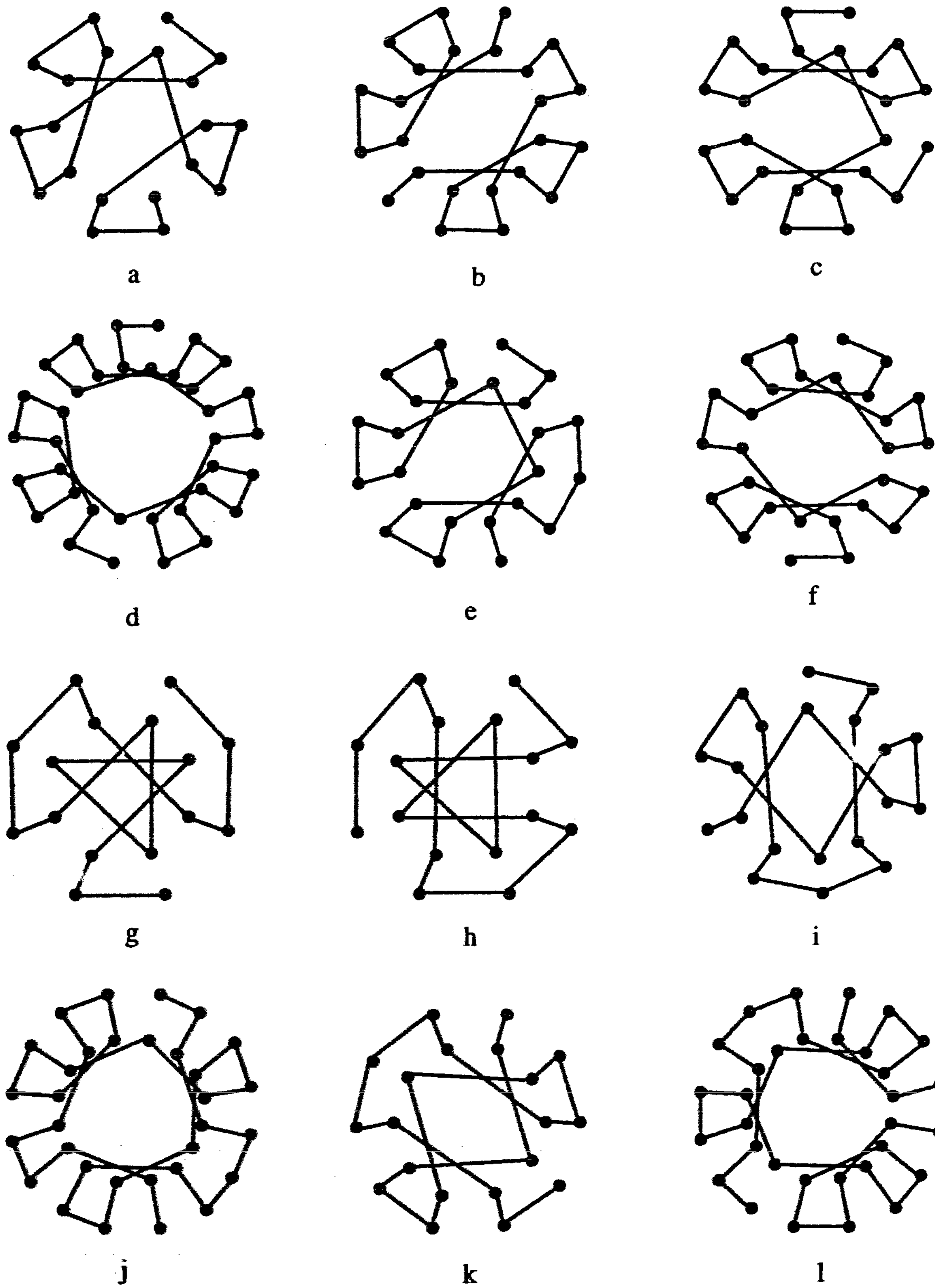


Figure 5.8: Figure 5.8a - Figure 5.8l

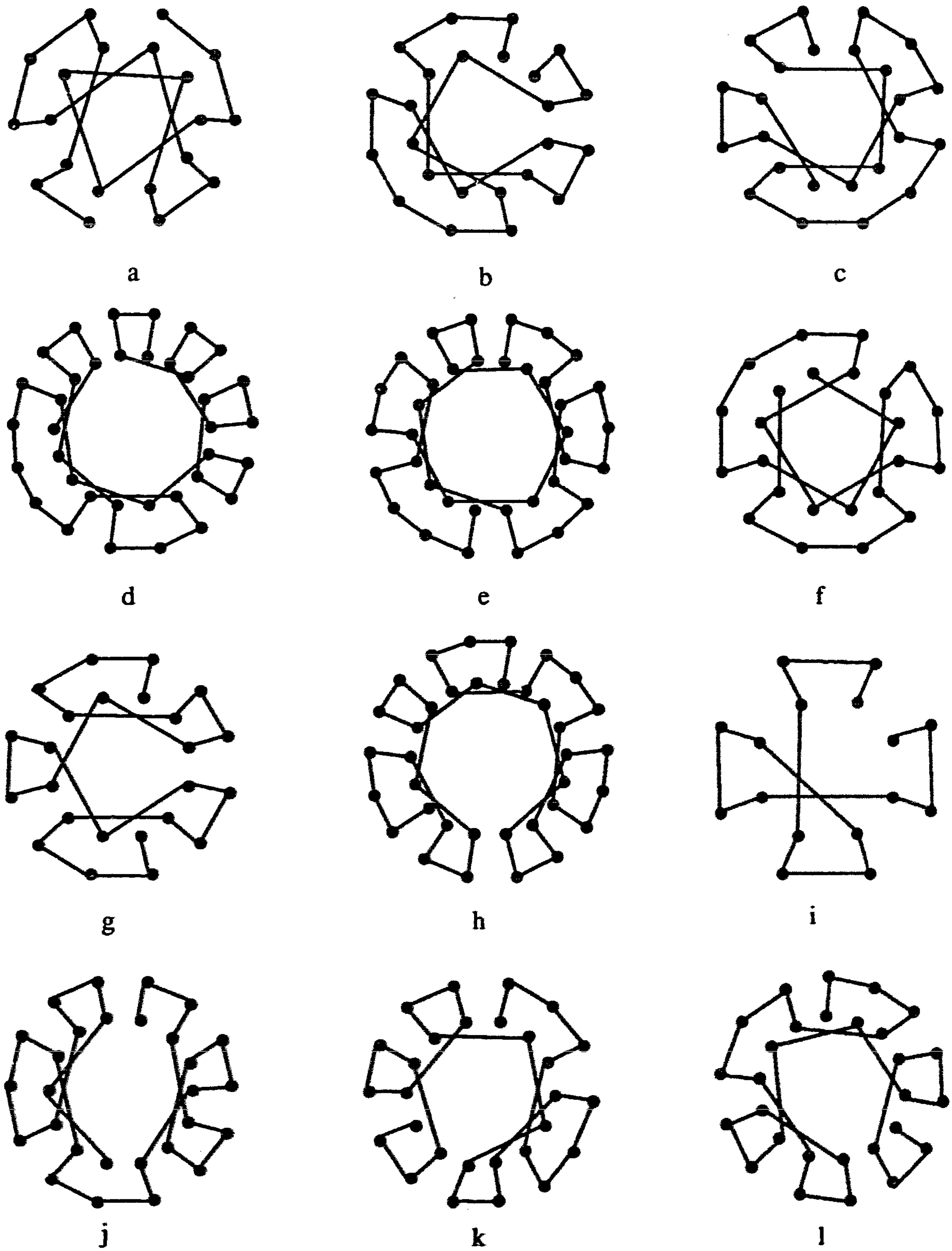


Figure 5.9: Figure 5.9a - Figure 5.9l

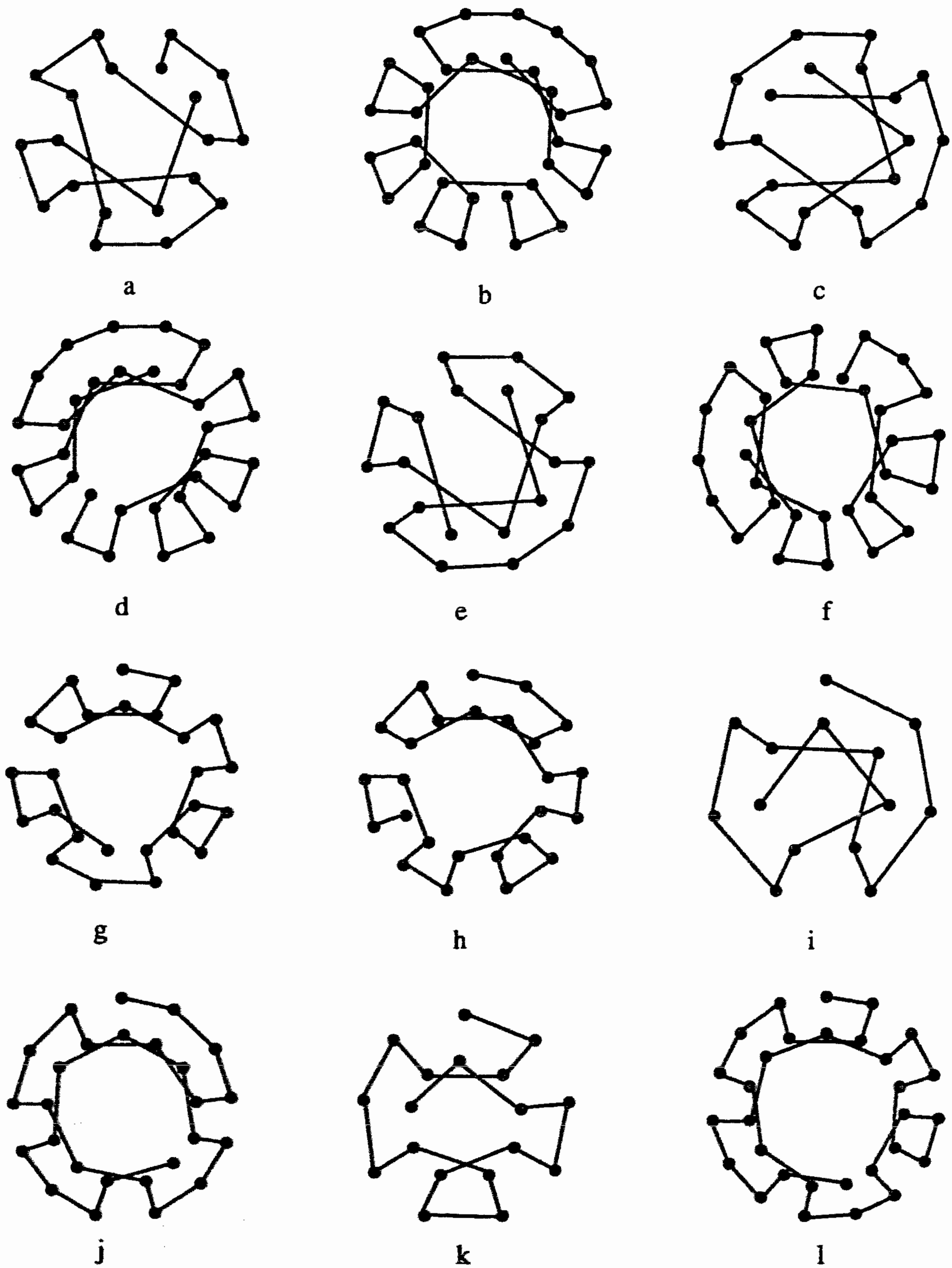


Figure 5.10: Figure 5.10a - Figure 5.10l

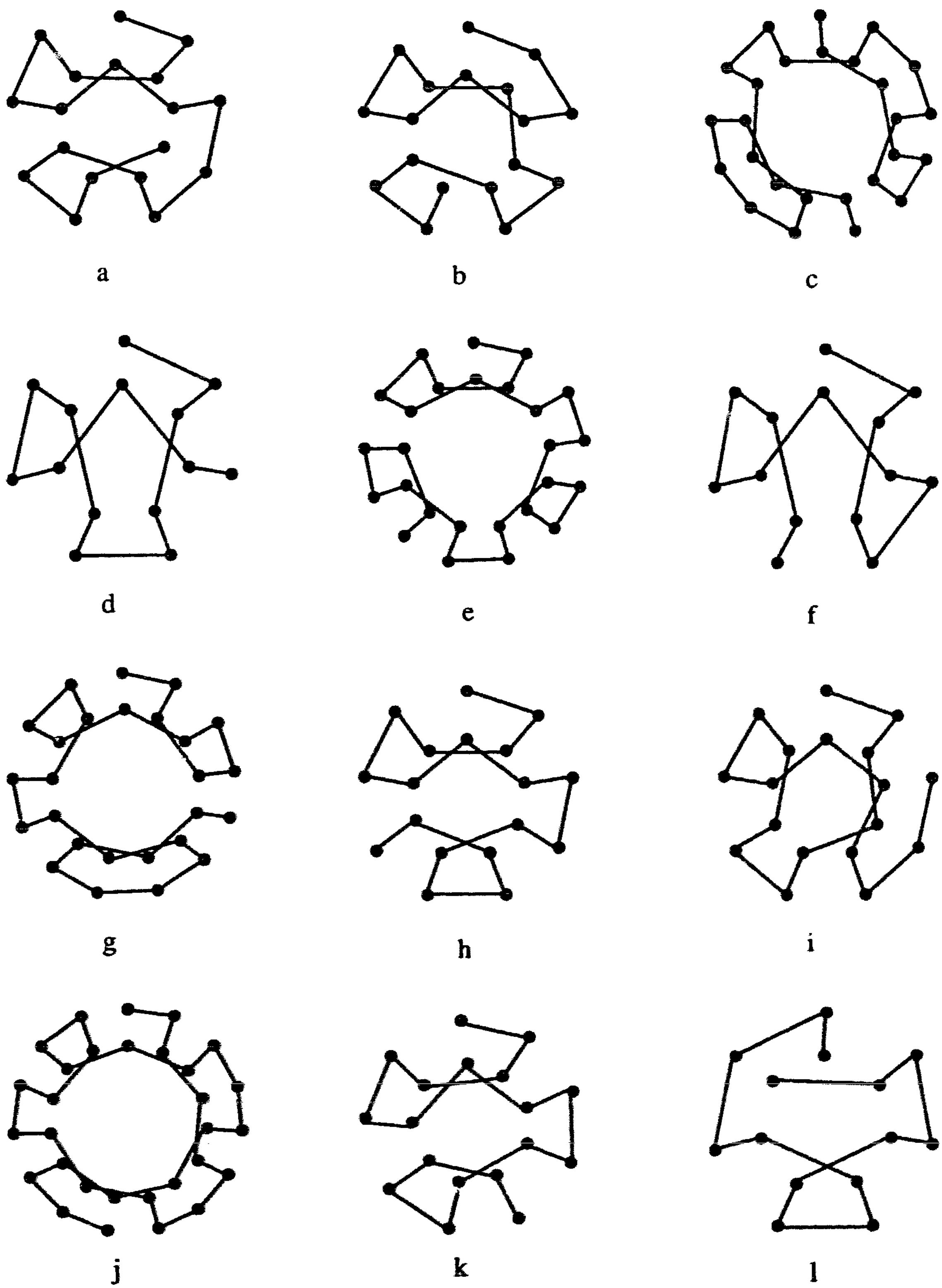


Figure 5.11: Figure 5.11a - Figure 5.11l

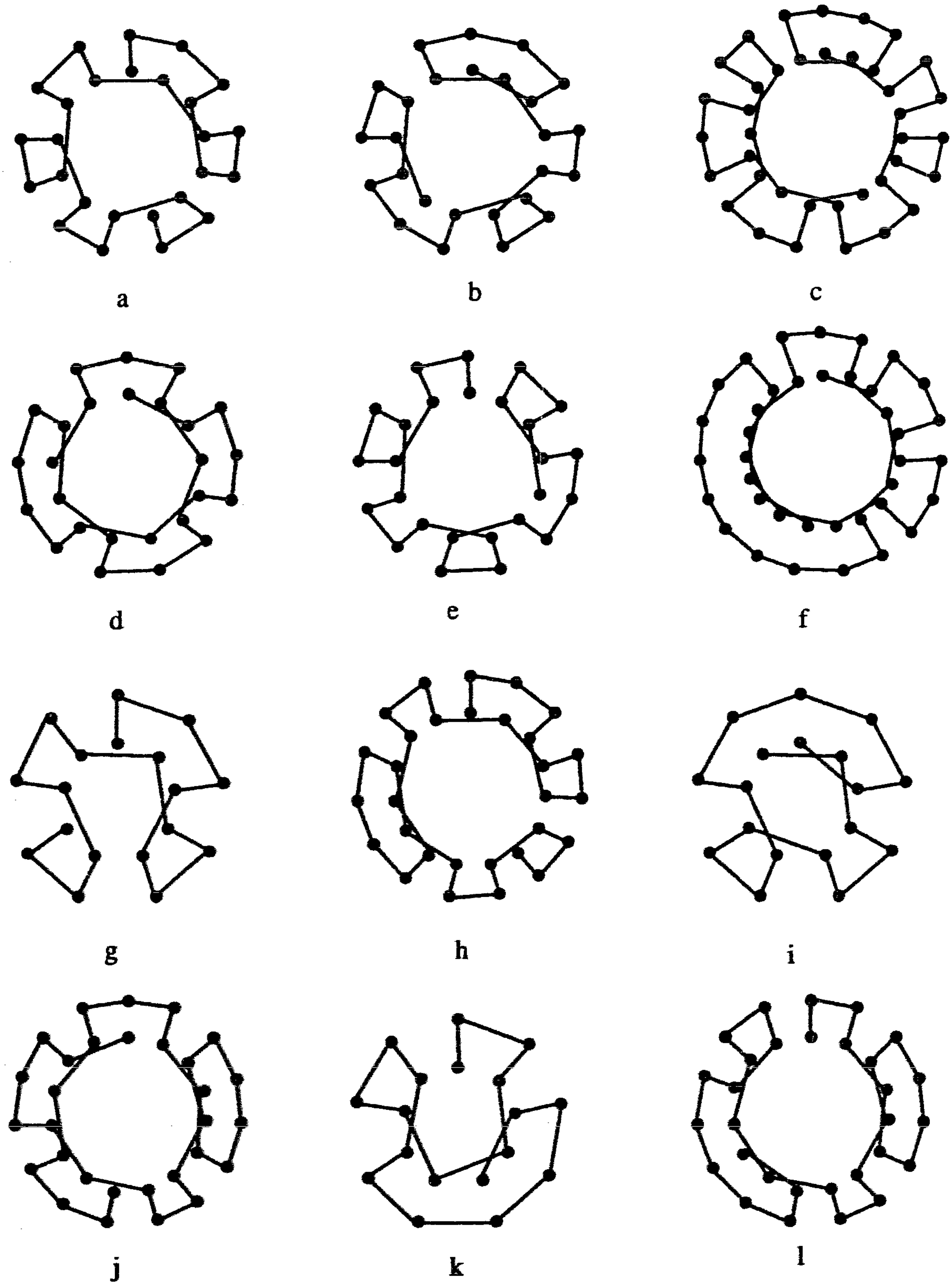


Figure 5.12: Figure 5.12a - Figure 5.12l

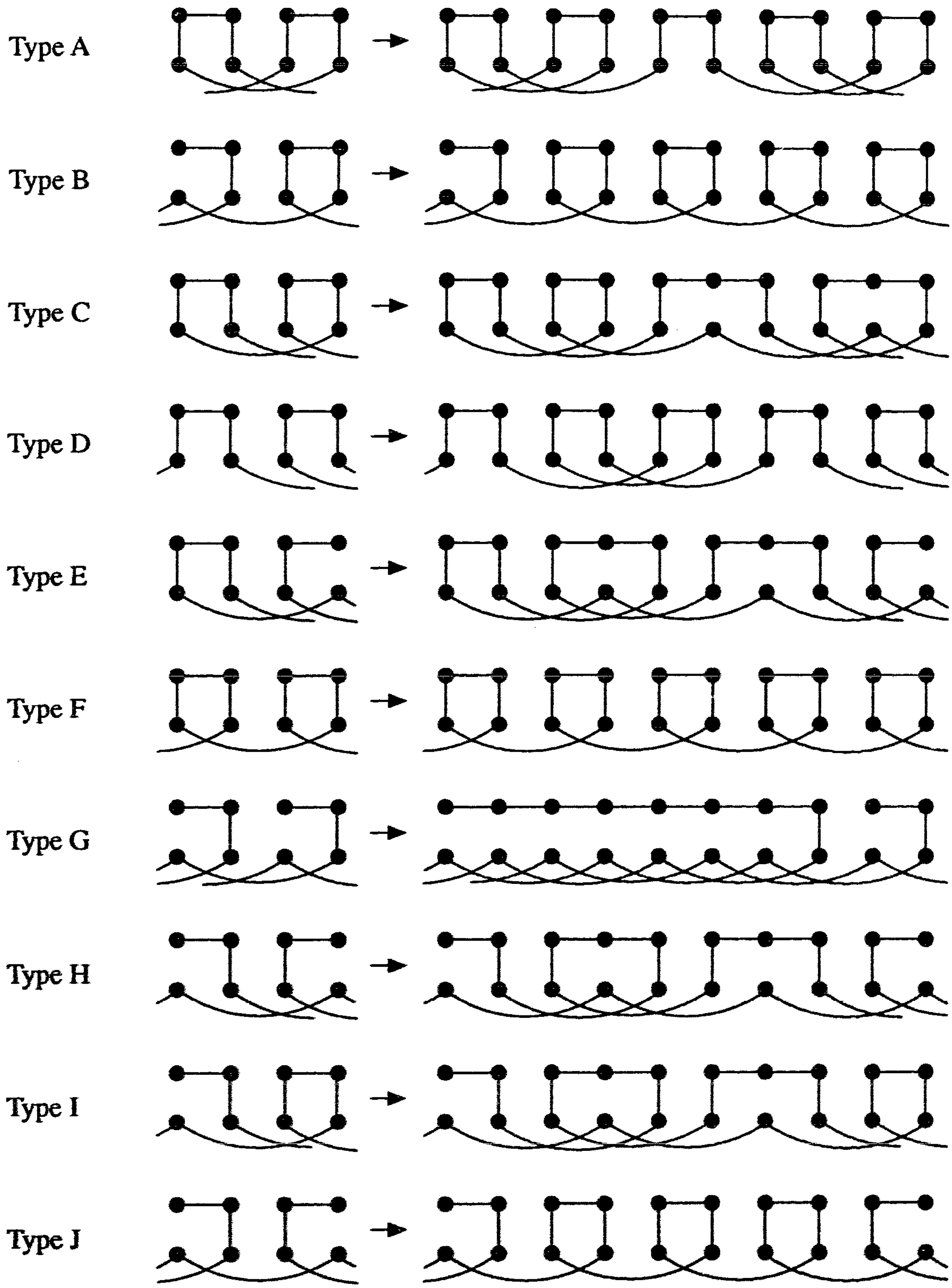


Figure 5.13: Type A - Type J

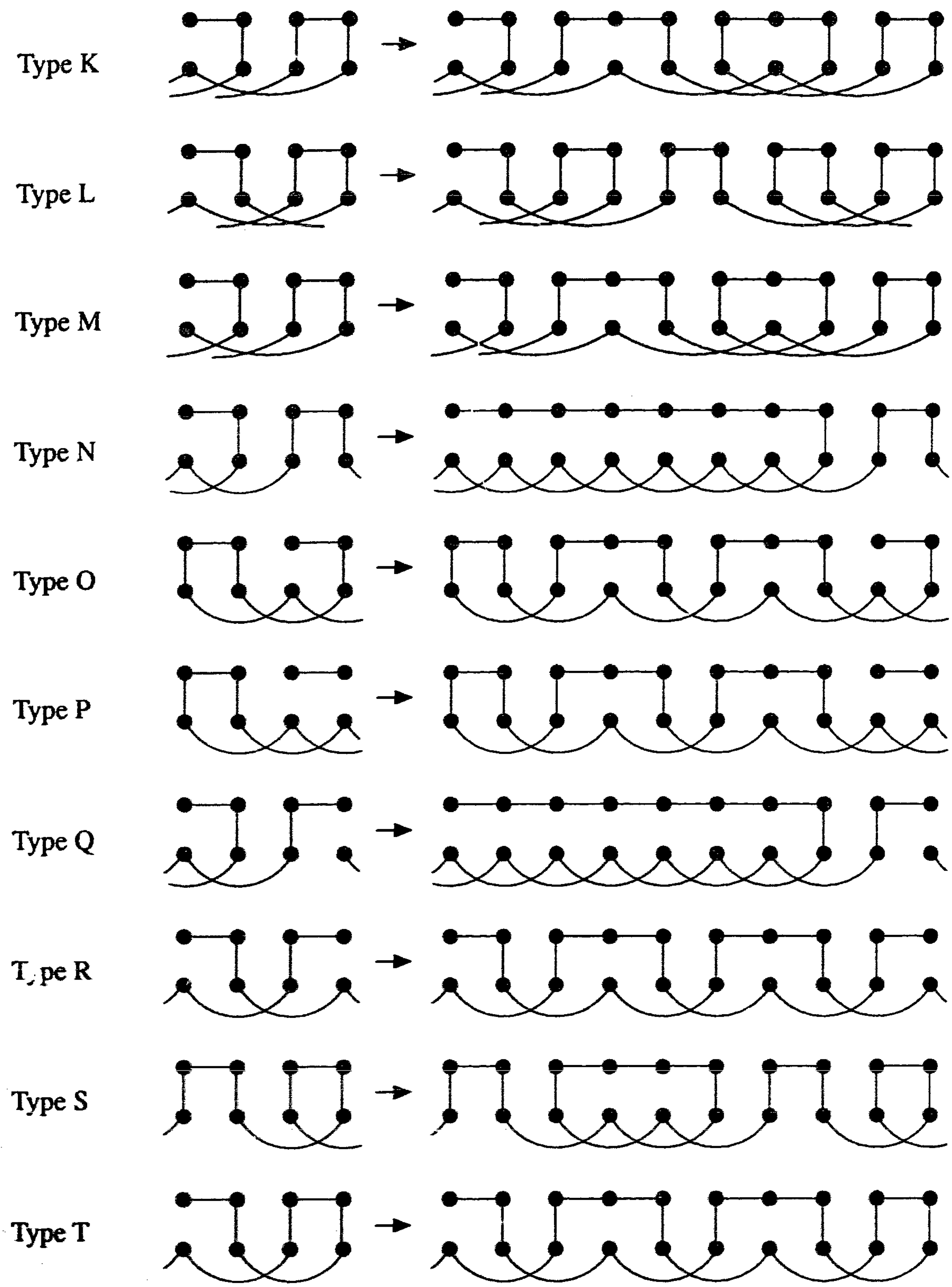


Figure 5.14: Type K - Type T

Chapter 6

Hamilton Decompositions of $C(2p, S)$

