SOME RESULTS

ON

ENUMERATION REDUCIBILITY

by

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ABSTRACT

Enumeration reducibility was defined by Friedberg and Rogers in 1959. Medvedev showed that there are partial degrees which are not total. Rogers in his book *Theory of Recursive Functions and Effective Computability* gives all the basic results and definitions concerning enumeration reducibility and the partial degrees. He mentions in this book that the existence of a minimal partial degree is an open problem. In this thesis it is shown that there are no minimal partial degrees. This leads naturally to the conjecture that the partial degrees are dense. This thesis leaves this question unanswered, but it is shown that there are no degrees minimal above a total degree, and there are at most countably many degrees minimal above a non-total degree. J. W. Case has proved several results about the partial degrees. He conjectured that there is no set in a total degree whose complement is in a non-total degree. In this thesis that conjecture is disproved. Finally, Case's result that there is a minimal pair of partial degrees is strengthened to show that there is a minimal pair of partial degrees which are total and form a minimal pair of r.e. degrees.

ACKNOWLEDGEMENTS

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CHAPTER I

INTRODUCTION AND TERMINOLOGY

§1.1 INTRODUCTION

In this thesis we consider the partial (enumeration) degrees as defined by Friedberg and Rogers [2]. In Chapter II we disprove a conjecture by Case [1] that there is no set A in a total degree whose complement is in a non-total degree. Case [1] has shown that there is a minimal pair of partial degrees. In Chapter III we strengthen this result to show that there is a pair of co-r.e. sets whose partial degrees form a minimal pair of partial degrees; and hence their Turing degrees form a minimal pair of r.e. degrees. In Chapter IV we show that there are no minimal partial degrees and demonstrate that this result relativizes to show that there are no degrees minimal above a total degree. This still leaves open the question as to whether the partial degrees are dense. In Chapter V we move closer to a solution of the density problem by showing that given any partial degree $\underline{a}$ there can be at most a countable number of partial degrees minimal above $\underline{a}$.

§1.2 TERMINOLOGY

In this section we intend to outline all the basic notation and definitions used in this thesis. For a detailed explanation and study of recursion theory the reader is referred to Rogers [5].

We will denote subsets of N by upper case letters, with D, E, F

being reserved for finite subsets. Members of N will be represented

by lower case letters except for f, g, h which are reserved for total

functions. Partial functions (that is, functions whose domain is a

subset of N) will be denoted by ψ and ϕ.

$\overline{A}$ will denote the complement of A, C_A will be used for the

characteristic function of A and A Δ B will denote the symmetric

difference of A and B. For notational convenience we will sometimes

use X[n] to denote $C_X(n)$. We will call a finite initial segment of

a characteristic function an *initial function*. A join B is the set

$$\{2x : x \in A\} \cup \{2x + 1 : x \in B\}.$$

We define the binary function τ by,

$$\tau(x, y) = \frac{1}{2}(x^2 + 2xy + y^2 + 3x + y),$$

and τ is a recursive, one to one mapping of N $\times$ N onto N (see

Rogers [5, p.64]). For each k we define the k-ary function τ^k as

follows:

$$\tau^1 = \lambda x [x]$$

$$\tau^{k+1} = \lambda x_1 \ldots x_k [\tau(\tau^k(x_1, \ldots, x_k), x_{k+1})].$$

We abbreviate $\tau^k(x_1, \ldots, x_k)$ by $\langle x_1, \ldots, x_k \rangle$. For each k we

define the projection functions π_i^k, $1 \le i \le k$, by

$$\pi_i^k(\tau^k(x_1, \ldots, x_k)) = x_i, \quad 1 \le i \le k.$$

We will usually drop the superscript on π_i^k when its value is clear from the context.

A set B is *single-valued* if

$$\langle n, j\rangle \in B \text{ and } \langle n, k\rangle \in B \to j = k.$$

A set B is *total* if for all n there is a m such that $\langle n, m\rangle \in B$. If D is finite, i.e. $D = \{x_1, \ldots, x_k\}$, $x_1 < x_2 < \ldots < x_k$, then the *canonical index* of D is $2^{x_1} + \ldots + 2^{x_k}$, if $D = \emptyset$ we let its canonical index be 0. We denote by D_i the finite set whose canonical index is i. We will often make no distinction between a finite set and its canonical index, as in the use of $\langle E, x\rangle$ for $\langle i, x\rangle$, where $D_i = E$.

We will assume that the reader is familiar with the *recursive partial functionals*, (see [6, p.150]).

We let $\langle \Theta_e\rangle_{e=0}^{\infty}$ be an effective enumeration of the recursive partial functionals. For each e we denote the eth-*partial recursive function* $\lambda n[\Theta_e(C_\emptyset, n)]$ by ϕ_e. The function f is *recursive* if for some e, ϕ_e is total and $f(n) = \phi_e(n)$ for all n. A set is *recursive* just if its characteristic function is recursive. A set is *recursively enumerable* if it is the range of some partial recursive function. We denote the range of ϕ_e by W_e. We say f is *Turing-reducible* to g ($f \le_T g$) if $f = \lambda n[\Theta_e(g, n)]$ for some e. We extend this reducibility to sets by defining $A \le_T B$ if $C_A \le_T C_B$. Two functions f and g

are *Turing equivalent* (f $\equiv_T$ g) if f $\leq_T$ g and g $\leq_T$ f. It can be shown

that $\equiv_T$ is an equivalence relation (see [5, p.137]). The equivalence

classes under $\equiv_T$ of total functions (or sets by considering their

characteristic function) are called the *Turing-degrees*. A Turing-degree

that contains an r.e. set is called an *r.e. degree*. We will use small

underlined letters for the Turing-degrees, for example $\underline{a}$, $\underline{b}$, $\underline{c}$. The

jump of a set A, denoted A', is the set

$$\{e : \Theta_e(C_A, e) \text{ is defined}\}.$$

The jump operator is well defined for the degrees and if $\underline{a}$ is the

degree of A then we denote by $\underline{a}'$ the degree of A'. We use $\underline{0}$ to

denote the smallest degree, that is the degree of all recursive functions.

We reserve the symbol K for the set $\{e : e \in W_e\}$ and the symbol K_0

for the set $\{<n, e> : n \in W_e\}$. Both K and K_0 are of degree $\underline{0}'$.

§1.3 ENUMERATION REDUCIBILITY

Enumeration reducibility is defined in Friedberg and Rogers [2]

and discussed in Rogers [5, p.146]. Intuitively, a set A is enumera-

tion reducible to B is there is an algorithm that will work on *any*

enumeration of B (the *input* enumeration) and produce an enumeration

of A (the *output* enumeration). The formal definitions are as follows:

1.3.1 W(A) = $\{x : \text{for some } D, D \subseteq A \text{ and } <D, x> \in W\}$.

1.3.2 A is *enumeration reducible* to B (A $\leq_e$ B) if there is an

r.e. set W such that A = W(B).

1.3.3 $A \equiv_e B$ if $A \leq_e B$ and $B \leq_e A$.

1.3.4 $A <_e B$ if $A \leq_e B$ and $B \not\leq_e A$.

Thus to every r.e. set W_e there is associated a function Φ_e, from 2^N to 2^N. We call this function an *enumeration operator*. In this thesis we shall not distinguish between an r.e. set W_e and its associated operator Φ_e, for example if Φ is the enumeration operator associated with the r.e. set W we will write $W(A)$ for $\Phi(A)$. Notice that the definition of $W(A)$ in 1.3.1 did not specify that W had to be r.e. In later chapters for purposes of relativization we shall sometimes use non-r.e. sets to operate on other sets.

It is easy to show that $\equiv_e$ is an equivalence relation on the subsets of N (see [5, p.153]). The equivalence classes are called *enumeration degrees*. One of the reasons for studying enumeration reducibility, other than the fact that it is a very natural relationship between enumerations, is that we can get a reduction between partial functions by defining

$$\psi \leq_e \phi \quad \text{if} \quad \tau(\psi) \leq_e \tau(\phi).$$

The equivalence classes of partial functions (or single-valued sets) are called *partial degrees*.

A partial degree is called *total* if it has as a member some total function f. We will use $\leq$ to denote the partial ordering of partial degrees and enumeration degrees induced by $\leq_e$. We will use underlined lower case letters to denote partial and enumeration degrees as well as

Turing degrees. We write $\underline{a} < \underline{b}$ if $\underline{a} \leq \underline{b}$ and $\underline{b} \nleq \underline{a}$. The structure obtained by restricting the partial degrees to the total partial degrees is order isomorphic to the Turing degrees (see [5, p.153]).

There is really no difference between the structure of the enumeration degrees under $\leq$ and the structure of the partial degrees under $\leq$. The map which takes ϕ to $\tau(\phi)$ induces an isomorphism between the partial degrees and the enumeration degrees. The inverse of this isomorphism is induced by a map which takes a set A into a constant function with domain A. Hence in this thesis the terms partial degree and enumeration degree will be used synonymously. We will use $\underline{0}$ to denote the smallest enumeration degree that is the degree of all r.e. sets (from a partial function viewpoint it is the degree of all partial recursive functions).

Because of the above isomorphism between the enumeration and partial degrees, and the isomorphic embedding of the Turing degrees into the partial degrees, we are justified in using common notation, such as $\leq$ and $\underline{0}$, for all three structures.

One interesting fact is that if A is r.e. then $\overline{A} \equiv_e C_A$. Hence the enumeration degrees of co-r.e. sets have a structure isomorphic to the r.e. degrees. Thus we will denote by $\underline{0}'$ the partial degree of $\overline{K}$.

For composition of operators we will often omit the parentheses, for example we will use $WV(B)$ for $W(V(B))$. Also we will often use lower case letters in parentheses where it would be more usual to use subscripts as in $V(k)$.

§1.4 INFINITE GAMES

Throughout this thesis we shall be using the method of infinite games as explained in Lachlan [4]. In this section we outline some of the basic ideas and terminology.

We consider games with two people, one called the *player* and the other the *opponent*. Each person enumerates a sequence of sets. That is each member of the player's sequence is a set that is enumerated during the game, and similarly for the opponent's sequence. To gain an intuitive feel for this method it is helpful to view these sets as receptacles into which numbers can be placed. The player and the opponent take alternate turns, with the opponent taking the first turn. For purposes of formal definition it is usual to insist that each person enumerate at most one number in one set during his turn. However in describing actual games we shall only keep the opponent to this restriction and allow the player any finite number of such actions in one turn (this does not make any change to the ability of the player to win a game). The game ends after ω turns.

We shall call the combined turns of the player and the opponent a *stage*. We number the stages starting with 0. The nth stage of the game is the portion of the game that begins at the start of the opponent's $n+1^{st}$ turn and ends when the player's $n+1^{st}$ turn is completed. If X is one of the sets being enumerated during the game we will use X^s to denote X as it appears at the end of stage s. We will normally use X to denote the set as it appears at the end of the game.

Occasionally, however, we will refer to X as a receptacle into which

numbers are placed, as in the phrase "the player puts n into X".

To specify a game we indicate the sets to be enumerated by the

player and those to be enumerated by the opponent. We also give a

recursively enumerable sequence of *requirements*. Each requirement

states a relationship between some of the sets being enumerated. For

example a requirement could be A ≠ B where the opponent is enumera-

ting A and the player is enumerating B. A requirement is said to

be *satisfied* if it holds at the end of the game, in our example

above the requirement would be satisfied it at the game's conclusion

the resulting set A is different from the resulting set B.   The

player is said to *win* the game if at the end of the game all the

requirements are satisfied.

After n stages of the game, n < ω, at most a finite number of

the sets being enumerated will be non-empty. A list of these sets

together with all the numbers in them and the stage each number was

put in is a *game situation*. A *strategy* is a map from game situations

into possible moves. A strategy S for the player is said to be

complete if no matter what strategy the opponent follows the player is

always able to follow S. A strategy S for the player is a *winning*

strategy if S is complete and every play of the game in which the

player follows S results in a win for the player. By encoding moves

and game situations into numbers we can consider a strategy as a

function from N to N. Thus *effective strategy*, *effective winning*

strategy and the *partial degree of a strategy* are defined.

Most of the games considered in this thesis will use recursion theory notation in their requirements. It is possible to rephrase these games in purely game theoretic terms and eliminate all notions special to recursion theory. Although this would emphasize Lachlan's result that recursion theory can be done by strictly game theoretic means [4], we have sacrificed this interesting point to make the results more concise.

In the discussions and proofs about the games we sometimes identify with the player and occasionally refer to the player's actions as our own.

All the strategies given for the games in this thesis are effective. We shall not prove this for any strategy as it is clear from their description.

CHAPTER II

A SET IN A TOTAL DEGREE WHOSE COMPLEMENT

IS IN A NON-TOTAL DEGREE

§2.1 INTRODUCTION

Case [1, p.426] has conjectured that there is no set A whose

partial degree is total and whose complement has a non-total partial

degree. In this chapter we will disprove this conjecture by showing

that there is a total function f, $\emptyset <_T f \leq_T K$, such that for all

single-valued B

$$B \leq_e \overline{\tau(f)} \quad \text{and} \quad B \text{ total} \; . \to . \; B \equiv_e \emptyset.$$

§2.2 DESCRIPTION OF THE GAME

Consider a game where the player defines a set A and enumerates

the sets $V(0)$, $V(1)$, ..., and where the opponent enumerates the sets

$W(0)$, $W(1)$, ...

The requirements are:

$R(-1)$: A is well defined, single valued, and total

$R(0)$: $W(0)(\overline{A})$ single valued and total $\to$ $W(0)(\overline{A}) \, \Delta \, V(0)$ finite

$R(1)$: $\overline{A} \neq W(0)$

R(2) : $W(1)(\overline{A})$ single valued and total $\rightarrow$ $W(1)(\overline{A}) \Delta V(1)$ finite

R(3) : $\overline{A} \neq W(1)$

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The opponent during his turn can put one number into one of his sets $W(0)$, $W(1)$, ... The player during his turn can add numbers to one of his sets $V(0)$, $V(1)$, ... The player is also allowed during his turn to remove a finite number of members of A and replace each one with another number.

We will allow the player to change the value of $C_A(n)$ only finitely many times for each n. This is equivalent to the player enumerating a series of initial functions, say $\langle C_A^s \rangle_{s=0}^{\infty}$, that converge to A. If at the beginning of the game A is recursive, the player's strategy is effective, and the opponent simultaneously enumerates all the r.e. sets in an effective manner then the Turing degree of A will be less than or equal to $\underline{0}'$.

We will assume that at the start of the game $A = \{\langle n, n \rangle : n \in N\}$. If all our changes to A that remove a member of the form $\langle n, m_1 \rangle$ also add a number of the form $\langle n, m_2 \rangle$ and occur below some stage $S(n)$ then the requirement $R(-1)$ will be satisfied.

Throughout the game the player will be putting labels on requirements. These are a "bookkeeping" device which make the strategy more concise. The labels persist until they are explicitly removed. The player also has an infinite number of labels, denoted 1-label, 2-label, ..., with which to label numbers. If a number becomes n-labelled at some

stage s then it is n-labelled for all stages t, t ≥ s.

§2.3 AN EFFECTIVE WINNING STRATEGY

At stage 0 all sets are empty except for A, which is
$\{<n, n> : n \in N\}$, and the player does nothing.

