THE CONSTRUCTIVE HAAR INTEGRAL

by

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ABSTRACT

The Haar integral is a positive integral which is invariant under a group of transformations on an integration space. There are several classical proofs which show that a Haar integral exists on every locally compact group. Errett Bishop in Foundations of Constructive Analysis has given a constructive proof of this result, based on the method of Henri Cartan.

In this paper we first discuss the constructivist view of analysis and give some examples of the differences between classical and constructive mathematics. In Chapter 2 we discuss the constructive Daniell integral and define a set measure from the integral. Chapter 3 applies the Daniell theory to integration on locally compact spaces. Finally, for any locally compact group X, we give a method of constructing the Haar integral on C(X)--the set of continuous functions on X with compact support. Using the Daniell theory, we can then extend the integral to $C_1(X)$, the "completion" of C(X).

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INTRODUCTION

Before the development of nineteenth century mathematical analysis, there was hardly any need to prefix any mathematical theory with the word "constructive." There were few examples of non-constructive theorems to be found. The proof of a mathematical conjecture generally proceeded in a way analogous to the steps in a scientific experiment. That is, it was understood that at each step in a proof, instructions should be stated in such a way as to be clearly humanly performable, and assertions ought to be verifiable by any reasonable independent observer.

However, with the birth of rigorous analysis, certain underlying assumptions of classical mathematics became problematic. In particular, there arose the questions of whether mathematical objects have an autonomous existence, independent of human construction, and whether "self-evident" axioms about finite sets and processes are equally valid when applied to infinite processes. In general, nineteenth century analysts answered these questions affirmatively, and in so doing, created a theory which was far removed in spirit from the familiar kind of finitary mathematics that preceded it.

To a constructivist, adjoining humanly performable processes and "ideally performable" processes in a single theory gives an ambiguous meaning to that theory. Thus, to a large extent, the purpose of constructivizing classical analysis is to separate those operations which can be humanly executed (e.g. "add 2 and 2") from those ideal operations having no known method of execution (e.g. "well-order the real numbers"). Then, in most cases, we can find constructive substitutes for the classical theorems whose proofs rely on ideal operations. The system which results at least has the virtue that every assertion in it is finitely verifiable (in principle) and therefore that the meaning of every claim is unambiguous to finite beings such as ourselves.

To say that a statement is true or false constructively does not mean that its truth-value is predetermined in some universe and needs only to be discovered. Rather, it means that its validity has been established or contradicted by an argument which is totally convincing to any reasonable individual. In practice, the requirement that a proof be convincing constructively is equivalent to a requirement that there be a finite mechanical routine (for instance a computer program) which, if performed, is guaranteed to verify the assertion in question.

An illustration of this viewpoint can be found in the constructive interpretation of the logical connectives and quantifiers. [In the following paragraphs, whenever we use the words "method" or "procedure" we will always mean a finite routine--such as a computer program.]

The connective "and" is treated the same way constructively as classically: that is, to prove "A and B" we must supply a method for proving $\underline{A}$ and a method for proving $\underline{B}$.

There are two ways of proving "A or B" : the first way is to provide a finite routine which will verify A ; the second is to give a finite routine for verifying B . Hence, if one asserts "A or B" one ought to be able to tell which of A or B is valid.

To prove "A implies B" ($A \Rightarrow B$) there must be a method which will produce from any proof of A , a proof of B . Of course, if it is impossible to prove A , (i.e. A is contradictory), then "A implies B" will be valid for any B .

The assertion "not A" ($\sim A$) we define to mean that A is contradictory. "Not A" is equivalent to the statement "A implies $0 = 1$".

Simply from the meaning given to these four connectives, it is clear that certain classical theorems are not constructively valid. To prove the law of the excluded middle (A or $\sim A$) requires that we have a finite, purely routine method for proving or disproving any arbitrary mathematical statement A. No one is optimistic about finding such a method. To assert ($\sim \sim A \Rightarrow A$) requires that we be able to find a method of proving A whenever we are given a proof that it is impossible that A is contradictory. Again there is little hope of finding this method.

To prove "$\forall x\, A(x)$" we must have a routine which will yield a proof of A(c) for each c in the range of the variable x. To be able to assert "$\exists x\, A(x)$" we require a method for constructing a mathematical object c in the range of the variable x, together with a proof of A(c).

Many classical theorems which do not hold constructively claim to show the (ideal) existence of objects. One example is the assertion that every bounded monotone sequence of real numbers has a limit. The constructive interpretation of this statement is that, given the sequence of numbers, we can begin to compute the decimal expansion of its limit in a finite number of steps. The consideration of a few examples of bounded monotone sequences should demonstrate the implausibility of this assertion.

Recalling Goldbach's conjecture that every even integer is the sum of two primes, we define the "Goldbach sequence", $\{\alpha_k\}_{k=1}^{\infty}$, by

$$\alpha_k = \begin{cases} 0 \text{ if Goldbach's conjecture holds for all integers between 4 and } k \\ 1 \text{ if Goldbach's conjecture fails for some number } m \leq k \end{cases}$$

This is certainly a bounded monotone sequence. The computation of its limit, however, depends on the solution of a problem, and we cannot guarantee that we will be able to solve the problem in a finite number of steps.

The sequence $\{\beta_k\}_{k=1}^{\infty}$ defined by

$$\beta_k = \begin{cases} 0 \text{ if the sequence 0123456789 has not occurred in the decimal} \\ \quad \text{expansion of } \pi \text{ before the k'th place in that expansion} \\ 1 \text{ if the sequence has appeared before the k'th place} \end{cases}$$

is another bounded monotone sequence whose limit is not known--and hence cannot be asserted to exist constructively.

It is easy to see that there are plenty of these types of sequences to be constructed. Even if all current problems in number theory were solved we could still define non-convergent bounded monotone sequences based on the results of coin flips or some other random process.

When we say that a classical theorem fails to hold constructively, it is in the sense alluded to above -- namely that when the statement of the classical theorem is interpreted constructively, the validity of the statement then hinges on the resolution of an unsolved problem. If we wish to be more formal, instead of producing constructive "counter-examples" (such as the Goldbach sequence above) to a classical but non-constructive theorem $\underline{A}$, we should be able to prove that "A implies the law of the excluded middle (E.M.)", or "A implies the limited principle of omniscience (L.P.O.)". [The limited principle of omniscience in its simplest form states that for any sequence $\{n_k\}_{k=1}^{\infty}$ of zeros and ones, either we can prove $n_k = 0$ for all $\underline{k}$, or we can find a $\underline{k}$ with $n_k = 1$.] However, since the production of constructive "counterexamples" is usually more amusing than giving formal proofs of " $A \Rightarrow$ E.M." or "$A \Rightarrow$ L.P.O.", we will usually do the former, with the understanding that such formal proofs should be available (and usually will be obvious).

Much of the resistance to constructive mathematics comes from the mistaken idea that its aims are to eliminate non-effectively constructed objects from mathematics altogether, and to "mutilate" what remains by restricting the methods of operation available to mathematicians. On the contrary, the purpose of constructivizing mathematics is to describe precisely how effectively constructable objects and non-effectively constructable objects (e.g. sets) can be defined, and how they really behave when viewed in strictly constructive terms. Thus, for instance, we must make a distinction between bounded monotone sequences and convergent sequences; and between sets in general and sets whose elements have been constructed.

Traditionally, mathematicians have been willing to implicitly set down a theorem A depending on the Axiom of Choice as "A.C. ⇒ A". Constructivists would ask that if they wish to state theorems whose proofs rely on Excluded Middle or L.P.O., they also write them as implications: "E.M. ⇒ A" or "L.P.O. ⇒ A". Then perhaps it will become more noticeable that a reasonable goal for mathematicians is to discover what types of theorems can be proven without non-constructive assumptions. The effect of making these distinctions is not to mutilate mathematics, but to deepen its meaning and to gain more insight into the nature of mathematical systems.

CHAPTER I

A CONSTRUCTIVE APPROACH TO REAL ANALYSIS

1. Sets and functions

The usual classical notion of a set is that of a collection of objects from some pre-existent (but not necessarily constructed) universe. This is clearly not compatible with the constructivist view that a mathematical object exists only if it has been constructed, and that the properties of that object are determined by its construction. Hence, to define a set constructively, it is necessary to state what must be done to construct an element of the set, and what else must be done to show that two elements of the set are equal. The equality relation on the set is required to be an equivalence relation.

For example, the set of integers, Z, can be defined as follows:

1. to construct an element of the set, one must specify, either explicitly or implicitly, a finite mechanical process which will give the decimal representation for an integer.

2. two elements are equal if their decimal representations are equal in the usual sense.

[Hopefully we can agree on what a decimal representation of an integer looks like. The problem of what an integer really <u>is</u>, is irrelevant, because we work only with its representation. With this in mind, we will use the terms "integer" and "representation for an integer" interchangeably.]

Thus, the specification "the smallest integer greater than 3" defines an integer, while the specification "the smallest integer which is a counterexample to Goldbach's conjecture, or 5 if no such integer exists" does not, since we cannot guarantee at present that there is a finite routine for writing down this quantity.

However, it is important to note the distinction between constructing a mathematical object and defining a set. To define a set it is necessary to know what must be done to construct its elements. It is not necessary to give a finite routine for actually constructing its elements or for deciding equality. For instance we could define a set A by stating: "Prove Goldbach's conjecture or find a counterexample. Then to construct an element of A, construct 5 if Goldbach's conjecture is true, or construct the counterexample if Goldbach's conjecture is false." So while the second definition in the last paragraph does not define an integer, it does implicitly define a set.

An operation, φ, between two sets, is a rule which provides a finite mechanical procedure for constructing $\varphi(a)$ when presented with a routine for constructing any a in the domain of φ. A function is an operation which assigns equal values to equal arguments.

Thus, for example, a (constructively defined) function f between the integers will when presented with a decimal representation of an integer n, furnish a finite routine for computing the decimal representation of f(n).

As usual, any function with domain Z^+ -the set of positive integers- is called a sequence.

A subset (A,i) of a set B consists of a set A and a function $i:A \to B$ with the property that $i(a_1) = i(a_2)$ if and only if $a_1 = a_2$. (Notice that the set A consisting of 5 if Goldbach's conjecture is true or the smallest counterexample to Goldbach's conjecture if it is false, is a subset of the integers since the ordinary inclusion map $i:A \to Z$ will provide a finite routine for constructing i(a) when the construction of a is specified.) Unless explicitly stated otherwise, we will write $A \subset B$ to mean $(A,i) \subset B$, if i is the ordinary inclusion map from A to B.

If (A,i_A) and (B,i_B) are two subsets of S, then the subsets $(A \cup B,i)$ and $(A \cap B,j)$ can be defined in the following way:

(1) to construct an element c of $A \cup B$, either construct an element a of A and let $i(c) = i_A(a)$, or construct an element b of B and let $i(c) = i_B(b)$. To show that $c_1 = c_2$ in $A \cup B$, show that $i(c_1) = i(c_2)$.

(2) to construct an element c of $A \cap B$, construct an element a of A and an element b of B and prove that $i_A(a) = i_B(b)$. Let $i(c) = i_A(a)$, and define $c_1 = c_2$ to mean that $i(c_1) = i(c_2)$.

Classically, the equality relation is supposed to formalize the notion of identity of objects in the universe. Constructively, of course, equality is a convention related to the construction of objects which form a particular set. Hence it is not meaningful to speak of equality between elements of two different sets A and B or to perform set operations like union or intersection, except when A and B are realized as subsets of another set S. In practice this restriction does not present any real difficulty.

2. The real number system

Rational numbers are defined as ordered pairs of integers, and manipulations of rationals are done in the traditional manner.

Once we have the set of rationals, we would like to complete the number line. We wish, therefore, to define a set R of real numbers, having the property that every Cauchy sequence of rational real numbers converges to some number in R. Hence R is defined as follows:

(1) to construct a real number x we must

(i) construct a sequence of rational numbers $\{x_n\}_{n=1}^{\infty}$

(ii) construct a sequence of positive integers $\{m_k\}_{k=1}^{\infty}$

(iii) prove that, whenever $i,j \geq m_k$,

$$|x_i - x_j| \leq \frac{1}{k}$$

[In case (i), (ii), and (iii) are satisfied, we write $x = \{x_n\}_{n=1}^{\infty}$ and call $\{x_n\}$ a representing sequence for x.]

(2) to show that two real numbers $x = \{x_n\}$ and $y = \{y_n\}$ are equal, construct a sequence of positive integers $\{N_j\}_{j=1}^{\infty}$ and prove that $|x_n - y_n| \leq \frac{1}{j}$ if $n \geq N_j$.

[Notice that a rational real number has a representing sequence of the form $\{ \frac{p}{q}, \frac{p}{q}, \frac{p}{q}, \ldots\}$, where $p,q \in Z$.]

Most simple operations with real numbers involve straightforward manipulations of their representations. For instance, if $x = \{x_n\}_{n=1}^{\infty}$ and $y = \{y_n\}_{n=1}^{\infty}$, then

(a) $x + y = \{x_n + y_n\}_{n=1}^{\infty}$

(b) $- x = \{- x_n\}_{n=1}^{\infty}$

(c) $xy = \{x_n y_n\}_{n=1}^{\infty}$

(d) $\max \{x,y\} = \{\max \{x_n,y_n\}\}_{n=1}^{\infty}$

(e) $|x| = \{\max \{x_n,- x_n\}\}_{n=1}^{\infty}$

[Proofs that these quantities are real numbers and that the operations are in fact functions can be found in Bishop [1].]

