CONTROLLED GROWTH RESTRICTION POLICIES FOR CERTAIN CLOSED FOOD-CHAIN SYSTEMS

by

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Controlled growth restriction policies for certain closed food-chain systems.

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ABSTRACT

Ecological systems of 2n interacting populations (consumers, resources) forming closed food-chain systems subjected to control are considered. The growth rate of the first population involves a constant source term and terms representing the decomposed biomass of some species which serve as a resource. The growth restriction policies for the closed food-chain systems are established through the utilization of a Lyapunov design methodology for controlled open food-chain systems, and are based on the concept of avoidance control. They are targeted towards ensuring population coexistence by restricting the fluctuations of the size of the populations in the chain to an allowable level from point of view of the participating populations (internal control) or the managers outside the system (external control). Numerical simulations for the controlled closed food-chain systems are presented to illustrate the growth restriction policies.

DEDICATION

To my parents

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INTRODUCTION

Studies, both experimental and theoretical, on the interaction between biological populations have been a topic of continuing interest in mathematical ecology. However, with the increasing threat to our environment, be it man-made or natural, there has been insufficient focus on the use of control theory to interacting populations.

We note here the works by Albrecht, Gatzke, Haddad and Nelson [1] and Goh, Leitmann and Vincent [2] using optimal control theory. The aim of these papers is to design optimal strategies for the control of prey-predator systems. The optimal strategies are implemented by alternating the control variable from the zero level to the maximum level. As a result, the response is steered from the initial state to the stable equilibrium point of the uncontrolled models.

Bojadziev and Skowronski [3] took a different approach introducing qualitative control policies for a predator-prey model (a particular case of a food-chain) of the Lotka-Volterra type

$$\begin{aligned} \dot{x}_1 &= x_1(\alpha_1 - \beta_1 x_1), \\ \dot{x}_2 &= x_2(-\alpha_2 + \beta_2 x_1) + u_2 x_2^2, \end{aligned} \qquad \textbf{(0.1)}$$

where $\dot{x}_s = dx_s/dt$, $s = 1, 2$, and the control $u_2(t)$ adjusts the number of predators so as to maintain a reasonable level of both populations.

Bojadziev [4,5,6], generalizing [3], established qualitative control policies for open food-chain systems of Lotka-Volterra type involving four or more interacting populations. In [7], Bojadziev included harvesting efforts in the model **(0.1)** and managed to obtain a similar type of control policy.

These growth restriction policies concerning control are discussed from the point of view of qualitative control. The rational is that bioeconomical systems having behaviour based on some kind of policy of qualitative nature could respond quickly to new changes of circumstances after a decision has been implemented. Qualitative control ensures flexible behaviour which is compatible with the nature of the evolutionary process.

The authors of [3 - 7] have adopted, with proper adjustments and modifications, the usage of a Lyapunov design avoidance control methodology developed by Leitmann and Skowronski [8]. The control policies in [3 - 7] have been based on the existence of a Lyapunov function for uncontrolled food-chain models.

However, for some systems in population dynamics which are of a more complicated nature than open food-chains, the Lyapunov functions are either difficult to find or they might be non-existent. A particular example is a closed food-chain system. Bojadziev [9] deals with the utilization of a Lyapunov function for open food-chains in cases where the system dynamics has a more general character in order to establish a growth restriction policy.

The main objective of this thesis is to provide a qualitative study of the effect of control on ecological systems of 2n interacting populations (consumers,resources) forming closed food-chain systems. The qualitative behaviour of the interacting closed food-chain models with control is compared to that in the absence of control. Since it is not known if closed food-chains possess a Lyapunov function, their growth restriction policies, based on the concept of avoidance control, are established using the methodology developed in Bojadziev [9]. The main target of these policies (also called ***balanced zone stabilization policies)*** is directed towards ensuring a desirable population coexistence by damping large fluctuations of population sizes. This is achieved by restricting the population growth in the chain to an allowable or manageable level from point of view of the participating populations (internal control) or from managers outside the system (external control).

The material in this thesis is divided in the following manner:

It begins with a preliminary chapter discussing some material which is necessary for the applications in later chapters. The chapter concludes with an introduction into the basic dynamics of the models under study.

The second chapter deals with equilibria, a Lyapunov function and its properties, and the stability conditions for equilibria.

Chapter three entitled Control Policies, begins with justification for the need of control. Important definitions are introduced and a theorem is proved, providing the bases for the design of the control policies.

The concluding Chapter Four is devoted to numerical simulations for the closed food-chain models, in particular of length 2 and 4, for the case of external control as well as for internal control.

At the end we present our concluding remarks and in 7 Appendices present the program for the numerical simulation, 4 tables with data and 2 proofs.

The References contain 22 titles.

CHAPTER ONE

PRELIMINARIES

This chapter discusses some of the material that is necessary for the applications in later chapters . Portions of Sections 1.2 and 1.3 have been adapted from the books by Sanchez [10], Jordan and Smith [11] and Hahn [12]. The chapter concludes with the introduction of the basic dynamics of the controlled closed food-chain models under study.

Section 1.1 The method of Runge-Kutta

We begin with some preliminaries that will be used in the later sections of this thesis. First, we will consider the numerical procedure or method used to obtain a solution to a general system of non-linear differential equations.

Ames [13] has commented in his book that there is no single numerical method that is applicable to every differential equation, much less to the smaller class of ordinary linear differential equations.

The Runge-Kutta method is one of the widely used quadrature methods for solving non-linear differential equations numerically.

Runge [14] developed this method as a means of avoiding the complications of successive differentiations while simultaneously preserving the increased accuracy furnished by the Taylor series. Later, Kutta [15] and Heun [16] improved on it.

We remark that the Runge-Kutta method is a single-step or one-step method in which the solution of the differential equation

$$\frac{dy}{dt} = f(t,y), \qquad y(t_o) = y_o, \qquad t \in [t_o,t_f], \qquad \textbf{(1.1)}$$

is approximated by calculating the solution of a related first order difference equation. As a result, as in any single-step method, we can approximate **(1.1)** in the form

$$y_{n+1} = y_n + h\Phi(t_n,y_n,h), \qquad n = 0,1,2,... \qquad \textbf{(1.2)}$$

where Φ is some reasonable function called the increment function and h is the step size. The true value of $y(t_n)$ satisfies

$$y(t_{n+1}) = y(t_n) + h\Phi(t_n,y_n,h) + T_n, \qquad n = 0,1,2, \qquad \textbf{(1.3)}$$

where T_n is the truncation error. The order of the single-step method is given by the largest integer p such that $|h^{-1}T_n| = O(h^p)$.

The general idea of the Runge-Kutta approach is to find the slope of the given function at t_n and at several other points; average these slopes, multiply by the step size h and add the result to y_n. Thus the Runge-Kutta method with m slopes can be written as

$$K_i = hf(t_n + c_i h,\, y_n + \sum_{j=1}^{i-1} a_{ij} K_j), \qquad c_1 = 0,\, i = 1,2,\ldots,m \qquad \textbf{(1.4)}$$

and

$$y_{n+1} = y_n + \sum_{i=1}^{m} w_i K_i\,, \qquad m = 1,2,.. \qquad \textbf{(1.5)}$$

where $n \in Z_0^+$ and the parameters $c_2,\ldots, c_m$, $m \geq 2$, $a_{2,j},\ldots, a_{m,(m-1)}$ and w_i are arbitrary with the sum of $w_i = 1$, for $i = 1,\ldots, m$. According to the suggestions by Kutta [15], we may choose the increment function Φ to be the linear combination of the slopes at t_n and at several points between t_n and t_{n+1}. The specific values for the parameters are obtained by expanding y_{n+1} in powers of the step size h and comparing it with the Taylor series expansion of the solution of the differential equation to a specified number of terms. These coefficients have been extensively calculated in numerous texts on numerical analysis (see Ames [13], Hilderbrand [17]) and its procedure will not be illustrated here.

Jain [18] has outlined the Runge-Kutta methods of different orders. However, these different orders involve a similar general procedure in obtaining a solution as outlined above. The main difference is that the low-order methods produce less accurate results as compared to their high-order counterparts. Furthermore, a high-order method as well as a diminishing step size results in an increase in accuracy of the solution.

In Chapter four, we will make use of the fourth-order Runge-Kutta method (this being the most common of all Runge-Kutta methods) to obtain numerical solutions for our system of differential equation.

In concluding this section, we will use the fact that a system of m first order initial value problems can be dealt with by a similar procedure as any m^{th} order initial value problem.

Consider a system of n differential equations of the form

$$\begin{aligned} \dot{x}_1 &= f_1(t,x_1,\dots,x_n), \\ \dot{x}_2 &= f_2(t,x_1,\dots,x_n), \\ &\vdots \\ \dot{x}_n &= f_n(t,x_1,\dots,x_n), \end{aligned} \tag{1.6}$$

with initial conditions

$$\mathbf{x}(t_o) = (x_{o1}(t_o), x_{o2}(t_o), \dots, x_{on}(t_o))^T. \tag{1.7}$$

Then, we can write **(1.5)** as

$$\mathbf{x}_{i+1} = \mathbf{x}_i + \frac{1}{6}\sum_{s=1}^{4} w_s K_s\,, \tag{1.8}$$

with coefficients

$$\begin{aligned} w_s &= 1, && \text{for } s = 1, 4, \\ w_s &= 2, && \text{for } s = 2, 3, \end{aligned} \tag{1.9a}$$

and with

$$K_s = (K_{1s}, K_{2s}, \ldots, K_{ns})^T, \qquad \text{for } s = 1, \ldots, 4, \tag{1.9b}$$

where

$$\begin{aligned}
K_{j1} &= hf_j(t_i, x_{1i}, x_{2i}, \ldots, x_{ni}), \\
K_{j2} &= hf_j(t_i + \frac{1}{2}h, x_{1i} + \frac{1}{2}K_{11}, x_{2i} + \frac{1}{2}K_{21}, \ldots, x_{ni} + \frac{1}{2}K_{n1}), \\
K_{j3} &= hf_j(t_i + \frac{1}{2}h, x_{1i} + \frac{1}{2}K_{12}, x_{2i} + \frac{1}{2}K_{22}, \ldots, x_{ni} + \frac{1}{2}K_{n2}), \\
K_{j4} &= hf_j(t_i + h, x_{1i} + K_{13}, x_{2i} + K_{23}, \ldots, x_{ni} + K_{n3}), \qquad j = 1, \ldots, n.
\end{aligned} \tag{1.10}$$

As a final remark about the Runge-Kutta method, we observe that the advantages of using the Runge-Kutta method are that

a) it is self-starting,

b) it provides approximations which converge to the solution as $h \rightarrow 0$, and

c) the method is iterative.

Its disadvantage is that the method involves far more computations per step than other finite difference method.

Section 1.2 Classical investigation of an equilibrium point

In this section, we review the classical methods for the investigation of the topological structure of an equilibrium point $E(x_1^o, x_2^o)$ of the two-dimensional system

$$\frac{dx_1}{dt} = P(x_1, x_2) \quad , \quad \frac{dx_2}{dt} = Q(x_1, x_2) \qquad \textbf{(1.11)}$$

where P and Q are known scalar functions and together with their first partial derivatives, are continuous in some domain G of the x_1x_2-plane.

Definition 1.1. An equilibrium point of **(1.11)** is any point (x_1^o, x_2^o) in G at which P and Q both vanish. That is,

$$P(x_1^o, x_2^o) = 0 \qquad \text{and} \qquad Q(x_1^o, x_2^o) = 0. \qquad \textbf{(1.12)}$$

These are points where the motion described by **(1.11)** is in a state of rest.

Let us consider the case of an isolated simple equilibrium point E, that is, an equilibrium point having no other equilibrium points in its neighbourhood. Without loss of generality, it is assumed that the equilibrium under study is at the origin O(0,0). The simple change of variables $x_1 \to x_1^o + x_1$; $x_2 \to x_2^o + x_2$ ensures this. System **(1.11)** can then be written as

$$\frac{dx_1}{dt} = a_1x_1 + a_2x_2 + F(x_1, x_2) \,,$$
$$\frac{dx_2}{dt} = c_1x_1 + c_2x_2 + G(x_1, x_2) \,, \qquad \textbf{(1.13)}$$

where a_s and c_s, s = 1, 2, denotes the partial derivatives of P and Q with respect to x_s, s = 1, 2, at the equilibrium point $E(x^o)$ respectively. The functions F and G are of class C^k, $k \geq 1$, defined in some closed bounded region G*. At this point O(0,0), F and G, together with their partial derivatives, vanish.

The Jacobian matrix, denoted by J, of the linear part of system **(1.13)** at the equilibrium point O is given by

$$J = \begin{bmatrix} a_1 & a_2 \\ c_1 & c_2 \end{bmatrix}. \qquad \textbf{(1.14)}$$

Since O(0,0) is a simple equilibrium point,

$$\Delta = |J| = \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} \neq 0. \qquad \textbf{(1.15)}$$

We make use of the characteristic equation of **(1.14)**

$$|J - \lambda I| = \lambda^2 - (a_1+c_2)\lambda + (a_1c_2 - a_2c_1) = 0, \qquad \textbf{(1.16)}$$

and the eigenvalues of **(1.14)** to investigate the nature of the stability of the equilibrium point O(0,0). The roots of **(1.16)** are given by

$$\lambda_{1,2} = \frac{\sigma}{2} \pm \frac{1}{2}\sqrt{\sigma^2 - 4\Delta}\ , \qquad \textbf{(1.17)}$$

where $\sigma = a_1 + c_2$ and $\Delta = a_1c_2 - a_2c_1$. Depending on the sign of the discriminant D where

$$D = \sigma^2 - 4\Delta, \qquad \textbf{(1.18)}$$

several cases concerning the stability of the equilibrium point O(0,0) can be obtained as illustrated in Table 1.

Case	Roots λ_1, λ_2	Equilibrium	Stability
$D = 0$	$\lambda_s < 0; \lambda_s \in R$ $s = 1, 2$	Node	Stable if $\sigma < 0$
	$\lambda_s > 0; \lambda_s \in R$ $s = 1, 2$		Unstable if $\sigma > 0$
$D < 0$	Complex roots with $Re(\lambda_s) < 0$ $s = 1,2.$	Focus if $\sigma \neq 0$	Stable if $\sigma < 0$
	Complex roots with $Re(\lambda_s) > 0$ $s = 1,2.$		Unstable if $\sigma > 0$
	λ_s pure imaginary $s = 1, 2$	Centre if $\sigma = 0$	
$D > 0$	$\lambda_s < 0; \lambda_s \in R$ $s = 1, 2$ Distinct	Node if $\Delta > 0$	Stable if $\sigma < 0$
	$\lambda_s > 0; \lambda_s \in R$ $s = 1, 2$ Distinct		Unstable if $\sigma > 0$
	$\lambda_s \in R$ $s = 1, 2$ Distinct and of opposite signs.	Saddle if $\Delta < 0$	

Table 1.

Section 1.3. The Lyapunov function

In this section we will introduce the Lyapunov Direct Method also called the 2^{nd} method of Lyapunov.

The Lyapunov Direct Method is one of the most powerful tools in the study of dynamical systems. It makes statements on the stability of the equilibrium without any knowledge of the solutions of the differential equation. The stability statements are made directly by using, in addition to the differential equation, suitable functions defined in the phase space. Such functions are known as Lyapunov functions and in general, the sign of the Liapunov function and its time derivative for the differential equation have to be taken into consideration.

We recall the definition and two important theorems of a Lyapunov function (see Jordan and Smith [11], Hahn [12] and Huang and Morowitz [19]).

Definition 1.2 A ***Lyapunov function*** $V(x_1,...,x_n)$ is a scalar function which satisfies the following conditions:

(i) $V(x_1,...,x_n)$ together with its first partial derivatives are continuous in a certain open domain Ω around the origin $x_1 = x_2 = ... = x_n = 0$.

(ii) V is non-negative in the domain Ω and vanishes only at the origin.

Lyapunov formulated two stability theorems on equilibria whose proofs are not presented here but can be found in Jordan and Smith [11] (Chapter 10).

Theorem 1.1 (Weak Lyapunov function) [11] Let $\mathbf{x}^*(t) = \mathbf{0}$, $t \geq t_o$, be the zero solution or equilibrium of the regular system $\dot{\mathbf{x}}(t) = \mathbf{X}(\mathbf{x})$, where $\mathbf{X}(\mathbf{0}) = \mathbf{0}$. Then $\mathbf{x}^*(t)$ is ***uniformly stable*** for $t \geq t_o$ if there exist $V(\mathbf{x})$ with the following properties in some neighbourhood ϑ^* of $\mathbf{x} = \mathbf{0}$:

(i) $V(\mathbf{x})$ and its partial derivatives are continuous,

(ii) $V(\mathbf{0}) = 0$ and $V(\mathbf{x}) > 0$ for all $\mathbf{x} \neq 0$ in $\Im$ in a neighbourhood $\Im$ of the origin,

(iii) $V(\mathbf{0}) = 0$ and $dV(\mathbf{x})/dt \leq 0$ for all $\mathbf{x} \neq 0$ in $\Im$ in a neighbourhood $\Im$ of the origin.

Theorem 1.2 (Strong Lyapunov function) [11] Let all the conditions of Theorem 1.1 apply, except that (iii) is replaced by

(iii) $V(\mathbf{0}) = 0$ and $dV(\mathbf{x})/dt < 0$ for all $\mathbf{x} \neq 0$ in $\Im$ in a neighbourhood $\Im$ of the origin.

Then the zero solution or equilibrium is ***asymptotically stable***.

Theorems 1.1 and 1.2 can be viewed from a geometrical perspective. Suppose that the Lyapunov function $V(\mathbf{x})$ is positive definite, that is, $V(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$ in a neighbourhood $\Im$, and $V(\mathbf{0}) = 0$. Then the level curves, $V(\mathbf{x}) = c$, a constant parameter, form a topological system consisting of a family of simple closed curves enclosing the neighbourhood of the origin. As $c \to \infty$, the further the curves are from the origin and as $c \to 0$, the origin is approached.

The geometrical interpretations of the level curves cannot be visualized in higher dimensions, however, when given a positive definite function $V(\mathbf{x})$ with continuous partial derivatives, then for all small enough positive c and for $\mathbf{x}$ in some sufficiently small neighbourhood of the origin, a similar result is achieved.

Figures 1 and 2 show a positive definite function V in R^3 and the family of level curves $V(x,y) = c$ in R^2 respectively.