Strategy at stage s + 1:

1. If $\pi_1(s + 1) = 2p$ then

1.1. for each $x \in W^s(p)(\overline{A^s})$ the player puts x into V(p), and

1.2. if R(2p) is not labelled at the end of stage s and for
some <D, <n, m>> and <D*, <n, m*>> in $W^s(p)$ we have

1.2.1. m ≠ m* and

1.2.2. $<x, y> \in (D \cup D*) \cap A^s \rightarrow <x, y>$ is not <r, 1>-labelled
for any r ≤ 2p

then the player takes the smallest such pair, say <E, <j, k>> and
<E*, <j, k*>>, and

1.2.3. <2p, 0>-labels all members of E ∪ E*,

1.2.4. unlabels all labelled requirements R(q), q > 2p,

1.2.5. labels R(2p), and

1.2.6. for any $<x, y> \in (E \cup E*) \cap A^s$ the player removes <x, y>
from A and puts the smallest <x, y'> such that

1.2.6.1. <x, y'> ∉ E ∪ E* and

1.2.6.2. <x, y'> is not <r, 0>-labelled or <r, 1>-labelled for
any r ≤ 2p,

into A and <2p, 1>-labels <x, y'>.

2. If $\pi_1(s + 1) = 2p + 1$, R(2p + 1) is not labelled at the end
of stage s, and there exists $<n, m> \in W^s(p)$ such that

2.1. no number <n, m'> is <r, 1>-labelled for some r ≤ 2p + 1,

and

2.2. <n, m> is not <r, 0>-labelled for some r ≤ 2p + 1,

then the player takes the smallest such number, say <j, k>, and

2.3. puts <j, k> into A,

2.4. <2p + 1, 1>-labels <j, k>,

2.5. removes and <2p + 1, 0>-labels all members of A of the
form <j, k'> with k' ≠ k,

2.6. unlabels all R(q), q > 2p + 1, and

2.7. labels R(2p + 1).

§2.4 A PROOF THAT THE STRATEGY IS A WINNING STRATEGY

A requirement R(p) labelled at the end of stage s can be
unlabelled at stage s + 1 only if for some q < p, R(q) is not
labelled at the end of stage s, but R(q) is labelled at the end
of stage s + 1. Thus for each p there exists an r(p) such that
either R(p) is labelled at all stages ≥ r(p) or R(p) is not
labelled at any stage ≥ r(p). Choose r(0), r(1), ... such that
r(0) ≤ r(1) ≤ ... . If <n, m(1)> is put into A at stage s(1)
then for some p(1), <n, m(1)> is <p(1), 1>-labelled at stage s(1).
If <n, m(2)>, m(2) ≠ m(1), is put into A at stage s(2) > s(1),
then <n, m(2)> must be <p(2), 1>- labelled at stage s(2) for some
p(2) < p(1). Therefore there must be a stage s(k) and a number
m(k) such that <n, m(k)> ∈ A^t for all t ≥ s(k), and consequently
A must be well-defined. A is single-valued and total since A^s is
single-valued and total for every s.

It now remains to be shown that R(q) is satisfied for every

$q \geq 0$. Consider a requirement of the type $R(2p)$, $p \geq 0$, i.e.

$W(p)(\overline{A})$ single-valued and total $\rightarrow$ $V(p) \Delta W(p)(\overline{A})$ finite.

Now suppose $W(p)(\overline{A})$ is single-valued and total.

Let $L = V^{r(2p)}(p)$. Clearly L is finite. To show that

$R(2p)$ is satisfied it will suffice to show that $V(p) - L \subset$

$W(p)(\overline{A})$ and $W(p)(\overline{A}) \subset V(p)$. Consider any $<n, m>$ that is put into

$V(p)$ at a stage $s + 1$, $s \geq r(2p)$. There must be a member of $W^s(p)$

of the form $<D, <n, m>>$ where $D \subset \overline{A^s}$. Now suppose, for proof by

contradiction, that $<n, m> \notin W(p)(\overline{A})$. As $W(p)(\overline{A})$ is total there

must be a stage $t + 1$, with $t > s$ and $\pi_1(t + 1) = 2p$, such that

for some D^* and $m^* \neq n$, $<D^*, <n, m^*>> \in W^t(p)$ and $D^* \subset \overline{A^t}$. We

claim that 1.2.1. and 1.2.2. hold for the pair $<D, <n, m>>$ and

$<D^*, <n, m^*>>$ at stage t. Clearly $m^* \neq m$, so 1.2.1. holds.

Now suppose $<x, y> \in (D \cup D^*) \cap \overline{A^t}$ and $<x, y>$ is $<r, 1>$-labelled

at stage t with $r \leq 2p$. Now as $D^* \subset \overline{A^t}$ then $<x, y> \in D \cap \overline{A^t}$.

However $D \subset \overline{A^s}$. Hence $<x, y>$ must be put into A at a stage t',

$t > t' > s \geq r(2p)$. This can occur at only two points in the player's

strategy, 1.2.6. or 2.3. Now if the player puts $<x, y>$ into A

at stage t' on behalf of 1.2.6., then as $t' > r(2p)$ we have

$\pi_1(t') > 2p$ and $<x, y>$ cannot be $<r, 1>$-labelled for any $r \leq \pi_1(t')$,

which contradicts our assumption about $<x, y>$. If the player puts

$<x, y>$ into A at stage t' on behalf of 2.3., then by 2.1.,

$\pi_1(t') < r \leq 2p$ and $R(\pi_1(t))$ must become labelled at a stage $t' >$

$r(2p) \geq r(\pi_1(t))$ which is impossible. Therefore $<D, <n, m>>$ and

$<D^*, <n, m^*>>$ satisfy 1.2.1. and 1.2.2. at stage $t + 1$. It

follows that $R(2p)$ is already labelled at stage t because otherwise

R(2p) would become labelled at stage $t + 1$ contradicting $t + 1 >$ r(2p). Let $u + 1$ be the largest stage $< t$ at which R(2p) becomes labelled. Then there must exist some least pair $<E, <j, k>>$ and $<E^*, <j, k^*>>$ in $W^u(p)$ that satisfy 1.2.1 and 1.2.2. at stage $u + 1$.

The player $<2p, 0>$-labels every member of $E \cup E^*$ at stage $u + 1$ and removes every member of $(E \cup E^*) \cap A^u$ from A at stage $u + 1$. Clearly then $<j, k>, <j, k^*> \in W^{u+1}(p)(\overline{A^{u+1}})$, and $k \neq k^*$. If at some stage $v + 1 > u + 1$ a member of $E \cup E^*$, say $<x, y>$, is put into A then, $R(\pi_1(v + 1))$ is labelled at stage $v + 1$ and as $<x, y>$ is $<2p, 0>$-labelled at stage $u + 1$ we have $\pi_1(v + 1) < 2p$. Therefore $r(\pi_1(v + 1)) \geq v + 1$ which contradicts $v + 1 > r(2p)$. As no such stage $v + 1$ can exist we have $<j, k>, <j, k^*> \in W(p)(\overline{A})$ and $k \neq k^*$. This in turn contradicts our assumption that $W(p)(\overline{A})$ is single-valued. Hence $<n, m> \in W(p)(\overline{A})$ and $V(p) - L \subset W(p)(\overline{A})$.

Now suppose $<n, m> \in W(p)(\overline{A})$, then for some s, $<n, m> \in W^t(p)(\overline{A^t})$ for all $t \geq s$. There must be a stage $u \geq s$ such that $\pi_1(u + 1) = 2p$, hence $<n, m> \in V^{u+1}(p) \subset V(p)$. Clearly $W(p)(\overline{A}) \subset V(p)$, and $W(p)(\overline{A}) \Delta V(p) \subset L$. Therefore all the requirements of the type R(2p), $p \geq 0$ must be satisfied.

Consider a requirement of the type R(2p + 1), $p \geq 0$, i.e.

$$\overline{A} \neq W(p).$$

As A is total and single-valued, for any n there is an m such that $<n, m> \in \overline{A}$. Thus we may suppose without loss of generality that for all n there is an m such that $<n, m> \in W(p)$.

Each time a number becomes $<r, 0>$-labelled or $<r, 1>$-labelled

the requirement $R(r)$ becomes labelled. If $r \leq 2p + 1$, then $R(r)$ cannot become labelled at any stage greater than $r(2p + 1)$. As only a finite set of numbers become labelled at any stage, there must be a $n(p)$ such that if $k > n(p)$ then for all y, $<k, y>$ is never $<r, 0>$-labelled or $<r, 1>$-labelled for any $r \leq 2p + 1$. Choose a $<k, y>$ such that $k > n(p)$ and $<k, y> \in W(p)$. There must be a stage $s + 1 > r(2p + 1)$ such that $\pi_1(s + 1) = 2p + 1$ and $<k, y> \in W^s(p)$. Since $k > n(p)$, 2.1. and 2.2 are satisfied at stage s. Thus $R(2p + 1)$ must be already labelled at stage s, as otherwise $R(2p + 1)$ would become labelled at stage $s + 1$ contradicting $s + 1 > r(2p + 1)$. Let t be the greatest stage $< s + 1$ at which $R(2p + 1)$ becomes labelled. Some $<j, x>$ is put into A at stage t and $<j, x>$ becomes $<2p + 1, 1>$-labelled at stage t. Now if $<j, x> \notin A$ then $<j, x>$ must become $<r, 0>$-labelled at some stage $t' > t$ where $r < 2p + 1$. We would then have that $t < t' < s + 1$ and the label on $R(2p + 1)$ is removed at stage t', which contradicts our choice of t. Therefore $<j, x> \in A$ and as $<j, x> \in W^t(p) \subset W(p)$ we have $W(p) \cap A \neq \emptyset$. Therefore $W(p) \neq \bar{A}$.

§2.5 CONCLUSIONS

Consider a play of the game in which the opponent follows an effective strategy whereby $W(0)$, $W(1)$, ... is an enumeration of all the r.e. sets and in which the player follows his effective winning strategy, then all the sets $V(0)$, $V(1)$, ... will be r.e. Now suppose $B \leq_e \bar{A}$ and the partial degree of B is total. Then there must be a single-valued total set $B*$, $B* \equiv_e B$. Thus for some j, $B* = W(j)(\bar{A})$ and as

the player's strategy is a winning one,

$$B^* \Delta V(j) \quad \text{is finite.}$$

Hence $B^* \equiv_e V(j)$ and $\emptyset \equiv_e B^* \equiv_e B$. Also $\overline{A} \neq W(j)$ for all j, whence $\overline{A} >_e \emptyset$. It follows that $A >_e \emptyset$, because A is single-valued and total. This completes the result claimed for this chapter because the partial degree of A is total and non-zero, but there is no non-zero partial degree, less than or equal to the partial degree of $\overline{A}$, that is total.

We also note that a theorem of Medvedev is a corollary to the above result.

Corollary 2.5.1. (Medvedev)

There is a non recursive ϕ such that for all f,

$$f \leq_e \phi \rightarrow f \text{ recursive.}$$

CHAPTER III

MINIMAL PAIRS OF PARTIAL DEGREES

§3.1 INTRODUCTION

If Γ is a set partially ordered by $\leq'$ with a least element α, then $\beta, \gamma \in \Gamma$ are said to be a *minimal pair* if $\beta, \gamma \neq \alpha$ and

$$\forall \delta \in \Gamma(\delta \leq' \beta \text{ and } \delta \leq' \gamma \rightarrow \delta = \alpha).$$

Case [1] has shown that there exists a minimal pair of partial degrees. Lachlan [3] and Yates [7] have shown independently that there is a minimal pair of r.e. degrees. In this chapter we will combine these two results to show that there are two non-r.e. but co-r.e. sets whose partial degrees form a minimal pair. The method used is similar to the one in the paper by Lachlan [3]. Whether every minimal pair of r.e. degrees forms a minimal pair of partial degrees is an open question.

§3.2 DESCRIPTION OF THE GAME

We consider a game where the player enumerates the sets $A, B, V(0), V(1), \ldots,$ and the opponent enumerates the sets $W(0), W(1), \ldots, U(0), U(1), \ldots$.

The requirements of the game are:

R(1): $W(0)\,(\overline{A}) = U(0)\,(\overline{B}) \rightarrow \exists i(V(i) \;\Delta\; W(0)\,(\overline{A})$ is finite)

R(2): $\overline{A} \neq W(0)$

R(3): $\overline{B} \neq U(0)$

R(4): $W(1)(\overline{A}) = U(1)(\overline{B}) \rightarrow \exists i (V(i) \Delta W(1)(\overline{A})$ is finite$)$.

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§3.3. BASIC IDEA BEHIND THE STRATEGY

Before giving the actual strategy and proof we outline the main ideas behind the strategy. Let us first consider the problem of constructing two non-r.e. but co-r.e. sets which satisfy the first requirement, i.e.

$$W(0)(\overline{A}) = U(0)(\overline{B}) \rightarrow \exists i (V(i) \Delta W(0)(\overline{A}) \text{ is finite}).$$

At each stage s of the game we define Level $(0, s)$ as the largest $n < s$ such that

$$\forall k \leq n (k \in W^s(0)(\overline{A^s}) \leftrightarrow k \in U^s(0)(\overline{B^s})).$$

We also consider the sequence $s_0 < s_1 < \ldots,$ defined by

$$s_i = \mu x ((\forall j < i)(s_j < x \text{ and Level } (0, s_j) < \text{Level } (0, x))).$$

This sequence is either finite or infinite. If it is finite then R(0) is clearly satisfied.

We reserve a set $V(0)$ for this requirement. For each i, we enumerate each number of $W^{s_i}(0)(\overline{A^{s_i}})$ that is $<$ Level $(0, s_i)$ into $V(0)$. At stage s we may wish to put some member x into A to ensure $\overline{A} \neq W(p)$ for some p. Putting x into A may remove some $k <$ Level $(0, s_j)$, with $s_j < s$, from $W^s(0)(\overline{A^s})$. Despite this we may enumerate x in A as long as we then keep out of B all numbers whose removal from $\overline{B}$ would remove k from $U^s(0)(\overline{B^s})$. If Level $(0, t) < k$

for all $t > s$ then $R(0)$ will be satisfied and the restriction on B persists from stage s through all subsequent stages. If for some stage $t > s$, Level $(0, t) \geq k$ then the restriction on B may be lifted at stage t and at stage t we may put any number y in A or B provided that an appropriate restriction is placed on B or A respectively. The effect of this procedure, if the sequence $s_0 < s_1 < \ldots$ is infinite, is to ensure for all k and all $t > s$ we have $k \in U^t(0)(\overline{B^t})$ or $k \in W^t(0)(\overline{A^t})$. However we want $\lim_{t \to \infty} W^t(0)(\overline{A^t})[k]$ to be defined, so we only allow k to leave $W^t(0)(\overline{A^t})$ when we place numbers into A to ensure that $\overline{A} \neq W(q)$ where $q < k$. This restriction also ensures that the set of numbers restricted from entry to A on behalf of k is finite. A similar method ensures the same for B.

Now if the sequence $s_0 < s_1 < \ldots$ is finite then let u be the maximum stage in the sequence. For each $k <$ Level $(0, u)$ we have seen that the set of all numbers restricted from entering A or B on behalf of k is finite. This allows the player to start playing the other requirements starting at stage $u + 1$ with only a finite interference from $R(0)$. If the sequence $s_0 < s_1 < \ldots$ is infinite then any restrictions on entry into A or B will be eventually lifted and other requirements may wait for this relaxation before they act.

At any stage s_i we assume that the sequence will be infinite, and play a strategy on these stages that conforms with this assumption. On all other stages we assume that we have passed the last member of the sequence and play accordingly. Thus for the requirement $R(1)$ we have two strategies being played. For each strategy we carry a distinct set from the sequence $V(0)$, $V(1)$, $\ldots$. When we are assuming that the

sequence $s_0 < s_1 < \ldots$ is infinite, that is when we are at a stage s_j, then we play as if the only stages that have occurred are $s_0 < s_1 < \ldots < s_j$. This policy is extended to all requirements. Thus for a requirement $R(p)$ there are p requirements above it and hence there are 2^p possible assumptions. For each of these assumptions we require a distinct set from the sequence $V(0)$, $V(1)$, ...

§3.4 A WINNING STRATEGY FOR THE GAME

We first give some definitions used in the strategy.