6.1 Introduction

Definition 6.1 Let G be a regular graph. It is said to have a Hamilton decomposition (or to be Hamilton-decomposable) if either

(i) $deg(G) = 2d$ and $E(G)$ can be partitioned into d Hamilton cycles, or

(ii) $deg(G) = 2d + 1$ and $E(G)$ can be partioned into d Hamilton cycles and a perfect matching.

Many known Cayley graphs on abelian groups are Hamilton-decomposable. This led Alspach [5] to ask the following question:

Does every connected Cayley graph on an abelian group have a Hamilton decomposition?

If the degree of the graph is 2, the answer is obviously yes. If the degree is 3, the answer is again yes since such a graph has a Hamilton cycle. The case of degree 4 has been solved by J-C. Bermond, O. Favaron and M. Maheo [16] and the answer is again yes. The answer is also yes for degree 5 [10]. Here we write these results as a theorem in the case when G is a circulant.

Theorem 6.1 *If $C(n, S)$ is a connected circulant of degree at most 5, then $C(n, S)$ is Hamilton-decomposable.*

The Hamilton decomposability of a graph some times depends on the Hamilton decomposability of the cartesian product of two graphs.

Definition 6.2 The cartesian product $G_1 \times G_2$ of G_1 and G_2 has vertex set $V(G_1) \times V(G_2)$ with (u_1, u_2) adjacent to (v_1, v_2) if and only if either $u_1 = v_1$ and u_2 is adjacent to v_2 in G_2 or $u_2 = v_2$ and u_1 is adjacent to v_1 in G_1.

The strongest result about the Hamilton decomposability of cartesian products was obtained by Stong [47] recently.

Theorem 6.2 *If G_1 has a decomposition into n_1 Hamilton cycles and G_2 has a decomposition into n_2 Hamilton cycles, $n_1 \leq n_2$, then $G_1 \times G_2$ has a Hamilton decomposition if any one of the following is true:*

(i) $n_2 \leq 3n_1$,

(ii) $n_1 \geq 3$,