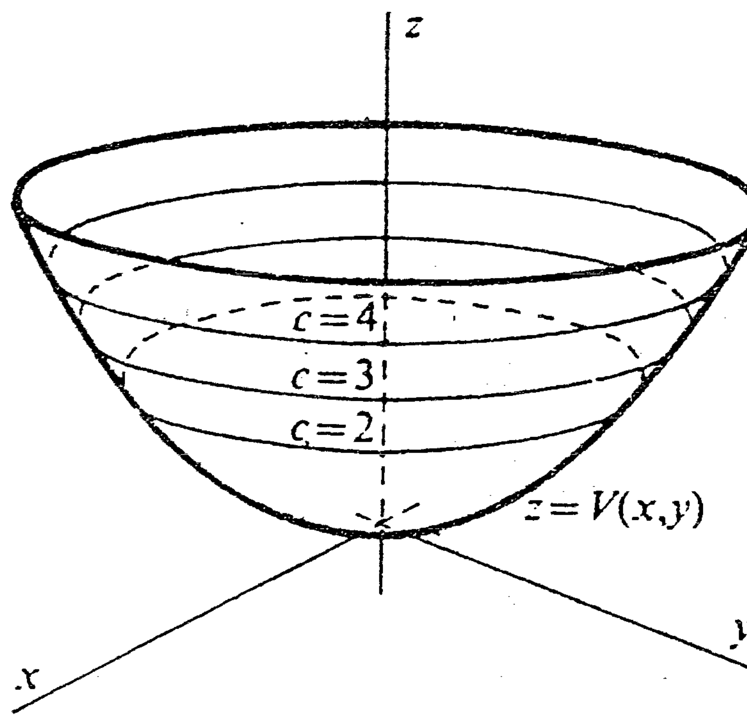

Figure 1. A positive definite Lyapunov function in R^3.

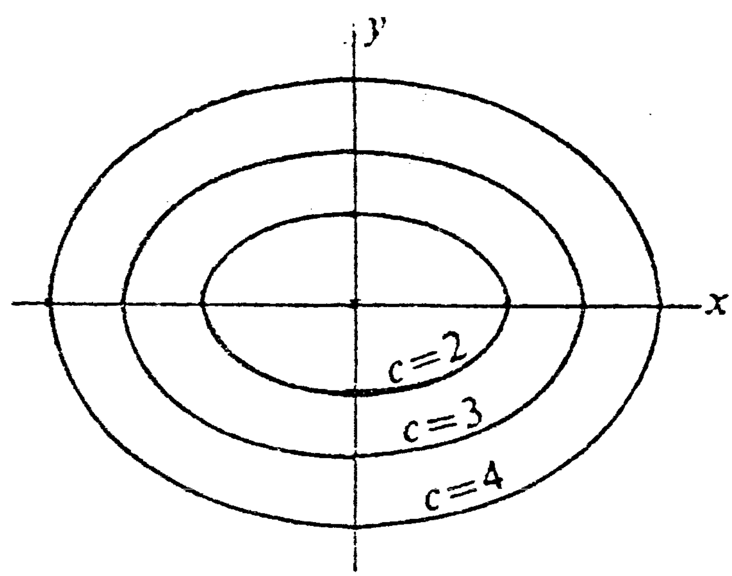

Figure 2. Family of level curves in R^2.

Section 1.4 Dynamics of the model

The dynamics of the general class of closed food-chain models of length 2n, the number of interacting populations, with control is modelled by the following equations

$$
\begin{aligned}
\dot{x}_1(t) &= Q - \frac{\beta_1}{\gamma_1} x_1 x_2 + \sum_{s=2}^{2n} (\alpha_s a_s x_s) + \varphi_1(\mathbf{u},\mathbf{x}), \\
\dot{x}_{2s}(t) &= x_{2s}(-\alpha_{2s} + \frac{\beta_{2s-1}}{\gamma_{2s}} x_{2s-1} - \frac{\beta_{2s}}{\gamma_{2s}} x_{2s+1}) + \varphi_{2s}(\mathbf{u},\mathbf{x}), \qquad \mathbf{(1.19)} \\
\dot{x}_{2s+1}(t) &= x_{2s+1}(-\alpha_{2s+1} + \frac{\beta_{2s}}{\gamma_{2s+1}} x_{2s} - \frac{\beta_{2s+1}}{\gamma_{2s+1}} x_{2s+2}) + \varphi_{2s+1}(\mathbf{u},\mathbf{x}), \\
\dot{x}_{2n}(t) &= x_{2n}(-\alpha_{2n} + \frac{\beta_{2n-1}}{\gamma_{2n}} x_{2n-1}) + \varphi_{2n}(\mathbf{u},\mathbf{x}), \qquad s = 1,\dots, n-1.
\end{aligned}
$$

Here t is the time variable and $\mathbf{x}(t) = (x_1,\dots,x_{2n})^T$ is the population vector. The parameters α_s, a_s, $s = 2,\dots, 2n$, β_s, $s = 1,\dots, 2n-1$, γ_s, $s = 1,\dots, 2n$, and Q are positive constants. They have the following biological meaning: α_s is the growth rate constant representing the growth when the s^{th} species is isolated in an environment free from other species; β_s is the interaction coefficient which measures the intensity or strength of the interaction between species; γ_s is the trophic weight factor where $\gamma_{2s+1}/\gamma_{2s}$ represents the gain-loss ratio when population 2s+1 interacts with population 2s (a predator-prey type interaction), and Q is the supply rate of an external resource. The term $\alpha_s a_s x_s$, $s = 2,\dots, 2n$ represents the dead biomass of the s^{th} species which serves as a resource for the first species x_1; hence $0 \le a_s \le 1$. The function $\varphi_s(\mathbf{u}(t),\mathbf{x}(t)) \in R^{2n}$ is a known continuous function. The control in vector form is denoted by $\mathbf{u}(t) = (u_1,\dots,u_{2n})^T$ where the vector $\mathbf{u}(t) \in U \subset R^{2n}$, U is a constraint set to be determined later. It is assumed that $\mathbf{u}(t)$ is piecewise constant on $[t_o,t_\infty)$, $t_\infty < +\infty$. Each choice of control, say $\mathbf{u}(t_o)$, a constant on some interval $[t_o,t_f]$, a subset of $[t_o,t_\infty)$, generates a solution or a response of **(1.19)** which is denoted by $\mathbf{x}(t) = \mathbf{x}(t,\mathbf{x}(t_o),\mathbf{u}(t_o))$ where $\mathbf{x}(t_o)$ is the initial state. We note that by solutions, we mean absolutely continuous functions that satisfy **(1.19)** almost everywhere on $[t_o,t_\infty)$ (see for example Roxin [20] and Fillipov [21]).

Each type of control presents a different meaning for the scalar function $\varphi(\mathbf{u},\mathbf{x})$ in **(1.19)**. Consider the s^{th} component of φ in **(1.19)**. If $\varphi_s > 0$, then the growth of the s^{th} population (consumer) with population density x_s is enhanced (increasing returns). For $\varphi_s < 0$, its growth is damped (diminishing returns). That is, we have a situation where the growth of the s^{th} species becomes so large that it begins to hinder its own growth, for example by competing with each other for limited resources. The specific way in which we select φ determines the nature of the control of our system.

For the case of an externally controlled system, for example managers outside the system, we have chosen

$$\varphi(\mathbf{u}(t),\mathbf{x}(t)) = (u_1,\ldots,u_{2n})^T = \mathbf{u}(t), \qquad n = 1,2,\ldots, \qquad \mathbf{(1.20)}$$

and for the internally controlled system, we have selected

$$\varphi(\mathbf{u}(t),\mathbf{x}(t)) = (0,u_2x_2^2,\ldots,u_{2n}x_{2n}^2)^T, \qquad n = 1,2,\ldots. \qquad \mathbf{(1.21)}$$

Here, the s^{th} population, considered as a consumer (predator) by controlling its own growth, affects the growth of the populations in the closed food-chain. This control is exhibited by all members of the chain with the exception of the first with density x_1, which is considered as a resource.

For $u_s = 0$, $s = 1,.., 2n$, $\varphi_s(\mathbf{u},\mathbf{x}) = 0$, hence **(1.19)** reduces to an uncontrolled food-chain model of closed type, a particular case of uncontrolled models investigated by Svirezhev and Logofet [22].

Following Bojadziev [9], we add and subtract the term $\alpha_1 x_1$ to Q in the first equation of **(1.19)** which gives

$$\begin{aligned}
\dot{x}_1(t) &= x_1(\alpha_1 - \frac{\beta_1}{\gamma_1} x_2) + W + \varphi_1(\mathbf{u},\mathbf{x}), \\
\dot{x}_{2s}(t) &= x_{2s}(-\alpha_{2s} + \frac{\beta_{2s-1}}{\gamma_{2s}} x_{2s-1} - \frac{\beta_{2s}}{\gamma_{2s}} x_{2s+1}) + \varphi_{2s}(\mathbf{u},\mathbf{x}), \qquad \textbf{(1.22)} \\
\dot{x}_{2s+1}(t) &= x_{2s+1}(-\alpha_{2s+1} + \frac{\beta_{2s}}{\gamma_{2s+1}} x_{2s} - \frac{\beta_{2s+1}}{\gamma_{2s+1}} x_{2s+2}) + \varphi_{2s+1}(\mathbf{u},\mathbf{x}), \\
\dot{x}_{2n}(t) &= x_{2n}(-\alpha_{2n} + \frac{\beta_{2n-1}}{\gamma_{2n}} x_{2n-1}) + \varphi_{2n}(\mathbf{u},\mathbf{x}), \qquad s = 1,\dots, n-1,
\end{aligned}$$

where

$$W = Q - \alpha_1 x_1 + \sum_{s=2}^{2n} a_s \alpha_s x_s. \qquad \textbf{(1.23)}$$

We remark that the coefficient $\alpha_1 > 0$ in **(1.22)** and **(1.23)** represents the growth rate of the first population in the open food-chain.

The model **(1.22)** with $W = 0$ represents the controlled open food-chain investigated in [5]. If we consider $\varphi = \mathbf{0}$ and $W = 0$ in **(1.22)**, we obtain the uncontrolled open food-chain

$$\begin{aligned}
\dot{x}_1(t) &= x_1(\alpha_1 - \frac{\beta_1}{\gamma_1} x_2), \\
\dot{x}_{2s}(t) &= x_{2s}(-\alpha_{2s} + \frac{\beta_{2s-1}}{\gamma_{2s}} x_{2s-1} - \frac{\beta_{2s}}{\gamma_{2s}} x_{2s+1}), \qquad \textbf{(1.24)} \\
\dot{x}_{2s+1}(t) &= x_{2s+1}(-\alpha_{2s+1} + \frac{\beta_{2s}}{\gamma_{2s+1}} x_{2s} - \frac{\beta_{2s+1}}{\gamma_{2s+1}} x_{2s+2}), \\
\dot{x}_{2n}(t) &= x_{2n}(-\alpha_{2n} + \frac{\beta_{2n-1}}{\gamma_{2n}} x_{2n-1}), \qquad s = 1,\dots, n-1.
\end{aligned}$$

In Chapter Two, we will review some properties of the open food-chain model **(1.24)**, as discussed in [5,6], concerning the equilibria and the Lyapunov function.

CHAPTER TWO

EQUILIBRIA AND LYAPUNOV FUNCTION OF THE CLOSED FOOD-CHAIN MODEL

Chapter two studies the equilibria and a Lyapunov function of the uncontrolled open food-chain model in addition to the equilibria of the uncontrolled closed food-chain model. It studies the properties of the Lyapunov function and investigates the nature and stability of the equilibria using the techniques of Sections 1.2 and Sections 1.3.

Section 2.1. Equilibrium of the open food-chain in R^{2n}, $n \geq 1$

In Section 1.2, we provided the definition of an equilibrium point and showed how this point could be found in R^2.

In a similar manner, we can generalize this definition to the case of R^{2n} for $n \geq 1$.

Consider our open food-chain system **(1.24)**. By setting $\dot{x}(t) = \mathbf{0}$ in **(1.24)** and solving the equations obtained, we find the coordinates of the nontrivial equilibrium $E(\mathbf{x}^o) \in R^{2n}$ as given below:

$$\overset{o}{x}_2 = \frac{\alpha_1\gamma_1}{\beta_1}, \qquad \overset{o}{x}_{2n-1} = \frac{\alpha_{2n}\gamma_{2n}}{\beta_{2n-1}},$$

(2.1)

$$\overset{o}{x}_{2s-1} = \frac{\alpha_{2s}\gamma_{2s} + \beta_{2s}\overset{o}{x}_{2s+1}}{\beta_{2s-1}}, \qquad \overset{o}{x}_{2s+2} = \frac{-\alpha_{2s+1}\gamma_{2s+1} + \beta_{2s}\overset{o}{x}_{2s}}{\beta_{2s+1}}, \qquad s = 1,\ldots, n-1.$$

We note that populations are never negative. Therefore, in order to have biological meaning, we require that $E(\mathbf{x}^o) \in \text{Int } R^{2n}_+$. From **(2.1)** we make the following observations:

(a) Clearly, $\overset{o}{x}_2 > 0$ since $E(\mathbf{x}^o) \in \text{Int } R^{2n}_+$.

(b) Since $\overset{o}{x}_{2n-1} > 0$, therefore $\overset{o}{x}_{2s-1} > 0$ for $s = 1,\ldots, n-1$.

(c) In order to secure the species density, $\overset{o}{x}_{2s+2} > 0$, $s = 1,\ldots, n-1$, for the food-chain system we must have that $\beta_{2s}\overset{o}{x}_{2s} > \alpha_{2s+1}\gamma_{2s+1}$.

For example, since $\overset{o}{x}_2 = \frac{\alpha_1\gamma_1}{\beta_1} > \frac{\alpha_3\gamma_3}{\beta_3}$, it follows that $\overset{o}{x}_4 = \frac{-\alpha_3\gamma_3\beta_1 + \alpha_1\gamma_1\beta_2}{\beta_1\beta_3} > 0$.

Section 2.2. Lyapunov function and its properties

In Section 1.3 we defined and stated two general theorems on the Lyapunov function.

It is well-known that (see for example Huang and Morowitz [19]) the Volterra-integral or V-function for a predator-prey system is indeed also a Lyapunov function and is everywhere concave upward.

The open food-chain model (1.24) has the Lyapunov function (see [6])

$$V(x) = \sum_{S=1}^{2n} \gamma_S x_s^o \left(\frac{x_s}{x_s^o} - 1 - \ln \frac{x_s}{x_s^o} \right) \tag{2.2}$$

with the following properties.

(i) $V(x)$ is minimum when $x(t) = x^o(t)$. That is, $\min V(x) = V(x^o) = 0$, where x^o is the equilibrium position.

(ii) $V(x)$ is monotonically increasing about the equilibrium point; $V(x) \to \infty$ as $\|x_s\| \to \infty$ and as $\|x_s\| \to 0$, $s = 1,\ldots, 2n$.

(iii) $$\frac{dV(x)}{dt} = \sum_{S=1}^{2n} \frac{\partial V}{\partial x_s} f_s(x) = 0. \tag{2.3}$$

where f_s represents the right-hand side of the s^{th} equation on (1.24). The proof of property (iii) is presented in Appendix 6.

From (2.3), using *Theorem 1.1* in Section 1.3, we can conclude that the equilibrium point $E(x^o)$ (obtained in Section 2.1) is stable.

Consider the first integral

$$V(x) = h, \qquad h \text{ constant} > 0 \tag{2.4}$$

of the model (1.24); it represents the family of level surfaces ϑ_h in R^{2n+1}. The orthogonal projection of ϑ_h onto R^{2n} generates 2n dimensional hypersurfaces $\aleph_h$ in R^{2n} which are closed, do not intersect, enclose the equilibrium $E(x^o)$, and accommodate orbits of (1.24). In addition, if $h_1 < h_2$, the hypersurface $\aleph_{h_1}$ is inside the hypersurface $\aleph_{h_2}$. The proof of (2.4) is given in Appendix 7.

As an illustration, consider the open food-chain system (1.24) of length two with parameters $\alpha_1 = \alpha_2 = \beta_1 = \gamma_1 = \gamma_2 = 1$, whose modelling equations are

$$\begin{aligned} \dot{x}_1(t) &= x_1(1 - x_2), \\ \dot{x}_2(t) &= x_2(-1 + x_1). \end{aligned} \tag{2.5}$$

Using Section 2.1 with n = 1, we obtain that the equilibrium point of the open food-chain model (2.5) is $E(\overset{o}{x}_1,\overset{o}{x}_2) = (1,1)^T$. Then the Lyapunov function (2.2) for the system (2.5) with γ_s, s = 1, 2, given above can be written as

$$V(x_1,x_2) = \sum_{s=1}^{2} x_s - 1 - \ln x_s. \tag{2.6}$$

We present the hypersurfaces of the Lyapunov function (2.6), a particular case of (2.2) in Figure 3. Here, $h_1 = V(0.8,0.8) = 0.046287$, $h_2 = V(0.7,0.7) = 0.113350$, $h_3 = V(0.5,0.5) = 0.436370$ and $h_4 = V(0.4,0.4) = 1.075884$.

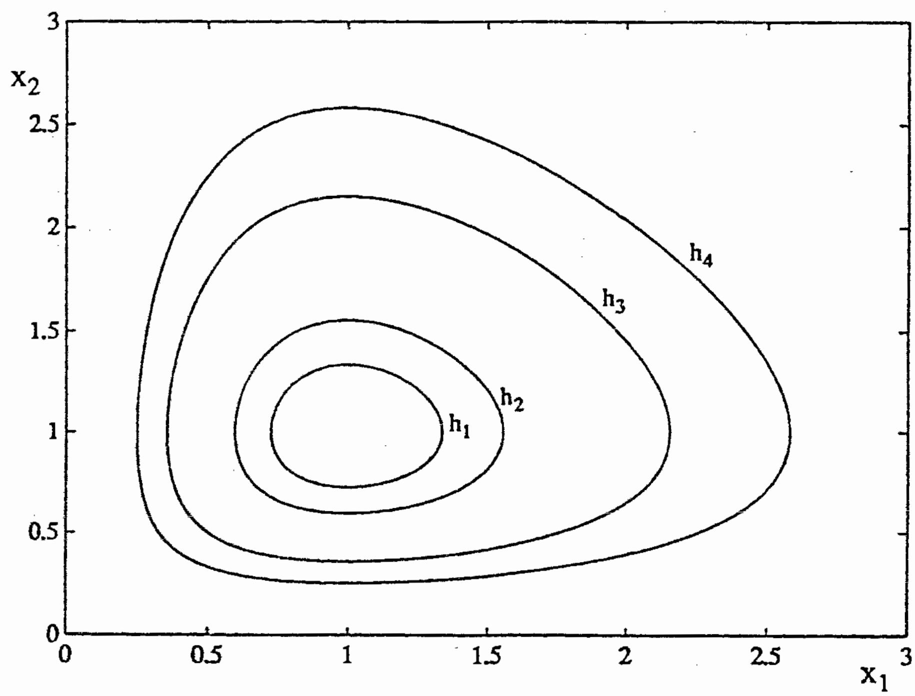

Figure 3. Hypersurfaces of the Lyapunov function in R^2.