Definition 3.4.1

.1. Level (p, s) is the largest $n < s$ such that

$$\forall k \leq n (k \in W^s(p) \overline{(A^s)} \leftrightarrow k \in U^s(p) \overline{(B^s)}).$$

.2. A *p-state* is a subset of $\{i : i < p\}$

.3. The p-states are linearly ordered by $\leq^*$ defined by

$$E \leq^* F \leftrightarrow \forall j (j \in F - E \rightarrow \exists i (i \in E - F \text{ and } i < j))$$

.4. We define $E(s) \subset \{i : i \leq s\}$ and finite sequences
$\langle N(i, s) \mid i \leq s \rangle$ and $\langle u(i, j, s) \mid i \leq s, \; j \leq N(i, s) \rangle$ by
the following stipulations:

.4.1. $N(0, s) = s$, $u(0, j, s) = j$ for $j \leq N(0, s)$

.4.2. $k \in E(s) \leftrightarrow$ Level $(k, s) > \max\limits_{i < N(k, s)}$ Level $(k, u(k, i, s))$

.4.3. $k \notin E(s)$ and $j \leq N(k, s) \rightarrow N(k + 1, s) = N(k, s)$ and
$u(k + 1, j, s) = u(k, j, s)$

.4.4. $k \in E(s) \rightarrow u(k + 1, 0, s) = 0$

.4.5. $k \in E(s)$ and $u(k + 1, j, s)$ defined and

$u(k + 1, j, s) < s$.→. $u(k + 1, j + 1, s) = u(k, y, s)$

where

$y = \mu z(\text{Level } (k, u(k, z, s)) > \text{Level } (k, u(k + 1, j, s)))$

.4.6. $k \in E(s)$ and $u(k + 1, j, s) = s$.→. $N(k + 1, s) = j$.

.5. We define $E(p, s)$ as $E(s) \cap \{i : i < p\}$.

.6. $R(3p + 1)$ is *frozen at stage* s if for some m,

$m \in W^s(p) \cap A^s$.

.7. $R(3p + 2)$ is *frozen at stage* s if for some m,

$m \in U^s(p) \cap B^s$.

In the strategy we will make use of the auxiliary function $L(p, E, s)$. It is assumed that if $L(p, E, s + 1)$ is not set explicitly it is equal to $L(p, E, s)$. At stage 0 all sets are empty and $L(p, E, 0) = 0$. At stage 0 the player does nothing.

Strategy at stage $s + 1$:

1. For each $p \in E(s)$ if there exist n and D such that

1.1 $<D, n> \in W^s(p)$

1.2 $n \notin \overline{V^s(<p, E(p, s)>)}$

1.3 $D \subset \overline{A}^s$

1.4 $n \leq \text{Level } (p, s)$

then the player chooses the smallest such n and

1.5 puts n into $V(p, E(p, s))$ and

1.6 sets $L(p, E(p, s), s + 1)$ to the maximum of $L(p, E(p, s), s)$ and n.

The player chooses the least $p < s$, if any, such that EITHER

2. $R(3p + 1)$ is not frozen at stage s and there exists m such that

2.1. $\langle m, p, 1\rangle \in W^s(p)$

2.2. if $q \leq p$ and for some q-state E we have $E \leq^* E(q, s)$ and further there is a D such that $\langle D, k\rangle \in W^s(q)$ and $D \subset \overline{A^s}$ for some $k \leq L(q, E, s + 1)$, then if D^* is the least such D

$$\langle m, p, 1\rangle \in D^* \rightarrow k \in U^s(q)(\overline{B^s})$$

(note that at this point in the strategy $L(q, E, s + 1)$ will have been set, either explicitly by part 1, or implicitly by our convention)

2.3. if $q \leq p$ and $k < p$ and if there is a D such that $D \subset \overline{A^s}$ and $\langle D, k\rangle \in W^s(q)$ then if D^* is the least such D we have $\langle m, p, 1\rangle \notin D^*$ OR

3. $R(3p + 2)$ is not frozen at stage s and there exists m such that

3.1. $\langle m, p, 2\rangle \in U^s(p)$

3.2. if $q \leq p$ and for some q-state E we have $E \leq^* E(q, s)$ and further there is a D such that $\langle D, k\rangle \in U^s(q)$ and $D \subset \overline{B^s}$ for some $k \leq L(q, E, s + 1)$, then if D^* is the least such D

$$\langle m, p, 2\rangle \in D^* \rightarrow k \in W^s(q)(\overline{A^s})$$

(note that at this point in the strategy $L(q, E, s + 1)$ will have been set, either explicitly by part 1, or implicitly by our convention)

3.3. if $q \leq p$ and $k < p$ and if there is a D such that $D \subset \overline{B^s}$ and $\langle D, k\rangle \in U^s(q)$ then if D^* is the least such D we have $\langle m, p, 2\rangle \notin D^*$.

If   p   exists satisfying   2.   then the player chooses the least   m   that satisfies   2.   for   p   and puts   <m, p, 1>   into   A.   If   p   exists but not satisfying   2.   then the player chooses the least   m   that satisfies   3.   for   p   and puts   <m, p, 2>   into   B.

§3.5   PROOF THAT THE STRATEGY IS A WINNING ONE

For each $p \geq 0$ we define the *final p-state*,   denoted by F_p,   as the smallest   p-state   under $\leq^*$,   such that the sequence of all stages   s   with $E(p, s) = F_p$,   say

$$v(p, 0) < v(p, 1) < v(p, 2) < \ldots ,$$

is infinite.   Now it is clear from the definition of   E(s)   that $F_p \subset F_{p+1}$.   We define the   *final game state*,   denoted by F_∞ as $\bigcup_{p=0}^{\infty} F_p$.

Consider a requirement of the type   R(3p + i), i = 1, 2.   We let   r(3p + i)   be the least stage   s   such that   R(3p + i)   is frozen at stage   s,   if no such stage exists we let   r(3p + i)   be   0.

Lemma 3.5.1   If for some   p, $p \geq 0$ we have

$$\forall n \, \exists s \, \forall t (t > s \rightarrow \text{Level} \ (p, t) \geq n)$$

then $p \in F_\infty$.

Proof: Suppose   p   satisfies the hypothesis but $p \notin F_\infty$.   Clearly then $p \notin F_{p+1}$.   Consider the infinite sequence

$$v(p + 1, 0) < v(p + 1, 1) < v(p + 1, 2) < \ldots ,$$

For any   k,   k > 0,   there must be a   j > k   such that

Level $(p, v(p + 1, j)) > k$. Therefore there is an infinite subsequence

$$t(0) < t(1) < \ldots$$

such that $E(p + 1, t(i)) = F_{p+1} \cup \{p\} <^* F_{p+1}$, $i = 0, 1, 2, \ldots$. This contradicts our definition of F_{p+1} and our result is shown.

Lemma 3.5.2 If for some $p \geq 0$, $W(p)(\overline{A}) = U(p)(\overline{B})$ then

$$\forall n \, \exists s \, \forall t (t > s \rightarrow \text{Level}(p, t) \geq n)$$

Proof: Consider $p \geq 0$, and suppose $W(p)(\overline{A}) = W(p)(\overline{B})$. Now for a proof by contradiction assume that there is an n such that for all s there is a stage $t'(s) > s$ such that Level $(p, t'(s)) < n$. Clearly there must be an infinite sequence $\langle t(i) \rangle_{i=0}^{\infty}$ such that for some m

$$m \in W^{t(i)}(p)(\overline{A^{t(i)}}) \, \Delta \, U^{t(i)}(p)(\overline{B^{t(i)}})$$

for all i.

Choose j sufficiently large so that $t(j) > r(3q + i)$, $q \leq m$, $i = 1, 2$. Now either $m \in W^{t(j)}(p)(\overline{A^{t(j)}})$ or $m \in U^{t(j)}(p)(\overline{B^{t(j)}})$. Suppose the former, then for some set D, $D \subset \overline{A^{t(j)}}$ and $\langle D, m \rangle \in W^{t(j)}(p)$. Therefore there must be some set D^* such that D^* is the least set such that for some $s \geq t(j)$ $D^* \subset \overline{A^s}$ and $\langle D^*, m \rangle \in W^s(p)$. If $D^* \not\subset \overline{A}$ then at some least stage $r > s$ a number of the form $\langle n, q', 1 \rangle$ is put into A. Hence $R(3q' + 1)$ becomes frozen at stage r and as $r > s \geq t(j) > r(3q + 1)$, $q \leq m$, $i = 1, 2$, we must have $q' > m$. However $\langle n, q', 1 \rangle$ must satisfy 2. of the player's strategy at stage r and hence $m > q'$ by part 2.3. of the player's strategy.

This is a contradiction so we must have $D^* \subset \overline{A}$.

Now as $<D^*, m> \in W^s(p) \subseteq W(p)$ we have $m \in W(p)(\overline{A})$. However $W(p)(\overline{A}) = U(p)(\overline{B})$. Hence $m \in W(p)(\overline{A}) \cap U(p)(\overline{B})$ and for some k we have $m \in W^{t(k)}(\overline{A^{t(k)}}) \cap U^{t(k)}(p)(\overline{B^{t(k)}})$. Similarly for $m \in U^{t(j)}(\overline{B^{t(j)}})$. This contradicts our choice of the sequence $<t(i)>_{i=0}^{\infty}$ and establishes our lemma.

Lemma 3.5.3 For every $p \geq 0$ there is a stage $z'(p)$ such that

$$s \geq z'(p) \to F_p \leq^* E(p, s).$$

Proof: As there are only finitely many p-states it suffices to show that for each p-state E, $E <^* F_p$, that there is a stage $t(E)$ such that

$$s > t(E) \to E(p, s) \neq E.$$

This is immediate from the definition of F_p.

For economy of notation we note that there is a stage $z(p)$ such that $z(p) > z'(p)$ and $z(p) > r(3q + i)$, $q \leq p$, $i = 1, 2$.

Lemma 3.5.4 For all $q \geq 0$ there is a $b(q)$ such that $L(q, E, s) \leq b(q)$ for all stages s and all q-states E such that $E <^* F_q' = F_q \cup \{q\}$.

Proof: Consider a q-state E with $E <^* F_q'$. It suffices to show that for some n, $L(q, E, s) < n$ for all s. Suppose for proof by contradiction that for all n there is a stage s such that $L(q, E, s) > n$. Hence there must be an infinite sequence of stages say

$$t(0) < t(1) < t(2) < \ldots,$$

such that $L(q, E, t(i) + 1) > L(q, E, t(i))$, $i = 0, 1, 2, \ldots$. Clearly then, as the auxiliary function L can be increased only on behalf of part 1. of the player's strategy, $E(q, t(i)) = E$, $i = 0, 1, 2, \ldots$. Therefore by our definition of F_q, $F_q' \leq^* E$. This contradicts our assumption that $E <^* F_q'$ and establishes our Lemma.

Lemma 3.5.5 If $q < p$ and $q \in F_\infty$ then for all $k > 0$.

$$L(q, F_q, v(p, k) + 1) \leq \text{Level } (q, v(p, k)).$$

Proof: As $<v(p, k)>_{k=0}^{\infty}$ is a subsequence of $<v(q, k)>_{k=0}^{\infty}$ it suffices to show that for all $k \geq 0$

1. $L(q, F_q, v(q, k) + 1) \leq \text{Level } (q, v(q, k))$.

Now $L(q, F_q, v(q, 0) + 1)$ either is equal to $L(q, F_q, v(q, 0))$ which has value 0 or is set to the maximum of $L(q, F_q, v(q, 0))$ and some $n \leq \text{Level } (q, v(q, 0))$. Clearly then $L(q, F_q, v(q, 0) + 1) \leq \text{Level } (q, v(q, 0))$. Now suppose 1. is true for $k = j$. Now $L(q, F_q, v(q, j + 1) + 1)$ either equals $L(q, F_q, v(q, j + 1))$ or is set to the maximum of $L(q, F_q, v(q, j + 1))$ and some $n \leq \text{Level } (q, v(q, j + 1))$. To show that 1. holds for $k = j + 1$ it clearly suffices to show that

$$L(q, F_q, v(q, j + 1)) \leq \text{Level } (q, v(q, j + 1)).$$

By our choice of $<v(q, i)>_{i=0}^{\infty}$ we have $L(q, F_q, v(q, j + 1)) =$

$L(q, F_q, v(q, j) + 1)$ whence by our induction hypothesis

$$L(q, F_q, v(q, j + 1)) \leq \text{Level } (q, v(q, j))$$

$$< \text{Level } (q, v(q, j + 1)).$$

This completes our proof by induction of the required result.

We now show that requirements of the type $R(3p)$ are satisfied. Consider $p \geq 0$, suppose $W(p)(\overline{A}) = U(p)(\overline{B})$. Let $L = W^{z(p)}(p)(N)$ and $V = V(<p, F_p>)$.

We first show by induction that $W(p)(\overline{A}) \subset V$. Suppose that for all $k < n$

$$k \in W(p)(\overline{A}) \to k \in V.$$

Now suppose $n \in W(p)(\overline{A})$. We can choose s sufficiently large so that if $k < n$ and $k \in W(p)(\overline{A})$ then $t \geq s \to k \in W^t(p)(\overline{A^t}) \cap V^t$ and for some $D \subset \overline{A}$, $<D, n> \in W^s(p)$. Now by Lemma 3.5.2 and Lemma 3.5.1 we have $p \in F_\infty$. As s was chosen arbitrarily high we can assume that $s = v(p + 1, i)$ for some i, hence $p \in E(s)$. Also by Lemma 3.5.2 we can choose s so that $n \leq \text{Level } (p, s)$. Therefore 1.1., 1.3., and 1.4., of the player's strategy hold for n and D at stage $s + 1$. Therefore either 1.2. does not hold for n and $n \in V^s$, or 1.2. holds for n and $n \in V^{s+1}$. Clearly $n \in V$, which completes our proof by induction that $W(p)(\overline{A}) \subset V$.

Suppose $n \in V - L$. Clearly there is a stage $s(1)$ such that $n \in W^{s(1)}(\overline{A^{s(1)}})$. Let $D(1)$ be the smallest set D such that $<D, n> \in W^{s(1)}$ and $D \subset \overline{A^{s(1)}}$. In the following construction of the sequences $<D(i)>_{i=1}^{\infty}$,

$\langle s(i)\rangle_{i=1}^{\infty}$, $\langle w(i)\rangle_{i=1}^{\infty}$ we will assume that if $w(i) = 1$ and $s(i+1) \geq s >$ $s(i)$ then $D(i)$ is the least set D with $\langle D, n\rangle \in W^s(\overline{A^s})$ and $D \subset \overline{A^s}$, similarly for $w(i) = 2$ by replacing A with B. Clearly this assumption is equivalent to dropping an argument s from each $D(i)$.

Case 1. $D(1) \subset \overline{A}$, in which case $n \in W^{s(1)}(\overline{A}) \subset W(p)(\overline{A})$. This is what we want to show. We call this a terminating case.

Case 2. $D(1) \not\subset \overline{A}$.

Let us suppose the latter case holds. Now a member of $D(1)$ can be put into A only on behalf of part 2. of the player's strategy. Let $s(2)$ be the smallest stage $> s(1)$ such that a member of $D(1)$, say $\langle m(2), q(2), w(2)\rangle$ where $w(2) = 1$, is placed into A at this stage. Let $t(2)$ be the stage immediately preceding $s(2)$. Now $s(2) > s(1) > z(p)$ as $n \in W^{s(1)}(\overline{A^{s(1)}}) - L$, hence as $R(3q(2) + 1)$ becomes frozen at stage $s(2)$ we have $q(2) > p$. Now by Lemma 3.5.3, $F_p \leq^* E(p, t(2))$. Also

$$n \leq L(p, F_p, s(1) + 1) \leq L(p, F_p, s(2)).$$

Hence by part 2.2. of the player's strategy,

$$n \in W^{t(2)}(p)(\overline{A^{t(2)} \cup \{\langle m(2), q(2), w(2)\rangle\}}) \cup U^{t(2)}(p)(\overline{B^{t(2)}}).$$

That is either

1. $n \in W^{s(2)}(p)(\overline{A^{s(2)}})$ or
2. $n \in U^{s(2)}(p)(\overline{B^{s(2)}})$.