(iii) G_1 has an even number of vertices, or

(iv) $\nu(G_2) \geq 6\frac{n_2}{n_1} - 3$.

For a general class of vertex-transitive graphs, Alspach [2] proved that every connected vertex-transitive graph of order $2p$, $p \equiv 3 (\mathrm{mod}\ 4)$ and p a prime, has a Hamilton decomposition.

It is expected that the same result holds for $p \equiv 1 (\mathrm{mod}\ 4)$ except for some special cases. In section 2, we shall show that this is true for circulants.

6.2 Main Result

Let $n = pq$, where p and q are distinct primes. Let $S_p = \{mp : mp \in S\}, S_q = \{mq : mq \in S\}$ and $S_u = \{s : s \in S, s$ is a unit of $Z_n\}$. Then $S = S_p \bigcup S_q \bigcup S_u$. Let $\frac{S_p}{p} = \{m : mp \in S_p\}$ and $\frac{S_q}{q} = \{m : mq \in S_q\}$. We have the following decomposition result.

Lemma 6.3 $C(pq, S) \cong (C(p, \frac{S_q}{q}) \times C(q, \frac{S_p}{p})) \oplus C(pq, S_u)$.

Proof. Consider $X = C(pq, S_p \cup S_q)$. We partition Z_{pq} into the left cosets of $< p >$, that is, $Z_{pq} =< p > \bigcup 1+< p > \bigcup \cdots \bigcup (p-1)+< p >$. On each coset $i+< p >$, the induced subgraph $X[i+< p >]$ is isomorphic to $C(q, \frac{S_p}{p})$, for $i = 0, 1, \ldots, p-1$.

