A COMPARATIVE STUDY OF
MODAL PROPOSITIONAL SEMANTICS

by

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A DISSERTATION SUBMITTED IN PARTIAL FULFILLMENT

OF THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

in the Department

of

Mathematics

SIMON FRASER UNIVERSITY

June 1974

Approval

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Degree: Doctor of Philosophy

Title of Dissertation: A Comparative Study of Modal Propositional Semantics

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Title of Thesis/Dissertation:

A Comparative Study of Modal Propositional Semantics

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26 JUNE 1974

(date)

Abstract

We define six semantics (systems of structures) for the modal language (the language obtained by adding to the language of classical propositional calculus the single unary connective □): the boolean semantics (derived from the algebraic structures of McKinsey and Tarski [9]), the neighbourhood (Scott-Montague) semantics, the relational (Kripke) semantics, the first-order semantics (Makinson's[10] generalized "relational models"), and finally the connected and superconnected semantics. The dissertation studies the strengths of these semantics in various ways: first, with respect to containment of one within another and then with respect to width and depth (measures of the "span" and "density" of the set of logics which are complete with respect to the semantics in question).

Previously known results in these areas are noted and some other results are proved. In particular, we note that the width of the relational semantics is that of normal logics and that any relational frame is in fact a neighbourhood frame, so that the "ground" common to both semantics is that of normal logics. Among the major results of the dissertation is the presentation of two normal logics, one between T and S4 and the other an extension of S4, which are incomplete with respect to the neighbourhood semantics, thus showing that, unlike the boolean and first-order semantics,

the neighbourhood semantics does not have maximal depth even with respect to normal logics. We also exhibit a neighbourhood frame which models T and is equivalent to no set of relational frames, thus showing that the relational semantics has even less depth than the neighbourhood. The connected and superconnected semantics are shown to have the same depth as the relational semantics.

Finally, we discuss some properties of structures in a semantics which can be "described" directly in the modal language; these are called modal properties. We note that all modal properties are second-order but that the converse in not true. We can, however, through a system of reductions, use the modal language to describe indirectly all second-order properties.

Acknowledgments

Among all the people and institutions who have helped me in my work on this thesis, four particularly must be mentioned. The first is the National Research Council of Canada, whose assistance through a Postgraduate Scholarship freed me to spend more time on research than otherwise would have been possible. The second is Steven Thomason, whose interest and work in Modal Logic inspired mine. The third is Ronald Harrop, whose help at all stages of my work, from the initial development of my research to the proof reading of my amateur typing, has been constant and invaluable. His willingness to plunge with me into areas quite new to him and his continued ability to make helpful suggestions have been admired and appreciated. To him I am particularly grateful. The fourth is Carole, who has managed to stagger her periods of insanity with mine so well that we have both been able to stay on our feet.

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1. Introduction: Some Terminology and Background

1.0 The study of Mathematical Logic depends on a number of basic concepts. First, there is the concept of a formal language: a set of symbols and formulae constructed from these symbols. Then there is the concept of proof and from this comes the notion of a theory or logic. These may be called the syntactical aspects of Mathematical Logic. We also have the semantical aspects of Logic. For each language we may have a particular concept of structures or models for that language.

In some of the major branches of Mathematical Logic we study different aspects of these concepts. We may restrict ourselves, for example, to the study of languages (or one particular language) and proofs in the language(s), and in this case we have Proof Theory. Or we may study the relationships between a class of languages, such as first-order languages, and various structures or models for them, without particular emphasis on the notion of proof. Here we have Model Theory. On the other hand, we may restrict ourselves to one particular language, one notion of proof in that language, and various models or structures for it, such as in Set Theory or Arithmetic.

In this thesis we study one particular language, the Modal Propositional Language, and compare various concepts of structure for it, without regard to the notion of provability.

1.1 We begin by trying to make precise some general terms that we shall use throughout.

By a *language* we shall mean a set of symbols together with a notion of *formulae* which will be certain finite sequences of the symbols. The set of formulae in a language is usually defined inductively.

By a *semantics* for a language we shall mean a class of "sets with structure" together with a notion of *validity*, the type of structure being fixed for a particular semantics. The elements of the class, i.e. the sets with structure, will be called simply *structures*. Together with each structure goes a set of formulae called the formulae *valid* in the structure. The notion of validity in structures is extended to classes of structures in the following way. A formula is valid in a class of structures (all in the same semantics) if and only if it is valid in each structure in the class. If $\mathfrak{A}$ is a semantics we shall often use a mild abuse of notation and write $S \in \mathfrak{A}$ to mean that S is a structure in the semantics $\mathfrak{A}$. If A is a formula, we write $S \models A$ to mean that A is valid in S; we also say in this case that S *models* A.

By a *logic* on a language we mean simply a particular set of formulae of the language. If K is a logic on a language $\mathcal{L}$ and $\mathfrak{A}$ is a semantics for $\mathcal{L}$ and $S \in \mathfrak{A}$, then we write $S \models K$ to mean that every member of K is valid in S.

If $\mathcal{S} \models K$ we say that $\mathcal{S}$ is a model of K or $\mathcal{S}$ models K. K is *the logic determined by* $\mathcal{S}$ if it is the set of all formulae valid in $\mathcal{S}$. Similarly, we refer to the logic determined by a class of structures in $\mathfrak{A}$. If K is a logic we write $K \vdash A$ to mean $A \in K$.

By a *system* we shall mean a pair $(\mathcal{L}, \mathfrak{A})$ where $\mathcal{L}$ is a language and $\mathfrak{A}$ a semantics for $\mathcal{L}$. If $\mathcal{C}$ is a set of logics on $\mathcal{L}$ then we can compare $\mathcal{C}$ with the set of logics determined by structures, or classes of structures, in $\mathfrak{A}$. If each logic in $\mathcal{C}$ is determined by some structure or class of structures in $\mathfrak{A}$, then we say that *$\mathfrak{A}$ is adequate for $\mathcal{C}$*. (We shall later indroduce the notion of "depth" and, at least for the modal language, shall say that *$\mathfrak{A}$ has maximal depth with respect to $\mathcal{C}$* in this case.) If K is a particular logic on $\mathcal{L}$ we say that *$\mathfrak{A}$ is adequate for K* or that *K is complete with respect to $\mathfrak{A}$* if for each formula A in $\mathcal{L}$ either $K \vdash A$ or there is $\mathcal{S} \in \mathfrak{A}$ with $\mathcal{S} \models K$ and $\mathcal{S} \not\models A$ (i.e. every formula not in K is not valid in some model of K in $\mathfrak{A}$).

1.2 As already stated, we shall, in this thesis, be concentrating on one particular language, the modal language $\mathcal{M}$, and on certain semantics for $\mathcal{M}$. The study of the modal language, its semantics and its proof theory is usually called Modal Logic. Modal Logic in its modern form was originated by Lewis and Langford [8] in 1932. Their concern was entirely syntactical. Early semantical studies in Modal Logic were made by McKinsey and Tarski [9] and later by Saul Kripke (whose name is usually attached to the relational semantics) and Krister Segerberg [14] (who fully developed the neighbourhood semantics). There are a number of surveys of Modal Logic made from an almost exclusively syntactical point of view, but the only extensive semantical studies or surveys are to our knowledge those of Segerberg[14] and of Hansson and Gärdenfors [6].

1.3 There are two approaches that we can take in studying the relationships between languages and semantics. We can accept the language as primary and think of each semantics as a way of interpreting the language. In this case we would study the relative strengths of the various semantics with respect to various things, including adequacy and completeness.

On the other hand, we may accept the semantics as primary and consider the language as something with which we can "talk about" the structures in the semantics. In the case of this approach it is interesting to see how powerful the language is for describing various properties of structures. We may wish somehow to compare the powers of various languages for describing certain structures.

We take the first approach in chapters 4 to 9 and the second in chapter 10 to 12.

1.4 A major problem with the development of the literature in Modal Logic has been one of terminology. Some Modal Logicians come from Mathematics, others from Philosophy, and each seems to bring the terminology of his background with him. Thus, for example, the word "model" is used in different and incompatible ways by different authors. What we are calling a "structure" is sometimes called a "model", what we are calling a "logic" is sometimes called a "theory", a "calculus", or a "system"; what some call "theorems" are called "theses" by others; and so on. The notation, also, has not been standardized. We use $\Box$ and $\Diamond$ as necessity and possibility operators, but many other authors use L and M. We therefore make an effort to define out terminology and notation as clearly as possible. We have attempted wherever possible to use terms and notation in a manner consistent with common usage among Mathematical Logicians.

1.5 We have tried to keep this thesis self-contained with respect to its Logic content, so that an intelligent reader with no more than an undergraduate background in Mathematical Logic and in Boolean Algebra should be able to follow it in its entirety without making outside references to the literature in Logic. A few major results from Boolean Algebra and Topology are used and references to major texts in those areas are given. Modal Logic results are given with complete proofs, with a couple of minor exceptions in chapters 10 and 12. The results of chapter 7 also appear in Gerson [3].

1.6 Sections are numbered beginning anew with each chapter and theorems, lemmas, corollaries, and definitions are numbered sequentially beginning anew with each section. Thus 4.6.8 refers to the eighth item (either a theorem, lemma, corollary, or definition) of the sixth section of chapter 4. Ends of proofs are noted with the symbol XXX. The last theorem in the thesis, 12.7.1, has a particularly long and tedious proof and so the details of this proof are given in an appendix.

2. The Modal Language

2.1 The (propositional) modal language, M, is the language obtained by adding the single unary connective $\Box$ to the classical propositional language. Thus, the symbols of the language comprise an infinite sequence of propositional variables, $p_1, p_2, \ldots$, together with the symbols $\sim$, $\rightarrow$, $\Box$, and parentheses, (and). The formulae are defined inductively: each propositional variable p_i is a formula; if A and B are formulae, then so are $\sim A$, $(A \rightarrow B)$, and $\Box A$.

To simplify our notation we use the defined symbols $\vee$, $\wedge$, $\leftrightarrow$, and $\Diamond$. If A and B are any formulae,we may write $(A \vee B)$ for $(\sim A \rightarrow B)$, $(A \wedge B)$ for $\sim(A \rightarrow \sim B)$, $(A \leftrightarrow B)$ for $((A \rightarrow B) \wedge (B \rightarrow A))$, and $\Diamond A$ for $\sim\Box\sim A$. We also omit parentheses where it is possible to do so without ambiguity. The usual hierarchy of connectives is observed: first $\leftrightarrow$, then $\rightarrow$, then $\vee$ and $\wedge$, and finally $\sim$, $\Box$, and $\Diamond$. (Those appearing later in the list bind closer.) Thus $\Box\sim p_1 \vee p_2 \leftrightarrow p_3 \rightarrow \Diamond p_4 \wedge p_5$ is unambiguously interpreted as $((\Box\sim p_1 \vee p_2) \leftrightarrow (p_3 \rightarrow (\Diamond p_4 \wedge p_5)))$. We often use lower case letters q, r, s, t with or without subscripts to represent arbitrary (but fixed and distinct in a given context) propositional variables. The connectives $\Box$ and $\Diamond$ are called <u>necessity</u> and <u>possibility</u> operators respectively, and we say "necessarily A" and "possibly A" for $\Box A$ and $\Diamond A$.

The following definition will be useful later.

2.1.1 Definition: If A is a modal formula then we say that the formula B occurs as an elementary subformula in A if and only if B is a propositional variable or is $\Box C$ for some C and B has an occurrance as a subformula of A which does not lie within the scope of any $\Box$ connective. In other words, B is not a proper subformula of any subformula of A of the form $\Box D$.

For example, if A is $\Box(\Box p \to q) \to (\Box r \to s)$ then $\Box(\Box p \to q)$ $\Box r$, s all have elementary subformula occurrences in A.

2.2 A formula of the modal language is a tautology if it is a substitution instance of a tautology of the classical propositional logic. A modal logic is a logic on the modal language which contains all tautologies and is closed under substitution (Sub) and modus ponens (MP). Thus if L is a modal logic, A is a formula such that $L \vdash A$, p_i is a variable with occurrences in A, C is any formula, and A is the formula obtained by replacing all occurrences of p_i in A with occurrences of C, then $L \vdash B$; if A and B are any modal formulae such that $L \vdash A \to B$ and $L \vdash A$, then $L \vdash B$.

2.3 A modal logic L is classical if it is closed under the rule of equivalence

$$\text{RE: if } L \vdash A \leftrightarrow B \text{ then } L \vdash \Box A \leftrightarrow \Box B.$$

The formula

$$\Box(p_1 \to p_2) \to (\Box p_1 \to \Box p_2)$$

will be denoted as [K]. A modal logic L is normal if L $\vdash$ K and L is closed under the rule of necessitation

RN: if L $\vdash$ A then L $\vdash$ $\Box$A.

2.3.1 Theorem: If L is normal and L $\vdash$ $A \to B$, then L $\vdash$ $\Box A \to \Box B$ and L $\vdash$ $\Diamond A \to \Diamond B$.

Proof: By RN, L $\vdash$ $\Box(A \to B)$. From K and Sub, L $\vdash$ $\Box(A \to B) \to (\Box A \to \Box B)$. Therefore by MP, L $\vdash$ $\Box A \to \Box B$. Now L $\vdash$ $A \to B$ $\Rightarrow$ L $\vdash$ $\sim B \to \sim A$ $\Rightarrow$ L $\vdash$ $\Box\sim B \to \Box\sim A$ $\Rightarrow$ L $\vdash$ $\sim\Box\sim A \to \sim\Box\sim B$ $\Rightarrow$ L $\vdash$ $\Diamond A \to \Diamond B$. XX

2.3.2 Theorem: Every normal logic is classical.

Proof: Suppose L is normal and L $\vdash$ $A \leftrightarrow B$. Then L $\vdash$ $A \to B$ and so by 2.3.1 L $\vdash$ $\Box A \to \Box B$. Similarly L $\vdash$ $\Box B \to \Box A$, and therefore L $\vdash$ $\Box A \leftrightarrow \Box B$. XX

2.4 We mention four particular modal logics which are of general philosophical interest and to which we shall have occasion to refer later. The smallest classical logic is called E; the smallest normal logic is called K (no ambiguity should arise from this double use of the symbol); the smallest normal logic containing the formula $\Box p_1 \to p_1$ is called T; the smallest normal logic containing T and the formula $\Box p_1 \to \Box\Box p_1$ is called S4.

3. Modal Semantics

3.0 We shall consider six semantics for the modal language: the boolean semantics $(\mathfrak{B})$, the neighbourhood or Scott-Montague semantics $(\mathfrak{S})$, the relational or Kripke semantics $(\mathfrak{R})$, the first-order relational semantics $(\mathfrak{F})$, and two "subsemantics" or restricted versions of $\mathfrak{R}$, the connected relational semantics $(\hat{\mathfrak{R}})$ and the superconnected relational semantics $(\bar{\mathfrak{R}})$. Structures in any of these semantics are commonly called *frames*, and we shall continue this practice.

3.1 A *boolean frame* is a sextuple $\mathcal{B} = (B,0,1,-,\cap,*)$ where $(B,0,1,-,\cap)$ is a non-trivial boolean algebra (i.e., 0 and 1 are distinct elements) and $*$ is any function from B to B. The function, or operator, $*$, is called an *interior* operator and if $b \in B$ then $*b$ will be called the interior of b. If $\mathcal{B} = (B,0,1,-,\cap,*)$ is a boolean frame, an *assignment*, V, on $\mathcal{B}$ is a function from $\mathbb{N}$, the set of (positive) natural numbers, to B. Each assignment V determines a function, also called V, from the set of modal formulae to B as follows: $V(p_i) = V(i)$ for each $i \in \mathbb{N}$; $V({\sim}A) = -V(A)$: $V(A \to B) = -(V(A) \cap -V(B))$; and $V(\Box A) = *V(A)$. A formula A is *valid in* $\mathcal{B}$ if $V(A) = 1$ for all assignments V.

The boolean semantics is a generalization of the closure algebras of McKinsey and Tarski [9]. "Boolean frames" was the term used by Hansson and Gärdenfors [6].

Makinson [10] refers to the same things as "modal algebras".

The boolean semantics is denoted by $\mathfrak{B}$.

The following lemma will be used frequently, usually without specific reference to it.

__3.1.1__ __Lemma:__ If $\mathcal{B} = (B,0,1,-,\cap,*)$ is a boolean frame and V an assignment on $\mathcal{B}$, then for any formulae A,B

(a) $V(A\wedge B) = V(A)\cap V(B)$,

(b) $V(A\vee B) = -(-V(A)\cap -V(B)) = V(A)\cup V(B)$,

(c) $V(A\rightarrow B) = 1$ if and only if $V(A) \leq V(B)$,

(d) $V(A\longleftrightarrow B) = 1$ if and only if $V(A) = V(B)$.

Proof: (a) $V(A\wedge B) = V(\sim(A\rightarrow\sim B)) = -V(A\rightarrow\sim B) = -(-(V(A)\cap -V(\sim B))) = --(V(A)\cap --V(B)) = V(A)\cap V(B)$.

(b) $V(A\vee B) = V(\sim A\rightarrow B) = -(V(\sim A)\cap -V(B)) = -(-V(A)\cap -V(B)) = V(A)\cup V(B)$.

(c) $V(A\rightarrow B) = 1 \Leftrightarrow -(V(A)\cap -V(B)) = 1 \Leftrightarrow V(A)\cap -V(B) = 0 \Leftrightarrow V(A) \leq V(B)$.

(d) $V(A\longleftrightarrow B) = 1 \Leftrightarrow V((A\rightarrow B)\wedge(B\rightarrow A)) = 1 \Leftrightarrow V(A\rightarrow B) = 1$ and $V(B\rightarrow A) = 1 \Leftrightarrow V(A) \leq V(B)$ and $V(B) \leq V(A)$ $\Leftrightarrow V(A) = V(B)$. XXX

__3.2__ A __neighbourhood frame__ is a pair $\mathcal{F} = (U,\mathcal{N})$ where U is any set and $\mathcal{N}$ any function from U to $\mathcal{P}(\mathcal{P}(U))$. If $u \in U$ we write $\mathcal{N}_u$ instead of $\mathcal{N}(u)$, and if $S \in \mathcal{N}_u$ we say that "S is a neighbourhood of u". Thus $\mathcal{N}$ assigns to each point a set of neighbourhoods of the point. An assignment, V, on $\mathcal{F}$ is a function from $\mathbb{N}$ to $\mathcal{P}(U)$. Each assignment V determines a function, also called V, from the set of formulae to $\mathcal{P}(U)$ as follows: $V(p_i) = V(i)$ for all $i \in \mathbb{N}$, $V(\sim A) = U-V(A)$, $V(A\rightarrow B) = - V(A) \cup V(B)$, and $V(\Box A) = \{ u \mid V(A) \in N_u \}$. For $u \in U$, we sometimes write $V(A,u) = T$ if $u \in V(A)$ and $V(A,u) = F$ if $u \notin V(A)$. We say that a formula A is <u>valid at a point</u> u if $V(A,u) = T$ for all assignments V, and that A is <u>valid</u> in $\mathcal{F}$ if A is valid at all points in U.

Thus, we think of each assignment as assigning truth or falsity to each propositional variable at each point in the frame, the assignment of truth or falsity being extended through the boolean connectives in the usual manner and a formula $\Box A$ being assigned truth at a point if A is true <u>exactly</u> on a neighbourhood of the point.

The idea of a semantics based on neighbourhoods is due originally to Scott [13] and Montague [11] and is more fully developed by Segerberg [14] who uses the term "neighbourhood frame". Hansson and Gärdenfors [6] refer to them as "Scott-Montague" frames.

The neighbourhood semantics is denoted by $\mathcal{S}$.

<u>3.3</u> A <u>relational frame</u> is a pair $\mathcal{W} = (W,<)$ where W is a set and $<$ is a binary relation on the set. If $u,v \in W$ and $u<v$ we say that v is a successor of u or that v is accessible from u. An assignment, V, on $\mathcal{W}$ is, as in the neighbourhood semantics, a function from $\mathbb{N}$ to $\mathcal{P}(W)$, and

determines a function, also called V, from the set of formulae to $\mathcal{P}(U)$ as follows: $V(p_i) = V(i)$ for all $i \in \mathbb{N}$, $V({\sim}A) = W - V(A)$, $V(A \to B) = -V(A) \cup V(B)$, and $V(\Box A) = \{ u \mid \{v \mid u<v\} \subset V(A) \}$.

As with the neighbourhood semantics, we write $V(A,u) = T$ or F according as $u \in V(A)$ or not. Again, a formula A is *valid at a point* u if $V(A,u) = T$ for all assignments V, and A is *valid* in $\mathcal{W}$ if A is valid at all u in W. So the assignment is again thought of as assigning truth or falsity to each variable at each point and hence to each formula at each point, this time $\Box A$ being true at u if A is true at all successors of u (and maybe at other points as well).

The relational semantics is due to Saul Kripke and, in fact, relational frames are often referred to by authors as "Kripke structures" or "Kripke models".

The relational semantics is denoted by $\mathbb{R}$.

3.4 A *first-order relational frame* is a triple, $\mathcal{U} = (U,<,\Pi)$ where $(U,<)$ is a relational frame and Π is a field of subsets of U (a boolean subalgebra of the boolean algebra of all subsets of U) which separates points (for $u,v \in U$ there is $S \in \Pi$ with $u \in S$ and $v \notin S$) and is closed under the operation $\overset{*}{<}$, where for $S \subset U$, $\overset{*}{<}S = \{ u \mid \{v \mid u<v\} \subset S \}$. In this case, an assignment, V, on $\mathcal{U}$ is a function from $\mathbb{N}$ to Π. Since $\Pi \subset \mathcal{P}(U)$, the assignments on $\mathcal{U}$ comprise a

subset of the set of assignments on the relational frame $(U,<)$. They are extended to functions from the set of all formulae to $\mathcal{P}(U)$ as with the relational semantics, and since Π is closed under $-$, $\cap$, and $\overset{*}{<}$ we see that such an extension will be a function from the set of formulae to Π.

We write $V(A,u) = T$ or F as before and define validity of a formula at a point or in a frame as with relational frames, keeping in mind that we have fewer assignments and that therefore validity is easier to obtain. That is,

<u>3.4.1</u> <u>Theorem</u>: If $\mathcal{U} = (U,<,\Pi)$ is a first-order frame, and if a formula A is valid in the relational frame $(U,<)$, then A is valid in $\mathcal{U}$. XXX

The first-order relational semantics is due to Makinson [10]. Makinson calls his frames, which are basically first-order frames, simply "relational models" and notes that they are generalizations of the usual (Kripke) relational models (our relational frames). Fine [2], studying Modal Logic with propositional quantifiers recognizes the possibility of treating the quantifiers as ranging over a possibly restricted class of sets of points in an otherwise Kripke-type semantics.

Thomason [22] defines a first-order semantics for Tense Logic, and it is from him that we take our terminology in this case.

The first-order semantics is denoted by $\mathcal{F}$.

3.5 Let $\mathcal{W} = (W,<)$ be a relational frame. If $u,v \in W$ we say that u and v are *directly connected* (via $<$) if either $u<v$, $v<u$, or $u = v$. We say that u and v are *connected* if there is a finite sequence $u=u_1, u_2, \ldots, u_m=v$ of points in W such that each is directly connected to the next. We say that $\mathcal{W}$ is connected if u and v are connected for each pair u,v of points in W. It is clear that connectedness is an equivalence relation on W; in fact, it is the smallest equivalence relation containing $<$.

The restriction of the relational semantics to connected relational frames is called the *connected* (*relational*) *semantics*. Thomason [19,20,21] on occasion considers connected frames for tense languages; this author has not discovered other references to them.

The connected semantics is denoted by $\hat{\mathcal{R}}$.

3.6 We shall later find it convenient to consider a further restriction of the relational semantics. If $\mathcal{W} = (W,<)$ is a *connected* relational frame, let W_E denote $\{ u \in W \mid \nexists v \in W \text{ with } u<v \}$; that is, W_E is the set of points in W with no successors. Now, if $\tilde{\mathcal{W}}$ is the frame $(W-W_E, \tilde{<})$ where $\tilde{<}$ is the restriction of $<$ to $W-W_E$, then we say that $\mathcal{W}$ is *superconnected* if $\tilde{\mathcal{W}}$ is connected.

The restriction of the relational semantics to superconnected frames is called the superconnected semantics and is denoted by $\overline{R}$.

4. The Semantics Hierarchy

4.0 In this chapter we shall study a strong relationship between the six semantics. We shall show that some of the semantics can be said to be actually contained in others in the following sense. If $\mathcal{C}$ and $\mathcal{D}$ are semantics for the same language and if there is a mapping from the structures in $\mathcal{C}$ to the structures in $\mathcal{D}$ which is one-to-one (up to isomorphism) and which preserves validity and non-validity of formulae, then we say that $\mathcal{C}$ is contained in or is a subsemantics of $\mathcal{D}$. In the cases we shall study, those of the six modal semantics already introduced, the mapping will be seen to be a natural one in a very real sense.

4.1 We have already noted the most obvious examples of subsemantics. Clearly, $\hat{\mathcal{R}}$ is a subsemantics of $\mathcal{R}$ and $\overline{\mathcal{R}}$ is a subsemantics of $\hat{\mathcal{R}}$. In these cases the mapping is simply the identity mapping since a frame in $\hat{\mathcal{R}}$ *is* in $\mathcal{R}$ and one in $\overline{\mathcal{R}}$ *is* in $\hat{\mathcal{R}}$.

4.2 Almost as straightforward is the case of $\mathcal{R}$ and $\mathcal{F}$. The mapping $\mathcal{W} = (W,<) \longrightarrow \mathcal{W}' = (W,<,\mathcal{P}(W))$ is clearly a mapping from the frames in $\mathcal{R}$ to those in $\mathcal{F}$ having the desired properties. Thus $\mathcal{R}$ is a subsemantics of $\mathcal{F}$.

4.3 We shall show that $\mathcal{R}$ is a subsemantics of $\mathcal{S}$. If $\mathcal{W} = (W,<)$ is a relational frame, then we define a function $N^{<}$ from W to $\mathcal{P}(\mathcal{P}(W))$ by $N^{<}_{u} = \{ S \subset W \mid \{v|u<v\} \subset S \}$. Let $F_{\mathcal{W}}$

be the neighbourhood frame $(W, N^{<})$. We must show that the mapping $\mathcal{W} \dashrightarrow \mathcal{F}_{\mathcal{W}}$ is one-to-one (up to isomorphism) and preserves validity and non-validity. If $\mathcal{F}_{\mathcal{W}} = \mathcal{F}_{\mathcal{U}}$ where $\mathcal{W} = (W, <)$ and $\mathcal{U} = (U, \Delta)$, then $W = U$ and $N^{<} = N^{\Delta}$. Thus for $u \in W$ $(=U)$ we have $\{ S \mid \{v|u<v\} \subset S \} = \{ S \mid \{v|u\Delta v\} \subset S \}$, thus $\{v|u<v\} = \{v|u\Delta v\}$, and so $u<v$ iff $u\Delta v$. Hence $< = \Delta$ and $\mathcal{W} = \mathcal{U}$.

Let V be an assignment on the relational frame $\mathcal{W} = (W, <)$. Then V can also be thought of as a corresponding assignment on $\mathcal{F}_{\mathcal{W}} = (W, N^{<})$, since the underlying sets of $\mathcal{W}$ and $\mathcal{F}_{\mathcal{W}}$ are the same and since V is simply a mapping from $\mathbb{N}$ to that underlying set in either case. We shall see that the extension of V to formulae is the same in $\mathcal{W}$ as in $\mathcal{F}_{\mathcal{W}}$, in other words that for any formula A, V(A) in $\mathcal{W}$ is the same subset of W as V(A) in $\mathcal{F}_{\mathcal{W}}$.

This is certainly true if A is a variable since in either case $V(p_i) = V(i)$ for all $i \in \mathbb{N}$. If A is $\sim B$ or $B \rightarrow C$ then we recall from 3.2 and 3.3 that in both $\mathcal{W}$ and $\mathcal{F}_{\mathcal{W}}$, $V(A) = -V(B)$ or $V(A) = -V(B) \cup V(C)$ respectively. If A is $\Box B$, then V(A) in $\mathcal{W}$ is $\{ u \mid \{v|u<v\} \subset V(B) \} = \{ u \mid V(B) \in N^{<}_{u} \}$ which is V(A) in $\mathcal{F}_{\mathcal{W}}$.

Since V ranges over all assignments on $\mathcal{F}_{\mathcal{W}}$ as it ranges over all assignments on $\mathcal{W}$, we see that $\mathcal{W} \models A$ iff $\mathcal{F}_{\mathcal{W}} \models A$. So the mapping preserves validity and lack of it. Hence, $\mathbb{R}$ is a subsemantics of $\mathbb{S}$. In the remainder of the thesis we shall often identify $\mathcal{W}$ with $\mathcal{F}_{\mathcal{W}}$ and consider a relational frame to be a certain type of neighbourhood frame.

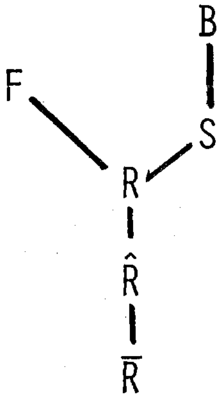
4.4 We now show that $\mathbb{S}$ is contained in $\mathbb{B}$. If $F = (U,N)$ is a neighbourhood frame then we define an operator $\overset{*}{_N}$ on $\mathcal{P}(U)$ as follows. If $S \subset U$, then $\overset{*}{_N}S = \{ u \in U \mid S \in N_u \}$. Let B_F be the boolean frame $(\mathcal{P}(U), \phi, U, -, \cap, \overset{*}{_N})$. Then we claim that the mapping $F \dashrightarrow B_F$ is one-to-one and preserves validity and non-validity. Suppose that $B_F = B_G$ where $F = (U,N)$ and $G = (W,M)$ are neighbourhood frames. Then $U = W$ and $\overset{*}{_N} = \overset{*}{_M}$. Thus, for all $S \subset U$ $(=W)$ $\{ u \in U \mid S \in N_u \} = \{ u \in U \mid S \in M_u \}$, hence for all $S \subset U$ and for all $u \in U$, $S \in N_u$ iff $S \in M_u$. So for all $u \in U$, $N_u = M_u$, therefore $N = M$ and thus $F = G$.

It remains to show that if A is any formula, then $F \models A$ iff $B_F \models A$. Let V be any assignment on F. Then V is a function from $\mathbb{N}$ to $\mathcal{P}(U)$ and so can also be thought of as an assignment on B_F. It suffices, then, to show that the extension in F of each assignment V to formulae is the same as the extension of V in B_F; that is, for each formula A, V(A) in F is the same subset of U (element of $\mathcal{P}(U)$) as V(A) in B_F.

If A is a variable then $V(p_i) = V(i)$ in F and in B_F and if A is $\sim B$ then $V(A) = -V(B)$ in F and in B_F. If A is $B \to C$ then in F, $V(A) = -V(B) \cup V(C) = -(V(B) \cap -V(C))$ which is V(A) in B_F. Finally, if A is $\Box B$ then in F, $V(A) = \{ u \mid V(B) \in N_u \} = \overset{*}{_N}V(B)$ which is V(A) in B_F. Thus V(A) in F is V(A) in B_F and so $F \models A$ iff $B_F \models A$.

We shall often identify F with B_F and thus consider a neighbourhood frame to be a particular type of boolean frame. Thus a relational frame is, in turn, also a particular type of boolean frame. In fact, we shall identify $\mathcal{W} = (W,<)$ with $B_{F_{\mathcal{W}}} = (\wp(U),\phi,U,-,\cap,\overset{*}{N}_{<})$ where we note that $\overset{*}{N}_{<}S =$ $\{ u \mid \{v|u<v\} \subset S \}$. We also note that lemma 3.1.1 can be applied to neighbourhood and relational frames, and, since it deals with single assignments, to first-order frames, as well as to boolean frames.

4.5 We have the following containment diagram for modal semantics.



We cannot show that F is a subsemantics of B (or vice versa), but we shall, in chapter 6, construct a mapping Ψ from the structures of F to the structures of B which preserves validity and non-validity but which is not one-to-one up to isomorphism.

4.6 In order to further understand the subsemantics relationship between the various semantics it will be necessary now to prove some general properties of structures in $\mathcal{B}$, $\mathcal{F}$, $\mathcal{R}$, and $\mathcal{S}$.

4.6.1 Definition: A frame in any modal semantics (any semantics for the modal language) is classical if it determines a classical logic.

4.6.2 Theorem: Every boolean (hence every neighbourhood, every relational, every connected and every superconnected) frame is classical.

Proof: Let $\mathcal{B} = (B, 0, 1, -, \cap, *)$ be a boolean frame and suppose that $\mathcal{B} \models A \leftrightarrow B$. Then by 3.1.1 for each assignment V on $\mathcal{B}$ $V(A) = V(B)$, so $*V(A) = *V(B)$, $V(\Box A) = V(\Box B)$, and (3.1.1 again) $V(\Box A \leftrightarrow \Box B) = 1$; hence $\mathcal{B} \models \Box A \leftrightarrow \Box B$. XXX

4.6.3 Definition: A frame in any modal semantics is normal if it determines a normal logic.

4.6.4 Lemma: If L is a classical logic closed under RN and such that $L \vdash \Box(p \wedge q) \leftrightarrow (\Box p \wedge \Box q)$ and $L \vdash A \rightarrow B$, then $L \vdash \Box A \rightarrow \Box B$.

Proof: $L \vdash A \rightarrow B \Rightarrow$ (taut) $L \vdash A \leftrightarrow (A \wedge B) \Rightarrow$ (RE) $L \vdash \Box A \leftrightarrow \Box(A \wedge B)$. But $L \vdash \Box(p \wedge q) \leftrightarrow (\Box p \wedge \Box q) \Rightarrow$ (Sub) $L \vdash \Box(A \wedge B) \leftrightarrow (\Box A \wedge \Box B) \Rightarrow L \vdash \Box(A \wedge B) \rightarrow \Box B$. Therefore $L \vdash \Box A \rightarrow \Box B$. XXX

4.6.5 Lemma: If L is a classical logic closed under RN, then $L \vdash K$ if and only if $L \vdash \Box(p \wedge q) \leftrightarrow (\Box p \wedge \Box q)$.

Proof: Suppose that $L \vdash K$. Then (taut.) $L \vdash (p\to(q\to(p\wedge q))$; thus (2.3.1) $L \vdash \Box p\to\Box(q\to(p\wedge q))$ and (K,sub) $L \vdash \Box(q\to(p\wedge q))\to(\Box q\to\Box(p\wedge q))$; hence $L \vdash \Box p\to(\Box q\to\Box(p\wedge q))$ and so (taut.) $L \vdash \Box p\wedge\Box q\to\Box(p\wedge q)$. Also $L \vdash p\wedge q\to p$ and $L \vdash p\wedge q\to q$ (taut.) and so (2.3.1) $L \vdash \Box(p\wedge q)\to\Box p$ and $L \vdash \Box(p\wedge q)\to\Box q$: thus $L \vdash \Box(p\wedge q)\to(\Box p\wedge\Box q)$ and $L \vdash \Box(p\wedge q)\leftrightarrow(\Box p\wedge\Box q)$.

Now suppose $L \vdash \Box(p\wedge q)\leftrightarrow(\Box p\wedge\Box q)$. Then (sub) $L \vdash \Box(p_1\to p_2)\wedge\Box p_1 \to \Box((p_1\to p_2)\wedge p_1))$; but also (taut.) $L \vdash ((p_1\to p_2)\wedge p_1)\to p_2$ and so (4.6.4) $L \vdash \Box((p_1\to p_2)\wedge p_1)\to\Box p_2$. Thus $L \vdash (\Box(p_1\to p_2)\wedge\Box p_1)\to\Box p_2$ and therefore $L \vdash \Box(p_1\to p_2)\to(\Box p_1\to\Box p_2)$, i.e. $L \vdash K$. XXX

4.6.6 **Theorem:** A boolean frame $\mathcal{B} = (B,0,1,-,\cap,*)$ is normal if and only if $*1 = 1$ and for all $b,c \in B$, $*(b\cap c) = *b\cap *c$.

Proof: Suppose $\mathcal{B}$ is normal. Then for all assignments V, $V(\Box(p_1\to p_1)) = 1$. But $V(\Box(p_1\to p_1)) = *V(p_1\to p_1) = *1$. So $*1 = 1$. If $b,c \in B$, pick V such that $V(p_1) = b$ and $V(p_2) = c$. Then (4.6.5 and 3.3.1(d)) $V(\Box(p_1\wedge p_2) = V(\Box p_1\wedge\Box p_2)$; so $*(V(p_1)\cap V(p_2)) = *V(p_1)\cap *V(p_2)$; thus $*(b\cap c) = *b\cap *c$.

Now suppose $*1 = 1$ and for all $b,c \in B$, $*(b\cap c) = *b\cap *c$. If $\mathcal{B} \models A$ then $V(A) = 1$ for all assignments V on $\mathcal{B}$. Thus $V(\Box A) = *V(A) = *1 = 1$ for all V on $\mathcal{B}$ and so $\mathcal{B} \models \Box A$. For any V on $\mathcal{B}$, $V(\Box(p_1\wedge p_2)\leftrightarrow(\Box p_1\wedge\Box p_2)) = 1 \Leftrightarrow *(V(p_1)\cap V(p_2)) = *V(p_1)\cap *V(p_2)$ which is true by hypothesis. Since by 4.6.2 $\mathcal{B}$ is classical, 4.6.5 applies, and so $\mathcal{B}$ is normal. XXX

4.6.7 **Theorem:** A neighbourhood frame $\mathcal{F} = (U, \mathcal{N})$ is normal if and only if for each $u \in U$, $\mathcal{N}_u$ is a filter.