Now if 1. is true we can repeat the above argument. Either this repetition will continue without end or it will terminate finitely, i.e. either we can get an infinite sequence

$$s(1) < s(2) < s(3) < \ldots$$

or a finite sequence

$$s(1) < s(2) < \ldots < s(j)$$

such that $n \in U^{s(j)}(\overline{B^{s(j)}})$. Suppose the latter (this case includes 2., as the special case $j = 2$), then there must be some smallest set $D(j)$, $D(j) \subset \overline{B^{s(j)}}$, such that $\langle D(j), n \rangle \in U^{s(j)}(p)$. Now as before we have two cases.

Case 1. $D(j) \subset \overline{B}$, in which case $n \in U(p)(\overline{B}) = W(p)(\overline{B})$ which is what we want to show. This is also a terminating case.

Case 2. $D(j) \not\subset \overline{B}$.

Let us suppose case 2. holds, then there must be a smallest stage $s(j + 1)$, $s(j + 1) > s(j)$, such that $D(j) \not\subset \overline{B^{s(j+1)}}$. Some number $\langle m(j + 1), q(j + 1), w(j + 1) \rangle$, $w(j + 1) = 2$, must be put into B at stage $s(j + 1)$. Now $s(j + 1) > s(1) > z(p)$, so $q(j + 1) > p$. Let $t(j + 1)$ be the stage immediately preceding $s(j + 1)$. By Lemma 3.5.3, $F_p \leq^* E(p, t(j + 1))$. Therefore by part 3.2. of the player's strategy which we can apply as $n \leq L(p, F_p, s(1)) \leq L(p, F_p, t(j + 1))$ we have

$$n \in U^{t(j+1)}(p)(\overline{B^{t(j+1)} \cup \{\langle m(j + 1), q(j + 1), w(j + 1) \rangle\}})$$

$$\cup\ U^{t(j+1)}(p)(\overline{A^{t(j+1)}}).$$

That is either

1. $n \in U^{s(j+1)}(p)(\overline{B^{s(j+1)}})$ or
2. $n \in W^{s(j+1)}(p)(\overline{A^{s(j+1)}})$.

Now if 1. holds we can repeat this argument and continue the sequence. As before we get either an infinite sequence

$$s(1) < s(2) < \ldots < s(j) < \ldots$$

or a finite sequence

$$s(1) < s(2) < \ldots < s(j) < \ldots < s(k)$$

such that $n \in W^{s(k)}(\overline{A^{s(k)}})$. In the latter case (2. is a special sub-case with $k = j + 1$) we can apply the argument used on $s(1)$ to continue the sequence. Thus either one of the terminating cases will hold and $n \in W(p)(\overline{A})$ or we get an infinite sequence

$$s(1) < s(2) < \ldots .$$

with the associated sequences

$$t(2) < t(3) < \ldots$$

$$q(2), q(3), \ldots$$

$$D(1), D(2), \ldots$$

$$w(2), w(3), \ldots .$$

Consider $i \geq 1$. Now if $w(i + 1) = 1$, then $D(i)$ is the smallest set such that $D(i) \subset \overline{A^{s(i)}}$ and $\langle D(i), n \rangle \in W^{s(i)}(p)$. By part 2.3. of the player's strategy $q(i + 1) \leq n$. Now if $w(i + 1) = 2$, then $D(i)$ is the smallest set such that $D(i) \subset \overline{B^{s(i)}}$ and $\langle D(i), n \rangle \in U^{s(i)}(p)$. By part 3.3. of the player's strategy $q(i + 1) \leq n$. Now $r(3q(i) + w(i))$ equals $s(i)$ for all $i \geq 2$, hence as at most one requirement becomes

frozen at any stage, $3q(i) + w(i) \neq 3q(j) + w(j)$ for all $i \neq j$, $i, j \geq 2$. This is a contradiction, hence one of our terminating cases must hold.

We have shown that $W(p)(\overline{A}) \subset V$ and $V - L \subset W(p)(\overline{A})$, therefore $W(p)(\overline{A}) \Delta V \subset L$, and all the requirements $R(3p)$, $p \geq 0$, are satisfied.

Now let us consider a requirement of the type $R(3p + 1)$, i.e.

$$\overline{A} \neq W(p).$$

If $R(3p + 1)$ is frozen at some stage t then there is an x such that $x \in A^t \cap W^t(p)$, and $\overline{A} \neq W(p)$. Now suppose for proof by contradiction that $R(3p + 1)$ is never frozen at any stage. Clearly then for all m, $\langle m, p, 1 \rangle \not\in A$. If at the game's conclusion there is an m such that $\langle m, p, 1 \rangle \not\in W(p)$ then $\langle m, p, 1 \rangle \in \overline{A} - W(p)$, and $\overline{A} \neq W(p)$. We are left with the case when for all m, $\langle m, p, 1 \rangle \in W(p) - A$.

We will first show that there is a number d_1 and a function h_1 such that if $m > d_1$ and $s > h_1(m)$ then $\langle m, p, 1 \rangle$ will satisfy part 2.1. of the player's strategy at stage s.

Let $d_1 = 0$ and define $h_1(m)$ to be the smallest stage s such that $\langle m, p, 1 \rangle \in W^s(p)$. Clearly if $m > d_1$ and $s > h_1(m)$ then $\langle m, p, 1 \rangle \in W^s(p)$ whence 2.1. is satisfied.

We will now show that there are numbers d_3, e_3 such that if $m > d_3$ and $s > e_3$ then $\langle m, p, 1 \rangle$ satisfies part 2.3. of the player's strategy at stage s.

Consider any $q \leq p$ and $k < p$. Suppose there is a set D such that for some stage t, $t \geq z(k)$, $D \subset \overline{A^t}$ and $<D, k> \in W^t(q)$. Let $D(k, q)$ be the smallest such set D, and let $t(k, q)$ be the smallest stage such that $D(k, q) \subset \overline{A^{t(k,q)}}$ and $<D(k, q), k> \in W^{t(k,q)}(q)$. If $D(k, q) \not\subset \overline{A}$ then some $<j, q', 1>, <j, q', 1> \in D(k, q)$, must be put into A at some stage $t > t(k, q)$. Now as $<j, q', 1>$ satisfies part 2.3. of the player's strategy at stage t we have $q' < k$. Now $R(3q' + 1)$ becomes frozen at stage t, and we have $t > t(k, q) > z(k) > r(3q' + 1)$. Thus we must have $D(k, q) \subset \overline{A}$. Now if there is no such set D such that for some $t \geq z(k)$, $D \subset \overline{A^t}$ and $<D, k> \in W^t(q)$, then we let $D(k, q) = \emptyset$ and $t(k, q) = z(k)$. We let d_3 be the maximum of all m such that $<m, p, 1> \in D(k, q)$ for some $k < p$, $q \leq p$, and we let e_3 be the maximum of all the stages $t(k, q)$, $k < p$ and $q \leq p$.

Now suppose $m > d_3$ and $s > e_3$. Consider any $q \leq p$ and $k < p$ such that there is a D, $D \subset \overline{A^s}$ and $<D, k> \in W^s(q)$. Now $s > t(k, q)$ hence if D^* is the smallest such D, $D^* = D(k, q)$. Clearly then, by our choice of d_3, $<m, p, 1> \not\in D^*$. Therefore d_3 and e_3 have the desired property.

We will now demonstrate that there are numbers d_2 and e_2 such that if $m > d_2$ and for some i, $v(p', i) > e_2$, $p' = p + 1$, then $<m, p, 1>$ satisfies part 2.3. of the player's strategy at stage $v(p', i) + 1$.

Consider any $q \leq p$ and $k \leq b(q)$. We can construct $D(k, q)$ and $t(k, q)$ exactly as above. Let d_2 be the maximum of all m such that $<m, p, 1> \in D(k, q)$, $k \leq b(q)$, $q \leq p$, and let e_2 be the maximum of all the stages $t(k, q)$, $k \leq b(q)$, $q \leq p$.

Let $m > d_2$ and let i be sufficiently large so that $v(p',i) > e_2$. Assume for proof by contradiction that part 2.2. of the player's strategy does not hold for $<m, p, 1>$ at stage $v(p',i) + 1$. Hence there must be a $q \leq p$, a q-state E with $E \leq^* E(p, v(p',i)) = F_p$, and a $k \leq L(q, E, v(p',i) + 1)$, such that if D^* is the smallest set such that $D^* \subset \overline{A^{v(p',i)}}$ and $<D^*, k> \in W^{v(p',i)}(q)$ then $<m, p, 1> \in D^*$ and $k \notin U^{v(p', i)}(\overline{B^{v(p',i)}})$. Now if $E <^* F_q \cup \{q\}$, then by Lemma 3.5.4 $k \leq b(q)$, and as $v(p',i) > t(k, q)$, we must have $D^* = D(k, q)$. Hence $<m, p, 1> \notin D^*$ as $m > d_2$. This leaves the case $E = F_q$ and $q \in F_\infty$. By Lemma 3.5.5, $L(q, E, v(p',i) + 1) \leq$ Level $(q, v(p',i))$, hence $k \leq$ Level $(q, v(p',i))$. Now as $k \in W^{v(p',i)}(q)(\overline{A^{v(p',i)}})$, we have $k \in U^{v(p',i)}(\overline{B^{v(p',i)}})$ which contradicts our assumption about k. Hence d_2 and e_2 have the desired property.

Now choose m such that m is greater than any of d_1, d_2 and d_3, and choose an i such that $v(p',i) > z(p)$, $v(p',i) > h(m)$ and $v(p',i)$ is greater than either e_2 or e_3. Now $<m, p, 1>$ must satisfy 2.1., 2.2., and 2.3. at stage $v(p',i) + 1$. By our assumption that $R(3p + 1)$ is never frozen there must be a $q < p$ such that either $R(3q + 1)$ or $R(3q + 2)$ becomes frozen at stage $v(p',i) + 1$. This contradicts our choice of $v(p',i) > z(p)$. Hence we have shown that all requirements of the form $R(3p + 1)$, $p \geq 0$ are satisfied. We can show that the requirements of the form $R(3p + 2)$, $p \geq 0$ are satisfied by a similar argument.

§3.6 CONCLUSIONS

Consider a play of the game where the opponent follows an effective

strategy so that $\{(W(i), U(i)) : i \in N\}$ is the set of all pairs of r.e.

sets. Therefore as the player's strategy is effective A, B and all

the sets $V(0)$, $V(1)$, ... will be r.e. Furthermore $\overline{A} >_e \emptyset$ and $\overline{B} >_e \emptyset$

as $\overline{A} \neq W(i)$ and $\overline{B} \neq U(i)$, $i = 0, 1, 2, \ldots$. Now if $C \leq_e \overline{A}$ and

$C \leq_e \overline{B}$ then for some pair of r.e. sets $(W(i), U(i))$, we will have

$C = W(i)(\overline{A}) = U(i)(\overline{B})$. As $R(3i)$ is satisfied, there is a set $V(k)$

such that $C \Delta V(k)$ is finite, and as $V(k)$ is r.e. $C \equiv_e \emptyset$. Thus the

partial degrees of $\overline{A}$ and $\overline{B}$ form a minimal pair. Now $\overline{A}$ and $\overline{B}$ are

co-r.e. hence their partial degrees are total. Therefore the Turing

degrees of $\overline{A}$ and $\overline{B}$ form a minimal pair of r.e. degrees.

CHAPTER IV

THERE ARE NO MINIMAL

PARTIAL DEGREES

§4.1   INTRODUCTION

In this chapter we will show that there are no minimal partial

degrees.

Definition 4.1.1.

.1.   A partial degree $\underline{a}$ is *minimal* if $\underline{0} < \underline{a}$ and $\underline{b} < \underline{a} \rightarrow \underline{b} = \underline{0}$.

.2.   A set B is *minimal* if its partial degree is minimal.

Mention of this problem occurs in Rogers [5, p.282] and Case [1].

The proof will be in two parts. First we will show that a minimal degree

must be $\leq \underline{0}'$ and then we shall show that there is an r.e. set V such

that for all B, $\emptyset <_e B \leq_e \overline{K}$, we have $\emptyset <_e V(B) <_e B$.

§4.2   TWO GAMES

In this section we describe two games that have an effective winning

strategy.

We consider the following game where the player enumerates V and

the opponent enumerates W(0), W(1), ... We will assume that the

partial degree of the opponent's strategy is less than or equal to the

partial degree of a given set A. If the player's strategy is effective

then any set W(i) that the opponent enumerates will be such that

$W(i) \leq_e A$. Also if the player's strategy is effective we will have

$V \leq_e A$. The requirements of the game are:

R_{-1}: for each n there exists an $m \geq 1$ such that for all j

$$\langle n, j \rangle \in V(X) \rightarrow j \leq m \quad \text{for all } X, \; X >_e A$$

$$0 < j < m \rightarrow \langle n, j \rangle \in V(X) \quad \text{for all } X, \; X >_e A$$

$$n \in X \leftrightarrow \langle n, m \rangle \in V(X) \quad \text{for all } X, \; X >_e A$$

R_0: $W(0)V(X) \neq X$ for all $X, \; X >_e A$

R_1: $W(1)V(X) \neq X$ for all $X, \; X >_e A$

R_2: $W(2)V(X) \neq X$ for all $X, \; X >_e A$

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Now if we have an effective winning strategy for this game it will

work when the opponent simultaneously effectively enumerates all the

r.e. sets. In the following discussion we will assume that this is the

case. We can take A to be $\emptyset$ and hence the requirements will be

satisfied by any non r.e. set X. Also the set V the player enumerates

will be r.e. Now suppose B is minimal. We must have $B >_e \emptyset$, hence

B is non r.e. and B satisfies all the requirements of the above game.

By definition $\emptyset \leq_e V(B) \leq_e B$, hence either $V(B) \equiv_e B$ or $V(B) \equiv_e \emptyset$.

$V(B) \equiv_e B$ would imply that $B \leq_e V(B)$, hence for some r.e. set $W(p)$

we would have $B = W(p)V(B)$. This is impossible as B satisfies $R(p)$.

Therefore, for every minimal B we must have $V(B)$ r.e., that is

$V(B) = W(q)$ for some q. Now if $\underline{b}$ is the partial degree of the join

of the characteristic function of V and the characteristic function of

$W(q)$, then we have by R_{-1} that the partial degree of $C_B \leq \underline{b}$. Having

an oracle for V would allow us to compute for each n the m whose

existence is assured by R_{-1}. We can then consult an oracle for $W(q)$

to see whether $\langle n, m \rangle$ is in $W(q)$ or not. In the former case $n \in B$,

in the latter $n \notin B$. Now clearly $\underline{b}$ is a total degree and $\underline{b} \leq \underline{0}'$,

hence we have the following:

Theorem 4.2.1. If X is minimal then X has T-degree $\leq \underline{0}'$.

As $V(B) = W(q)$, an enumeration of V and $W(q)$ will give us a

sequence of initial functions $\{C_B^s\}_{s=0}^{\infty}$ which will converge to C_B. For

given partial enumerations of V and $W(q)$, say V^s and $W^s(q)$, we

take $C_B^s(n)$ to be 1, if there is $\langle n, j \rangle \in W^s(q)$ such that

$\langle n, k \rangle \in W^s(q) \to k \leq j$ and $k < j \leftrightarrow \langle n, k \rangle \rangle \in V^s$, and 0 otherwise.