If there is an edge between $i+< p >$ and $j+< p >$ with symbol in S_q, then there is a perfect matching between $i+< p >$ and $j+< p >$ with the same symbol. The edges with the same symbol in S_q between the cosets consist of p-cycles.

There is at most one symbol which belongs to S_q contributing to edges between $i+< p >$ and $j+< p >$. Otherwise, we will have $j-i+m_0p, j-i+m_1p \in S_q$ for some m_0 and m_1, $m_0 \neq m_1$. This implies that $j-i+m_0p = k_0q$ and $j-i+m_1p = k_1q$ for some k_0 and k_1. Therefore, $(m_0 - m_1)p = (k_0 - k_1)q$ which is a contradiction.

If we let $\{< p >, 1+< p >, \ldots, (p-1)+< p >\}$ be a vertex set and $\frac{S_q}{q}$ be a symbol set, we obtain a circulant $C(p, \frac{S_q}{q})$. Clearly,

$$C(pq, S_p \bigcup S_q) \cong C(p, \frac{S_q}{q}) \times C(q, \frac{S_p}{p}).$$

Therefore,

$$C(pq, S) \cong (C(p, \frac{S_q}{q}) \times C(q, \frac{S_p}{p})) \oplus C(pq, S_u).$$

This completes the proof. ∎

For example, $C(15, \{3, 6, 5, 12, 9, 10\}) \cong C(3, \{1, 2\}) \times C(5, \{1, 2, 3, 4\})$.

Corollary 6.4 *If p and q are odd primes, and $0 < |S_p| \leq |S_q| \leq 3|S_p|$ or $|S_p| \geq 6$, then $C(pq, S)$ has a Hamilton decomposition.*

Proof. The proof follows from Lemma 6.3 and Theorem 6.2. ∎

Theorem 6.5 *$C(2p, S)$ is Hamilton-decomposable.*

Proof. Recall that $S_p = \{mp : mp \in S\}$ and $S_2 = \{2m : 2m \in S\}$. There are two cases to consider.

Case 1. $S_p \neq \emptyset$.

In this case, we have that $S_p = \{p\}$ and $C(2, \frac{S_p}{p}) \cong K_2$. Therefore,

$$C(2p, S) \cong (K_2 \times C(p, \frac{S_2}{2})) \oplus C(2p, S_u)$$

by Lemma 6.3.

If $S_2 \neq \emptyset$, let $\frac{S_2}{2} = \{a_1, a_2, \ldots, a_m, p - a_1, \ldots, p - a_m\}$. Take an m-matching of $C(p, \frac{S_2}{2})$, say $\{(x_1, y_1), \ldots, (x_m, y_m)\}$ such that $y_i - x_i = a_i$ or $p - a_i$.

There are two parts in $K_2 \times C(p, \frac{S_2}{2})$, each of which is isomorphic to $C(p, \frac{S_2}{2})$, and there is a perfect matching between the two parts. We can label the vertices of one part by $\{x_1, x_2, \ldots, x_p\}$, and the other by $\{x_1', x_2', \ldots, x_p'\}$, such that $(x_i, x_i') \in E(K_2 \times C(p, \frac{S_2}{2}))$.

Now we can give the Hamilton decomposition as follows. Let

$$H_i = (E_{a_i} - (x_i, y_i)) \bigcup (E_{a_i}' - (x_i', y_i')) \bigcup \{(x_i, x_i'), (y_i, y_i')\}$$

for $i = 1, 2, \ldots, m$, where E_{a_i}' is the image of E_{a_i} under the prime map. Then each H_i is a Hamilton cycle of $K_2 \times C(p, \frac{S_2}{2})$, and $H_i \bigcap H_j = \emptyset$. What remains in $K_2 \times C(p, \frac{S_2}{2})$ is a perfect matching $\{(x_1, y_1), \ldots, (x_m, y_m)\} \bigcup \{(x_1', y_1'), \ldots, (x_m', y_m')\} \bigcup \{(x_i, x_i'), (y_i, y_i') : i \neq 1, 2, \ldots, m\}$.

If $S_2 = \emptyset$, then $C(2p, S) \cong E_p \oplus C(2p, S_u)$. E_p is a perfect matching of $C(2p, S)$. $C(2p, S_u)$ is Hamilton-decomposable if $S_u \neq \emptyset$. We also have that $C(2p, S)$ is Hamilton-decomposable.

Case 2. $S_p = \emptyset$.

Since $C(2p, S)$ is connected, there is at least one $a \in S_u$. The map $a^{-1} : C(2p, S) \to C(2p, a^{-1}S)$ defined by $a^{-1}(s) = a^{-1}s$ for any $s \in Z_{2p}$ is an isomorphism. So we can assume that $1 \in S$.

Let $S' = S_u - \{1, -1\}$. Then we have that $C(2p, S')$ is Hamilton-decomposable if S' is nonempty, and $C(2p, S) \cong C(2p, S_2 \bigcup \{-1, 1\}) \oplus C(2p, S')$.