Order relations in R are not so straightforward. Let $a = \{a_n\}$ and $b = \{b_n\}$ be two real numbers. We define "$a < b$" to mean that we can find two positive integers k and N such that $a_n \leq b_n - \frac{1}{k}$ for all $n \geq N$, and "$a \leq b$" to mean that we can construct a sequence $\{N_m\}_{m=1}^{\infty}$ of positive integers with the property that $a_n \leq b_n + \frac{1}{m}$ whenever $n \geq N_m$. Then "$a \neq b$" is defined as "$a < b$ or $a > b$" (or, equivalently, that there are two positive integers k and N with $|x_n - y_n| \geq \frac{1}{k}$ for all $n \geq N$).

We don't wish to define "$a \leq b$" as "$a < b$ or $a = b$". This is because we can define real numbers r which, for instance, are clearly non-negative in any reasonable sense of the word, but which cannot be asserted to have the property "$r = 0$ or $r > 0$". For example, recall the Goldbach sequence of integers, $\{\alpha_k\}_{k=1}^{\infty}$, defined by

$$\alpha_k = \begin{cases} 0 \text{ if Goldbach's conjecture is true for all integers between 4 and } k \\ 1 \text{ if Goldbach's conjecture is false for some integer } m \le k \end{cases}$$

and let $r = \sum_{k=1}^{\infty} \frac{\alpha_k}{2^k}$. It is easily seen that $\underline{r}$ is a real number, (and ought to be called non-negative), but to show "$r = 0$ or $r > 0$" it is necessary to decide Goldbach's conjecture. However, the constructive definition of " $\le$ " gives $\underline{r}$ the property that $0 \le r$, since $\{r_n\} = \{ \sum_{k=1}^{n} \frac{\alpha_k}{2^k} \}_{n=1}^{\infty}$ is a representation for $\underline{r}$, and $0 \le r_n + \frac{1}{m}$ for every positive integer m.

On the other hand, there seems to be no way to give a constructive proof of the classical law of trichotomy. Consider the real number $r^* = \sum_{1}^{\infty} (-1)^k \frac{\alpha_{2k}}{2^k}$. We cannot claim that "$r^* < 0$, $r^* = 0$, or $r^* > 0$", nor can we claim "$r^* < 0$ or $r^* \ge 0$". The classical theorem "$a \not\le b$ implies $a > b$" also fails constructively, because the impossibility of producing a sequence $\{N_m\}$ with $a_n \le b_n + \frac{1}{m}$ for $n \ge N_m$ does not guarantee the existence of a method to produce integers m and N with $b_n \le a_n - \frac{1}{m}$ for all $n \ge N$. [See Heyting [4], Sections 7.3 and 8.1.]

Nevertheless, there is a constructive substitute for trichotomy which may be suggested by the above examples. It is: For any real numbers a, b, and ε, with $\varepsilon > 0$, either $a < b$ or $a > b - \varepsilon$. The proof of this assertion involves just computing sufficiently many terms of $\{a_n\}$ and $\{b_n\}$ to decide which relation holds. Also, "$a \not< b$ implies $a \ge b$" is a theorem, since for any $m \in Z$, either $a_n \le b_n - \frac{1}{m}$ or $a_n > b_n - \frac{1}{m}$ (the law of trichotomy is valid for the rational numbers a_n and b_n), and the former case can be ruled out for all sufficiently large $\underline{n}$, (say, $n \ge N_m$), thus proving that $b_n < a_n + \frac{1}{m}$ for all $n \geq N_m$.

Sequences and series of real numbers are defined in the usual manner. A sequence of real numbers $\{x_n\}_{n=1}^{\infty}$ converges if a real number x_0 and a sequence of positive integers $\{N_k\}_{k=1}^{\infty}$ can be constructed with

$$|x_n - x_0| \leq \frac{1}{k}$$

for all $n \geq N_k$. This means, of course, that both the limit of the sequence and the rate of convergence must be known before we can say a sequence converges. Most of the familiar classical properties of sequences and series are valid constructively. In particular, a sequence of real numbers converges if and only if it is a Cauchy sequence. Hence, substituting the constructive form of trichotomy for the classical one, we can say that the constructive real numbers form a complete ordered field.

As we saw before, one important classical theorem which fails is the one claiming that all bounded monotone sequences of real numbers converge. A (constructive) counterexample to this is the Goldbach sequence. Here, as you would expect, the problem lies in the fact that, classically, the "rate of convergence" is not known, and so constructively, it has not been proven that the sequence converges.

The set of values $\{\alpha_1, \alpha_2, \alpha_3, \ldots\}$ of the Goldbach sequence also provides a counterexample to the classical theorem that all bounded sets in R have a least upper bound, since we cannot say whether 0 or 1 is the least upper bound of this set.

3. Functions defined on the real numbers

Let $[a,b]$ be a non-empty closed interval in R. A function $f: [a,b] \to R$ is a rule, which, when given a representing sequence $\{x_n\}$ for a number x in the domain, will compute a sequence $\{z_n\}$ for $f(x) \in R$, in such a way that $x = y$ implies $f(x) = f(y)$. A function $f: R \to R$ is defined similarly.

A function $f: [a,b] \to R$ is continuous if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - f(y)| \leq \varepsilon \qquad \text{whenever } |x - y| \leq \delta, \qquad (a \leq x,y \leq b)$$

[Notice that this definition means we must have an operation $\omega: R \to R$ which, when given $\varepsilon > 0$ will supply a number $\delta > 0$ in a finite, mechanical manner.] A function $f: R \to R$ is continuous if it is continuous on every closed interval in R.

A little reflection will show the difficulty of defining discontinuous functions between the real numbers. For example,

$$f_1(x) = \begin{cases} 0 \text{ if } x \text{ is rational} \\ 1 \text{ if } x \text{ is irrational (i.e. } x \neq r \text{ for any rational number } r) \end{cases}$$

or

$$f_2(x) = \begin{cases} 0 \text{ if } x < 0 \\ 1 \text{ if } x \geq 0 \end{cases}$$

are not constructively defined functions on the whole real line. If there were a finite mechanical method of computing $f_1(x)$ or $f_2(x)$ for every $x \in R$, that method would also decide whether each x was rational or irrational, negative or non-negative. In fact no such method presently exists, and so neither of these classical functions are defined constructively on all the real numbers.

Since we don't wish to eliminate the classical discontinuous functions from constructive analysis, we are led to enlarge the class of functions on R to include partial mappings, whose domains may be any subset of the real numbers. Continuity and discontinuity are defined in the obvious manner. Then the classical function f_1 defined above is a discontinuous partial function from R to R whose domain is the union of the set of rationals with the set of irrationals; f_2 is a discontinuous partial function whose domain is $(-\infty,0) \cup [0,\infty)$. [Notice that neither of these set unions is equal to the whole real line. For example, the number r^*, defined in Section 2, is not in $(-\infty,0) \cup [0,\infty)$, since putting r^* into one of $(-\infty,0)$ or $[0,\infty)$ requires a proof that $r^* < 0$ or a proof that $r^* \geq 0$.]

We might note here that if f is a continuous partial function whose domain is a dense subset of R, then f has a unique continuous extension to all of R. Thus we could have defined the absolute value function as the unique continuous extension of the partial function

$$|x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

Of course, no such extension principle is available for discontinuous partial functions.

4. Partial functions

As one might guess from the above considerations, partial functions are more basic in constructive analysis than total functions. The definition of a partial function can be extended to sets other than R.

Let S be any non-empty set. We formally define a real-valued partial function on S as an ordered pair $(f, D(f))$, where $D(f) \subset S$ and f is a function mapping $D(f)$ to R. (We will usually call the function simply f, whenever its domain $D(f)$ is understood.) Two partial functions f and g are equal if $D(f) = D(g)$ and $f(x) = g(x)$ for all $x \in D(f)$. A function f is non-negative if $f(x) \geq 0$ for all $x \in D(f)$.

Functional operations are defined in the following way: Let $(f, D(f))$ and $(g, D(g))$ be two partial functions on S. Then $(f + g, D(f + g))$ is defined to be that function with $D(f + g) = D(f) \cap D(g)$ and $(f + g)(x) = f(x) + g(x)$ for all $x \in D(f + g)$. The partial function $(fg, D(fg))$ has $D(fg) = D(f) \cap D(g)$ and $fg(x) = f(x)g(x)$ for all $x \in D(fg)$. Max $\{f,g\}$ and min $\{f,g\}$ are defined in a similar way. $(|f|, D(|f|))$ is that partial function with $D(|f|) = D(f)$ and $|f|(x) = |f(x)|$ whenever $x \in D(|f|)$. Max $\{f,\alpha\}$, min $\{f,\alpha\}$, and αf can be defined similarly, for any given $\alpha \in R$.

5. Complemented sets

The properties of the order relations in R ought to illustrate that there is rarely any useful purpose to be served by using negative definitions. For instance, we could have defined "$a \leq b$" to mean "$a \not> b$", (i.e. $a > b \Rightarrow 0 = 1$). This proves to be equivalent to the definition we did give, and yet does not describe as well the property that we want two numbers a and b with $a \le b$ to have: namely that there is a sequence $\{N_m\}_{m=1}^{\infty}$ of positive integers with $a_n \le b_n + \frac{1}{m}$ for $n \ge N_m$. To define "$a < b$" as "$a \not\ge b$" only invites confusion, since there is no obvious general method of obtaining the property we would certainly like a and b to have (i.e., $a < b$ iff $\exists\, k,N \in Z^+$ with $a_n \le b_n - \frac{1}{k}$ for $n \ge N$) from a proof of "$a \ge b \Rightarrow 0 = 1$". And using these two negative definitions still does not retrieve the property "$a = b$ or $a \ne b$".

With this in mind, we define an inequality relation on any set S, not as the negation of equality, but in the following positive sense:

Let " = " be the equality relation on S. A relation " $\ne$ " is an inequality relation if, for all x, y, z $\in$ S,

(i) $x = y$ and $x \ne y \Rightarrow 0 = 1$

(ii) $x = y$ and $y \ne z \Rightarrow x \ne z$

(iii) $x \ne y \Rightarrow y \ne x$

(iv) $x \ne y \Rightarrow x \ne z$ or $y \ne z$

Then a complemented set in S is an ordered pair (A,B) of subsets of S with the property that for any $x \in A$ and $y \in B$, $x \ne y$. If (A,B) is a complemented set in S, we define its complement $-(A,B)$ to be (B,A), which is also a complemented set. We write $x \in (A,B)$ if $x \in A$, and $x \in -(A,B)$ if $x \in B$.

Notice that complemented sets have the property that $--(A,B) = (A,B)$. This would not be the case if we defined complementation in terms of negation. [For instance, the interval $[0,\infty)$ would be the complement of $(-\infty,0)$ if the complement of $(-\infty,0)$ were defined to be $\{x \in R \mid x \not< 0\}$.

We could not, however, prove that $(-\infty,0)$ was the complement of $[0,\infty)$ if that complement were defined as $\{x \in R \mid x \not\geq 0\}$.] In addition, the general notion of complementation given here is flexible enough to cover structures such as metric complements (discussed in Chapter 3), which would not be incorporated in the usual negative formulation of complementation.

If (A,B) is a complemented set in S, then its characteristic function χ_A is a partial function on S with domain $A \cup B$, such that

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in (A,B) \\ 0 & \text{if } x \in -(A,B) \end{cases}$$

We will be defining a measure on complemented sets in terms of their characteristic functions. Therefore we define the set operations on complemented sets to correspond to operations on their characteristic functions. If $A = (A_1,A_2)$ and $B = (B_1,B_2)$ are complemented sets in S, then:

(a) their "intersection", $A \wedge B$, is

$$(A_1 \cap B_1,\ (A_1 \cap B_2) \cup (A_2 \cap B_1) \cup (A_2 \cap B_2))$$

(b) their "union", $A \vee B$, is

$$((A_1 \cap B_1) \cup (A_1 \cap B_2) \cup (A_2 \cap B_1),\ A_2 \cap B_2)$$

(c) $A < B$ ("A is a subset of B") if $A_1 \subset B_1$ and $B_2 \subset A_2$

(d) $A - B = A \wedge (-B)$

Notice that $\chi_{A \wedge B}$ has domain $(A_1 \cap B_1) \cup (A_1 \cap B_2) \cup (A_2 \cap B_1) \cup (A_2 \cap B_2)$ $= [A_1 \cap (B_1 \cup B_2)] \cup [A_2 \cap (B_1 \cup B_2)] = (A_1 \cup A_2) \cap (B_1 \cup B_2)$, and so

$\chi_{A \wedge B} = \min\{\chi_A, \chi_B\} = \chi_A \cdot \chi_B$. Similarly, $\chi_{A \vee B} = \chi_A + \chi_B - \chi_{A \wedge B}$. [Note: In Bishop [1], the union of complemented sets $A = (A_1, A_2)$ and $B = (B_1, B_2)$ is defined as $A \cup B = (A_1 \cup B_1, A_2 \cap B_2)$, and their intersection as $A \cap B = (A_1 \cap B_1, A_2 \cup B_2)$. While these definitions are simpler, they present a problem in dealing with characteristic functions. Since the functions, and not the sets, are basic to the type of measure theory that will be done in this paper, we have simplified the "function theory" at the expense of the set theory.]

CHAPTER II

THE DANIELL INTEGRAL

The theory of integration plays a central role in mathematical analysis and geometry. It is customary in analysis to study first the Riemann integral and then its generalization and extension to the Lebesgue integral via the theory of measure. A constructive treatment of this subject can be found in Bishop [1]. The Daniell theory, on the other hand, provides a development of integration which focuses directly on functions without any preliminary discussion of measures on sets.

1. Integration spaces

Definition 2.1. An integration space is a triple (X,L,I), where X is a non-empty set with equality and inequality relations, L is a set of partial functions from X to R, and I is a function from L to R, with the following properties:

(1) If f is in L, so are $|f|$ and $\min\{f,1\}$. If $f,g \in L$ and $\alpha,\beta \in R$, then $\alpha f + \beta g$ is in L and $I(\alpha f + \beta g) = \alpha I(f) + \beta I(g)$.