Now recall that V satisfies R_{-1} and thus for each n there exists an

$m \geq 1$ such that $\langle n, j \rangle \in W^s(q) \to j \leq m$ and $j < m \to \langle n, j \rangle \in W^s(q)$

and $\langle n, m \rangle \in V(B) \leftrightarrow n \in B$. For any n either $\langle n, m \rangle$ will appear

in $W^s(q)$ for some $s > n$ in which case $t \geq s \to C_B^t(n) = 1$, or there

will be an $s > n$ such that $t \geq s \to C_B^t(n) = 0$. As $W(q) = V(B)$ and

$\langle n, m \rangle \in V(B) \leftrightarrow n \in B$ these final values will agree with C_B. Now each

of these initial functions C_B^s is finite and hence has a canonical index,

thus we can consider the sequence $\{C_B^s\}_{s=0}^{\infty}$ as a function from N to N,

say g. As game one is played we are essentially getting an enumeration of countably many such functions, and if a minimal B existed at least one of these functions would have to converge to B in the manner described above. To show that a non r.e. set B is not minimal it is sufficient to find an effective procedure that, applied to the above function g enumerated by game one, will enumerate a set V such that $\emptyset <_e V(B) <_e B$.

Let us consider the following game where the player enumerates V and the opponent enumerates $W(0)$, $W(1)$, ... As before we assume that the opponent's strategy has partial degree less than or equal to the partial degree of a given set A. We will also assume that at each stage of the game we have an initial function C_B^s such that $C_B = \lim_{s \to \infty} C_B^s$, $B >_e A$ and the function that encodes this sequence of initial functions has partial degree less than or equal to the partial degree of A.

The requirements are:

R(0): $W(0) \neq V(B)$

R(1): $W(1)V(B) \neq B$

R(2): $W(2) \neq V(B)$

R(3): $W(3)V(B) \neq B$

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Now suppose that we have an effective winning strategy for the above game. Applying that strategy when the opponent simultaneously effectively enumerates all the r.e. sets such that $W(2p) = W(2p + 1)$ and $A = \emptyset$

we see that there exists an r.e. set V such that all the requirements
are satisfied. By definition $\emptyset \leq_e V(B)$. If $\emptyset \equiv_e V(B)$ then for some
r.e. set W(2r) we have V(B) = W(2r), which would imply that B does
not satisfy R(2r). If $V(B) \equiv_e B$ then for some r.e. set W(2r + 1)
we have B = W(2r + 1)V(B) and B does not satisfy R(2r + 1). Both
of these assumptions contradict the existence of an effective winning
strategy. Hence if we can find an effective winning strategy for both
games we have:

Theorem 4.2.2. For every $B >_e \emptyset$ there is an r.e. set V such that
$\emptyset <_e V(B) <_e B$.

   We notice that the V depends upon the choice of B and hence the
result is not uniform. Now suppose $B >_e C$ and the partial degree of
C is total (we may then regard C as being the graph of a characteristic
function). We now re-examine the games and this time in the first game
let the opponent play so that his strategy has partial degree less than
or equal to the partial degree of C, and W(i) = U(i)(C) i = 0, 1, 2, ...
where U(0), U(1), ... are all the r.e. sets. We also take A to be C.
Hence as our winning strategy is effective and C is total $V \leq_e C$,
that is for some r.e. set U_0, $V = U_0(C)$.

   This is not necessarily the case if the partial degree of C, say
c, is not total. As c is total we can assume that the order of
enumeration of $\{<n, j> : n \in W(j)(C)\}$ is fixed. Because given any
enumeration of C there is an effective procedure that will output a
fixed enumeration of C. Hence as our game takes a fixed enumeration
of $\{<n, j> : n \in W(j)(C)\}$ and outputs an enumeration of a set V, we

can combine these two effective procedures and get an effective procedure such that for any enumeration of C it will output an enumeration of V. However, if $\underline{c}$ is non-total we cannot effectively extract a fixed enumeration of C from an arbitrary enumeration of C, see Case [1], and so we cannot conclude that $V \leq_e C$ unless $\underline{c}$ is total.

Now $V(B) = U_0(C)(B)$ but $B >_e C$ hence for some r.e. set U_1, $C = U_1(B)$. Therefore $V(B) = U_0(U_1(B))(B)$ thus there is an r.e. set U_2 such that $V(B) = U_2(B) \leq_e B$. Finally there must be some r.e. set U_3 such that $U_3(B) = V(B)$ join C. Now suppose there is no set strictly between C and B. If $U_3(B) \equiv_e B$, then for some r.e. set W_1, $W_1 U_3(B) = B$ and so $W_1(V(B)$ join C) = B. Therefore for some q, $W(q)V(B) = B$. This would contradict the fact that an effective winning strategy exists for game one. Hence $U_3(B) \equiv_e C$, that is $U_3(B) = W(q)$ for some q. Now $U_3(B) = V(B)$ join C, hence we can extract a sequence of initial functions as before, the only difference being that this time the function that encodes them will have a partial degree that is less than or equal to the partial degree of C. Hence we can apply the second game to this sequence where the opponent's strategy is $\leq_e C$ and where $W(2i + 1) = W(2i) = U(i)(C)$, i = 0, 1, 2, ... Again we take A to be C. Now consider V(B). Clearly $V \leq_e C$, hence as in the analysis of game one, there is an r.e. set U^* such that $U^*(B) =$ V(B) join C, and $C \leq_e U^*(B) \leq_e B$. If $U^*(B) \equiv_e B$ then for some r.e. set W^*, $W^* U^*(B) = B$. However, $W^* U^*(B) = W^*(V(B)$ join C) thus as $B >_e C$ there is a q such that $W^* U^*(B) = W(2q + 1)V(B)$. Therefore $B = W^* U^*(B) = W(2q + 1)V(B)$ which contradicts R(2q + 1).

The only other alternative is $U^*(B) \equiv_e C$, that is for some r.e. set

W^*, $W^*(C) = U^*(B) = V(B)$ join C. Thus there must be an r.e. set W^{**}

such that $W^{**}(C) = V(B)$ but $W^{**}(C) = W(2q)$ for some q and hence

$W(2q) = V(B)$. This is impossible by $R(2q)$ hence our assumption that

there is no set strictly between B and C must be false. We have

Theorem 4.2.3. If $\underline{a}$ and $\underline{b}$ are partial degrees with $\underline{a}$ total and

$\underline{a} < \underline{b}$ then there is a partial degree $\underline{c}$ such that $\underline{a} < \underline{c} < \underline{b}$.

§4.3 STRATEGIES FOR THE GAMES

We now present an effective strategy for the first game and prove

that it is a winning one. The player will construct a function $f(n, s)$

at each stage s. It is intended that the value of $f(n, s)$ should

remain fixed for all sufficiently large stages. This final value of

$f(n, s)$ will be the m required for n by R_{-1}. If, during the

construction the player does not explicitly set the value of $f(n, s + 1)$

it is assumed to be $f(n, s)$. At stage 0, all sets are empty and

$f(n, 0) = 1$ for all n. A superscript s on a set being enumerated

will indicate that set as it appears after s stages of the game.

The basic idea behind the strategy is that each number n is issued

a flag $\langle n, 1 \rangle$ by putting $\langle \{n\}, \langle n, 1 \rangle \rangle$ into V. (At stage s the

flag of n will be $\langle n, f(n, s) \rangle$). We call $\langle n, 1 \rangle$ a flag since the

presence of $\langle n, 1 \rangle$ in $V(X)$ would indicate the presence of n in X.

To satisfy R_0 we ensure that the only solution of $X = W(0)V(X)$ is

$\bigcup_{n=0}^{\infty} (W(0)V)^n(\emptyset)$, where $(W(0)V)^0$ is $W(0)V$ and $(W(0)V)^{n+1}$ is

$W(0)V((W(0)V)^n)$. For the first instruction of the form $\langle E, 0 \rangle$ that is

placed in $W(0)$ we put $\langle \emptyset, \langle m, j \rangle \rangle$ into V for any $\langle m, j \rangle \in E$.

Thus $0 \in X \leftrightarrow 0 \in W(0)V(\emptyset)$. Now $\langle m, j \rangle \in E \subset V(X)$ will no longer indicate the presence of m in X, so we issue a new flag $\langle m, j + 1 \rangle$ by putting $\langle \{m\}, \langle m, j + 1 \rangle \rangle$ into V. For the first instruction of the form $\langle E, 1 \rangle$ in $W(0)$ we proceed as we would for 0 but we do not disturb any flags of 0. Thus $1 \in X \leftrightarrow 1 \in W(0)V(\emptyset) \cup (W(0)V)^1(\emptyset)$. It is clear that we can ensure that $n \in \bigcup_{j=0}^{n} (W(0)V(0))^j(\emptyset) \leftrightarrow n \in X$. Notice also that this strategy requires that n has at most $n + 1$ flags, as we only issue a new flag for n to handle a number less than or equal to n. This will satisfy $R(0)$. For $R(k)$, $k > 0$, the method is the same except that we never repeal a flag of any $n < k$. This means that any solution of $X = W(k)V(X)$ has the form $\bigcup_{n=0}^{\infty} (W(k)V)^n(F)$ where F is a subset of $\{0, 1, \ldots, k - 1\}$ and hence will be $\leq_e A$.

Player's strategy at stage $s + 1$:

1. For every member $\langle D, r \rangle$ of $W^s(\pi_1(s))$ and for every sequence $\langle F_0, \ldots, F_r \rangle$ with $F_i \subset \{x : x < \pi_1(s)\}$ such that

1.1. $D \subset V^s(N)$

1.2. $\langle m, k \rangle \in D$ and $\pi_1(s) \leq m < r \to \langle m, k \rangle \in V^s(\emptyset)$ or m is $\langle \pi_1(s), F_m \rangle$-labelled

1.3. r is not $\langle \pi_1(s), F_r \rangle$-labelled

1.4. $F_r = \{n < \pi_1(s) : \langle n, j \rangle \in D - V^s(\emptyset)$ for some $j\} \cup \bigsqcup \{F_m : \pi_1(s) \leq m < r$ and $\langle m, k \rangle \in D - V^s(\emptyset)$ for some $k\}$

the player $\langle \pi_1(s), F_r \rangle$-labels r and for each $\langle n, j \rangle \in D$ such that $n \geq \pi_1(s)$ and $n \geq r$ the player

1.5. puts $\langle \emptyset, \langle n, j \rangle \rangle$ and $\langle \{n\}, \langle n, j + 1 \rangle \rangle$ into V

1.6. sets $f(n, s + 1)$ to $f(n, s) + 1$ if $j = f(n, s)$.

(He proceeds through the members $<D, r>$ of $W^s(\pi_1(s))$ and the subsets $F_0, \ldots, F_r$ of $\{n : n < \pi_1(s)\}$ in numerical order with respect to the index $<F_0, F_2, \ldots, F_r <D, r>>$.)

2. The player puts $<\{\pi_1(s)\}, <\pi_1(s), 1>>$ into V and sets $f(\pi_1(s), s + 1) = 1$ if $f(\pi_1(s), s) = 0$.

Now consider any X, $X >_e A$. We show that R_{-1} is satisfied. Consider any $n \in N$. Clearly by our construction if $f(n, s + 1) > f(n, s) \geq 1$ then $\pi_1(s) \leq n$ and there is an $r \leq n$ that becomes $<\pi_1(s), E>$-labelled at stage $s + 1$ for some $E \subseteq \{x : x < \pi_1(s)\}$. However a number may become $<\pi_1(s), E>$-labelled at most once. Hence for all t, $f(n, t) \leq 2^n \cdot (n + 1)^2 + 1$. Thus there must be a smallest t_n such that $s \geq t_n \to f(n, s) = f(n, t_n)$. We denote $f(n, t_n)$ by $f(n)$. By induction on s:

1. $\exists t(t < s$ and $\pi_1(t) = s) \to f(n, s) \geq 1$

2. $<n, j> \in V^s(X) \to j \leq f(n, s)$

3. $0 < j < f(n, s) \to <\emptyset, <n, j>> \in V^s$

4. $f(n, s) > 0$ and $n \in X \leftrightarrow <n, f(n, s)> \in V^s(X)$.

Therefore we can conclude:

5. $<n, j> \in V(X) \to j \leq f(n)$

6. $0 < j < f(n) \to <\emptyset <n, j>> \in V$

7. $n \in X \leftrightarrow <n, f(n)> \in V(X)$.

Thus R_{-1} is satisfied for all X.

Now suppose for some $X >_e A$ and some $W(q)$ we have $W(q)V(X) = X$. We define U as follows:

$$X(0) = \{x : x < q\} \cap X$$

$$X(m + 1) = W(q)V(X(m)) \cup X(m)$$

$$U = \overset{\infty}{\underset{m=0}{\bigcup}} X(m).$$

Since the player's strategy is effective and the opponent's strategy has degree $\le_e A$ we must have $U \le_e A$. We show by induction on m that for every m, $X(m) \subset X$. Clearly $X(0) = \{x : x < q\} \cap X \subset X$. Suppose $X(m) \subset X$, then

$$X(m + 1) = W(q)V(X(m)) \cup X(m) \subset W(q)V(X) \cup X \subset X.$$

We now show by induction on n that

IH1. $n \in X \to$ there is a set $F_n \subset X$ and a stage t_n such that the player $<q, F_n>$-labels n at stage t_n

IH2. if the conclusion of the implication in IH1. holds then $n \in U$.

Suppose this is true for all $n < k$ and suppose $k \in X$. Clearly $k \in W^s(q)V^s(X)$ for some stage s. Thus there must be a member of $W^s(q)$, say $<D^*, k>$ such that $D^* \subset V^s(X)$. Let F_k be the set

$$\{n < q : <n, j> \in D^* - V(\emptyset) \text{ for some } j\} \cup$$

$$\bigcup \{F_m : q \le m < k \text{ and } <m, j> \in D^* - V(\emptyset) \text{ for some } j\}.$$

We see that $F_k \subset X$, by our induction hypothesis and noting that $n < q$ and $<n, j> \in D^* - V(\emptyset) \subset V(X) \to j = f(n)$ and $n \in X$. Now conditions 1.1., 1.2. and 1.4. given in the player's strategy clearly hold for

$\langle D*, k \rangle$ and F_k at any sufficiently large stage t with $\pi_1(t) = q$.
Hence there must be a stage t_k such that the player $\langle q, F_k \rangle$-labels
k at stage t_k. Thus we have shown part IH1. of our induction result.
Now suppose the conclusion of part IH1. holds about k. Let $\langle D, k \rangle$
be the member of $W^{t_k}(q)$ that causes k to be $\langle q, F_k \rangle$-labelled at
stage t. Now consider $\langle n, j \rangle \in D$.

Case a) $n < q$: in which case either $\langle n, j \rangle \in V(\emptyset) \subseteq V(U)$ or

$n \in F_k \subseteq X(0)$ whence $\langle n, j \rangle \in V(X(0)) \subseteq V(U)$.

Case b) $n \geq q$: we have two sub-cases

i) $n \geq k$, in which case $\langle \emptyset, \langle n, j \rangle \rangle \in V^{t_k+1}$ and

hence $\langle n, j \rangle \in V(\emptyset \subseteq V(X)$,

ii) $n < k$, in which case either $\langle n, j \rangle \in V^{t_n}(\emptyset) \subseteq V(U)$

or n is $\langle q, F* \rangle$-labelled for some $F* \subseteq F_k \subseteq X$

and hence by our induction hypothesis $n \in U$ and

$\langle n, j \rangle \in V(U)$.

Thus in all cases $\langle n, j \rangle \in V(U)$. Therefore $D \subseteq V(U)$ so $k \in W(q)V(U) = U$. This completes our induction.

Thus we have shown that $X \subseteq U$ and $U \subseteq X$ hence $X = U$. This
would contradict our hypothesis that $X >_e A$ as $U \leq_e A$. Hence our
assumption that $W(q)V(X) = X$ is false. Thus any X such that $X >_e A$
must satisfy all the requirements.