Let $Y = C(2p, S_2 \bigcup \{-1, 1\})$. We partition Z_{2p} into $< 2 >$ and $1+< 2 >$. The edges with symbols in S_2 induce subcirculants on $< 2 >$ and $1+< 2 >$, both of which are isomorphic to $C(p, \frac{S_2}{2})$. The edges with symbol 1 form two 'parallel' perfect matchings between $< 2 >$ and $1+< 2 >$: one is $\{(0, 1), (2, 3), \ldots, (2p - 2, 2p - 1)\}$, denoted by M_1, and the other is $\{(2, 1), (4, 3), \ldots, (2p - 1, 0)\}$, denoted by M_{-1}.

Let $S_2 = \{b_1, b_2, \ldots, b_m, 2p - b_1, \ldots, 2p - b_m\}$, where $b_1 > b_2 \cdots > b_m$. To decompose $C(2p, S_2 \bigcup \{-1, 1\})$ into Hamilton cycles, we need to find a special matching.

Claim. There is an m-matching $M_0 = \{(x_1, y_1), \ldots, (x_m, y_m)\}$ in $X[< 2 >]$ such that

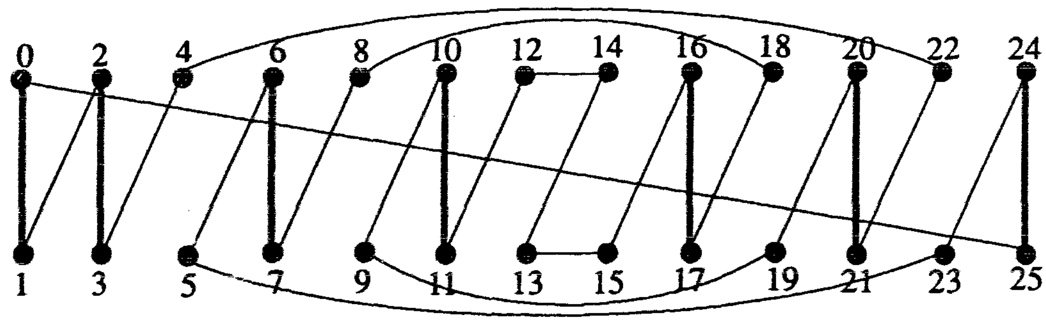


Figure 6.1:

(i) $y_i - x_i = b_i, i = 1, 2, \ldots, m$, and

(ii) $0 < x_1 < x_2 < \cdots < x_m < y_m < y_{m-1} < \cdots < y_2 < y_1$.

To prove the claim, let K_p be a complete graph with vertex set Z_p. Then $M(0) = \{(1, p-1), (2, p-2), \ldots, (\frac{p-1}{2}, \frac{p+2}{2})\}$ is a near perfect matching of K_p.

Let $2M(p) = \{(2, 2(p-1)), (4, 2(p-2)), \ldots, (p-1, p+1)\}$. Then

$$M_0 = 2M(p) \bigcap E(X[< 2 >])$$

has the required properties.

Let $H_i' = E_{b_i} \bigcap E(X[< 2 >])$. We know that $H_1', \ldots, H_m'$ is a Hamilton decomposition of $X[< 2 >]$, and $(x_i, y_i) \in H_i'$. Let $H_1'', H_2'', \ldots, H_m''$ be the corresponding Hamilton decomposition of $X[1+< 2 >]$. Note also that $(1 + x_i, 1 + y_i) \in H_i''$.

Now let

$$H_i = (H_i' - (x_i, y_i)) \bigcup (H_i'' - (1 + x_i, 1 + y_i)) \bigcup \{(x_i, 1 + x_i), (y_i, 1 + y_i)\}$$

for $i = 1, 2, \ldots, m$. Then H_i is a Hamilton cycle of $C(2p, S_2 \bigcup \{-1, 1\})$, and $H_i \bigcap H_j = \emptyset$, if $i \neq j$.

The remaining edges are

$$H_{m+1} = M_0 \bigcup (1 + M_0) \bigcup (M_1 - \{(x_1, 1 + x_1), \ldots, (x_m, 1 + x_m)\}) \bigcup M_{-1}.$$

To show that H_{m+1} is a Hamilton cycle of $C(2p, S_2 \bigcup \{-1, 1\})$, let $x' = 1 + x$. Then $C = 00'22' \ldots (x_m - 1)' x_m y_m (y_m - 1)' (y_m - 1)(y_m - 2)' \ldots y_{m-1}' x_{m-1}' (x_{m-1} + 1)(x_{m-1} +$

$1)' \ldots x_{m-2} y_{m-2} (y_{m-2} - 1)' (y_{m-2} - 1) \ldots x_i y_i y'_{i-1} y_{i-1} y'_{i-2} y_{i-2} \cdots y'_{i+1} x'_{i+1} (x_{i+1} + 1)(x_{i+1} + 1)' \ldots x_{i+2} y_{i+2} (y_{i+2} - 1)' (y_{i+2} - 1) \ldots x_2 (x_2 - 1)' (x_2 - 1) \ldots x'_1 y'_1 (y_1 + 1) \ldots (p-1)0$ is a Hamilton cycle. But $E(C) = H_{m+1}$ (see Figure 6.1) and therefore, H_{m+1} is a Hamilton cycle and hence $C(2p, S)$ is Hamilton-decomposable. This completes the proof of the theorem. ∎

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