Proof: Suppose $\mathcal{F}$ is normal. We must show that for each $u \in U$: (a) $\mathcal{N}_u \neq \phi$, (b) $N, N' \in \mathcal{N}_u \Rightarrow N \cap N' \in \mathcal{N}_u$, and (c) $N \in \mathcal{N}_u$ and $N' \supset N \Rightarrow N' \in \mathcal{N}_u$. Now, by tautology and RN $\mathcal{F} \models \Box(p \to p)$. Thus for all $u \in U$ and any assignment V on $\mathcal{F}$, $u \in V(\Box(p \to p))$ and so $U = V(p \to p) \in \mathcal{N}_u$. Thus $\mathcal{N}_u \neq \phi$.

Suppose now that $N \in \mathcal{N}_u$ and $N' \in \mathcal{N}_u$. Let V be such that $V(p) = N$ and $V(q) = N'$. Then $u \in V(\Box p) \cap V(\Box q) = V(\Box p \wedge \Box q)$. Then by 4.6.5, $u \in V(\Box(p \wedge q))$ and so $N \cap N' = V(p \wedge q) \in \mathcal{N}_u$.

Suppose $N \in \mathcal{N}_u$ and $N' \supset N$. Let V be such that $V(p) = N'$ and $V(q) = N$. Then $V(p \wedge q) = N' \cap N = N \in \mathcal{N}_u$. Thus $u \in V(\Box(p \wedge q))$ and, since by taut. and 2.3.1 $V(\Box(p \wedge q) \to \Box p) = 1$, $u \in V(\Box p)$. Thus $N' = V(p) \in \mathcal{N}_u$ and so $\mathcal{N}_u$ is a filter.

Now suppose that $\forall u \in U$, $\mathcal{N}_u$ is a filter; then (a), (b), and (c) hold. By 4.6.2 and 4.6.5 it suffices to show that RN preserves validity and that $\mathcal{F} \models \Box(p \wedge q) \leftrightarrow (\Box p \wedge \Box q)$. Suppose that $\mathcal{F} \models A$. Then for all V on $\mathcal{F}$, $V(A) = U$ and so $V(\Box A) = U$ since, by (a) and (c), $U \in \mathcal{N}_u$ for all $u \in U$; hence $\mathcal{F} \models \Box A$ and RN preserves validity.

Suppose that $u \in V(\Box(p \wedge q))$. Then $V(p \wedge q) \in \mathcal{N}_u$. Now $V(p) \supset V(p \wedge q)$ and $V(q) \supset V(p \wedge q)$; so by (c), $V(p)$ and $V(q)$ are in $\mathcal{N}_u$; hence $u \in V(\Box p) \cap V(\Box q) = V(\Box p \wedge \Box q)$. If, on the other hand, $u \in V(\Box p \wedge \Box q)$, then $u \in V(\Box p)$ and $u \in V(\Box q)$ and so $V(p)$ and $V(q)$ are in $\mathcal{N}_u$ and so by (b) $V(p \wedge q) \in \mathcal{N}_u$; thus $u \in V(\Box(p \wedge q))$.

Therefore $V(\Box(p\wedge q)) = V(\Box p\wedge\Box q)$; hence $V(\Box(p\wedge q)\leftrightarrow(\Box p\wedge\Box q)) = U$ and $\mathcal{F} \models \Box(p\wedge q)\leftrightarrow(\Box p\wedge\Box q)$. Thus $\mathcal{F}$ is normal. **XXX**

4.6.8 Theorem: Every first order frame, hence every relational, every connected, and every superconnected frame, is normal.

Proof: Let $\mathcal{U} = (U,<,\Pi)$ be a first order frame. By proof similar to that of 4.6.2 with $\overset{*}{<}$ for $*$, $\mathcal{U}$ is classical. Thus by 4.6.5, it suffices to show that RN preserves validity and that $\mathcal{U} \models \Box(p\wedge q)\leftrightarrow(\Box p\wedge\Box q)$. Suppose $\mathcal{U}\models A$. Then for any assignment V on $\mathcal{U}$, $V(A) = U$. Thus for $u \in U$, $V(A) \supset \{v|u<v\}$ and so $u \in V(\Box A)$. Thus $V(\Box A) = U$; hence $\mathcal{U} \models \Box A$ and RN preserves validity.

Suppose that for an assignment V on $\mathcal{U}$ and $u \in U$, $u \in V(\Box(p\wedge q))$. Then for all $v>u$, $v \in V(p\wedge q)$. So for all $v>u$, $v \in V(p)$ and $v \in V(q)$ and so $u \in V(\Box p)\cap V(\Box q) = V(\Box p\wedge\Box q)$. Suppose, on the other hand, that $u \in V(\Box p\wedge\Box q)$. Then for all $v>u$, $v \in V(p)$ and $v \in V(q)$; so $v \in V(p\wedge q)$; hence $u \in V(\Box(p\wedge q))$. Therefore $V(\Box(p\wedge q)) = V(\Box p\wedge\Box q)$ and so $\mathcal{U} \models \Box(p\wedge q)\leftrightarrow(\Box p\wedge\Box q)$. **XXX**

The following results will be useful later.

4.6.9 Theorem: If $\mathcal{W} = (W,<)$ is a relational frame, then $\mathcal{W} \models T$ iff $<$ is reflexive ($x<x$ for all $x \in W$).

Proof: If $<$ is reflexive then for any assignment V, $V(\Box p,u) = T \Leftrightarrow V(p,v) = T$ for all v such that $u<v \Rightarrow V(p,u) = T$. Thus $V(\Box p\rightarrow p,u) = T\ \forall u\in U$ and $\mathcal{W} \models \Box p\rightarrow p$. Since $\mathcal{W}$ is normal by 4.6.8, $\mathcal{W} \models T$.

If $\mathcal{W} \models T$ then $\mathcal{W} \models \Box p \rightarrow p$. Pick $u \in U$. Let V be such that $V(p) = \{v | u<v\}$. Then $V(\Box p,u) = T$. Since $\mathcal{W} \models \Box p \rightarrow p$ we must have $V(p,u) = T$, and so $u<u$. Thus $<$ is reflexive. **XXX**

<u>4.6.10</u> <u>Corollary</u>: If $\mathcal{F} = (U,<,\Pi)$ is a first-order frame and $<$ is reflexive, then $\mathcal{F} \models T$.

Proof: By 4.6.9, $(U,<) \models T$ and so by 3.4.1, $\mathcal{F} \models T$.**XXX**

<u>4.6.11</u> <u>Theorem</u>: If $\mathcal{W} = (W,<)$ is a relational frame, then $\mathcal{W} \models S4$ iff $<$ is reflexive and transitive.

Proof: If $<$ is reflexive and transitive then, by 4.6.9, $\mathcal{W} \models T$. Let V be an assignment on $\mathcal{W}$ and suppose that $V(\Box p,u) = T$. But then $V(\Box\Box p,u) = T$. Therefore $V(\Box p \rightarrow \Box\Box p,u) = T \ \forall u \in U$ and so $\mathcal{W} \models \Box p \rightarrow \Box\Box p$ and $\mathcal{W} \models S4$.

If $\mathcal{W} \models S4$ then $\mathcal{W} \models T$ and so by 4.6.9 $<$ is reflexive. Also $\mathcal{W} \models \Box p \rightarrow \Box\Box p$. Let $u,v,w \in W$ such that $u<v$ and $v<w$. Let V be an assignment on $\mathcal{W}$ such that $V(p) = \{x | u<x\}$. Then $V(\Box p,u) = T$. Since $\mathcal{W} \models \Box p \rightarrow \Box\Box p$, we have $V(\Box\Box p,u) = T$, hence $V(\Box p,v) = T$, hence $V(p,w) = T$. Therefore $u<w$ and so $<$ is transitive. **XXX**

<u>4.6.12</u> <u>Corollary</u>; If $\mathcal{F} = (U,<,\Pi)$ is a first-order frame and $<$ is reflexive and transitive, then $\mathcal{F} \models S4$.

Proof: By 4.6.11, $(U,<) \models S4$ and so by 3.4.1, $\mathcal{F} \models S4$. **XXX**

4.7 Clearly there are neighbourhood frames which are not normal, since we can always define the neighbourhood function on a set so that the set of neighbourhoods of some point is not a filter. Therefore, when we consider relational frames as neighbourhood frames we see that not all neighbourhood frames are relational frames. If, however, we restrict ourselves to normal frames the question remains: are there normal neighbourhood frames which are not (isomorphic to) relational frames and if so how can we characterize the neighbourhood frames which are (isomorphic to) relational frames?

4.7.1 Theorem: A neighbourhood frame $\mathcal{F} = (U,\mathcal{N})$ is isomorphic to a relational frame if and only if for each $u \in U$, $\mathcal{N}_u$ is a (proper or improper) principal filter.

Proof: Suppose $\mathcal{F} = (U,\mathcal{N})$ is isomorphic to a relational frame. In other words, we are saying that it is isomorphic to $\mathcal{F}_{\mathcal{W}}$ for some relational frame $\mathcal{W} = (W,<)$. We may as well, in fact, assume that $\mathcal{F}$ is $\mathcal{F}_{\mathcal{W}}$, that is, that $U = W$ and that $\mathcal{N} = \mathcal{N}^<$. Then for $u \in U$, $\mathcal{N}_u = \mathcal{N}^<_u = \{ S \subset U = W \mid S \supset \{v \mid u<v\} \}$ which is the principal filter generated by $\{v \mid u<v\}$.

Suppose that for each $u \in U$, $\mathcal{N}_u$ is a principal filter. Define a relation $<$ on U by $u<v$ iff $v \in N \; \forall N \in \mathcal{N}_u$. But $\mathcal{N}_u$ is a principal filter, so there is an $M \in \mathcal{N}_u$ such that $N \in \mathcal{N}_u$ iff $N \supset M$. Thus $u<v$ iff $v \in M$. Then $N \in \mathcal{N}^<_u$ iff $N \supset \{v \mid u<v\} = \{v \mid v \in M\} = M$. So $\mathcal{N}^<_u = \mathcal{N}_u$ and therefore $\mathcal{N}^< = \mathcal{N}$ and, if $\mathcal{U} = (U,<)$, $\mathcal{F} = \mathcal{F}_{\mathcal{U}}$. Hence $\mathcal{F}$ is (isomorphic to) a relational frame. XXX

Thus when we refer to a particular neighbourhood frame as being a relational frame, we simply mean that it is a neighbourhood frame which has the property that the set of neighbourhoods of any point is a principal filter.

4.8 We can ask the same question of boolean and neighbourhood frames: are there boolean frames which are not neighbourhood frames and if so how can we characterize the boolean frames which are (isomorphic to) neighbourhood frames?

4.8.1 Theorem: A boolean frame $\mathcal{B} = (B,0,1,-,\cap,*)$ is isomorphic to a neighbourhood frame if and only of the boolean algebra $(B,0,1,-,\cap)$ is isomorphic to a boolean algebra of all subsets of some set, or, in the language of boolean algebras, if and only if $(B,0,1,-,\cap)$ is a complete atomic algebra.

Proof: A complete boolean algebra is one in which every set of elements has a meet (or sup) and an atomic boolean algebra is one in which every non-zero element is greater than some minimal non-zero element. It is a standard result of boolean algebra that an algebra is isomorphic to an algebra of all subsets of a set if and only if it is complete and atomic. (See, for example, Halmos [5] p.70).

Now, if $\mathcal{B}$ is isomorphic to $\mathcal{B}_{\mathcal{F}}$ for some neighbourhood frame $\mathcal{F} = (U,\mathcal{N})$, then $(B,0,1,-,\cap)$ is isomorphic to the algebra $(\mathbb{P}(U),\phi,U,-,\cap)$ of all subsets of U.

If, on the other hand, $(B,0,1,-,\cap)$ is isomorphic to the algebra $(\mathcal{P}(U),\phi,U,-,\cap)$ of all subsets of the set U then we shall define a neighbourhood function N on U such that * corresponds to $\underset{N}{*}$ under the isomorphism.

To simplify notation, we may assume that B is $\mathcal{P}(U)$, 0 is ϕ, 1 is U, so that we are beginning with the frame $\mathcal{B} = (\mathcal{P}(U),\phi,U,-,\cap,*)$ and wish to find N such that $* = \underset{N}{*}$. Define N by $N_u = \{ S\subset U \mid u \in *S \}$. Then for $S \subset \mathcal{P}(U)$, $\underset{N}{*}S = \{ u\in U \mid S \in N_u \} = \{ u\in U \mid u \in *S \} = *S$. Therefore $\underset{N}{*} = *$ and so $\mathcal{B} = \mathcal{B}_F$. XXX

Thus when we refer to a particular boolean frame as being a neighbourhood frame we simply mean that the boolean algebra part of it is complete and atomic.

4.8.2 Theorem: A boolean frame $\mathcal{B} = (B,0,1,-,\cap,*)$ is (isomorphic to) a relational frame if and only if the algebra $(B,0,1,-,\cap)$ is complete and atomic, $*1 = 1$, and for every set $\{ b_i \mid i \in I \}$ of elements of B, $*\bigcap_{i\in I} b_i = \bigcap_{i\in I} *b_i$.

Proof: Every relational frame is a normal neighbourhood frame and so is a complete, atomic, boolean frame in which $*1 = 1$ and in which $*\bigcap_{i\in I} b_i = \bigcap_{i\in I} *b_i$ for any finite set I. It remains to show that if $F = (U,N)$ is a neighbourhood frame then $\underset{N}{*}\bigcap_{i\in I} S_i = \bigcap_{i\in I} \underset{N}{*}S_i$ for arbitrary sets I if and only if for all $u \in U$, N_u has a minimal element (hence, is a principal filter since we already know that it is a filter).

Suppose that $\underset{N}{*}\bigcap_{i\in I} S_i = \bigcap_{i\in I} \underset{N}{*}S_i$ for all index sets

I. Fix $u \in U$ and pick I so that $\{ S_i \mid i \in I \} = N_u$. Then since $S_i \in N_u$, $u \in \overset{*}{N}S_i$ for $i \in I$ and so $u \in \bigcap_{i \in I} \overset{*}{N}S_i = \overset{*}{N} \bigcap_{i \in I} S_i$. But $u \in \overset{*}{N} \bigcap_{i \in I} S_i$ implies that $\bigcap_{i \in I} S_i \in N_u$ and since it is the intersection of all sets in N_u, must be minimal in N_u.

Now suppose that for each $u \in U$, N_u has a minimal element. Let $\{ S_i \mid i \in I \}$ be an arbitrary set of subsets of U. Suppose that $u \in \overset{*}{N} \bigcap_{i \in I} S_i$. Then $\bigcap_{i \in I} S_i \in N_u$ and since we know that N_u is a filter and since for each $j \in I$, $S_j \supset \bigcap_{i \in I} S_i$, $S_j \in N_u$ for all $j \in I$. Thus $u \in \overset{*}{N}S_j$ for all $j \in I$ and hence $u \in \bigcap_{i \in I} \overset{*}{N}S_i$. Now suppose that $u \in \bigcap_{i \in I} \overset{*}{N}S_i$. Then for each $j \in I$, $u \in \overset{*}{N}S_j$ and so $S_j \in N_u$. Let N be the minimal element of N_u. Then $S_j \supset N$ for each $j \in I$ and so $\bigcap_{i \in I} S_i \supset N$ and hence $\bigcap_{i \in I} S_i \in N_u$. But then $u \in \overset{*}{N} \bigcap_{i \in I} S_i$. Therefore we have shown that $\overset{*}{N} \bigcap_{i \in I} S_i = \bigcap_{i \in I} \overset{*}{N}S_i$. XX

4.9 We can, in summary, say the following about boolean frames. If $\mathcal{B} = (B,0,1,-,\cap,*)$ is a boolean frame then

(a) $\mathcal{B}$ is normal iff for all $b,c \in B$, $*(b \cap c) = *b \cap *c$ and $*1 = 1$

(b) $\mathcal{B}$ is a neighbourhood frame iff $(B,0,1,-,\cap)$ is a complete atomic boolean algebra

(c) $\mathcal{B}$ is a relational frame iff it is a complete atomic frame in which $*1 = 1$ and for each arbitrary set $\{ b_i \mid i \in I \} \subset B$, $* \bigcap_{i \in I} b_i = \bigcap_{i \in I} *b_i$. (We note that if we adopt the usual convention that an empty intersection is the universe then we can drop the condition that $*1 = 1$

since it is taken up by $* \bigcap_{i \in I} b_i = \bigcap_{i \in I} *b_i$ in the case when I is empty.)

4.10 We shall take this opportunity to present a couple of interesting properties of the neighbourhood semantics which will be useful later.

4.10.1 Theorem: If $\mathcal{F} = (U,\mathcal{N})$ is a normal neighbourhood frame then $\mathcal{F} \models T$ iff every point in U is an element of all its own neighbourhoods, i.e. iff $\forall u \in U,\ N \in \mathcal{N}_u \Rightarrow u \in N$.

Proof: Since $\mathcal{F}$ is normal we know that $\mathcal{F} \models T$ iff $\mathcal{F} \models \Box p \to p$. Suppose $\mathcal{F} \models \Box p \to p$. Then pick $u \in U$ and $N \in \mathcal{N}_u$ and let V be any assignment on $\mathcal{F}$ such that $V(p) = N$. Now $\mathcal{F} \models \Box p \to p$ implies that $V(\Box p \to p, u) = T$. But since $V(p) = N \in \mathcal{N}_u$, $V(\Box p, u) = T$. Thus $V(p,u) = T$ and so $u \in V(p) = N$.

Now suppose that $\forall u \in U,\ N \in \mathcal{N}_u \Rightarrow u \in N$. Then let V and u be arbitrary. If $u \notin V(\Box p)$ then $V(\Box p \to p, u) = T$. If $u \in V(\Box p)$ then $N = V(p) \in \mathcal{N}_u$. Thus $u \in N = V(p)$ and so $V(\Box p \to p, u) = T$. Either way, $\mathcal{F} \models T$. XXX

Now if $\mathcal{F} = (U,\mathcal{N})$ is a neighbourhood frame then we can consider the pair $(U, \underset{\mathcal{N}}{*})$ where $\underset{\mathcal{N}}{*}$ is the function from $\mathcal{P}(U)$ to $\mathcal{P}(U)$ induced by $\mathcal{N}$. We shall refer to * as the <u>interior operator on U induced by $\mathcal{N}$</u>. We shall see that

4.10.2 Theorem: $(U, \underset{\mathcal{N}}{*})$ is a topological space where $\underset{\mathcal{N}}{*}$ is the interior operator of the space, if and only if $\mathcal{F} = (U,\mathcal{N})$ models S4.

Proof: It is known that $(U, \underset{N}{*})$ is a topological space with $\underset{N}{*}$ the interior operator iff for all $S, S' \subset U$ (a) $\underset{N}{*}S \subset S$, (b) $\underset{N}{*}(S \cap S') = \underset{N}{*}S \cap \underset{N}{*}S'$, (c) $\underset{N}{*}S \subset \underset{N}{*}\underset{N}{*}S$, and (d) $\underset{N}{*}U = U$. (See, for example, Kuratowski [7] p.61) We saw in 4.4 that the boolean frame $\mathcal{B}_F = (\wp(U), \phi, U, -, \ , \underset{N}{*})$ has the same valid formulae as F. Now, by 4.6.6, $\mathcal{B}_F$, and hence F, is normal iff (d) and (b) hold. $\mathcal{B}_F \models \Box p \to p \Leftrightarrow V(\Box p) \subset V(p)$ for all V on $\mathcal{B}_F \Leftrightarrow \underset{N}{*}V(p) \subset V(p) \ \forall V$ on $\mathcal{B}_F \Leftrightarrow$ (a) holds for all $S \subset U$. Finally, $\mathcal{B}_F \models \Box p \to \Box\Box p \Leftrightarrow V(\Box p) \subset V(\Box\Box p) \ \forall V$ on $\mathcal{B}_F$ $\Leftrightarrow \underset{N}{*}V(p) \subset \underset{N}{*}\underset{N}{*}V(p) \ \forall V$ on $\mathcal{B}_F \Leftrightarrow$ (c) holds $\forall S \subset U$. Therefore $\mathcal{B}_F$, and hence F, models S4 iff $(U, \underset{N}{*})$ is a topological space. XXX

5. Width and Depth

5.0 In the previous chapter we studied one way of comparing semantics; namely, saying that one semantics is greater than another if the class of frames in the one actually contains the class of frames in the other in some meaningful way. In chapters 5 to 9 we discuss a weaker but perhaps more significant method of comparing semantics.

5.1 Hansson and Gärdenfors refer to the width and depth of a semantics. The width of a semantics is measured by the logic determined by the class of all frames (structures) in the semantics. (This definition applies to languages and semantics in general, although we are only interested in the modal language and the modal semantics.) One semantics has greater width than another for the same language if the logic which is a measure of the width of the first is contained in the logic which is a measure of the second.

The concept of depth is intuitively a measurement of the "density" of the set of logics which are complete with respect to the semantics in question. It is, perhaps, vaguer than width since we have no way of denoting the depth of most semantics, but shall use the term in a relative way; thus we shall speak of one semantics as having greater depth than another, or of a semantics as having maximal depth. Specifically, one semantics will have greater depth than another with respect to some specified set of logics if each of those logics which is complete with respect to the other semantics is also complete with respect to the first. A semantics has *maximal* depth with respect to a certain set of logics if every logic in the set is complete with respect to the semantics. It is obvious that if one semantics is a subsemantics of another then the other has greater (or equal) width and greater (or equal) depth than the first.

In the remainder of this chapter we begin to discuss the widths of our six modal semantics; in chapters 6 to 9 we further discuss width and also discuss depth with respect to classical and with respect to normal logics.

5.2 We shall see that both $\mathcal{B}$ and $\mathcal{S}$ have width of measure E (see 2.4) and that $\mathcal{F}$, $\mathcal{R}$, $\hat{\mathcal{R}}$, and $\overline{\mathcal{R}}$ all have less width: of measure K. By 4.6.2 we see that the widths of $\mathcal{B}$ and of $\mathcal{S}$ have measure which is an extension of E (since every boolean frame is classical, hence determines an extension of E). To show that both $\mathcal{B}$ and $\mathcal{S}$ have width of measure E it suffices, then, to show that for each formula A such that $E \nvdash A$ there is a frame $\mathcal{F}$ in $\mathcal{S}$ such that $\mathcal{F} \not\models A$. To do this we use Segerberg's [14] construction of a "canonical frame".

5.3 A set Γ of formula is said to be consistent if the closure under the single rule MP of Γ together with the set of all tautologies (the smallest set of formulae containing Γ, containing all tautologies, and closed under MP) does not contain the negation of any tautology. By a simple application of Zorn's lemma (or Lindenbaum's Lemma) we see that each consistent set of formulae Γ is contained in a maximal consistent set (m.c.s.) and that each m.c.s. contains one of A, $\sim$A for each formula A.

For each classical logic L let $\mathcal{F}_L = (U_L, \mathcal{N}_L)$ be the following frame. U_L is the set of all m.c.s.'s containing L. If $u \in U_L$ and A is a formula such that $\Box A \in u$ then a typical neighbourhood of u is the set of all m.c.s.'s v in U_L such that $A \in v$. Formally, for $u \in U_L$

$$\mathcal{N}_u = \{ \{v \in U_L | A \in v\} \mid \Box A \in u \}.$$

$\mathcal{F}_L$ is called the <u>canonical frame from L</u>.

Let V_L be the assignment on $\mathcal{F}_L$ given by $V_L(i) = \{ u \mid p_i \in u \}$. We claim that for each $u \in U_L$ and formula A, $V_L(A,u) = T$ if and only if $A \in u$. The proof is by induction on the length of A.

If A is a variable p_i then $V_L(A,u) = T \Leftrightarrow V_L(p_i,u) = T \Leftrightarrow u \in V_L(p_i) \Leftrightarrow u \in V_L(i) \Leftrightarrow p_i \in u$. Now assume that for all formulae shorter than A the claim holds. If A is $\sim B$ then $V_L(A,u) = T \Leftrightarrow V_L(B,u) = F \Leftrightarrow B \notin u \Leftrightarrow A \in u$ (since u is an m.c.s.). If A is $B \to C$ then $V_L(A,u) = T \Leftrightarrow V_L(B,u) = F$ or $V_L(C,u) = T \Leftrightarrow B \notin u$ or $C \in u \Leftrightarrow \sim B \in u$ or $C \in u \Leftrightarrow A = B \to C \in u$ (since u is closed under MP). It remains to show that if A is $\Box B$ then $V_L(A,u) = T$ iff $A \in u$.

First assume that $A = \Box B \in u$. Then $\{v | B \in v\} \in \mathcal{N}_u$ and so, by induction hypothesis, $\{ v \mid V_L(B,v) = T \} \in \mathcal{N}_u$. Hence, $V_L(A,u) = V_L(\Box B,u) = T$. Now assume that $V_L(A,u) =$

T. Then $\{ v \mid V_L(B,v) = T \} \in \mathcal{N}_u$ and hence, by the induction hypothesis, $\{ v \mid B \in v \} \in \mathcal{N}_u$. Then, by the definition of $\mathcal{N}_u$, there is a formula C such that $\Box C \in u$ and $\{v|B\in v\} = \{v|C\in v\}$, i.e. $B \in v \Leftrightarrow C \in v$. We claim that, therefore, $L \vdash B\leftrightarrow C$. For suppose that $L \nvdash B\rightarrow C$. Then the set of formulae $L \cup \{B,\sim C\}$ is consistent and thus contained in an m.c.s. which would be in $\{v|B\in v\}$ but not in $\{v|C\in v\}$. Thus $L \vdash B\rightarrow C$; similarly $L \vdash C\rightarrow B$; and therefore $L \vdash B\leftrightarrow C$. Since L is classical, $L \vdash \Box B\leftrightarrow\Box C$ and so, if $\Box C \in u$ then $A = \Box B \in u$ also. The proof of the claim that $V_L(A,u) = T$ iff $A \in u$ is complete. Since we know that for each formula A such that $L \nvdash A$ there is an m.c.s. u such that $A \notin u$, we have:

5.3.1 *Theorem:* If L is a classical logic and $\mathcal{F}_L$ the canonical frame from L, then for all formulae A, $L \nvdash A$ implies $\mathcal{F}_L \nvDash A$. XXX

It must be stressed that for a classical logic L the canonical frame *from* L is not necessarily a model of L (which is sometimes called a frame *for* L). We do know that the logic that $\mathcal{F}_L$ determines is contained in L, but it may or may not be all of L. In fact, Segerberg[14] calls a classical logic *natural* if the converse of 5.3.1 holds for it, that is if $\mathcal{F}_L \vDash L$, and asks whether all classical logics are natural. In chapter 7 we shall give a negative answer to his question by displaying two non-natural logics.

5.4 As an immediate consequence of 5.3.1 we have

5.4.1 Theorem: S (and hence B) has width of measure E.

Proof: We have already noted that every formula of E is in all classical logics, hence valid in all neighbourhood frames. 5.3.1, on the other hand, ensures that a formula which is not in E is not valid in $\mathcal{F}_E$. The theorem follows. XX

What we have really shown here is that E is a natural logic.

5.5 By 4.6.8 we see that the widths of F, R, $\hat{\mathsf{R}}$, and $\overline{\mathsf{R}}$ have measure which is an extension of K. To show that the widths of F, R, $\hat{\mathsf{R}}$, and $\overline{\mathsf{R}}$ have measure exactly K we require techniques which we shall develop in chapters 6 and 8.

6. Maximal Depth of $\mathfrak{B}$ and $\mathfrak{F}$

6.0 We shall show that the boolean semantics, $\mathfrak{B}$, has maximal depth with respect to (or, is adequate for) classical modal logics and that the first-order semantics, $\mathfrak{F}$, has maximal depth with respect to normal logics.

6.1 Let L be any classical modal logic. We shall let $\mathcal{B}_L$ denote the "Lindenbaum-Tarski" frame for L constructed as follows. If A and B are modal formulae then we write $A \simeq B$ iff $L \vdash A \leftrightarrow B$. Clearly "$\simeq$" is an equivalence relation. Let B_L be the set of $\simeq$-equivalence classes of formulae and let $|A|$ denote the $\simeq$-equivalence class of A. If we define $|A| \wedge |B| = |A \wedge B|$ and $-|A| = |{\sim}A|$ then, since $L \vdash A_1 \leftrightarrow A_2$ and $L \vdash B_1 \leftrightarrow B_2$ implies $L \vdash {\sim}A_1 \leftrightarrow {\sim}A_2$ and $L \vdash A_1 \wedge B_1 \leftrightarrow A_2 \wedge B_2$, $\wedge$ and $-$ are well-defined. We can easily see, then, that $(B_L, |p_1 \wedge {\sim}p_1|, |p_1 \vee {\sim}p_1|, -, \wedge)$ is a boolean algebra.

Now we define the operator $*$ on B_L by $*|A| = |\Box A|$. Then the fact that L is classical implies that if $L \vdash A_1 \leftrightarrow A_2$ then $L \vdash \Box A_1 \leftrightarrow \Box A_2$ and so $*$ is well defined. Thus $\mathcal{B}_L = (B_L, |p_1 \wedge {\sim}p_1|, |p_1 \vee {\sim}p_1|, -, \wedge, *)$ is a boolean frame.

6.1.1 Lemma: $\mathcal{B}_L \models L$.

Proof: Let V be any assignment on $\mathcal{B}_L$. For each $i \in \mathbb{N}$, let D_i denote some formula in V(i), i.e. choose D_i such that $V(i) = |D_i|$. For each formula A let A' denote the formula obtained from A by simultaneously substituting, for each i $\in \mathbb{N}$, D_i for each occurrence of p_i in A. We show by induction on the length of A that $V(A) = |A'|$.

If A is a variable p_i then $V(A) = V(i) = |D_i| = |A'|$. If A is $B\to C$, then A' is $B'\to C'$ and $V(A) = -(V(B)\wedge -V(C)) = -(|B'|\wedge -|C'|) = |A'|$. If A is $\sim B$ then A' is $\sim B'$ and $V(A) = -V(B) = -|B'| = |A'|$. And if A is $\Box B$ then A' is $\Box B'$ and $V(A) = {*V(B)} = {*|B'|} = |\Box B'| = |A'|$.

By sub, if $L \vdash A$ then $L \vdash A'$ and hence $L \vdash A' \leftrightarrow p \vee\sim p$. Thus $|A'| = |p_1\vee\sim p_1|$ and therefore if $L \vdash A$ then $V(A) = |A'| = |p_1\vee\sim p_1|$. So $\mathcal{B}_L \models L$. XXX

6.1.2 Lemma: If $L \nvdash A$ then $\mathcal{B}_L \not\models A$.

Proof: Let V_0 be the assignment on $\mathcal{B}_L$ given by $V_0(i) = |p_i|$. Then by the previous proof, $V_0(A) = |A|$. Now if $L \nvdash A$ then $L \nvdash A\leftrightarrow p_1\vee\sim p_1$ and so $|A| \neq |p_1\vee\sim p_1|$ and therefore $\mathcal{B}_L \not\models A$. XXX

6.1.3 Theorem: $\mathfrak{B}$ has maximal depth with respect to classical logics.

Proof: By 6.1.1 and 6.1.2, $\mathcal{B}_L$ determines L for any classical logic L and hence any classical logic is complete with respect to the boolean semantics. XXX

6.2 We have already seen that each first-order frame determines a normal logic and that hence the measure of the width of $\mathfrak{F}$ is an extension of K. Therefore it would be impossible to expect $\mathfrak{F}$ to have **maximal** depth with respect to classical logics. We see, in fact, that we do get the best result that we could possibly expect for $\mathfrak{F}$; namely, that $\mathfrak{F}$ has maximal depth with respect to normal logics.

In order to do this we shall use constructions of Makinson's[10] of maps Φ and Ψ between frames in $\mathfrak{F}$ and <u>normal</u> frames in $\mathfrak{B}$.

Let $\mathcal{B} = (B,0,1,-,\cap,*)$ be any normal boolean frame. We define the first-order frame $\Phi(\mathcal{B}) = (U,<,\Pi)$ as follows. Let U be the set of all ultrafilters u in the boolean algebra $(B,0,1,-,\cap)$. $u<v$ iff for all $a \in B$, if $*a \in u$ then $a \in v$. If $S \subset U$, then $S \in \Pi$ iff there is an element $a \in B$ such that $S = \{ u \in U \mid a \in u \}$.

<u>6.2.1</u> <u>Lemma</u>: If $\mathcal{B} = (B,0,1,-,\cap,*)$ is a normal boolean frame and if $a\cap -b = 0$, then $*a \cap -*b = 0$.

Proof: $a\cap -b = 0 \Rightarrow a\cap b = a \Rightarrow *a\cap *b = *a \Rightarrow *a\cap -*b = 0$. XXX

<u>6.2.2</u> <u>Lemma</u>: If $\mathcal{B}$ is a boolean frame and $\Phi(\mathcal{B}) = (U,<,\Pi)$ is the structure defined above, and $\underset{<}{*}$ is the operation defined in 3.4 (recall that $\underset{<}{*}S = \{ v \in U \mid \{w \mid v<w\} \subset S \}$), then for all $a \in B$, $\underset{<}{*}(\{ u \in U \mid a \in u \}) = \{ u \in U \mid *a \in u \}$.

Proof: Let $S = \{ u \mid a \in u \}$. Then we must show that $w \in \underset{<}{*}S$ iff $*a \in w$.

Suppose that $*a \in w$ and let $w<v$. Then $a \in v$ and so $v \in S$. Hence $\{ v \mid w<v \} \subset S$ and so $w \in \underset{<}{*}S$.

Now suppose that $w \in \underset{<}{*}S$; thus if $w<v$ then $a \in v$. So if $v \in U$ such that $b \in v$ whenever $*b \in w$, then $a \in v$.

Suppose $*a \notin w$. Then $-*a \in w$. Let $P, P' \subset B$ be such that $P = \{-a\} \cup \{b \mid *b \in w\}$ and $P' = \{-*a\} \cup \{*b \mid *b \in w\}$. Then since $-*a \in w$, every finite intersection (meet) of elements of P' must be nonzero; but then by 6.2.1, every finite intersection of elements of P must be nonzero. Therefore P is contained in an ultrafilter v and we have $w<v$ and $a \notin v$; a contradiction. Thus we must have $*a \in w$. XXX

6.2.3 Lemma: $\Phi(\mathcal{B}) = (U,<,\Pi)$ is, in fact, a first-order frame.

Proof: Clearly U is a nonempty set and $<$ a binary relation on U. It remains only to show that Π is a boolean algebra of subsets of U which separates points in U and is closed under the operation $\underset{<}{*}$.

Suppose that $S_1, S_2 \in \Pi$. Then there are $a_1, a_2 \in B$ such that $S_i = \{ u \mid a_i \in u \}$ $(i=1,2)$. We have $U-S_1 = \{ u \mid a_1 \notin u \} = \{ u \mid -a_1 \in u \} \in \Pi$ and $S_1 \cap S_2 = \{ u \mid a_1 \in u \text{ and } a_2 \in u \} = \{ u \mid a_1 \cap a_2 \in u \} \in \Pi$; so Π is, in fact, a boolean algebra. It is well known that if $a \neq b \in B$ then there is $u \in U$ with $a \in u$ and $b \notin u$ (or vice versa) and so $\{ u \mid a \in u \} \neq \{ v \mid b \in v \}$; hence, Π separates points. By 6.2.2, Π is closed under $\underset{<}{*}$; therefore $(U,<,\Pi)$ is, in fact, a first-order frame. XXX

6.2.4 Lemma: If A is any formula then $\mathcal{B} \models A$ if and only if $\Phi(\mathcal{B}) \models A$. Thus $\mathcal{B}$ and $\Phi(\mathcal{B})$ are equivalent.

Proof: It is a well known property of ultrafilters on boolean algebras that $\{ u \in U \mid a \in u \} = \{ u \in U \mid b \in u \}$

iff $a = b$. Thus the correspondence $a \longrightarrow \{ u \mid a \in u \}$ is one-to-one from B <u>onto</u> Π. Now if V is an assignment on $\mathcal{B}$ then for each i, $V(i) \in B$; and so the mapping $V \longrightarrow V_\Phi$, where $V_\Phi(i) = \{ u \mid V(i) \in u \} \in \Pi$, is one-to-one from the set of assignments on $\mathcal{B}$ to the set of assignments on $\Phi(\mathcal{B})$. It suffices, therefore, to show that if V is any assignment on $\mathcal{B}$ then for all formulae A, $V(A) = 1$ iff $V_\Phi(A) = U$. Since $U = \{ u \mid 1 \in u \}$ it suffices to show that for all A, $V_\Phi(A) = \{ u \mid V(A) \in u \}$. We do this by induction on the length of A.