Section 2.3. Existence of Simple Equilibria

Using the methodology in Section 1.2, we will investigate the nature of the stability of the equilibrium point of the closed food-chain **(1.19)** in R^2, a particular case of **(1.19)** with $\varphi = 0$. For a chain of length two, **(1.19)** is given by

$$\begin{aligned} \dot{x}_1(t) &= Q - \frac{\beta_1}{\gamma_1} x_1 x_2 + \alpha_2 a_2 x_2, \\ \dot{x}_2(t) &= x_2(-\alpha_2 + \frac{\beta_1}{\gamma_2} x_1). \end{aligned} \tag{2.7}$$

The equilibrium positions of the system **(2.7)** are solutions of the equations

$$\begin{aligned} & Q - \frac{\beta_1}{\gamma_1} x_1 x_2 + \alpha_2 a_2 x_2 = 0, \\ & x_2(-\alpha_2 + \frac{\beta_1}{\gamma_2} x_1) = 0. \end{aligned} \tag{2.8}$$

From **(2.8)**, the non-trivial equilibrium $E(\xi^o) \in R_+^2$ has coordinates

$$\xi_1^o = \frac{\alpha_2 \gamma_2}{\beta_1} \quad , \quad \xi_2^o = \frac{Q\gamma_1}{\alpha_2(\gamma_2 - a_2\gamma_1)} , \tag{2.9}$$

provided that $\gamma_2 - \alpha_2\gamma_1 > 0$ (since populations are never negative).

Hence, we note that **(2.7)** has only one nontrivial solution ξ^o satisfying **(2.8)**, where $\xi_s^o > 0, \;\; s = 1, 2.$

The Jacobian matrix J_o of the linearized system of **(2.7)** enables us to study the nature and stability of the equilibrium point $E(\xi^o)$. We have

$$J_o = \begin{bmatrix} -\frac{\beta_1}{\gamma_1}\xi_2^o & -\frac{\beta_1}{\gamma_1}\xi_1^o + a_2\alpha_2 \\ \frac{\beta_1}{\gamma_2}\xi_2^o & -\alpha_2 + \frac{\beta_1}{\gamma_2}\xi_1^o \end{bmatrix}. \tag{2.10}$$

The eigenvalues of **(2.10)** are

$$\lambda_{1,2} = \frac{\sigma}{2} \pm \sqrt{\sigma^2 - 4\Delta}, \tag{2.11}$$

where

$$\sigma = \frac{-Q\beta_1}{\alpha_2(\gamma_2 - a_2\gamma_1)}, \tag{2.12}$$

and

$$\Delta = \frac{Q\beta_1}{\gamma_2}. \tag{2.13}$$

Since the parameters in **(2.12)** and **(2.13)** are positive and $\gamma_2 - a_2\gamma_1 > 0$, we have that $\sigma < 0$ and $\Delta > 0$. Depending on the choice of numerical values of these parameters, we have several cases to consider, either the discriminant $D = \sigma^2 - 4\Delta$, is greater than 0, is less than 0 or is equal to 0. In any case, since $\sigma < 0$, we can conclude that the equilibrium point is stable and is not a centre. Therefore the equilibrium point $E(\xi^o)$ of the uncontrolled food-chain model **(2.7)** is either a stable spiral or a stable node (see Table 1, Section 1.2).

In Chapter 3, with the appropriate choice of the numerical values for the parameters, we can ascertain the nature of the stability of the point $E(\xi^o)$ of (2.7).

CHAPTER THREE

CONTROL POLICIES

In this chapter, we consider first the reasons as to why a control is needed. We will then introduce important definitions and a theorem concerning the concept of avoidance control and establish control policies.

Section 3.1 Justification for a Control

In this section we discuss the justification for introducing control terms into the closed food-chain systems.

Consider a closed food-chain model in R^2, a particular case of **(1.19)** without control, given by the equations

$$\dot{x}_1 = Q - \frac{\beta_1}{\gamma_1} x_1 x_2 + \alpha_2 a_2 x_2,$$
$$\dot{x}_2 = \frac{\beta_1}{\gamma_2} x_1 x_2 - \alpha_2 x_2. \tag{3.1}$$

We recall that in Section 2.3 of Chapter 2, we showed that the equilibrium $E(\xi_1^o, \xi_2^o)$ for the uncontrolled closed food-chain system **(3.1)** given by **(2.7)** is stable, and made the assumption that for $E(\xi^o) \in \text{Int}\, R_+^2$, $\gamma_2 > \gamma_1 a_2$.

The solution of **(3.1)** with initial state $\mathbf{x}_o(t_o) \in R_+^2$, describes the behaviour of the interacting populations in the chain with initial sizes $x_{o1}(t_o)$ and $x_{o2}(t_o)$. Depending on the values of the parameters Q, γ_1, γ_2, β_1, α_2 and a_2, the solution may move on an orbit of **(3.1)** which winds away from $\mathbf{x}_o$. This may endanger the existence of an acceptable size of the population which may result in an extinction or 'explosion' of a species.

We will illustrate this by the following numerical simulation. Let us set the parameters in **(3.1)** to

$$Q = 0.25,\ \gamma_1 = \gamma_2 = \beta_1 = \alpha_2 = 1.0,\ a_2 = 0.5. \tag{3.2}$$

Then **(3.1)** is simplified to

$$\dot{x}_1 = \frac{1}{4} - x_1 x_2 + \frac{x_2}{2},$$
$$\dot{x}_2 = x_1 x_2 - x_2. \tag{3.3}$$

Solving (3.3) for its equilibrium (using Section 2.3), we obtain that $E(\xi_1^o, \xi_2^o) = (1.0, 0.5)^T$. We select the initial point x_o to be $x_o(0) = (1.2, 1.0)^T$. The solution of (3.3), starting from the initial point $x_o(0)$ on the time interval [0,10] in the phase plane is presented in Figure 4 by the curve *l*. From this figure, we observe that as time progresses, that is, for $t > 10$, the solution will approach the equilibrium $(1.0, 0.5)^T$ of the system (3.3). We note here that with the choice of these parameters (3.2), the equilibrium $E(\xi_1^o, \xi_2^o)$ (as obtained in Section 2.3) is an asymptotically stable focus. The fluctuations of the population densities x_1 and x_2 with time are illustrated in Figure 5.

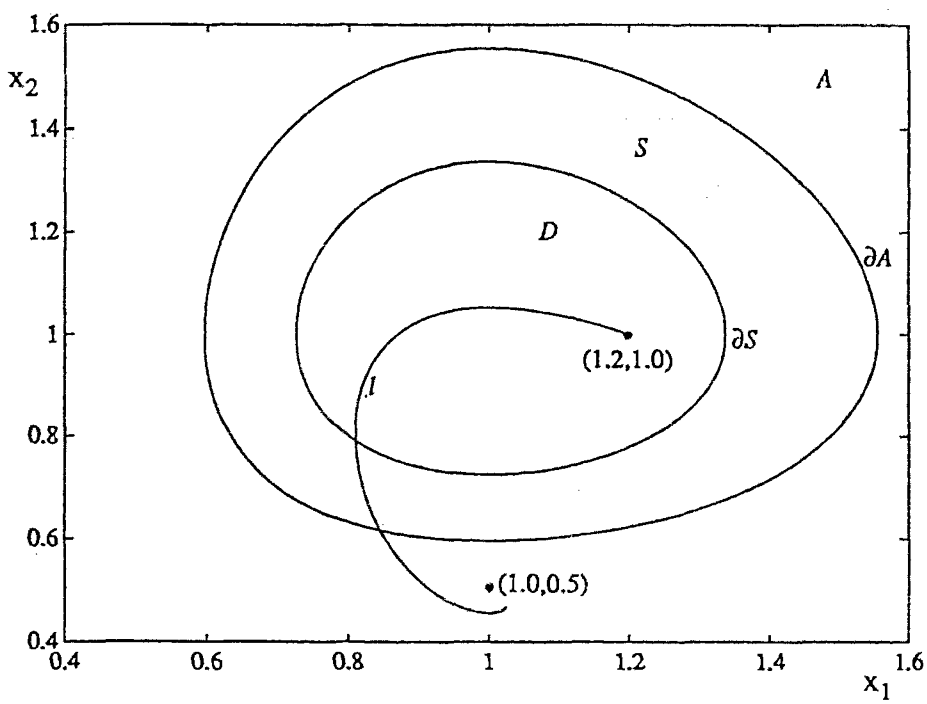

Figure 4. Solution curve *l* in the phase plane.

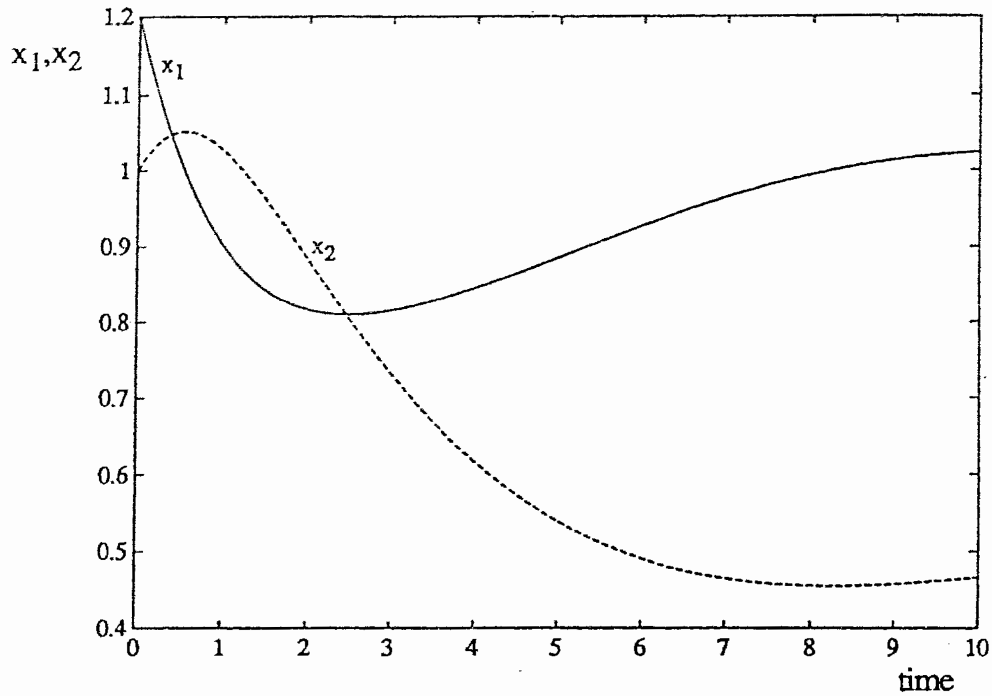

Figure 5. Fluctuations of x_1 and x_2 in R^2 without control.

The prey-population x_2 decreases fast and we have a stabilization of **x** around $E(\xi^o)$ which may not be desirable from the managerial point of view or it may not be in the interest of the population itself. Such a situation or situations where the population x_1 becomes small, or the sizes of both populations become small could be avoided if the manager opts to introduce control and to stabilize the populations in a desirable zone not necessarily close to $E(\xi^o)$.

Having justified our reason for a control, we will in the next section use the Lyapunov function **(2.2)** for the open food-chain **(1.24)** to define an avoidance region $A \in R_+^{2n}$, and a safety zone $S \in R_+^{2n}$, between the boundaries ∂S and ∂A which guards the solution of the system **(1.19)** in R_+^{2n}, from entering A through the boundary ∂A. (Figure 4 illustrates A, S, ∂A and ∂S for the model **(3.3)** with **(3.2)** in R^2).

As an additional illustration, we present the fluctuations of the population densities in a closed food-chain of length four against time in the absence of control (see Figure 6) with initial point $x(0) = (1.8,1.2,1.0,1.0)^T$; the modelling equations are

$$\begin{aligned}
\dot{x}_1(t) &= x_1(\alpha_1 - \frac{\beta_1}{\gamma_1}x_2) + Q - \alpha_1 x_1 + \sum_{s=2}^{4} a_s\alpha_s x_s,\\
\dot{x}_2(t) &= x_2(-\alpha_2 + \frac{\beta_1}{\gamma_2}x_1 - \frac{\beta_2}{\gamma_2}x_3),\\
\dot{x}_3(t) &= x_3(-\alpha_3 + \frac{\beta_2}{\gamma_3}x_2 - \frac{\beta_3}{\gamma_3}x_4),\\
\dot{x}_4(t) &= x_4(-\alpha_4 + \frac{\beta_3}{\gamma_4}x_3),
\end{aligned} \tag{3.4}$$

with parameters chosen to have values

$$Q = 1.5,\ \alpha_1 = 2,\ \alpha_s = a_s = 1,\ s = 2,..,4,\ \beta_s = 1,\ s = 1,..,3,\ \gamma_s = 1,\ s = 1,..,4. \tag{3.5}$$

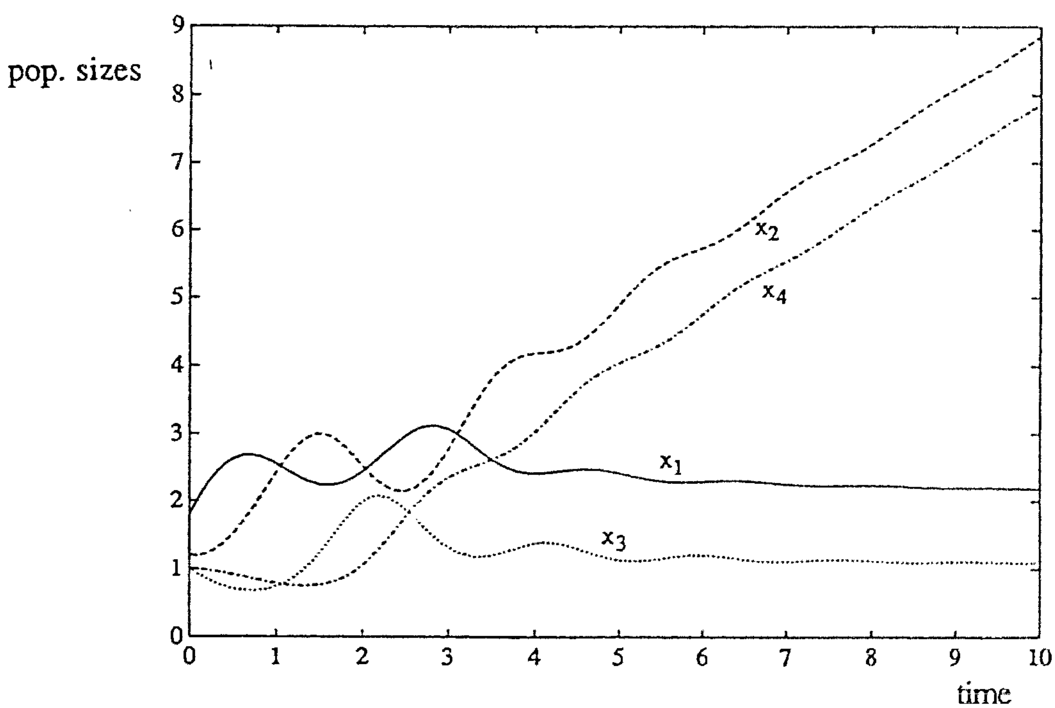

Figure 6. Fluctuations of x in R^4 without control.

Section 3.2 Avoidance Control

In the designing of an avoidance control policy, it is essential that the following definitions and theorem concerning the closed food-chain model **(1.19)** be introduced. We acknowledge that the papers by Bojadziev and Skowronski [3] and Bojadziev [9] play an important role.

Consider the constant vectors $\varepsilon \in R_{+}^{2n}$ and $\delta \in R_{+}^{2n}$, having components

$$0 < \varepsilon_s < \delta_s < x_s^o, \qquad s = 1,.., 2n, \qquad \textbf{(3.6)}$$

where x_s^o is given by **(2.1)**. We know that V(**x**) is monotonically increasing about the equilibrium point (property (ii) of V(**x**), Section 2.2), therefore with **(3.6)**, we can conclude that

$$V(\varepsilon) = h_\varepsilon > V(\delta) = h_\delta. \qquad (3.7)$$

We refer to ε_s as the avoidance parameter and δ_s as the security parameter. The choices of ε_s and δ_s are arbitrary as long as **(3.6)** is observed.

Now, making use of **(3.7)**, we introduce the following basic definitions.

Definition 3.1. ***Avoidance region A.***

$$A \triangleq \{\mathbf{x} \in R_{+}^{2n} : V(\mathbf{x}) \geq h_\varepsilon\}. \qquad \textbf{(3.8)}$$

Definition 3.2. ***Safety zone S***

$$S \triangleq \{\mathbf{x} \in R_{+}^{2n} : h_\delta \leq V(\mathbf{x}) < h_\varepsilon\}. \qquad \textbf{(3.9)}$$

Definition 3.3. ***Admissible region Ω***

$$\Omega \triangleq \{\mathbf{x} \in R_{+}^{2n} : V(\mathbf{x}) < h_\varepsilon\}. \qquad \textbf{(3.10)}$$

Definition 3.4. ***Desirable zone D***

$$D \triangleq \{\mathbf{x} \in R_+^{2n} : V(\mathbf{x}) \leq h_\delta\}. \quad (3.11)$$

Definition 3.5. ***Boundaries*** ∂A and ∂S, of the avoidance region A and the safety zone S respectively are defined correspondingly as

a) $$\partial A = \aleph_{h_\varepsilon} \triangleq \{\mathbf{x} \in R_+^{2n} : V(\mathbf{x}) = h_\varepsilon\}, \quad (3.12)$$

b) $$\partial S = \aleph_{h_\delta} \triangleq \{\mathbf{x} \in R_+^{2n} : V(\mathbf{x}) = h_\delta\} \text{ and } \partial A. \quad (3.13)$$

We note that the admissible region Ω is enclosed by ∂A, the desirable zone D, is enclosed by ∂S and $\Omega = S \cup D$.

Definition 3.6. The region A as defined in **(3.8)** is avoidable for the response $\mathbf{x}(t)$ of **(1.19)** if there is a control $\mathbf{u}(t) \in U$ such that for all states $\mathbf{x}(t_1) \in S$, S as defined in **(3.9)**, the response $\mathbf{x}(t)$ cannot enter A. That is,

$$\mathbf{x}(t, \mathbf{x}(t_1), \mathbf{u}(t_1)) \cap A = \emptyset \qquad \forall\, t. \quad (3.14)$$

Sufficient conditions for the avoidance of A are established below.

Theorem 3.1. ***(Avoidance Control Theorem)*** The response $\mathbf{x}(t)$ of the closed food-chain model **(1.19)** is controllable in the zone Ω for the avoidance of A if there is a control $\mathbf{u}(t) \in U$ for which

$$\frac{dV(\mathbf{x})}{dt} = \sum_{S=1}^{2n} \frac{\partial V}{\partial x_S} \dot{x}_S \leq 0, \quad (3.15)$$

where $\dot{x}_S$ is given by **(1.22)** and $V(\mathbf{x})$ is the Lyapunov function **(2.2)** for the open food-chain.

Proof (by contradiction):

Let us assume that the region A is not avoidable. That is, **(3.14)** is violated. Hence the response $\mathbf{x}(t) = \mathbf{x}(t, \mathbf{x}(t_1), \mathbf{u}(t_1))$ originating from $\mathbf{x}(t_1) \in S$ enters A, for some $t > t_1$. As a result, there is a time $t_2 > t_1$ for which $\mathbf{x}(t_2) = \mathbf{x}(t_2, \mathbf{x}(t_1), \mathbf{u}(t_1)) \in \partial A$.