To construct a winning strategy for the second game we rely heavily
on the sequence of characteristic functions. At any stage we act as if
the characteristic function given to us at that stage is correct. The
negations of some requirements, which are equalities, will be evidently

false if we know that our current characteristic function is correct.
We only deal with the smallest requirement that could possibly be unsat-
isfied (i.e. the equality could hold). If this requirement is an even
one, say R(2p), we issue flags of the form <n, 2p> for larger and
larger n. If the requirement is odd, say R(2p + 1), we force
W(2p + 1)V(X) to be W(2p + 1)V($\emptyset$) for any X. If the opponent allows
us to work on a requirement infinitely many times any solution to its
equation will have to have partial degree less than or equal to the
partial degree of A. Thus for any requirement he must make the equality
in that requirement contradict the characteristic functions for all
sufficiently large stages. However as the sequence of characteristic
functions converges to C_B it will be impossible for any such equality
to hold at the game's conclusion.

A superscript s on one of the sets being enumerated by the
opponent or the player will denote that set as it appears after s stages.
At stage 0 all sets are empty and the player does nothing. We make the
following definitions to simplify the strategy and the proof.

4.3.1. s(n) is the smallest stage such that

$$s \geq s(n) \rightarrow C_B^s(n) = C_B^{s(n)}(n) = C_B(n).$$

4.3.2. t(q, s) is the largest stage < s such that R(q) is the
first requirement not passive at stage t(q, s) and is 0
if no such stage exists.

4.3.3. R(2q) is *passive at stage* s if s > 0 and for some
$n < t(2q, s)$, $<n, 2q> \notin V^s(\emptyset)$ and $W^s(2q)[<n, 2q>] \neq C_B^s(n)$.

4.3.4. $D(2q + 1, s) = \{<n, 2p> \in V^s(N) : 2p < 2q + 1$ and $c_B^s(n) = 1\}$

$\cup \{<n, 2p> \in V^s(\emptyset) : 2p < 2q + 1\}.$

4.3.5. $Z(2q + 1) = \{<n, 2p> : n \in N, \ 2p > 2q + 1\}$

4.3.6. $R(2q + 1)$ is *passive at stage* s if $s > 0$ and for some

$n < t(2q + 1, s)$ either

i) $1 = W^s(2q + 1)(D(2q + 1, s) \cup V^s(\emptyset))[n] \neq c_B^s(n)$ or

ii) $0 = W^s(2q + 1)(D(2q + 1, s) \cup Z(2q + 1))[n] \neq c_B^s(n).$

4.3.7. $R(q)$ is active at stage s if it is not passive at stage s.

Player's strategy at stage $s + 1$:

Let q be the smallest number $< s$ such that $R(q)$ is active at

stage s (if there is no such q the player does nothing at stage $s + 1$).

Case 1. $q = 2p$

for each $n < s$ the player puts $<\{n\}, <n, 2p>>$ into V.

Case 2. $q = 2p + 1$

for all n, r such that $s > 2r > 2p + 1$ and $n < s$ the

player puts $<\emptyset, <n, 2r>>$ into V.

We now show that this strategy is a winning one. In particular we

shall show that for each q, $R(q)$ is satisfied and there is a stage

$r(q)$ such that $s \geq r(q) \rightarrow R(q)$ is passive at stage s. Suppose there

is a q for which this does not hold and let q^* be the smallest such

q. Let t' be chosen $> r(p)$ for all $p < q^*$ and let $t^* > t'$ be

chosen such that $t^* > s(m)$ for all $m < t'$.

We consider two cases.

Case 1. $q^* = 2p^*$ for some p^*.

Suppose for each stage t there is an $s \geq t$ such that $R(2p*)$ is active at stage s. Let $U = \{n : \langle n, 2p* \rangle \in W(2p*)\}$, and let $L = \{n : n < t*\}$. Clearly $U \leq_e W(2p*) \leq_e A$. We shall prove that the symmetric difference of B and U is a subset of L and hence finite. We can then conclude that $B \leq_e U \leq_e A$ which is impossible, hence $r(2p*)$ must exist. Suppose $n \in B - L$. Let us choose an $r > 0$ such that $r > s(n)$, $r > t*$, $t(2p*, r) > n$ and $R(2p*)$ is active at stage r. Consider any $2q + 1 < 2p*$. Since $t* > r(2q + 1)$, $R(2q + 1)$ cannot be active after stage $t*$ and since $n > t*$ the player never puts $\langle \emptyset, \langle n, 2p* \rangle \rangle$ into V on behalf of $R(2q + 1)$. Thus $\langle n, 2p* \rangle \notin V(\emptyset)$, and as $R(2p*)$ is active at stage r we must have $W^r(2p*)[\langle n, 2p* \rangle] = C_B^r(n)$. Now $r > s(n)$ and $n \in B$, hence $C_B^r(n) = 1$. Therefore $\langle n, 2p* \rangle \in W^r(2p*) \subseteq W(2p*)$ which implies that $n \in U$. This shows that $B - L \subseteq U$. Suppose $n \in U - L$ and $n \notin B$. Choose r as above. As $n \notin B$ and $r > s(n)$ we have $0 = C_B(n) = C_B^r(n)$. By the same reasoning as above $\langle n, 2p* \rangle \notin V(\emptyset)$ hence as $r(2p*)$ is active at stage r, $\langle n, 2p* \rangle \notin W^r(2p*)$. Now as out choice of r could have been arbitrarily large we must have $\langle n, 2p* \rangle \notin W(2p*)$ which contradicts the fact that $n \in U$. Hence $U - L \subseteq B$ and thus the symmetric difference of B and U is included in L. By our remarks above $r(2p*)$ must exist. We must then have by our assumption about $q*$ that $R(2p*)$ is not satisfied, that is $W(2p*) = V(B)$.

Choose $r > s(m)$ for all $m < t* + r(2p*)$. Now as $r > r(2p*)$ we have $R(2p*)$ passive at stage r, hence for some $n < t(2p*, r) < r(2p*)$, $\langle n, 2p* \rangle \notin V^r(\emptyset)$ and $W^r(2p*)[\langle n, 2p* \rangle] \neq C_B^r(n)$. Now as $n < r(2p*)$, $r > s(n)$ hence $C_B(n) = C_B^r(n)$. Also as we could have

chosen r arbitrarily large we can assume that $W^r(2p^*)[<n, 2p^*>] = W(2p^*)[<n, 2p^*>]$. Hence $C_B(n) \neq W(2p^*)[<n, 2p^*>]$. As $n < t(2p^*, r)$ the player must have put $<\{n\}, <n, 2p^*>>$ into V at stage $t(2p^*, r)$. However $<n, 2p^*> \notin V(\emptyset)$, thus $C_B(n) = V(B)[<n, 2p^*>]$. By our assumption that $R(2p^*)$ is not satisfied we have $V(B) = W(2p^*)$ which would imply that $C_B(n) = W(2p^*)[<n, 2p^*>]$ contradicting $W^r(2p^*)[<n, 2p^*>] \neq C_B(n)$. Therefore our assumption about q^* cannot hold in this case, and Case 1 is impossible.

Case 2. $q^* = 2p^* + 1$ for some p^*.

Suppose for each stage t there is an $s \geq t$ such that $R(2p^* + 1)$ is active at stage s. Let

$$U = W(2p^* + 1)(D(2p^* + 1, t^*) \cup V(\emptyset)).$$

Clearly $U \leq_e A$. We shall show $B = U$ and hence $r(2p + 1)$ must exist. Suppose $n \in B$. Choose $r > 0$ such that $r > s(n)$, $r > t^*$, $t(2p^* + 1, r) > n$ and $R(2p^* + 1)$ is active at stage r. Now suppose $<m, 2p> \in D(2p^* + 1, t^*)$. Thus $<m, 2p> \in V^{t^*}(N)$, hence $m < r(2p)$. Therefore $t^* > s(m)$ and $1 = C_B^{t^*}(m) = C_B^r(m)$. Also $V^{t^*}(N) \subset V^r(N)$, therefore $D(2p^* + 1, t^*) \subset D(2p^* + 1, r)$. Notice that if $<m, 2p> \in V^r(N)$ then $m < r(p) < t^*$ hence $<m, 2p> \in V^{t^*}(N)$ and $C_B^r(m) = 1 \rightarrow C_B^{t^*}(m) = 1$ as $r > t^*$, therefore $D(2p^* + 1, r) \subset D(2p^* + 1, t^*)$. This shows that $D(2p^* + 1, r) = D(2p^* + 1, t^*)$. Now suppose $<m, 2q> \in Z(2p^* + 1)$, we can choose $r' > r$ so that $r' > 2q$, $r' > m$ and $R(2p^* + 1)$ is active at stage r'. Hence as $r' > r > r(2p)$ for all $2p < 2p^* + 1$ and as $R(2p^* + 1)$ is active at stage r' the player puts $<\emptyset, <m, 2q>>$ into V at stage r'. Thus $Z(2p^* + 1) \subset V(\emptyset)$

hence

$$W^r(2p* + 1)(D(2p* + 1, r) \cup Z(2p* + 1))$$

$$\subset W(2p* + 1)(D(2p* + 1, t*) \cup V(\emptyset)) = U$$

Now as $R(2p* + 1)$ is active at stage r and $r > s(n)$ then $1 = C_B(n) = C_B^r(n) = W^r(2p* + 1)(D(2p* + 1, r) \cup Z(2p* + 1))[n]$ hence $n \in U$. Now suppose $n \in U$, we can choose r as before but with the added condition that

$$n \in W^r(2p* + 1)(D(2p* + 1, t*) \cup V^r(\emptyset)).$$

Now as before we have $D(2p* + 1, r) = D(2p* + 1, t*)$. Thus as $R(2p* + 1)$ is active at stage r we have

$$1 = W^r(2p* + 1)(D(2p* + 1, r) \cup V^r(\emptyset))[n] = C_B^r(n)$$

and as $r > s(n)$, $C_B(n) = 1$. Thus $B = U$ which is impossible as $U \leq_e A$, hence $r(2p* + 1)$ must exist. Therefore, by our assumption about $q*$, $R(2p* + 1)$ cannot be satisfied, that is $B = W(2p* + 1)V(B)$. Choose r such that $r > r(2p* + 1)$, $r > t*$ and such that $r > s(m)$ for all $m < r(2p* + 1)$. As above we have $D(2p* + 1, r) = D(2p* + 1, t*)$. Also $R(2p* + 1)$ is passive at stage r hence for some $n < t(2p* + 1, r) < r(2p* + 1)$ we have two cases:

Case a) $1 = W^r(2p* + 1)(D(2p* + 1, r) \cup V^r(\emptyset))[n] \neq C_B^r(n)$.

Consider $\langle m, 2q \rangle \in D(2p* + 1, r)$. Either $\langle m, 2q \rangle \in V^r(\emptyset) \subset V(B)$ or $C_B^r(m) = 1$. Now as $m < r(2q)$ we have $r > s(m)$ hence $C_B(m) = 1$ and thus $\langle m, 2q \rangle \in V(B)$. Therefore $D(2p* + 1, r) \cup V^r(\emptyset) \subset V(B)$ and

thus $n \in W^r(2p* + 1)(D(2p* + 1, r) \cup V^r(\emptyset)) \subset W(2p* + 1)V(B) = B$ which is impossible.

Case b) $0 = W^r(2p* + 1)(D(2p* + 1, r) \cup Z(2p* + 1))[n] \neq C_B^r(n)$.

Now as $n < t(2p* + 1, r) < r(2p* + 1)$ we have $r > s(n)$, and thus $n \in B$. As $B = W(2p* + 1)V(B)$, $n \in W(2p* + 1)V(B)$ and there must be a member of $W(2p* + 1)$ of the form $<D, n>$ where $D \subseteq V(B)$. Consider $<m, 2q> \in D$, there are two possibilities:

      i) $2q < 2p* + 1$, in which case either $<m, 2q> \in V(\emptyset)$ or $m \in B$. Suppose $m \in B$, we must have $m < r(2q)$, by our choice of r we have $r > s(m)$ hence $C_B^r(m) = 1$ and $<m, 2q> \in D(2p* + 1, r)$. Thus for sufficiently large r, $<m, 2q> \in D$ and $2q < 2p* + 1 \rightarrow <m, 2q> \in D(2p* + 1, r)$.

     ii) $2q > 2p* + 1$, in which case $<m, 2q> \in Z(2p* + 1)$. Thus we can choose r sufficiently large so that $D \subseteq D(2p* + 1, r) \cup Z(2p* + 1)$ and $<D, n> \in W^r(2p* + 1)$. Hence $n \in W^r(2p* + 1)(D(2p* + 1, r) \cup Z(2p* + 1))$ which is impossible.

Thus neither case a) nor b) can hold and hence $R(2p* + 1)$ must be satisfied. Therefore there can be no such stage $q*$ and the player's strategy must be a winning one. This completes the proof of our result.

CHAPTER V

DEGREES MINIMAL ABOVE A

NON-TOTAL DEGREE

§5.1 INTRODUCTION

In this chapter we will extend some of the results of the previous chapter. In particular we will show that for a given partial degree $\underline{a}$ there are at most countably many partial degrees minimal above $\underline{a}$, that is, for any B there are at most countably many C such that $C >_e B$ and for no A is $C >_e A >_e B$. In the previous chapter we were able to do this for any B that belonged to a total degree. However for sets that belonged to non-total partial degrees we were not able to ensure that the enumeration operator we constructed was independent of the particular order in which the opponent enumerated his sets. We will give a new strategy for the first game of the previous chapter that will itself be an enumeration operator and hence will produce an enumeration operator that is independent of the order of the opponent's enumeration.

§5.2 EXTENDED ENUMERATION OPERATORS

In order to demonstrate this new strategy we will need operators that consider not only the input enumeration but also their own output enumeration. We will show that such operators can be replaced by enumeration operators, and hence are only a notational convenience.

Definition 5.2.1.

.1. a triple of the form $\langle n, e, D \rangle$ is called a *positive e-*
    condition

.2. a triple is a *positive condition* if it is a positive e-
    condition for some e

.3. an *extended enumeration operator* S is an r.e. set of
    positive conditions

.4. a set T is said to *satisfy* S *for* X if for every positive
    e-condition $\langle n, e, D \rangle$ in S, we have

$$n \in W_e(T \text{ join } X) \rightarrow D \subseteq T$$

.5. S$\langle$X$\rangle$ is the intersection of all sets T which satisfy S for
    X.

Theorem 5.2.2. If S is an extended enumeration operator then S$\langle$X$\rangle$
satisfies S for X.

Proof: Suppose L(1), L(2), ... is an enumeration of S. Also suppose
L(j) = $\langle$n, e, D$\rangle$. Then we define

$$Y(0, j) = \begin{cases} D & n \in W_e(\emptyset \text{ join } X) \\ \\ \emptyset & \text{otherwise .} \end{cases}$$

Let $Z(0) = \bigcup_{j=0}^{\infty} Y(0, j)$. Now we define Z(k + 1) by induction. Suppose
Z(k) has been defined, also suppose that L(j) = $\langle$n, e, D$\rangle$, then we let

$$Y(k + 1, j) = \begin{cases} D & \text{if } n \in W_e(Z(k) \text{ join } X) \\ \\ \emptyset & \text{otherwise .} \end{cases}$$

We then set $Z(k + 1) = \bigcup_{j=0}^{\infty} Y(k + 1, j) \cup Z(k)$. Finally we set $Z = \bigcup_{k=0}^{\infty} Z(k)$.

Consider any $L(j)$, say $\langle n, e, D \rangle$, and suppose $n \in W_e(Z \text{ join } X)$. Now $Z(k) \subset Z(k + 1)$ for all k, hence there must be m such that $n \in W_e(Z(m) \text{ join } X)$. By our construction of $Y(m + 1, j)$ we have $D = Y(m + 1, j) \subset Z(m + 1) \subset Z$. Therefore Z satisfies S for X and thus $S\langle X \rangle \subset Z$.