(2) If f is in L and $\{f_n\}_{n=1}^{\infty}$ is a sequence of non-negative functions in L such that $\sum_{n=1}^{\infty} I(f_n)$ converges and $\sum_1^{\infty} I(f_n) < I(f)$, then there is a point $x \in \bigcap_1^{\infty} D(f_n) \cap D(f)$ where $\sum_1^{\infty} f_n(x)$ converges, and $\sum_1^{\infty} f_n(x) < f(x)$.

(3) There is a function p in L with $I(p) = 1$.

(4) For each $f \in L$, $\lim_{n\to\infty} I(\min\{f,n\}) = I(f)$, and $\lim_{n\to\infty} I(\min\{|f|, \frac{1}{n}\}) = 0$.

Notice that if $f \in L$, we may assume $\min\{f,n\}$ and $\min\{|f|, \frac{1}{n}\}$ are also in L, since $\min\{f,n\} = n\cdot\min\{\frac{1}{n}\cdot f,1\}$ and $\min\{|f|, \frac{1}{n}\} = \frac{1}{n}\cdot\min\{n\cdot|f|,1\}$. Properties (2) and (3) ensure that all functions in L have non-empty domains: if $f \in L$, so is $|f|$, and $I(|f|) < I(|f| + p) = I(|f|) + 1$; hence there must be a point x in $D(f) \cap D(p)$.

Proposition 2.2.

(i) If f and g are in L, so are $\max\{f,g\}$ and $\min\{f,g\}$.

(ii) $f \in L$ if and only if f^+ and f^- are in L. $I(f) = I(f^+) - I(f^-)$ for every $f \in L$.

(iii) If $f \in L$ and $f(x) \geq 0$ for all $x \in D(f)$, then $I(f) \geq 0$.

Proof: (i) The function $(f - g)^+$ is in L, since $(f - g)^+ = \frac{1}{2}(f - g + |f - g|)$. Therefore $\max\{f,g\} = g + (f - g)^+$ and $\min\{f,g\} = -\max\{-f, -g\}$ are also in L.

(ii) $f^+ = \max\{f,0\cdot f\}$ and $f^- = \max\{-f, 0\cdot f\}$. Clearly f, f^+, and f^- must have the same domain, so $f = f^+ - f^-$ and $I(f) = I(f^+) - I(f^-)$.

(iii) Suppose $f \geq 0$ and $I(f) < 0$. Then $I(f^+) < I(f^-)$, hence by Property (2), there is an $x \in D(f)$ with $f^+(x) < f^-(x)$. But then $f(x) = f^+(x) - f^-(x) < 0$, which is contrary to the assumption. Therefore $I(f) \geq 0$.□

Proposition 2.3. Let $\{f_n\}$ be any sequence of functions in L. Then $\bigcap_1^\infty D(f_n)$ is non-empty.

Proof: $\{f_n - f_n\}_{n=1}^\infty$ is a sequence of non-negative functions in L, and

$\sum_{n=1}^{\infty} I(f_n - f_n) = 0 < I(p) = 1$. Hence, by Property (2), there is a point $x \in \bigcap_1^{\infty} D(f_n) \cap D(p) \subset \bigcap_1^{\infty} D(f_n)$.□

The following definition enlarges the set L to form a set L_1, of integrable functions, which will be in a sense "complete" with respect to I.

Definition 2.4. Let (X,L,I) be an integration space. An integrable function is an ordered pair $(f, \{f_n\}_{n=1}^{\infty})$, where f is a partial function on X and $\{f_n\}$ is a sequence of functions in L such that $\sum_{n=1}^{\infty} I(|f_n|)$ exists and $\sum_{n=1}^{\infty} f_n(x) = f(x)$ whenever $\sum_1^{\infty} |f_n(x)|$ converges. The integral of f is defined to be $I(f) = \sum_1^{\infty} I(f_n)$. Two integrable functions $(f, \{f_n\})$ and $(g, \{g_n\})$ are equal if $f = g$ as partial functions.

If $f \in L$, then $(f, \{f, 0\cdot f, 0\cdot f, \ldots\})$ is in L_1, so L is a subset of L_1 under the inclusion mapping $f \to (f, \{f, 0\cdot f, 0\cdot f, \ldots\})$. We will usually denote a function $(f, \{f_n\})$ in L_1 by its first element f, and call $\{f_n\}$ a representing sequence (or representation) for f. It will be seen later that (X, L_1, I) is in fact an integration space, and consequently that every function in L_1 has a non-empty domain.

Operations on integrable functions are defined in the following way: Let $(f, \{f_n\})$ and $(g, \{g_n\})$ be in L_1. Then

(a) $f + g = (f + g, \{f_n + g_n\}) \in L_1$.

(b) $\alpha f = (\alpha f, \{\alpha f_n\}) \in L_1$ for every real number α.

(c) $|f| = (|f|, \{\varphi_n\}) \in L_1$, where $\{\varphi_n\} = \{|f_1|, f_1, -f_1, |f_1 + f_2| - |f_1|, f_2, -f_2, |f_1 + f_2 + f_3| - |f_1 + f_2|, \ldots\}$.

(d) $\min\{f,1\} = (\min\{f,1\}, \{\psi_n\}) \in L_1$, where $\{\psi_n\} = \{\min\{f_1,1\},$
$f_1, -f_1, \min\{f_1+f_2,1\} - \min\{f_1,1\}, f_2, -f_2, \ldots\}$.

In (c), the terms $f_1, -f_1, \ldots, f_n, -f_n, \ldots$ are included in the representation $\{\varphi_n\}$ in order to ensure that $\Sigma\,|\varphi_n(x)|$ does not converge outside of the domain of f. (If it did, then obviously we could not say $|f|(x) = \Sigma\,\varphi_n(x)$ whenever $\Sigma|\varphi_n(x)|$ converges.) If we defined $\{\varphi_n\} = \{|f_1|,$ $|f_1+f_2| - |f_1|, \ldots\}$, $\Sigma|\varphi_n(x)|$ might converge when $\Sigma|f_n(x)|$ did not-- for instance, consider $f_1 \neq 0$ and $f_{n+1} = -2(f_1 + \ldots + f_n)$. The same consideration applies in (d).

To show (X,L_1,I) is an integration space, it is necessary first to establish that I is a function on L_1. This can be done following a discussion of the properties of I on L_1.

<u>Lemma 2.5.</u> Suppose $(f, \{f_n\}) \in L_1$ and $f(x) \geq 0$ whenever $\sum_1^\infty |f_n(x)|$ converges. Then $I(f) \geq 0$.

<u>Proof:</u> Let $A = \{x \in X: \sum_1^\infty |f_n(x)| \text{ converges}\}$. The two series $\sum_1^\infty f_n^+$ and $\sum_1^\infty f_n^-$ also converge on A and $f(x) = \Sigma f_n^+(x) - \Sigma f_n^-(x)$ for all $x \in A$. $\sum_1^\infty I(f_n^+)$ and $\sum_1^\infty I(f_n^-)$ both converge, since the sum $\sum_1^\infty I(|f_n|)$ exists. Let $N \in Z^+$ and suppose $\sum_1^\infty I(f_n^+) < \sum_{n=1}^N I(f_n^-)$. Since $\sum_1^\infty I(f_n^-)$ converges there is an $M \in Z^+$ with

$$\sum_1^\infty I(f_n^+) + \sum_{n=M}^\infty I(f_n^-) < \sum_{n=1}^N I(f_n^-)$$

Then by Property (2), there must be an $x \in A$ with

$$\sum_1^\infty f_n^+(x) + \sum_{n=M}^\infty f_n^-(x) < \sum_{n=1}^N f_n^-(x)$$

which implies that $\sum_1^\infty f_n^+(x) < \sum_1^\infty f_n^-(x)$. But this is impossible because $f \geq 0$ on A and so we must have $\sum_1^\infty I(f_n^+) \geq \sum_{n=1}^N I(f_n^-)$ for every $N \in Z^+$, and hence $\sum_1^\infty I(f_n^+) \geq \sum_1^\infty I(f_n^-)$. Then $I(f) = \Sigma\, I(f_n) = \Sigma\, I(f_n^+) - \Sigma\, I(f_n^-) \geq 0$.□

[In particular, we might note here that $|I(f)| \leq I(|f|)$ for each $f \in L_1$.]

Definition 2.6. A subset of X which contains a countable intersection of domains of integrable functions is called a full set.

Lemma 2.7 Every full set contains the domain of some integrable function.

Proof: Let $\{f_n\}$ be a sequence of functions in L_1 with $\bigcap_{n=1}^\infty D(f_n)$ contained in the full set A. Each f_n has a representing sequence $\{f_{nk}\}_{k=1}^\infty$ of functions in L. Let $\delta_n = 1 + \sum_{k=1}^\infty I(|f_{nk}|)$, and let $\{\varphi_m\}$ be a rearrangement of the double sequence $\{ \frac{f_{nk}}{2^n \delta_n} \}_{k,n=1}^\infty$ into a single sequence. Since $\sum_{k=1}^\infty I(|f_{nk}|)$ converges for each $n \in Z^+$, $\sum_{m=1}^\infty I(|\varphi_m|)$ converges. Hence there is a function $f \in L_1$ whose domain, $D(f) = \{x \in X : \Sigma\, |\varphi_m(x)| \text{converges}\}$, and whose value at each $x \in D(f)$ is $\Sigma\, \varphi_m(x)$. Clearly $D(f) \subset \bigcap_1^\infty D(f_n) \subset A$.□

Proposition 2.8. If f and g are in L_1 and $f(x) \leq g(x)$ for all x in some full set A, then $I(f) \leq I(g)$.

Proof: There is a function $h \in L_1$ with $D(h) \subset A$. Then $F = f + h - h$ and $G = g + h - h$ are both in L_1, and $D(F) = D(G) = D(h) \subset A$. Therefore $F(x) \leq G(x)$ on $D(F) = D(G)$, and so $I(F) \leq I(G)$ by Lemma 2.5. Any representing sequence $\{F_n\}$ for F will also be a representing sequence for f, hence $I(F) = I(f)$, and similarly, $I(G) = I(g)$. Thus $I(f) \leq I(g)$.□

Corollary 2.9. I is a function on L_1.

Proof: We need only show that $f = g$ implies $I(f) = I(g)$ for $f,g \in L_1$. But if $f,g \in L_1$, D(f) and D(g) are full sets, and so $f = g$ on a full set.

Hence by Proposition 2.8, $I(f) = I(g)$.□

More generally, if $f = g$ on any full set, then $I(f) = I(g)$. Consequently we can define <u>$f = g$ almost everywhere</u> [a.e.] to mean that $f = g$ on a full set. Similarly, $f \le g$ [a.e.] will mean $f \le g$ on a full set.

The following corollary will be used later in our discussion of integrable sets.

<u>Corollary 2.10.</u> Let f be a partial function on X and g a function in L_1. If $f = g$ [a.e.] then f is also in L_1, and $I(f) = I(g)$.

<u>Proof</u>: $f = g$ on some full set A. By Lemma 2.7, there is a function $h \in L_1$ with $D(h) \subset A$. Then $F = g + h - h$ is in L_1 and is equal to f on $D(F) = D(h)$. If $\{\varphi_n\}$ is any representation of F, it must also be a representation of f, hence $f \in L_1$, and $I(f) = \sum_1^\infty I(\varphi_n) = I(F) = I(g)$.□

We can now state the completeness theorem for L_1.. The theorem asserts that if we redefine equality on L_1 as equality almost everywhere, and apply the limit processes in Definition 2.4 to L_1, we do not obtain a larger class of integrable functions. The proof of this assertion can be found in Bishop and Cheng [2].

<u>Theorem 2.11.</u> [Completeness Theorem] Suppose $\{f_n\}$ is a sequence of functions in L_1, and $\sum_1^\infty I(|f_n|)$ converges. Then there is a full set A and a function $f \in L_1$ such that $\sum_1^\infty |f_n(x)|$ converges on A, $f(x) = \sum_1^\infty f_n(x)$ for all $x \in A$, and $I(f) = \sum_1^\infty I(f_n)$.□

We need two more lemmas before we can prove that (X,L_1,I) is an integration space.

Lemma 2.12. If f is a non-negative function in L_1, then for each $\varepsilon > 0$, f has a representing sequence $\{f_n\}$ with

$$\sum_1^\infty I(|f_n|) < I(f) + \varepsilon$$

Proof: Let $\varepsilon > 0$ be given, and let $\{\varphi_n\}$ be any representation of f. Since $\sum_1^\infty I(|\varphi_n|)$ converges and $\sum_1^\infty I(\varphi_n) = I(f) \geq 0$, we can find a $k \in Z^+$ with $\sum_{n=k+1}^\infty I(|\varphi_n|) < \frac{\varepsilon}{2}$ and $I(\,|\sum_{n=1}^k \varphi_n|) \leq I(f) + \frac{\varepsilon}{2}$. [$\lim_{k\to\infty} I(|\sum_{n=1}^k \varphi_n|) = I(|f|) = I(f)$.]

Let $f_1 = \sum_{n=1}^k \varphi_n$, $f_2 = \varphi_{k+1}$, $f_3 = \varphi_{k+2}$,...etc. Then $\{f_n\}$ is a representation of f and $\sum_1^\infty I(|f_n|) < I(f) + \varepsilon$.□

Lemma 2.13. If $f \in L_1$ and $\{f_n\}$ is any representing sequence for f, then $\lim_{N\to\infty} I(|f - \sum_{n=1}^N f_n|) = 0$.