If A is p_i, then $V_\Phi(A) = V_\Phi(i) = \{ u \mid V(i) \in u \} = \{ u \mid V(A) \in u \}$. If A is $\sim B$, then $V_\Phi(A) = U - V_\Phi(B) = \{ u \mid V(B) \notin u \} = \{ u \mid -V(B) \in u \} = \{ u \mid V(A) \in u \}$. If A is $B \rightarrow C$, then $V_\Phi(A) = -V_\Phi(B) \cup V_\Phi(C) = U-(V_\Phi(B)-V_\Phi(C)) = \{ u \mid -(V(B) \cap -V(C)) \in u \} = \{ u \mid V(\sim(B \wedge \sim C)) \in u \} = \{ u \mid V(A) \in u \}$. Finally, if A is $\Box B$, then $V_\Phi(A) = \underset{<}{*} V_\Phi(B) = \underset{<}{*} \{ u \mid V(B) \in u \} = \{ u \mid *V(B) \in u \}$ (by 6.2.2) $= \{ u \mid V(A) \in u \}$. The induction is complete. XXX

We see that if we restrict $\mathfrak{B}$ to the semantics of <u>normal</u> boolean frames and if we apply Φ to the frames in this restricted semantics, then, since it is easy to see that Φ is one-to-one up to isomorphism, Φ satisfies the conditions of the mapping called for in 4.0 and so, strictly speaking, we could say that the restriction $\overline{\mathfrak{B}}$ of $\mathfrak{B}$ to normal frames is a subsemantics of $\mathfrak{F}$. The mapping Φ is, however, not really a natural or canonical one and so it would offend our intuitive notion of sub-semantics to consider $\overline{\mathfrak{B}}$ as a subsemantics of $\mathfrak{F}$, and therefore we shall not.

The mapping Φ does have an important application, however. Since $\mathfrak{B}$ has maximal depth with respect to classical logics it certainly has maximal depth with respect to normal logics; hence, since for normal frames $\mathcal{B}$ and $\Phi(\mathcal{B})$ are equivalent, we have

6.2.5 Theorem: $\mathfrak{F}$ has width of measure K.

Proof: We already know that the measure of the width of $\mathfrak{F}$ is an extension of K. But now $\Phi(\mathcal{B}_K) \models A$ iff $K \vdash A$ so the measure of the width of $\mathfrak{F}$ is K. XXX

6.2.6 Theorem: $\mathfrak{F}$ has maximal depth with respect to normal logics. XXX

6.3 We noticed that through the function Φ, the semantics, $\overline{\mathfrak{B}}$, of normal boolean frames was strictly speaking a subsemantics of $\mathfrak{F}$, although this offended our intuition about subsemantics. We now investigate Makinson's other mapping, Ψ, from $\mathfrak{F}$ to $\mathfrak{B}$. We shall see that Ψ has almost all the requirements for the mapping in the subsemantics definition and, contrary to Φ, is "natural" or intuitive, but is, however, not one-to-one.

If $F = (U,<,\Pi)$ is a first-order frame, then $\Psi(F) =$

$(\Pi, \phi, U, -, \cap, \underset{<}{*})$. Since we know that Π is closed under the boolean operations and under $\underset{<}{*}$, we see immediately that $\Psi(F)$ is a boolean frame.

<u>6.3.1</u> <u>Lemma</u>: For each formula A, $F \models A$ if and only if $\Psi(F) \models A$. Thus F and $\Psi(F)$ are equivalent.

Proof: An assignment V on F is a mapping from $\mathbb{N}$ to Π and so is at once an assignment on $\Psi(F)$. It is clear that it ranges over all assignments on $\Psi(F)$ also. We also know that the extension of V on F to a function from the set of all formulae to Π is the same as that on $\Psi(F)$. The lemma follows. XXX

<u>6.3.2</u> <u>Lemma</u>: For each normal boolean frame B, $\Psi(\Phi(B))$ is boolean frame-isomorphic to B.

Proof: If $B = (B, 0, 1, -, \cap, *)$ then $\Phi(B) = (U, <, \Pi)$ where U is the set of ultrafilters on $(B, 0, 1, -, \cap)$, $u < v$ iff $a \in v$ whenever $*a \in u$, and for $S \subset U$, $S \in \Pi$ iff there is $a \in U$ with $S = \{ u \mid a \in u \}$. Hence, $\Psi(\Phi(B)) = (\Pi, \phi, U, -, \cap, \underset{<}{*})$. We have already seen in the proof of 6.2.3 that $U - \{ u \mid a \in u \} = \{ u \mid -a \in u \}$ and that $\{ u \mid a \in u \} \cap \{ u \mid b \in u \} = \{ u \mid a \cap b \in u \}$, and in 6.2.2 that $\underset{<}{*}\{ u \mid a \in u \} = \{ u \mid *a \in u \}$. We also see easily that $\phi = \{ u \mid 0 \in u \}$ and $U = \{ u \mid 1 \in u \}$. Thus the mapping $a \dashrightarrow \{ u \mid a \in u \}$ is an isomorphism from B to $\Psi(\Phi(B))$. XXX

We observe that it is not in general true that as first-order frames $\mathcal{F}$ and $\Phi(\Psi(\mathcal{F}))$ are isomorphic. Let $\mathcal{F} = (W,<,\Pi)$ be any first-order frame where W is an infinite set, $<$ is any binary relation on W, and $\Pi = \mathcal{P}(W)$. Then $\Psi(\mathcal{F}) = (\mathcal{P}(W),\phi,W,-,\cap,\overset{*}{<})$ and $\Phi(\Psi(\mathcal{F})) = (U,<',\Pi')$ where U is the set of ultrafilters on the boolean algebra $\mathcal{P}(W)$ and $<'$ and Π' are defined appropriately. It is a known fact of set theory that $\text{card}(U) = 2^{2^{\text{card}(W)}}$ (see Gillman & Jerison [4]p.130, Thm.9.2) and so there can't possibly be a first-order frame-isomorphism from $\mathcal{F}$ to $\Phi(\Psi(\mathcal{F}))$.

Finally, if we let $\mathcal{F}$ be any first-order frame such that $\mathcal{F}$ and $\Phi(\Psi(\mathcal{F}))$ are not isomorphic, then by 6.3.2, $\Psi(\mathcal{F})$ and $\Psi(\Phi(\Psi(\mathcal{F})))$ <u>are</u> isomorphic as boolean frames and so we see that as a mapping of isomorphism-types, Ψ is not one-to-one.

7. Non-Maximal Depth of $\mathcal{S}$

7.0 We have already seen (5.4.1) that $\mathcal{S}$ has width of measure E. It is known, in fact, that E, K, T, S4, and virtually all other classical modal logics of particular philosophical interest are complete with respect to $\mathcal{S}$. It might be expected, then, that all classical logics be complete with respect to $\mathcal{S}$. We shall see, however, that this is not the case. Thomason[16] and Fine[1] independently constructed normal logics, which we shall, in this chapter, denote by L_1 and L_2 respectively, which they showed to be incomplete with respect to $\mathcal{R}$. We shall show that these two logics are incomplete with respect to $\mathcal{S}$, thus showing that $\mathcal{S}$ does not have maximal depth with respect to classical logics and hence is "weaker" than $\mathcal{B}$.

Of course, only one of these logics is really necessary to obtain our result, but the logics are quite different and so add enough strength to the result to make the presentation of both worthwhile. The logic L_1 is an extension of T and contained in S4; in fact, we shall see that L_1 is properly contained in S4 but that all neighbourhood frames which model L_1 also model S4. L_1 is a relatively simple logic and its position, "so close to" S4, makes it interesting in itself. If we knew only of the existence of L_1, however, it would then be natural to ask whether $\mathcal{S}$ had maximal depth with respect to classical, or perhaps normal, extensions of S4. But L_2 is an extension of S4, and although it is considerably more complicated that L_1, its existence shows that even "above" S4, $\mathcal{S}$ does not have maximal depth, and so its presentation here is indeed worthwhile.

7.1 Before continuing with the presentation of L_1 it will be convenient first to prove some general results on neighbourhood frames. In the remainder of this section $\mathcal{F} = (U,\mathcal{N})$ will be an arbitrary neighbourhood frame and A a modal formula.

7.1.1 Definition: If $S \subset U$ then we define $\overline{S}^{\mathcal{N}}$ to be $-\underset{\mathcal{N}}{*}-S$.

7.1.2 Lemma: For any assignment V, $\overline{V(A)}^{\mathcal{N}} = V(\Diamond A)$.

Proof: Immediate from 7.1.1 and section 4.4. XXX

Throughout the remainder of this chapter we shall usually omit the subscript "$\mathcal{N}$" and simply write $*S$ for $\underset{\mathcal{N}}{*}S$ and $\overline{S}$ for $\overline{S}^{\mathcal{N}}$.

7.1.3 Lemma: If $\mathcal{F}$ is normal and $S \subset U$, then $\overline{S} = \{ x \mid N \in \mathcal{N}_x \Rightarrow N \cap S \neq \phi \}$.

Proof: $x \in \overline{S} \Leftrightarrow x \notin *-S \Leftrightarrow$ there is no set $N \in \mathcal{N}_x$ such that $N \subset -S$ (by 4.4 and 4.6.7) $\Leftrightarrow \forall N \in \mathcal{N}_x, N \cap S \neq \phi$. XXX

7.1.4 Corollary: In normal $\mathcal{F}$, $\overline{\phi} = \phi$. XXX

7.1.5 Lemma: If $\mathcal{F}$ is normal and $P \subset S \subset U$, then $*P \subset *S$

and $\overline{P} \subset \overline{S}$.

Proof: $y \in {*P} \Rightarrow P \in \mathcal{N}_y \Rightarrow$ (by 4.6.7) $S \in \mathcal{N}_y \Rightarrow y \in {*S}$. Thus ${*P} \subset {*S}$. Now $P \subset S \Rightarrow -S \subset -P \Rightarrow {*-S} \subset {*-P} \Rightarrow -{*-P} \subset -{*-S} \Rightarrow \overline{P} \subset \overline{S}$. XXX

7.1.6 Lemma: Let $\mathcal{F}$ be normal. Then $\mathcal{F}$ models $\Diamond^2 p \rightarrow \Diamond p$ iff for all nonempty $S,P,Q \subset U$, if $S \subset \overline{P}$ and $P \subset \overline{Q}$ then $S \subset \overline{Q}$.

Proof: Suppose $\mathcal{F}$ models $\Diamond^2 p \rightarrow \Diamond p$ and $S \subset \overline{P}$ and $P \subset \overline{Q}$. Let V be an assignment such that $V(p) = Q$. Then by lemma 7.1.2, $V(\Diamond p) = \overline{Q}$ and $V(\Diamond^2 p) = \overline{\overline{Q}}$, and so $\overline{\overline{Q}} \subset \overline{Q}$. Also from $P \subset \overline{Q}$ and lemma 7.1.5, we have $\overline{P} \subset \overline{\overline{Q}}$ and so $S \subset \overline{P} \subset \overline{\overline{Q}} \subset \overline{Q}$.

Suppose, on the other hand, that for all nonempty $S,P,Q \subset U$, if $S \subset \overline{P}$ and $P \subset \overline{Q}$ then $S \subset \overline{Q}$. Let V be any assignment on $\mathcal{F}$. If $V(\Diamond^2 p) = \phi$, then $V(\Diamond^2 p) \subset V(\Diamond p)$. So we shall assume that $V(\Diamond^2 p) \neq \phi$. Then since $V(\Diamond^2 p) = \overline{V(\Diamond p)}$ and $V(\Diamond p) = \overline{V(p)}$ we must have $V(\Diamond p) \neq \phi$ and $V(p) \neq \phi$ (by corollary 7.1.4). Now by our supposition, $V(\Diamond^2 p) \subset \overline{V(p)}$ (by 7.1.2). Thus for any V, $V(\Diamond^2 p) \subset V(\Diamond p)$. Hence, $V(\Diamond^2 p \rightarrow \Diamond p) = U$ and so $\mathcal{F}$ models $\Diamond^2 p \rightarrow \Diamond p$. XXX

7.1.7 Lemma: If M is a modal logic ($\mathcal{F}$ any frame), then $\Box p \rightarrow \Box^2 p$ is included in M (valid in $\mathcal{F}$) if and only if $\Diamond^2 p \rightarrow \Diamond p$ is included in M (valid in $\mathcal{F}$).

Proof: $M \vdash \Box p \rightarrow \Box^2 p \Rightarrow M \vdash \Box \sim p \rightarrow \Box^2 \sim p \Rightarrow M \vdash \sim\Box^2 \sim p \rightarrow \sim\Box \sim p \Rightarrow M \vdash \Diamond^2 p \rightarrow \Diamond p$. Similarly for the opposite implication. Now repeat the above, replacing "$M \vdash$" with

with "$\mathcal{F} \models$". XXX

7.1.8 Corollary: If $\mathcal{F} = (U,\mathcal{N})$ models T, then $\mathcal{F}$ models S4 if and only if for all nonempty $S,P,Q \subset U$, if $S \subset \overline{P}$ and $P \subset \overline{Q}$, then $S \subset \overline{Q}$. XXX

7.1.8a Lemma: If S4 $\vdash A \to \Box(B \to C)$ then S4 $\vdash A \to \Box(\Diamond B \to \Diamond C)$.

Proof: S4 $\vdash (B \to C) \to ({\sim}C \to {\sim}B)$, thus by 2.3.1 S4 $\vdash \Box(B \to C) \to \Box({\sim}C \to {\sim}B)$. But S4 $\vdash \Box({\sim}C \to {\sim}B) \to (\Box{\sim}C \to \Box{\sim}B)$ and S4 $\vdash (\Box{\sim}C \to \Box{\sim}B) \to ({\sim}\Box{\sim}B \to {\sim}\Box{\sim}C)$ and ${\sim}\Box{\sim}B \to {\sim}\Box{\sim}C = \Diamond B \to \Diamond C$. Therefore S4 $\vdash \Box(B \to C) \to (\Diamond B \to \Diamond C)$. Then by 2.3.1, S4 $\vdash \Box^2(B \to C) \to \Box(\Diamond B \to \Diamond C)$. Also S4 $\vdash \Box(B \to C) \to \Box^2(B \to C)$. Thus S4 $\vdash \Box(B \to C) \to \Box(\Diamond B \to \Diamond C)$. The lemma follows. XXX

7.1.8b Lemma: If S4 $\vdash A \to \Box B$ and S4 $\vdash B \to C$, then S4 $\vdash A \to \Box C$.

Proof: If S4 $\vdash B \to C$ then S4 $\vdash \Box B \to \Box C$, by 2.3.1. This, with S4 $\vdash A \to \Box B$, yields S4 $\vdash A \to \Box C$. XXX

7.1.8c Lemma: If S4 $\vdash A \to \Box B$ and S4 $\vdash A \to \Box C$ and S4 $\vdash B \wedge C \to D$, then S4 $\vdash A \to \Box D$.

Proof: If S4 $\vdash B \wedge C \to D$ then by 2.3.1 S4 $\vdash \Box(B \wedge C) \to \Box D$. By 4.6.5, S4 $\vdash \Box(B \wedge C) \leftrightarrow (\Box B \wedge \Box C)$. Thus S4 $\vdash \Box B \wedge \Box C \to \Box D$. This, with S4 $\vdash A \to \Box B$ and S4 $\vdash A \to \Box C$, yields S4 $\vdash A \to \Box D$. XXX

7.1.9 Lemma: If $\mathcal{F} \models T$ and $S \subset U$, then $*S \subset S \subset \overline{S}$.

Proof: Since $\mathcal{F} \models T$, $\mathcal{F} \models \Box p \to p$. Thus for all assignments V, $V(\Box p) \subset V(p)$. Pick $S \subset U$ and V such that $V(p) = S$. Then $*S = *V(p) = V(\Box p) \subset V(p) = S$. Since this is true for arbitrary $S \subset U$ we also have $*{-S} \subset -S$ and so $S = --S \subset -*-S = \overline{S}$. XXX

7.1.10 Lemma: If $\mathcal{F} \models T$, V is an assignment on $\mathcal{F}$, $x \in U$, A a formula, and $i \leq k$, then $V(\Box^k A,x) = T \Rightarrow V(\Box^i A,x) = T$.

Proof: $\mathcal{F} \models \Box p \to p$. Let $j = k-i$. By an obvious induction on j, $\mathcal{F} \models \Box^j p \to p$, hence $\mathcal{F} \models \Box^j(\Box^i A) \to \Box^i A$, i.e. $\mathcal{F} \models \Box^k A \to \Box^i A$. Thus $V(\Box^k A,x) = T \Rightarrow V(\Box^i A,x) = T$. XXX

7.2 We now proceed with the construction of the logic L_1, due to Thomason[16], and show that it is incomplete with respect to the neighbourhood semantics. Consider the formulae:

$$A_i = \Box(q_i \to r) \qquad (i = 1,2)$$

$$B_i = \Box(r \to \Diamond q_i) \qquad (i = 1,2)$$

$$C_1 = \Box\sim(q_1 \wedge q_2)$$

$$A = r \wedge \Box p \wedge \sim\Box^2 p \wedge A_1 \wedge A_2 \wedge B_1 \wedge B_2 \wedge C_1 \to \Diamond(r \wedge \Box(r \to q_1 \vee q_2))$$

$D = (p \wedge \Diamond^2 q) \rightarrow (\Diamond q \vee \Diamond^2 (q \wedge \Diamond p))$

$E = (\Box p \wedge \sim\Box^2 p) \rightarrow \Diamond(\Box^2 p \wedge \sim\Box^3 p)$

$F = \Box p \rightarrow \Box^2 p$.

Let L_1 be the smallest normal logic containing T, A, D, and E (here we are referring to the formula E, above -- T, however, refers to the logic T). We see easily that A, D, and E are in S4 so that L_1 is between T and S4. (To see that D is in S4, note that by 7.1.7, S4 $\models \Diamond^2 q \rightarrow \Diamond q$.)

<u>7.2.1</u> <u>Lemma</u>: The formula F is valid in all neighbourhood frames modelling L_1.

Proof: Assume that $\mathcal{F} = (U, \mathcal{N})$ is a particular neighbourhood frame which models T, models $D \wedge E$, but does not model F; we show that $\mathcal{F}$ also fails to model A, thus establishing that F is valid in all frames modelling L_1.

Claim: If $S \subset \overline{P}$ and $P \subset \overline{Q}$ and $S \not\subset \overline{Q}$, then there is a point $z \in Q \cap \overline{S}$.

Let V be an assignment such that $V(p) = S$ and $V(q) = Q$. Then $S \subset V(p \wedge \Diamond^2 q)$. Since $\mathcal{F} \models D$, $V(p \wedge \Diamond^2 q) \subset V(\Diamond q \vee \Diamond^2 (q \wedge \Diamond p))$. Thus $S \subset V(\Diamond q \vee \Diamond^2 (q \wedge \Diamond p)) = \overline{Q} \cup \overline{\overline{Q \cap \overline{S}}}$. But $S \not\subset \overline{Q}$ so that $S \cap \overline{\overline{Q \cap \overline{S}}} \neq \phi$. Thus $\overline{\overline{Q \cap \overline{S}}} \neq \phi$ and so by 7.1.4 $Q \cap \overline{S} \neq \phi$. Let $z \in Q \cap \overline{S}$. The claim is proved.

We now define an assignment V on $\mathcal{F}$ and inductively choose a pairwise disjoint sequence of nonempty sets

$W_1, W_2, \ldots$ so that for all i,j

$$i<j \Rightarrow W_i \subset \overline{W_j}$$
$$W_i \subset V(\Box^i p \wedge \sim\Box^{i+1} p).$$

Since $\mathcal{F}$ does not model F, there are, by corollary 7.1.8, nonempty sets $S,P,Q \subset U$ such that $S \subset \overline{P}$, $P \subset \overline{Q}$ and $S \not\subset \overline{Q}$. Let $w \in S-\overline{Q}$. Let V be such that $V(p) = U-Q$. Then $w \in -\overline{Q} = *-Q = V(\Box p)$. Now $V(\sim p) = Q$ and so, since $P \subset \overline{Q}$, $P \subset V(\Diamond\sim p)$. Thus $\{w\} \subset S \subset \overline{P} \subset \overline{V(\Diamond\sim p)} = V(\Diamond^2\sim p) = V(\sim\Box^2 p)$; and therefore $\{w\} \subset V(\Box p \wedge \sim\Box^2 p)$. Let $W_1 = \{w\}$; then the required conditions are satisfied by W_1.

Suppose $W_1, \ldots, W_n$ have been chosen to satisfy the conditions. Since $\mathcal{F}$ models E, $\mathcal{F}$ models $(\Box^n p \wedge \sim\Box^{n+1} p) \rightarrow \Diamond(\Box^{n+1} p \wedge \sim\Box^{n+2} p)$. Since $W_n \subset V(\Box^n p \wedge \sim\Box^{n+1} p)$, $W_n \subset V(\Diamond(\Box^{n+1} p \wedge \sim\Box^{n+2} p))$. So let $W_{n+1} = V(\Box^{n+1} p \wedge \sim\Box^{n+2} p)$; then $W_n \subset \overline{W_{n+1}}$ by 7.1.2. Since by the induction hypothesis $W_n \neq \phi$, $\overline{W_{n+1}} \neq \phi$, and so by 7.1.4, $W_{n+1} \neq \phi$. For $i<n+1$, $x \in W_i \Rightarrow V(\Box^{i+1} p,x) = F$, but $x \in W_{n+1} \Rightarrow V(\Box^{n+1} p,x) = T \Rightarrow$ (by 7.1.10) $V(\Box^{i+1} p,x) = T$; thus $W_i \cap W_{n+1} = \phi$.

We already have $W_n \subset \overline{W_{n+1}}$. For $i<n$, $W_i \subset \overline{W_n}$ by the induction hypothesis; thus, by the claim, either

$$\text{(a) } W_i \subset \overline{W_{n+1}}$$

or

$$\text{(b) there is } y \in W_{n+1} \cap \overline{W_i}$$

But $W_i \subset V(\sim\Box^{i+1} p)$, so $\overline{W_i} \subset V(\Diamond\sim\Box^{i+1} p) = V(\sim\Box^{i+2} p)$. Thus in case (b), $V(\Box^{i+2} p,y) = F$ and so, since $i<n \Rightarrow i+2 \leq n+1$,

we have $V(\Box^{n+1}p,y) = \mathbf{F}$, by 7.1.10, contradicting the fact that $W_{n+1} \subset V(\Box^{n+1}p)$. Hence $W_i \subset \overline{W_{n+1}}$ and the induction is complete.

Now let V also be such that $V(r) = \bigcup_{n=1}^{\infty} W_n$; $V(q_i) = \bigcup_{n=1}^{\infty} W_{3n+i}$ $(i = 1,2)$. Then $w \in W_1 \subset V(r)$ and $w \in W_1 \subset V(\Box p \wedge \sim\Box^2 p)$. $V(q_i) \subset V(r)$, hence $V(A_i) = V(\Box(q_i \rightarrow r)) = {}^*V(q_i \rightarrow r) = {}^*U = U$ for $i = 1,2$. For $x \in V(r)$ we have $x \in W_j$ for some j and if we choose k so that $j<3k+i$, then $W_j \subset \overline{W_{3k+i}}$ and so $x \in \overline{W_{3k+i}} \subset \overline{V(q_i)}$ (by 7.1.5). So $V(r) \subset \overline{V(q_i)} = V(\Diamond q_i)$; hence, $V(B_i) = V(\Box(r \rightarrow \Diamond q_i)) = {}^*V(r \rightarrow \Diamond q_i) = {}^*U = U$ for $i = 1,2$. Finally $V(C_1) = V(\Box\sim(q_1 \wedge q_2)) = {}^*-(V(q_1) \cap V(q_2)) = {}^*- \phi = {}^*U = U$. So we have $w \in V(r \wedge \Box p \wedge \sim\Box^2 p \wedge A_1 \wedge A_2 \wedge B_1 \wedge B_2 \wedge C_1)$. Suppose that $y \in V(r)$. Then $y \in W_i$ for some i. Let k be such that $3k>i$. Then $y \in \overline{W_{3k}} \subset \overline{V(\sim(r \rightarrow q_1 \vee q_2))}$; so $y \in W_i \subset \overline{W_{3k}} \subset \overline{V(\sim(r \rightarrow q_1 \vee q_2))} = V(\Diamond\sim(r \rightarrow q_1 \vee q_2)) = U - V(\Box(r \rightarrow q_1 \vee q_2))$ and hence $V(r \wedge \Box(r \rightarrow q_1 \vee q_2)) = \phi$. So $V(\Diamond(r \wedge \Box(r \rightarrow q_1 \vee q_2))) = \phi$ and thus $w \notin V(A)$. Therefore A is not valid in $\mathcal{F}$ and the lemma is proved. XXX

7.2.2 Lemma: $L_1 \not\vdash F$.

Proof: This result and the proof we give here are due to Thomason[16]. It would suffice to find a frame in any semantics which modelled L_1 and not F. We use a first-order frame $\mathcal{H} = (H,\Delta,\Pi)$. $H = \mathbb{N}$, the set of natural numbers; for $a,b \in H$, $a \Delta b$ iff $a \leq b+1$ (where $<$ is the usual ordering on the natural numbers); Π is the set of all finite and cofinite subsets of H. We know that Π is a boolean algebra of subsets of H. If $S \subset \Pi$ then we can see that if S is finite, $\underset{\Delta}{*}S$ is empty, if S is cofinite then if b is the greatest number in H-S, $\underset{\Delta}{*}S = \{ a \mid a \geq b+2 \}$ and $\underset{\Delta}{*}H = H$.

We must show that $\mathcal{H}$ models T and A, D, and E but not F. Clearly, Δ is reflexive; and so by 4.6.10, $\mathcal{H} \models T$.

Let V be an assignment on $\mathcal{H}$ such that $V(p) = \{ a \mid a \geq 2 \}$. Then $V(\Box p) = \underset{\Delta}{*}V(p) = \{ a \mid a \geq 3 \}$ and $V(\Box^2 p) = \{ a \mid a \geq 4 \}$. So $V(\Box p) \not\subset V(\Box^2 p)$; $V(\Box p \rightarrow \Box^2 p) \neq H$; hence $\mathcal{H} \not\models F$.

Now let V be any assignment on $\mathcal{H}$ and suppose that $a \in V(r \wedge \Box p \wedge \sim\Box^2 p \wedge A_1 \wedge A_2 \wedge B_1 \wedge B_2 \wedge C_1)$. Now for all c such that $a \Delta c$, if $c \in V(r)$ then, since $V(\Box(r \rightarrow \Diamond q_i), a) = T$ and therefore $V(r \rightarrow \Diamond q_i, c) = T$ for $i = 1,2$, we must have $V(\Diamond q_1, c) = V(\Diamond q_2, c) = T$; and so there must be $d_i \geq c-1$ with $d_i \in V(q_i)$ for $i = 1,2$. So we see that if V(r) is infinite then both $V(q_1)$ and $V(q_2)$ are infinite and hence both cofinite. But $V(\Box\sim(q_1 \wedge q_2), a) = T$ implies that $\{c \mid c \geq a-1\} \cap V(q_1) \cap V(q_2) = \phi$ which would contradict the cofiniteness of $V(q_1)$ and $V(q_2)$. So V(r) must be finite.

Now let b be the greatest element of V(r). Since $a \in V(r)$, such an element must exist and $a \leq b$; hence for all c such that $b \Delta c$, $a \Delta c$ and, in particular, $a \Delta b$. Thus, $V(r \wedge (r \rightarrow \Diamond q_1) \wedge (r \rightarrow \Diamond q_2), b) = T$ and so $V(\Diamond q_1 \wedge \Diamond q_2, b) = T$. Now, if $b < c$ then $a \Delta c$ and so $V((q_1 \rightarrow r) \wedge (q_2 \rightarrow r), c) = T$, so we could not have $V(q_i,c) = T$ for that would contradict our choice of b. We also have $a\Delta b$ and $a\Delta b-1$ and therefore $V(\sim(q_1\wedge q_2),b) = V(\sim(q_1\wedge q_2),b-1)=T$. So we must have $b \in V(q_1)$ and $b-1 \in V(q_2)$ or vice versa. Then we have $V(r,b) = T$ and for $b\Delta d$ we have $V(r\to q_1\vee q_2,d) = T$ (since either $d = b-1 \in V(q_1)\cup V(q_2)$ or $d = b \in V(q_1)\cup V(q_2)$ or $b<d \notin V(r)$); thus $V(r\wedge\Box(r\to q_1\vee q_2),b) = T$. Therefore, since $a\Delta b$, $V(\Diamond(r\wedge\Box(r\to q_1\vee q_2)),a) = T$, and so $V(A,a) = T$. Hence $V(A,x) = T$ for all $x \in H$ and $\mathcal{H} \models A$.

Now suppose that $V(p\wedge\Diamond^2 q,a) = T$. Then $a \in V(p)$ and there is $b\geq a-2$ such that $b \in V(q)$. Now if $b\geq a-1$ we have $V(\Diamond q,a) = T$ and so $V(\Diamond q\vee\Diamond^2(q\wedge\Diamond p),a) = T$. So if $V(\Diamond q,a) = F$ we must have $b = a-2$. But then $V(\Diamond p,b) = T$ and so $V(\Diamond q\vee\Diamond^2(q\wedge\Diamond p),a) = V(\Diamond^2(q\wedge\Diamond p),a) = T$. In either case we have $V(D,a) = T$. Thus $\mathcal{H} \models D$.

Finally, suppose that $a \in V(\Box p\wedge\sim\Box^2 p)$. Then $\{ c \mid c\geq a-1\} \subset V(p)$ but $\{ c \mid c\geq a-2 \} \not\subset V(p)$, thus $a-2 \notin V(p)$. Then we have $a+1 \in V(\Box^2 p)$ but $a+1 \notin V(\Box^3 p)$ and so $V(\Box^2 p\wedge\sim\Box^3 p,a+1) = T$. Therefore $V(\Diamond(\Box^2 p\wedge\sim\Box^3 p),a) = T$ and so $V(E,a) = T$. Thus $\mathcal{H} \models E$ and the lemma is proved. XXX

7.2.3 Theorem: L_1 is a normal logic between T and S4 which is incomplete with respect to $\mathcal{S}$.

Proof: Lemmas 7.2.1 and 7.2.2. XXX

7.2.4 Corollary: $\mathcal{S}$ does not have maximal depth with respect to classical or even normal logics. XXX

<u>7.3</u> We now continue to the construction of L_2, due to Fine[1], and show that it, also, is incomplete with respect to the neighbourhood semantics. Consider the following formulae where p_0, p_1, q_0, q_1, r_0, r_1, t, and s are distinct propositional variables and $m \geq 0$.

$$B_0 = q_0 \qquad\qquad B_1 = q_1$$

$$C_0 = r_0 \qquad\qquad C_1 = r_1$$

$$B_{m+2} = \Diamond B_{m+1} \wedge \Diamond C_m \wedge \sim\Diamond C_{m+1}$$

$$C_{m+2} = \Diamond C_{m+1} \wedge \Diamond B_m \wedge \sim\Diamond B_{m+1}$$

$$A_m = \Diamond B_{m+1} \wedge \Diamond C_{m+1} \wedge \sim\Diamond B_{m+2} \wedge \sim\Diamond C_{m+2}$$

$$J_1 = \Box((p_0 \rightarrow \sim p_1 \wedge \Diamond p_1) \wedge (p_1 \rightarrow \Diamond p_0))$$

$$J_2 = \Box(\sim(p_0 \vee p_1) \rightarrow \Box\sim(p_0 \vee p_1))$$

$$J_3 = \Box(B_1 \rightarrow \Diamond B_0 \wedge \sim\Diamond C_0)$$

$$J_4 = \Box(C_1 \rightarrow \Diamond C_0 \wedge \sim\Diamond B_0)$$

$$J_5 = \Box(B_0 \rightarrow \sim\Diamond(B_1 \vee C_1))$$

$$J_6 = \Box(C_0 \rightarrow \sim\Diamond(C_1 \vee B_1))$$

$$D = (p_0 \vee p_1) \wedge J_1 \wedge J_2$$

$$K = J_3 \wedge J_4 \wedge J_5 \wedge J_6$$

$$E = D \wedge \Diamond A_0 \wedge K$$

$$F = \Diamond((p_0 \vee p_1) \wedge \sim\Diamond A_0 \wedge \Diamond A_1)$$

$$G = E \rightarrow F$$

$$H = \sim(s \wedge \Box(s \rightarrow \Diamond(\sim s \wedge t \wedge \Diamond((\sim s \wedge \sim t) \wedge \Diamond s))))$$

(In the remainder of section 7.3, the letters assigned above to formulae will always denote those formulae, even though some of these letters, notably K and E, are commonly used to denote other particular formulae or logics.)

The logic L_2 is the smallest normal logic containing S4 and G and H.

7.3.1 Lemma: The formula $\sim E$ is valid in all neighbourhood frames which model L_2.

Proof: The proof requires two preliminary results. For $m \geq 0$ and any formula A let A^m be the result of substituting, for i = 0,1, B_{m+i} for B_i (= q_i) and C_{m+i} for C_i (= r_i) in A.

Claim 1: $B^m_n = B_{m+n}$, $C^m_n = C_{m+n}$, and $A^m_n = A_{m+n}$ for $m,n \geq 0$.

Proof of claim: We first show that $B^m_n = B_{m+n}$ and simultaneously that $C^m_n = C_{m+n}$ by induction on n. If n = 0 or 1 then the result follows by the definition of A^m. Suppose now that $r \geq 1$ and that for all $n \leq r$ and $m \geq 0$, $B^m_n = B_{m+n}$ and $C^m_n = C_{m+n}$. Then $B^m_{r+1} = (\Diamond B_r \wedge \Diamond C_{r-1} \wedge \sim \Diamond C_r)^m = \Diamond B^m_r \wedge \Diamond C^m_{r-1} \wedge \sim \Diamond C^m_r$ = (by induction hypothesis) $\Diamond B_{m+r} \wedge \Diamond C_{m+r-1} \wedge \sim \Diamond C_{m+r} = B_{m+r+1}$. Similarly, $C^m_{r+1} = C_{m+r+1}$. The induction is complete.

Now $A^m_n = (\Diamond B_{n+1} \wedge \Diamond C_{n+1} \wedge \sim \Diamond B_{n+2} \wedge \sim \Diamond C_{n+2})^m = \Diamond B^m_{n+1} \wedge \Diamond C^m_{n+1} \wedge \sim \Diamond B^m_{m+2} \wedge \sim \Diamond C^m_{n+2}$ = (by above induction) $B_{m+n+1} \wedge \Diamond C_{m+n+1} \wedge \sim \Diamond B_{m+n+2} \wedge \sim \Diamond C_{m+n+2} = A_{m+n}$. The claim is proved.

Claim 2: For any $m \geq 0$, S4 $\vdash K^m \rightarrow K^{m+1}$. Thus if $m' \geq m$, S4 $\vdash K^m \rightarrow K^{m'}$.

Proof of claim: The second statement clearly follows from the first. We have

(a) $K^m = J_3^m \wedge J_4^m \wedge J_5^m \wedge J_6^m = \Box(B_{m+1} \to \Diamond B_m \wedge \sim\Diamond C_m) \wedge \Box(C_{m+1} \to \Diamond C_m \wedge \sim\Diamond B_m)$
$\wedge \Box(B_m \to \sim\Diamond(B_{m+1} \vee C_{m+1})) \wedge \Box(C_m \to \sim\Diamond(C_{m+1} \vee B_{m+1}))$.

(b) $B_{m+2} = \Diamond B_{m+1} \wedge \Diamond C_m \wedge \sim\Diamond C_{m+1}$.

(c) $C_{m+2} = \Diamond C_{m+1} \wedge \Diamond B_m \wedge \sim\Diamond B_{m+1}$.

(d) $S4 \vdash K^m \to J_3^{m+1}$. (from (b), RN, and taut.)

(e) $S4 \vdash K^m \to J_4^{m+1}$. (from (c), RN, and taut.)

(f) $S4 \vdash K^m \to \Box(C_{m+2} \to \Diamond C_{m+1})$. (from (e), 7.1.8b)

(g) $S4 \vdash K^m \to \Box(C_{m+1} \to \Diamond C_m)$. ((a), 7.1.8b)

(h) $S4 \vdash K^m \to \Box(\Diamond C_{m+1} \to \Diamond^2 C_m)$. ((g), 7.1.8a)

(i) $S4 \vdash \Diamond^2 C_m \to \Diamond C_m$. (7.1.7)

(j) $S4 \vdash K^m \to \Box(\Diamond C_{m+1} \to \Diamond C_m)$. ((h),(i), 7.1.8b)

(k) $S4 \vdash K^m \to \Box(C_{m+2} \to \Diamond C_m)$. ((f),(j), 7.1.8c)

(l) $S4 \vdash K^m \to \Box(B_{m+2} \to \Diamond C_m)$. ((a), 7.1.8b)

(m) $S4 \vdash K^m \to \Box(B_{m+2} \vee C_{m+2} \to \Diamond C_m)$ ((k),(l), 7.1.8c)

(n) $S4 \vdash K^m \to \Box(\Diamond(B_{m+2} \vee C_{m+2}) \to \Diamond^2 C_m)$. ((m), 7.1.8a)

(o) $S4 \vdash K^m \to \Box(\Diamond(B_{m+2} \vee C_{m+2}) \to \Diamond C_m)$. ((n),(i), 7.1.8b)

(p) $S4 \vdash K^m \to \Box(B_{m+1} \to \sim\Diamond C_m)$. ((a), 7.1.8b)

(q) $S4 \vdash K^m \to \Box(B_{m+1} \to \sim\Diamond(B_{m+2} \vee C_{m+2}))$. ((o),(p), 7.1.8c)

Thus, $S4 \vdash K^m \to J_5^{m+1}$. Interchanging B's and C's in (f) to (q) yields $S4 \vdash K^m \to J_6^{m+1}$, and with (d) and (e) we have $S4 \vdash K^m \to K^{m+1}$. The claim is proved.