We will now show that $Z \subset S\langle X \rangle$. This will prove that $Z = S\langle X \rangle$ and as we have shown that Z satisfies S for X, we will have proved the theorem.

Let us consider a T such that T satisfies S for X. If $x \in Z(0)$ then $x \in Y(0, j)$ for some j, where $L(j) = \langle n, e, D \rangle$, and by our construction of $Y(0, j)$ we have $n \in W_e(\emptyset \text{ join } X)$ and $x \in D$. However $W_e(\emptyset \text{ join } X) \subset W_e(T \text{ join } X)$, and as T satisfies S for X we can conclude that $D \subset T$. Thus $x \in T$ and hence $Z(0) \subset T$.

Now suppose $Z(k) \subset T$. If $x \in Z(k + 1) - Z(k)$ then $x \in Y(k + 1, j)$ for some $L(j) = \langle n, e, D \rangle$. Therefore $n \in W_e(Z(k) \text{ join } X)$ and $x \in D$. Now $Z(k) \subset T$, hence $n \in W_e(Z(k) \text{ join } X) \subset W_e(T \text{ join } X)$ and as T satisfies S for X, $x \in D \subset T$. Thus by induction $Z \subset T$. Now as T was an arbitrary set that satisfied S for X, we must have $Z \subset S\langle X \rangle$.

This theorem tells us that $S\langle X \rangle$ is a solution to all the conditions in S. We will now show that an extended enumeration operator can be replaced by an enumeration operator.

Theorem 5.2.3. If S is an extended enumeration operator then there exists an enumeration operator V such that for all X, $V(X) = S\langle X \rangle$.

Proof: Let $L(1), L(2), \ldots$ be a recursive enumeration of S. We will

assume that this enumeration is being given, together with a simultaneous

effective enumeration of all the r.e. sets. We will construct V in

stages. In effect we will be showing that V is enumeration reducible

to these input enumerations and hence is r.e. We will denote by W_e^s,

the finite subset of W_e that has been enumerated at stage s.

Let $V(0) = \emptyset$. Consider some stage $s + 1$. Suppose $\pi_1(s + 1) = j$.

Let $U = \bigcup_{k=0}^{s} V(k)$, and suppose $L(j) = <n, e, D>$. Then we take as $V(s + 1)$

the set of all pairs $<F_0 \cup F_1, x>$ such that $x \in D$, $F_0 \subset \{i : i < s\}$

and for some $F_2 \subset U(F_0)$ we have $<F_2$ join $F_1, n> \in W_e^s$. Let $V = \bigcup_{k=0}^{\infty} V(k)$.

Clearly each $V(k)$ is finite and $V \leq_e S$ and hence V is r.e.

We now complete the proof by showing that $V(X) = S<X>$ for all X.

Consider any set X. Clearly $V(0)(X) \subset S<X>$. Suppose $V(t)(X) \subset$

$S<X>$ for all $t < s + 1$. Let $n \in V(s + 1)(X)$. Suppose $\pi_1(s + 1) = j$

and $L(j) = <m, e, D>$. Now as $n \in V(s + 1)(X)$ there must be some pair

$<F_0 \cup F_1, n> \in V(s + 1)$ with $F_0 \cup F_1 \subset X$, $n \in D$ and for some F_2,

$F_2 \subset (\bigcup_{k=0}^{s} V(k))(F_0)$ and $<F_2$ join $F_1, m> \in W_e$. By the induction hypothesis

$V(k)(X) \subset S<X>$ for all $k \leq s$ hence

$$F_2 \subset (\bigcup_{k=0}^{s} V(k))(X) \subset \bigcup_{k=0}^{\infty}(V(k)(X)) \subset S<X>$$

and

$$m \in W_e(S<X> \text{ join } X).$$

As $S<X>$ satisfies S for X by Theorem 5.2.2., $D \subset S<X>$ and thus

$n \in S<X>$. Clearly then

$$V(X) \subset S<X>.$$

Referring to our proof of Theorem 5.2.2., we see that $S\langle X\rangle = \bigcup_{k=0}^{\infty} Z(k)$.

Suppose $n \in Z(0)$, then $n \in Y(0, j)$ for some j, where $L(j) = \langle m, e, D\rangle$. By the construction of $Y(0, j)$, $m \in W_e(\emptyset \text{ join } X)$ and $Y(0, j) = D$. Now choosing s sufficiently large with $\pi_1(s + 1) = j$ we have $m \in W_e^s(\emptyset \text{ join } X)$, that is there is a finite set F_1 such that $F_1 \subseteq X$ and $\langle \emptyset \text{ join } F_1, m\rangle \in W_e^s$. Now clearly $\emptyset \subseteq V(s)(\emptyset)$, hence $\langle \emptyset \cup F_1, n\rangle \in V(s + 1)$ and $n \in V(s + 1)(X) \subseteq V(X)$. Therefore $Z(0) \subseteq V(X)$. Now suppose $Z(k) \subseteq V(X)$. Let $n \in Z(k + 1) - Z(k)$, then $n \in Y(k + 1, j)$ for some j, where $L(j) = \langle m, e, D\rangle$. We see that $m \in W_e(Z(k) \text{ join } X)$ and $Y(k + 1, j) = D$. We choose s sufficiently large so that $m \in W_e^s(Z(k) \text{ join } X)$ and $\pi_1(s + 1) = j$. There must be a member of W_e^s of the form $\langle F_2 \text{ join } F_1, m\rangle$ where $F_1 \subseteq X$ and $F_2 \subseteq Z(k)$. Now $Z(k) \subseteq V(X)$, hence we can assume that s is sufficiently large so that $F_2 \subseteq (\bigcup_{t=0}^{s} V(t))(X)$. Thus there is a finite set $F_0 \subseteq X$ such that $F_2 \subseteq (\bigcup_{t=0}^{s} V(t))(F_0)$. We can also assume that $F_0 \subseteq \{i : i < s\}$. By our construction of $V(s + 1)$ we have $\langle F_0 \cup F_1, n\rangle \in V(s + 1)$, and $n \in V(X)$ as $F_0 \cup F_1 \subseteq X$. Therefore $Z = S\langle X\rangle \subseteq V(X)$ by induction and so

$$S\langle X\rangle = V(X).$$

§5.3 AN ENUMERATION STRATEGY

We will now reconsider the first game in the previous chapter. One of the conclusions drawn from that game was that there are at most a countable number of partial degrees minimal above a total degree. If the opponent changes the order of enumeration of his sets, the strategy given in the previous chapter will produce a different set V (unless the

opponent is enumerating all sets enumeration reducible to a total set, in which case it can always be assumed that the order is fixed). Let K_C denote the set $\{<n, e> : n \in W_e(C)\}$. We will demonstrate an extended enumeration operator S such that for any A if $V = S<K_A>$ and $W(i) = W_i(A)$ for all i then for $A = W(0)$ the requirements R_0, R_1, R_2, ... listed on page 37 hold together with

R_{-1}': $\forall Y \forall Z (Y \neq Z \rightarrow V(Y) \neq V(Z))$.

We will show that this is sufficient to prove our result.

The idea behind the construction of our extended enumeration operator is similar to the strategy for the first game in Chapter IV. To illustrate this and to clarify the construction we will first construct an S such that $V = S<W(0)>$ satisfies the requirements R_0, R_{-1}'.

Let f be the function from $N \times N$ to 2^N defined by letting $f(n, j)$ be the set $\{i : i \leq n\} - D_{j-1}$ if $j \geq 1$ and $\emptyset$ otherwise. Let g be the function from $N \times N$ to N defined by letting $g(j, i)$ be 1 plus the canonical index of $D_{j-1} \cup \{i\}$ if $j \geq 1$ and 0 otherwise.

Consider the following four sentences concerning a pair $<D, r>$:

1. $<D, r> \in Y$

2. $D \subset S<Y>(N)$

3. $(<n, j> \in D$ and $n \geq r) \rightarrow r \in f(n, j)$

4. $(<n, j> \in D$ and $n < r) \rightarrow$ (there is a set D^* such that $<D^*, n>$ satisfies 1. – 4. or $<n, j> \in S<Y>(\emptyset)$).

When we take Y to be $W(0)$, then these will be the conditions on a pair $<D, r>$ in $W(0)$ that must hold before we repeal some flags in D. The first condition asks that $<D, r>$ be in $W(0)$ and the second

that $D \subseteq S<W(0)>(N)$. Both of these conditions are clearly necessary.
The third condition is to avoid a process in the strategy of the first
game that cannot be handled by an enumeration operator. In that strategy
once we have forced D, from a pair $<D, r>$ into $W(0)$, into
$\bigcup_{k=0}^{\infty} (W(0)V)^k(\emptyset)$ we then label r so as not to upset more flags to achieve
an already accomplished result. Let us look at an example of how we
achieve a similar result in an enumeration operator. Now 0 will have
two possible flags $<0, 1>$ and $<0, 2>$. We will construct S so that
$<\{0\}, <0, 1>>$ is always in $S<W(0)>$. If some $<E, 0>$ appears in $W(0)$
with $E \subseteq S<W(0)>$, then we will repeal all the flags in E, that is,
force each $<n, j>$ in E into a solution of $X = W(0)S<W(0)>(X)$ by
putting $<\emptyset, <n, j>>$ into $S<W(0)>$. We will then issue new flags to all
the n such that $<n, j> \in E$ which will encode the fact that 0 has
been forced into a solution of $X = W(0)S<W(0)>(X)$. The set D_{j-1} is
the set of all $r \leq n$ that have already been forced into a solution of
our equation, $g(j, 0)$ will then be one plus the canonical index of
$D_{j-1} \cup \{0\}$, and we will issue a new flag $<n, g(j, 0)>$ to each $<n, j>$
in E. Hence if $<0, 1> \in E$ then we will put $<\emptyset, <0, 1>>$ and $<0, <0, 2>>$
into $S<W(0)>$. Now $<0, 2>$ will never be repealed because as in the
strategy for the first game in Chapter IV we never repeal a flag of r to
force in a number $n > r$, and as $<0, 2>$ has coded into it the fact that
0 has been forced into a solution of our equation, we have exhausted all
the possible reasons to repeal a flag of 0. In general there are 2^{n+1}
possible flags of n.

In the first strategy given in the last chapter much the same result
as the above was gained by labelling numbers to indicate that they had

been forced into a solution of $X = W(0)V(X)$. The player could then

check the label of a number to see if he wanted to repeal flags on its

behalf or not. This technique cannot be used in a strategy which is an

enumeration operator as one cannot inhibit instructions from working

after they have been put in the operator.

The intent of the fourth condition is to ensure that if there is a

set $E \subset S<W(0)>(N)$ and $<E, r> \in W(0)$ and if we force r into any

solution of $X = W(0)S<W(0)>(X)$ then for all $<n, j> \in E$ with $n < r$,

which we cannot repeal for r, we have that n has been forced into a

solution of $X = W(0)S<W(0)>(X)$.

If we consider the set of all $<D, r>$ that satisfy 1. to 4. above,

we see as f is recursive that this set is $\leq_e S<Y>$ join Y and hence is

$W_d(S<Y>$ join $Y)$ for some d.

Now if $<D, r>$ satisfies 1. to 4. we will want to repeal all flags

$<n, j>$, $<n, j> \in D$, such that $n \geq r$. Thus we let

$$E_1(x) = \{<\emptyset,<n, j>> : <n, j> \in D_{\pi_1(x)} \text{ and } n \geq \pi_2(x)\}$$

We will also need to issue new flags which will indicate that r has been

forced into our solution. We let

$$E_2(x) = \{<\{n\}, <n, g(j, \pi_2(x))>> : <n, j> \in D_{\pi_1(x)} \text{ and}$$

$$n \geq \pi_2(x)\}.$$

We let S_1 be the set of all positive conditions $<x, d, E_1(x) \cup E_2(x)>$.

Clearly $\lambda x[E_1(x) \cup E_2(x)]$ is recursive hence S_1 is r.e.

We also need instructions to issue initial flags to all n. Let S_2

be the set of all positive conditions $\langle n, d', D \rangle$ where d' is such that $W_{d'}(\emptyset) = N$ and $D = \{\langle\{n\}, \langle n, 1\rangle\rangle\}$.

Finally let $S = S_1 \cup S_2$. Clearly S is r.e. We shall show that $V = S\langle W(0)\rangle$ satisfies R_0 and R'_{-1}. We denote by P1., P2., P3. and P4. the instances of 1., 2., 3. and 4. respectively that result from replacing Y by $W(0)$, and the references to 1., 2., 3. and 4. by references to P1., P2., P3., and P4.

Consider R_0,

$$W(0)V(X) \neq X, \quad \text{for all} \quad X >_e A.$$

Consider a particular X so that

$$W(0)V(X) = X.$$

We will show that $X \leq_e W(0)$. Let

$$X(0) = \emptyset$$

$$X(m + 1) = W(0)V(X(m)) \cup X(m)$$

$$U = \overset{\infty}{\underset{m=0}{\cup}} X(m).$$

Now by Theorem 5.2.3 $V = S\langle W(0)\rangle \leq_e W(0)$, hence $U \leq_e W(0)$.

We claim that $X = U$. Clearly $U \subset X$ as

$$X(0) = \emptyset \subset X$$

$$X(m) \subset X \rightarrow X(m + 1) = W(0)V(X(m)) \cup X(m) \subset X.$$

We will show by induction that $X \subset U$. Suppose for all $k < r$ we have

IH1. $k \in X \rightarrow$ there is a D^* such that $\langle D^*, k \rangle$ satisfies P1. to P4.

IH2. if the conclusion of the implication in IH1. holds then $k \in U$.

We first show that part IH1. of our induction hypothesis holds for r.

Suppose $r \in X$, then for some $D \subset V(X)$, $\langle D, r \rangle \in W(0)$. We claim that P1. to P4. hold for the pair $\langle D, r \rangle$ or for some pair $\langle D^*, r \rangle$.

As $\langle D, r \rangle \in W(0)$ and $D \subset V(X) \subset V(N)$ clearly P1. and P2. hold for $\langle D, r \rangle$. Now suppose $\langle n, j \rangle \in D$ and $n \geq r$. If $r \notin f(n, j)$ then as $\langle n, j \rangle \in V(X)$ and V satisfies S for $W(0)$ we must have for some j', $\langle \{n\}, \langle n, g(j', r) \rangle \rangle \in V$ and hence for some D^*, $\langle D^*, r \rangle \in W_d(V$ join $W(0))$. That is $\langle D^*, r \rangle$ satisfies P1. to P4. which would show IH1. was true for r. Now suppose $\langle n, j \rangle \in D$ and $n < r$, then as $\langle n, j \rangle \in V(X)$ either $\langle n, j \rangle \in V(\emptyset)$ or $n \in X$ whence by our induction hypothesis P1. to P4. hold for some $\langle D^*, n \rangle$.

We now show that part IH2. holds for r. Suppose for some D, $\langle D, r \rangle$ satisfies P1. to P4. We claim that $D \subset V(U)$. Consider $\langle n, j \rangle \in D$:

Case 1. $n < r$

Now as $\langle D, r \rangle$ satisfies P4. we have either for some D^*, $\langle D^*, n \rangle$ satisfies P1. to P4. hence by our induction hypothesis $n \in U$, and thus $\langle n, j \rangle \in V(U)$, or $\langle n, j \rangle \in V(\emptyset)$ whence $\langle n, j \rangle \in V(\emptyset) \subset V(U)$.