However, from property (ii) of $V(\mathbf{x})$ in Section 2.2, we observe that

$$V(\mathbf{x}(t_1)) < V(\mathbf{x}(t_2)) \tag{3.16}$$

Therefore, the function $V(\mathbf{x})$ is increasing hence $dV(\mathbf{x})/dt > 0$.

On the other hand, we have from **(3.15)** that $V(\mathbf{x})$ is a non-increasing function along the responses of **(1.19)**. Therefore, we have arrived at a contradiction. ♦

This theorem plays an important role in the designing of an avoidance control policy for the region A by the response of **(1.19)** which will be discussed in the next section.

Section 3.3. Avoidance Control Policy

With the use of ***Theorem 3.1*** (***Avoidance Control Theorem***) in the last section, we will design a growth restriction or avoidance control policy.

Incorporating our model (**1.22**) into the inequality (**3.15**) of the ***Avoidance Control Theorem***, we rewrite (**3.15**) as

$$\frac{dV(t)}{dt} = \frac{\partial V}{\partial x_1}\dot{x}_1 + \sum_{s=2}^{2n}\frac{\partial V}{\partial x_s}\dot{x}_s \leq 0. \qquad (3.17)$$

Using (**1.23**) and (**2.3**) we get that

$$\frac{\partial V}{\partial x_1}(W + \varphi_1(\mathbf{u},\mathbf{x})) + \sum_{s=2}^{2n}\frac{\partial V}{\partial x_s}\varphi_s(\mathbf{u},\mathbf{x}) \leq 0. \qquad (3.18)$$

By differentiating (**2.2**), the elaboration of (**3.18**) reads

$$\gamma_1 x_1^o\left(\frac{1}{x_1^o} - \frac{1}{x_1}\right)(W + \varphi_1(\mathbf{u},\mathbf{x})) + \sum_{s=2}^{2n}\gamma_s x_s^o\left(\frac{1}{x_s^o} - \frac{1}{x_s}\right)\varphi_s(\mathbf{u},\mathbf{x}) \leq 0, \qquad (3.19)$$

where φ_s is defined in Section 1.4.

With (**3.19**), we have established an inequality relationship between the control vector $\mathbf{u}(t)$ and the population vector $\mathbf{x}(t)$. All vectors $\mathbf{u}$ for which (**3.19**) is satisfied form the set U. We will consider (**3.19**) as a control law of qualitative nature in its implicit form.

Having established the qualitative control law for the closed food-chain system, we are now able to formulate an ***Avoidance Control Policy*** with the aid of the ***Avoidance Control Theorem.***

Avoidance Control Policy. Our aim is to assure that the response $\mathbf{x}(t) = \mathbf{x}(t,\mathbf{x}(t_o),\mathbf{u}(t_o))$ of the closed food-chain model (**1.19**) with initial state $\mathbf{x}(t_o) \in D$ and a fixed control $\mathbf{u}(t_o)$ (which may or may not be zero) does not go beyond the safety zone S, or in other words, does not enter the avoidance region A.

In order to achieve this, we monitor the value of the Lyapunov function $V(\mathbf{x}(t))$. Suppose that for some time $t_1 > t_o$, we obtain that $h_\delta < V(\mathbf{x}(t_1)) < h_\varepsilon$, which indicates that the response $\mathbf{x}(t_1)$ has crossed through the boundary ∂S and has entered into the safety zone S. At this stage, in order to prevent the response $\mathbf{x}(t_1)$ from crossing the boundary ∂A and entering into the avoidance region A, we introduce a new control $\mathbf{u}(t_1)$ at the switching point $\mathbf{x}(t_1) \in S$ which obeys the control law **(3.19)**. The new response $\mathbf{x}(t)$ to $\mathbf{u}(t_1)$ is steered "down" through decreasing levels of the Lyapunov function $V(\mathbf{x}(t)) = h < h_\delta$ into the desirable region D.

If for any future time, say $t^* > t_1$, the response $\mathbf{x}(t^*)$ under the control $\mathbf{u}(t_1)$ enters the safety zone S again, the same control policy is applied with a new control $\mathbf{u}(t^*)$ which should satisfy **(3.19)**.

Growth restriction policy. The avoidance control policy is essentially a growth restriction policy (also known as the ***Balanced zone stabilization policy***, see [6]) since the response vector $\mathbf{x}(t)$ of the closed food-chain system **(1.19)** represents the change of the population sizes $\mathbf{x}_s$, $s = 1,\ldots, 2n$. The avoidance control policy ensures that a response of **(1.19)** that originates in D cannot enter A and as a result, the population sizes are restricted in the region D which can be made small if so desired by proper selection of the boundary parameters ε_s and δ_s.

In the last chapter, by the means of numerical simulations, we will illustrate the growth restriction policy for the closed food-chain systems.

CHAPTER FOUR

NUMERICAL SIMULATIONS OF THE CLOSED FOOD-CHAIN MODELS

We have studied in the previous chapters the growth restriction policy or avoidance control policy for a general controlled closed food-chain system with 2n interacting populations.

In this last chapter, particular cases of the general controlled food-chain model **(1.19)** are studied using numerical simulations in order to illustrate the growth restriction policies. A fourth-order Runge-Kutta numerical method (explained in Chapter 1, Section 1) with step-size $h = 0.1$ is used to integrate the modelling equations so as to find the response of the closed food-chain system.

Section 4.1. Closed food-chain model with external control of length two

In discussing the dynamics of the model in Section 1.4, we remarked that the meaning of the known function $\varphi(\mathbf{u},\mathbf{x})$ depends on the kind of control involved.

Consider the model **(1.19)** of length two with $\varphi(\mathbf{u},\mathbf{x}) = (u_1,u_2)^T$ as given in **(1.20)**. That is,

$$\begin{aligned} \dot{x}_1(t) &= Q - \frac{\beta_1}{\gamma_1} x_1 x_2 + \alpha_2 a_2 x_2 + u_1, \\ \dot{x}_2(t) &= x_2(-\alpha_2 + \frac{\beta_1}{\gamma_2} x_1) + u_2. \end{aligned} \qquad \textbf{(4.1)}$$

This is the case of an externally controlled closed food-chain system where the control comes from outside the system such as external environmental influences.

The presentation of **(4.1)** in the form of **(1.22)** is

$$\begin{aligned} \dot{x}_1(t) &= x_1(\alpha_1 - \frac{\beta_1}{\gamma_1} x_2) + W + u_1, \\ \dot{x}_2(t) &= x_2(-\alpha_2 + \frac{\beta_1}{\gamma_2} x_1) + u_2, \end{aligned} \qquad \textbf{(4.2)}$$

where

$$W = Q - \alpha_1 x_1 + \alpha_2 a_2 x_2. \qquad \textbf{(4.3)}$$

For a numerical simulation, we select the parameters as follows:

$$Q = 0.25,\ \gamma_1 = \gamma_2 = \beta_1 = \alpha_2 = 1,\ a_2 = 0.5. \qquad \textbf{(4.4)}$$

With these values, we obtain that the equilibrium point for the corresponding open food-chain to **(4.2)** ($W = 0$, $(u_1,u_2)^T = \mathbf{0}$) as illustrated in Section 2.1, is given by $(x_1^o, x_2^o)^T = (1.0,1.0)^T$. Also, the Lyapunov function **(2.2)** reduces to

$$V(\mathbf{x}) = \sum_{S=1}^{2} (x_s - 1 - \ln x_s). \qquad \textbf{(4.5)}$$

We perform the numerical simulations with initial state $\mathbf{x}_0(0) = (1.2,1.0)^T$, the same initial state that was used in Section 3.1 in justifying the need for a control.

The qualitative control law **(3.19)** can be expressed as

$$\gamma_1 x_1^o\left(\frac{1}{x_1^o} - \frac{1}{x_1}\right)(W + u_1) + \gamma_2 x_2^o\left(\frac{1}{x_2^o} - \frac{1}{x_2}\right) u_2 \leq 0, \qquad \textbf{(4.6)}$$

where W is given by **(4.3)** and $\gamma_s = 1$, $s = 1, 2$. This is implied by the validity of the following inequalities

$$\begin{aligned}
&W + u_1 > 0 && \text{if } x_1 < x_1^o = 1, \\
&W + u_1 < 0 && \text{if } x_1 > x_1^o = 1, \\
&u_2 \quad > 0 && \text{if } x_2 < x_2^o = 1, \\
&u_2 \quad < 0 && \text{if } x_2 > x_2^o = 1.
\end{aligned} \qquad \textbf{(4.7)}$$

Now the constant vectors $\varepsilon \in R_+^2$ and $\delta \in R_+^2$ are appropriately chosen to have values $\varepsilon_s = 0.7$ and $\delta_s = 0.8$, $s = 1,2$, satisfying **(3.6)**. Using these values together with **(3.7)** and **(4.5)**, we obtain that $h_\delta = 0.046287$ and $h_\varepsilon = 0.113350$. Thus for this externally controlled closed food-chain model **(4.2)** in R^2, we have determined the boundaries of *A* and *S* respectively with **(3.12)** and **(3.13)**. The controlled behaviour of the response in the phase plane is illustrated in Figure 7.

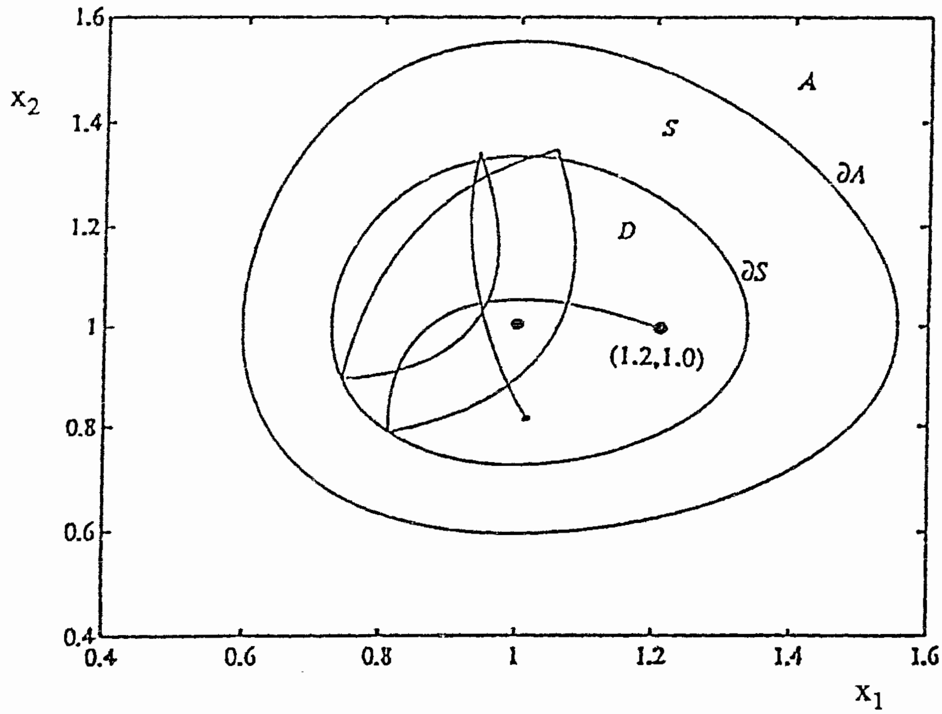

Figure 7. Controlled behaviour of the response in the phase plane.

Consider the response of (**4.2**) starting at the initial point $\mathbf{x}_0(0) = (1.2,1.0)^T$ with zero initial control, that is $\mathbf{u}(0) = \mathbf{0}$. The situation at this stage is identical to the response described in Section 3.1 (Justification for a control). We compare Figure 7 with Figure 4 (Section 3.1) and observe that initially the response moves along the curve l (Figure 4). According to the ***Avoidance Control Policy*** (Section 3.3), we monitor the level of the Lyapunov function (**4.5**) and note that at time t_1 = 2.6, the Lyapunov function (**4.5**) has the value $V(\mathbf{x}(2.6))$ = 0.046567 (see Appendix 2) which exceeds h_δ = 0.046287. Hence the response of (**4.2**) has entered into the safety zone S. Now, since $x_1 < 1.0$ and $x_2 < 1.0$ (see Appendix 2), then with the implementation of the control law (**4.7**), we select at time t_1, a control (using lines 1 and 3 of (**4.7**)) $\mathbf{u}(t_1) = (0.41585,0.25000)^T$. As a result, the response changes abruptly and re-enters the desirable region D. The Lyapunov function is monitored continuously and for each time, say $t_s^* > t_1$, $s \in R$, that $V(\mathbf{x}(t_s^*)) > h_\delta$, we introduce a new control vector $\mathbf{u}(t_s^*)$ satisfying **(4.7)** at the switching point $\mathbf{x}(t_s^*) \in S$, thus ensuring that the response returns to the desirable region D. On the time interval [0,10], the control has been changed four times, hence restricting the growth of the closed food-chain **(4.2)** to D. The fluctuations of the population densities x_1 and x_2 against time are given in Figure 8 and the levels of the Lyapunov function in Figure 9.

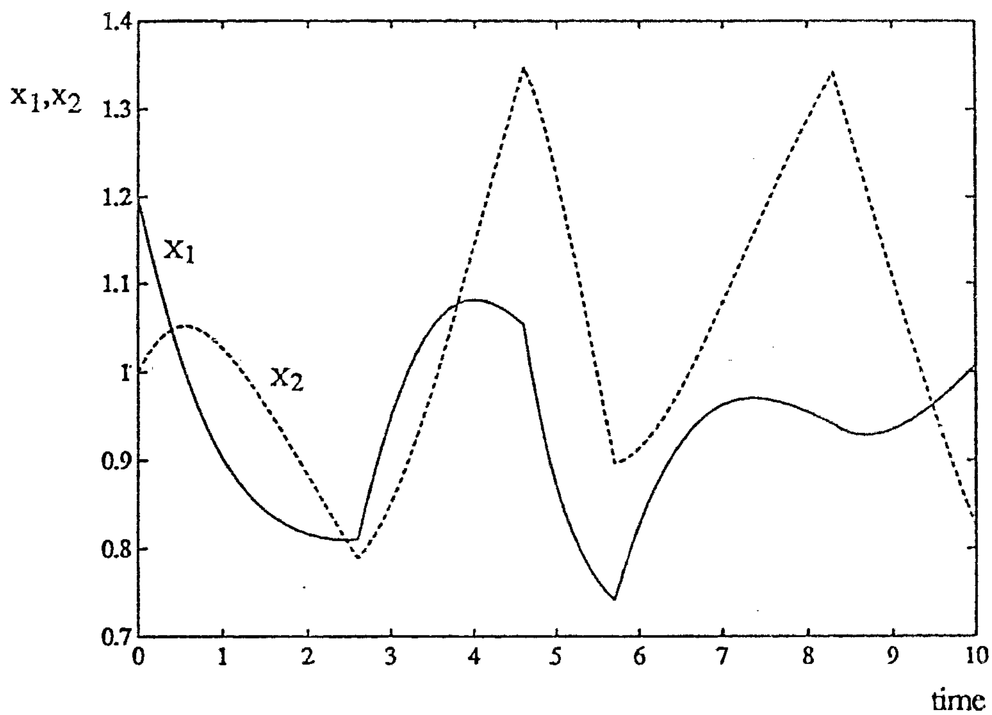

Figure 8. Fluctuations of the species densities in R^2.

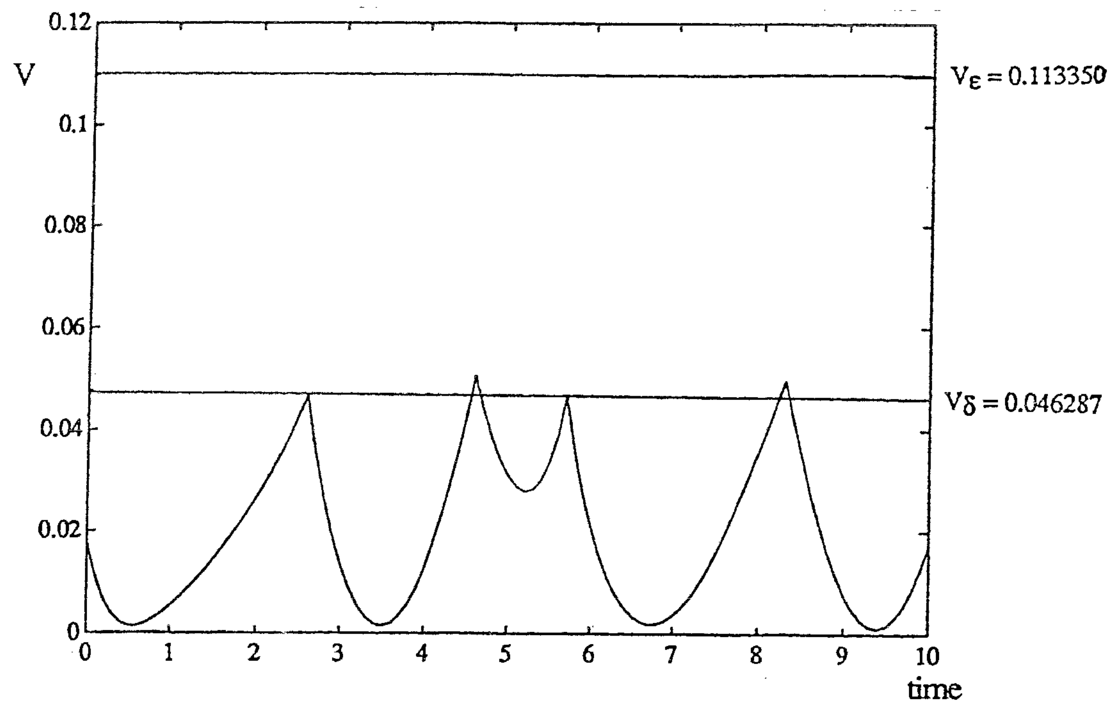

Figure 9. Changes of the levels of $V(x_1,x_2)$.

Numerical data for the externally controlled closed food-chain model in R^2 are presented in the table in Appendix 2.