Proof: Let $A = \{x \in X : \Sigma|f_n(x)| < \infty\}$. A is a full set since we can define a function $F \in L_1$ with domain A such that $I(F) = \Sigma I(f_n)$ and $F(x) = \Sigma f_n(x)$ for all $x \in A$. Since $\Sigma I(|f_n|)$ converges, there exists, for any given $\varepsilon > 0$, an $N \in Z^+$ with $\sum_{n=N+1}^\infty I(|f_n|) < \varepsilon$.

Now $|F(x) - \sum_1^N f_n(x)| \leq \sum_{n=N+1}^\infty |f_n(x)|$ on the full set A, and hence $I(|F - \sum_1^N f_n|) \leq \sum_{N+1}^\infty I(|f_n|) < \varepsilon$.

But $F - \sum_1^N f_n = f - \sum_1^N f_n$ on A, and so

$$I(|f - \sum_{n=1}^N f_n|) = I(|F - \sum_1^N f_n|) < \varepsilon$$

and $\lim_{N\to\infty} I(|f - \sum_{n=1}^N f_n|) = 0$.□

Theorem 2.14. (X,L_1,I) is an integration space.

Proof: We must show that properties (1) to (4) of Definition 2.1 are valid for (X,L_1,I).

(1) has already been shown.

(2) Suppose $f \in L_1$ and $\{f_n\}$ is a sequence of non-negative functions in L_1 with $\sum_1^\infty I(f_n) < I(f)$. Then there is an $\varepsilon > 0$ with the property:

$$\sum_1^\infty I(f_n) + 3\varepsilon < I(f)$$

and

(a) there is a representing sequence $\{f_{nk}\}_{k=1}^\infty$ for each f_n, with $\sum_{k=1}^\infty I(|f_{nk}|) < I(f_n) + \frac{\varepsilon}{2^n}$ [Lemma 2.12]

(b) if $\{\varphi_k\}$ is any representation of f, there exists an $N \in Z^+$ with $I(f) \leq \sum_{k=1}^N I(\varphi_k) + \varepsilon$ and $\sum_{k=N+1}^\infty I(|\varphi_k|) \leq \varepsilon$.

Thus, $\sum_{n=1}^\infty \sum_{k=1}^\infty I(|f_{nk}|) + \sum_{k=N+1}^\infty I(|\varphi_k|) < \sum_{n=1}^\infty (I(f_n) + \frac{\varepsilon}{2^n}) + \varepsilon$
$\leq \sum_1^\infty I(f_n) + 2\varepsilon < I(f) - \varepsilon \leq \sum_{k=1}^N I(\varphi_k)$.

Since Property (2) is valid for functions in L, we have

$$\sum_{n=1}^\infty \sum_{k=1}^\infty |f_{nk}(x)| + \sum_{k=N+1}^\infty |\varphi_k(x)| < \sum_{k=1}^N \varphi_k(x)$$

for some $x \in X$. Then

$$\sum_{n=1}^\infty f_n(x) < \sum_{k=1}^N \varphi_k(x) - \sum_{k=N+1}^\infty |\varphi_k(x)| \leq \sum_{k=1}^\infty \varphi_k(x) = f(x)$$

(3) If $p \in L$ and $I(p) = 1$, then $(p, \{p, 0\cdot p, 0\cdot p, \ldots\}) \in L_1$ and $I(p) = 1$ in L_1.

(4) By Lemma 2.13, if $f \in L_1$ and $\{\varphi_n\}$ is any representation of f, then, for any $\varepsilon > 0$, there is a function $f_k \in L$ with $f_k = \sum_{n=1}^k \varphi_n$ and $I(|f - f_k|) < \frac{\varepsilon}{3}$. Since property (4) holds for functions in L, there

exists an $N \in Z^+$ with $I(f_k) - I(\min\{f_k,N\}) < \frac{\varepsilon}{3}$. Then

$$|I(\min\{f,N\}) - I(\min\{f_k,N\})| \le I(|\min\{f,N\} - \min\{f_k,N\}|)$$
$$\le I(|f - f_k|) < \frac{\varepsilon}{3}$$

Therefore, $I(f) - I(\min\{f,N\})$

$$\le |I(f) - I(f_k)| + |I(f_k) - I(\min\{f_k,N\})| + |I(\min\{f_k,N\}) - I(\min\{f,N\})|$$
$$< I(|f - f_k|) + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon$$

and so $\lim_{n\to\infty} I(\min\{f,n\}) = I(f)$.

Similarly, if $f \in L_1$, for any $\varepsilon > 0$ there is an $f_k \in L$ and $N \in Z^+$ with $I(|\,|f| - |f_k|\,|) < \frac{\varepsilon}{2}$ and $I(\min\{|f_k|, \frac{1}{N}\}) < \frac{\varepsilon}{2}$. Then

$$I(\min\{|f|, \tfrac{1}{N}\})$$
$$\le |I(\min\{|f|, \tfrac{1}{N}\}) - I(\min\{|f_k|, \tfrac{1}{N}\})| + |I(\min\{|f_k|, \tfrac{1}{N}\})|$$
$$< \varepsilon$$

so $\lim_{n\to\infty} I(\min\{|f|, \frac{1}{n}\}) = 0$.□

2. Integrable sets

Definition 2.15. A complemented set $A = (A,B)$ is *integrable* if its characteristic function χ_A is in L_1. The *measure* of A is defined to be $\mu(A) = I(\chi_A)$..

Proposition 2.16. (i) If A and B are integrable sets, then $A \wedge B$ and $A \vee B$ are also integrable, and $\mu(A) + \mu(B) = \mu(A \vee B) + \mu(A \wedge B)$.

(ii) If A and $A \wedge B$ are integrable, so is $A - B$, and $\mu(A - B) = \mu(A) - \mu(A \wedge B)$.

Proof: (i) We saw in Section 5 of the first chapter that $\chi_{A\wedge B} =$

min $\{\chi_A, \chi_B\}$. Hence by Proposition 2.2 and Theorem 2.14, $A \wedge B$ is integrable with measure $\mu(A \wedge B) = I(\min\{\chi_A, \chi_B\})$. Consequently, $\chi_{A \vee B} = \chi_A + \chi_B - \chi_{A \wedge B}$ is also integrable, and $\mu(A \vee B)$ $= \mu(A) + \mu(B) - \mu(A \wedge B)$.

(ii) $\chi_{A-B} = \min\{\chi_A, 1 - \chi_B\} = \chi_A(1 - \chi_B) = \chi_A - \chi_{A \wedge B}$.□

Proposition 2.17. (i) If $A = (A,B)$ is integrable, then $A \cup B$ is a full set.

(ii) If $A = (A,B)$ is integrable, then $\mu(A) = 0$ if and only if B is a full set.

Proof: (i) $A \cup B$ is the domain of the integrable function χ_A .

(ii) If B is full, then $\chi_A = 0$ on a full set, and hence by Corollary 2.10, $\mu(A) = I(0 \cdot \chi_A) = 0 \cdot I(\chi_A) = 0$.

If $\mu(A) = I(\chi_A) = 0$, then $I(n \cdot \chi_A) = n \cdot I(\chi_A) = 0$ for every $n \in Z^+$. Thus $\sum_{n=1}^{\infty} I(|n \cdot \chi_A|)$ converges, and so there is a function $f \in L_1$ with $f(x) = \sum_1^{\infty} n \cdot \chi_A(x)$ for every $x \in D(f) = \{x \in X : \sum_1^{\infty} |n \cdot \chi_A(x)|$ converges$\}$. Now $x \in D(f)$ implies $x \in D(\chi_A)$ and $\chi_A(x) = 0$, so x must be in B. Therefore $D(f) \subset B$ and so B is a full set.□

Corollary 2.18. If $A = (A_1, A_2)$ and $B = (B_1, B_2)$ are integrable sets with $A < B$, then $\mu(A) \leq \mu(B)$.

Proof: $A < B$ means that $A_1 \subset B_1$ and $B_2 \subset A_2$. Both χ_A and χ_B are defined on $(A_1 \cup A_2) \cap (B_1 \cup B_2)$, and if $\chi_A(x) = 1$ for any x in this set, then clearly $\chi_B(x) = 1$ also. It follows that $\chi_A \leq \chi_B$ on $(A_1 \cup A_2) \cap (B_1 \cup B_2)$. But this is a full set, and hence by Proposition 2.8, $\mu(A) \leq \mu(B)$.□

Now that we have established these facts about integrable sets, it is natural to ask whether we can find many integrable sets in an integration space (X,L_1,I). Bishop and Cheng [2] have proved that if $\underline{f}$ is any function in L_1, then for all but a countable number of $t \in R$, the complemented sets $A_t = (\{x \in X : f(x) \geq t\}, \{x \in X : f(x) < t\})$ and $B_t = (\{x \in X : f(x) > t\}, \{x \in X : f(x) \leq t\})$ are integrable and have the same measure. The proof of this theorem is rather long, and so we shall not repeat it here. It should be noted, though, that in the proof, a sequence of real numbers $\{\alpha_n\}_{n=1}^{\infty}$ is constructed, with the property that whenever $t \neq \alpha_n$ for any $n \in Z^+$, A_t and B_t are integrable. The fact that we can find such a $\underline{t}$ in any non-empty interval in R is a consequence of the uncountability of the real numbers. This can be formalized as follows:

<u>Proposition 2.19.</u> Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence of real numbers and let (a,b) be any non-empty open interval in R. Then there exists a number x in $[a,b]$ with $x \neq \alpha_n$ for any $n \in Z^+$.

<u>Proof</u>: We construct two sequences of rational numbers $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ by induction. They will have the properties:

(i) $a \leq x_m \leq x_n < y_n \leq y_m \leq b$ for $n \geq m \geq 1$

(ii) $x_n > \alpha_n$ or $y_n < \alpha_n$ for each $n \geq 1$

(iii) $y_n - x_n < \frac{1}{n}$ for all $n \geq 1$

Let $x_0 = a$, $y_0 = b$, and for $k \geq 1$, suppose $x_0, \ldots, x_{k-1}$ and $y_0, \ldots, y_{k-1}$ have been constructed. Since $x_{k-1} < y_{k-1}$, we can prove that $\alpha_k > x_{k-1}$ or $\alpha_k < y_{k-1}$. Construct x_k and y_k in one of the following ways:

(1) If $\alpha_k > x_{k-1}$, let x_k be any rational number with $x_{k-1} < x_k < \min\{\alpha_k, y_{k-1}\}$, and let y_k be any rational with $x_k < y_k < \min\{\alpha_k, y_{k-1}, x_k + \frac{1}{k}\}$.

(2) If $\alpha_k < y_{k-1}$, let y_k be any rational number with $\max\{\alpha_k, x_{k-1}\} < y_k < y_{k-1}$, and let x_k be any rational with $\max\{\alpha_k, x_{k-1}, y_k - \frac{1}{k}\} < x_k < y_k$.

It is easy to see that (i), (ii), and (iii) are satisfied.

Then for any $n \geq m \geq 1$, $|x_m - x_n| = x_n - x_m < y_m - x_m < \frac{1}{m}$ and $|y_m - y_n| = y_m - y_n < y_m - x_m < \frac{1}{m}$. Hence $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ are the representing Cauchy sequences for two real numbers x and y. Since $|y_n - x_n| < \frac{1}{n}$ for each $n \geq 1$, $x = y$. Also $\alpha_n < x_n$ or $y_n < \alpha_n$ for each n, and since $x_n \leq x$ and $x \leq y_n$, we have $x \neq \alpha_n$ for any $n \in Z^+$.□

Therefore, it follows from Bishop and Cheng's theorem in [2] that the set $\{t \in R : A_t$ and B_t are integrable$\}$, is dense in R, and that we can effectively construct an element of this set in any non-empty open interval in R.

CHAPTER III

INTEGRATION ON LOCALLY COMPACT SPACES

1. Metric spaces

A metric space (X,ρ) consists of a set X and a total function $\rho: X \times X \to R$ with the properties:

(a) $\rho(x,y) \geq 0$ for all $x,y \in X$

(b) $\rho(x,y) = 0$ if and only if $x = y$

(c) $\rho(x,y) = \rho(y,x)$ for all $x,y \in X$

(d) $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$ for all $x,y,z \in X$

Also, if x and y are two elements of (X,ρ), then $x \neq y$ if and only if $\rho(x,y) > 0$.

Functions, uniform continuity, sequences, and converges of sequences in metric spaces are defined in the usual manner. [An important example of a uniformly continuous function is $f(x) = \rho(x,x_0)$, which maps (X,ρ) to R (with the Euclidean metric). This is continuous for any fixed $x_0 \in X$ since, if $\rho(x,y) \leq \varepsilon$, then $|f(x) - f(y)| = |\rho(x,x_0) - \rho(y,x_0)|$ $\leq |\rho(x,y) + \rho(y,x_0) - \rho(y,x_0)| = \rho(x,y) \leq \varepsilon$.]

A subset Y of a metric space (X,ρ) is also a metric space when given the metric ρ_Y - the restriction of ρ to Y. We will usually denote an induced metric space of this type simply as (Y,ρ).

A subset Y of a metric space (X,ρ) is <u>closed</u> if every Cauchy sequence in Y converges to a point in Y.

A metric space (X,ρ) is <u>bounded</u> if there exists a constant $C \in R$ with $\rho(x,y) \leq C$ for every x and y in X. If (X,ρ) is bounded by C, we say that the <u>diameter</u> of X is at most C.