We are now ready to conclude the proof of 7.3.1. Suppose that $F = (U, N)$ is a particular neighbourhood frame which models S4 and in which G is valid but $\sim E$ is not valid.

We shall show that H is not valid in $\mathcal{F}$ and so $\mathcal{F}$ is not a model of L_2. This will show that $\sim E$ is valid in all neighbourhood models of L_2.

Since $\sim E$ is not valid in $\mathcal{F}$, there is an assignment V on $\mathcal{F}$ such that $V(E) \neq \phi$. Let $w \in V(E)$. We shall construct by induction an infinite sequence of pairwise disjoint subsets $W_0, W_1, \ldots$ of U such that for $m \geq 0$, $W_m \subset \overline{W_{m+1}}$ and $W_m \subset V(E^m)$.

Let $W_0 = \{w\}$. Assume that $W_0, \ldots, W_n$ have been chosen to satisfy the conditions. Now $G^n = E^n \rightarrow F^n$. Thus, since $W_n \subset V(E^n)$, $W_n \subset V(F^n)$, i.e. $V(\Diamond((p_0 \vee p_1) \wedge \sim\Diamond A_n \wedge \Diamond A_{n+1}), x) = T \;\; \forall x \in W_n$.

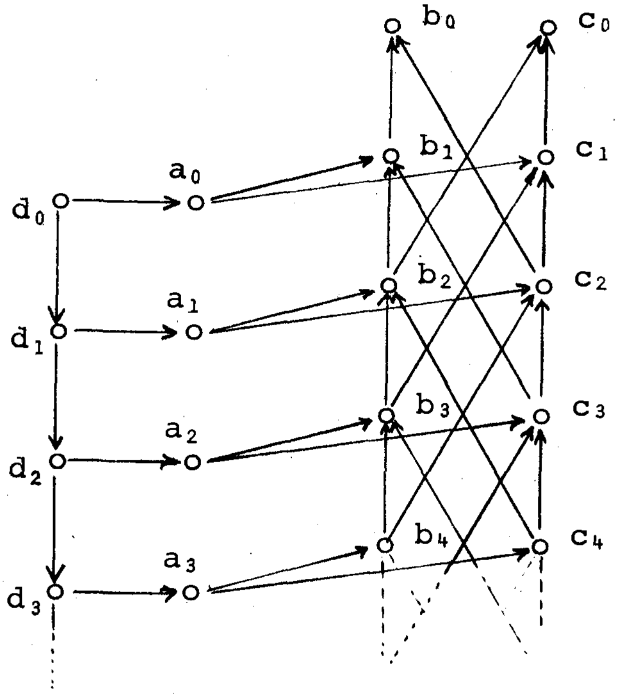
Let $P_{n+1} = V((p_0 \vee p_1) \wedge \sim\Diamond A_n \wedge \Diamond A_{n+1})$. Then $W_n \subset \overline{P_{n+1}}$ by 7.1.2. We shall show by induction on i that if $0 \leq i \leq n$, then $W_{n-i} \subset V(\Diamond A_n)$. Suppose $i = 0$. $W_n \subset V(E^n)$ and so, since $\Diamond A_n$ is a conjunct of E^n, $W_{n-i} = W_n \subset V(\Diamond A_n)$. Suppose now that $0 < i \leq n$ and $W_{n-(i-1)} \subset V(\Diamond A_n)$. Then since $W_{n-i} \subset \overline{W_{n-(i-1)}}$, $W_{n-i} \subset V(\Diamond^2 A_n)$ by 7.1.2 and 7.1.5. Thus, since $S4 \vdash \Diamond^2 A_n \rightarrow \Diamond A_n$ by 7.1.7, $W_{n-i} \subset V(\Diamond A_n)$. The induction on i is complete; thus for all i, $0 \leq i \leq n$, $W_i \subset V(\Diamond A_n)$. But $P_{n+1} \subset U - V(\Diamond A_n)$. Thus $P_{n+1} \cap W_i = \phi \;\; \forall i \leq n$.

Now $W_n \subset V(E^n)$, and so $W_n \subset V(K^n)$. Thus by claim 2, $W_n \subset V(K^{n+1})$. Now let $S_i^{n+1} = V(J_i^{n+1})$ for $1 \leq i \leq 6$. (Note that for $i = 1$ or 2, $J_i^n = J_i \;\forall n$, hence $S_i^m = S_i^n \;\forall m,n$.) Let $W_{n+1} = (\bigcap_{i=1}^{6} S_i^{n+1}) \cap P_{n+1}$. Then from definitions of P_{n+1}, S_i^{n+1}, and E^{n+1} we have $W_{n+1} \subset V(E^{n+1})$. Let $y \in W_n$; then since each J_i is a conjunct of E, $y \in S_i^{n+1}$ for $1 \leq i \leq 6$. Since each J_i^{n+1} begins with $\Box$, and since $\mathcal{F}$ models S4, $\mathcal{F}$ models $\Box J_i^{n+1} \leftrightarrow J_i^{n+1}$; thus $S_i^{n+1} = V(J_i^{n+1}) = V(\Box J_i^{n+1})$; and so, since $y \in S_i^{n+1}$, $S_i^{n+1} \in \mathcal{N}_y$. Let $N \in \mathcal{N}_y$. Then by 4.6.7 $(\bigcap_{i=1}^{6} S_i^{n+1}) \cap N \in \mathcal{N}_y$; and so, since $y \in W_n \subset \overline{P_{n+1}}$, $(\bigcap_{i=1}^{6} S_i^{n+1}) \cap N \cap P_{n+1} \neq \phi$ by 7.1.3. Thus $N \cap W_{n+1} \neq \phi$; so $W_n \subset \overline{W_{n+1}}$ by 7.1.3. Since also $W_{n+1} \subset V(E^{n+1})$, the construction of our required sequence of sets is complete.

It remains to show that H is not valid in $\mathcal{F}$. Let V' be the assignment on $\mathcal{F}$ such that $V'(s) = \bigcup_{i=0}^{\infty} W_{3i}$, $V'(t) = \bigcup_{i=0}^{\infty} W_{3i+1}$, $V'(p) = \phi$ for all propositional variables p other than s and t. Then for each i, $W_{3i+2} \subset \overline{W_{3i+3}} \subset \overline{V'(s)} = V'(\Diamond s)$; thus $W_{3i+2} \subset V'(\sim s \wedge \sim t \wedge \Diamond s)$; and so $W_{3i+1} \subset -V'(s) \cap V'(t) \cap \overline{V'(\sim s \wedge \sim t \wedge \Diamond s)} = V'(\sim s \wedge t \wedge \Diamond(\sim s \wedge \sim t \wedge \Diamond s))$. Therefore $W_{3i} \subset \overline{W_{3i+1}} \subset V'(\Diamond(\sim s \wedge t \wedge \Diamond(\sim s \wedge \sim t \wedge \Diamond s)))$. So $V'(s) \subset V'(\Diamond(\sim s \wedge t \wedge \Diamond(\sim s \wedge \sim t \wedge \Diamond s)))$ and therefore $V'(s \rightarrow \Diamond(\sim s \wedge t \wedge \Diamond(\sim s \wedge \sim t \wedge \Diamond s))) = U$. Hence $w \in W_0 \subset V'(s \wedge \Box(s \rightarrow \Diamond(\sim s \wedge t \wedge \Diamond(\sim s \wedge \sim t \wedge \Diamond s))))$ and so $w \notin V'(H)$. Therefore H is not valid in $\mathcal{F}$ and the lemma is proved. XX

7.3.2 Lemma: $L_2 \nvdash \sim E$.

Proof: This lemma is due to Fine[1]. The proof we give here is a modification of his proof. Let $\mathcal{F} = (W, \Delta, \Pi)$ be the first-order frame with the following diagram, where $u \Delta v$ if $v = u$ or if one can reach v from u by travelling along a finite number of arrows always in the direction of the arrow.



Formally, we let { a_n | $n \geq 0$ }, { b_n | $n \geq 0$ }, { c_n | $n \geq 0$ }, and { d_n | $n \geq 0$ } be four countably infinite, pairwise disjoint sets and let W be the union of these four sets. Now, for $u,v \in W$ let $u \Delta v$ if and only if

(1) $u = b_i$ and either $v = b_j$ some $j \leq i$ or $v = c_j$ some $j \leq i-2$

or (2) $u = c_i$ and either $v = c_j$ some $j \leq i$ or $v = b_j$ some $j \leq i-2$

or (3) $u = a_i$ and either $v = a_i$ or $v = b_j$ some $j \leq i+1$ or $v = c_j$ some $j \leq i+1$

or (4) $u = d_i$ and either $u = d_j$ some $j \geq i$ or $v = a_j$ some $j \geq i$ or $v = b_j$ or $v = c_j$ for some j.

We note that the relation Δ is reflexive and transitive and so by 4.6.12, $F \models S4$.

Let Π be the smallest family of subsets of W closed under the operations of complementation, intersection, and $\overset{*}{\Delta}$ and containing the sets $\{b_0\}$, $\{b_1\}$, $\{c_0\}$, $\{c_1\}$, $\{\, d_{2n} \mid n \geq 0 \,\}$, and $\{\, d_{2n+1} \mid n \geq 0 \,\}$.

Let V_0 denote the particular assignment on F such that $V_0(q_i) = \{b_i\}$, $V_0(r_i) = \{c_i\}$, $V_0(p_i) = \{\, d_{2n+i} \mid n \geq 0 \,\}$ for i = 0,1, and $V_0(p) = \phi$ for all propositional variables p other than q_0, q_1, r_0, r_1, p_0, and p_1.

It will be helpful, before continuing with the proof proper, to make and prove the following claims.

<u>Claim 1</u>: For any $m \geq 0$, $V_0(B_m) = \{b_m\}$ and $V_0(C_m) = \{c_m\}$.

Proof of claim: By induction on m. For m = 0 and m = 1 the result follows by definitions of V_0, B_i, C_i (i = 0,1). Suppose the result holds for all $m \leq k$ where $k \geq 1$. Then $V_0(B_{k+1},w) = T \Leftrightarrow V_0(\Diamond B_k \wedge \Diamond C_{k-1} \wedge \sim\Diamond C_k, w) = T \Leftrightarrow$ there are $x,y \in W$ such that $x \in V_0(B_k)$, $y \in V_0(C_{k-1})$, $w\Delta x$, and $w\Delta y$, and there is no z such that $z \in V_0(C_k)$ and $w\Delta z$. Since $V_0(B_k) = \{b_k\}$, $V_0(C_{k-1}) = \{c_{k-1}\}$, $V_0(C_k) = \{c_k\}$ by the induction hypothesis, we see that $V_0(B_{k+1},w) = T$ if and only if $w\Delta b_k$, $w\Delta c_{k-1}$, and $w\not\Delta c_k$. But we can easily see that this only happens if $w = b_{k+1}$. Similarly, interchanging B and C, b and c in the above, we see that $V_0(C_{k+1},w) = T$ if and only if $w = c_{k+1}$. The claim is proved.

Claim 2: Let V be any assignment on F and $w \in W$ such that $V(D,w) = T$. Then $w = d_m$ for some m and for all $j \geq m$, $d_j \in V(p_0 \vee p_1)$.

Proof of claim: We show that if $V(D,w) = T$ then either a cycle or an infinite chain with respect to Δ is accessible from w, or, to be precise, there is an infinite sequence $w_0, w_1, w_2, \ldots$ of points in W such that $w = w_0$ and for $i \geq 0$, $w_i \Delta w_{i+1}$ and $w_i \neq w_{i+1}$. It is easy to see that we can only have this if each $w_i = d_j$ for some j, and in particular, $w = d_m$ for some m.

We construct this infinite sequence by induction. Let $w_0 = w$. Since $V(D,w) = T$ we have $w \in V(p_0 \vee p_1)$. Suppose that $w_1, \ldots, w_k$ have been chosen to satisfy the preceeding conditions and such that each $w_i \in V(p_0 \vee p_1)$. Since $w = w_0 \;\Delta\; w_k$ and $V(D,w) = T$ we have $V((p_0 \rightarrow \Diamond p_1) \wedge (p_1 \rightarrow \Diamond p_0) \wedge \sim(p_0 \wedge p_1), w_k) = T$. Therefore, if $w_k \in V(p_0)$ then $V(\Diamond p_1, w_k) = T$ in which case we choose w_{k+1} so that $w_k \Delta w_{k+1}$ and $w_{k+1} \in V(p_1)$, and if $w_k \in V(p_1)$ then $V(\Diamond p_0, w_k) = T$ in which case we choose w_{k+1} so that $w_k \Delta w_{k+1}$ and $w_{k+1} \in V(p_0)$. Since $V(\sim(p_0 \wedge p_1), w_k) = T$ we know that $w_k \neq w_{k+1}$. The induction is complete.

Since there are not $u,v \in W$ with $u \neq v$, $u \Delta v$, and $v \Delta u$ (i.e., there are no non-trivial cycles in F), we see that we must have an infinite set of d_j's in $V(p_0 \vee p_1)$. Thus for each $j \geq m$ there is a $j' \geq j$ such that $d_{j'} \in V(p_0 \vee p_1)$. Then d_j

$\in V(\Diamond(p_0 \vee p_1))$. If $d_j \notin V(p_0 \vee p_1)$ then $V(\sim(p_0 \vee p_1), d_j) = T$. But $V(J_2, d_m) = T$ and since $d_m \Delta d_j$, $V(\sim(p_0 \vee p_1) \rightarrow \Box\sim(p_0 \vee p_1), d_j) = T$. Thus $V(\Box\sim(p_0 \vee p_1), d_j) = T$ and so $V(\sim\Diamond(p_0 \vee p_1), d_j) = T$, a contradiction. Therefore we must have $d_j \in V(p_0 \vee p_1)$ for all $j \geq m$ and the claim is proved.

Claim 3: Let V be any assignment on $\mathcal{F}$ and $w \in W$ such that $V(H,w) = F$. Then $w = d_m$ for some m.

Proof of claim: The proof is similar to that of claim 2. By a similar induction we construct an infinite sequence satisfying similar conditions as the one in the proof of claim 2, and so $w = d_m$ for some m and the claim is proved.

Claim 4: If A is any formula and V any assignment on $\mathcal{F}$, then there is a formula A' which is a substituted case of A such that $V_0(A') = V(A)$.

Proof of claim: We shall first show that for any $S \in \Pi$ there is a formula B such that $V_0(B) = S$. It suffices to show that Π is equal to Π', where we define Π' by $\Pi' = \{ V_0(B) \mid B \text{ is a formula} \}$. We know that $V_0(p) \in \Pi$ for each propositional variable p and therefore that $V_0(B) \in \Pi$ for each formula B. Thus $\Pi' \subset \Pi$. Now since $U - V_0(B) = V_0(\sim B)$, $V_0(B) \cap V_0(C) = V_0(B \wedge C)$ and $*V_0(B) = V_0(\Box B)$, we see that Π' is closed under the operations of complementation, intersection, and $*$, and from the definition of V_0 we see that Π' contains $\{b_0\}$, $\{b_1\}$, $\{c_0\}$, $\{c_1\}$, $\{ d_{2n} \mid n \geq 0 \}$, $\{ d_{2n+1} \mid n \geq 0 \}$ and therefore $\Pi \subset \Pi'$.

Now let p_{i_1}, ... , p_{i_m} be a list of the propositional variables occuring in A and let B_{i_1}, ... , B_{i_m} be formulae such that $V_0(B_{i_k}) = V(p_{i_k})$ for k = 1, ... , m. Let A′ be the substituted case of A obtained by replacing each occurrence of p_{i_k} by B_{i_k}. Then $V(A) = V_0(A')$. The claim is proved.

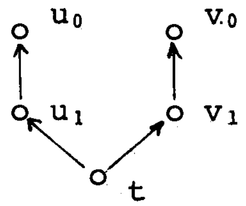
We now continue with the proof of 7.3.2.

We first show that $\mathcal{F} \not\models \sim E$ by showing that $V_0(E,d_0) = T$. We have $d_0 \in V_0(p_0) \subset V_0(p_0 \vee p_1)$. If $w \in V_0(p_0)$ then $w = d_{2j}$ for some j and so $w \in V_0(\sim p_1)$, but since $w \Delta d_{2j+1}$ and $d_{2j+1} \in V_0(p_1)$, $w \in V_0(\Diamond p_1)$. Therefore $V_0(p_0 \to \Diamond p_1) = W$. Clearly $V_0(\sim(p_0 \wedge p_1)) = W$. Therefore $d_0 \in V_0(\Box((p_0 \to \Diamond p_1) \wedge (p_1 \to \Diamond p_0) \wedge \sim(p_0 \wedge p_1))) = V_0(J_1)$. If $w \in V_0(\sim(p_0 \vee p_1))$ then $w \neq d_i$ for any i and so if $w \Delta v$ then $v \neq d_j$ for any j and therefore $w \in V_0(\Box\sim(p_0 \vee p_1))$. Thus $V_0(\sim(p_0 \vee p_1) \to \Box\sim(p_0 \vee p_1)) = W$ and so $d_0 \in V_0(\Box(\sim(p_0 \vee p_1) \to \Box\sim(p_0 \vee p_1))) = V_0(J_2)$. Therefore $d_0 \in V(D)$.

Now $a_0 \Delta b_1$, $a_0 \Delta c_1$, $a_0 \not\Delta b_0$, and $a_0 \not\Delta c_0$. Thus, by claim 1, $V_0(A_0, a_0) = T$ and, since $d_0 \Delta a_0$, $V_0(\Diamond A_0, d_0) = T$. We note that $b_1 \Delta b_0$ and $b_1 \not\Delta c_0$, $c_1 \Delta c_0$ and $c_1 \not\Delta b_0$, $b_0 \not\Delta b_1$, $c_0 \not\Delta c_1$, $b_0 \not\Delta c_1$, $c_0 \not\Delta b_1$; so by claim 1 again, $V_0(B_1 \to \Diamond B_0 \wedge \sim\Diamond C_0) = V_0(C_1 \to \Diamond C_0 \wedge \sim\Diamond B_0) = V_0(B_0 \to \sim\Diamond(B_1 \vee C_1)) = V_0(C_0 \to \sim\Diamond(C_1 \vee B_1) = W$. Hence $d_0 \in V_0(J_3) \cap V_0(J_4) \cap V_0(J_5) \cap V_0(J_6) = V_0(K)$. Thus $d_0 \in V_0(E)$.

Next we show that $\mathcal{F} \models G$. Let V be any assignment on $\mathcal{F}$ and $w \in W$ such that $w \in V(E)$. Since D is a conjunct of E,

$V(D,w) = T$, and hence, by claim 2, $w = d_m$ for some m. $\Diamond A_0$ is a conjunct of E and so, for some $t \in W$, $w\Delta t$ and $V(A_0,t) = T$: thus $V(\Diamond B_1 \wedge \Diamond C_1 \wedge \sim\Diamond B_2 \wedge \sim\Diamond C_2, t) = T$. Thus for some u_1 and v_1, $t\Delta u_1$, $t\Delta v_1$, $V(B_1,u_1) = T$ and $V(C_1,v_1) = T$. Now J_3 is a conjunct of E, and so $w \in V(J_3) = V(\Box(B_1 \rightarrow \Diamond B_0 \wedge \sim\Diamond C_0))$. Since $w\Delta u_1$ and $V(B_1,u_1) = T$, we must have $V(\Diamond B_0 \wedge \sim\Diamond C_0, u_1) = T$. Thus there is u_0 such that $u_1 \Delta u_0$ and $V(B_0,u_0) = T$. Similarly, from $w \in V(J_4)$ we see that $V(\Diamond C_0 \wedge \sim\Diamond B_0, v_1) = T$ and so there is v_0 such that $v_1 \Delta v_0$ and $V(C_0,v_0) = T$. Furthermore, since $V(\sim\Diamond C_0,u_1) = V(\sim\Diamond B_0,v_1) = T$, we have $v_1 \not\Delta u_0$ and $u_1 \not\Delta v_0$. Now J_5 is also a conjunct of E; so $V(\Box(B_0 \rightarrow \sim\Diamond(B_1 \vee C_1)),w) = T$ and since $w\Delta u_0$ and $V(B_0,u_0) = T$, $V(\sim\Diamond(B_1 \vee C_1),u_0) = T$. Similarly from $V(J_6,w) = T$ we get $V(\sim\Diamond(C_1 \vee B_1),v_0) = T$. Thus $u_0 \not\Delta u_1$, $u_0 \not\Delta v_1$, $v_0 \not\Delta v_1$, $v_0 \not\Delta u_1$. We see now that the structure (W,Δ) restricted to $\{t,u_1,v_1,u_0,v_0\}$ has the diagram



where the only "Δ-accessibility" is that marked by the arrows. Now by comparing with the diagram of (W,Δ) we see that the only possibility for such a substructure is to have $t = a_n$, $u_i = b_{n+i}$, and $v_i = c_{n+i}$ for some n (i = 1,2) and that, in fact, $V(B_i) = \{b_{n+i}\}$ and $V(C_i) = \{c_{n+i}\}$ for i = 0 and 1.

From this it follows, by a routine induction argument, that for each $i \geq 0$ $V(B_i) = \{b_{n+i}\}$, $V(C_i) = \{c_{n+i}\}$, and $V(A_i) = \{a_{n+i}\}$. Now $d_{n+1} \Delta a_{n+1}$ and $d_{n+1} \not\Delta a_n$ and so $V(\sim\Diamond A_1 \wedge \Diamond A_0, d_{n+1}) = T$. Since $d_m = w \Delta t = a_n$ we must have $n \geq m$ and therefore $w = d_m \Delta d_{n+1}$. Hence, by the second part of claim 2, $d_{n+1} \in V(p_0 \vee p_1)$. Therefore we have $V((p_0 \vee p_1) \wedge \sim\Diamond A_0 \wedge \Diamond A_1, d_{n+1}) = T$ and so $V(\Diamond((p_0 \vee p_1) \wedge \sim\Diamond A_0 \wedge \Diamond A_1), w) = T$, i.e. $V(F,w) = T$. Now we have $V(G,w) = T$ for all $w \in W$, V on $\mathcal{F}$ and so $\mathcal{F} \models G$.

Finally we show that $\mathcal{F} \models H$. Suppose otherwise. Then for some $w \in W$ and some V on $\mathcal{F}$, $V(H,w) = F$. By claim 3, $w = d_m$ for some m and by claim 4 there are formulae A and B such that $V_0(A \wedge \Box(A \rightarrow \Diamond(\sim A \wedge B \wedge \Diamond(\sim A \wedge \sim B \wedge \Diamond A))), w) = T$. Now let $\Box D_1, \ldots, \Box D_k$ be all the formulae beginning with $\Box$ which have elementary subformula occurrences in either A or B (see 2.1.1). If $j > i$ and $V_0(\Box D_h, d_i) = T$ then $V_0(\Box D_h, d_j) = T$ for $h = 1,2,\ldots,k$ and $i \geq 0$. So for some $n \geq m$ we have for all $h = 1,\ldots,k$ and all $n' \geq n$, $V_0(\Box D_h, d_{n'}) = V_0(\Box D_h, d_n)$.

If $i,j \geq n$ and $i-j$ is even then for each propositional variable p, $V_0(p,d_i) = V_0(p,d_j)$; therefore $V_0(A,d_i) = V_0(A,d_j)$, $V_0(\sim A \wedge B, d_i) = V_0(\sim A \wedge B, d_j)$, and $V_0(\sim A \wedge \sim B, d_i) = V_0(\sim A \wedge \sim B, d_j)$. It is easy to see that for some $i_1 > i_2 > i_3 \geq n$, $V(A, d_{i_3}) = V(\sim A \wedge B, d_{i_2}) = V(\sim A \wedge \sim B, d_{i_1}) = T$. But one of $(i_1 - i_2)$, $(i_1 - i_3)$, $(i_2 - i_3)$ must be even and so either $V(A \wedge (\sim A \wedge B), d_{i_2}) = T$ or $V((\sim A \wedge B) \wedge (\sim A \wedge \sim B), d_{i_1}) = T$ or $V(A \wedge (\sim A \wedge \sim B), d_{i_1}) = T$; an impossibility. So we must have

$\mathcal{F} \models H$. XX

7.3.3 **Theorem:** L_2 is a normal extension of S4 which is incomplete with respect to $\mathcal{S}$.

Proof: Lemmas 7.3.1 and 7.3.2. XX

7.3.4 **Corollary:** $\mathcal{S}$ does not have maximal depth even with respect to normal extensions of S4. XX

7.4 We now have a negative answer to Segerberg's question mentioned in 5.3: "Are all classical modal logics natural?" Since we have already seen (5.3.1) that the canonical frame $\mathcal{F}_L$ from a logic L models *no more* than L, if we also had $\mathcal{F}_L \models L$ we would have L determined by the single frame $\mathcal{F}_L$ and thus complete with respect to $\mathcal{S}$. Therefore as a corollary of 7.2.3 and 7.3.3 we cannot have $\mathcal{F}_{L_1} \models L_1$ or $\mathcal{F}_{L_2} \models L_2$, and hence

7.4.1 **Theorem:** Neither L_1 nor L_2 is a natural logic. XX

8. Width and Depth of $\mathcal{R}$

8.0 In this chapter we shall study the width and depth of the relational semantics. As we have already mentioned, we shall see that the relational semantics has width of measure K and has less depth than the neighbourhood semantics.

8.1 We have already seen in 5.5 that the measure of the width of $\mathcal{R}$ is an extension of K. To show that it is K it suffices to construct a relational frame $\mathcal{W}_0$ such that for each formula A, $\mathcal{W}_0 \models A$ if and only if $K \vdash A$.

Let $\mathcal{B}_K$ be the Lindenbaum-Tarski boolean frame for K described in 6.1. By 6.1.1 and 6.1.2, $\mathcal{B}_K \models A$ iff $K \vdash A$. Now let $\Phi(\mathcal{B}_K) = (U_0, <_0, \Pi_0)$ be the first-order relational frame where Φ is Makinson's mapping described in 6.2. By 6.2.4, $\Phi(\mathcal{B}_K) \models A$ iff $\mathcal{B}_K \models A$. Now let $\mathcal{W}_0 = (U_0, <_0)$. Then by 3.4.1, if $\mathcal{W}_0 \models A$ then $\Phi(\mathcal{B}_K) \models A$. Therefore if $\Phi(\mathcal{B}_K) \not\models A$ then $\mathcal{W}_0 \not\models A$. Thus if $K \not\vdash A$ then $\mathcal{W}_0 \not\models A$, and since we know by 4.6.8 that $\mathcal{W}_0 \models K$, we have $\mathcal{W}_0 \models A$ iff $K \vdash A$. Hence

8.1.1 Theorem: $\mathcal{R}$ has width of measure K. XXX

8.2 We shall show that even when restricted to normal frames, the neighbourhood semantics has strictly greater depth than the relational semantics. We do this by displaying a normal neighbourhood frame $\mathcal{F} = (U, \mathcal{N})$ which is not equivalent to any class of relational frames. Then, if $L_{\mathcal{F}}$ is the logic determined by $\mathcal{F}$, $L_{\mathcal{F}}$ will be complete with respect to $\mathcal{S}$ but incomplete with respect to $\mathcal{R}$. Since we already know that the depth of $\mathcal{S}$ is at least as great as that of $\mathcal{R}$ this result will show that $\mathcal{S}$ in fact has strictly greater depth. We show that $\mathcal{F}$ is not equivalent to any class of relational frames by showing that a particular set Γ of formulae is valid in $\mathcal{F}$ but that another particular formula G is not. We then show that in every relational frame which models Γ, G is valid. Thus no class of relational frames can determine the logic $L_{\mathcal{F}}$.

Frequent reference will be made to the following formulae.

A_i	$\Box(q_i \to r)$	$(1 \le i \le 3)$
B_{ij}	$\Box^2 \sim(q_i \wedge q_j)$	$(1 \le i < j \le 3)$
C_i	$\Box(r \to \Diamond q_i)$	$(1 \le i \le 3)$
A	$r \wedge \bigwedge A_i \wedge \bigwedge B_{ij} \wedge \bigwedge C_i$	
B	$(A \wedge \Box p \wedge \sim\Box^2 p) \to \Diamond(r \wedge \Box(r \to q_1 \vee q_2 \vee q_3))$	
C	$(q \wedge \Box^2 r \wedge \sim\Box^3 r) \to \Box((\Box r \wedge \sim\Box^2 r) \to \Diamond q)$	
D	$(\Box p \wedge \sim\Box^2 p \wedge \Diamond q \wedge \Diamond r) \to (\Diamond(q \wedge \Diamond r) \vee \Diamond(r \wedge \Diamond q))$	
E	$\Box p \to p$	
F_n	$(A \wedge \Box p \wedge \sim\Box^2 p) \to \Diamond(\Box^n p \wedge \sim\Box^{n+1} p)$	$(n \ge 2)$
G	$A \to (\Box p \to \Box^2 p)$	

Let Γ denote the infinite set $\{B, C, D, E, F_n \mid n \ge 2\}$.

8.2.1 Lemma: Any relational frame that models Γ also models G.

Proof: The formulae Γ, G are very similar to those used by Thomason in [17] and the proof that we give of this lemma is an adaptation of his proof of the corresponding fact in [17].

Suppose that $\mathcal{W} = (W,<)$ models C, D, E, and F_n for all $n\geq 2$ but that it fails to model G. We show that $\mathcal{W}$ fails to model B, and thus any relational frame that models Γ must also model G. Since $\mathcal{W} \not\models G$ there is $w \in W$ and an assignment V on $\mathcal{W}$ such that $V(A,w) = T$ and $V(\Box p\to\Box^2 p,w) = F$. Thus $V(A\wedge\Box p\wedge\sim\Box^2 p,w) = T$. For each $n\geq 2$, $V(F_n,w) = T$ and so $V(\Diamond(\Box^n p\wedge\sim\Box^{n+1} p),w) = T$. Thus there is a point, call it w_n, in W such that $w<w_n$ and $V(\Box^n p\wedge\sim\Box^{n+1} p,w_n) = T$. Let $w = w_1$. We shall show that for $i\leq j$, $w_i<w_j$.

Since $\mathcal{W} \models E$, $\mathcal{W} \models T$; so by 4.6.9, < is reflexive. Choose positive integers i and j. Then $w<w_i$ and $w<w_j$. Let V_1 be such that $V_1(\Box p\wedge\sim\Box^2 p,w) = T$ and also such that $V_1(q) = \{w_i\}$ and $V_1(r) = \{w_j\}$. Then $V_1(\Box p\wedge\sim\Box^2 p\wedge\Diamond q\wedge\Diamond r,w) = T$ and since also $V_1(D,w) = T$, we have $V_1(\Diamond(q\wedge\Diamond r)\vee\Diamond(r\wedge\Diamond q),w) = T$. If $V_1(\Diamond(q\wedge\Diamond r),w) = T$ we must have $w_i<w_j$ and if $V_1(\Diamond(r\wedge\Diamond q),w) = T$ we must have $w_j<w_i$. So either $w_i<w_j$ or $w_j<w_i$.

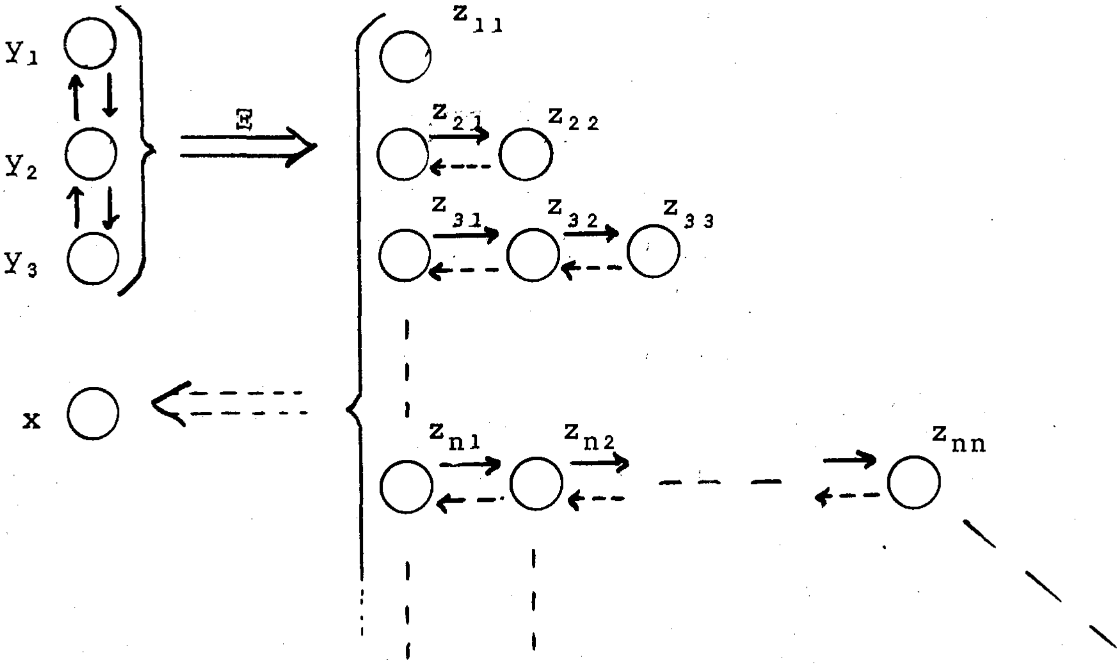
If $i=j$, then since < is reflexive, $w_i<w_j$. If $i+2\leq j$ and if $w_j<w_i$ then since $V(\Box^i p\wedge\sim\Box^{i+1} p,w_i) = T$, we have

$V(\sim\Box^{i+1}p,w_i) = T$ and thus $V(\Diamond\sim\Box^{i+1}p,w_j) = T$. Therefore $V(\sim\Box^{i+2}p,w_j) = T$. But $V(\Box^j p\wedge\sim\Box^{j+1}p,w_j) = T$ and so $V(\Box^j p,w_j) = T$ and by 7.1.10 $V(\Box^{i+2}p,w_j) = T$; a contradiction. So we can't have $w_j<w_i$ and therefore must have $w_i<w_j$.

Now suppose $i+1 = j$. If $w_j\not<w_i$ then $w_i<w_j$. So, suppose that $w_j<w_i$. Then let V_2 be such that $V_2(q) = \{w_j\}$, $V_2(r) = V(\Box^{i-1}p)$ (of course, $\Box^0 p$ would simply be p). We know that $V(\Box^j p\wedge\sim\Box^{j+1}p,w_j) = V(\Box^{i+1}p\wedge\sim\Box^{i+2}p,w_j) = T$ and so $V_2(\Box^2 r\wedge\sim\Box^3 r,w_j) = T$. Thus $V_2(q\wedge\Box^2 r\wedge\sim\Box^3 r,w_j) = T$, and since $V_2(C,w_j) = T$, we have $V_2(\Box((\Box r\wedge\sim\Box^2 r)\rightarrow\Diamond q),w_j) = T$. Now $V_2(\Box r\wedge\sim\Box^2 r,w_i) = V(\Box^i p\wedge\sim\Box^{i+1}p,w_i) = T$ and $w_j<w_i$ and so $V_2(\Diamond q,w_i) = T$. Therefore $w_i<w_j$. All cases are taken care of, and so, if $i\leq j$ then $w_i<w_j$.

Now let V_3 be such that $V_3(p) = V(p)$, $V_3(r) = \{ w_n \mid n\geq 1 \}$, $V_3(q_i) = \{ w_{4n-i} \mid n\geq 1 \}$ $(i = 1,2,3)$. Then it is easy to see that $V_3(A\wedge\Box p\wedge\sim\Box^2 p,w) = T$. Now let u be such that $w<u$ and $u \in V_3(r)$. Then $u = w_j$ for some $j\geq 1$. Let k be such that $4k\geq j$. Then $w_j<w_{4k}$ and since for each $i = 1,2,3$, $w_{4k} \notin V_3(q_i)$ but $w_{4k} \in V_3(r)$, we have $V_3(r\rightarrow q_1\vee q_2\vee q_3, w_{4k}) = F$. Thus $V_3(\Box(r\rightarrow q_1\vee q_2\vee q_3),w_j) = F$. Thus for all u such that $w<u$, $V_3(r\wedge\Box(r\rightarrow q_1\vee q_2\vee q_3),u) = F$ and so $V_3(\Diamond(r\wedge\Box(r\rightarrow q_1\vee q_2\vee q_3)),w) = F$. Therefore $V_3(B,w) = F$ and so $w \not\models B$. XXX

Now let $F = (U,N)$ be the particular neighbourhood frame with the following diagram.



Ξ is any non-principal ultrafilter on the natural numbers (beginning with 1). The arrows are to be interpreted as follows. If v can be reached from u by a sequence of solid single arrows or by one dotted arrow (double or single -- note that x can be reached from each z_{n1}) then v is in every neighbourhood of u. In addition, the solid double arrow with "Ξ" above it indicates that the neighbourhoods of y_1, y_2, and y_3 are those sets which in addition to containing $\{y_1, y_2, y_3\}$ contain, for some $K \in \Xi$, all the rows of z's corresponding to the numbers in K. Also every element is in all its own neighbourhoods.