Section 4.2 The closed food-chain of length four with external control

Here, we consider the same closed food-chain system with external control **(1.22)** of length four.

$$
\begin{aligned}
\dot{x}_1(t) &= x_1(\alpha_1 - \frac{\beta_1}{\gamma_1} x_2) + W + \varphi_1(\mathbf{u},\mathbf{x}), \\
\dot{x}_{2s}(t) &= x_{2s}(-\alpha_{2s} + \frac{\beta_{2s-1}}{\gamma_{2s}} x_{2s-1} - \frac{\beta_{2s}}{\gamma_{2s}} x_{2s+1}) + \varphi_{2s}(\mathbf{u},\mathbf{x}), \\
\dot{x}_{2s+1}(t) &= x_{2s+1}(-\alpha_{2s+1} + \frac{\beta_{2s}}{\gamma_{2s+1}} x_{2s} - \frac{\beta_{2s+1}}{\gamma_{2s+1}} x_{2s+2}) + \varphi_{2s+1}(\mathbf{u},\mathbf{x}), \\
\dot{x}_{2n}(t) &= x_{2n}(-\alpha_{2n} + \frac{\beta_{2n-1}}{\gamma_{2n}} x_{2n-1}) + \varphi_{2n}(\mathbf{u},\mathbf{x}), \qquad s = 1,\dots, n-1,
\end{aligned} \tag{4.8}
$$

Our model **(1.22)** with **(1.20)** and parameters

$$Q = 1.5,\ \alpha_1 = 2,\ \alpha_s = a_s = 1,\ s = 2,..,4,\ \beta_s = 1,\ s = 1,..,3,\ \gamma_s = 1,\ s = 1,..,4, \tag{4.9}$$

gives us

$$
\begin{aligned}
\dot{x}_1(t) &= x_1(2 - x_2) + W + u_1, \\
\dot{x}_2(t) &= x_2(-1 + x_1 - x_3) + u_2, \\
\dot{x}_3(t) &= x_3(-1 + x_2 - x_4) + u_3, \\
\dot{x}_4(t) &= x_4(-1 + x_3) + u_4,
\end{aligned} \tag{4.10}
$$

where $W = Q - 2x_1 + x_2 + x_3 + x_4$ (a particular case of **(1.23)**).

We need the equilibrium point of the uncontrolled open food-chain of **(4.10)**, that is, with $W = 0$ and $\varphi_s = 0$. Following Section 2.1, by setting $\dot{\mathbf{x}}(t) = \mathbf{0}$ in

(1.24) with (4.9) and n = 2, we obtain the equilibrium of the open food-chain system in R^4 to be

$$E(x_1^o, x_2^o, x_3^o, x_4^o) = (2,2,1,1)^T. \tag{4.11}$$

Therefore, the Lyapunov function (2.2) of the system (4.10) with (4.9) and (4.11) is given by

$$V(\mathbf{x}) = \sum_{S=1}^{2n} \gamma_S x_S^o \left(\frac{x_S}{x_S^o} - 1 - \ln\frac{x_S}{x_S^o}\right)$$

$$\Rightarrow \quad V(\mathbf{x}) = \sum_{S=1}^{2} 2\left(\frac{x_S}{2} - 1 - \ln\frac{x_S}{2}\right) + \sum_{S=3}^{4} (x_S - 1 - \ln x_S)$$

$$\Rightarrow \quad V(\mathbf{x}) = \sum_{S=1}^{4} (x_S) - 6 - 2\ln\frac{x_1 x_2}{4} - \ln x_3 x_4. \tag{4.12}$$

We express the qualitative control law (3.19) with (4.9) and (4.12) as

$$2\left(\frac{1}{2} - \frac{1}{x_1}\right)(W + u_1) + 2\left(\frac{1}{2} - \frac{1}{x_2}\right)(u_2) + \sum_{S=3}^{4} \left(1 - \frac{1}{x_S}\right)u_s \le 0. \tag{4.13}$$

This inequality (4.13) is satisfied if the following inequalities hold

$$\begin{array}{lll} Q - 2x_1 + x_2 + x_3 + x_4 + u_1 > 0 & \text{if } x_1 < x_1^o, & \\ Q - 2x_1 + x_2 + x_3 + x_4 + u_1 < 0 & \text{if } x_1 > x_1^o, & \\ u_s > 0 & \text{if } x_s < x_s^o, & \\ u_s < 0 & \text{if } x_s > x_s^o, & s = 2,\ldots, 4. \end{array} \tag{4.14}$$

We consider (4.14) as the control law for the system (4.10). In order that (3.6) is satisfied, we have selected $\varepsilon_s = 1.2$, $\delta_s = 1.7$ for s = 1, 2, and $\varepsilon_s = 0.4$, $\delta_s = 0.5$ for s = 3, 4, as our appropriate boundary parameters. Thus with (3.7) and (4.12), we determine that the boundary ∂S has value $h_\delta = 0.436370$ and the avoidance boundary ∂A is given by $h_\varepsilon = 1.075884$.

The fluctuations of population densities x_1, x_2, x_3 and x_4 with initial size densities $x_1(0) = 1.8$, $x_2(0) = 1.2$, $x_s(0) = 1.0$, $s = 3, 4$, are presented in Figure 10. Let us compare Figure 10 with Figure 6 (Section 3.1) where the fluctuations of the uncontrolled model of **(4.10)** ($\mathbf{u} = \mathbf{0}$) have been given. We observe that the difference occurs at time $t_1 = 2.100$ (see Appendix 3). At this time t_1, the Lyapunov function for the controlled system **(4.10)** is $V(\mathbf{x}(t_1)) = 0.443317$, which exceeds $h_S = 0.436370$. This indicates that the response has entered the safety region S. The response $\mathbf{x}$ at t_1 is $\mathbf{x}(t_1) = (2.535087, 2.386631, 2.057807, 1.170476)^T$. From this, we observe that $x_s(t_1) > x_s^o(t_1)$, $s = 1,..., 4$, the corresponding equilibrium value. By applying our control law **(4.14)**, we select at this time t_1 the control (using lines 2 and 4 of **(4.14)**) $u_1(t_1) = -3.04474$, $u_s(t_1) = -1.00000$, $s = 2,..., 4$, which as a result steers the response back into the desirable region D. On the time interval [0,10], we have changed the control a total of nine times so as to secure the presence of the response in D.

In contrast, after time $t_1 = 2.100$, the corresponding uncontrolled model (Figure 6, Section 3.1) appears to diverge to the extreme situations.

We present the levels of the Lyapunov function $V(\mathbf{x})$ in Figure 11. The numerical output for this uncontrolled model in R^4 is given in Appendix 3.

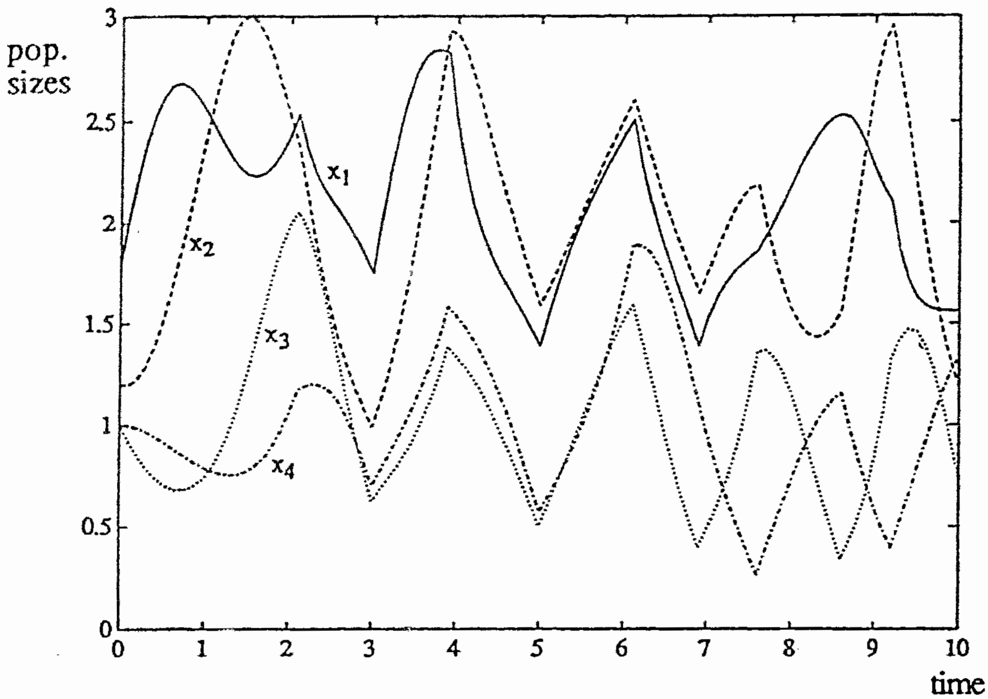

Figure 10. Fluctuations of x_1, x_2, x_3 and x_4.

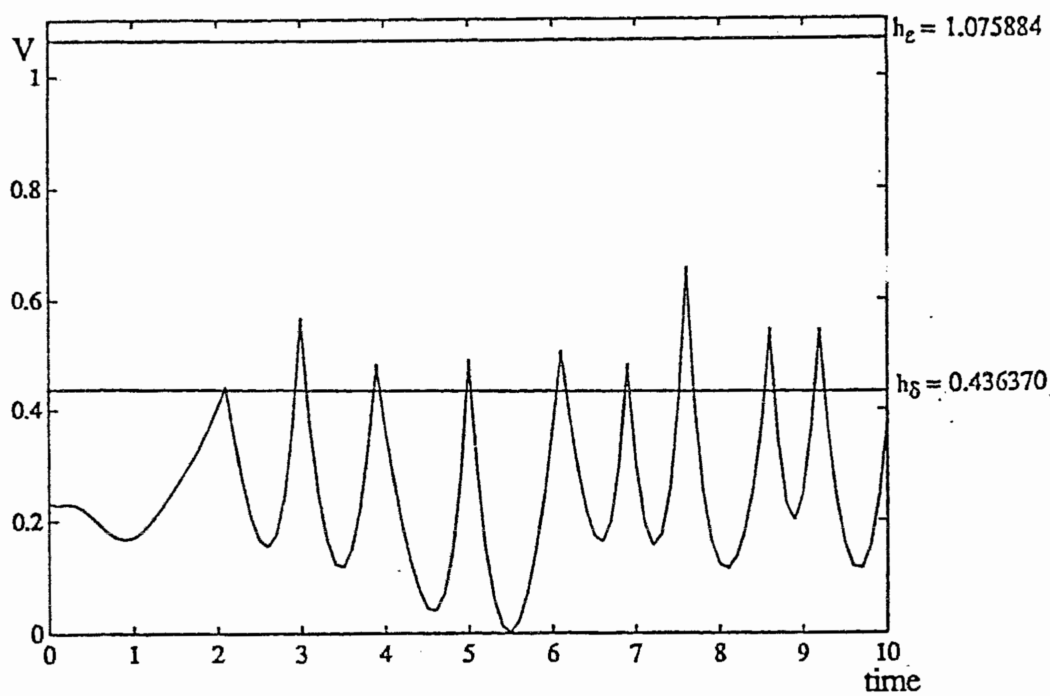

Figure 11. Levels of the Lyapunov function in R^4.

Section 4.3 Closed food-chain system with internal control in R^2

In the next two sections we will study numerically the internally controlled closed food-chain system of length two and four whose dynamics are a particular case of the general model **(1.19)**.

We recall from Section 1.4 that the function $\varphi(\mathbf{u},\mathbf{x})$ for an internally controlled system has been chosen to be of the form $\varphi(\mathbf{u},\mathbf{x}) = (0,u_2x_2^2,\ldots,u_{2n}x_{2n}^2)^T$.

For a chain of length 2, we have that $\varphi(\mathbf{u},\mathbf{x}) = (0,u_2x_2^2)$. That is, $\varphi_1 = 0$ which implies that the prey population with size x_1 can neither exercise control nor can it control its own growth directly. Hence the prey population is considered as a resource. Setting $\varphi_2 = u_2x_2^2$, suggests that the predator population with size x_2 indirectly controls the growth of the resource by controlling its own growth (self control). Thus the predator population is referred to as a consumer.

Consider the model **(1.22)** with **(1.21)**, **(4.4)** and n = 1; we obtain that

$$\begin{aligned} \dot{x}_1(t) &= x_1(1-x_2) + W, \\ \dot{x}_2(t) &= x_2(-1+x_1) + u_2x_2^2, \end{aligned} \qquad \mathbf{(4.15)}$$

where

$$W = Q - x_1 + \frac{x_2}{2}. \qquad \mathbf{(4.16)}$$

Similar to the closed food-chain with external control in R^2, the equilibrium of the corresponding uncontrolled open food-chain model ($W = 0$, $\varphi = \mathbf{0}$) in **(4.15)** has been found to be $E(1,1)^T \in \text{Int}\, R_+^2$ and its Lyapunov function **(2.2)** is given by

$$V(x_1,x_2) = \sum_{s=1}^{2}(x_s - 1 - \ln x_s). \qquad \mathbf{(4.17)}$$

The control law **(3.19)** together with **(4.16)** and $\varphi(\mathbf{u},\mathbf{x}) = (0,u_2x_2^2)$ after expansion appears as

$$\gamma_1 x_1^o \left(\frac{1}{x_1^o} - \frac{1}{x_1}\right)W + \gamma_2 x_2^o \left(\frac{1}{x_2^o} - \frac{1}{x_2}\right)u_2x_2^2 \leq 0. \qquad \textbf{(4.18)}$$

By substituting $\gamma_1 = \gamma_2 = 1$ and $(x_1^o, x_2^o) = (1,1)^T$, **(4.18)** simplifies to

$$\left(1 - \frac{1}{x_1}\right)W + \left(1 - \frac{1}{x_2}\right)u_2x_2^2 \leq 0. \qquad \textbf{(4.19)}$$

This inequality **(4.19)** is satisfied if the following set of inequalities, which we will consider as our control law for the internally controlled closed food-chain in R^2, holds. That is,

$$\begin{array}{lll}
\left(1 - \frac{1}{x_1}\right)W + \left(1 - \frac{1}{x_2}\right)u_2x_2^2 & < 0 & \text{if } x_2 > 1 \text{ and } \left(1 - \frac{1}{x_1}\right)W > 0, \\
\left(1 - \frac{1}{x_1}\right)W + \left(1 - \frac{1}{x_2}\right)u_2x_2^2 & > 0 & \text{if } x_2 < 1 \text{ and } \left(1 - \frac{1}{x_1}\right)W > 0, \\
u_2 & < 0 & \text{if } x_2 > 1 \text{ and } \left(1 - \frac{1}{x_1}\right)W < 0, \\
u_2 & > 0 & \text{if } x_2 < 1 \text{ and } \left(1 - \frac{1}{x_1}\right)W < 0.
\end{array} \qquad \textbf{(4.20)}$$

The avoidance and safety parameters ε_s and δ_s, s = 1, 2, respectively, satisfy **(3.6)** and are chosen to take the values $\varepsilon_1 = \varepsilon_2 = 0.7$ and $\delta_1 = \delta_2 = 0.8$. These values with **(3.7)**, **(3.12)**, **(3.13)** and **(4.17)** gives us $h_\delta = 0.046287$ and $h_\varepsilon = 0.113350$ which defines the boundaries of the security zone and the avoidance zones respectively (see Figure 12).

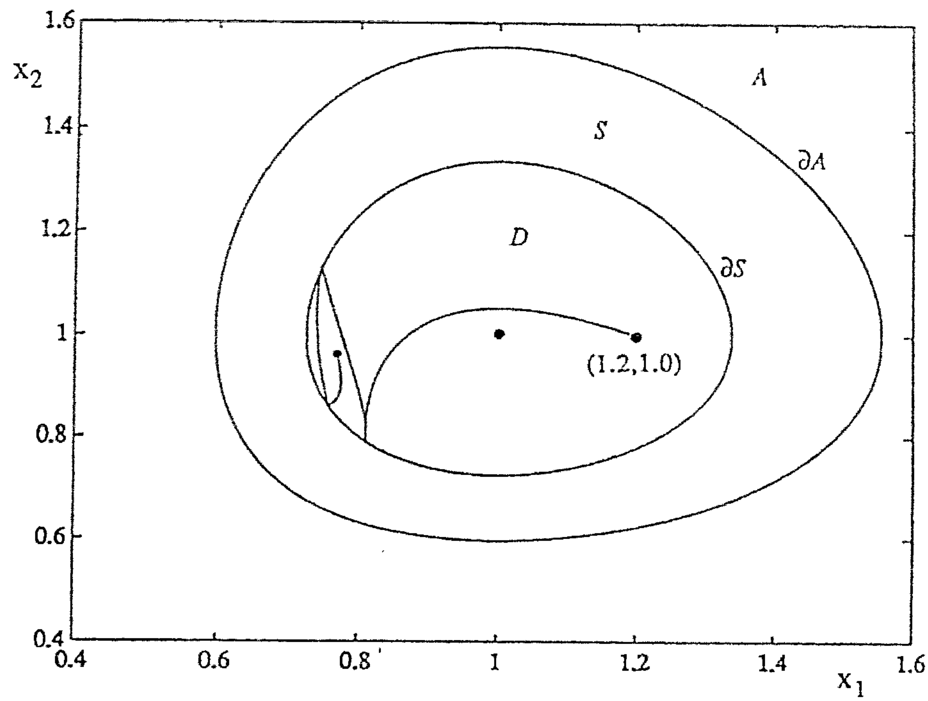

Figure 12. Controlled behaviour of the response in the phase plane.

We begin by considering the response of **(4.15)** with initial populations $x_1(0) = 1.2$, $x_2(0) = 1.0$ and fixed control $u_2 = 0$. Following the control policy established in Section 3.3, for each time t, $t \in [0.10]$, we calculate the components x_s, $s = 1, 2$, of the response $\mathbf{x}(t)$ and the value of the Lyapunov function $V(\mathbf{x}(t))$. In so far as $V(\mathbf{x}(t)) < h_\delta$, we continue the calculations of the response $\mathbf{x}(t)$ in the desirable region D with the same control u_2, each time monitoring the value of the Lyapunov function. If for any time, for example $t = t^*$, $t^* > t_o$ where $V(\mathbf{x}(t^*)) > h_\delta$, we implement our control policy using the control law **(4.20)**. We observe that at time $t_1 = 2.6$, $V(\mathbf{x}(t_1)) = 0.046567 > h_\delta = 0.046287$, hence the response $\mathbf{x}(t)$ has crossed the safety boundary ∂S and entered into the safety zone S. At this stage, we prevent the response from advancing into the avoidance region A by imposing our control law **(4.20)**. We select at time t_1, the control $u_2(t_1) = 0.33316$ making use of the fourth line of **(4.20)** with $x_1 = 0.810432$ and $x_2 = 0.789158$. This change in control from $u_2 = 0$ to

$u_2 = 0.33316$ at the switching point $\mathbf{x}(t_1)$ with components $x_1(t_1) = 0.810432$ and $x_2(t_1) = 0.789158$ (see Appendix 4), results in an abrupt change in the response steering it "downwards" towards the desirable region D. The Lyapunov function $V(\mathbf{x}(t))$ is monitored continunously for each t ensuring that the function is less than h_δ, otherwise the control law is introduced at each switching point $\mathbf{x}(t^*)$.

The ***avoidance control policy*** for $t \in [0,10]$ is illustrated in Figure 12 and a graphical representation of the fluctuations of the species densities, x_1 and x_2 against time is shown in Figure 13. The levels of the Lyapunov function $V(\mathbf{x}(t))$ are presented in Figure 14.