A non-empty subset Y of (X,ρ) is <u>located</u> if $\rho(x,Y) = \inf\{\rho(x,y) : y \in Y\}$ exists for every $x \in X$. If Y is located, then its <u>metric complement</u> is defined to be the set $-Y = \{x \in X : \rho(x,Y) > 0\}$. [Note: A subset X of R has a least upper bound or supremum (respectively, greatest lower bound or infimum) if there exists a number $c \in R$ such that $x \leq c$ (resp. $c \leq x$) for all $x \in X$, and for each $\varepsilon > 0$ there exists an $x \in X$ with $c - x < \varepsilon$ (resp. $x - c < \varepsilon$).]

2. Locally compact metric spaces

A set <u>A</u> is an initial segment of Z^+ if $A = \phi$ or $A = \{1,\ldots, n\}$ for some $n \in Z^+$. A set X is finite if there is a one-to-one function mapping X onto an initial segment of Z^+. X is called subfinite if there is an operation φ mapping X onto an initial segment A of Z^+ and a function f from A to X such that $f(\varphi(x)) = x$ for every $x \in X$. Intuitively, a set is finite if it has exactly <u>n</u> elements, and subfinite if it has at most <u>n</u> elements, for some non-negative integer <u>n</u>. [Not all subfinite sets can be proven to be finite. For example, the set consisting of zero and the Goldbach number $r = \sum_1^\infty \frac{\alpha_k}{2^k}$ (where $\{\alpha_k\}$ is the Goldbach sequence) has at most two elements, but we do not know exactly how many elements it has.]

Definition 3.1. A subset A of a metric space (X,ρ) is totally bounded if, for each $\varepsilon > 0$, there exists an $N \in Z^{0+}$ and a finite subset $\{x_1,\ldots, x_N\}$ of A with the property that if $a \in A$, then at least one of the numbers $\rho(x_1,a)$, $\rho(x_2,a),\ldots, \rho(x_N,a)$ is less than ε. The set $\{x_1,\ldots, x_N\}$ is called an ε approximation to A.

It is sometimes easier to find subfinite ε approximations to sets. The following proposition indicates that a set is totally bounded if, for every $\varepsilon > 0$, it has a subfinite ε approximation.

Proposition 3.2. If a metric space (A,ρ) has a subfinite ε approximation for every $\varepsilon > 0$, then it also has a finite ε approximation for every $\varepsilon > 0$.

Proof: Let $\varepsilon > 0$ be given, and let $X = \{x_1,\ldots, x_n\}$ be a subfinite $\frac{\varepsilon}{2}$ approximation to A. For $1 \leq i < k \leq n$, either $\rho(x_i,x_k) \geq \frac{\varepsilon}{4}$ or $\rho(x_i,x_k) < \frac{\varepsilon}{2}$. If $\rho(x_1,x_k) < \frac{\varepsilon}{2}$ for any $k > 1$, discard x_k from X. If x_2 has not been discarded and $\rho(x_2,x_k) < \frac{\varepsilon}{2}$ for any $k > 2$, discard x_k from the set. Continue this process for successive i's until x_n is reached. The set $Y = \{x_1,\ldots,x_m\}$ which remains is finite since $i \neq j$ implies $x_i \neq x_j$ for any x_i and x_j in Y. If $a \in A$ either $\rho(a,x_k) < \frac{\varepsilon}{2}$ for some x_k Y or else there exist elements $x_i \in X$ and $x_k \in Y$ such that $\rho(a,x_i) < \frac{\varepsilon}{2}$ and $\rho(x_i,x_k) < \frac{\varepsilon}{2}$. In either case $\rho(a,x_k) < \varepsilon$. Hence Y is an ε approximation to A.□

Definition 3.3. A subset of a metric space is compact if it is closed and totally bounded.

Definition 3.4. A metric space (X,ρ) is locally compact is every bounded subset of X is contained in a compact set. A (total) function on a locally compact space (X,ρ) is continuous if it is uniformly continuous on every compact subset (equivalently, every bounded subset) of X.

We next establish some properties of compact and locally compact spaces which will be useful later.

Proposition 3.5. Let f be a continuous function from a locally compact space (X,ρ) to a metric space (Y,ρ^*). If A is a totally bounded subset of X, then its image, f(A), is also totally bounded.

Proof: If A is totally bounded, then it is bounded and hence contained in a compact set K. f is uniformly continuous on K. For any given $\varepsilon > 0$, let $\delta > 0$ be such that $\rho^*(f(x), f(y)) < \varepsilon$ whenever $\rho(x,y) < \delta$ and $x,y \in K$. Let $\{x_1,\ldots, x_n\}$ be a δ approximation to A. Then for any f(a) in f(A), there is an x_i with $\rho(a,x_i) < \delta$ and $\rho^*(f(a), f(x_i)) < \varepsilon$. Thus $\{f(x_1),\ldots, f(x_n)\}$ is an ε approximation to f(A).□

Corollary 3.6. Let f be a continuous function from a locally compact space X to a locally compact space Y. If A is a bounded set in X, then f(A) is bounded in Y [and hence contained in a compact set].

Proof: If A is bounded, it is contained in a compact set K. f(K) is totally bounded, by Proposition 3.5. Therefore f(A) is contained in the bounded set f(K), which is contained in a compact set in Y.□

Proposition 3.7. Let $f: (X,\rho) \to R$ be continuous on the locally compact space X. If K is any non-empty compact subset of X, then $\sup \{f(x) : x \in K\}$ and $\inf \{f(x) : x \in K\}$ exist.

Proof: By Proposition 3.5, f(K) is totally bounded in R. For each $k \in Z^+$ choose a $\frac{1}{k}$ approximation $\{x_1, \ldots, x_n\}$ to f(K). For some m, $1 \le m \le n$, we have $x_m \ge \max \{x_1, \ldots, x_n\} - \frac{1}{k}$. Write $c_k = x_m$.

For any $j,k \in Z^+$, $|c_k - c_j| \le \frac{2}{j} + \frac{2}{k}$, therefore $\{c_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Write $c = \lim_{n\to\infty} c_n$. Then for any $x \in f(K)$, $x - c \le \lim_{n\to\infty} (x - c_n) \le \lim_{n\to\infty} \frac{2}{n} = 0$. Therefore $x \le c$ for each $x \in f(K)$. Since $c = \lim_{n\to\infty} c_n$ and each $c_n \in f(K)$, c is the least upper bound of f(K).

A similar proof shows that $\inf \{f(x) : x \in K\}$ exists.□

Proposition 3.8. (i) A non-empty compact subset of an arbitrary metric space (X,ρ) is closed and located.

(ii) A closed and located subset of a compact space (X,ρ) is compact.

Proof: (i) If K is a compact subset of (X,ρ) it is closed. Let x_0 be any point in X. Then $f(x) = \rho(x,x_0)$ is uniformly continuous on X, and hence on K. By Proposition 3.7, $\inf \{f(x) : x \in K\}$ exists, so $\rho(x_0,K) = \inf \{\rho(x,x_0) : x \in K\}$ exists for each point $x_0 \in X$.

(ii) Let Y be a closed located subset of the compact space (X,ρ). Fix $\varepsilon > 0$ and let $\{x_1, \ldots, x_n\}$ be an $\frac{\varepsilon}{3}$ approximation to X. For each i, $1 \le i \le n$, we can choose a $y_i \in Y$ with $\rho(x_i,y_i) < \rho(x_i,Y) + \frac{\varepsilon}{3}$.

For any $y \in Y$, there is an x_i with $\rho(x_i,y) < \frac{\varepsilon}{3}$. We chose y_i with $\rho(x_i,y_i) < \rho(x_i,Y) + \frac{\varepsilon}{3} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$. Thus $\rho(y_i,y) \le \rho(y_i,x_i) + \rho(x_i,y) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$, and the subfinite set $\{y_1, \ldots, y_n\}$ is an ε approximation to Y. Since ε was any arbitrary positive number, it follows that Y is totally bounded and hence compact.□

Proposition 3.8 (ii) is one substitute for the classical result that a closed subset of a compact space is compact. Of course, classically, every subset of a metric space is "located" since the greatest lower bound of a bounded set of numbers is not required to be effectively computable. It is easy to find an example of a closed subset of a compact space which is not provably compact (constructively). Let $f: [0,1] \to [0,1]$ be the unique continuous extension of the partial function φ, defined by

$$\varphi(x) = \begin{cases} \frac{3}{2}x & \text{for } 0 \le x \le \frac{1}{3} + r \\ \frac{1}{2} + \frac{3}{2}r & \text{for } \frac{1}{3} + r \le x \le 1 \end{cases}$$

where $r = \sum_{1}^{\infty} \frac{\alpha_k}{2^k}$ and $\{\alpha_k\}$ is the Goldbach sequence. Then $\{x \in [0,1] : f(x) \le \frac{1}{2}\}$ is certainly closed, but it is not located since we do not know whether this set is equal to $[0, \frac{1}{3}]$ or $[0,1]$.

Theorem 3.10 provides another set of sufficient conditions for a closed subset of a compact space to be compact.

Lemma 3.9. Let (X,ρ) be a compact metric space. Then for every $\varepsilon > 0$, there exist a finite number of compact subsets of X whose diameters are ar most ε, and whose union is X.

Proof: Let $\{x_1,\ldots,x_N\}$ be an $\frac{\varepsilon}{9}$ approximation to X. We define by induction N sequences $\{X_1^i\}_{i=1}^{\infty},\ldots, \{X_N^i\}_{i=1}^{\infty}$ of subfinite subsets of X such that:

(i) $X_j^1 \subset X_j^2 \subset \ldots$ for $j = 1, \ldots, N$

(ii) $\rho(x, X_j^i) < \frac{\varepsilon}{3^i}$ if $x \in X_j^{i+1}$, $1 \le j \le N$

(iii) $\rho(x, X_j^{i+1}) < \frac{\varepsilon}{3^{i+2}}$ if $\rho(x, X_j^i) < \frac{\varepsilon}{3^{i+1}}$, $1 \le j \le N$

Let $X_1^1 = \{x_1\}$, $X_2^1 = \{x_2\}, \ldots, X_N^1 = \{x_N\}$, and suppose $X_1^i, \ldots, X_N^i$ have been defined for $i \ge 1$. Let $\{y_1, \ldots, y_m\}$ be an $\frac{\varepsilon}{3^{i+2}}$ approximation to X. Then for each j and k, $1 \le j \le N$, $1 \le k \le m$, either $\rho(y_k, X_j^i) < \frac{\varepsilon}{3^i}$ or $\rho(y_k, X_j^i) > \frac{\varepsilon}{2 \cdot 3^i}$. For $1 \le j \le N$, let

$$X_j^{i+1} = X_j^i \cup \{y_k : \rho(y_k, X_j^i) < \frac{\varepsilon}{3^i}, 1 \le k \le m\}.$$

(i) and (ii) are clearly satisfied. Suppose $\rho(x, X_j^i) < \frac{\varepsilon}{3^{i+1}}$. There is a y_k in $\{y_1, \ldots, y_m\}$ with $\rho(y_k, x) < \frac{\varepsilon}{3^{i+2}}$, and consequently $\rho(y_k, X_j^i) \le \rho(y_k, x) + \rho(x, X_j^i) < \frac{\varepsilon}{3^{i+2}} + \frac{\varepsilon}{3^{i+1}} < \frac{\varepsilon}{2\ 3^i}$, and $y_k \in X_j^{i+1}$. Therefore $\rho(x, X_j^{i+1}) < \rho(x, y_k) < \frac{\varepsilon}{3^{i+2}}$, and (iii) is satisfied.

Now let $Y_j = \bigcup_{i=1}^{\infty} X_j^i$ for $j = 1, \ldots, N$. If $y \in Y_j$ then $\rho(y, X_j^i) \le \sum_{k=i}^{\infty} \frac{\varepsilon}{3^k} = \frac{\varepsilon}{2 \cdot 3^{i-1}}$ for every $i \in Z^+$. It follows that for any $i \in Z^+$ we can choose an element x in the subfinite subset X_j^{i+2} of Y_j, with $\rho(x, y) < \frac{\varepsilon}{2 \cdot 3^i}$. Therefore Y_j is totally bounded.

Let X_j be the closure of Y_j. Then X_j is compact, and by (ii), it has diameter at most $2 \cdot \sum_{i=1}^{\infty} \frac{\varepsilon}{3^i} = \varepsilon$.

Suppose $x \in X$. Since $\{x_1, \ldots, x_N\}$ was an $\frac{\varepsilon}{9}$ approximation to X, there is an x_j with $\rho(x, x_j) = \rho(x, X_j^1) < \frac{\varepsilon}{9}$. Then by (iii) we have $\rho(x, X_j^i) < \frac{\varepsilon}{3^{i+1}}$ for every $i \in Z^+$. Therefore $x \in X_j$ and $\bigcup_{j=1}^{N} X_j = X$.□

Theorem 3.10. Let (X,ρ) be a compact space and $f: X \to R$ a uniformly continuous function on X. Then for all except countably many $\alpha \in R$, the set $X_\alpha = \{x \in X : f(x) \leq \alpha\}$ is compact. [Similarly, for all except countably many $\alpha \in R$, the set $X_\alpha^* = \{x \in X : f(x) \geq \alpha\}$ is compact.]

Proof: If X is empty, the theorem is trivial.