Thus we have, formally:

Ξ any non-principal ultrafilter on the natural numbers (positive integers)

$$U = \{x, y_1, y_2, y_3\} \cup \{ z_{ij} \mid 1 \leq j \leq i \}$$

for $i = 1,2,3$ $\quad N \in \mathcal{N}_{y_i}$ iff there is $K \in \Xi$ such that

$$N \supset \{y_1, y_2, y_3\} \cup \{ z_{kj} \mid j\le k,\ k \in K \}$$

$N \in \mathcal{N}_x$ iff $x \in N$

$N \in \mathcal{N}_{z_{n1}}$ iff $N \supset \{x, z_{n1}, \dots, z_{nn}\}$

for $2\le m\le n$ $\quad N \in \mathcal{N}_{z_{nm}}$ iff $N \supset \{z_{n\,m-1}, \dots, z_{nn}\}$

Since Ξ is a filter, $\mathcal{N}_{y_i}$ is itself a filter for each i. It is clear that the set of neighbourhoods of each other point in U is a filter. So $\mathcal{F}$ is normal by 4.6.7. Since, also, every element of U is in all its own neighbourhoods, $\mathcal{F}$ models T, by 4.10.1, and so

<u>8.2.2</u> <u>Lemma</u>: $\mathcal{F} \models E$. XXX

<u>8.2.3</u> <u>Lemma</u>: If $V(A,u) = T$ then $u = y_i$ for some i $(1\le i\le 3)$.

Proof: Suppose $V(r\wedge\bigwedge C_i,x) = T$. Then since $\{x\} \in \mathcal{N}_x$, $V(q_i,x) = T$ for all i $(1\le i\le 3)$. Thus $V(B_{ij},x) = F$ for $1\le i<j\le 3$. So $V(A,x) = F$.

Now suppose that $V(A,z_{nk}) = T$. Let h be the greatest such that $V(r,z_{nh}) = T$. Such an h must exist since r is a conjunct of A and hence $V(r,z_{nk}) = T$. Thus $k\le h\le n$. Now $z_{nh} \in N$ for every $N \in \mathcal{N}_{z_{nk}}$ and so, since $\Box(r\rightarrow\Diamond q_i)$ is a conjunct of A for $1\le i\le 3$, $V(r\rightarrow\Diamond q_i, z_{nh}) = T$ and hence $V(\Diamond q_i,z_{nh}) = T$ for $1\le i\le 3$. We have $\{z_{n\,h-1}, z_{nh}, \dots, z_{nn}\} \in \mathcal{N}_{z_{nh}}$ (if $h = k = 1$ then $z_{n\,h-1}$ simply denotes x), and so for $1\le i\le 3$, $V(q_i) \cap \{z_{n\,h-1}, \dots, z_{nn}\} \neq \phi$. But $z_{nm} \in N$ for every $N \in \mathcal{N}_{z_{nh}}$ if $h-1\le m\le n$, and so $V(q_i\rightarrow r,z_{nm}) = T$ for such m and

$1 \leq i \leq 3$. Since h is the largest such that $V(r, z_{nh}) = T$, we must have $V(q_i, z_{nm}) = F$ for $h+1 \leq m \leq n$. Thus $V(q_i) \cap \{z_{n\ h-1}, z_{nh}\} \neq \phi$ for i = 1, 2, and 3 and so for some i,j such that $1 \leq i < j \leq 3$ either $V({\sim}(q_i \wedge q_j), z_{n\ h-1}) = F$ or $V({\sim}(q_i \wedge q_j), z_{nh}) = F$. In either case this contradicts $V(\Box^2 {\sim}(q_i \wedge q_j, z_{nk}) = T$, and so $V(A, z_{nk}) = F$. XX

<u>8.2.4</u> <u>Lemma</u>: If H is any formula, V any assignment, and $1 \leq i \leq 3$, then (i) if $V(\Box^2 H, y_i) = T$ then $V(H,x) = T$, (ii) if $V(\Box H, y_i) = T$ and $V(H,x) = T$ then $V(\Box^2 H, y_i) = T$, and (iii) if $V(\Box^2 H, y_i) = T$ then $V(\Box^j H, y_i) = T$ for all $j \geq 0$.

Proof: (i) $V(\Box^2 H, y_i) = T$ implies that $V(\Box H, u) = T$ for all u in some neighbourhood of y_i; in particular, $V(\Box H, z_{n1}) = T$ for some n. But then $V(H,x) = T$ since x is in every neighbourhood of z_{n1}.

(ii) If $V(\Box H, y_i) = T$ then there is some $K \in \Xi$ such that $V(H,u) = T$ for $u \in \{y_1, y_2, y_3\} \cup \{z_{nm} \mid n \in K\}$. Then $V(\Box H, y_j) = T$ for $1 \leq j \leq 3$. Also, since $V(H,x) = T$ and $\{x, z_{n1}, \ldots, z_{nn}\} \in N_{z_{nm}}$, $V(\Box H, z_{nm}) = T$ for all $n \in K$ and $m \leq n$. Thus $V(\Box H, u) = T$ for $u \in \{y_1, y_2, y_3\} \cup \{z_{nm} \mid n \in K, m \leq n\}$ and so $V(\Box^2 H, y_i) = T$.

(iii) If $V(\Box^2 H, y_i) = T$, then by 8.2.2 $V(\Box^j H, y_i) = T$ for j = 0 or 1. Suppose $j > 2$ and that if $V(\Box^2 H, y_i) = T$ then $V(\Box^{j-1} H, y_i) = T$. Then by (i) with H replaced by $\Box^{j-3} H$, $V(\Box^{j-3} H, x) = T$, and so, since $\{x\} \in N_x$, $V(\Box^{j-2} H, x) = T$. But then by (ii), with H replaced by $\Box^{j-2} H$, $V(\Box^j H, y_i) = T$. XX

<u>8.2.5</u> <u>Lemma</u>: $\mathcal{F} \models B$.

Proof: Let V be an assignment such that for some u, $V(A,u) = T$. Then by 8.2.3, $u = y_k$ for some k $(1 \leq i \leq 3)$.

Let $S_0 = \{ n \mid \text{for every } m \leq n,\ z_{nm} \notin V(r) \}$

$S_i = \{ n \mid \text{for every } m \leq n,\ V(q_i \rightarrow r, z_{nm}) = T \}$ $(1 \leq i \leq 3)$

$S_{1+i+j} = \{ n \mid \text{for every } m \leq n,\ V(\sim(q_i \wedge q_j), z_{nm}) = T \}$ $(1 \leq i < j \leq 3)$

$S_{6+i} = \{ n \mid \text{for every } m \leq n,\ V(r \rightarrow \Diamond q_i, z_{nm}) = T \}$ $(1 \leq i \leq 3)$

Then since $V(A,y_k) = T$; $S_1, \ldots, S_9 \in \Xi$. We shall show that also $S_0 \in \Xi$.

Let $Q = -S_0 \cap S_1 \cap \ldots \cap S_9$ and suppose that $Q \neq \phi$. Pick $n \in Q$ and let m be the greatest such that $z_{nm} \in V(r)$. Then since $\{z_{n\ m-1}, z_{nm}, \ldots, z_{nn}\} \in \mathcal{N}_{z_{nm}}$ (if $m = 1$ then $z_{n\ m-1}$ denotes x) and since for $m < h \leq n$, $V(q_i, z_{nh}) = V(r, z_{nh}) = F$, $V(q_i) \cap \{z_{nm}, z_{n\ m-1}\} \neq \phi$ for each i $(1 \leq i \leq 3)$. Thus for some i,j $(1 \leq i < j \leq 3)$, $V(\sim(q_i \wedge q_j), z_{nm}) = F$ or $V(\sim(q_i \wedge q_j), z_{n\ m-1}) = F$. But $V(\sim(q_i \wedge q_j), z_{nm}) = F$ contradicts the fact that $n \in Q$, as does $V(\sim(q_i \wedge q_j), z_{n\ m-1}) = F$ if $m \geq 2$. On the other hand, if $m = 1$ the impossibility of $V(\sim(q_i \wedge q_j), x) = F$ follows from the fact that $V(A,y_k) = T$ and and from 8.2.4(i). Thus we must have $Q = \phi$. Therefore $Q \notin \Xi$ and so $-S_0 \notin \Xi$ and, since Ξ is an ultrafilter, $S_0 \in \Xi$. Now let $R = S_0 \cap S_1 \cap \ldots \cap S_9$. Then $R \in \Xi$.

Now, since $V(A,y_k) = T$, $V(\Diamond q_j, y_k) = T$ for $1 \leq j \leq 3$. Thus every neighbourhood of y_k includes points v_1, v_2, and v_3

such that $v_j \in V(q_j)$ $(1 \leq j \leq 3)$. One particular neighbourhood of y_k is the set $M = \{y_1, y_2, y_3\} \cup \{ z_{nm} \mid n \in R \}$. But if $n \in R$ then $n \in S_0$ and so $z_{nm} \notin V(r)$ for all $m \leq n$, and $n \in S_1 \cap S_2 \cap S_3$ and so $z_{nm} \notin V(q_j)$ for $j = 1, 2,$ and 3. Thus, since we can't have $v_j = z_{nm}$ we must have $\{v_1, v_2, v_3\} = \{y_1, y_2, y_3\}$. And so for $1 \leq j \leq 3$, we have $y_j \in V(q_1) \cup V(q_2) \cup V(q_3)$. Thus for $v \in M$ we have $V(r \rightarrow q_1 \vee q_2 \vee q_3, v) = T$, and so $V(r \wedge \Box(r \rightarrow q_1 \vee q_2 \vee q_3), y_k) = T$; hence $V(\Diamond(r \wedge \Box(r \rightarrow q_1 \vee q_2 \vee q_3)), y_k) = T$ and so $V(B, y_k) = T$. Thus $\mathcal{F} \models B$.

<u>8.2.6</u> <u>Lemma</u>: If H is any formula, V any assignment, $1 \leq m \leq n$, and $h \geq m$, then if $V(\Box^h H, z_{nm}) = T$ then for all $u \in \{x, z_{n1}, \ldots, z_{nn}\}$ and all $j \geq 0$, $V(\Box^j H, u) = T$.

Proof: Primary induction on m. Secondary induction on j. If $m = 1$, then since $h \geq m$ and $\mathcal{F} \models E$, $V(\Box H, z_{n1}) = T$. Since for all $N \in \mathcal{N}_{z_{n1}}$, $\{x, z_{n1}, \ldots, z_{nn}\} \subseteq N$, we must have $V(H, u) = T$ for $u \in \{x, z_{n1}, \ldots, z_{nn}\}$. But for all such u, $\{x, z_{n1}, \ldots, z_{nn}\} \in \mathcal{N}_u$, and so for such u $V(\Box H, u) = T$. The induction step yielding $V(\Box^j H, u) = T$ for $j > 1$ for all such u is obvious.

Now suppose $m > 1$, that the result holds for $m-1$, and that $V(\Box^h H, z_{nm}) = T$ where $h \geq m$. Then since for all $N \in \mathcal{N}_{z_{nm}}$, $z_{n\,m-1} \in N$, we have $V(\Box^{h-1} H, z_{n\,m-1}) = T$. But $h-1 \geq m-1$ and so by induction hypothesis $V(\Box^j H, u) = T$ for all $u \in \{x, z_{n1}, \ldots, z_{nn}\}$. XXX

8.2.7 **Lemma**: If V is any assignment and k any number such that $\{z_{k1}, \ldots, z_{kk}\} \subset V(p)$ and $x \notin V(p)$, then, for $1 \leq m \leq k$, $V(\square^{m-1}p \wedge \sim\square^{m}p, z_{km}) = T$.

Proof: Induction on m. $V(p, z_{k1}) = T$, but since $x \in N$ for all $N \in \mathcal{N}_{z_{k1}}$, $V(\square p, z_{k1}) = F$. Thus $V(\square^{0}p \wedge \sim\square^{1}p, z_{k1}) = T$. Also since $j \geq 2$ implies $\{z_{k1}, \ldots, z_{kk}\} \in \mathcal{N}_{z_{kj}}$, $V(\square p, z_{kj}) = T$ $\forall j \geq 2$.

Suppose that $m > 1$ and $V(\square^{m-2}p \wedge \sim\square^{m-1}p, z_{k\ m-1}) = T$ and that for all $j \geq m$, $V(\square^{m-1}p, z_{kj}) = T$. Then in particular, $V(\square^{m-1}p, z_{km}) = T$. But for all $N \in \mathcal{N}_{z_{km}}$, $z_{k\ m-1} \in N$, and since $V(\square^{m-1}p, z_{k\ m-1}) = F$, $V(\square^{m}p, z_{km}) = F$. Thus $V(\square^{m-1}p \wedge \sim\square^{m}p, z_{km}) = T$. Also, since $\{z_{km}, \ldots, z_{kk}\} \in \mathcal{N}_{z_{kj}}$ for all $j \geq m+1$, $V(\square^{m}p, z_{kj}) = T$ for all $j \geq m+1$. The induction is complete. XX

8.2.8 **Lemma**: For all $n \geq 2$, $\mathcal{F} \models F_n$.

Proof: Pick $n \geq 2$. Suppose $V(A \wedge \square p \wedge \sim\square^{2}p, u) = T$. By 8.2.3, $u = y_i$ for some i $(1 \leq i \leq 3)$, and by 8.2.4(ii), $x \notin V(p)$. If $R = \{\, k \mid \{z_{k1}, \ldots, z_{kk}\} \subset V(p) \,\}$, then $R \in \Xi$. Let $N \in \mathcal{N}_{y_i}$. Then if $S_N = \{\, k \mid \{z_{k1}, \ldots, z_{kk}\} \subset N \,\}$ then $S_N \in \Xi$. Thus $S_N \cap R \in \Xi$ and so $S_N \cap R$ is infinite. Pick k such that $k \in S_N \cap R$ and $k > n$. Then the hypotheses of 8.2.7 hold for V and k and so $V(\square^{n}p \wedge \sim\square^{n+1}p, z_{k\ n+1}) = T$ with $z_{k\ n+1} \in N$. Thus $V(\Diamond(\square^{n}p \wedge \sim\square^{n+1}p), y_i) = T$ and so $V(F_n, y_i) = T$. Hence $\mathcal{F} \models F_n$. XX

8.2.9 **Lemma**: $\mathcal{F} \models C$.

Proof: We show that for any assignment V and any $u \in U$, $V(C,u) = T$.

If $u = z_{n1}$ or z_{n2} for some n, then, by 8.2.6, $V(\Box^2 r \wedge \sim\Box^3 r, u) = F$; if $u = y_i$ $(1 \leq i \leq 3)$ then by 8.2.4(iii), $V(\Box^2 r \wedge \sim\Box^3 r, u) = F$. If $u = x$ and $V(\Box^2 r, u) = T$, then since $\{x\} \in \mathcal{N}_x$, $V(\Box^3 r, x) = T$. Thus $V(C,x) = T$.

Finally, suppose $u = z_{nk}$ for some n and some $k \geq 3$. Suppose further that $V(q \wedge \Box^2 r \wedge \sim\Box^3 r, u) = T$. Then $V(\Box r, z_{n\ k-1}) = T$ since $z_{n\ k-1} \in N$ for all $N \in \mathcal{N}_{z_{nk}}$. Thus $z_{nj} \in V(r)$ for all $j \geq k-2$. But $z_{n\ k-3} \notin V(r)$ (if $k = 3$ then $z_{n\ k-3}$ denotes x) since otherwise we would have $V(\Box^3 r, z_{nk}) = T$. Now $\{z_{n\ k-1}, \ldots, z_{nn}\} \in \mathcal{N}_{z_{nk}}$ and $V(\Box r, z_{nj}) = T$ for all j such that $k-1 \leq j \leq n$, and so $V(\Box^2 r, z_{nj}) = T$ for all j such that $k \leq j \leq n$. Thus if $V(\Box r \wedge \sim\Box^2 r, z_{nj}) = T$ for some j $(k-1 \leq j \leq n)$ we must have $j = k-1$. But $z_{nk} \in N$ for all $N \in \mathcal{N}_{z_{n\ k-1}}$ and $z_{nk} \in V(q)$, thus $V(\Diamond q, z_{n\ k-1}) = T$. Hence $V(\Box((\Box r \wedge \sim\Box^2 r) \rightarrow \Diamond q), z_{nk}) = T$ and so $V(C, z_{nk}) = T$. XXX

8.2.10 Lemma: $\mathcal{F} \models D$.

Proof: Let V be any assignment. Then for any n, $V(\Box p \wedge \sim\Box^2 p, z_{n1}) = F$ by 8.2.6, and so $V(D, z_{n1}) = T$; since $\{x\} \in \mathcal{N}_x$, $V(\Box p \wedge \sim\Box^2 p, x) = F$, and so $V(D,x) = T$.

Now let V be an assignment such that $V(\Diamond q \wedge \Diamond r, z_{nm}) = T$ where $2 \leq m \leq n$. Then $\{z_{n\ m-1}, \ldots, z_{nn}\} \cap V(q) \neq \phi$ and $\{z_{n\ m-1}, \ldots, z_{nn}\} \cap V(r) \neq \phi$. So let j and k be such that $z_{nj} \in \{z_{n\ m-1}, \ldots, z_{nn}\} \cap V(q)$ and $z_{nk} \in$

$\{z_{n\,m-1}, \ldots, z_{nn}\} \cap V(r)$ and let $h = \min(j,k)$. Then for all $N \in \mathcal{N}_{z_{nm}}$, $z_{nh} \in N$ and either $V(q\wedge\Diamond r, z_{nh}) = T$ or $V(r\wedge\Diamond q, z_{nh}) = T$ depending on whether $h = j$ or $h = k$. Thus $V(\Diamond(q\wedge\Diamond r)\vee\Diamond(r\wedge\Diamond q), z_{nm}) = T$ and hence $V(D, z_{nm}) = T$.

Finally, suppose that $1\le i\le 3$ and $V(\Diamond q\wedge\Diamond r, y_i) = T$. If for any j ($1\le j\le 3$) $V(q, y_j) = T$, then since $\mathcal{N}_{y_i} = \mathcal{N}_{y_j}$, $V(\Diamond r, y_j) = T$; and so $V(\Diamond(q\wedge\Diamond r), y_i) = T$ and so $V(D, y_i) = T$. Similarly, if $V(r, y_j) = T$ for some j. Thus we can assume that for $1\le j\le 3$, $V(r, y_j) = V(q, y_j) = F$.

Then, if we let $S_q = \{n \mid \text{for some } m\le n,\ z_{nm} \in V(q)\}$ and let $S_r = \{n \mid \text{for some } m\le n,\ z_{nm} \in V(r)\}$, we must have $S_q, S_r \in \Xi$. Now let $S_1 = \{ n \mid \text{for some } k,m \text{ with } k\le m\le n,$ $z_{nk} \in V(r)$ and $z_{nm} \in V(q)\}$ and let $S_2 = \{ n \mid$ for some k,m with $k\le m\le n$, $z_{nk} \in V(q)$ and $z_{nm} \in V(r)\}$. Then $S_1 \cup S_2 = S_r \cap S_q$. Since $S_r, S_q \in \Xi$, $S_r \cap S_q \in \Xi$ and so since Ξ is an ultrafilter, either $S_1 \in \Xi$ or $S_2 \in \Xi$. If $S_1 \in \Xi$ then $V(\Diamond(r\wedge\Diamond q), y_i) = T$ and if $S_2 \in \Xi$ then $V(\Diamond(q\wedge\Diamond r), y_i) = T$. Thus $V(\Diamond(r\wedge\Diamond q)\vee\Diamond(r\wedge\Diamond q), y_i) = T$ and so $V(D, y_i) = T$. XX

8.2.11 Lemma: $\mathcal{F} \models \Gamma$ but $\mathcal{F} \not\models G$.

Proof: Let V be an assignment such that $V(p) = U-\{x\}$, $V(r) = \{y_1, y_2, y_3\}$ and $V(q_i) = \{y_i\}$ for $1\le i\le 3$. Then it is easy to see that $V(G, y_i) = F$ for $1\le i\le 3$ and so $\mathcal{F} \not\models G$. This together with 8.2.2, 8.2.5, 8.2.8, 8.2.9, and 8.2.10 gives the result. XX

8.2.12 **Theorem:** $\mathcal{F}$ is a normal neighbourhood frame with no equivalent class of relational frames.

Proof: 8.2.1 and 8.2.11. XXX

8.2.13 **Corollary:** $\mathcal{R}$ has strictly less depth than $\mathcal{S}$, even if $\mathcal{S}$ is restricted to normal frames. XXX

9. Width and Depth of $\hat{R}$ and $\bar{R}$

9.0 We introduced the connected relational semantics, $\hat{R}$, and the superconnected relational semantics, $\bar{R}$, in 3.5 and 3.6 and noted in 4.1 that $\bar{R}$ is a subsemantics of $\hat{R}$ which in turn is a subsemantics of R. We shall see in this chapter that in a very minor sense $\hat{R}$ is weaker than R and $\bar{R}$ is weaker than $\hat{R}$, but that the three semantics all have the same width and depth and so are equally strong for all intents and purposes.

9.1 It is certainly clear that there are relational frames that are not connected. It may be true, however, that for every relational frame there is a connected frame which determines the same logic, i.e. an equivalent connected frame. We show that this is not the case.

Let $\mathcal{W}_1 = (W_1, <_1)$ where $W_1 = \{x,y\}$ and for $u,v \in W_1$, $u<v$ iff $u = v = y$. Thus $\mathcal{W}_1$ has the diagram

Let $B = p\leftrightarrow\Box p$ and $C = \Box(p\wedge\sim p)$. Then $\mathcal{W}_1 \models B\vee C$, $\mathcal{W}_1 \not\models B$, and $\mathcal{W}_1 \not\models C$.

At any point in any relational frame, C is true under one assignment at that point iff it is true under all assignments at that point iff the point has no successors (not even itself). If a point has a successor and is in a frame which models $B \vee C$, then B must be true under <u>all</u> assignments and so the point is its own sole successor. Thus any connected frame which models $B \vee C$ has exactly one point which either is or is not its own successor. But then the frame either models B or models C, and so is not equivalent to $\mathcal{W}_1$.

So we see that it is not the case that every relational frame is equivalent to a single connected frame. In this very limited sense we might say that R is stronger than $\hat{\mathsf{R}}$.

<u>9.2</u> Since we have a relational frame equivalent to no single connected frame, it seems natural to ask next whether for every relational frame there is a class of connected frames which determines the same logic. The answer is yes.

If $\mathcal{W} = (W,<)$ is a relational frame, then for $u \in W$ we define the connected component containing u to be $(U,<_U)$ where $U = \{v \in W \mid u \text{ and } v \text{ are connected}\}$ and $<_U$ is the restriction of $<$ to U. It is clear that if $\{\mathcal{U}_i \mid i \in I\}$ is the set of connected components of then for each formula A, $\mathcal{W} \models A$ if and only if $\mathcal{U}_i \models A$ for each $i \in I$. Thus the logic determined by $\mathcal{W}$ is the same as that determined by the set $\{\mathcal{U}_i \mid i \in I\}$ of connected frames.

Now for each logic L that is complete with respect to

$\mathcal{R}$ there is a set $\{ \mathcal{W}_j \mid j \in J \}$ of relational frames which determines L. If we let $\{ \mathcal{U}_i \mid i \in I_j \}$ denote the set of connected components of $\mathcal{W}_j$, then, by what we have just shown, $\bigcup_{j \in J} \{ \mathcal{U}_i \mid i \in I_j \}$ will also determine L. And so L is complete with respect to $\hat{\mathcal{R}}$. We have

<u>9.2.1</u> <u>Theorem</u>: A logic L is complete with respect to $\mathcal{R}$ if and only if it is complete with respect to $\hat{\mathcal{R}}$.

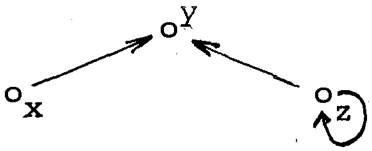
<u>9.2.2</u> <u>Corollary</u>: $\hat{\mathcal{R}}$ has the same width and depth as $\mathcal{R}$.

This essentially means that anything we can do with relational frames we can do with connected frames, although we may sometimes need more connected frames to do the same thing.

<u>9.3</u> We shall, later on, find it convenient to restrict our semantics even further, to that of superconnected frames: $\overline{\mathcal{R}}$. We should, then, ask the same questions comparing $\overline{\mathcal{R}}$ with $\hat{\mathcal{R}}$ as we asked to compare $\hat{\mathcal{R}}$ with $\mathcal{R}$.

It is certainly clear that there are connected frames that are not superconnected. It may be true, however, that for every connected frame there is an equivalent superconnected frame. We show that this is not the case.

Let $\mathcal{W}_2 = (W_2, <_2)$ where $W_2 = \{x,y,z\}$ and for $u,v \in W_2$, $u<v$ iff either $u = x$ and $v = y$ or $u = z$ and $v = y$ or z. Thus $\mathcal{W}_2$ has the diagram



Let $B = (p\leftrightarrow\Box(p\vee\Box(p\wedge\sim p)))\vee\Box(p\wedge\sim p)$ and $C = \Box\Box(p\wedge\sim p)$ It is easy to see that if V is any assignment on $\mathcal{W}_2$ we have $V(C,x) = V(\Box(p\wedge\sim p),y) = V(p\leftrightarrow\Box(p\vee\Box(p\wedge\sim p)),z) = T$ and thus $V(B\vee C) = W$. So we have $\mathcal{W}_2 \models B\vee C$. But we can also easily see that $\mathcal{W}_2 \not\models B$ and $\mathcal{W}_2 \not\models C$. We also note that $\mathcal{W}_2$ is connected but not superconnected. ($(W_2)_E = \{y\}$ and if we remove $\{y\}$ from $\mathcal{W}_2$ we get a disconnected frame.)

Let $\mathcal{W} = (W,<)$ be any superconnected frame such that $\mathcal{W} \models B\vee C$ and $\mathcal{W} \not\models C$. Then there is $w \in W$ and an assignment V on $\mathcal{W}$ such that $V(C,w) = F$. Thus there exist $w_1,w_2 \in W$ such that $w<w_1<w_2$. Let V' be an assignment on $\mathcal{W}$ such that $V'(p) = \{w\}$. Then $V'(C,w) = F$ and so $V'(B,w) = T$, and since $V'(\Box(p\wedge\sim p),w)= F$ and $V'(p,w) = T$, we must have $V'(\Box(p\vee\Box(p\wedge\sim p)),w) = T$. But since $V'(\Box(p\wedge\sim p),w_1) = F$ we must have $V'(p,w_1) = T$. Thus $w_1 = w$ (since $V'(p) = \{w\}$).

Now let $E = W_E = \{\ u\in W \mid$ for <u>no</u> v do we have $u<v\ \}$. Since $\mathcal{W}$ is superconnected we know that the frame $\mathcal{W}'$ obtained by restricting $\mathcal{W}$ to W-E is connected. But we just showed that any point in W at which C fails is in W-E and is its own sole successor in $\mathcal{W}'$. Since $\mathcal{W}'$ is connected, this point must be the only point in W-E. So the only points in W are this point (at which $p\leftrightarrow\Box(p\vee\Box(p\wedge\sim p))$ is true under any assignment) and points in E (at which $\Box(p\wedge\sim p)$ is true under

any assignment); so $\mathcal{W} \models B$.

We have seen that if $\mathcal{W}$ is superconnected and $\mathcal{W} \models B \vee C$ and $\mathcal{W} \not\models C$ then $\mathcal{W} \models B$, and therefore cannot be equivalent to $\mathcal{W}_2$. So $\mathcal{W}_2$ is a connected frame with no equivalent single superconnected frame.

We can see, then, that it is not the case that every connected frame is equivalent to a single superconnected frame. In this very limited sense we might say that $\hat{R}$ is stronger than $\overline{R}$.

9.4 Since we now have a connected frame equivalent to no single superconnected frame, it seems natural to ask whether for every connected frame there is a class of superconnected frames which determines the same logic. As in 9.2, the answer again is yes. The proof of this, however, is not quite the simple affair that the corresponding proof in 9.2 was.

We show, in fact, that every relational frame is equivalent to a set of superconnected frames. We begin by introducing the concept of a generated subframe. The concept of a generated subframe is derived from Kripke and developed more fully by Segerberg [14], among others. We shall see that generated subframes are superconnected and that the set of all generated subframes of a relational frame determines the same logic as the relational frame itself. It will follow, then, that the superconnected semantics has the same width and depth as the relational and connected semantics.

Suppose that $\mathcal{W} = (W,<)$ is a relational frame. Then we define the relation $<<$ on W (called the ancestral of $<$) as the transitive reflexive closure of $<$; namely, for $u,v \in W$, $u<<v$ if

i) $u = v$,

ii) $u<v$,

or iii) there is $w \in W$ such that $u<<w$ and $w<v$.

For $w \in W$ we define the subframe generated by w to be the frame $\mathcal{W}^w = (W^w,<^w)$ where $W^w = \{\ u \in W \mid w<<u\ \}$ and $<^w = < \cap\ (W^w \times W^w)$. Now for each assignment V on $\mathcal{W}$ let V^w be the assignment on $\mathcal{W}^w$ given by $V^w(n) = V(n) \cap W^w$ for each $n \in \mathbb{N}$.

9.4.1 Lemma: For each formula A and $u \in W^w$, $V(A,u) = V^w(A,u)$.

Proof: By induction on the length of A. If A is a variable p_i then since $V^w(i) = V(i) \cap W^w$, $V(A,u) = V^w(A,u)$. If A is $\sim B$ or $B \rightarrow C$ then the induction goes through in the obvious manner. Suppose, then, that A is $\Box B$.

Now, $\{\ v \in W \mid u<v\ \} = \{\ v \in W^w \mid u <^w v\ \}$ and by the induction hypothesis, for such v, $V(B,v) = V^w(B,v)$. Therefore $V(A,u) = V^w(A,u)$. XXX

It is certainly the case that every generated frame is superconnected. Let $\mathcal{W}^w$ be the subframe of $\mathcal{W}$ generated by w and, as before, let $W_E^w = \{ u \in W^w \mid \text{there is } \underline{\text{no}}\ v \in W^w$ such that $u <^w v \}$, the set of points in W^w with no successors. If we let $\widetilde{\mathcal{W}}^w$ denote the restriction of $\mathcal{W}^w$ to $W^w - W_E^w$ then we must show that $\widetilde{\mathcal{W}}^w$ is connected.

Let u and v be points in $W^w - W_E^w$. Since u and v are in W^w there are sequences $u_0, \ldots, u_n$ and $v_1, \ldots, v_m$ of points in W^w such that for each i, $u_i <^w u_{i+1}$ and $v_i <^w v_{i+1}$, $u_0 = w = v_0$, $u_n = u$, and $v_m = v$. Clearly, then, all the points in the sequences are in $W^w - W_E^w$ and so $u = u_n, \ldots, u_1, w, v_1, \ldots, v_m = v$ is a sequence in $W^w - W_E^w$ connecting u and v.

<u>9.4.2</u> <u>Lemma</u>: For each relational frame $\mathcal{W} = (W, <)$, the set $\{ \mathcal{W}^w \mid w \in W \}$ of all generated subframes of $\mathcal{W}$ determines the same logic as $\mathcal{W}$.

Proof: Suppose $\mathcal{W} \models A$. We show that for each $w \in W$, $\mathcal{W}^w \models A$. We note that as V ranges over all assignments on $\mathcal{W}$, V^w ranges over all assignments on $\mathcal{W}^w$. Pick $u \in W^w$. Then $u \in W$ and for all V on $\mathcal{W}$, $V(A,u) = T$. Thus $V^w(A,u) = T$ by 9.4.1 for all V^w on $\mathcal{W}^w$. Hence, $\mathcal{W}^w \models A$.

Now suppose $\mathcal{W} \not\models A$. Then there is a point $x \in W$ and an assignment V on $\mathcal{W}$ such that $V(A,x) = F$. But then by 9.4.1, in $\mathcal{W}^x$, $V^x(A.x) = F$ and so $\mathcal{W}^x \not\models A$. XXX

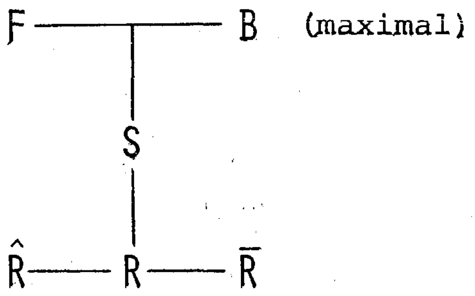
Now suppose that L is a logic complete with respect to R. Then there is a set $\{ \mathcal{W}_j \mid j \in J \}$ of relational frames which determines L. Then by 9.4.2, $\bigcup_{j \in J} \{ \mathcal{W}_j^{\,w} \mid w \in W_j \}$ will also determine L. And so L is complete with respect to $\overline{\mathcal{R}}$. We have

<u>9.4.3</u> <u>Theorem</u>: A logic L is complete with respect to $\mathcal{R}$ if and only if it is complete with respect to $\overline{\mathcal{R}}$. XXX

<u>9.4.4</u> <u>Corollary</u>: $\overline{\mathcal{R}}$ has the same width and depth as $\mathcal{R}$, and hence as $\hat{\mathcal{R}}$. XXX

So anything we can do with relational frames or with sets of relational frames we can, in fact, do with sets of superconnected frames. And so if sometimes instead of considering $\mathcal{R}$ with the modal language, $\mathcal{M}$, we consider the subsemantics $\hat{\mathcal{R}}$ or even $\overline{\mathcal{R}}$ with $\mathcal{M}$, as we shall later on, we are not really losing anything.

<u>9.5</u> In chapters 5 through 9, then, we have seen that $\mathcal{B}$ and $\mathcal{S}$ have width of measure E while $\mathcal{F}$, $\mathcal{R}$, $\hat{\mathcal{R}}$, and $\overline{\mathcal{R}}$ all have width of measure K. If we restrict $\mathcal{B}$ and $\mathcal{S}$ to frames which model K, i.e. to normal frames, so that they are on "common ground" with $\mathcal{F}$, $\mathcal{R}$, $\hat{\mathcal{R}}$, and $\overline{\mathcal{R}}$, we see that $\mathcal{B}$ and $\mathcal{F}$ have maximal depth, $\mathcal{S}$ has strictly less depth and $\mathcal{R}$ has strictly less depth again. $\hat{\mathcal{R}}$ and $\overline{\mathcal{R}}$ have, however, the same depth as $\mathcal{R}$. We have the following "depth diagram" for normal frames.



9.6 While we have answered the questions about the relative depths of our various modal semantics, we have left a large question unanswered: the question of the absolute depths of the various semantics. In other words, we may ask for a non-semantic characterization of the set of logics complete with respect to the respective semantics.

The question is, of course, answered for B and F: the answers being "all classical logics" and "all normal logics" respectively. The problems for S and for R, however, remain unsolved. These are, perhaps, the outstanding unsolved problems in this area at this time.

10. Modal Properties

10.0 So far in this thesis we have taken the approach of starting with $\ddot{\mathsf{M}}$, the modal language, and studying the strengths of various semantics for $\ddot{\mathsf{M}}$. For the remainder of the thesis we shall take the opposite approach. We shall be looking at the strength of $\ddot{\mathsf{M}}$ as a language which we may use to discuss properties of structures in the various modal semantics. In this chapter we shall be looking particularly at properties of structures in B, $\mathcal{S}$, and R. In later chapters, when we discuss reductions, we shall be particularly interested in the system $(\mathsf{M},\overline{\mathsf{R}})$.

10.1 Let D denote an arbitrary modal semantics. If P is a property which may or may not be held by frames in D, then P will be called a modal property of D if there is a set M_P of formulae in M such that for each frame $\mathcal{F}$ in D, $\mathcal{F}$ has property P if and only if $\mathcal{F} \models A$ for each $A \in M_P$. In such a case, P and M_P are said to correspond. P will be called a strongly modal property of D if and only if there is some corresponding M_P which is finite. Clearly, then, P is strongly modal if and only if there is a single formula A_P such that for each frame $\mathcal{F}$ in D, $\mathcal{F}$ has property P if and only if $\mathcal{F} \models A_P$.

In the remainder of chapter 10, we shall consider certain properties of boolean, neighbourhood, or relational frames and consider whether or not they are modal or strongly modal.

10.2 Various properties of relational frames are known to be strongly modal. For example, if $\mathcal{W} = (W,<)$ then

(a) $\mathcal{W}$ is reflexive (i.e., $<$ is reflexive) iff

$$\mathcal{W} \models \Box p \to p \quad (4.6.9)$$

(b) $\mathcal{W}$ is transitive iff $\mathcal{W} \models \Box p \to \Box^2 p$

thus (c) $\mathcal{W}$ is pre-ordered iff $\mathcal{W} \models (\Box p \to p) \wedge (\Box p \to \Box^2 p)$

(4.6.11) iff $\mathcal{W} \models S4$.

(d) $<$ is an equivalence relation iff $\mathcal{W} \models (\Box p \to p) \wedge$ $(\Box p \to \Box\ p) \wedge (\Diamond p \to \Box \Diamond p)$. (The smallest normal logic containing this formula is called S5).

10.3 Sahlqvist[12] has shown that the property of being irreflexive (i.e. $u \not< u\ \forall u \in W$) is not modal. In fact, he has shown much more.

If $\mathcal{W} = (W,<)$ and $u,v \in W$, then we write

$$u <^0 v \text{ iff } u = v$$

$$u <^{m+1} v \text{ iff } \exists x \in W\ (u <^m x \text{ and } x < v).$$

Thus $u <^m v$ iff v can be reached from u in exactly m steps.