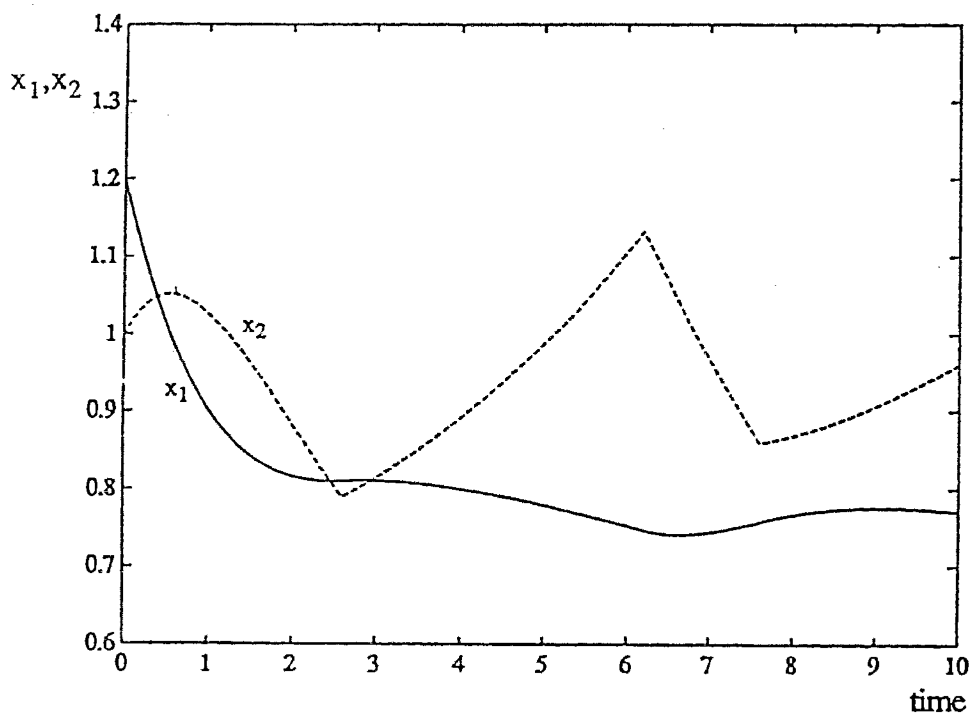

Figure 13. Fluctuations of x_1 and x_2 in R^2.

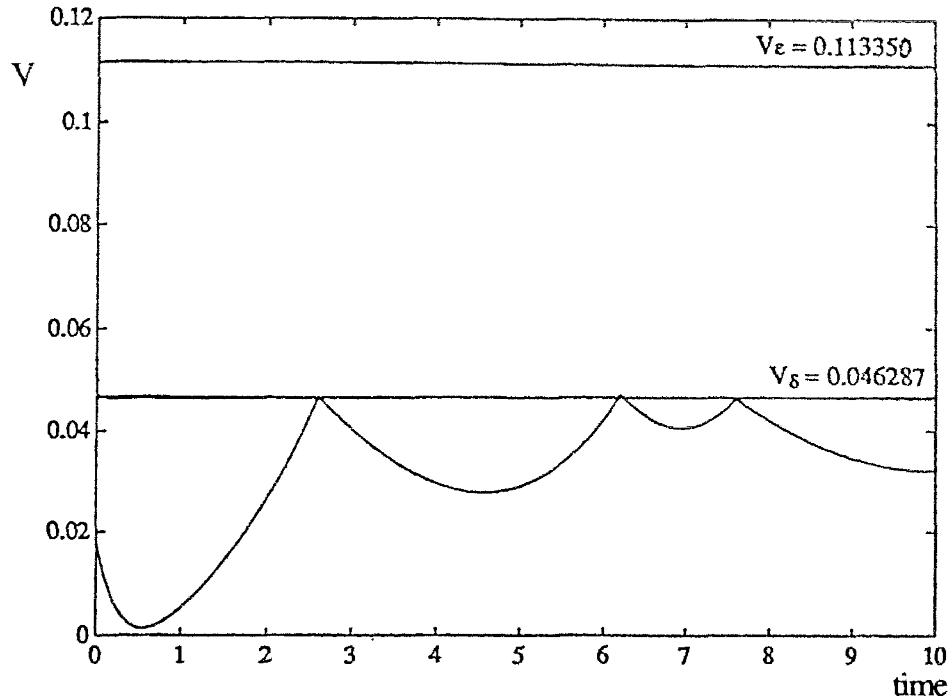

Figure 14. Changes of the levels of $V(x_1,x_2)$.

From Figure 14, we observe that on the time interval [0,10], the control has been changed three times thus restraining of the growth of the internally controlled closed food-chain **(4.15)** into the desirable region *D*. This yields a manageable population level in R^2.

Appendix 4 provides the numerical results obtained in the monitoring process pertaining to this internally controlled closed food-chain in R^2.

Section 4.4 An internally controlled closed food chain system of length four

For the closed food-chain system of length four with internal control, we consider the model **(1.22)** with **(1.21)** and **(4.9)** setting n = 2. Thus we have that

$$\begin{aligned}\dot{x}_1(t) &= x_1(2 - x_2) + W,\\ \dot{x}_2(t) &= x_2(-1 + x_1 - x_3) + u_2x_2^2,\\ \dot{x}_3(t) &= x_3(-1 + x_2 - x_4) + u_3x_3^2,\\ \dot{x}_4(t) &= x_4(-1 + x_3) + u_4x_4^2,\end{aligned} \qquad \textbf{(4.21)}$$

where $W = \frac{3}{2} - 2x_1 + x_2 + x_3 + x_4$ ((**1.23**) with n = 2).

Similar to Section 4.2, the equilibrium of the corresponding uncontrolled open food-chain ($W = 0$, $\varphi = \mathbf{0}$) in **(4.21)** is given by $E(\mathbf{x}^o) = (2,2,1,1)^T$, and its Lyapunov function (after simplification) by

$$V(\mathbf{x}) = \sum_{s=1}^{4} x_s - 6 - 2\ln\frac{x_1x_2}{4} - \ln x_3x_4. \qquad \textbf{(4.22)}$$

The implicit form of the control law **(3.19)** for this system **(4.21)** with W given above reduces to

$$x_1^o\left(\frac{1}{x_1^o} - \frac{1}{x_1}\right)W + \sum_{s=2}^{3} x_s^o\left(\frac{1}{x_s^o} - \frac{1}{x_s}\right)u_sx_s^2 + x_4^o\left(\frac{1}{x_4^o} - \frac{1}{x_4}\right)u_4x_4^2 \le 0. \qquad \textbf{(4.23)}$$

Substituting the equilibrium points $E(\mathbf{x}^o)$, **(4.23)** simplifies to

$$2\left(\frac{1}{2} - \frac{1}{x_1}\right)W + \left(1 - \frac{1}{x_4}\right)u_4x_4^2 + 2\left(\frac{1}{2} - \frac{1}{x_2}\right)u_2x_2^2 + \left(1 - \frac{1}{x_3}\right)u_3x_3^2 \le 0, \qquad \textbf{(4.24)}$$

which can be written as

$$\frac{(x_1 - 2)}{x_1} W + (x_4 - 1)u_4x_4 + (x_2 - 2)u_2x_2 + (x_3 - 1)u_3x_3 \le 0. \qquad \textbf{(4.25)}$$

This qualitative control law holds if the following set of equalities and inequalities are satisfied.

$$
\begin{aligned}
u_4 &= \frac{(2 - x_1)W}{x_1x_4(x_4 - 1)} && \text{if } x_1 \neq x_1^o \text{ and } x_4 \neq x_4^o, \\
u_4 &= 0 && \text{if } x_1 = x_1^o \text{ or } x_4 = x_4^o, \qquad \textbf{(4.26)} \\
u_s &> 0 && \text{if } x_s < x_s^o, \\
u_s &< 0 && \text{if } x_s > x_s^o, \qquad s = 2, 3.
\end{aligned}
$$

Our selection of $\delta = (1.7,1.7,0.5,0.5)^T$ and $\varepsilon = (1.2,1.2,0.4,0.4)^T$ (satisfying **(3.6)**) along with **(3.7)** and **(4.22)** defines ∂S, the boundary of the safety zone with value $h_\delta = 0.436370$ and ∂A, the boundary of the avoidance region calculated to be $h_\varepsilon = 1.075884$.

We commence our numerical simulation for the system **(4.21)** at time $t = 0$ with initial population size densities $x_1(0) = 1.8$, $x_2(0) = 1.2$, $x_s(0) = 1.0$, $s = 3, 4$, and with no control. Reading from the table in Appendix 5, we observe that the response remains in the desirable zone between times $t = 0.000$ and $t = 2.000$. However, at time $t = t_1 = 2.100$, the Lyapunov function is $V(\mathbf{x}(t_1)) = 0.443317$ which exceeds $h_\delta = 0.435370$. This shows that the response $\mathbf{x}(t)$ has moved through the boundary ∂S and has entered the safety zone S. In order to restrict the growth of our response to the desirable region D, we introduce the control law **(4.26)** at this time t_1. Since each component of $\mathbf{x}(t_1)$ (see Appendix 5) is greater than its corresponding equilibrium value $x_s^o(t_1)$, $s = 1, \ldots, 4$, we make use of lines 1 and 4 of the control law **(4.26)** to select at time t_1, a control $\mathbf{u}(t_1) = (0, -0.25000, -0.25000, -2.16294)^T$. The choice of the control $\mathbf{u}(t_1)$ alters the behaviour of the population growth of the system from $\mathbf{x}(t_1) = (2.535087,2.386631,2.057807,1.170476)^T$ to $\mathbf{x}(t_2) = (2.637427,2.158512,1.990413,1.023744)^T$ where $t_2 = 2.200$. At time t_2, the Lyapunov function (**4.22**) due to the implementation of the control law (**4.26**) has been reduced to $V(\mathbf{x}(t_2)) = 0.392430$, which is less than $h_\delta = 0.436370$. This indicates that the response has been driven back into the desirable region *D*. Hence the control growth policy has been effective in restraining the fluctuations of the population sizes to a manageable level.

As time progresses from t = 0 to t = 10 with a 0.1 unit interval, we continuously monitor the value of the Lyapunov function (**4.22**), ensuring that $V(\mathbf{x})$ does not override h_δ. Should this happen, the control law (**4.26**) is implemented in order to reduce the value of the Lyapunov function and also to attract or pull the population sizes to a desirable level.

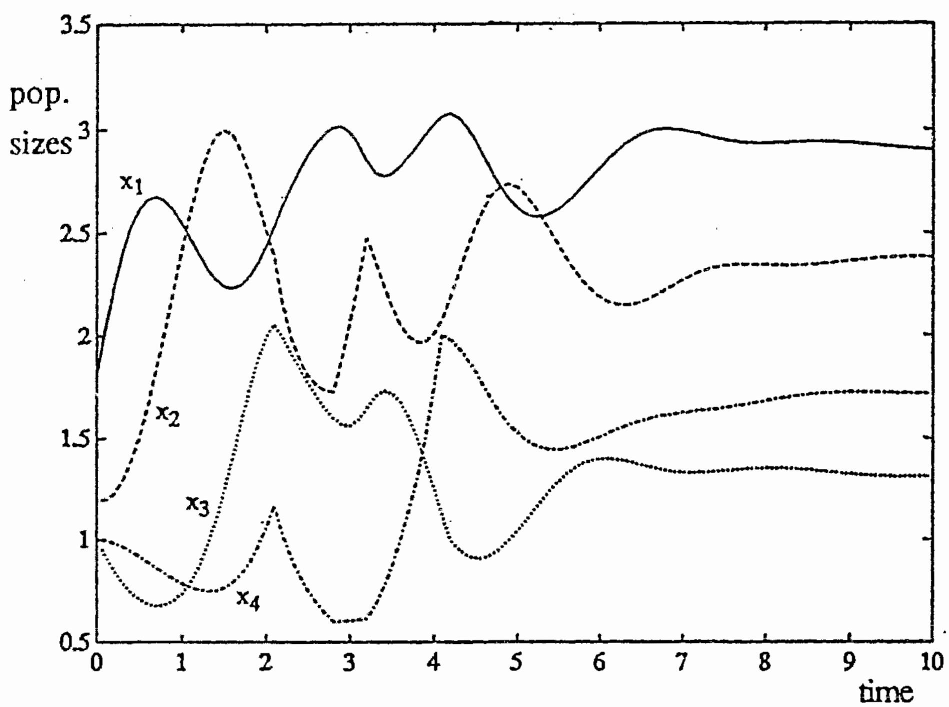

Figure 15. Fluctuations of the species densities in R^4.

The fluctuations of the population sizes x_1, x_2, x_3 and x_4 against time are shown in Figure 15. The changes of the levels of the Liapunov function $V(\mathbf{x}(t))$ are represented in Figure 16. From these figures, in particular in Figure 16, we observe that the control has been changed eight times in the time interval [0,10], in order to secure the presence of the response in *D*.

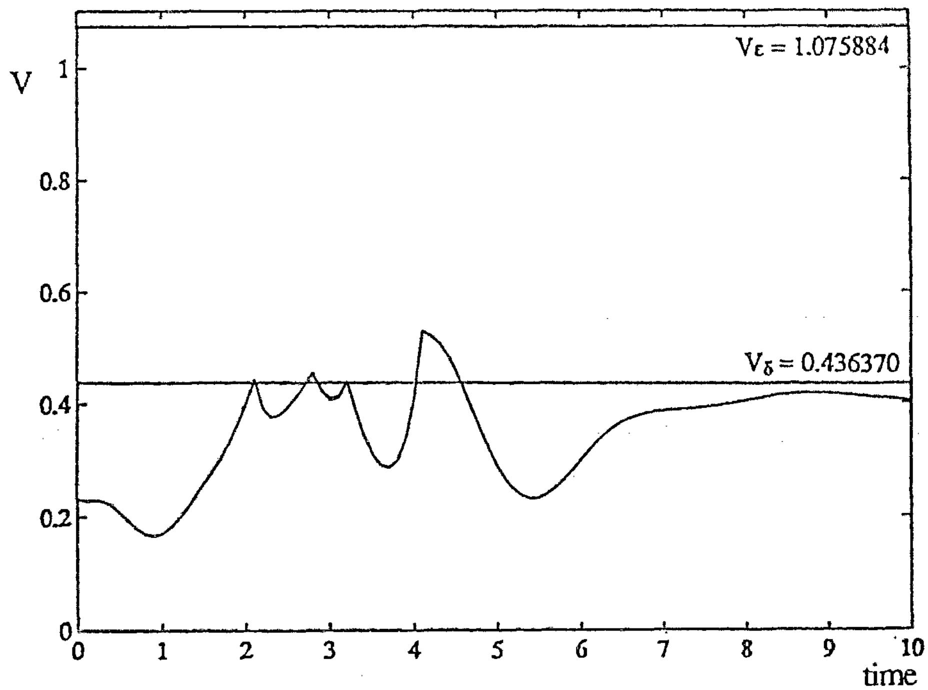

Figure 16. Changes of the levels of the Lyapunov function.

The numerical data obtained in the monitoring process concerning t, **u**, **x**(t) and V(**x**) are presented in the table in Appendix 5.

CONCLUSIONS

CONCLUSIONS

We began this thesis with a discussion on the material necessary for the applications in our topic of interest and have progressed to the introduction of the dynamics of the different models to be studied. We have differentiated between the open and closed food-chains and have analyzed the equilibrium for the open food-chain (in particular) and its Lyapunov function. In addition, we have also compared the response of the systems with and without the presence of a control.

We have remarked that the dynamics of the closed food-chain system are of a more complicated nature than their open system counterparts. However, we have managed to use the Lyapunov function for the latter system in analyzing the response and behaviour of the former.

Further, using the concept of avoidance control, we have proved an avoidance control theorem for the controlled closed food-chain system and have established a growth restricion policy (balanced zone stabilization policy). We have provided numerical simulations to support our growth policies and theory and have shown how the control involved in each of our systems has been selected according to a control law capable of restraining the growth behaviour of the populations to a manageable level.

We have illustrated how the growth restriction policy dampens the large fluctuations of population sizes and restricts the population growth to an allowable region. We have also seen that this growth policy could be achieved not only by the managers outside the system (external control) but also by the participating populations in the chain (internal control). Hence by the introduction of a control and by stabilizing the population in a desirable region not necessarily close to the equilibrium of the system, it is possible to avoid the situation or similar situations where the populations exhibit extreme fluctuations.

I believe that this topic of controlled food-chain systems with emphasis on the concept of growth control and zone stabilization is of importance and may have valuable applications in the fields of ecology, economics, epidemiology, and pest management.

APPENDICES AND REFERENCES

In the 7 Appendices that follow, we present the program for the numerical simulations, 4 tables with data and 2 proofs.

APPENDIX 1

The numerical simulations for the Runge-Kutta fourth order method was written using Fortran although a software package, ODE could have been used instead. We present here the written program for the closed food-chain with external control of length two and note that the programs for the other food-chain systems are generalizations of this.

```
      Program Foodchain
c
      Implicit double precision(a-h, o-z)
      dimension w(10), rk(10,10)
c
      open(8,file='fd1')
c
c STEP 1: INITIALIZATION
c
      n = 100
      neqn = 2
      delta = 0.8d0
      eps = 0.7d0
      tin = 0.0
      tout = 10.0
      q = 0.25
      h = (tout - tin)/n
c
      w(1) = 1.2
      w(2) = 1.0
      write(8,2)
      do 5 i = 1, neqn
         u(i) = 0.0d0
5     continue
c
c STEP 2: VALUES FOR THE BOUNDARY PARAMETERS
c
      hliap = hl(w(1),w(2))
      hdelta = hl(delta,delta)
      heps = hl(eps,eps)
c
c STEP 3: FINDS RESPONSES USING THE RUNGE-KUTTA METHOD
c
      do 6 i = 1, n
         rk(1,1) = h*f1(t,w(1),w(2),u(1))
         rk(1,2) = h*f2(t,w(1),w(2),u(2))
```

```
        rk(2,1) = h*f1(t+h/2,w(1)+0.5d0*rk(1,1),w(2)+0.5d0*rk(1,2),u(1))
        rk(2,2) = h*f1(t+h/2,w(1)+0.5d0*rk(1,1),w(2)+0.5d0*rk(1,2),u(2))
        rk(3,1) = h*f1(t+h/2,w(1)+0.5d0*rk(2,1),w(2)+0.5d0*rk(2,2),u(1))
        rk(3,2) = h*f1(t+h/2,w(1)+0.5d0*rk(2,1),w(2)+0.5d0*rk(2,2),u(2))
        rk(4,1) = h*f1(t+h/2,w(1)+0.5d0*rk(3,1),w(2)+0.5d0*rk(3,2),u(1))
        rk(4,2) = h*f1(t+h/2,w(1)+0.5d0*rk(3,1),w(2)+0.5d0*rk(3,2),u(2))
       do 7 j = 1, neqn
          w(j) = w(j) + (rk(1,j) + 2*(rk(2,j)+rk(3,j)) + rk(4,j))/6.0d0
7 continue
c
c STEP 4: CALCULATES THE LYAPUNOV FUNCTION
c
   v = 0.0d0
     do 9 j = 1, neqn
        v = v + (w(j) - dlogw(j)) - 1.0d0)
9 continue
c
c STEP 5: CHECKS IF V(X) IS IN THE SAFETY REGION (< HDELTA)
c
   p = -(q - w(1) + 0.5d0*w(2))
   if (v .gt. hdelta) then
     if (w(1) .lt. 1.0d0) then
       u(1) = p + 0.25d0
     else
       u(1) = p - 0.25d0
     endif
     if (w(2) .lt. 1.0d0) then
       u(2) = 0.25d0
     else
       u(2) = -0.25d0
     endif
   endif
c
   write (8,3) a+i*h, u(1),u(2),w(1),w(2),v
   t = a+i*h
6 continue
2  format (4x, 't', 10x, 'u1', 11x, 'u2', 9x, 'x1', 10x, 'x2', 7x, 'v(x1,x2)')
3  format (2x, f6.3, 2(4x,f8.5), 2(4x, f8.6), 4x, f8.6)
   end
c
c STEP 6: SUBPROGRAMS FOR SYS. OF EQNS AND LYAP. FUNCTION
c
   function hl(x,y)
   implicit double precision(a-h, o-z)

     hl = x – dlog(x) – 2.0d0 + y – dlog(y)

   end
c
```

```
      function f1(t,x,y,u)
      implicit double precision(a-h,o-z)

        q = 0.25d0
        f1 = x*(1.0d0 – y) + (0.5d0*y – x + q) + u

      end
c
      function f2(t,x,y,u)
      implicit double precision(a-h,o-z)

        f2 = y*(–1.0d0 + x) + u

      end
```

For the internally controlled closed food-chain system, we modify the program above slightly by

(i) setting u(1) = 0 throughout the program,

(ii) rewriting step 5 to read as follows:

```
p = (w(1) - 1.0d0)*(0.5d0*w(2) - w(1) + q)
prod = w(1)*w(2)(w(2) - 1.0d0)
if (v .gt. hdelta) then
  if (w(2) .lt. 1.0d0) then
    if (p .gt. 0.0d0) then
      u(2) = (p/prod) + 0.1d0
    else
      u(2) = -(-p/-prod) + 0.1d0
    endif
  else
    if (p .gt. 0.0d0) then
      u(2) = -(p/prod) - 0.1d0
    else
      u(2) = (-p/prod) - 0.1d0
    endif
endif
```

(iii) Making the necessary change of control components in step 6 in the function subprograms f1 and f2.