If X is non-empty, then for each $k \in Z^+$ we can find non-empty compact sets $X_1^k, \ldots, X_{N(k)}^k$ such that $\bigcup_{i=1}^{N(k)} X_i^k = X$ and the diameter of each X_i^k is less than $\frac{1}{k}$. Let $c_{jk} = \inf \{f(x) : x \in X_j^k, 1 \leq j \leq N(k)\}$.. By Proposition 2.19, we can find a number α in any non-empty open interval in R with $\alpha \neq c_{jk}$ for every j, $1 \leq j \leq N(k)$, and every $k \in Z^+$. For each such α, and each $k \in Z^+$, we can construct a $\frac{1}{k}$ approximation to X_α, in the following way: If $c_{jk} < \alpha$, $(1 \leq j \leq N(k))$, pick a point $x_j \in X_j^k$. Let A_k be the set containing all such x_j's. (Note that A_k is subfinite.) Now if $x \in X_\alpha$, then $x \in X_j^k$ for some j, and then $c_{jk} \leq f(x) \leq \alpha$. But $c_{jk} \neq \alpha$ for all j and k, hence $c_{jk} < \alpha$, and so $x_j \in A_k$ and $\rho(x, x_j) < \frac{1}{k}$. Thus A_k is a $\frac{1}{k}$ approximation to X_α. X_α is closed since f is a uniformly continuous function. Therefore X_α is compact. The proof that X_α^* is compact is similar.□

Corollary 3.11. Let (X,ρ) be a locally compact space and let K be a compact subspace of X. Then for all except countably many $\alpha \in R$, the set $K_\alpha = \{x \in X : \rho(x, K) \leq \alpha\}$ is compact.

Proof: We shall show that for any $n \in Z^+$, the sets K_α are compact for all except countably many α in $(-\infty, n)$.

Let $n \in Z^+$ be given. The function $f(x) = \rho(x, K)$ is uniformly continuous on X since K is located. $K_n = \{x \in X : \rho(x, K) \leq n\}$ is bounded, and hence is contained in a compact set K^*. Now $f(x) = \rho(x, K)$ is uniformly continuous on the compact space (K^*,ρ), and therefore for all except countably many $\alpha \in (-\infty, n)$, the set $K_\alpha = \{x \in K^* : \rho(x, K) \leq \alpha\}$ is compact.□

3. Positive Integrals

Definition 3.12. Let $f: X \to R$ be a continuous function on the locally compact space (X,ρ). A compact set $K \subset X$ is a support for f if $f(x) = 0$ for all $x \in -K$ (the metric complement of K). The set of all continuous functions on X with compact support is denoted $C(X)$.

Proposition 3.13. Every function in $C(X)$ is uniformly continuous.

Proof: Let K be a compact support for $f \in C(X)$, and choose any $\alpha > 0$. f is uniformly continuous on the bounded set $K_\alpha = \{x \in X : \rho(x,K) \leq \alpha\}$. Therefore for every $\varepsilon > 0$ there is a δ, $0 < \delta < \frac{\alpha}{2}$, so that for all $x,y \in K_\alpha$, $|f(x) - f(y)| \leq \varepsilon$ whenever $\rho(x,y) \leq \delta$. Now let x and y be in X with $\rho(x,y) \leq \delta$. Either $\rho(x,K) \leq \alpha - \delta$ or $\rho(x,K) > \delta$. In the first case, both x and y are in K_α, and so $|f(x) - f(y)| \leq \varepsilon$. In the second case, both x and y are in $-K$, so $|f(x) - f(y)| = |0 - 0| \leq \varepsilon$. Hence f is uniformly continuous on X.□

Definition 3.14. A positive integral, I, on a locally compact space (X,ρ) is a linear real-valued functional on $C(X)$ such that

(a) if $f \in C(X)$ and $f \geq 0$, then $I(f) \geq 0$

(b) there is a function $f \in C(X)$ with $I(f) \neq 0$.

Theorem 3.15. If (X,ρ) is a locally compact space and I is a positive integral on X, then (X, C(X), I) is an integration space.

Proof: We must check Properties (1) through (4) of Definition 2.1:

(1) If f has compact support, so do $|f|$ and $\min\{f,1\}$. If f and g have compact support, then so does $\alpha f + \beta g$ and, by the linearity of I, $I(\alpha f + \beta g) = \alpha I(f) + \beta I(g)$ for every $\alpha,\beta \in R$. [Note: if $f,g \in C(X)$, so is fg.]

(2) The proof that (X, C(X), I) satisfies (2) requires several technical lemmas which we will not present here. They may be found in Bishop and Cheng [2], p. 67 - 74. The idea of the proof is that, given $g \in C(X)$ and the sequence $\{f_n\}_1^\infty$ in C(X) with $\sum_1^\infty I(f_n) < I(g)$, we can construct a Cauchy sequence $\{x_n\}_1^\infty$ of points of X, in such a way that

(a) there is a sequence $\{\lambda_n\}$ of functions in C(X) with $\lambda_n(x_n) > 0$, but $\lambda_n(y) = 0$ whenever $\rho(x_n,y) > \frac{1}{n^2}$;

(b) there is a strictly increasing sequence $\{M_n\}_{n=1}^\infty$ of positive integers;

(c) there is a sequence of real numbers $\{\delta_n\}$, with $\delta_n \le \frac{1}{2^n}$;

and for all $n \in Z^+$ and some $\varepsilon > 0$,

$$\sum_{k=M_{n-1}}^{M_n-1} I(f_k\lambda_n) < \prod_{k=1}^{n-1} \delta_k(1 - \delta_n)I(\lambda_n(g - \varepsilon))$$

If $x = \lim_{n\to\infty} x_n$, then, after some work, we can conclude that

$$\sum_{k=M_{n-1}}^{M_n-1} f_k(x) \le \prod_{k=1}^{n-1} \delta_k(1 - \delta_n)[g(x) - \frac{\varepsilon}{4}]$$

and since $\sum_{n=1}^\infty [(1 - \delta_n) \prod_{k=1}^{n-1} \delta_k] \le 1$, it follows that $\sum_{n=1}^\infty f_n(x) < g(x)$. The details of all these constructions can be found in Bishop and Cheng [2], p. 70 - 74.

(3) By assumption there is an $f \in C(X)$ with $I(f) \neq 0$. Then the function $\frac{f}{I(f)} \in C(X)$ and $I(\frac{f}{I(f)}) = 1$.

(4) Let K be a compact support for $f \in C(X)$. Then $\sup \{f(x) : x \in X\} = \sup \{f(x) : x \in K\}$, and, by Proposition 3.7, this quantity exists and is finite. Pick $N \in Z^+$ with $\sup \{f(x) : x \in X\} \leq N$. Then

$$\lim_{n\to\infty} I(\min \{f,n\}) = \lim_{n\geq N} I(\min \{f,n\}) = I(f)$$

If K supports f, then the continuous function $g(x) = [1 - \rho(x,K)]^+$ is supported by any compact set containing the bounded set $K_1 = \{x \in X : \rho(x,K) \leq 1\}$. Let $M = \sup \{|f(x)| : x \in K\} + 1$. Then for every $x \in X$ and every $n \in Z^+$,

$$0 \leq \min \{|f|(x), \frac{1}{n}\} \leq \frac{M}{n} g(x)$$

and consequently, $0 \leq I(\min \{|f|, \frac{1}{n}\}) \leq \frac{M}{n} I(g)$. Therefore

$$0 \leq \lim_{n\to\infty} I(\min \{|f|, \frac{1}{n}\}) \leq \lim_{n\to\infty} \frac{M}{n} I(g) = 0$$

hence $\lim_{n\to\infty} I(\min \{|f|, \frac{1}{n}\}) = 0$.□

The locally compact integration space (X, C(X), I) can be enlarged, using Definition 2.4, to form another integration space, $(X, C_1(X), I)$. [We can also modify Definition 3.14 in the obvious way to define a positive integral on $C_1(X)$.] $C_1(X)$ will include integrable partial functions. In particular, there will be many integrable compact subsets of X in $(X, C_1(X), I)$. [Note: A complemented set in X is compact if its first element is compact as a subset of X.] This is because, if X is locally compact and $f \in C_1(X)$, the set

$$A_t = (\{x \in X : f(x) \geq t\}, \{x \in X : f(x) < t\})$$

is compact and integrable for all except countably many $t \in R$. For example, if $N \in Z^+$ and K is any compact subset of the locally compact space X, then the function $[N - \rho(x,K)]^+$ is in $C(X)$; hence $K_{N-t} = (\{x \in X : \rho(x,K) \leq t\}, \{x \in X : \rho(x,K) > t\})$ is compact and integrable for all except countably many t in $(-\infty, N]$.

Classically, of course, every compact set K in a locally compact space is integrable. This follows from the fact that each function $f_n(x) = [1 - n\rho(x,K)]^+$, $(n \in Z^+)$, is integrable, and $\{f_n\}$ converges monotonically (pointwise) to χ_K. Thus $\{I(f_n)\}$ is a monotone decreasing sequence converging to $I(\chi_K)$. However, constructively, we require the series $\sum_1^\infty I(|f_{n+1} - f_n|)$ to be convergent (with a known rate of convergence) before we can say that K is integrable and $\mu(K) = \lim_{n\to\infty} I(f_n)$. This may be restated in the following way.

<u>Proposition 3.16.</u> A compact set $K = (K, -K)$ in a locally compact integration space is integrable if there exists a constant $c \in R$, such that for all $\varepsilon > 0$ there is a $\delta > 0$ with $|I(f) - c| < \varepsilon$, whenever $f \in C(X)$, $0 \leq f \leq 1$, and $f(x) = 1$ on K, $f(x) = 0$ if $\rho(x,K) > \delta$.

<u>Proof</u>: By assumption we can find a sequence $\{f_n\}$ of functions in $C(X)$ with $\lim_{n\to\infty} f_n = \chi_K$ and $|I(f_n) - c| < \frac{1}{2^n}$ for some constant c. Then $I(|f_1|) + \sum_1^\infty I(|f_{n+1} - f_n|)$ must converge, and hence K is integrable and $\mu(K) = I(f_1) + \sum_1^\infty I(f_{n+1} - f_n) = c.$ □

An important example of a locally compact integration space is (R, C(R), I). If $f \in C(R)$, then I(f) is defined to be the ordinary Riemann integral $\int_a^b f(x)dx$, where [a,b] is any compact interval supporting f. The Lebesgue integral on $C_1(R)$ is then defined as in Definition 2.4.

As an illustration of the method in Proposition 3.16, we can show that the set of rationals (Q, Q') is Lebesgue integrable. [$Q' = \{x \in R : x \neq q$ for any rational number $q\}$]. Let $\{q_n\}_{n=1}^{\infty}$ be an enumeration of Q. Then each singleton set $\{q_m\}$ is compact and $\chi_{\{q_m\}}$ can be approximated by a sequence $\{f_j^m\}_{j=1}^{\infty}$ of functions in C(R) with $\lim_{j\to\infty} f_j^m = \chi_{\{q_m\}}$ and $|I(f_j^m)| < \frac{1}{2^j}$. Therefore $\chi_{\{q_m\}}$ is integrable, and $\mu(\{q_m\}) = 0$. Then by the Completeness Theorem, $\chi_Q = \sum_{n=1}^{\infty} \chi_{\{q_n\}}$ is also integrable, and hence Q also has measure zero.

CHAPTER IV

THE HAAR INTEGRAL

In this chapter we prove that every locally compact group G admits a left-invariant positive integral and that this integral is unique up to a constant of proportionality. [The proof can be easily modified to show that a right-invariant integral also exists on G.] The construction of the Haar integral is basic to the study of certain properties of locally compact Abelian groups. A constructive treatment of some of the applications of the Haar integral can be found in Bishop [1], Chapter 10.

1. Locally compact groups

Definition 4.1. A locally compact metric space G is a locally compact group if G is a group and the mapping $(x,y) \to x^{-1}y$ from $G \times G$ to G is continuous.

[If ρ denotes the metric on G, then the product metric ρ^* on $G \times G$ is defined by $\rho^*((x_1,x_2), (y_1,y_2)) = \rho(x_1,y_1) + \rho(x_2,y_2)$.]

The identity element of G will be denoted by $\underline{e}$.

Proposition 4.2. Let G be a locally compact group, and let x,y be elements of G. Then

(i) the operation $x \to x^{-1}$ is continuous

(ii) the operation $(x,y) \to xy$ is continuous

(iii) for each $a \in G$, the transformations $x \to ax$ and $x \to xa$ are continuous

(iv) if H and K are bounded subsets of G, then the sets HK, $H^{-1}K$ and HK^{-1} are also bounded. Similarly, HK, $H^{-1}K$ and HK^{-1} are totally bounded if H and K are totally bounded subsets of G.

Proof: (i) The composite function $x \to (x,e) \to x^{-1}e = x^{-1}$ is continuous.

(ii) Since $x \to x^{-1}$ is continuous, so is $(x,y) \to (x^{-1},y) \to (x^{-1})^{-1}y = xy$.

(iii) $x \to xa$ is equivalent to the composite mapping $x \to (x^{-1},a) \to (x^{-1})^{-1}a = xa$, which is continuous for any fixed $a \in G$. Similarly, $x \to (a^{-1},x) \to (a^{-1})^{-1}x = ax$ is continuous for fixed $a \in G$.

(iv) The functions $(x,y) \to xy$, $(x,y) \to x^{-1}y$, and $(x,y) \to (x^{-1},y^{-1}) \to \to xy^{-1}$ from $G \times G$ to G are continuous mappings from one locally compact space to another. Hence, by Proposition 3.5 and Corollary 3.6, they take bounded sets into bounded sets and totally bounded sets into totally bounded sets.□

Proposition 4.3. Let G be a locally compact group. If H is any bounded subset of G, then

(i) for each $\varepsilon > 0$ there is a $\delta > 0$ such that $\rho(x^{-1}y,e) \le \varepsilon$ whenever $x,y \in H$ and $\rho(x,y) \le \delta$.

(ii) for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $\rho(x,y) \leq \varepsilon$ whenever $x,y \in H$ and $\rho(x^{-1}y,e) \leq \delta$.

Proof: Recall that a function on a locally compact space G is continuous if and only if it is uniformly continuous on every bounded subset of G.

(i) The composite function $(x,y) \to x^{-1}y \to (x^{-1}y,e)$ is continuous on $G \times G$, and hence uniformly continuous on $H \times H$.