For $m \neq n$, we say that $<$ (or $\mathcal{W}$) is m,n-intransitive iff

$$\forall u,v \in W\ (\sim(u <^m v \text{ and } u <^n v)) .$$

and we say that $<$ (or $\mathcal{W}$) is m,m-intransitive iff

$\forall u,v,w_1, \ldots , w_{m-1}, x_1, \ldots , x_{m-1} \in W\ (u<w_1<\ldots<w_{m-1}<v$ and $u<x_1<\ldots<x_{m-1}<v$ implies $\underset{1<i<m-1}{\&} (w_i = x_i))$.

Thus, $\mathcal{W}$ is m,n-intransitive iff v can never be reached from u both in exactly m steps and in exactly n steps, and $\mathcal{W}$ is m,m-intransitive iff whenever v can be reached from u in exactly m steps, then there is only one path of that length from u to v.

Sahlqvist[12] (chapter 7) uses Segerberg's "unravelling technique" to show that for $m \neq n$, m,n-intransitivity is not modal, and for $m \geq 2$, m,m-intransitivity is not modal. It is easy to see that irreflexivity is simply 1,0-intransitivity, asymmetry($\forall w \in W$, $\sim(u<v \ \& \ v<u)$) is 2,0-intransitivity, and intransitivity ($\forall u,v,w \in W$, $u<v \ \& \ v<w \Rightarrow \sim(u<w)$) is 2,1-intransitivity. Thus none of these properties is modal.

10.4 The following lemmas will be useful.

10.4.1 Lemma: If P is a property which may or may not be held by frames in $\mathbb{D}$, and if there is some logic J such that for every formula A either $J \vdash A$ or there is a frame $\mathcal{F}$ in $\mathbb{D}$ with property P such that $\mathcal{F} \not\models A$, and if there is a frame without property P which models J, then P is not modal.

Proof: Suppose P were modal and M_P the corresponding set of formulae. Since there is a frame, $\mathcal{F}_0$, without property P which models J, at least one of the formulae in M_P, say B, must not be valid in $\mathcal{F}_0$. Thus $J \not\vdash B$. But then there is a frame $\mathcal{F}$ with property P such that $\mathcal{F} \not\models B$ contrary to our assumption that $B \in M_P$. XXX

<u>10.4.2</u> <u>Lemma</u>: If $\mathcal{D}_2$ is a subsemantics of $\mathcal{D}_1$, and if P_1, P_2 are properties which may be held by frames in $\mathcal{D}_1$, $\mathcal{D}_2$ respectively such that a frame in $\mathcal{D}_2$ has property P_2 iff it has property P_1 (when considered as a frame in $\mathcal{D}_1$), then if P_1 is a modal property, P_2 is also.

Proof: A frame in $\mathcal{D}_1$ has property P_1 iff it models M_{P_1}. Since every frame in $\mathcal{D}_2$ is in $\mathcal{D}_1$ and it has property P_1 iff it has property P_2, then it has property P_2 iff it models M_{P_1}. Thus P_2 is a modal property in $\mathcal{D}_2$ and we may take $M_{P_2} = M_{P_1}$. XXX

<u>10.5</u> Quite a few interesting properties of neighbourhood frames are modal.

(a) Normality is strongly modal in $\mathcal{B}$, hence, by 10.4.2, in $\mathcal{S}$. We claim that $\mathcal{B}$ is normal iff $\mathcal{B} \models \Box(p\to p)\wedge(\Box(p\to q)\to(\Box p\to\Box q))$. Let $L_{\mathcal{B}}$ be the logic determined by a boolean frame $\mathcal{B}$. Now if $\mathcal{B}$ is normal then $L_{\mathcal{B}}$ is normal. Since $L_{\mathcal{B}}$ is closed under RN and $p\to p$ is a tautology, $L_{\mathcal{B}} \vdash \Box(p\to p)$. Also $L_{\mathcal{B}} \vdash$ K and so $L_{\mathcal{B}} \vdash \Box(p\to q)\to(\Box p\to\Box q)$. Therefore $\mathcal{B} \models \Box(p\to p)\wedge(\Box(p\to q)\to(\Box p\to\Box q))$. If, on the other hand, $\mathcal{B} \models \Box(p\to p)\wedge(\Box(p\to q)\to(\Box p\to\Box q))$, then $L_{\mathcal{B}} \vdash \Box(p\to p)$. Since $\mathcal{B}$, hence $L_{\mathcal{B}}$, is classical (4.6.2), $L_{\mathcal{B}} \vdash \Box(p\to p)\leftrightarrow\Box A$ if $L_{\mathcal{B}} \vdash p\to p\leftrightarrow A$. Thus $L_{\mathcal{B}} \vdash \Box A$ if $L_{\mathcal{B}} \vdash A$. Therefore $L_{\mathcal{B}}$ is closed under RN and contains $\Box(p\to q)\to(\Box p\to\Box q)$ and so $\mathcal{B}$ is normal.

(b) We saw in 4.10.2 that a neighbourhood frame was a topological space in a natural way iff it modeled S4. By

(a) that will happen iff it models $\Box(p\to p)\wedge(\Box(p\to q)\to(\Box p\to\Box q)\wedge(\Box p\to p)\wedge(\Box p\to\Box^2 p)$. Thus the property of being a topological space is a strongly modal property in $\mathcal{S}$.

<u>10.6</u> In chapter 4 we studied particular embeddings of $\mathcal{R}$ in $\mathcal{S}$ and of $\mathcal{S}$ in $\mathcal{B}$ and decided to say that a neighbourhood frame was a relational frame if it was isomorphic to the image under this embedding of a relational frame, and similarly that a boolean frame was a neighbourhood frame if it was isomorphic to the image under this embedding of a neighbourhood frame. We shall have this interpretation in mind in this and following sections.

In $\mathcal{B}$, the property of being a neighbourhood frame is not a modal property. We have seen that K is complete with respect to $\mathcal{R}$ (8.1.1) and so it is certainly complete with respect to $\mathcal{S}$. Thus every formula not in K is not valid in some neighbourhood frame which models K. It is easy to see that there are boolean frames which model K but are not neighbourhood frames. (The Lindenbaum-Tarski frame for K, for example, is a countably infinite boolean frame and therefore cannot be a neighbourhood frame.) Thus by 10.4.1, the property of being a neighbourhood frame is not modal in $\mathcal{B}$.

<u>10.7</u> We shall see that in $\mathcal{S}$, the property of being a relational frame is not a modal property. We begin by showing

that for each relational frame $\mathcal{W} = (W,\lhd)$, if there are some $u,v \in W$ with $u \lhd v$, then there is a normal neighbourhood frame $G_{\mathcal{W}}$ which is not a relational frame and is such that $\mathcal{W} \not\models A$ implies $G_{\mathcal{W}} \not\models A$.

Let Ξ be a non-principal ultrafilter on $\mathbb{N}$, the natural numbers. Let $G_{\mathcal{W}} = (W\times\mathbb{N}, \mathcal{N})$ where $S \in \mathcal{N}_{(u,m)}$ if and only if there is a set $K \in \Xi$ such that if $u \lhd v$ and $m \in K$ then $(v,m) \in S$.

First, we see that $G_{\mathcal{W}}$ is not a relational frame. By 4.7.1 it suffices to show that there is a point $(u,n) \in W\times\mathbb{N}$ such that $\mathcal{N}_{(u,n)}$ is not a principal filter. Pick u such that there is a v with $u \lhd v$ and let $n = 1$. Since Ξ is a non-principal ultrafilter, for each $m \in \mathbb{N}$ there is a $K \in \Xi$ such that $m \notin K$. Thus there is no point in $W\times\mathbb{N}$ which is in every $S \in \mathcal{N}_{(u,1)}$. But since $\phi \notin \Xi$ and there is v with $u \lhd v$, $\phi \notin \mathcal{N}_{(u,1)}$. Thus $\mathcal{N}_{(u,1)}$ is not a principal filter, and hence $G_{\mathcal{W}}$ is not a relational frame.

Next, we see that $G_{\mathcal{W}}$ is normal. Let $S_1, S_2 \in \mathcal{N}_{(u,n)}$. Then there are $K_1, K_2 \in \Xi$ such that, for $i = 1,2$, if $u \lhd v$ and $m \in K_i$ then $(u,m) \in S_i$. Thus if $u \lhd v$ and $m \in K_1 \cap K_2$ then $(v,m) \in S_1 \cap S_2$, and therefore, since $K_1 \cap K_2 \in \Xi$, $S_1 \cap S_2 \in \mathcal{N}_{(u,n)}$. Also, since $\Xi \neq \phi$, $W\times\mathbb{N} \in \mathcal{N}_{(u,n)}$ and so $G_{\mathcal{W}}$ is normal.

Finally, suppose that $\mathcal{W} \not\models A$. Then there is a point $u \in W$ and an assignment V on $\mathcal{W}$ such that $V(A,u) = F$. Let the assignment V_0 on $G_{\mathcal{W}}$ be such that $V_0(p) = V(p)\times\mathbb{N}$ for each propositional variable p. We claim that for each formula B, and for each $(v,m) \in W\times\mathbb{N}$, $V_0(B,(v,m)) = V(B,v)$. The proof is by induction on the length of B.

If B is a propositional variable then the claim is true by definition of V_0. If B is $\sim C$ or $C\to D$ then the claim follows trivially. So, suppose B is $\Box C$. Then $V(B,v) = T$ $\Leftrightarrow$ $V(C,w) = T$ for all w such that $v \lhd w$ $\Leftrightarrow$ (by induction hypothesis) $V_0(C,(w,m')) = T$ for all w such that $v \lhd w$ and for all $m' \in \mathbb{N}$ $\Leftrightarrow$ $V_0(B,(v,m)) = T$ for all $m \in \mathbb{N}$ since $\{ (w,m') \mid v \lhd w \text{ and } m' \in \mathbb{N} \} \in N_{(v,m)}$. Therefore $V_0(B,(v,m)) = V(B,v)$ and so, in particular, $V_0(A,(u,1)) = V(A,u) = F$. Hence $G_{\mathcal{W}} \not\models A$.

Now let $\mathcal{W} = (W,<)$ be a particular relational frame such that there are $u,v \in W$ with $u<v$ (i.e., the relation $<$ is nonempty). Then $G_{\mathcal{W}}$ is normal and hence $G_{\mathcal{W}} \models K$, but is not relational. We know (8.1) that K is complete with respect to R, so every formula is either in K or not valid in a relational frame which models K. But there is a non-relational frame, $G_{\mathcal{W}}$, which models K. So by 10.4.1 the property of being a relational frame is not a modal property in S.

We can use our techniques of this section to get another result. We have shown that every formula not valid in some relational frame is not valid in some non-relational neighbourhood frame which models K, and therefore every formula is either in K or not valid in a non-relational frame which models K. But there is a relational frame which models

K, and hence by 10.4.1 the property of being a non-relational frame is also not modal in $\mathcal{S}$.

10.8 The property of being *equivalent to* a neighbourhood frame (determining the same logic as a neighbourhood frame) is not a modal property in $\mathcal{B}$. Since K is complete with respect to $\mathcal{S}$, every formula is either in K or not valid in some neighbourhood frame which models K. But we have seen in chapter 7 that there are boolean frames which model K and which are equivalent to no neighbourhood frames. Therefore, by 10.4.1, the property of being equivalent to a neighbourhood frame is not modal in $\mathcal{B}$.

10.9 The property of being *equivalent to* a relational frame is not modal in $\mathcal{S}$. Since K is complete with respect to $\mathcal{R}$, and since we have seen in chapter 8 that there is a neighbourhood frame which models K and which is equivalent to no relational frame, the result follows as above from 10.4.1.

10.10 Connectedness is not a modal property in $\mathcal{R}$. K is complete with respect to $\mathcal{R}$ and so, by 9.2.1, with respect to $\hat{\mathcal{R}}$. Thus each formula is either in K or not valid in some connected frame. Clearly, however, there are disconnected relational frames and we know that they model K, Therefore by 10.4.1 connectedness is not a modal property in $\mathcal{R}$.

10.11 Superconnectedness is not a modal property in either $\mathcal{R}$ or $\hat{\mathcal{R}}$. The proofs are analogous to that in 10.10.

11. Higher-Order Languages and Semantics and Second-Order Properties of Frames

11.0 In this chapter we introduce other languages that can be used to describe frames in our various modal semantics. These are the natural higher-order languages for the various types of frames we are dealing with. We compare properties of frames that can be described in these languages with modal properties. We concentrate on neighbourhood and relational frames and are particularly interested in second-order properties, for we shall see that all modal properties in S and R are second-order.

11.1 A higher-order language is a monadic n^{th}-order language (for some $n \geq 1$) with finitely many predicate symbols each taking a finite number of arguments of specified orders. More precisely, L is an n^{th}-order language if L has variables

$$x_1^1,\ x_2^1,\ \ldots;\ x_1^2,\ x_2^2,\ \ldots;\ \ldots;\ x_1^n,\ x_2^n,\ \ldots$$

(the superscript denotes the order of the variable); symbols $\rightarrow$, $\sim$, $\forall$, $=$, $\in$; predicates $Q_1, \ldots, Q_m$ -- we say that Q_i is *of type* $(a_{i1},\ldots,a_{ik_i})$, where $a_{i1}, \ldots, a_{ik_i}$ are positive integers $\leq n$, if Q_i takes k_i arguments, the first of order a_{i1}, the second of order a_{i2}, ... , the k_i^{th} of order a_{ik_i}. Formulae of L are defined as follows.

An atomic formula is of one of the following forms:

(a) of the form $Q_i(y_1,\ldots,y_{k_i})$ where, if Q_i is of type$(a_{i1},\ldots,a_{ik_i})$ then for $1 \le j \le k_i$, y_j is a variable of order a_{ij}.

(b) of the form $y = y'$ where y and y' have same order.

(c) of the form $y \in y'$ where the order of y' is one greater that the order of y.

If A and B are formulae, so are $A \to B$, $\sim A$, and $\forall yA$, where y is any variable (of order $\le n$). We shall use the "defined symbols" $\wedge$, $\vee$, $\exists$; so that $A \wedge B$ denotes $\sim(A \to \sim B)$, $A \vee B$ denotes $\sim A \to B$, $\exists yA$ denotes $\sim\forall y \sim A$.

We are now ready to define a semantics for L, the nth-order semantics for the nth-order language. First, if S is a set, we define $\mathcal{P}^0(S) = S$ and $\mathcal{P}^{i+1}(S) = \mathcal{P}(\mathcal{P}^i(S))$. ($\mathcal{P}$ denotes the power set operation.)

If A denotes the nth-order semantics for L, then $\mathcal{S}$ is a structure in A if $\mathcal{S} = (S, \mathcal{Q}_i, \ldots, \mathcal{Q}_m)$, where S is a nonempty set and if Q_i is of type $(a_{i1},\ldots,a_{ik_i})$ then so is $\mathcal{Q}_i$, i.e. $\mathcal{Q}_i \subset \mathcal{P}^{a_{i1}-1}(S) \times \ldots \times \mathcal{P}^{a_{ik_i}-1}(S)$.

An assignment Λ on $\mathcal{S}$ is a mapping from the set of variables of L to $\bigcup_{i=0}^{n-1} \mathcal{P}^i(S)$ such that $\Lambda(x_i^j) \in \mathcal{P}^{j-1}(S)$. Then Λ induces a function, also called Λ, from the formulae of L to $\{T,F\}$ as follows:

$\Lambda(Q_i(y_1,\ldots,y_{k_i})) = T$ iff $(\Lambda(y_1),\ldots,\Lambda(y_{k_i})) \in \mathcal{Q}_i$

$\Lambda(y_1 = y_2) = T$ iff $\Lambda(y_1) = \Lambda(y_2)$

$\Lambda(y_1 \in y_2) = T$ iff $\Lambda(y_1) \in \Lambda(y_2)$

$\Lambda(A \to B) = T$ iff $\Lambda(A) = F$ or $\Lambda(B) = T$

$\Lambda(\sim A) = T$ iff $\Lambda(A) = F$

and $\Lambda(\forall y A) = T$ iff $\Lambda'(A) = T$ for all assignments Λ' which agree with Λ on all variables except y.

If $\mathcal{S}$ is a structure in $\mathcal{A}$ and A a formula in $\mathcal{L}$ then $\mathcal{S}$ models A ($\mathcal{S} \models A$) if and only if $\Lambda(A) = T$ for all assignments Λ on $\mathcal{S}$.

We shall often be dealing with second-order languages. In this case we shall often use lower case letters $x_1, x_2, \ldots$ in place of $x_1^1, x_2^1, \ldots$ and upper case letters $X_1, X_2, \ldots$ in place of $x_1^2, x_2^2, \ldots$.

<u>11.2</u> We shall now consider the second-order language $\mathcal{L}_s$ for neighbourhood frames. $\mathcal{L}_s$ will be the monadic second-order language with a single predicate symbol N of type (2,1). Let $\mathcal{A}_s$ be the second-order semantics for $\mathcal{L}_s$. Then each structure $\mathcal{U}$ is a pair $\mathcal{U} = (U, \mathcal{N})$ where U is a nonempty set and $\mathcal{N} \subset \mathcal{P}(U) \times U$. There is a natural correspondence between neighbourhood frames (structures in the semantics $\mathcal{S}$ for the language $\mathcal{L}$) and the structures in $\mathcal{A}_s$. With the frame $\mathcal{F} = (U, \mathcal{N})$ in $\mathcal{S}$ we associate the structure $\mathcal{U}_{\mathcal{F}} = (U, \overline{\mathcal{N}})$ in $\mathcal{A}_s$ where $(S,u) \in \overline{\mathcal{N}}$ iff $S \in \mathcal{N}_u$. If $\mathcal{U} = (U, \mathcal{M})$ is a structure in $\mathcal{A}_s$, then define the neighbourhood frame $\mathcal{F}_{\mathcal{U}} = (U, \mathcal{N}^{\mathcal{M}})$ by $\mathcal{N}^{\mathcal{M}}{}_u = \{ S \in U \mid (S,u) \in \mathcal{M} \}$. Clearly, then, $\overline{\mathcal{N}^{\mathcal{M}}} = \mathcal{M}$ so that $\mathcal{U}_{\mathcal{F}_{\mathcal{U}}} = \mathcal{U}$. Thus the correspondence $\mathcal{F} \dashrightarrow \mathcal{U}_{\mathcal{F}}$ is one-to-one and onto. Therefore we can identify neighbourhood frames with structures in the semantics $\mathcal{A}_s$ for $\mathcal{L}_s$ and vice versa. In this manner we may think of $\mathcal{L}_s$ as a second-order language

for neighbourhood frames.

11.3 Next, we consider the second-order language L_r for relational frames. L_r will be the monadic second-order language with a single predicate symbol Q of type (1,1). Let $\mathcal{A}_r$ be the second-order semantics for L_r. Then a structure $\mathcal{W}$ in $\mathcal{A}_r$ is a pair $\mathcal{W} = (W,<)$ where W is a nonempty set and $< \subset W\times W$. Thus the class of structures in $\mathcal{A}_r$ is exactly the same as the class of structures in $\mathcal{R}$, i.e. each relational frame is a second-order structure for L_r and vice versa. In this way we can speak of L_r as a second-order language for relational structures.

11.4 If P is a property which may or may not be held by neighbourhood (relational) frames, then P is called a second-order property if there is a set S_P of formulae in L_s (L_r) such that for each neighbourhood (relational) frame F

F has property P if and only if $F \models B \; \forall B \in S_P$.

(if F is a neighbourhood frame we identify it with $\mathcal{U}_F$).

P is strongly second-order if it is second-order and there is a corresponding S_P which is finite.

11.5 In $\mathcal{S}$, if P is the property of being a relational frame, then we have already seen that P is not modal. However, P is strongly second-order. Recall (4.7.1) that a neighbourhood frame is relational iff the set of neighbourhoods of each point is a proper or improper principal filter. Thus we let $S_p = \{A\}$ where A is

$$\forall x_1 \exists X_1 (N(X_1, x_1) \wedge \forall X_2 (N(X_2, x_1) \leftrightarrow \forall x_2 (x_2 \in X_1 \rightarrow x_2 \in X_2))).$$

11.6 We have already noted that the property in $\mathcal{R}$ of being irreflexive is not modal. However, irreflexivity is strongly second order. Let $S_p = \{A\}$ where A is

$$\forall x_1 (\sim Q(x_1, x_1)).$$

It is, in fact, not hard to see from our description in 10.3 of m,n-intransitivity that for each m and n, m,n-intransitivity is a strongly second-order property.

11.7 Connectedness is not a modal property in $\mathcal{R}$ (10.10), but it is a second-order property. Let $S_p = \{A\}$ where A is

$$\forall X_1 ((\exists x_1 (x_1 \in X_1) \wedge \forall x_1 \forall x_2 ((x_1 \in X_1 \wedge (Q(x_1, x_2) \vee Q(x_2, x_1))) \\ \rightarrow x_2 \in X_1)) \rightarrow \forall x_1 (x_1 \in X_1))$$

11.8 We shall show that every modal property in $\mathcal{S}$ is second-order. Let T be the transformation from formulae in $\mathcal{M}$ to formulae in L_S defined inductively as follows:

$T(p_i) = x_1 \in X_i$ for each propositional variable p_i

$T(\sim A) = \sim TA$

$T(A \rightarrow B) = TA \rightarrow TB$

$T(\Box A) = \exists X_n (N(X_n, x_1) \wedge \forall x_m (x_m \in X_n \leftrightarrow TA(x_m/x_1))$

where n and m are the least numbers such that X_n and x_m respectively do not occur in TA and where $TA(x_m/x_1)$ is the formula obtained by replacing all free occurrences of x_1 in TA by x_m.

We note the following two facts, which can immediately be seen to be true for all modal formulae from the definition of T.

1. The only first-order variable with free occurrences in TA is x_1 and x_1 has only free occurrences in TA.

2. X_i occurs free in TA iff p_i occurs in A.

We now show that $\mathcal{F} \models A$ (in $\mathcal{S}$) iff $\mathcal{F} \models TA$ (in $\mathcal{A}_s$). Pick a neighbourhood frame $\mathcal{F} = (U,\mathcal{N})$ and an assignment V on $\mathcal{F}$ in $\mathcal{S}$. Then for each $u \in U$ we let Λ_u be an assignment on $\mathcal{F}$ in $\mathcal{A}_s$ such that $\Lambda_u(X_i) = V(p_i)$ and $\Lambda_u(x_1) = u$. We claim that $V(A,u) = \Lambda_u(TA)$ for each modal formula A.

Since x_1 is the only free first-order variable in TA and since for assignments V' and V'' on $\mathcal{F}$, if V' and V'' agree on all variables occurring in A then $V'(A) = V''(A)$, we know that as V runs through all assignments on $\mathcal{F}$ in $\mathcal{S}$ and u runs through all elements of U, Λ_u runs through all the assignments on $\mathcal{F}$ in $\mathcal{A}_s$ which are significant in determining the validity of TA in $\mathcal{F}$; i.e. TA is valid in $\mathcal{F}$ if $\Lambda_u(TA) = T$ for all such Λ_u. From the claim then, we see that $V(A,u) = T$ for all $u \in U$ and for all V on $\mathcal{F}$ in $\mathcal{S}$ iff $\Lambda(TA) = T$ for all Λ on $\mathcal{F}$ in $\mathcal{A}_s$. Thus, to prove that $\mathcal{F} \models A$ iff $\mathcal{F} \models TA$ it suffices to prove the claim.

Proof of claim by induction on length of A:

If A is p_i then $TA = x_1 \in X_i$. We have $V(A,u) = T \Leftrightarrow V(p_i,u) = T \Leftrightarrow u \in V(p_i) \Leftrightarrow u \in \Lambda_u(X_i) \Leftrightarrow \Lambda_u(x_1,X_i) = T \Leftrightarrow \Lambda_u(TA) = T$.

If A is $\sim B$ then $TA = \sim TB$. We have by induction hypothesis, $V(B,u) = \Lambda_u(TB)$, thus $V(\sim B,u) = \Lambda_u(\sim TB)$, i.e. $V(A,u) = \Lambda_u(TA)$.

If A is $B \to C$ then we have $V(A,u) = T \Leftrightarrow V(B\to C,u) = T \Leftrightarrow V(B,u) = F$ or $V(C,u) = T \Leftrightarrow$ (by induction hypothesis) $\Lambda_u(TB) = F$ or $\Lambda_u(TC) = T \Leftrightarrow \Lambda_u(TB \to TC) = T \Leftrightarrow \Lambda_u(TA) = T$.

If A is $\Box B$ then $V(A,u) = T \Leftrightarrow V(\Box B,u) = T \Rightarrow$ for some $S \in \mathcal{N}_u$ ($V(B,v) = T$ iff $v \in S$) $\Leftrightarrow$ for some S such that $(S,u) \in \mathcal{N}$, ($\Lambda_v(TB) = T$ iff $v \in S$) $\Leftrightarrow$ $\Lambda_u(\exists X_n(N(X_n,x_1) \wedge \forall x_m(x_m \in X_n \longleftrightarrow TB(x_m/x_1)))) = T \Leftrightarrow \Lambda_u(TA) = T$.

The induction is complete and the claim is proved. Thus $\mathcal{F} \models A$ iff $\mathcal{F} \models TA$ for all modal formulae A.

Now suppose P is a modal property in $\mathcal{S}$. Then there is a set M_P of modal formulae such that for each neighbourhood frame $\mathcal{F}$, $\mathcal{F}$ has property P iff $\mathcal{F} \models A$ for all $A \in M_P$. Let $S_P = \{ TA \mid A \in M_P \}$. Then $\mathcal{F}$ has property P iff $\mathcal{F} \models A$ for all $A \in S_P$. Therefore P is a second-order property. If M_P is a finite set then so is S_P and so every strongly modal property in $\mathcal{S}$ is strongly second-order.

<u>11.9</u> We shall show that every modal property in $\mathcal{R}$ is also second-order. Let R be the transformation from formulae in

$\mathcal{M}$ to formulae in L_r defined inductively as follows:

$R(p_i) = x_1 \in X_i$ for each propositional variable p_i

$R(\sim A) = \sim RA$

$R(A \rightarrow B) = RA \rightarrow RB$

$R(\Box A) = \forall x_n(Q(x_1, x_n) \rightarrow RA(x_n/x_1))$

The proof that $\mathcal{W} \models A$ (in $\mathcal{R}$) iff $\mathcal{W} \models RA$ (in $\mathcal{A}_r$) is essentially the same as the proof that $\mathcal{F} \models A$ (in $\mathcal{S}$) iff $\mathcal{F} \models TA$ (in $\mathcal{A}_s$) in 11.8. It differs only in the last case of the inductive proof of $V(A,u) = \Lambda_u(RA)$ and so we shall do only that here:

If A is $\Box B$ then $V(A,u) = T \Leftrightarrow V(\Box B,u) = T \Leftrightarrow$ for all v such that $u<v$, $V(B,v) = T \Leftrightarrow$ for all v such that $u<v$, $\Lambda_v(RB) = T \Leftrightarrow \Lambda_u(\forall x_n(Q(x_1, x_n) \rightarrow RB(x_m/x_1))) = T \Leftrightarrow \Lambda_u(RA) = T$.

Thus $\mathcal{W} \models A$ iff $\mathcal{W} \models RA$. If P is a modal property in $\mathcal{R}$ then, as in 11.8, we can let $S_P = \{ RA \mid A \in M_P \}$ and we see that P is a second-order property. If P is strongly modal then it is strongly second-order.

12. Reductions and Quasi-Modal Properties

12.0 We saw in chapter 11 that all modal properties in $\mathcal{S}$ or $\mathcal{R}$ were second-order properties but that the converse was not the case. This would seem to indicate that the second-order languages are more powerful, or more expressive, than the modal language for these particular semantics. This is certainly true if we are using the language to "say things" directly about the frames in the various semantics. But we shall see that we can use the languages through a series of reductions to say considerably more than we could say directly.

Our reductions are devices through which we can "talk about" structures in one semantics in a language which belongs with another semantics; or, as we shall eventually do, in a language which belongs with the semantics in question but which can be strengthened considerably by using the reduction. Our reduction is a development of the reductions used by Thomason [19,20,21] which in turn were inspired by the notion of interpretations as in Shoenfield [15](p.61)

12.1 If $(L,\mathcal{A})$ and $(L',\mathcal{A}')$ are systems then a *reduction of* $(L,\mathcal{A})$ *into* $(L',\mathcal{A}')$ is a device through which we can use the language L' to describe properties of structures in $\mathcal{A}$.

(At this point we shall begin to make use of the abuse of notation mentioned in 1.1 and shall write $M \in \mathcal{A}$ if M is a structure in the semantics $\mathcal{A}$.)

A reduction ρ of $(\mathcal{L},\mathcal{A})$ into $(\mathcal{L}',\mathcal{A}')$ is a quadruple $(\nabla_\rho, T_\rho, \mathcal{C}_\rho, \mathcal{D}_\rho)$ satisfying:

(i) ∇_ρ is an association between isomorphism classes of structures in $\mathcal{A}$ and isomorphism classes of structures in $\mathcal{A}'$. (As a further abuse of notation we write $M \nabla_\rho M'$ if $M \in \mathcal{A}$ and $M' \in \mathcal{A}'$ and the isomorphism class of M is associated via ∇_ρ with the isomorphism class of M'.) ∇_ρ must satisfy the condition that for $M \in \mathcal{A}$ the class of isomorphism classes in $\mathcal{A}'$ associated via ∇_ρ with the isomorphism class of M in $\mathcal{A}$ is a set, or more precisely, any class of structures in $\mathcal{A}'$ which contains a unique representative from each isomorphism class in $\mathcal{A}'$ associated via ∇_ρ with the isomorphism class of M in $\mathcal{A}$ and which contains only such structures is a set. The further condition that distinct isomorphism classes in $\mathcal{A}$ are associated with disjoint sets of isomorphism classes in $\mathcal{A}'$ must also be satisfied. ∇_ρ, then, is almost a function from a subclass of $\mathcal{A}'$ onto $\mathcal{A}$ which is "set-to-one" -- except, of course, that neither $\mathcal{A}'$ nor $\mathcal{A}$ are usually sets and so cannot really have functions defined on them.

(ii) T_ρ is an effectively computable one-to-one transformation of formulae of $\mathcal{L}$ into formulae of $\mathcal{L}'$.

(iii) $\mathcal{C}_\rho$ and $\mathcal{D}_\rho$ are particular finite sets of formulae of $\mathcal{L}'$.

(iv) For all formulae A of $\mathcal{L}$, $M \in \mathcal{A}$, and $M' \in \mathcal{A}'$ such that $M \nabla_\rho M'$,

$$\mathcal{M} \models A \Leftrightarrow \mathcal{M}' \models T_{\rho} A$$

(v) For all $\mathcal{M}' \in \mathfrak{A}'$

$$(\exists \mathcal{M} \in \mathfrak{A} \text{ such that } \mathcal{M} \underset{\rho}{\nabla} \mathcal{M}') \Leftrightarrow (\mathcal{M}' \models C \ \forall C \in \mathcal{C}_{\rho} \text{ and } \mathcal{M}' \not\models D \ \forall D \in \mathcal{D}_{\rho}).$$

Intuitively, then, each structure $\mathcal{M}$ in $\mathfrak{A}$ is associated with structures $\mathcal{M}'$ in $\mathfrak{A}'$ such that for any formula A in the language L, $\mathcal{M}$ models A if and only if $\mathcal{M}'$ models its transform in L'. The formulae $\mathcal{C}_{\rho}$ and $\mathcal{D}_{\rho}$ are used to tell whether or not a structure in $\mathfrak{A}'$ is associated with any structure in $\mathfrak{A}$ (is in the "range" of $\underset{\rho}{\nabla}$). A structure in $\mathfrak{A}'$ is so associated if and only if it models all formulae in $\mathcal{C}_{\rho}$ and none in $\mathcal{D}_{\rho}$. Thus the reduction must be such that the "range" of $\underset{\rho}{\nabla}$ is describable in L' in this weak sense.

We note that a reduction ρ is particularly interesting if $\underset{\rho}{\nabla}$ can be effectively described, in the sense that given $\mathcal{M} \in \mathfrak{A}$ we can describe in terms of $\mathcal{M}$ the set of isomorphism classes associated with $\mathcal{M}$ via $\underset{\rho}{\nabla}$, and if $\underset{\rho}{\nabla}$ is effectively invertible, in the sense that we can describe the "range" of $\underset{\rho}{\nabla}$ in $\mathfrak{A}'$ and for each $\mathcal{M}'$ in the range we can describe the $\mathcal{M}$ in $\mathfrak{A}$ such that $\mathcal{M} \underset{\rho}{\nabla} \mathcal{M}'$. All the reductions we shall use in this thesis have these properties to a certain extent. They will also have the additional properties that $\mathcal{C}_{\rho}$ and $\mathcal{D}_{\rho}$ are either empty or singletons.

12.2 We shall use reductions in the following way. Suppose there is a reduction ρ of $(\mathsf{L}, \mathfrak{A})$ to $(\mathsf{L}', \mathfrak{A}')$. Then we can use the language $\mathbb{L}'$ to describe properties of frames in $\mathbb{A}$ as follows. Let P be a property in $\mathbb{A}$. If there is a set S_P of formulae in $\mathbb{L}'$ such that for $\mathcal{M} \in \mathbb{A}$, $\mathcal{M}$ has property P if and only if $\mathcal{M}'$ models each formula in S_P for each structure $\mathcal{M}' \in \mathbb{A}'$ such that $\mathcal{M} \underset{\rho}{\nabla} \mathcal{M}'$, then we could say that S_P describes property P through the reduction ρ.

We see immediately that any property in $\mathbb{A}$ describable in $\mathbb{L}$ is also describable in $\mathbb{L}'$. Suppose that P is describable in $\mathbb{L}$. Then there is a set S_P of formulae in $\mathbb{L}$ such that $\mathcal{M} \models A$ for all $A \in S_P$ iff $\mathcal{M}$ has property P. Let S_P' be the set $\{ T_\rho A \mid A \in S_P \}$ of formulae in $\mathbb{L}'$. Then, since ρ is a reduction, S_P' describes property P through the reduction ρ.

12.3 As previously, $\mathbb{M}$ will be the modal language and $\bar{\mathbb{R}}$ the superconnected relational semantics for $\mathbb{M}$. Let $\mathbb{D}$ be any semantics for $\mathbb{M}$. We say that a property P which may or may not be held by frames in $\mathbb{D}$ is quasi-modal if there is a reduction ρ of $(\mathbb{M},\mathbb{D})$ to $(\mathbb{M},\bar{\mathbb{R}})$ and a set Q_P of modal formulae such that for each frame $\mathcal{F}$ in $\mathbb{D}$ and each $\mathcal{W}$ in $\bar{\mathbb{R}}$ such that $\mathcal{F} \underset{\rho}{\nabla} \mathcal{W}$

$\mathcal{F}$ has property P if and only if $\mathcal{W} \models B \ \forall B \in Q_P$.

A property P is strongly quasi-modal if it is quasi-modal and has an associated Q_P which is finite.

Thus, P is quasi-modal if P can be described in $\mathbb{M}$, not necessarily directly, but through some reduction to $(\mathbb{M},\bar{\mathbb{R}})$.

It would

be nice if it were immediately obvious that every modal property is quasi-modal. While this fact is true, its proof will take a considerable amount of work and so will come later.

We shall restrict our study of quasi-modal properties to those in the semantics $\mathcal{S}$ and $\mathcal{R}$. Our object is to show that every second-order property is quasi-modal. Since we know that every modal property is second-order (11.9) it will follow that every modal property is quasi modal.

12.4 The following theorem shows that reductions can be composed.

12.4.1 Theorem: Let ρ be a reduction of (L_1, A_1) to (L_2, A_2) and σ a reduction of (L_2, A_2) to (L_3, A_3). Then $\sigma\circ\rho$ is a reduction of (L_1, A_1) to (L_3, A_3) where

(a) $\underset{\sigma\circ\rho}{\nabla} = \underset{\sigma}{\nabla}\circ\underset{\rho}{\nabla}$ (if $\mathcal{U}_1$ and $\mathcal{U}_3$ are isomorphism classes in A_1 and A_3 respectively, then $\mathcal{U}_1 \underset{\sigma\circ\rho}{\nabla} \mathcal{U}_3$ iff there is an isomorphism class $\mathcal{U}_2$ in A_2 such that $\mathcal{U}_1 \underset{\rho}{\nabla} \mathcal{U}_2$ and $\mathcal{U}_2 \underset{\sigma}{\nabla} \mathcal{U}_3$).

(b) $T_{\sigma\circ\rho} = T_\sigma \circ T_\rho$.

(c) $\mathcal{C}_{\sigma\circ\rho} = \mathcal{C}_\sigma \cup \{ T_\sigma C \mid C \in \mathcal{C}_\rho \}$

(d) $\mathcal{D}_{\sigma\circ\rho} = \mathcal{D}_\sigma \cup \{ T_\sigma D \mid D \in \mathcal{D}_\rho \}$

Proof: We must show that $\underset{\sigma\circ\rho}{\nabla}$ satisfies (i) - (v) in the definition of reduction.