Case 2. $n \geq r$

Now as $\langle D, r \rangle$ satisfies P1. to P4. we have $\langle D, r \rangle \in W_d(V$ join $W(0))$. Hence as V satisfies S for $W(0)$, $E_1(\cdot D, r \rangle) \subset V$, and $\langle \emptyset, \langle n, j \rangle \rangle \in V$. Therefore $\langle n, j \rangle \in V(\emptyset) \subset V(U)$.

This completes our proof that $D \subseteq V(U)$. Clearly $r \in W(0)V(U) = U$. Hence we have shown by induction that $X \subseteq U$. It follows that $X = U$ and thus $X \leq_e W(0)$.

We now show that V satisfies R'_{-1}. Suppose for proof by contradiction that $Z \neq Z^*$ (we can assume $Z - Z^* \neq \emptyset$) but $V(Z) = V(Z^*)$. Let $n \in Z - Z^*$. Consider $\langle n, j(1) \rangle \in V(Z)$ (there must be at least one such number in $V(Z)$ as $n \in W_d.(W(0)) = N$ and hence $\langle \{n\}, \langle n, 1 \rangle \rangle \in S\langle W(0) \rangle$). As $n \notin Z^*$ we have $\langle n, j(1) \rangle \in V(\emptyset)$. Then $\langle \emptyset, \langle n, j(1) \rangle \rangle \in V$, whence some $\langle D, r(1) \rangle \in W_d$, $n \geq r(1)$, and $\langle n, j(1) \rangle \in D$. Now as $\langle D, r(1) \rangle$ satisfy P1. to P4. we have $r(1) \in f(n, j(1))$. Also $\langle \{n\}, \langle n, g(j(1), r(1)) \rangle \rangle \in V$. Let $j(2) = g(j(1), r(1))$. Now $\langle n, j(2) \rangle \in V(Z)$ hence we can repeat our argument with $j(2)$ instead of $j(1)$. Thus we can construct two sequences $\langle j(i) \rangle_{i=1}^{\infty}$, $\langle r(i) \rangle_{i=1}^{\infty}$. Now as $j(k + 1) = g(j(k), r(k))$ we have $r(k) \notin f(n, j(k + 1))$. However, $r(k) \in f(n, j(k))$. Therefore for some k, $f(n, j(k)) = \emptyset$ and $r(k) \in f(n, j(k))$ which is impossible. Hence $Z \neq Z^* \rightarrow V(Z) \neq V(Z^*)$ and R'_{-1} is satisfied.

We will now show that it is possible to extend S so that $V = S\langle K_A \rangle$ will satisfy all the requirements R'_{-1}, R_0, R_1, $\ldots$. The basic idea, as in the first game of Chapter IV, is to not let any requirement $R(k)$, $k > n$, disturb any flag of n. This, however, adds a complication, specifically that it is not sufficient to force some r into $(W(q)V(0))^k(F)$ for some k by repealing some members of a set D, and then consider r as "handled" for the remainder of the game. Attention must be paid to the flags of numbers $< q$ that were in the set D. We must be prepared to take action on behalf of r for each possible subset of numbers $< q$. To provide for this action we let our second co-ordinate

code not only the numbers less than n that have been acted on but also
the set of high priority numbers that was involved in this action. We
re-define f as a function from $N \times N$ into $2^{N \times F}$, where F is the
set of all finite subsets of N. We let f(n, j) be the set $\{<r, F> :$
$r \leq n$ and $F \subset \{i : i < n\}\} - D_{j-1}$ when $j \geq 1$ and $\emptyset$ otherwise.
We re-define g as a function from $N \times N \times F$ to N by setting
g(j, r, F) to the canonical index of the set $D_{j-1} \cup \{<r, F>\}$, if
$j \geq 1$ and 0 otherwise.

For each j, j = 0, 1, 2, ..., consider the following six
sentences concerning a triple <D, r, F>:

j.1. $<D, r> \in W_{h(j)}(Y)$

j.2. $D \subseteq S<Y>(N)$

j.3. $<n, k> \in D$ and $n \geq r$ and $n \geq j \rightarrow <r, F> \in f(n, k)$

j.4. $<n, k> \in D$ and $j \leq n < r .\rightarrow.$ there is an E*, $E* \subseteq F$ and
a set D* such that <D*, n, E*> satisfy j.1. to j.6. or $<n,k> \in S<Y>(\emptyset)$.

j.5. $m \in \{n : \exists k(<n, k> \in D$ and $n < j)\} - F \rightarrow m \in S<Y>(\emptyset)$

j.6. $F \subseteq \{n : n < j\}$.

Let us suppose that h(j) is recursive. Then clearly there is a
recursive function d(j) such that <D, r, F> satisfies j.1. to j.6.
if and only if <D, r, F> $\in W_{d(j)}$(S<Y> join Y).

We let

$$E_3(x, j) = \{<\emptyset, <n, k>> : <n, k> \in \pi_1(x) \text{ and}$$

$$n \geq \pi_2(x) \text{ and } n \geq j\}$$

$$E_4(x, j) = \{<n, <n, g(k, \pi_2(x), \pi_3(x))>> : <n, k> \in \pi_1(x) \text{ and}$$

$$n \geq \pi_2(x) \text{ and } n \geq j\}.$$

Let S(j) be the set of all positive d(j) conditions

$$\langle x,\ d(j),\ E_3(x,\ j)\ \cup\ E_4(x,\ j)\rangle.$$

Now choose c so that $W_c(\emptyset) = N$ and let S(-1) be the set of all triples

$$\langle n,\ c,\ \{\langle n,\ \langle n,\ 1\rangle\rangle\}\rangle.$$

Finally let $S = \bigcup_{k=-1}^{\infty} S(k)$. Clearly each S(j), $j \geq 0$ is r.e. as $\lambda xy[E_3(x,\ y)\ \cup\ E_4(x,\ y)]$ is recursive, S(-1) is clearly r.e., and hence as d(j) is recursive S is r.e.

Now in the game the opponent enumerates the set W(0), W(1), ...
Let $W^* = \{\langle n,\ j\rangle : n \in W(j)\}$, and let us consider an h'(j) so that

$$W_{h'(j)}(W^*) = W(j).$$

Clearly we can take h'(j) to be recursive. Let $V = S\langle W^*\rangle$. We shall show that V satisfies the requirements:

$$R'_{-1},\ R_0,\ R_1,\ \ldots,\quad \text{for all}\quad X >_e W^*.$$

We let Pj.1. to Pj.6. be the instances of j.1. to j.6. that result from replacing Y by W^*, h(j) by h'(j), and all the references to j.1. - j.6. by references to Pj.1. to Pj.6. respectively.

Now consider a requirement of the type R_j, $j > 0$ i.e.

$$W(j)V(X) \neq X.$$

Suppose that R_j is not satisfied, i.e.

$$W(j)V(X) = X.$$

We shall show that in this event $X \leq_e W^*$. We define U by

$$X(0) = X \cap \{i : i < j\}$$

$$X(m + 1) = W(j)V(X(m)) \cup X(m)$$

$$U = \bigcup_{m=0}^{\infty} X(m).$$

Clearly $U \leq_e W^*$ as $V \leq_e W^*$ by Theorem 5.2.3 and $W(j) \leq_e W^*$. Now

$$X(0) \subset X$$

$$X(m) \subset X \rightarrow X(m + 1) = W(j)V(X(m)) \cup X(m) \subset X$$

hence $U \subset X$.

Now if $k \in X$ and $k < j$ then $k \in U$. Therefore if we can show that $k \in X$ and $k \geq j \rightarrow k \in U$ then we will have $X \subset U$, and hence $X = U \leq_e W^*$. This will show that R_j is satisfied.

We prove the following two statements for all $k \geq j$ by induction:

IH3. $k \in X$ and $k \geq j \rightarrow$ there is a set D and a set F_k such that $F_k \subset X(0)$ and Pj.1. to Pj.6. hold for $<D, k, F_k>$

IH4. if the conclusion of IH3. holds then $k \in U$.

Suppose IH3. and IH4. are true for all $k < r$. We first show that IH3. holds for r.

Suppose $r \in X$ and $r \geq j$. Now as $X = W(j)V(X)$ there is a $<D, r>$ such that $<D, r> \in W(j)$ and $D \subset V(X)$. Let $F_r = \{n : \exists k, <n, k> \in D$ and $n < j$ and $n \notin V(\emptyset)\} \cup \bigsqcup \{F_m : <m, k> \in D$ and $j \leq m < r\}$. Now if $k \in F_r$, then either $k \in F_m$ for some m, $j \leq m < r$, and $k \in X$ by IH3. or

$\langle n, i\rangle \in D$ for some i and $n < j$ and $n \notin V(\emptyset)$. Now if $n < j$ and $\langle n, i\rangle \in D \subseteq V(X)$ then either $n \in X(0)$ or $\langle n, i\rangle \in V(\emptyset)$, hence $F_r \subseteq X(0)$. We will now show that Pj.1. to Pj.6. hold for some $\langle D, r, F_r\rangle$.

Clearly $\langle D, r\rangle \in W(j) = W_{h'(j)}(W^*)$, so Pj.1. holds. Also $D \subseteq V(X) \subseteq S\langle W^*\rangle(N)$, hence Pj.2. holds. Now suppose $\langle n, k\rangle \in D$ and $n \geq r$ and $n \geq j$. Now if $\langle r, F_r\rangle \notin f(n, k)$ then there must be a $\langle n, k'\rangle \in V(N)$ such that $\langle r, F_r\rangle \in f(n, k')$ and for some D^*, $\langle D^*, r, F_r\rangle$ satisfies Pj.1. to Pj.6., which establishes IH3. Thus either for $\langle D, r, F_r\rangle$ Pj.3. holds or IH3. is true for r. Now suppose $\langle n, k\rangle \in D$ and $j \leq n < r$, then either $\langle n, k\rangle \in V(\emptyset)$ or $n \in X$ whence Pj.4. holds by our induction hypothesis. Pj.5. and Pj.6. are immediate from our definition of F_r.

We now show that IH4. is true for r. Suppose the conclusion of IH3. is true for r. That is there is a set D and a set F_r such that $F_r \subseteq X(0)$ and $\langle D, r, F_r\rangle$ satisfies Pj.1. to Pj.6. Now we claim that $D \subseteq V(U)$. Consider $\langle n, k\rangle \in D$:

Case 1. $n < j$

Now either $\langle n, k\rangle \in V(\emptyset) \subseteq V(U)$ or by Pj.5. $n \in F_r$, whence $n \in X(0)$ by our induction hypothesis and $\langle n, k\rangle \in V(X(0)) \subseteq V(U)$.

Case 2. $n \geq j$

Sub-case 2.1. $n < r$

Now as Pj.1. to Pj.6. holds for $\langle D, r, F_r\rangle$ then Pj.4. holds for $\langle n, k\rangle$. Hence either $\langle n, k\rangle \in V(\emptyset) \subseteq V(U)$ or there is a set $E^* \subseteq F_r$ and a set D^* such that $\langle D^*, n, E^*\rangle$ satisfy Pj.1. to Pj.6. Now $F_r \subseteq X(0)$ hence $E^* \subseteq X(0)$. Thus as $n < r$ by our induction hypothesis $n \in U$ and hence $\langle n, k\rangle \in V(U)$.

Sub—case 2.2. $n \geq r$

Now as $\langle D, r, F_r \rangle$ satisfies Pj.1. to Pj.6. we have $\langle D, r, F_r \rangle \in W_{d(j)}$ and as V satisfies S for W* we have $E_3(\langle D, r, F_r \rangle, j) \subset V$. Hence for all $\langle n, k \rangle \in D$ with $n \geq r$, $n \geq j$ we have $\langle \emptyset, \langle n, k \rangle \rangle \in V$ hence $\langle n, k \rangle \in V(\emptyset) \subset V(U)$.

We have shown that $D \subset V(U)$, hence $r \in W(j)V(U) = U$. This completes our induction proof that $X \subset U$. Hence $X = U$, and therefore $X \leq_e W^*$.

Now we will show that R'_{-1} is satisfied. Suppose not; that is, suppose there are two sets Z, Z^* such that $Z \neq Z^*$ but $V(Z) = V(Z^*)$. We can assume without loss of generality that for some n, $n \in Z - Z^*$. Now there must be a flag on n, say $\langle n, j(1) \rangle$ in $V(Z)$ ($\langle n, 1 \rangle$ is in $V(Z)$). Hence $\langle n, j(1) \rangle \in V(Z^*)$. However $n \notin Z^*$, thus $\langle n, j(1) \rangle \in V(\emptyset)$. Hence there must be a triple $\langle D(1), r(1), F(1) \rangle$ that satisfies Pj.1. to Pj.6. for some j with $\langle n, j(1) \rangle \in D(1)$ and $\langle r(1), F(1) \rangle \in f(n, j(1)$. Hence $\langle n, g(j(1), r(1), F(1)) \rangle \in V(Z)$. We can repeat the above argument to extract the sequences

$$\langle j(i) \rangle_{i=1}^{\infty}, \quad \langle F(i) \rangle_{i=1}^{\infty}, \quad \langle D(i) \rangle_{i=1}^{\infty}, \quad \langle r(i) \rangle_{i=1}^{\infty},$$

where $g(j(k), r(k), F(k)) = j(k + 1)$ and $\langle r(k), F(k) \rangle \in f(n, j(k))$. Thus for $t \geq k + 1$, $\langle r(k), F(k) \rangle \notin f(n, j(t))$. Now

$$\left| f(n, j(1)) \right| \leq 2^{\ell} \text{ where } \ell = 2^n \times (n + 1).$$

Hence there must be a k such that $f(n, j(k)) = \emptyset$, and $\langle r(k), F(k) \rangle \in f(n, j(k))$ which is impossible.

This completes our proof that our strategy will satisfy R'_{-1}, R_0, ...

§5.4 CONCLUSIONS

Consider a play of the game where the opponent's strategy has partial degree $\leq_e A$ and he enumerates all the sets $\leq_e A$, that is $W(i) = W_e(A)$. Then $W* = \{<n, j> : n \in W(j)\} = K_A \equiv_e A$. Let the player enumerate the set $V = S<K_A>$. Now by Theorem 5.2.3 we have $V \leq_e K_A \equiv_e A$. We have shown that V satisfies R'_{-1}, R_0, R_1, ..., for all $X >_e W* \equiv_e K_A$.

Consider any set $B, C,$ such that $B >_e C$. Consider a play of the game as described above where $A = C$. As the resulting V satisfies R_j for B, we have $W(j)V(B) \neq B$, for all $j \geq 0$. Now as $C <_e B$ we have $V <_e B$ and hence $V(B) \leq_e B$. Let $M = V(B)$ join C. Clearly $M \leq_e B$. Suppose $M \equiv_e B,$ then for some r.e. set U we have $U(M) = U(V(B)$ join $C) = B$. Thus there must be some $W(j),$ such that $W(j)V(B) = B$. This is impossible as V satisfies $R(j),$ hence $V(B)$ join $C <_e B$. Now suppose $B_1 >_e C,$ then by an identical argument to the above we have $V(B_1)$ join $C <_e B_1$. Now suppose both B_1 and B are minimal above $C,$ that is for all L

$$B_1 > L \geq C \rightarrow L \equiv_e C$$

and

$$B > L \geq C \rightarrow L \equiv_e C .$$

Hence $V(B)$ join $C \equiv_e C$ and $V(B_1)$ join $C \equiv_e C,$ so for some j and i

$$V(B) \text{ join } C = W(j) \quad \text{and} \quad V(B_1) \text{ join } C = W(i).$$

Now as V satisfies R'_{-1}

$$B_1 \neq B \rightarrow V(B_1) \neq V(B) \rightarrow V(B_1) \text{ join } C \neq V(B) \text{ join } C$$

$$\rightarrow W(i) \neq W(j)$$

$$\rightarrow i \neq j.$$

Therefore we have the following.

Theorem 5.4.1.   There can be at most a countable number of partial

degrees minimal above a partial degree.

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