APPENDIX 2

Numerical data for the closed food-chain model of length two with external control.

Initial conditions: $(x_1(0),x_2(0)) = (1.2,1.0)^T$

$(u_1(0),u_2(0)) = (0.0,0.0)^T$

$V(\mathbf{x}(0)) = (0.017678)$, $V(\delta) = 0.046287$ and $V(\varepsilon) = 0.113350$.

time	u_1	u_2	x_1	x_2	$v(\mathbf{x})$
0.000	0.00000	0.00000	1.200000	1.000000	0.017678
0.100	0.00000	0.00000	1.156572	1.017962	0.011271
0.200	0.00000	0.00000	1.116350	1.031920	0.006784
0.300	0.00000	0.00000	1.079366	1.042040	0.003852
0.400	0.00000	0.00000	1.045584	1.048543	0.002150
0.500	0.00000	0.00000	1.014915	1.051693	0.001402
0.600	0.00000	0.00000	0.987229	1.051780	0.001378
0.700	0.00000	0.00000	0.962367	1.049109	0.001894
0.800	0.00000	0.00000	0.940153	1.043985	0.002806
0.900	0.00000	0.00000	0.920402	1.036711	0.004005
1.000	0.00000	0.00000	0.902924	1.027575	0.005414
1.100	0.00000	0.00000	0.887534	1.016848	0.006983
1.200	0.00000	0.00000	0.874053	1.004783	0.008679
1.300	0.00000	0.00000	0.862310	0.991612	0.010486
1.400	0.00000	0.00000	0.852147	0.977543	0.012399
1.500	0.00000	0.00000	0.843415	0.962674	0.014422
1.600	0.00000	0.00000	0.835978	0.947444	0.016562
1.700	0.00000	0.00000	0.829711	0.931730	0.018831
1.800	0.00000	0.00000	0.824502	0.915752	0.021239
1.900	0.00000	0.00000	0.820246	0.899623	0.023799
2.000	0.00000	0.00000	0.816852	0.883440	0.026521
2.100	0.00000	0.00000	0.814236	0.867289	0.029413
2.200	0.00000	0.00000	0.812322	0.851240	0.032482
2.300	0.00000	0.00000	0.811042	0.835355	0.035731
2.400	0.00000	0.00000	0.810336	0.819686	0.039162
2.500	0.00000	0.00000	0.810149	0.804275	0.042775
2.600	0.41585	0.25000	0.810432	0.789158	0.046567
2.700	0.41585	0.25000	0.850719	0.800722	0.035356
2.800	0.41585	0.25000	0.887464	0.815176	0.026378
2.900	0.41585	0.25000	0.920742	0.832307	0.019178
3.000	0.41585	0.25000	0.950618	0.851921	0.013444
3.100	0.41585	0.25000	0.977156	0.873834	0.008964
3.200	0.41585	0.25000	1.000421	0.897869	0.005600
3.300	0.41585	0.25000	1.020491	0.923850	0.003263
3.400	0.41585	0.25000	1.037453	0.951594	0.001895
3.500	0.41585	0.25000	1.051412	0.980914	0.001462

3.600	0.41585	0.25000	1.062490	1.011613	0.001942
3.700	0.41585	0.25000	1.070828	1.043487	0.003315
3.800	0.41585	0.25000	1.076586	1.076322	0.005563
3.900	0.41585	0.25000	1.079942	1.109899	0.008665
4.000	0.41585	0.25000	1.081088	1.143993	0.012588
4.100	0.41585	0.25000	1.080231	1.178377	0.017295
4.200	0.41585	0.25000	1.077585	1.212827	0.022736
4.300	0.41585	0.25000	1.073373	1.247125	0.028851
4.400	0.41585	0.25000	1.067816	1.281060	0.035573
4.500	0.41585	0.25000	1.061135	1.314437	0.042824
4.600	-0.12000	-0.25000	1.053541	1.347074	0.050523
4.700	-0.12000	-0.25000	0.996466	1.325328	0.043674
4.800	-0.12000	-0.25000	0.947613	1.296587	0.038274
4.900	-0.12000	-0.25000	0.906049	1.262162	0.034047
5.000	-0.12000	-0.25000	0.870815	1.223244	0.030878
5.100	-0.12000	-0.25000	0.841061	1.180883	0.028773
5.200	-0.12000	-0.25000	0.816029	1.135985	0.027819
5.300	-0.12000	-0.25000	0.795056	1.089320	0.028165
5.400	-0.12000	-0.25000	0.777572	1.041530	0.029990
5.500	-0.12000	-0.25000	0.763087	0.993146	0.033494
5.600	-0.12000	-0.25000	0.751189	0.944601	0.038881
5.700	0.29340	0.25000	0.741527	0.896244	0.046357
5.800	0.29340	0.25000	0.772763	0.899467	0.035966
5.900	0.29340	0.25000	0.801191	0.905271	0.027639
6.000	0.29340	0.25000	0.826956	0.913389	0.020942
6.100	0.29340	0.25000	0.850184	0.923585	0.015564
6.200	0.29340	0.25000	0.870989	0.935648	0.011279
6.300	0.29340	0.25000	0.889477	0.949381	0.007925
6.400	0.29340	0.25000	0.905751	0.964603	0.005383
6.500	0.29340	0.25000	0.919914	0.981142	0.003569
6.600	0.29340	0.25000	0.932068	0.998834	0.002418
6.700	0.29340	0.25000	0.942318	1.017520	0.001882
6.800	0.29340	0.25000	0.950772	1.037045	0.001923
6.900	0.29340	0.25000	0.957542	1.057260	0.002507
7.000	0.29340	0.25000	0.962744	1.078019	0.003606
7.100	0.29340	0.25000	0.966494	1.099181	0.005190
7.200	0.29340	0.25000	0.968913	1.120609	0.007230
7.300	0.29340	0.25000	0.970122	1.142171	0.009696
7.400	0.29340	0.25000	0.970242	1.163743	0.012553
7.500	0.29340	0.25000	0.969394	1.185207	0.015768
7.600	0.29340	0.25000	0.967696	1.206453	0.019302
7.700	0.29340	0.25000	0.965262	1.227380	0.023116
7.800	0.29340	0.25000	0.962202	1.247897	0.027170
7.900	0.29340	0.25000	0.958621	1.267921	0.031423
8.000	0.29340	0.25000	0.954616	1.287381	0.035833
8.100	0.29340	0.25000	0.950278	1.306217	0.040361
8.200	0.29340	0.25000	0.945691	1.324377	0.044965
8.300	0.27002	-0.25000	0.940930	1.341820	0.049610
8.400	0.27002	-0.25000	0.934920	1.308570	0.041849

8.500	0.27002	-0.25000	0.931013	1.274891	0.034525
8.600	0.27002	-0.25000	0.928942	1.241065	0.027746
8.700	0.27002	-0.25000	0.928478	1.207323	0.021604
8.800	0.27002	-0.25000	0.929424	1.173850	0.016175
8.900	0.27002	-0.25000	0.931615	1.140798	0.011521
9.000	0.27002	-0.25000	0.934907	1.108282	0.007686
9.100	0.27002	-0.25000	0.939181	1.076396	0.004706
9.200	0.27002	-0.25000	0.944333	1.045209	0.002602
9.300	0.27002	-0.25000	0.950274	1.014774	0.001387
9.400	0.27002	-0.25000	0.956928	0.985127	0.001067
9.500	0.27002	-0.25000	0.964231	0.956294	0.001639
9.600	0.27002	-0.25000	0.972127	0.928290	0.003097
9.700	0.27002	-0.25000	0.980566	0.901120	0.005428
9.800	0.27002	-0.25000	0.989508	0.874786	0.008617
9.900	0.27002	-0.25000	0.998916	0.849282	0.012647
10.000	0.27002	-0.25000	1.008758	0.824598	0.017495

APPENDIX 3

Numerical data for the closed food-chain with external control of length four.

Initial conditions: $\mathbf{x}(0) = (1.8,1.2,1.0,1.0)^T$
$\mathbf{u}(0) = (0,0,0,0)^T$
$V(\mathbf{x}(0)) = 0.232372$, $V(\delta) = 0.435870$ and $V(\varepsilon) = 1.075884$.

time	u_1	u_2	u_3	u_4
0.000 - 2.000	0.00000	0.00000	0.00000	0.00000
2.100 - 2.900	-3.04474	-1.00000	-1.00000	-1.00000
3.000 - 3.800	0.68194	1.00000	1.00000	1.00000
3.900 - 4.900	-2.75065	-1.00000	-1.00000	-1.00000
5.000 - 6.000	-0.37880	1.00000	1.00000	1.00000
6.100 - 6.800	-3.55897	-1.00000	-1.00000	-1.00000
6.900 - 7.500	-0.87148	1.00000	1.00000	-1.00000
7.600 - 8.500	-0.59446	-1.00000	-1.00000	1.00000
8.600 - 9.100	-0.48917	1.00000	1.00000	-1.00000
9.200 - 10.000	-2.96006	-1.00000	-1.00000	1.00000

time	x_1	x_2	x_3	x_4	$v(\mathbf{x})$
0.000	1.800000	1.200000	1.000000	1.000000	0.232372
0.100	2.035959	1.195219	0.992729	0.996094	0.228330
0.200	2.236175	1.225698	0.853202	0.984922	0.229984
0.300	2.400279	2.400279	1.288369	0.793751	0.967601
0.400	2.527284	1.381531	0.745704	0.945481	0.221402
0.500	2.616318	1.503828	0.709859	0.919995	0.209101
0.600	2.667306	1.653315	0.686820	0.892560	0.194232
0.700	2.681665	1.826562	0.677247	0.864526	0.180135
0.800	2.662869	2.017941	0.682026	0.837165	0.170048
0.900	2.616698	2.219250	0.702372	0.811680	0.166365
1.000	2.551020	2.419884	0.739859	0.789238	0.170147
1.100	2.475058	2.607590	0.796335	0.771017	0.180988
1.200	2.398325	2.769714	0.873696	0.758265	0.197304
1.300	2.329538	2.894675	0.973438	0.752349	0.216979
1.400	2.275829	2.973409	1.095939	0.754824	0.238142
1.500	2.242377	3.000630	1.239524	0.767469	0.259801
1.600	2.232425	2.975807	1.399467	0.792301	0.282104
1.700	2.247475	2.903687	1.567313	0.831525	0.306143
1.800	2.287479	2.794118	1.731026	0.887377	0.333426
1.900	2.350940	2.660922	1.896297	0.961841	0.365219
2.000	2.434928	2.519919	1.988952	1.056200	0.402015

2.100	2.535087	2.386631	2.057807	1.170476	0.443317
2.200	2.371583	2.156527	1.976297	1.191120	0.347883
2.300	2.259937	1.933312	1.847130	1.200673	0.267983
2.400	2.178952	1.732577	1.682439	1.192610	0.205870
2.500	2.113184	1.559154	1.497405	1.162361	0.165827
2.600	2.051946	1.411648	1.305905	1.108132	0.153555
2.700	1.988208	1.285795	1.118250	1.030904	0.176323
2.800	1.917613	1.176487	0.940733	0.933850	0.243579
2.900	1.837689	1.078799	0.776252	0.821470	0.368023
3.000	1.747224	0.988432	0.625302	0.698776	0.567482
3.100	2.011600	1.112619	0.680441	0.773269	0.381366
3.200	2.247230	1.263099	0.736165	0.849628	0.251435
3.300	2.448143	1.442382	0.794996	0.928796	0.166939
3.400	2.609107	1.651482	0.859880	1.012041	0.122730
3.500	2.726756	1.888669	0.934155	1.101124	0.117104
3.600	2.800885	2.148138	1.021375	1.198500	0.150193
3.700	2.835529	2.419124	1.124874	1.307565	0.222575
3.800	2.839256	2.686178	1.246930	1.432921	0.334152
3.900	2.824276	2.931215	1.387386	1.580602	0.483503
4.000	2.505512	2.917769	1.337978	1.537155	0.371285
4.100	2.265151	2.837128	1.286619	1.484452	0.278008
4.200	2.085772	2.713171	1.227375	1.421967	0.197440
4.300	1.950450	2.566141	1.157800	1.348829	0.129123
4.400	1.844963	2.410675	1.078107	1.264409	0.076199
4.500	1.758262	2.256018	0.990106	1.168704	0.043872
4.600	1.682155	2.107152	0.896231	1.062485	0.038732
4.700	1.610760	1.966101	0.798830	0.947268	0.068811
4.800	1.539949	1.833041	0.699757	0.825146	0.144248
4.900	1.466889	1.707125	0.600238	0.698576	0.278648
5.000	1.389653	1.587036	0.500905	0.570169	0.491669
5.100	1.541432	1.673225	0.602192	0.643033	0.286279
5.200	1.681566	1.766234	0.704710	0.719493	0.146619
5.300	1.809711	1.864437	0.808900	0.801075	0.058340
5.400	1.925998	1.965795	0.914970	0.889478	0.012132
5.500	2.031082	2.067954	1.022693	0.986634	0.001712
5.600	2.126121	2.168464	1.131167	1.094730	0.022676
5.700	2.212684	2.265114	1.238602	1.216200	0.071812
5.800	2.292554	2.356375	1.342142	1.353648	0.146656
5.900	2.367474	2.441852	1.437852	1.509662	0.245232
6.000	2.438834	2.522670	1.520946	1.686510	0.365879
6.100	2.507370	2.601670	1.586342	1.885698	0.507164
6.200	2.287555	2.471635	1.433856	1.882137	0.390270
6.300	2.115558	2.334485	1.271125	1.848117	0.293584
6.400	1.973707	2.201435	1.105796	1.782451	0.219377
6.500	1.849156	2.076978	0.944507	1.686851	0.173023
6.600	1.733076	1.961486	0.791639	1.565394	0.162496
6.700	1.619659	1.853220	0.649213	1.423605	0.198807
6.800	1.505220	1.749636	0.517391	1.267552	0.297563
6.900	1.387474	1.648130	0.395143	1.103156	0.482572

7.000	1.518770	1.751319	0.481164	0.945501	0.300390
7.100	1.618644	1.859029	0.577743	0.804191	0.195454
7.200	1.692827	1.963914	0.688861	0.676857	0.155380
7.300	1.747487	2.057714	0.818706	0.561404	0.175696
7.400	1.789141	2.131669	0.971515	0.455837	0.257987
7.500	1.824428	2.177269	1.151229	0.358090	0.411071
7.600	1.859803	2.187293	1.360942	0.265829	0.656911
7.700	1.932574	1.988787	1.365443	0.377616	0.406651
7.800	2.016683	1.817092	1.328382	0.492818	0.253835
7.900	2.106700	1.676367	1.253529	0.608934	0.164698
8.000	2.197628	1.567979	1.148121	0.722355	0.121451
8.100	2.284850	1.491616	1.021247	0.828925	0.113507
8.200	2.364113	1.446220	0.882224	0.924635	0.134734
8.300	2.431538	1.430652	0.739309	1.006247	0.182838
8.400	2.483631	1.444080	0.598889	1.071701	0.259934
8.500	2.517328	1.486133	0.465174	1.120220	0.374511
8.600	2.530079	1.556896	0.340277	1.152157	0.546467
8.700	2.516092	1.851636	0.426669	0.986095	0.341277
8.900	2.369930	2.442391	0.663388	0.706949	0.200763
9.000	2.270915	2.689956	0.832628	0.590241	0.247306
9.100	2.178647	2.869361	1.050149	0.486667	0.363069
9.200	2.108616	2.956965	1.325827	0.394497	0.546210
9.300	1.865501	2.744014	1.418837	0.511548	0.367062
9.400	1.716553	2.471212	1.468818	0.637311	0.242469
9.500	1.631738	2.185167	1.460635	0.770348	0.159843
9.600	1.588136	1.916431	1.390939	0.906377	0.116741
9.700	1.569104	1.680489	1.267621	1.038664	0.114210
9.800	1.562832	1.481773	1.105943	1.159324	0.154448
9.900	1.561101	1.318105	0.923579	1.261081	0.240811
10.000	1.558349	1.184192	0.736510	1.338808	1.379125

APPENDIX 4

Numerical results for the internally controlled closed food-chain of length two.