(i) Clearly $\underset{\sigma\circ\rho}{\nabla}$ is an association between isomorphism classes of structures of A_1 and isomorphism classes of structures of A_3. If $\mathcal{U}_1$ is an isomorphism class in A_1, then the class of isomorphism classes $\mathcal{U}_3$ in A_3 such that $\mathcal{U}_1 \underset{\sigma\circ\rho}{\nabla} \mathcal{U}_3$ is $\bigcup_{\mathcal{U}_2 \in I} \{ \mathcal{U}_3 \mid \mathcal{U}_2 \underset{\sigma}{\nabla} \mathcal{U}_3 \}$ (where $I = \{ \mathcal{U}_2 \mid \mathcal{U}_2$ is an isomorphism class in A_2 and $\mathcal{U}_1 \underset{\rho}{\nabla} \mathcal{U}_2 \}$), and as such is the union of a set of sets and is therefore a set.

Suppose that $\mathcal{U}_3$ is an isomorphism class in A_3 and that $\mathcal{U}_1$ and $\mathcal{U}_1'$ are isomorphism classes in A_1 such that $\mathcal{U}_1 \;{}_{\sigma}\nabla_{\rho}\; \mathcal{U}_3$ and $\mathcal{U}_1' \;{}_{\sigma}\nabla_{\rho}\; \mathcal{U}_3$. Then there are isomorphism classes $\mathcal{U}_2, \mathcal{U}_2'$ in A_2 such that $\mathcal{U}_1 \underset{\rho}{\nabla} \mathcal{U}_2$, $\mathcal{U}_2 \underset{\sigma}{\nabla} \mathcal{U}_3$, $\mathcal{U}_1' \underset{\rho}{\nabla} \mathcal{U}_2'$, and $\mathcal{U}_2' \underset{\sigma}{\nabla} \mathcal{U}_3$. Since σ is a reduction and $\mathcal{U}_2 \underset{\sigma}{\nabla} \mathcal{U}_3$ and $\mathcal{U}_2' \underset{\sigma}{\nabla} \mathcal{U}_3$ we must have $\mathcal{U}_2 = \mathcal{U}_2'$. But then $\mathcal{U}_1 \underset{\rho}{\nabla} \mathcal{U}_2$, $\mathcal{U}_1' \underset{\rho}{\nabla} \mathcal{U}_2$ and since ρ is a reduction we must have $\mathcal{U}_1 = \mathcal{U}_1'$. Thus distinct isomorphism classes in A_1 are associated via ${}_{\sigma}\nabla_{\rho}$ with disjoint sets of isomorphism classes in A_3.

(ii) Clearly $T_{\sigma\circ\rho} = T_\sigma \circ T_\rho$ is effective and one-to-one since both T_σ and T_ρ are.

(iii) is immediate.

(iv) Suppose A is a formula of L_1, $M_1 \in A_1$ and $M_3 \in A_3$ with $M_1 \;{}_{\sigma}\nabla_{\rho}\; M_3$. Then there is $M_2 \in A_2$ with $M_1 \underset{\rho}{\nabla} M_2$ and $M_2 \underset{\sigma}{\nabla} M_3$. Now $M_1 \models A \Leftrightarrow M_2 \models T_\rho A \Leftrightarrow M_3 \models T_\sigma(T_\rho A) = T_{\sigma\circ\rho} A$.

(v) Let $M_3 \in A_3$. Suppose there is $M_1 \in A_1$ with $M_1 \;{}_{\sigma}\nabla_{\rho}\; M_3$. Then there is M_2 in A_2 with $M_1 \underset{\rho}{\nabla} M_2$ and $M_2 \underset{\sigma}{\nabla} M_3$. Thus $M_3 \models \mathcal{C}_\sigma$ and $M_2 \models \mathcal{C}_\rho$, and therefore $M_3 \models \mathcal{C}_\sigma \cup \{ T_\sigma C \mid C \in \mathcal{C}_\rho \} = \mathcal{C}_{\sigma\circ\rho}$. Also $M_3 \not\models D \cdot \forall D \in \mathcal{D}_\sigma$ and $M_2 \not\models D\ \forall D \in \mathcal{D}_\rho$, and therefore $M_3 \not\models D\ \forall D \in \mathcal{D}_\sigma \cup \{ T_\sigma B \mid B \in \mathcal{D}_\rho \} =$

$\mathcal{D}_{\sigma o \rho}$.

Now suppose $M_3 \models \mathcal{C}_{\sigma o \rho}$ and $M_3 \not\models D \; \forall D \in \mathcal{D}_{\sigma o \rho}$. Then $M_3 \models \mathcal{C}_{\sigma}$ and $M_3 \not\models D \; \forall D \in \mathcal{D}_{\sigma}$ and so there is $M_2 \in \mathsf{A}_2$ with $M_2 \underset{\sigma}{\nabla} M_3$. Also $M_3 \models \{ T_{\sigma}C \mid C \in \mathcal{C}_{\rho}\}$ and so $M_2 \models \mathcal{C}_{\rho}$. Further, $M_3 \not\models T_{\sigma}D$ for all $D \in \mathcal{D}_{\rho}$ and so $M_2 \not\models D \; \forall D \in \mathcal{D}_{\rho}$. Thus there is $M_1 \in \mathsf{A}_1$ with $M_1 \underset{\rho}{\nabla} M_2$ and so $M_1 \underset{\sigma o \rho}{\nabla} M_3$. XXX

12.5 We have already, in fact, shown in 11.2 and 11.8 that there is a reduction from (M,S) to $(\mathsf{L}_s,\mathsf{A}_s)$. Let $\underset{\gamma}{\nabla}$ be such that for $F \in \mathsf{S}$ and $U \in \mathsf{A}_s$, $F \underset{\gamma}{\nabla} U$ iff U is isomorphic to the structure U_F described in 11.2. Let T_{γ} be the transformation T of 11.8. Let $\mathcal{C}_{\gamma} = \mathcal{D}_{\gamma} = \phi$. It is easy to see that conditions (i) to (v) of reductions are satisfied by $\gamma = (\underset{\gamma}{\nabla}, T_{\gamma}, \mathcal{C}_{\gamma}, \mathcal{D}_{\gamma})$.

Similarly there is a reduction from (M,R) to $(\mathsf{L}_r,\mathsf{A}_r)$. Let $\underset{\delta}{\nabla}$ be such that for $W \in \mathsf{R}$ and $U \in \mathsf{A}_r$, $W \underset{\delta}{\nabla} U$ iff U is isomorphic to W in A_r (hence, also in R). Let T_{δ} be the transformation R of 11.9, and let $\mathcal{C}_{\delta} = \mathcal{D}_{\delta} = \phi$. Then it is easy to see that $\delta = (\underset{\delta}{\nabla}, T_{\delta}, \mathcal{C}_{\delta}, \mathcal{D}_{\delta})$ is a reduction.

12.6 Thomason [19,20,21] developed his notion of reduction for a purpose quite different from ours. He was studying the logical consequence relation of a system: if (L,A) is a system then the logical consequence relation R for (L,A) is a binary relation such that $(\Gamma, A) \in R$ if A is a formula in L, Γ a set of formulae in L and for each structure $M \in \mathsf{A}$, if $M \models \Gamma$ then $M \models A$.

If L is such that we can effectively code formulae with natural numbers, then we can consider the logical consequence relation as a binary relation between sets of numbers and numbers, and, as such, can discuss the recursive complexity of the relation (i.e., its position in the arithmetic or analytic hierarchies.)

Thomason shows that if there is a reduction from (L,A) to (L',A') then the logical consequence relation of (L,A) is at least as complex as that of (L',A'), i.e. the logical consequence relation of (L',A') is recursive in that of (L,A). This relative recursivity hinges on the fact that the transformation of formulae in a reduction must be effective and therefore recursive.

It should be pointed out that Thomason's notion of reduction is somewhat looser than ours, but that our notion would suffice for Thomason's work.

12.7 We shall conclude with the following theorem.

12.7.1 Theorem: If L is any higher-order language of the type described in 11.1 and if A is the higher-order semantics for L also described in 11.1, then there is a reduction of $(L\ A)$ to $(M,\bar{R})$.

12.7.1 Corollary: Any second-order property in S or R is quasi-modal and any strongly second-order property is strongly quasi-modal.

Proof of corollary; If σ is the reduction promised by the theorem 12.7.1 from $(L_s, \mathcal{A}_s)$ (or $(L_r, \mathcal{A}_r)$) to $(\mathcal{M}, \bar{\mathcal{R}})$ then let $Q_P = \{ T_\sigma A \mid A \in S_P \}$. XXX

The theorem tells us more than we originally sought: if P is a property in $\mathcal{S}$ or $\mathcal{R}$ (or any other semantics) which can be described in an appropriate language of *any* finite order, then P is quasi-modal and so can be described in the modal language $\mathcal{M}$ through an appropriate reduction. It follows immediately from 12.7.2 and the results of 12.5 that

12.7.3 *Corollary*: Every (strongly) modal property in $\mathcal{S}$ or $\mathcal{R}$ is (strongly) quasi-modal. XXX

It also follows from 12.7.2 and from 11.5, 11.6, 11.7 that the property of being a relational frame in $\mathcal{S}$ and the properties in $\mathcal{R}$ of being irreflexive or connected are all not modal but strongly quasi-modal.

The proof of 12.7.1 is a modification of Thomason's proofs in [19,20,21] and is rather long and tedious. For this reason the details of the proof will be given in an appendix and only a brief outline will appear in the main body of the thesis.

The reduction, σ, that we need is constructed as the composition of a sequence of other reductions. These reductions involve various *tense systems*, n-tense languages with the corresponding n-tense connected relational semantics.

The concept of the n-tense language is an extension and generalization of the concept of the modal language. The n-tense language, T_n, is the language with propositional variables $p_1, p_2, \ldots$, and classical connectives $\rightarrow$ and $\sim$ as in $\mathcal{M}$, and additional unary connectives $G_1, \ldots, G_n$ and $H_1, \ldots, H_n$. We use defined connectives $\vee$, $\wedge$, and $\leftrightarrow$ as before, in addition to $F_1, \ldots, F_n$ and $P_1, \ldots, P_n$ where for a formula A, $F_iA = \sim G_i \sim A$ and $P_iA = \sim H_i \sim A$.

An n-tense relational frame is an n+1 -tuple $\mathcal{W} = (W, <_1, \ldots, <_n)$ where W is a nonempty set and $<_i$ is a binary relation on W (for $1 \leq i \leq n$). As with the relational semantics for $\mathcal{M}$, a valuation is a mapping V from $\mathbb{N}$ to $\mathcal{P}(W)$ and induces a mapping, also called V, from the set of n-tense formulae to $\mathcal{P}(W)$ according to the rules:

$$
\begin{aligned}
V(p_i) &= V(i) \\
V(\sim A) &= W - V(A) \\
V(A \rightarrow B) &= V(\sim A) \cup V(B) \\
V(G_iA) &= \{ w \in W \mid \{v \mid w <_i v\} \subset V(A) \} \\
V(H_iA) &= \{ w \in W \mid \{v \mid v <_i w\} \subset V(A) \}
\end{aligned}
$$

We see immediately that $V(A \vee B) = V(A) \cup V(B)$, $V(A \wedge B) = V(A) \cap V(B)$, $V(A \leftrightarrow B) = (V(A) \cap V(B)) \cup (V(\sim A) \cap V(\sim B))$, $V(F_iA) = \{ w \mid \text{there is } v \in V(A) \text{ with } w <_i v \}$, $V(P_iA) = \{ w \mid \text{there is } v \in V(A) \text{ with } v <_i w \}$. As before, we write $V(A,w) = T$ or F if $w \in V(A)$ or $w \notin V(A)$ respectively.

If we think of $<_i$ as being the i^{th} future relation, i.e., $w <_i v$ if v is in the i^{th} future from w, then $V(F_iA,w)$

= T if on w, A is true at some point in the i_{th} future; $V(P_iA,w)$ = T if on w, A is true at some point in the i^{th} past; $V(G_iA,w)$ = T if on w, A is true at all points in the i^{th} future; and $V(H_iA,w)$ = T if on w, A is true at all points in the i^{th} past. The comparison with the modal language is obvious. The n-tense concept is an extension of the modal concept in that we have a past tense as well as a future and it is a generalization in that we may have more than only one "temporal ordering".

A formula A is valid in a frame $\mathcal{W}$ if $V(A) = W$ for all valuations V on $\mathcal{W}$. If $w,v \in W$, then w and v are directly connected via $<_i$ if $w<_iv$ ar $v<_iw$. Two elements w and v are connected in $\mathcal{W}$ if there is a sequence $w=w_0,w_1,\ldots,w_n=v$ of elements of W such that for each $j = 1,2,\ldots,m$ there is an i $(1 \leq i \leq n)$ such that w_{j-1} and w_j are directly connected via $<_i$. A frame $\mathcal{W} = (W,<_1,\ldots,<_2)$ is connected if every pair of elements in W is connected. The semantics of n-tense connected relational frames together with the above notion of validity is denoted $\mathcal{R}_n$.

Our construction of the reduction σ of $(\mathcal{L},\mathcal{A})$ to $(\mathcal{M},\overline{\mathcal{R}})$ proceeds in three stages. In the first stage we construct a reduction μ of $(\mathcal{L},\mathcal{A})$ to $(\mathcal{T}_h,\mathcal{R}_h)$ where $h = 6+2n+m+e+\sum_{i=1}^{m} k_i$, n is the order of the system $(\mathcal{L},\mathcal{A})$, m is the number of relations Q_i in $\mathcal{L}$, k_i is the arity of Q_i, and $e = 1 +$ maximum$(\{k_i \mid 1 \leq i \leq m\} \cup \{3\})$ (thus $e \geq 4$). The second stage consists of the construction, for each $n\geq 2$, of a reduction ν_n of (T_n,R_n) to (T_{n-1},R_{n-1}), and in the final stage we construct a reduction ν_1 of (T_1,R_1) to $(M,\overline{R})$.

The reasons for our use of $\overline{R}$ in all of this rather than simply R or $\hat{R}$ will become apparent when the actual reduction ν_1 is presented. Our techniques would not yield a proper reduction at this stage if we were to use (M,R) or $(M,\hat{R})$ instead of $(M,\overline{R})$.

It is not known whether or not other techniques could be used to get a reduction of (T_1,R_1) to (M,R), or even to $(M\ \hat{R})$, or whether we could more directly construct a reduction of a higher-order system (L,A) to (M,R) or $(M,\hat{R})$. If we could, we could re-define quasi-modal properties to obtain what would perhaps be a more natural concept of quasi-modality.

Appendix

Proof of Theorem 12.7.1: Construction of a reduction $\sigma = (\mathcal{G}_\sigma, T_\sigma, \mathcal{C}_\sigma, \mathcal{D}_\sigma)$ of $(\mathsf{L}, \mathcal{A})$ to $(\mathsf{M}, \bar{\mathcal{R}})$ where L is the n^{th} order language with m predicate symbols $Q_1, \ldots, Q_m$, Q_i being of type $(a_{i1}, \ldots, a_{ik_i})$ for $1 \le i \le m$, and where $\mathcal{A}$ is the n^{th} order semantics for L. In the construction of μ which follows, n, m, k_i, and $a_{i1}, \ldots, a_{ik_i}$ for $1 \le i \le m$ will be fixed at these numbers.

Let $e = 1 + \max(\{k_i \mid 1 \le i \le m\} \cup \{3\})$; thus e is either 4 or 1 plus the greatest arity of the Q_i, whichever is greater. Now let $I = \{1,2,3,4,9,10,12,13\} \cup \{(j,i) \mid 5 \le j \le 6, 1 \le i \le n-1\} \cup \{(7,i,j) \mid 1 \le i \le m, 1 \le j \le k_i\} \cup \{(8,i) \mid 1 \le i \le m\} \cup \{(11,j) \mid 1 \le j \le e\}$. If $h = 6 + 2n + m + e + \sum_{i=1}^{m} k_i$, then $\mathrm{card}(I) = h$. If we are considering the system (T_h, R_h) it really makes no difference whether we subsctipt the G's and H's and $<$'s with integers from 1 to h or with elements from I. Thus, we shall take this latter course, keeping in mind that the subscripts that we are using are really just substitutions for the numbers 1 to h.

Ap.1 The first step in our construction of σ is the construction of a reduction μ of $(\mathsf{L}, \mathcal{A})$ to (T_h, R_h). We begin by associating with each formula A of L a finite sequence $\tau(A) = (\tau_1(A), \tau_2(A), \ldots, \tau_r(A))$ according to the following definition.

$$\tau(\sim A) = (0,A) \qquad \tau(A\to B) = (1,A,B)$$

$$\tau(x_i^k = x_j^k) = (2,k,i,j) \qquad \tau(x_i^k \in x_j^{k+1}) = (3,k,i,j)$$

$$\tau(\forall x_i^k A) = (4,k,i,A) \qquad \tau(Q_i(x_{j_1}^{a_{i1}}, x_{j_2}^{a_{i2}}, \ldots, x_{j_{k_i}}^{a_{ik_i}})) = (4+i, j_1, j_2, \ldots, j_{k_i})$$

Thus, for all formulae A in L, $\tau_1(A)$ is a number between 0 and 4+m, $\tau_2(A)$, $\tau_3(A)$, $\tau_4(A)$ are either numbers or subformulae of A, and $\tau_j(A)$ is a number for $j\geq 5$. We note that e is the maximum length of the τ-sequences. If A is a formula such that $\tau(A)$ is a sequence of length $e'<e$ then define $\tau_j(A) = 0$ for $e'<j\leq e$, so that for $1\leq j\leq e$, $\tau_j(A)$ is defined for all formulae A.

Let $S = (S, Q_1, \ldots, Q_m)$ be a structure in A. Then there is a structure $W = (W, <_{i(i\in I)})$ in R_h such that W is the union of disjoint nonempty sets U, N, R, D, and $K_1, \ldots, K_m$ and conditions (1) - (11) below are satisfied.

(1) $<_1$ is an irreflexive well-ordering of W.

(2) $<_2$ is an irreflexive well-ordering of N of order type ω. Thus N can and will be identified with the natural numbers beginning with 0.

(3) $<_3$ is a function from U to the subset $\{1,2,\ldots,n\}$ of N. Hence $U = U^1 \cup U^2 \cup \ldots \cup U^n$ disjointly, where $U^i = \{u \mid u <_3 i\}$.

(4) $<_4$ is a function from U to N. Therefore, for $1\leq i\leq n$,

$<_4|U^i$ (the restriction of the function $<_4$ to the domain U^i) is a function from U^i to N, and hence for $1 \le i \le n$, $U^i = \bigcup_{j=0}^{\infty} U^i_j$ disjointly, where $U^i_j = \{ u \in U^i \mid u <_4 j \}$.

(5) $U^i_0 = \wp^{i-1}(S)$ $(1 \le i \le n)$

$u <_{5,i} v$ iff $u \in U^i_0$, $v \in U^{i+1}_0$, and $u \in v$ $(1 \le i \le n-1)$

$u <_{6,i} v$ iff $u \in U^i_0$, $v \in U^{i+1}_0$, and $u \notin v$ $(1 \le i \le n-1)$

(6) For $1 \le i \le m$, $K_i = \wp^{a_{i1}-1}(S) \times \dots \times \wp^{a_{ik_i}-1}(S)$ $(= U^{a_{i1}}_0 \times \dots \times U^{a_{ik_i}}_0)$. (Even if Q_i and Q_j have the same type, we assume that K_i and K_j are disjoint.) Thus $\mathcal{Q}_i \subset K_i$.

$<_{7,i,j}$ is the projection from K_i to $U^{a_{ij}}_0$ $(= \wp^{a_{ij}-1}(S))$

$(1 \le i \le m,\ 1 \le j \le k_i)$

$<_{8,i}$ is the identity function on $\mathcal{Q}_i$; i.e., $u <_{8,i} v$ iff $u \in \mathcal{Q}_i \subset K_i$ and $u = v$. $(1 \le i \le m)$

(7) $<_9$ is a function from U into U such that for each i $(1 \le i \le n)$ and each $j \in \mathbb{N}$, $<_9|U^i_j$ is a one-to-one function from U^i_j onto U^i_0, and $<_9|U^i_0$ is the identity function on U^i_0.

(8) R is the set of all formulae of $\mathcal{L}$, and $<_{10}$ is a function from R to N such that $u <_{10} v$ iff $v = ht(u)$, where $ht(A) = 0$ if A is atomic, $ht({\sim}A) = ht(\forall x^j_i A) = 1 + ht(A)$, and $ht(A \to B) = 1 + \max(ht(A), ht(B))$.

(9) $u <_{11,j} v$ iff $u \in R$ and $\tau_j(u) = v$. $(1 \le j \le e)$

(10) D is the set of all assignments on $\mathcal{S}$. $u <_{12} v$

iff $v \in D$ and for some i $(1 \leq i \leq n)$, $u \in U^i_j$, and $u <_9 v(x^i_j)$. (Note that since v is an assignment, $v(x^i_j) \in \mathcal{P}^{i-1}(S) = U^i_0$; thus, each $v \in D$ has a unique $<_{12}$-predecessor in each of the subsets U^i_j of U.)

(11) $u <_{13} v$ iff $u \in R$, $v \in D$, and $v(u) = T$ in (i.e., the formula u is satisfied by the valuation v on $\mathcal{S}$).

Let C_μ be the conjunction of the formulae numbered (12) - (78) below. Note that some numbers are subscripted and really are associated with a finite sequence of formulae. In the lines marked (Def.), A, B, and C are used to denote arbitrary h-tense formulae.

(12) $\bigvee_{i \in I}(F_i p) \to p \vee F_1 p \vee P_1 p$

(13) $F_1 F_1 p \to F_1 p$

(Def.) $MA = A \vee F_1 A \vee P_1 A$

(Def.) $LA = \sim M \sim A$

(14) $F_1 P_1 p \vee P_1 F_1 p \to Mp$

(15) $P_1 p \to P_1(p \wedge \sim P_1 p)$

(Def.) $\mathrm{Unit}(A) = MA \wedge L(A \to \sim(F_1 A \vee P_1 A))$

(Def.) $N = F_2(p \vee \sim p)$

(16) MN

(17) $N \wedge M(N \wedge p) \to p \vee F_2 p \vee P_2 p$

(18) $P_2 N \to N$

(19) $F_2 F_2 p \to F_2 p$

(20) $F_2P_2p \vee P_2F_2p \rightarrow p \vee F_2p \vee P_2p$

(21) $P_2p \rightarrow P_2(p \wedge \sim P_2p)$

(22) $p \wedge P_2N \rightarrow P_2(G_2(p \vee P_2p))$

(Def.) $\mathrm{Nmbr}(A) = \mathrm{Unit}(A) \wedge M(N \wedge A)$

(Def.) $\mathrm{Sr}(A) = N \wedge P_2A \wedge H_2(A \vee F_2A)$

(Def.) $\{0\} = N \wedge H_2(p \wedge \sim p)$

(Def.) $\{k+1\} = \mathrm{Sr}(\{k\})$ $(k = 0,1,\dots)$

(Def.) $U = F_3N$

(Def.) $U^A = U \wedge F_3A$

(Def.) $U_B = U \wedge F_4B$

(Def.) $U^A_B = U \wedge F_3A \wedge F_4B$

(23) $F_3(p \vee \sim p) \longleftrightarrow U$

(24) $P_3(p \vee \sim p) \longleftrightarrow (F_2\{n+1\} \wedge \sim \{0\})$

(25) $F_4(p \vee \sim p) \longleftrightarrow U$

(26) $P_4(p \vee \sim p) \longleftrightarrow N$

(27) $F_3p \rightarrow G_3p$

(28) $F_4p \rightarrow G_4p$

(29) $MU^{\{1\}}_{\{0\}}$

$$\left.\begin{array}{ll} (30_i) & F_{5,i}p \rightarrow U^{\{i\}}_{\{0\}} \\ (31_i) & P_{5,i}p \rightarrow U^{\{i+1\}}_{\{0\}} \\ (32_i) & F_{6,i}p \rightarrow U^{\{i\}}_{\{0\}} \\ (33_i) & P_{6.i}p \rightarrow U^{\{i+1\}}_{\{0\}} \end{array}\right\} \quad (1 \leq i \leq n-1)$$

(Def.) $\mathrm{Ord}^i(A) = \mathrm{Unit}(A) \wedge M(U^{\{i\}}_{\{0\}} \wedge A)$ $(1 \leq i \leq n)$

$$\left.\begin{array}{ll}
(34_i) & \mathrm{Ord}^{i}(p)\wedge \mathrm{Ord}^{i+1}(q) \rightarrow L(p\rightarrow (F_{5,i}q \leftrightarrow \sim F_{6,i}q)) \\
(35_i) & \mathrm{Ord}^{i+1}(p)\wedge \mathrm{Ord}^{i+1}(q)\wedge M(p\wedge H_{5,i}r\wedge H_{6,i}\sim r)\wedge \\
 & \qquad M(q\wedge H_{5,i}r\wedge H_{6,i}\sim r) \rightarrow L(p\leftrightarrow q) \\
(36_i) & M(U_{\{0\}}^{\{i+1\}}\wedge H_{5,i}p\wedge H_{6,i}\sim p)
\end{array}\right\}(1\le i\le n-1)$$

$$\left.\begin{array}{ll}
(\mathrm{Def.}) & K_i = F_{7,i,1}(p\vee \sim p) \\
\begin{array}{l} (37_{ij}) \\ (38_{ij}) \\ (39_{ij}) \end{array} & \left.\begin{array}{l}
F_{7,i,j}(p\vee\sim p) \leftrightarrow K_i \\
P_{7,i,j}(p\vee\sim p) \leftrightarrow U_{\{0\}}^{\{a_{ij}\}} \\
F_{7,i,j}p \rightarrow G_{7,i,j}p
\end{array}\right\} \quad (1\le j\le k_i) \\
(40_i) & \bigwedge_{j=1}^{k_i}(\mathrm{Ord}^{a_{ij}}(p_j)) \rightarrow M(\bigwedge_{j=1}^{k_i}(F_{7,i,j}p_j)) \\
(\mathrm{Def.}) & \mathrm{Prod}_i(A) = \mathrm{Unit}(A)\wedge M(K_i\wedge A) \\
(41_i) & \mathrm{Prod}_i(q)\wedge \mathrm{Prod}_i(r)\wedge \bigwedge_{j=1}^{k_i}(\mathrm{Ord}^{a_{ij}}(p_j))\wedge \\
 & \qquad M(q\wedge \bigwedge_{j=1}^{k_i}(F_{7,i,j}p_j))\wedge M(r\wedge \bigwedge_{j=1}^{k_i}(F_{7,i,j}p_j) \\
 & \qquad \rightarrow L(q\leftrightarrow r) \\
(\mathrm{Def.}) & \mathcal{Q}_i = F_{8,i}(p\vee\sim p) \\
(42_i) & \mathcal{Q}_i \rightarrow K_i \\
(43_i) & F_{8,i}p \rightarrow p\wedge G_{8,i}p
\end{array}\right\}(1\le i\le m)$$

(44) $F_9 p \rightarrow U$

(45) $P_9 p \rightarrow U_{\{0\}}$

(46) $F_9 p \rightarrow G_9 p$

(47) $\mathrm{Nmbr}(p)\wedge \mathrm{Nmbr}(q)\wedge U_{\{0\}}^{p} \rightarrow P_9 U_q^p$

(48) $\mathrm{Nmbr}(p)\wedge U^p \rightarrow F_9 U_{\{0\}}^{p}$

(49) $\mathrm{Nmbr}(p)\wedge \mathrm{Nmbr}(q)\wedge P_9(U_q^p\wedge r) \rightarrow H_9(U_q^p\rightarrow r)$

(50) $U_{\{0\}} \rightarrow (p\leftrightarrow F_9 p)$

(Def.) $R = F_{10}N$

(Def.) $R^{A} = R \wedge F_{10}A$

(51) $F_{10}p \rightarrow R$

(52) $P_{10}p \rightarrow N$

(53) $F_{10}p \rightarrow G_{10}p$

(54) $R \rightarrow \bigwedge_{j=1}^{e}(F_{11,j}(p \vee \sim p))$

(55) $\bigvee_{j=1}^{e}(F_{11,j}p) \rightarrow R$

(56$_j$) $F_{11,j}p \rightarrow G_{11,j}(p \wedge (N \vee R))$ $(1 \leq j \leq e)$

(Def.) $\mathrm{Fmla}(A) = \mathrm{Unit}(A) \wedge M(R \wedge A)$

(Def.) $\mathrm{Fmla}^{B}(A) = \mathrm{Fmla}(A) \wedge M(A \wedge F_{10}B)$

(57) $\bigwedge_{j=1}^{e}(\mathrm{Unit}(p_j)) \wedge \mathrm{Fmla}(s) \wedge \mathrm{Fmla}(t) \wedge M(s \wedge \bigwedge_{j=1}^{e}(F_{11,j}p_j))$
$M(t \wedge \bigwedge_{j=1}^{e}(F_{11,j})) \rightarrow L(s \leftrightarrow t)$

(58) $\mathrm{Nmbr}(p) \wedge \mathrm{Nmbr}(q) \wedge \mathrm{Nmbr}(r) \wedge L(r \rightarrow (F_{2}\{n+1\} \wedge \sim\{0\}))$
$\rightarrow M(R^{\{0\}} \wedge F_{11,1}\{2\} \wedge F_{11,2}r \wedge F_{11,3}p \wedge F_{11,4}q \wedge$
$\bigwedge_{j=5}^{e}(F_{11,j}\{0\}))$

(59) $\mathrm{Nmbr}(p) \wedge \mathrm{Nmbr}(q) \wedge \mathrm{Nmbr}(r) \wedge L(r \rightarrow (F_{2}\{n\} \wedge \sim\{0\}))$
$\rightarrow M(R^{\{0\}} \wedge F_{11,1}\{3\} \wedge F_{11,2}r \wedge F_{11,3}p \wedge F_{11,4}q \wedge$
$\bigwedge_{j=5}^{e}(F_{11,j}\{0\}))$

(60$_i$) $\bigwedge_{h=1}^{k_i}(\mathrm{Nmbr}(p_h)) \rightarrow M(R^{\{0\}} \wedge F_{11,1}\{4+i\} \wedge \bigwedge_{h=1}^{k_i}(F_{11,1+h}p_h) \wedge$
$\bigwedge_{j=2+k_i}^{e}(F_{11,j}\{0\}))$ $(1 \leq i \leq m)$

(61) $\mathrm{Nmbr}(p) \wedge \mathrm{Fmla}^{p}(q) \rightarrow M(R^{Sr(p)} \wedge F_{11,1}\{0\} \wedge F_{11,2}q \wedge$
$\bigwedge_{j=3}^{e}(F_{11,j}\{0\}))$

(62)

$$\begin{aligned}&\mathrm{Nmbr}(p)\wedge\mathrm{Fmla}^{(p\vee F_2p)}(q)\wedge\mathrm{Fmla}^{(p\vee F_2p)}(r)\wedge\\&\quad(\mathrm{Fmla}^{p}(q)\vee\mathrm{Fmla}^{p}(r))\\&\quad\rightarrow\ M(R^{Sr(p)}\wedge F_{11,1}\{1\}\wedge F_{11,2}q\wedge F_{11,3}r\wedge\\&\quad\bigwedge_{j=4}^{e}(F_{ii,j}\{0\}))\end{aligned}$$

(63)

$$\begin{aligned}&\mathrm{Nmbr}(p)\wedge\mathrm{Nmbr}(q)\wedge\mathrm{Nmbr}(r)\wedge\mathrm{Fmla}^{r}(s)\wedge L(p\rightarrow(F_2\{n+1\}\wedge\sim\{0\}))\\&\quad\rightarrow\ M(R^{Sr(r)}\wedge F_{11,1}\{4\}\wedge F_{11,2}p\wedge F_{11,3}q\wedge F_{11,4}s\wedge\\&\quad\bigwedge_{j=5}^{e}(F_{11,j}\{0\}))\end{aligned}$$

(64)

$$\begin{aligned}R^{\{0\}}\ \rightarrow\ &([((F_{11,1}\{2\}\wedge F_{11,2}(F_2\{n+1\}\wedge\sim\{0\}))\\&\quad\vee(F_{11,1}\{3\}\wedge F_{11,2}(F_2\{n\}\wedge\sim\{0\})))\\&\quad\wedge F_{11,3}N\wedge F_{11,4}N\wedge\bigwedge_{j=5}^{e}(F_{11,j}\{0\})]\ \vee\\&\quad\bigvee_{i=1}^{m}[F_{11,1}\{4+i\}\wedge\bigwedge_{h=2}^{1+k_i}(F_{11,h}N)\wedge\bigwedge_{j=2+k_i}^{e}(F_{11,j}\{0\})])\end{aligned}$$

(65)

$$\begin{aligned}&\mathrm{Nmbr}(p)\wedge R^{Sr(p)}\quad\rightarrow\\&\quad([F_{11,1}\{0\}\wedge F_{11,2}R^{p}\wedge\bigwedge_{j=3}^{3}(F_{11,j}\{0\})]\\&\quad\vee[F_{11,1}\{4\}\wedge F_{11,2}(F_2\{n+1\}\wedge\sim\{0\})\wedge F_{11,3}N\wedge F_{11,4}R^{p}\\&\qquad\wedge\bigwedge_{j=5}^{e}(F_{11,j}\{0\})]\\&\quad\vee[F_{11,1}\{1\}\wedge F_{11,2}R^{(p\vee F_2p)}\wedge F_{11,3}R^{(p\vee F_2p)}\\&\qquad\wedge(F_{11,2}R^{p}\vee F_{11,3}R^{p})\wedge\bigwedge_{j=4}^{e}(F_{11,j}\{0\})])\end{aligned}$$

(Def.) $D = P_{12}(p\vee\sim p)$

(Def.) $\mathrm{Asst}(A) = \mathrm{Unit}(A)\wedge M(D\wedge A)$

(66) $D\rightarrow H_{12}U$

(67)

$$\begin{aligned}&\mathrm{Asst}(p)\wedge\mathrm{Nmbr}(q)\wedge\mathrm{Nmbr}(r)\wedge L(q\rightarrow(F_2\{n+1\}\wedge\sim\{0\}))\ \rightarrow\\&\quad\mathrm{Unit}(F_{12}p\wedge F_3q\wedge F_4r)\end{aligned}$$

(68) $L(N\rightarrow\bigwedge_{i=1}^{n}(P_4(p\wedge U^{\{i\}})))\ \rightarrow\ M(D\wedge H_{12}p)$

(69) $\mathrm{Asst}(p)\wedge\mathrm{Asst}(q)\wedge M(p\wedge H_{12}F_{12}q) \rightarrow L(p\leftrightarrow q)$

(Def.) ${}_{A}X^{B}_{C} = U^{B}_{\{0\}}\wedge P_{9}(F_{3}B\wedge F_{4}C\wedge F_{12}A)$

(Def.) $\tau_{j}(B) = P_{11,j}B$ $(1\leq j\leq e)$

(70) $P_{13}p \rightarrow D$

(71) $F_{13}p \rightarrow R$

(Def.) $"Q_{i}"(A_{1},\ldots,A_{k_{i}}) = M(Q_{i}\wedge\bigwedge_{j=1}^{k_{i}}(F_{7,i,j}A_{j}))$ $(1\ i\ m)$

(Def.) $A"="B = M(A\wedge B)$

(Def.) $A"\in"B = M(\bigvee_{i=1}^{n-1}(A\wedge F_{5,i}B))$

(Def.) $\mathrm{Stfy}(A,B) = \mathrm{Asst}(A)\wedge\mathrm{Fmla}(B)\wedge L(B\rightarrow F_{13}A)$

(72) $\mathrm{Asst}(p)\wedge\mathrm{Fmla}(q)\wedge(\tau_{1}(q)"="\{2\}) \rightarrow$

$(\mathrm{Stfy}(p,q)\leftrightarrow {}_{p}X^{\tau_{2}(q)}_{\tau_{3}(q)}"=" {}_{p}X^{\tau_{2}(q)}_{\tau_{4}(q)})$

(73) $\mathrm{Asst}(p)\wedge\mathrm{Fmla}(q)\wedge(\tau_{1}(q)"="\{3\}) \rightarrow$

$(\mathrm{Stfy}(p,q)\leftrightarrow {}_{p}X^{\tau_{2}(q)}_{\tau_{3}(q)}"\in" {}_{p}X^{Sr(\tau_{2}(q))}_{\tau_{4}(q)})$

(74_{i}) $\mathrm{Asst}(p)\wedge\mathrm{Fmla}(q)\wedge(\tau_{1}(q)"="\{4+i\}) \rightarrow$

$(\mathrm{Stfy}(p,q)\leftrightarrow "Q_{i}"({}_{p}X^{\{a_{i1}\}}_{\tau_{2}(q)},{}_{p}X^{\{a_{i2}\}}_{\tau_{3}(q)},\ldots,{}_{p}X^{\{a_{ik_{i}}\}}_{\tau_{k_{i}+1}(q)}))$

$(1\leq i\leq m)$

(75) $\mathrm{Asst}(p)\wedge\mathrm{Fmla}(q)\wedge(\tau_{1}(q)"="\{0\}) \rightarrow$

$(\mathrm{Stfy}(p,q)\leftrightarrow\sim\mathrm{Stfy}(p,\tau_{2}(q)))$

(76) $\mathrm{Asst}(p)\wedge\mathrm{Fmla}(q)\wedge(\tau_{1}(q)"="\{1\}) \rightarrow$

$(\mathrm{Stfy}(p,q)\leftrightarrow(\mathrm{Stfy}(p,\tau_{2}(q))\rightarrow\mathrm{Stfy}(p,\tau_{3}(q))))$

(77) $\mathrm{Asst}(p)\wedge\mathrm{Fmla}(q)\wedge(\tau_{1}(q)"="\{4\}) \rightarrow$

$(\mathrm{Stfy}(p,q)\leftrightarrow L((D\wedge H_{12}(U^{\tau_{2}(q)}_{\tau_{3}(q)}\vee F_{12}p))\rightarrow P_{13}\tau_{4}(q)))$

$$(78)\quad (U\vee N\vee R\vee D\vee \bigvee_{i+1}^{m}(K_i))\wedge\sim(U\wedge N)\wedge\sim(U\wedge R)\wedge\sim(U\wedge D)\wedge\sim(N\wedge R)\wedge\sim(N\wedge D)$$
$$\wedge\sim(R\wedge D)\wedge\bigwedge_{i=1}^{m}(\sim(U\wedge K_i)\wedge\sim(N\wedge K_i)\wedge\sim(R\wedge K_i)\wedge\sim(D\wedge K_i))$$
$$\wedge\bigwedge_{1\le i<j\le m}(\sim(K_i\wedge K_j))$$

Let $\mathcal{W} = (W,<_{i(i\in I)})$ be such that $\mathcal{W} \models C_\mu$.