(ii) From Proposition 4.2 (ii) and 4.2 (iv), we know that the mapping $(x,y) \to xy$ is uniformly continuous on $H \times H^{-1}H$. Therefore, for all $\varepsilon > 0$, there is a $\delta > 0$ such that if (x_1,y_1) and (x_2,y_2) are in $H \times H^{-1}H$ and $\rho^*((x_1,y_1), (x_2,y_2)) \leq \delta$, [i.e. $\rho(x_1,x_2) + \rho(y_1,y_2) \leq \delta$], then $\rho(x_1y_1, x_2y_2) \leq \varepsilon$. Substituting x for x_1 and x_2, e for y_1, and $x^{-1}y$ for y_2, we have $\rho(x, x(x^{-1}y)) = \rho(x,y) \leq \varepsilon$ whenever x and y are in H and $\rho(e, x^{-1}y) \leq \delta$.□

Corollary 4.4. Let G be a locally compact group. A subset K of G is totally bounded if and only if for each $\varepsilon > 0$ there exist $x_1,\ldots, x_n$ in K such that for any x in K at least one of the numbers $\rho(e,x_1^{-1}x),\ldots$ $\ldots, \rho(e,x_n^{-1}x)$ is less than ε.

Proof: If K is totally bounded, then it is bounded. Then by Proposition 4.3 (i), we can choose a δ approximation to K with the property desired.

Conversely, let $\varepsilon > 0$ and suppose $x_1,\ldots, x_n$ exist so that for each $x \in K$, there is an x_i with $\rho(e, x_i^{-1}x) < \varepsilon$. The set $\{x_i^{-1}x \in G : \rho(e, x_i^{-1}x) < \varepsilon\}$ is bounded, and the map $z \to x_iz$ is continuous for each fixed x_i. Hence $\{x \in G : \rho(e, x_i^{-1}x) < \varepsilon\}$ is also bounded, since continuous functions take bounded sets into bounded sets. Since $K \subset \bigcup_{i=1}^{n} \{x \in G : \rho(e, x_i^{-1}x) < \varepsilon\}$, K is a bounded set. We can then apply

Proposition 4.3 (ii) to obtain a λ approximation to K for any given $\lambda > 0$.□

Lemma 4.5. Let $f \in C(G)^{0+}$-- the set of all non-negative functions in C(G). Then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(y)| \leq \varepsilon$ if $\rho(xy^{-1}, e) \leq \delta$.

Proof: Since $z \to z^{-1}$ is continuous, we can choose for any $\lambda > 0$ a $\nu > 0$ with $\rho(e, z^{-1}) \leq \lambda$ whenever $\rho(e,z) \leq \nu$. Let E be any compact set containing $\{z \in G : \rho(e,z) \leq \lambda\}$ and let K be a compact support for f. Write $(EK)_1 = \{x \in G : \rho(x, EK) \leq 1\}$. The function $(x,y) \to xy$ is uniformly continuous on $E \times (EK)_1$, and f is uniformly continuous on G. Hence, for any $\varepsilon > 0$ we can choose a $\gamma > 0$ and δ with $0 < \delta < \nu$, such that $\rho^*((z,y), (e,y)) = \rho(z,e) \leq \delta$ implies $\rho(zy, y) \leq \gamma$ and $|f(zy) - f(y)| \leq \varepsilon$ for all $z \in E$ and $y \in (EK)_1$.

Now we claim that for all z in $\{z \in G : \rho(e,z) \leq \delta\}$ and every $y \in G$, $|f(zy) - f(y)| \leq \varepsilon$:

(1) If $y \in (EK)_1$, then $|f(zy) - f(y)| \leq \varepsilon$ by definition of δ.

(2) If $y \in -(EK)$, then since $e \in E$, $y \in -K$ and hence $f(y) = 0$. If $f(zy) > 0$, $zy \in K$ and $y \in z^{-1}K$. But since $\delta < \nu$, $\rho(e, z^{-1}) \leq \lambda$, and so $y \in EK$, which contradicts our assumption. Hence $f(zy) = 0$ also.

Now write $x = zy$, and $z = xy^{-1}$. Then we have for each $\varepsilon > 0$ a $\delta > 0$ such that $|f(x) - f(y)| \leq \varepsilon$ if $\rho(e, xy^{-1}) \leq \delta$.□

2. Construction of the Haar Integral

Definition 4.6. Let G be a locally compact group. For each function $f: G \to R$ and each $s \in G$, we define the left translate of f by s, $f \circ Ts$, by

$$f \circ Ts(x) = f(sx)$$

for all $x \in G$. The right translate is defined similarly.

Definition 4.7. Let G be a locally compact group and (G,L,I) an integration space. The integral I is said to be left-invariant, or invariant under left translations if $f \in L$ implies $f \circ Ts \in L$, and $I(f) = I(f \circ Ts)$ for every $f \in L$ and $s \in G$. The definition of right invariance is similar. A left-invariant positive integral on G is called a left Haar integral.

We begin the construction of the Haar integral by defining the Haar covering function, $(f:\varphi)$, which is a rough measure of the "size" of the function f, compared to another function φ.

Let $C(G)^{0+}$ denote the set of all non-negative functions in C(G), and let $C(G)^+$ be the set of all non-zero elements of $C(G)^{0+}$. Then for each $f \in C(G)^{0+}$ and $\varphi \in C(G)^+$ there exists a set S, consisting of all functions ξ with

(i) $\xi = \sum_{i=1}^{n} c_i \, \varphi \circ Ts_i$ for some $n \in Z^+$, where $c_i \geq 0$ and $s_i \in G$, $(1 \leq i \leq n)$,

and (ii) $f \leq \xi$.

We define

$$(f:\varphi) = \inf \{ \Sigma c_i : \xi = \Sigma (c_i \, \varphi \circ Ts_i) \text{ is in } S\}$$

whenever this infimum exists.

Lemma 4.8. Let $f \in C(G)^{0+}$ and $\varphi \in C(G)^{+}$. Then there is a compact set K such that if $f \le \sum_{i=1}^{n} c_i\ \varphi \circ Ts_i$, then $f \le \sum_{i \in A} c_i\ \varphi \circ Ts_i$, where $A \subset \{1,\ldots, n\}$ and $\{s_i : i \in A\} \subset K$.

Proof: Let H and J be compact supports for f and φ, respectively. By Proposition 4.2 (iv), JH^{-1} is totally bounded; hence its closure is compact. Choose $\alpha > 0$ so that

$$K = \{x \in G : \rho(x, JH^{-1}) \le \alpha\}$$

is compact. [The distance from a point x to a set A is the same as the distance from x to the closure of A, if either of these quantities exists.]

Suppose $f \le \sum_{i=1}^{n} c_i\ \varphi \circ Ts_i$. For each s_i, either $\rho(s_i, JH^{-1}) \le \alpha$ or $\rho(s_i, JH^{-1}) > 0$. Let $A = \{i : \rho(s_i, JH^{-1}) \le \alpha, 1 \le i \le n\}$.

If $f(x) > \sum_{i \in A} c_i\ \varphi \circ Ts_i(x)$ for any $x \in G$, then $x \in H$, and $\sum_{k \notin A} c_k\ \varphi \circ Ts_k(x) > 0$ $(1 \le k \le n)$. But for each $x \in H$ and $k \notin A$, $(1 \le k \le n)$, $\rho(s_k, JH^{-1}) > 0$. Hence $\rho(s_k x, J) > 0$ and $\varphi(s_k x) = \varphi \circ Ts_k(x) = 0$, since J supports φ. Therefore $f \le \sum_{i \in A} c_i\ \varphi \circ Ts_i$. □

We can now show that for any $f \in C(G)^{0+}$ and $\varphi \in C(G)^{+}$, the set $S = \{\xi : \xi = \sum c_i\ \varphi \circ Ts_i$, and $\xi \le f\}$ is non-empty.

Let H be a compact support for f and choose K as in the proof of the last lemma. There exists a $t \in G$ and $\gamma > 0$ with $\varphi(t) > \gamma$. Since $y \to ty$ is continuous, we can choose $\delta > 0$ so that $\varphi(ty) > \gamma$ whenever $\rho(e,y) < \delta$. Pick $s_1^{-1},\ldots, s_N^{-1}$ in K such that for each $x \in K$, we have $\rho(e, s_j x) < \delta$ for some j $(1 \le j \le N)$ [Proposition 4.3 (i)]. Then

$\varphi(ts_j x) > \gamma$ and

$$1 \le \frac{1}{\gamma} \sum_{i=1}^{N} \varphi(ts_i x)$$

for every $x \in H$. Let $M_f = \sup \{f(x) : x \in H\}$. Then

$$f(x) \le \frac{1}{\gamma} M_f \cdot \sum_{i=1}^{N} \varphi \circ Tts_i(x)$$

for all $x \in G$, and hence $\xi = \sum_{i=1}^{N} \frac{M_f}{\gamma} \cdot \varphi \circ Tts_i$ is in S.

Classically, then, the number $(f:\varphi)$ must exist because it is the infimum of a non-empty bounded set of numbers. However, for the constructive proof of the existence of $(f:\varphi)$, we require more information about the set $\{ \Sigma c_i : \Sigma\, c_i\, \varphi \circ Tt_i \in S\}$.

<u>Lemma 4.9.</u> The quantity $(f:\varphi)$ exists for every $f \in C(G)^{0+}$ and $\varphi \in C(G)^{+}$.

<u>Proof</u>: Given f and φ, let K be chosen as in Lemma 4.7, and let $\xi = \sum_{i=1}^{N} \frac{M_f}{\gamma}\, \varphi \circ Tts_i$ be as above. Write

$$A = \{\psi : \psi = \Sigma\, c_i\, \varphi \circ Tt_i \in S \text{ and } t_i \in K \text{ for each } i\}$$

and

$$A_c = \{ \Sigma c_i : \Sigma\, c_i\, \varphi \circ Tt_i \in A \text{ and } \Sigma\, c_i \le \frac{NM_f}{\gamma} \}$$

Then to show that $(f:\varphi) = \inf \{ \Sigma c_i : \Sigma\, c_i\, \varphi \circ Tt_i \in S\}$ exists, we need only show that $\inf \{ \Sigma c_i : \Sigma c_i \in A_c\}$ exists.

To do this, we provide a method of constructing, for any $\varepsilon > 0$, a subfinite set $B_\varepsilon \subset \{ \Sigma c_i : \Sigma\, c_i\, \varphi \circ Tt_i \in S\}$ such that, if Σc_k is in A_c, then there is a Σb_i in B_ε with $\Sigma b_i \le \Sigma c_k + \varepsilon$. From this construction, we can find $(f:\varphi)$ by a procedure similar to that in the proof of Proposition 3.7.

Let $\varepsilon > 0$ be given, and let H be a compact support for f. Write

$$\delta = \frac{\varepsilon}{\frac{N^2 M_f + \gamma N}{\gamma^2} + 2} \quad .$$ To construct B_ε:

(1) Choose elements $x_1, \ldots, x_J$ in K such that, for any $x \in K$, there is an x_j $(1 \leq j \leq J)$ with $\varphi(xy) \leq \varphi(x_j y) + \delta$ for every $y \in H$. [Since φ is uniformly continuous [Proposition 3.13], there is a $\lambda > 0$ such that $|\varphi(a) - \varphi(b)| \leq \delta$ whenever $\rho(a,b) \leq \lambda$ and $a,b \in G$. By the same method as in the proof of Proposition 4.3 (ii), we can find $x_1,\ldots,x_J$ in K such that for each $x \in K$ there is an x_j with $\rho(xy, x_j y) \leq \lambda$ for every $y \in H$. Then $|\varphi(xy) - \varphi(x_j y)| \leq \delta$.]

(2) Pick $n \in Z^+$ with $\frac{JNM_f}{\gamma\delta} \leq n$.

(3) Let B be the subfinite set consisting of all functions of the form

$$\zeta_k = \frac{\delta}{J} \sum_{i=1}^{J} (k_i + 2)\ \varphi\circ Tx_i + \frac{\delta}{\gamma}\left(\frac{NM_f}{\gamma} + 1\right) \sum_{i=1}^{N} \varphi\circ Tts_i$$

where $k = (k_1, \ldots, k_J)$ and $0 \leq k_i \leq n$ for each $i = 1,\ldots,J$.

(4) Divide B into two sets B' and B" in such a way that

(i) $f \leq \zeta_m$ for any $\zeta_m \in B'$

(ii) for each $\zeta_m \in B''$, there is a $y \in H$ with $f(y) > \zeta_m(y) - \delta$

[Since H is totally bounded, there is a finite method of assigning each element of B to at least one of B' and B".]

Then let $B_\varepsilon = \{\Sigma b_i : \zeta_m = \Sigma\, b_i\ \varphi\circ Tt_i \in B'\}$.