Initial conditions: $(x_1(0),x_2(0)) = (1.2,1.0)^T$

$(u_1(0),u_2(0)) = (0,0)^T$

$V(\mathbf{x}(0)) = 0.017678$, $V(\delta) = 0.046287$ and $V(\varepsilon) = 0.113350$

time	u	x_1	x_2	$v(x_1,x_2)$
0.100	0.00000	1.156572	1.017962	0.011271
0.200	0.00000	1.116350	1.031920	0.006784
0.300	0.00000	1.079366	1.042040	0.003852
0.400	0.00000	1.045584	1.048543	0.002150
0.500	0.00000	1.014915	1.051693	0.001402
0.600	0.00000	0.987229	1.051780	0.001378
0.700	0.00000	0.962367	1.049109	0.001894
0.800	0.00000	0.940153	1.043985	0.002806
0.900	0.00000	0.920402	1.036711	0.004005
1.000	0.00000	0.902924	1.027575	0.005414
1.100	0.00000	0.887534	1.016848	0.006983
1.200	0.00000	0.874053	1.004783	0.008679
1.300	0.00000	0.862310	0.991612	0.010486
1.400	0.00000	0.852147	0.977543	0.012399
1.500	0.00000	0.843415	0.962764	0.014422
1.600	0.00000	0.835978	0.947444	0.016562
1.700	0.00000	0.829711	0.931730	0.018831
1.800	0.00000	0.824502	0.915752	0.021239
1.900	0.00000	0.820246	0.899623	0.023799
2.000	0.00000	0.816852	0.883440	0.026521
2.100	0.00000	0.814236	0.867289	0.029413
2.200	0.00000	0.812322	0.851240	0.032482
2.300	0.00000	0.811042	0.835355	0.035731
2.400	0.00000	0.810336	0.819686	0.039162
2.500	0.00000	0.810149	0.804275	0.042775
2.600	0.33316	0.810432	0.789158	0.046567
2.700	0.33316	0.810826	0.795063	0.044925
2.800	0.33316	0.811010	0.801195	0.043331
2.900	0.33316	0.810993	0.807549	0.041789
3.000	0.33316	0.810784	0.814119	0.040305
3.100	0.33316	0.810393	0.820901	0.038883
3.200	0.33316	0.809827	0.827890	0.037527
3.300	0.33316	0.809095	0.835081	0.036242
3.400	0.33316	0.808206	0.842471	0.035032
3.500	0.33316	0.807167	0.850056	0.033901
3.600	0.33316	0.805986	0.857834	0.032854
3.700	0.33316	0.804672	0.865802	0.031894
3.800	0.33316	0.803232	0.873958	0.031025

3.900	0.33316	0.801672	0.882301	0.030251
4.000	0.33316	0.800001	0.890831	0.029575
4.100	0.33316	0.798225	0.899546	0.029001
4.200	0.33316	0.796350	0.908447	0.028532
4.300	0.33316	0.794383	0.917534	0.028172
4.400	0.33316	0.792330	0.926810	0.027924
4.500	0.33316	0.790198	0.936276	0.027791
4.600	0.33316	0.787991	0.945935	0.027776
4.700	0.33316	0.785716	0.955790	0.027883
4.800	0.33316	0.783376	0.965845	0.028115
4.900	0.33316	0.780978	0.976104	0.028477
5.000	0.33316	0.778525	0.986574	0.028970
5.100	0.33316	0.776022	0.997261	0.029600
5.200	0.33316	0.773473	1.008171	0.030371
5.300	0.33316	0.770882	1.019313	0.031286
5.400	0.33316	0.768251	1.030695	0.032351
5.500	0.33316	0.765586	1.042329	0.033571
5.600	0.33316	0.762887	1.054224	0.034951
5.700	0.33316	0.760158	1.066393	0.036498
5.800	0.33316	0.757402	1.078848	0.038217
5.900	0.33316	0.754619	1.091606	0.040117
6.000	0.33316	0.751813	1.104681	0.042205
6.100	0.33316	0.748985	1.118090	0.044489
6.200	0.05911	0.746136	1.131854	0.046980
6.300	0.05911	0.743682	1.110676	0.045531
6.400	0.05911	0.741975	1.089532	0.044199
6.500	0.05911	0.740928	1.068511	0.043025
6.600	0.05911	0.740465	1.047688	0.042044
6.700	0.05911	0.740521	1.027124	0.041284
6.800	0.05911	0.741038	1.006872	0.040765
6.900	0.05911	0.741966	0.986974	0.040503
7.000	0.05911	0.743261	0.967464	0.040510
7.100	0.05911	0.744885	0.948370	0.040791
7.200	0.05911	0.746802	0.929714	0.041349
7.300	0.05911	0.748984	0.911513	0.042184
7.400	0.05911	0.751404	0.893779	0.043292
7.500	0.05911	0.754037	0.876521	0.044666
7.600	0.30511	0.756863	0.859745	0.046300
7.700	0.30511	0.759636	0.861540	0.045126
7.800	0.30511	0.762131	0.863617	0.044011
7.900	0.30511	0.764364	0.865962	0.042951
8.000	0.30511	0.766350	0.868562	0.041945
8.100	0.30511	0.768100	0.871405	0.040989
8.200	0.30511	0.769629	0.874479	0.040082
8.300	0.30511	0.770947	0.877775	0.039223
8.400	0.30511	0.772066	0.881281	0.038411
8.500	0.30511	0.772997	0.884990	0.037646
8.600	0.30511	0.773750	0.888892	0.036928
8.700	0.30511	0.774334	0.892980	0.036257
8.800	0.30511	0.774758	0.897246	0.035634
8.900	0.30511	0.775031	0.901684	0.035058
9.000	0.30511	0.775162	0.906287	0.034531
9.100	0.30511	0.775159	0.911049	0.034054
9.200	0.30511	0.775028	0.915966	0.033626

9.300	0.30511	0.774778	0.921031	0.033249
9.400	0.30511	0.774415	0.926241	0.032924
9.500	0.30511	0.773945	0.931592	0.032652
9.600	0.30511	0.773376	0.937081	0.032433
9.700	0.30511	0.772712	0.942703	0.032268
9.800	0.30511	0.771960	0.948456	0.032158
9.900	0.30511	0.771125	0.954338	0.032105
10.000	0.30511	0.770213	0.960347	0.032109

APPENDIX 5

Numerical data for the closed food-chain model with internal control in R^4.

Initial conditions: $(x_1(0),x_2(0),x_3(0),x_4(0)) = (1,8,1.2,1.0,1.0)^T$

$(u_1(0),u_2(0),u_3(0),u_4(0)) = (0,0,0,0)^T$

$V(\mathbf{x}(0)) = 0.232372$, $V(\delta) = 0.436370$ and $V(\varepsilon) = 1.075884$.

Note: u_1 has been set to zero throughout this simulation.

time	u_2	u_3	u_4
0.000 - 2.000	0.00000	0.00000	0.00000
2.100 - 2.700	-0.25000	-0.25000	-2.16294
2.800 - 3.100	0.25000	-0.25000	-0.84316
3.200 - 4.000	-0.25000	-0.25000	0.66484
4.100	-0.25000	-0.25000	-0.09956
4.200	-0.25000	0.25000	-0.09173
4.300	-0.25000	0.25000	-0.11047
4.400	-0.25000	0.25000	-0.14275
4.500 - 10.000	-0.25000	0.25000	-0.18683

time	x_1	x_2	x_3	x_4	$V(\mathbf{x})$
0.100	2.035959	1.195219	0.922729	0.996094	0.228330
0.200	2.236178	1.225698	0.853202	0.984922	0.229984
0.300	2.400279	1.288369	0.793751	0.967601	0.228586
0.400	2.527284	1.381531	0.745704	0.945481	0.221402
0.500	2.616318	1.503828	0.709859	0.919995	0.209101
0.600	2.667306	1.653315	0.686820	0.892560	0.194232
0.700	2.681665	1.826562	0.677247	0.864526	0.180135
0.800	2.662869	2.017941	0.682026	0.837165	0.170048
0.900	2.616698	2.219250	0.702372	0.811680	0.166365
1.000	2.551020	2.419884	0.739859	0.789238	0.170147
1.100	2.475058	2.607590	0.796335	0.771018	0.180988
1.200	2.398325	2.769714	0.873696	0.758265	0.197304
1.300	2.329538	2.894675	0.973438	0.752349	0.216979
1.400	2.275829	2.973409	1.095939	0.754824	0.238142
1.500	2.242377	3.000630	1.239524	0.767469	0.259801
1.600	2.232425	2.975807	1.399467	0.792301	0.282104
1.700	2.247475	2.903687	1.567313	0.831525	0.306143
1.800	2.287479	2.794118	1.731026	0.887377	0.333426
1.900	2.350940	2.660922	1.876297	0.961841	0.365219
2.000	2.434928	2.519919	1.988952	1.056200	0.402015
2.100	2.535087	2.386631	2.057807	1.170476	0.443317
2.200	2.637427	2.158512	1.990413	1.023744	0.392430

2.300	2.730410	1.995214	1.915718	0.913736	0.377382
2.400	2.812255	1.881079	1.839959	0.826688	0.381490
2.500	2.881578	1.804533	1.767707	0.755161	0.395429
2.600	2.937608	1.756788	1.701935	0.694860	0.413893
2.700	2.980204	1.730979	1.644356	0.643157	0.433945
2.800	3.009819	1.721565	1.595767	0.598345	0.454091
2.900	3.015093	1.881265	1.564783	0.602646	0.423904
3.000	2.986593	2.065799	1.561249	0.605740	0.408487
3.100	2.930088	2.267138	1.587246	0.609358	0.412666
3.200	2.853937	2.473861	1.644645	0.615311	0.439487
3.300	2.795944	2.362539	1.703281	0.687460	0.388194
3.400	2.779645	2.245140	1.727968	0.775416	0.345985
3.500	2.793930	2.138203	1.716330	0.880832	0.313754
3.600	2.828804	2.052462	1.670252	1.005312	0.293358
3.700	2.875722	1.994145	1.594534	1.150623	0.287689
3.800	2.927617	1.966507	1.495418	1.319072	0.300973
3.900	2.978817	1.971113	1.379381	1.514072	0.339285
4.000	3.024994	2.008745	1.252298	1.741007	0.411350
4.100	3.063202	2.079975	1.118953	2.008619	0.529864
4.200	3.077014	2.183212	0.999274	1.980261	0.520334
4.300	3.056553	2.302913	0.952968	1.940056	0.507611
4.400	3.010651	2.422882	0.923274	1.887492	0.487240
4.500	2.946869	2.533110	0.909804	1.822649	0.458874
4.600	2.873243	2.624842	0.911979	1.747030	0.422980
4.700	2.798080	2.691629	0.928373	1.678398	0.387391
4.800	2.728743	2.730311	0.956765	1.618058	0.352923
4.900	2.670390	2.740966	0.994817	1.566744	0.320633
5.000	2.626037	2.726245	1.040066	1.524749	0.291770
5.100	2.596914	2.690494	1.089948	1.492028	0.267609
5.200	2.582921	2.638908	1.141866	1.468264	0.249231
5.300	2.583049	2.576849	1.193294	1.452922	0.237333
5.400	2.595706	2.509371	1.241913	1.445286	0.232123
5.500	2.618954	2.440932	1.285736	1.444483	0.233301
5.600	2.650662	2.375245	1.323229	1.449519	0.240117
5.700	2.688607	2.315233	1.353386	1.459304	0.251483
5.800	2.730553	2.263046	1.375762	1.472700	0.266113
5.900	2.774322	2.220110	1.390455	1.488563	0.282670
6.000	2.817858	2.187206	1.398040	1.505795	0.299892
6.100	2.859299	2.164544	1.399472	1.523402	0.316692
6.200	2.897041	2.151835	1.395959	1.540536	0.332226
6.300	2.929807	2.148371	1.388842	1.556534	0.345918
6.400	2.956693	2.153091	1.379471	1.570938	0.357457
6.500	2.977206	2.164657	1.369105	1.583497	0.366766
6.600	2.991261	2.181532	1.358837	1.594152	0.373955
6.700	2.999168	2.202075	1.349543	1.603007	0.379265
6.800	3.001573	2.224636	1.341855	1.610295	0.383018
6.900	2.999381	2.247655	1.336161	1.616329	0.385570
7.000	2.993663	2.269756	1.332615	1.621468	0.387276
7.100	2.985552	2.289827	1.331167	1.626072	0.388462
7.200	2.976151	2.307065	1.331598	1.630474	0.389412
7.300	2.966450	2.321005	1.333562	1.634954	0.390359
7.400	2.957267	2.331504	1.336630	1.639723	0.391476
7.500	2.949214	2.338704	1.340337	1.644917	0.392879
7.600	2.942686	2.342980	1.344219	1.650592	0.394626

7.700	2.937870	2.344864	1.347852	1.656732	0.396723
7.800	2.934767	2.344980	1.350878	1.663256	0.399128
7.900	2.933227	2.343973	1.353031	1.670033	0.401761
8.000	2.932991	2.342457	1.354139	1.676898	0.404516
8.100	2.933726	2.340968	1.354132	1.683670	0.407273
8.200	2.935070	2.339937	1.353032	1.690165	0.409908
8.300	2.936660	2.339671	1.350938	1.696213	0.412307
8.400	2.938168	2.340353	1.348012	1.701669	0.414374
8.500	2.939315	2.342044	1.344455	1.706424	0.416037
8.600	2.939893	2.344701	1.340487	1.710406	0.417250
8.700	2.939767	2.348193	1.336328	1.713582	0.417995
8.800	2.938878	2.352326	1.332184	1.715960	0.418280
8.900	2.937236	2.356867	1.328232	1.717580	0.418134
9.000	2.934911	2.361570	1.324613	1.718508	0.417606
9.100	2.932020	2.366191	1.321428	1.718831	0.416755
9.200	2.928710	2.370515	1.318738	1.718646	0.415647
9.300	2.925145	2.374361	1.316564	1.718056	0.414351
9.400	2.921490	2.377598	1.314897	1.717161	0.412935
9.500	2.917899	2.380141	1.313697	1.716053	0.411460
9.600	2.914507	2.381959	1.312907	1.714815	0.409980
9.700	2.911420	2.383063	1.312456	1.713512	0.408539
9.800	2.908714	2.383505	1.312268	1.712200	0.407173
9.900	2.906437	2.383366	1.312267	1.710914	0.405905
10.000	2.904604	2.382744	1.312382	1.709681	0.404749

APPENDIX 6

We will prove property (iii) of the Lyapunov function (**2.2**) for the open food-chain system (**1.24**) in R^2.

Proof:

$$\frac{dV(\mathbf{x}(t))}{dt} = \sum_{s=1}^{2} \frac{\partial V}{\partial x_s} f_s(\mathbf{x})$$

$$= \frac{\partial V}{\partial x_1} f_1(\mathbf{x}) + \frac{\partial V}{\partial x_2} f_2(\mathbf{x})$$

$$= \gamma_1 x_1^o \left(\frac{1}{x_1^o} - \frac{1}{x_1}\right) x_1 \left(\alpha_1 - \frac{\beta_1}{\gamma_1} x_2\right) + \gamma_2 x_2^o \left(\frac{1}{x_2^o} - \frac{1}{x_2}\right) x_2 \left(-\alpha_2 + \frac{\beta_1}{\gamma_2} x_1\right)$$

$$= (x_1 - x_1^o)(\alpha_1\gamma_1 - \beta_1 x_2) + (x_2 - x_2^o)(-\alpha_2\gamma_2 + \beta_1 x_1)$$

Now, the equilibrium points for the open food-chain system (**1.24**) is given (in Section 2.1) by

$$E(x_1^o, x_2^o) = \left(\frac{\alpha_2\gamma_2}{\beta_1}, \frac{\alpha_1\gamma_1}{\beta_1}\right)^T.$$

Therefore,

$$\frac{dV}{dt} = \left(x_1 - \frac{\alpha_2\gamma_2}{\beta_1}\right)(\alpha_1\gamma_1 - \beta_1 x_2) + \left(x_2 - \frac{\alpha_1\gamma_1}{\beta_1}\right)(-\alpha_2\gamma_2 + \beta_1 x_1)$$

$$= \frac{1}{\beta_1} \{(x_1\beta_1 - \alpha_2\gamma_2)(\alpha_1\gamma_1 - \beta_1 x_2) + (x_2\beta_1 - \alpha_1\gamma_1)(-\alpha_2\gamma_2 + \beta_1 x_1)\}$$

$$= \frac{1}{\beta_1} (x_1\beta_1 - \alpha_2\gamma_2)[\alpha_1\gamma_1 - \beta_1 x_2 + x_2\beta_1 + \alpha_1\gamma_1]$$

$$= 0. \blacklozenge$$

APPENDIX 7

We will illustrate the proof of the first integral (Section 2.2)

$$V(x) = h, \qquad h \text{ constant} > 0. \qquad \textbf{(2.4)}$$

for an uncontrolled open food-chain in R^2.

Consider the uncontrolled open food-chain model **(1.24)** of length 2; the modelling equations are

$$\dot{x}_1 = x_1(\alpha_1 - \frac{\beta_1}{\gamma_1} x_2),$$

$$\dot{x}_2 = x_2(-\alpha_2 - \frac{\beta_1}{\gamma_2} x_1).$$

The solution to this system can be obtained by solving $\frac{dx_2}{dx_1}$. This gives us that

$$\frac{dx_2}{dx_1} = \frac{x_2(-\alpha_2 - \frac{\beta_1}{\gamma_2} x_1)}{x_1(\alpha_1 - \frac{\beta_1}{\gamma_1} x_2)}$$

$$\Rightarrow \int_{x_2^o}^{x_2} \frac{(\alpha_1 - \frac{\beta_1}{\gamma_1} x_2)}{x_2} dx_2 = \int_{x_1^o}^{x_1} \frac{(-\alpha_2 - \frac{\beta_1}{\gamma_2} x_1)}{x_1} dx_1$$

$$\Rightarrow \alpha_1 \ln \frac{x_2}{x_2^o} - \frac{\beta_1}{\gamma_1}(x_2 - x_2^o) = -\alpha_2 \ln \frac{x_1}{x_1^o} + \frac{\beta_1}{\gamma_2}(x_1 - x_1^o) - h$$

$$\Rightarrow \alpha_1 \ln \frac{x_2}{x_2^o} + \alpha_2 \ln \frac{x_1}{x_1^o} - \frac{\beta_1}{\gamma_1} x_2^o(\frac{x_2}{x_2^o} - 1) - \frac{\beta_1}{\gamma_2} x_1^o(\frac{x_1}{x_1^o} - 1) = h$$

$$\Rightarrow \frac{\alpha_1\gamma_1\gamma_2}{\beta_1} \ln \frac{x_2}{x_2^o} + \frac{\alpha_2\gamma_1\gamma_2}{\beta_1} \ln \frac{x_1}{x_1^o} + \gamma_2 x_2^o(\frac{x_2}{x_2^o} - 1) - \gamma_1 x_1^o(\frac{x_1}{x_1^o} - 1) = h$$

$$\Rightarrow \quad \gamma_1 x_1^o \left(\frac{x_1}{x_1^o} - 1\right) - \gamma_1 x_1^o \ln \frac{x_1}{x_1^o} + \gamma_2 x_2^o \left(\frac{x_2}{x_2^o} - 1\right) - \gamma_2 x_2^o \ln \frac{x_2}{x_2^o} = h$$

$$\Rightarrow \quad \sum_{s=1}^{2} \gamma_s x_s^o \left(\frac{x_s}{x_s^o} - 1 - \ln \frac{x_s}{x_s^o}\right) = h. \blacklozenge$$

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