<u>Claim</u>: $\mathcal{W}$ is isomorphic to some $\mathcal{W}'$ satisfying (1) - (11).

(1). By definition $\mathcal{W}$ is connected by $\{<_i \mid i\in I\}$. (12) ensures that $\mathcal{W}$ is connected by $<_1$; (13) that $<_1$ is transitive; (14) that $<_1$ is linear; (15) that $<_1$ is irreflexive and every nonempty subset of W has a $<_1$-least element. Thus $<_1$ is an irreflexive well-ordering of W.

(2). For every assignment V, V(MA) is W or ϕ according as V(A) is nonempty or empty, because $\mathcal{W}$ satisfies (1). Also, $V(N) = \{x \mid$ for some y, $x <_2 y\}$ is independent of V; we write N for V(N). Then (16) and (17) ensure that $<_2 \subset N\times N$. (19) - (21) ensure that $<_2$ is an irreflexive well-ordering of N, and (22) that the order-type of $<_2$ is ω.

(3). & (4). V(U) is independent of V, so we write U for V(U). Similarly, we write i for V({i}). (23) and (24) ensure that $<_3 \subset U\times\{1,\dots,n\}$; (25) and (26) that $<_4 \subset U\times N$; (27) and (28) that $<_3$ and $<_4$ are functions.

(5). By (29) $U_0^0 \ne \phi$, and by (30_i) - (33_i), $<_{5,i} \subset U_0^i\times U_0^{i+1}$ and $<_{6,i} \subset U_0^i\times U_0^{i+1}$ for $1\le i\le n-1$. For any assignment V,

$V(\mathrm{Ord}^i(A)) = W$ if $V(A) = \{u\}$, $u \in U_0^i$, and $V(\mathrm{Ord}^i(A)) = \phi$ otherwise. By (34_i), for each $u \in U_0^i$, $v \in U_0^{i+1}$, exactly one of $u <_{5,i} v$, $u <_{6,i} v$ holds. Define φ_i from U_0^{i+1} to $\mathcal{P}(U_0^i)$ by $\varphi_i(v) = \{ u \mid u <_{5,i} v \}$. By (35_i) φ_i is one-to-one, and by (36_i) φ_i is onto. Thus $\mathcal{W}$ is isomorphic to some $\mathcal{W}'$ satisfying (5) for some $\mathcal{S}$ where S is in one-to-one correspondence with U_0^0.

(6). Fix i, $1 \leq i \leq m$. We write K_i for $V(K_i)$ (which is independent of V). For each j $(1 \leq j \leq k_i)$, (37_{ij}) and (38_{ij}) ensure that $<_{7,i,j} \subset K_i \times U_0^{aij}$. (39_{ij}) ensures that $<_{7,i,j}$ is a function and (37_{ij}) and (38_{ij}) ensure that $<_{7,i,j}$ is from K_i onto U_0^{aij}. Define θ_i from K_i to $\times_{j=1}^{ki}(U_0^{aij})$ by $\theta_i(u) = (v_1, \ldots, v_{k_i})$ where $u <_{7,i,j} v_j$. The fact that the $<_{7,i,j}$ are functions ensures that θ_i is well-defined; (40_i) that θ_i is onto; and (41_i) that it is one-to-one. We write $\mathcal{Q}_i$ for $V(\mathcal{Q}_i)$ (also independent of V) and note that (42_i) and (43_i) tell us that $\mathcal{Q}_i \subset K_i$ and that $<_{8,i}$ is the identity function on $\mathcal{Q}_i$. Since in any structure $\mathcal{S}$ in $\mathcal{A}$ the relations $\mathcal{Q}_i$ may be arbitrarily chosen as subsets of the proper product sets, the above confirms that $\mathcal{W}$ is isomorphic to some $\mathcal{W}'$ satisfying (6).

(7). By (44) – (50).

(8). & (9). Write R for V(R) (independent of V). By (51) – (53) $<_{10}$ is a function from R to N. By (54) – (56_j) each

$<_{11,j}$ is a function from R to $N\cup R$. If $u <_{10} v$ we write ht(u) for v; if $u <_{11,j}$ we write $(u)_j$ for v. By (57), if $(u)_j = (v)_j$ for all j, $1 \le j \le e$, then u = v. Hence the following defines a function ψ from a set of formulae of L into R:

$\psi(x_i^k = x_j^k)$ = u if $(u)_1 = 2$, $(u)_2 = k$, $(u)_3 = i$, $(u)_4 = j$,
and $(u)_h = 0$ for $5 \le h \le e$

$\psi(X_i^k \in x_j^{k+1})$ = u if $(u)_1 = 3$, $(u)_2 = k$, $(u)_3 = i$, $(u)_4 = j$,
and $(u)_h = 0$ for $5 \le h \le e$

$\psi(Q_i(x_{j_1}^{a_{i1}}, \ldots, x_{j_{k_i}}^{a_{ik_i}}))$ = u if $(u)_1 = 4+i$, $(u)_{1+h} = j_h$ for
$1 \le h \le k_i$, $(u)_h = 0$ for $k_i \le h \le e$

$\psi(\sim A)$ = u if $(u)_1 = 0$, $(u)_2 = \psi(A)$ and $(u_h) = 0$ for $3 \le h \le e$

$\psi(A \to B)$ = u if $(u)_1 = 1$, $(u)_2 = \psi(A)$, $(u)_3 = (B)$, $(u)_h = 0$
for $4 \le h \le e$

$\psi(\forall x_i^k A)$ = u if $(u)_1 = 4$, $(u)_2 = k$, $(u)_3 = i$, $(u)_4 = \psi(A)$,
$(u)_h = 0$ for $5 \le h \le e$

From the fact that for $u \in R$ and $1 \le i \le e$, $(u)_i$ is unique, we see by an obvious induction on length of formulae that ψ is one-to-one. From (58) - (63) we can prove by induction on ht(A) that $\psi(A)$ is defined for every formula A of L. (For example, (58) tells us that for numbers i,j,and k with $1 \le k \le n$, there is a $u \in R$ such that ht(u) = 0, $(u)_1 = 2$, $(u)_2 = i$, $(u)_4 = k$, and $(u)_h = 0$ for $5 \le h \le e$; thus $\psi(x_i^k = x_j^k)$ is defined. ...Assuming that $\psi(B)$ is defined for all formulae B with ht(B)$\le$r, then (63) tells us that for numbers k,i and formula A with $ht(A) = r$ and $1 \leq k \leq n$ there is a $u \in R$ such that $ht(u) = r+1$, $(u)_1 = 4$, $(u)_2 = k$, $(u)_3 = i$, $(u)_4 = \psi(A)$, and $(u)_h = 0$ for $5 \leq h \leq e$; thus $\psi(\forall x_i^k A)$ is defined and $ht(\forall x_i^k A) = r+1$. ...) By (64) and (65) it can be proved by induction on $ht(u)$ that every $u \in R$ is $\psi(A)$ for some formula A. Therefore ψ is a one-to-one correspondence from the formulae of L onto R such that if $u = \psi(A)$ then $ht(u) = ht(A)$. So $\mathcal{W}$ is isomorphic to some $\mathcal{W}'$ satisfying (8) and (9).

(10). By (66) and (67), each u in D (= V(D) for arbitrary V on $\mathcal{W}$) has exactly one $<_{12}$-predecessors in each U_j^i ($1 \leq i \leq n$, $0 \leq j$) and no other $<_{12}$-predecessors, so v may be associated with the assignment Λ_v on $\mathcal{S}$ defined by $\Lambda_v(x_j^i) = u$ iff there is u' in U_j^i such that $u' <_{12} v$ and $u' <_9 u$. Thus $\Lambda_v(x_j^i) \in U_0^i$ (which may be thought of as $\mathcal{P}^{i-1}(S)$) as we would expect. (68) tells us that for each assignment Λ on $\mathcal{S}$ there is $v \in D$ such that $\Lambda = \Lambda_v$ and (69) tells us that such a v is unique. Hence $\mathcal{W}$ is isomorphic to some $\mathcal{W}'$ satisfying (10).

(11). (70) and (71) tell us that $<_{13} \subset D \times R$. If we make the identification indicated in the previous paragraphs (D with assignments on $\mathcal{S}$, R with formulae of L, U_0^i with $\mathcal{P}^{i-1}(S)$, etc.) then the following hold. If V is any assignment on $\mathcal{W}$, $A_1, A_2, \ldots$ formulae in T_h, and $u_1, u_2, \ldots \in W$ such that $V(A_i) = \{u_i\}$ for $i \geq 1$, then $V(\text{"}Q_i\text{"}(A_1,\ldots,A_{k_i})) = W$ or ϕ according as, for $1 \leq j \leq k_i$, $u_j \in U_0^{a_{ij}}$ and $(u_1,\ldots,u_{k_i}) \in \mathcal{Q}_i$ or not;

$V(A_i "=" A_j) = W$ or ϕ according as $u_i = u_j$ or not; $V(A_i "\in" A_j) = W$ or ϕ according as for some h $(1 \leq h \leq n-1)$ $u_i \in U_0^h$, $u_j \in U_0^{h+1}$, and $u_i \in u_j$ or not; $V(Stfy(A_i, A_j)) = W$ or ϕ according as u_i is an assignment Λ_{u_i} on $\mathcal{S}$, u_j is a formula of L and $u_j <_{13} u_i$ or not. Also $V({}_{A_i}X^{A_j}_{A_h}) = \Lambda_{u_i}(x^{u_j}_{u_h})$ if u_i is an assignment Λ_{u_i} on $\mathcal{S}$, u_j and $u_h \in \mathbb{N}$ with $1 \leq u_j \leq n$, and ϕ otherwise, and $V(\tau_j(A_i)) = \tau_j(u_i)$ if u_i is a formula of L, and ϕ otherwise. In this light we see that by induction on formulae, (72) - (77) tell us that if $u_i \in D$ and $u_j \in R$ then $u_j <_{12} u_i$ if and only if the formula u_j is satisfied by the assignment Λ_{u_i} on $\mathcal{S}$. (For example, if u_j is $x^{h_1}_{h_2} = x^{h_1}_{h_3}$ then by (72) $u_j <_{12} u_i$ iff ${}_{u_i}(x^{h_1}_{h_2}) = {}_{u_i}(x^{h_1}_{h_3})$ ... assuming that the claim holds for all subformulae of u_j and that u_j is $\forall x^{h_1}_{h_2} A$, then by (77) $u_j <_{12} u_i$ iff for every assignment Λ on $\mathcal{S}$ which agrees with u_i on all variables other than $x^{h_1}_{h_2}$, Λ satisfies A. ...) Hence, $\mathcal{W}$ is isomorphic to some $\mathcal{W}'$ satisfying (11).

(78) ensures that W is, in fact, the union of the pairwise disjoint sets U, N, R, D, and $K_1, \ldots, K_m$.

We shall write $\mathcal{S} \underset{\mu}{\veebar} \mathcal{W}$ if $\mathcal{W}$ is isomorphic to some $\mathcal{W}'$ which satisfies (1) - (11) with respect to $\mathcal{S}$. We have shown that if $\mathcal{W} \models C_\mu$ then $\mathcal{S} \underset{\mu}{\veebar} \mathcal{W}$ for some $\mathcal{S}$. Conversely, given a structure $\mathcal{S}$ in $\mathcal{A}$, there are structures $\mathcal{W}$ such that $\mathcal{S} \underset{\mu}{\veebar} \mathcal{W}$ and they are simply those structures, or structures isomorphic to those, which satisfy (1) - (11) with respect to S.

The claim is proved

Define the transformation φ from formulae of L to formulae of T_h as follows.

$$\varphi(x_i^k = x_j^k) = R \wedge F_{11,1}\{2\} \wedge F_{11,2}\{k\} \wedge F_{11,3}\{i\} \wedge F_{11,4}\{j\} \wedge \bigwedge_{h=5}^{e} (F_{11,h}\{0\})$$

$$\varphi(x_i^k \in x_i^{k+1}) = R \wedge F_{11,1}\{2\} \wedge F_{11,2}\{k\} \wedge F_{11,3}\{i\} \wedge F_{11,4}\{j\} \wedge \bigwedge_{h=5}^{e} (F_{11,h}\{0\})$$

$$\varphi(Q_i(x_{j_1}^{a_{i1}}, \ldots, x_{j_{k_i}}^{a_{ik_i}})) = R \wedge F_{11,1}\{4+i\} \wedge \bigwedge_{h=2}^{k_i+1} (F_{11,h}\{j\}) \wedge \bigwedge_{h=k_i+2}^{e} (F_{11,h}\{0\})$$

$$\varphi(\sim A) = R \wedge F_{11,1}\{0\} \wedge F_{11,2}\varphi(A) \wedge \bigwedge_{h=3}^{e} (F_{11,h}\{0\})$$

$$\varphi(A \to B) = R \wedge F_{11,1}\{1\} \wedge F_{11,2}\varphi(A) \wedge F_{11,3}\varphi(B) \wedge \bigwedge_{h=4}^{e} (F_{11,h}\{0\})$$

$$\varphi(\forall x_i^k A) = R \wedge F_{11,1}\{4\} \wedge F_{11,2}\{k\} \wedge F_{11,3}\{i\} \wedge F_{11,4}\varphi(A) \wedge \bigwedge_{h=4}^{e} (F_{11,h}\{0\})$$

and let

$$T_\mu(A) = L(D \to P_{13}\varphi(A)).$$

If $S \underset{\mu}{\nabla} \mathcal{W}$ then by (8) - (11) $S \models A$ if and only if $\mathcal{W} \models T_\mu(A)$. Let $\mathcal{C}_\mu = \{C_\mu\}$ and let $\mathcal{D}_\mu = \phi$. Then we have shown that $\mu = (\underset{\mu}{\nabla}, T_\mu, \mathcal{C}_\mu, \mathcal{D}_\mu)$ is a reduction of (L, A) to (T_h, R_h).

<u>Ap.2</u> For $n \geq 2$ we shall construct a reduction ν_n of (T_n, R_n) to (T_{n-1}, R_{n-1}).

Let $\mathcal{W} = (W, <_1, \ldots, <_n)$ be a structure in R_n. Then there is a structure $\mathcal{W}' = (W', <_1', \ldots, <_{n-1}')$ in R_{n-1} such that W' is the union of disjoint nonempty sets E, U, U' and conditions

(1) - (9) below are satisfied.

(1) $U = W$.

(2) $<'_1 \cap (E\times W') = \phi$.

(3) $<'_1 \cap (U\times E)$ is a one-to-one correspondence from U onto E.

(4) $<'_1 \cap (U'\times E) = $.

(5) $<'_1 \cap (U'\times U)$ is a one-to-one correspondence from U' onto U. If $u \in U$ then we write $v = \bar{u}$ if $v \in U'$ and $v <'_1 u$.

(6) $<'_1 \cap (U\times U) = <_1$. (Recall that $U = W$ and $<_1 \subset W\times W$)

(7) $<'_1 \cap (U\times U') = \phi$.

(8) If $u',v' \in U'$, then $u' <'_1 v'$ if and only if $u' = \bar{u}$, $v' = \bar{v}$ for $u,v \in U$, and $u <_n v$ in $\mathcal{W}$.

(9) For $2\leq i\leq n-1$, $<'_i \subset U\times U$ ($= W\times W$) and $<'_i = <_i$.

Let C_{ν_n} be the conjunction of (10) - (18) below.

(Def.) $E = G_1(p\wedge\sim p)$

(Def.) $U = F_1E$

(Def.) $U' = \sim E\wedge\sim U$

(10) $E \rightarrow P_1U$

(11) $E\wedge P_1p \rightarrow H_1p$

(12) $U\wedge F_1(E\wedge p) \rightarrow G_1(E\rightarrow p)$

(13) $U \rightarrow P_1U$

(14) $U\wedge P_1(U'\wedge p) \rightarrow H_1(U'\rightarrow p)$

(15) $U' \rightarrow F_1U$

(16) $U' \wedge F_1(U \wedge p) \rightarrow G_1(U \rightarrow p)$

(17) $U' \rightarrow \sim P_1 U$

(18) $\bigvee_{i=2}^{n-1} (F_i p \vee P_i p) \rightarrow U$

Let $\mathcal{W}' \models C_{\nu_{\nu_n}}$.

<u>Claim</u>: $\mathcal{W}'$ satisfies (1) - (9) with respect to some $\mathcal{W} \in \mathcal{R}_n$.

Proof: For any assignment V on $\mathcal{W}'$, $u \in V(E)$ if and only if there is no $v \in W$ such that $u <_1' v$; thus V(E) is independent of V and we shall refer to the set V(E) as simply E. $u \in V(U)$ if and only if there is $v \in E$ such that $u <_1 v$, thus V(U) is independent of V and shall be called simply U. $V(U') = W - (V(E) \cup V(U)) = W' - (E \cup U)$ and shall be called U'. Clearly W' is the union of the disjoint sets E, U, U'.

(2) and (4) follow from the definitions of E, U, and U' above.

(3) follows from the definition of U and from (10) - (12). The definition of U ensures that each point in U has a successor in E and (10) that it has at most one. (11) ensures that each point in E has a predecessor in U and (12) that it has at most one.

(5) follows from (13) - (16) in a similar way.

(7) follows from (17).

(18) ensures that for $2 \leq i \leq n-1$, $<_i' \subset U \times U$.

To satisfy (1), (6), (8), and the rest of (9), we let

$W = U$, let $<_i$ = the restriction of $<'_i$ to $U \times U$ for $1 \le i \le n-1$ and for $u,v \in W$ (= U), let $u <_n v$ if and only if $\bar{u} <'_1 \bar{v}$.

It remains to show only that U, U′, and E are nonempty and that $\mathcal{W}$ is connected.

There are one-to-one correspondences between U, U′, and E, which together comprise W. If one of U, U′, E were empty, so would the other two be and hence W′ would be empty; an impossibility.

Let $u,v \in W$ (= U). We must show that u and v are connected in $\mathcal{W}$; we know that they are connected in $\mathcal{W}'$. Let $u=u_1, u_2, \ldots, u_{m-1}, u_m=v$ be a sequence of points in W′

such that for $1 \le i \le n-1$, u_i is directly connected to u_{i+1} via one of $<'_1, \ldots, <'_{n-1}$. We shall obtain a new sequence of points in W (= U) such that each point in the new sequence is directly connected to its successor via one of $<_1, \ldots, <_n$, by starting at the beginning of the existing sequence and replacing portions of it by new portions according to the following procedure.

Let U_i be the first point in the sequence not already dealt with. We can assume that $u_i \in U$ (= W). If i = 1 this will certainly be true. If u_i is directly connected to u_{i+1} via $<'_j$ where $2 \le i \le n-1$ then $u_{i+1} \in U$ and u_i and u_{i+1} are directly connected via $<'_j$. Do nothing to u_i in the sequence. The first point now not dealt with in the sequence is u_{i+1}. If u_i is directly connected to u_{i+1} via $<'_1$ we consider several subcases.

(i) $u_{i+1} \in U$. In this case u_i and u_{i+1} are directly connected via $<_1$ and so we do nothing to u_i in the sequence. The first point now not dealt with u_{i+1}.

(ii) $u_{i+1} \in E$. In this case the only point directly connected to u_{i+1} is u_i and so u_{i+2} is the same point as u_i. Omit u_i and u_{i+1} from the new sequence. The first point now not dealt with is u_{i+2}.

(iii) $u_{i+1} \in U'$. In this case we must have $u_{i+1} = \overline{u_i}$. Let u_j be the next point in the sequence (after u_i) in U. Then $u_{i+1}, \ldots, u_{j-1}$ must all be in U'. Let $v_{i+1}, \ldots, v_{j-1}$ be the points in U such that $u_{i+1} = \overline{v_{i+1}}, \ldots , u_{j-1} = \overline{v_{j-1}}$. Each of $u_{i+1}, \ldots, u_{j-1}$ must be directly connected to its successor via $<_1'$, thus each of $v_{i+1}, \ldots, v_{j-2}$ is directly connected to its successor in $\mathcal{W}$ via $<_n$ and we must have $u_{j-1} = \overline{u_j}$, thus $v_{j-1} = u_j$. So we replace the portion $u_1, \ldots, u_{j-1}$ with $v_{i+1}, \ldots, v_{j-2}$ and the first point now not dealt with is u_j.

When $i = m$, then $u_i = v$ and our new sequence is constructed.

Thus u and v are connected in $\mathcal{W}$ and so $\mathcal{W}$ is connected. The claim is proved.

Now, if $\mathcal{W} \in \mathcal{R}_n$ and if $\mathcal{W}''$ is isomorphic to some $\mathcal{W}'$ in $\mathcal{R}_{n-1}$ which satisfies (1) - (9) with respect to some $\mathcal{W}$, we write $\mathcal{W} \underset{\sim}{\nabla}_n \mathcal{W}''$. It is clear that if $\mathcal{W}$ and $\overline{\mathcal{W}}$ are isomorphic in $\mathcal{R}_n$ and if $\mathcal{W}'$ and $\overline{\mathcal{W}}'$ satisfy (1) - (9) with respect to

$\mathcal{W}$ and $\overline{\mathcal{W}}$ respectively, then $\mathcal{W}'$ and $\overline{\mathcal{W}}'$ are isomorphic in $\mathcal{R}_{n-1}$. Hence, $\mathcal{V}_{\nu_n}$ is well-defined on isomorphism classes of structures.

Define a transformation ψ from formulae of T_n to formulae of T_{n-1} by

$\psi p_i = p_i$ for each variable p_i

$\psi(\sim A) = \sim(U \wedge \sim A)$

$\psi(A \to B) = \psi A \to \psi B$

$$\left.\begin{aligned}\psi(G_i A) &= G_i(U \to \psi A)\\ \psi(H_i A) &= H_i(U \to \psi A)\end{aligned}\right\} \quad (1 \leq i \leq n-1)$$

$\psi(G_n A) = H_1(U' \to G_1(U' \to G_1(U \to \psi A)))$

$\psi(H_n A) = H_1(U' \to H_1(U' \to G_1(U \to \psi A)))$

Now, for a formula A of T_n, let

$$T_{\nu_n} A = U \to \psi A.$$

We must show that if $\mathcal{W} \in \mathcal{R}_n$, $\mathcal{W}' \in \mathcal{R}_{n-1}$ and $\mathcal{W}\ \mathcal{V}_{\nu_n}\ \mathcal{W}'$, then for a formula A of T_n

$$\mathcal{W} \models A \text{ if and only if } \mathcal{W}'' \models T_{\nu_n} A$$

Let U, U', and E be subsets of W' as before. We know that W is in some sense isomorphic to U and so shall assume that W = U. Now as V runs through all valuations on $\mathcal{W}'$, $V' = V|_W$ (the restriction on V to W) runs through all valuations on $\mathcal{W}$ (probably more than once, in fact). It suffices, then, to show that for each $u \in W' - U$, $V(T_{\nu_n} A, u) = T$ for all V on $\mathcal{W}$; and for each $u \in U$ (= W), $V(T_{\nu_n} A, u) = V'(A, u)$, for all V on $\mathcal{W}'$. If $u \notin U$ then $V(U, u) = F$ and so

$V(T_{\nu_n}A,u) = V(U\rightarrow\psi A,u) = T$. We show by induction on the length of A that for $u \in U$ $(= W)$, $V(T_{\nu_n}A,u) = V'(A,u)$ for all V on $\mathcal{W}'$.

If A is a variable p_i then $V(T_{\nu_n}A,u) = V(U\rightarrow p_i,u) = V(p_i,u)$ (since $V(U,u) = T$) $= V'(A,u)$. Now assume that the result holds for all formulae shorter than A.

If A is $\sim B$ then $V(T_{\nu_n}A,u) = V(U\rightarrow\psi A,u) = V(U\rightarrow\sim(U\wedge\psi B),u) = V(\sim\psi B,u) = V(\sim(U\rightarrow\psi B),u) = V(\sim T_{\nu_n}B,u) = V'(\sim B,u) = V'(A,u)$. If A is $B\rightarrow C$ then $V(T_{\nu_n}A,u) = V(U\rightarrow(\psi B\rightarrow\psi C),u) = V(\psi B\rightarrow\psi C,u)$. This is T iff either $V(\psi B,u) = F$ or $V(\psi C,u) = T$, which happens if and only if either $V(T_{\nu_n}B,u) = F$ or $V(T_{\nu_n}C,u) = T$, which happens iff either $V'(B,u) = F$ or $V'(C,u) = T$, which happens iff $V'(A,u) = T$.

If A is G_iB $(1\leq i\leq n-1)$ then $V(T_{\nu_n}A,u) = V(\psi A,u) = V(G_i(U\rightarrow\psi B),u) = T$ iff $V(U\rightarrow\psi B,v) = T$ for all v such that $u <_i' v$. But $V(U\rightarrow\psi B,v) = V(T_{\nu_n}B,v)$, and if $v \in U$, this, by the induction hypothesis, equals $V'(B,v)$, and if $v \notin U$ then $V(U\rightarrow\psi B,v) = T$. And so $V(U\rightarrow\psi B,v) = T$ for all v such that $u <_i' v$ iff $V'(B,v) = T$ for all v in W (= U) such that $u <_i v$. Therefore $V'(A,u) = V'(T_{\nu_n}A,u)$. Similarly, $V(T_{\nu_n}H_iB,u) = V'(H_iB,u)$ $(1\leq i\leq n-1)$.

Now, if A is G_nB then $V(T_{\nu_n}A,u) = V(\overline{\psi}A,u) = V(H_1(U'\rightarrow G_1(U'\rightarrow G_1(U\rightarrow\psi B))),u)$. This is T if and only if $V(G_1(U'\rightarrow G_1(U\rightarrow\psi B)),\overline{u}) = T$; if and only if for all $\overline{v} \in U$ such that $\overline{u} <_1' \overline{v}$, $V(G_1(U\rightarrow\psi B),\overline{v}) = T$; if and only if for all

$v \in U$ such that $u <_n v$ in $\mathcal{W}$, $V(T_{\nu_n} B, v) = T$; by induction hypothesis, if and only if $V'(B,v) = T$ for all such v; and hence, if and only if $V'(A,u) = V'(G_n B, u) = T$. Finally, $V(T_{\nu_n} H_n B, u) = V'(H_n B, u)$ follows similarly.

Let $\mathcal{C}_{\nu_n} = \{C_{\nu_n}\}$ and $\mathcal{D}_{\nu_n} = \phi$. Then we have shown that for $n \geq 2$, $\nu_n = (\mathcal{V}_{\nu_n}, T_{\nu_n}, \mathcal{C}_{\nu_n}, \mathcal{D}_{\nu_n})$ is a reduction of (T_n, R_n) to (T_{n-1}, R_{n-1}).

<u>Ap.3</u> Finally, we construct a reduction ν_1 (which we shall usually call simply ν) of (T_1, R_1) to $(M, \overline{R})$.

Let $\mathcal{W} = (W, <)$ be a structure in R_1. Then there is a superconnected structure $\mathcal{W}' = (W', <')$ in $\overline{R}$ such that W' is the union of disjoint nonempty sets U, U', and $\{e\}$ and conditions (1) - (8) below are satisfied.

(1) $U = W$.

(2) $<' \cap (\{e\} \times W') = \phi$.

(3) $<' \cap (U' \times \{e\}) = \phi$.

(4) $<' \cap (U \times U')$ is a one-to-one correspondence from U onto U'. If $u \in U$ then we write $v = \overline{u}$ if $v \in U'$ and $u <' v$.

(5) $<' \cap (U' \times U) = [<' \cap (U \times U')]^{-1}$.

(6) $<' \cap (U \times \{e\}) = U \times \{e\}$.

(7) $<' \cap (U \times U) = <$. (Recall that $U = W$ and $< \subset W \times W$.)

(8) If $u, v \in U$ then $\overline{u}, \overline{v} \in U'$ and we have $\overline{u} <' \overline{v}$ if and only if $v <' u$.

Let C be the conjunction of (9) - (18) below.

(Def.) $e = \Box(p\wedge{\sim}p)$

(Def.) $U = \Diamond e$

(Def.) $U' = {\sim}e\wedge{\sim}U$

(9) $U \to \Diamond U'$

(10) $U\wedge\Diamond(U'\wedge p) \to \Box(U'\to p)$

(11) $U' \to \Diamond U$

(12) $U'\wedge\Diamond(U\wedge p) \to \Box(U\to p)$

(13) $U\wedge p \to \Box(U'\to\Diamond p)$

(14) $U'\wedge p \to \Box(U\to\Diamond p)$

(15) $U\wedge p\wedge\Diamond(U\wedge q) \to \Diamond(q\wedge\Diamond U'\wedge\Diamond(U'\wedge\Diamond p)))$

(16) $U'\wedge p\wedge\Diamond(U'\wedge q) \to \Diamond(q\wedge\Diamond U\wedge\Diamond(U\wedge\Diamond p)))$

(17) $\Diamond(e\wedge p) \to \Box(e\to p)$

(18) $\Diamond(e\wedge p) \to \Box\Box(e\to p)$

Let $\mathcal{D}_\nu = \{e\}$.

<u>Claim</u>: If $\mathcal{W}' = (W',<') \in \bar{R}$, $\mathcal{W}' \models C_\nu$ and $\mathcal{W}' \not\models e$, then $\mathcal{W}'$ satisfies (1) - (8) with respect to some $\mathcal{W} \in R_1$.

Proof: It is easy to see from the definitions of e, U, and U that V(e), V(U) and V(U′) are independent of the assignment V, and so, as before, we shall call these three sets of points simply E, U, and U , respectively. It is clear from the definitions that they are disjoint and that W′ is their union.

The fact that $\mathcal{W}' \not\models e$ ensures that either U or U′ is nonempty. If $U \neq \phi$ then (9) ensures that also $U' \neq \phi$. If $U' \neq \phi$ then (11) ensures that also $U \neq \phi$. So both U and U′ are nonempty. But then the definition of the formula U ensures that E is nonempty.

(9) and (10) ensure that $<' \cap (U \times U')$ is a function from U into U′; (11) and (12) that $<' \cap (U' \times U)$ is a function fram U′ into U. Call the first function f and the second g. Then (13) ensures that for all $u \in U$, $g(f(u)) = u$; (14) that for all $v \in U$, $f(g(v)) = v$. Thus $g = f^{-1}$ and so f is a one-to-one function from U onto U′. (4) and (5) are therefore both satisfied.

Suppose $u,v \in U$ and $u <' v$. Then (15) ensures that in U , $\overline{v} <' \overline{u}$. Suppose, on the other hand, that in U′, $\overline{v} <' \overline{u}$. Then (16) ensures that in U, $u <' v$. Thus (8) is satisfied.

We shall now show that E is a singleton. We already know that $E \neq \phi$. Suppose that E is not a singleton. Then we shall show that there are distinct elements e and e′ in E such that there are points $u,v \in U$ with $u <' e$, $v <' e'$ and either $u <' v$ or $v <' u$.

Since $\mathcal{W}'$ is superconnected it is surely connected. Let $e=u_1, u_2, \ldots , u_m=e' \neq e$ be a sequence of points in W′ such that each is directly connected to the next via $<'$. We can without loss of generality assume that $u_2,\ldots,u_{m-1}$

$\notin$ E, for otherwise redesignate e′ to be the first point in the sequence $\neq$ e but in E, and remove all of the sequence before the last occurrence of e preceeding this new e′, and after this new e′. If not all of $u_2, \ldots, u_{m-1}$ are in U then we can replace this sequence by a similar one in which they are all in U according to the following inductive procedure.

u_2 must be in U since $u_1 \in$ E and so we leave u_2 as it is.

Let u_i be the first point in the sequence not yet dealt with. If $u_i \in$ U then leave it as it is in the sequence and u_{i+1} is now the first point not dealt with. If $u_i \in U'$ then $u_i = \overline{u_{i-1}}$. Let u_j be the next point after u_i that is in U. Then $u_i, \ldots, u_{j-1}$ are all in U′ and are directly connected via <′. Thus there are $v_i, \ldots, v_{j-1}$ in U such that they are all directly connected via <′, $u_i = \overline{v_i}, \ldots, u_{j-1} = \overline{v_{j-1}}$, and $v_i = u_{i-1}$ and $v_{j-1} = u_j$. Then replace $u_i, \ldots, u_j$ with $v_{i+1}, \ldots, v_{j-1}$. The first point not now dealt with is u_{j+1}.

Now we can assume that $u_2, \ldots, u_{m-1} \in$ U. The definition of the formula U and (17) ensure that each of $u_2, \ldots, u_{m-1}$ have exactly one <′- successor in E. Clearly the <′-successor of u_2 is e since u_2 is directly connected via <′ to u_1 (= e). Let u_i now be the first point in the sequence whose <′-successor in E is not e. Such a point must exist since the <′-successor in E of u_{m-1} is e′. Now redesignate

e' to be the successor in E of u_i if it is not already. Let $u = u_{i-1}$ and $v = u_i$. Then we have $u <' e$, $v <' e'$, and either $u <' v$ or $v <' u$. Without loss of generality assume $u <' v$.

Now let V be an assignment on $\mathcal{W}'$ such that $V(p) = \{e\}$. Then $V(\Diamond(e \wedge p), u) = T$ but $V(\Box\Box(e \rightarrow p), u) = F$ since $V(e, e') = T$ but $V(p, e') = F$. However this contradicts our original assumption that (18) is valid in $\mathcal{W}'$. So E must be a singleton after all. Let $E = \{e\}$.

(2), (3), and (6) now follow from the definitions of e, U, and U′.

If we let $W = U$ and $< \; = \; <'|_{W\,W}$ then (1) and (7) are satisfied. It remains only to show that $\mathcal{W}$ is connected.

Let $u, v \in W$ (= U). Then since $\mathcal{W}'$ is superconnected there is a sequence $u=u_1, \ldots, u_m=v$ of points in W′ such that each is directly connected to the next via $<'$ and such that none of $u_2, \ldots, u_{m-1}$ is e. But then, as we have seen before, we can reconstruct this sequence so that all the $u_2, \ldots, u_{m-1}$ are in U. (This reconstruction is exactly the same as the corresponding one in the preceeding section.) Thus $u=u_1, \ldots, u_m=v$ is a sequence of points in W such that each is directly connected to the next via $<$. So $\mathcal{W}$ is connected.

Let $\mathcal{W} \; \mathfrak{y} \; \mathcal{W}'$ if $\mathcal{W}' \in \overline{\mathcal{R}}$ is isomorphic to a structure which satisfies (1) – (8) with respect to $\mathcal{W} \in \mathcal{R}_1$. Clearly, if $\mathcal{W}'$ is isomorphic to a structure which satisfies (1) – (8) with respect to $\mathcal{W}$

and is also isomorphic to a structure which satisfies (1) - (8) with respect to $\mathcal{W}''$, then $\mathcal{W}$ and $\mathcal{W}''$ are isomorphic. So $\underset{\nu}{\nabla}$ is well-defined on isomorphism classes.

Define a transformation ψ from formulae of T_1 to formulae of M by

$\psi p_i = p_i$ for all propositional variables p_i

$\psi(\sim A) = \sim(U \wedge \psi A)$

$\psi(A \to B) = \psi A \to \psi B$

$\psi(G_1 A) = \Box(U \to \psi A)$

$\psi(H_1 A) = \Box(U' \to \Box(U' \to \Box(U \to \psi A)))$

Now, for a formula A of T_1, let

$$T_\nu A = U \to \psi A.$$

We must show that if $\mathcal{W} \in \mathrm{R}_1$, $\mathcal{W}' \in \overline{\mathrm{R}}$ and $\mathcal{W} \underset{\nu}{\nabla} \mathcal{W}'$ then for a formula A in T_1

$$\mathcal{W} \models A \text{ if and only if } \mathcal{W}' \models T_\nu A.$$

The proof is almost exactly word for word that of the corresponding property in the reduction ν_n and so won't be repeated.

Finally, we let $\mathcal{C}_\nu = \{C_\nu\}$. Recall that $\mathcal{D}_\nu = \{e\}$.

Then $\nu = (\underset{\nu}{\nabla}, T_\nu, \mathcal{C}_\nu, \mathcal{D}_\nu)$ is a reduction of $(\mathrm{T}_1, \mathrm{R}_1)$ to $(\mathrm{M}, \overline{\mathrm{R}})$..

If we let σ be the composite $\nu \circ \nu_2 \circ \cdots \circ \nu_h \circ \mu$, then σ is the desired reduction of (L, A) to $(\mathrm{M}, \overline{\mathrm{R}})$.

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