Now suppose $\psi = \Sigma\, c_k\ \varphi\circ Tu_k$ is in A and Σc_k is in A_c. From the definition of A, we know that each u_k is in K. We picked $x_1, \dots, x_J$ so that for any $u_k \in K$ there is an x_j with $\varphi(u_k y) \leq \varphi(x_j y) + \delta$ for all $y \in H$. This implies that $c_k\ \varphi \circ Tu_k(y) \leq c_k\ \varphi \circ Tx_j(y) + c_k\delta$ for every $y \in H$. Hence we can find non-negative integers $a_1, \dots, a_J$ (which are linear combinations of the c_k's) such that $\sum_{i=1}^{J} a_i = \Sigma c_k$, and

$$\psi(y) = \Sigma\ c_k\ \varphi \circ Tu_k(y) \leq \sum_{i=1}^{J} a_i\ \varphi \circ Tx_i(y) + \delta \Sigma c_k \qquad (y \in H)$$

Since $\Sigma c_k \in A_c$ and $1 \leq \frac{1}{\gamma} \sum_{i=1}^{N} \varphi \circ Tts_i(y)$ $[y \in H]$,

$$f(y) + \delta \leq \psi(y) + \delta \leq \sum_{i=1}^{J} a_i\ \varphi \circ Tx_i(y) + [\delta \cdot \frac{NM_f}{\gamma} + \delta]\ \frac{1}{\gamma} \sum_{i=1}^{N} \varphi \circ Tts_i(y)$$

for all $y \in H$. Now for $1 \leq j \leq J$, $a_j \leq \Sigma c_k \leq \frac{NM_f}{\gamma} \leq \frac{n\delta}{J}$, by our choice of $\underline{n}$. Hence there is an $m = (m_1, \dots, m_J)$ with $0 \leq m_j \leq n$ $(1 \leq j \leq J)$ and

$$\frac{\delta}{J}\, m_j \leq a_j \leq \frac{\delta}{J}\,(m_j + 2)$$

$$\text{Then } f(y) + \delta \leq \sum_{i=1}^{J} \frac{\delta}{J}\,(m_i + 2)\ \varphi \circ Tx_i(y) + \frac{\delta}{\gamma}\,\left(\frac{NM_f}{\gamma} + 1\right) \sum_{i=1}^{N} \varphi \circ Tts_i(y)$$

$$= \zeta_m(y) \qquad [y \in H]$$

and so $\zeta_m \in B'$.

We then have a $\Sigma b_i \in B$ such that

$$\Sigma b_i = \frac{\delta}{J} \sum_{i=1}^{J} (m_i + 2) + \frac{\delta}{\gamma}\left(\frac{NM_f}{\gamma} + 1\right)N$$

$$\leq \sum_{i=1}^{J} a_i + 2\delta + \delta\left(\frac{N^2M_f + \gamma N}{\gamma^2}\right)$$

$$= \Sigma c_k + \varepsilon. \square$$

For $f \in C(G)^{0+}$ and $\varphi \in C(G)^{+}$, the function $(f:\varphi)$ has the following properties:

(1) If $f \in C(G)^{+}$, then $(f:\varphi) > 0$. [If $f(x) \le \Sigma\, c_i\, \varphi\circ Ts_i(x)$ for all $x \in G$, then $f(x) \le \Sigma\, c_i \sup_{y\in G} \varphi(y)$ for all $x \in G$, and $\sup_{y\in G} f(y) \le \sup_{y\in G} \varphi(y)\cdot\Sigma c_i$. Thus $0 < \frac{\sup f}{\sup \varphi} \le \Sigma c_i$, and it follows that $0 < \frac{\sup f}{\sup \varphi} \le (f:\varphi)$.]

(2) If $\alpha \ge 0$, $(\alpha f:\varphi) = \alpha(f:\varphi)$.

(3) $(f\circ Ts:\varphi) = (f:\varphi)$ for every $s \in G$. [If $f(x) \le \Sigma\, c_i\, \varphi\circ Ts_i(x)$ for every $x \in G$, then $f\circ Ts(x) \le \Sigma\, c_i\, \varphi\circ Ts_i s(x)$ for all $x \in G$. Conversely, if $f\circ Ts(x) \le \Sigma\, d_i\, \varphi\circ Tt_i(x)$ for all $x \in G$, then, for each $y \in G$ there is an x with $y = sx$, and $f(y) \le \Sigma\, d_i\, \varphi\circ Tt_i s^{-1}(y)$.]

(4) If f_1 and f_2 are in $C(G)^{0+}$, then

(a) $(f_1 + f_2:\varphi) \le (f_1:\varphi) + (f_2:\varphi)$. [For every $\varepsilon > 0$ there exist non-negative integers $c_1,\ldots,c_n,d_1,\ldots,d_k$, and elements of G $s_1,\ldots,s_n,t_1,\ldots,t_k$, such that

$$f_1 \le \Sigma\, c_i\, \varphi\circ Ts_i \quad\text{and } \Sigma c_i < (f_1:\varphi) + \frac{\varepsilon}{2}$$
$$f_2 \le \Sigma\, d_j\, \varphi\circ Tt_j \quad\text{and } \Sigma d_j < (f_2:\varphi) + \frac{\varepsilon}{2}$$

Then $(f_1 + f_2:\varphi) \le \Sigma c_i + \Sigma d_j < (f_1:\varphi) + (f_2:\varphi) + \varepsilon$, and since this holds for any $\varepsilon > 0$, it follows that $(f_1 + f_2:\varphi) \le (f_1:\varphi) + (f_2:\varphi)$.]

(b) $(f_1:\varphi) - (f_2:\varphi) \le ((f_1 - f_2)^{+}:\varphi)$. [For each $\varepsilon > 0$ there are $e_1,\ldots,e_k,f_1,\ldots f_n$ in Z^{0+} and $s_1,\ldots,s_k,t_1,\ldots,t_n$ in G such that

$$(f_1 - f_2)^{+} \le \Sigma\, e_i\, \varphi\circ Ts_i \quad\text{and } \Sigma e_i < ((f_1 - f_2)^{+}:\varphi) + \frac{\varepsilon}{2}$$

$$f_2 \le \Sigma\, f_j\, \varphi\circ Tt_j \quad\text{and } \Sigma f_j < (f_2:\varphi) + \frac{\varepsilon}{2}$$

Then $f_1 \le \Sigma\, e_i\, \varphi\circ Ts_i + \Sigma\, f_j\, \varphi\circ Tt_j$ and $(f_1:\varphi) \le \Sigma e_i + \Sigma f_j$, so $(f_1:\varphi) - (f_2:\varphi) \le (\Sigma e_i + \Sigma f_j) - (\Sigma f_j - \frac{\varepsilon}{2}) = \Sigma e_i + \frac{\varepsilon}{2} < ((f_1 - f_2)^+:\varphi) + \varepsilon$.]

(5) If $\varphi_1, \varphi_2 \in C(G)^+$, then $(f:\varphi_2) \le (f:\varphi_1)(\varphi_1:\varphi_2)$, (and consequently, if $f \in C(G)^+$, $\frac{1}{(f:\varphi_1)} \le \frac{(\varphi_1:\varphi_2)}{(f:\varphi_2)}$). [For all $\varepsilon > 0$, there exist non-negative integers $c_1,\ldots,c_k,d_1,\ldots d_n$, and elements of G $s_1,\ldots,s_k,t_1,\ldots t_n$ such that

$$f \le \Sigma\, c_i\, \varphi_1\circ Ts_i \quad \text{and } \Sigma c_i < (f:\varphi_1) + \varepsilon$$
$$\varphi_1 \le \Sigma\, d_j\, \varphi_2\circ Tt_j \quad \text{and } \Sigma d_j < (\varphi_1:\varphi_2) + \varepsilon$$

Then for $i = 1,\ldots,k$,

$$\varphi_1(s_i x) \le \Sigma\, d_j\, \varphi_2\circ Tt_j s_i(x)$$

for all $x \in G$, hence

$$f(x) \le \Sigma\, c_i(\Sigma\, d_j\, \varphi_2\circ Tt_j s_i(x))$$

for all $x \in G$. Therefore $(f:\varphi_2) \le \Sigma c_i(\Sigma d_j) < [(f:\varphi_1) + \varepsilon][(\varphi_1:\varphi_2) + \varepsilon]$, and since ε can be arbitrarily small, the result follows.]

Now fix (for the rest of this section) a particular function f_0 in $C(G)^+$ and, for each f in $C(G)^{0+}$ and φ in $C(G)^+$, write

$$I_\varphi(f) = \frac{(f:\varphi)}{(f_0:\varphi)}$$

The functional I_φ has the following properties:

(1) If $f \in C(G)^+$ then $I_\varphi(f) > 0$.

(2) If $\alpha \geq 0$, $I_\varphi(\alpha f) = \alpha I_\varphi(f)$.

(3) $I_\varphi(f_1 + f_2) \leq I_\varphi(f_1) + I_\varphi(f_2)$.

(4) $I_\varphi(f \circ T s) = I_\varphi(f)$ for each $s \in G$. [$I_\varphi(f \circ T s) = \dfrac{(f \circ Ts:\varphi)}{(f_0:\varphi)} = \dfrac{(f:\varphi)}{(f_0:\varphi)} = I_\varphi(f)$]

(5) If $f \in C(G)^+$, then $\dfrac{1}{(f_0:f)} \leq I_\varphi(f) \leq (f:f_0)$. [For all $f \in C(G)^{0+}$, $(f:\varphi) \leq (f:f_0)(f_0:\varphi)$, hence $I_\varphi(f) \leq (f:f_0)$. If $f \in C(G)^+$, then $(f_0:\varphi) \leq (f_0:f)(f:\varphi)$, and so $\dfrac{1}{(f_0:f)} \leq I_\varphi(f)$.]

Lemma 4.10. Let ε and M be any positive numbers, and let $f_1,\ldots,f_n$ be functions in $C(G)^{0+}$. Then there exists a $\delta > 0$ such that

$$\sum_{i=1}^{n} \alpha_i I_\varphi(f_i) \leq I_\varphi\left(\sum_{i=1}^{n} \alpha_i f_i\right) + \varepsilon$$

for any real numbers $\alpha_1,\ldots,\alpha_n$ with $0 \leq \alpha_i \leq M$ $(1 \leq i \leq n)$, and any $\varphi \in C(G)^+$ such that $\varphi(x) = 0$ whenever $\rho(e,x) \geq \delta$.

Proof: It is sufficient to show that for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\sum_{1}^{n} \alpha_i I_\varphi(f_i) \leq \left(I_\varphi\left(\sum_{1}^{n} \alpha_i f_i\right) + \varepsilon\right)(1 + \varepsilon)$$

for every $\varphi \in C(G)^+$ with $\varphi(x) = 0$ whenever $\rho(e,x) \geq \delta$. [$I_\varphi(\sum_1^n \alpha_i f_i) \leq I_\varphi(\sum_1^n M f_i)$, so the functionals $I_\varphi(\sum_1^n \alpha_i f_i)$ are bounded independently of the choice of $\alpha_1,\ldots,\alpha_n$.]

Suppose $\varepsilon > 0$, M, and $f_1,\ldots,f_n$ have been given. Let K be a compact support for all of the functions $f_1,\ldots,f_n$, and let $g \in C(G)^+$ be any function with $g(x) = 1$ for all $x \in K$.

Write $\lambda = \dfrac{\varepsilon}{(g:f_0)}$ and $h_j = \dfrac{f_j}{\sum_{i=1}^{n} \alpha_i f_i + \lambda g}$. By Lemma 4.5, we can choose $\delta > 0$ so that $|h_j(s^{-1}) - h_j(x)| \le \frac{\varepsilon}{n}$, for $1 \le j \le n$, whenever $\rho(e,sx) < 2\delta$. [The choice of δ is independent of the values of $\alpha_1,\ldots,\alpha_n$ because $|h_j(u) - h_j(v)| = \left|\frac{f_j(u)}{\Sigma\alpha_i f_i(u) + \lambda g(u)} - \frac{f_j(v)}{\Sigma\alpha_i f_i(v) + \lambda g(v)}\right|$

$\le \frac{1}{\lambda^2}|\lambda(f_j(u)g(v) - f_j(v)g(u)) + f_j(u)\sum_1^n M(f_i(v) - f_i(u)) + \sum_1^n Mf_i(u)(f_i(u) - f_i(v))|$]

Now let $\varphi \in C(G)^+$ be such that $\varphi(x) = 0$ for all x with $\rho(e,x) \ge \delta$. By Lemma 4.9, there exist $c_1,\ldots,c_m \in R^{0+}$ and $s_1,\ldots,s_m \in G$ with

$$\sum_{i=1}^{n}(\alpha_i f_i) + \lambda g \le \sum_{j=1}^{m} c_j\ \varphi\circ Ts_j$$

Without loss of generality, we may assume $\rho(s_i x,e) < 2\delta$ for $i = 1,\ldots,m$ and all $x \in K$. [If $\rho(s_i x,e) > \delta$, then $\varphi(s_i x) = 0$.] Then $h_j(x) \le h_j(s_i^{-1}) + \frac{\varepsilon}{n}$ $(1 \le j \le n,\ 1 \le i \le m)$. This gives

$$\begin{aligned} f_j(x) &= h_j(x)\Big(\sum_{i=1}^{n}(\alpha_i f_i(x)) + \lambda g(x)\Big) \\ &\le h_j(x) \sum_{i=1}^{m} c_i\ \varphi\circ Ts_i(x) \\ &\le \sum_{i=1}^{m}\Big(h_j(s_i^{-1}) + \frac{\varepsilon}{n}\Big)(c_i\ \varphi\circ Ts_i(x)) \end{aligned}$$

Thus $(f_j:\varphi) \le \sum_{i=1}^{m}(h_j(s_i^{-1}) + \frac{\varepsilon}{n})c_i$, and $I_\varphi(f) \le \sum_{i=1}^{m} \frac{c_i(h_j(s_i^{-1}) + \frac{\varepsilon}{n}}{(f_0:\varphi)}$.

Summing, we have

$$\begin{aligned} \sum_{j=1}^{n} \alpha_j I_\varphi(f) &\le \sum_{j=1}^{n} \alpha_j\Big(\sum_{i=1}^{m} c_i\Big(\frac{h_j(s_i^{-1}) + \frac{\varepsilon}{n}}{(f_0:\varphi)}\Big)\Big) \\ &= \sum_{i=1}^{m} c_i \sum_{j=1}^{n} \alpha_j\Big(\frac{h_j(s_i^{-1}) + \frac{\varepsilon}{n}}{(f_0:\varphi)}\Big) \\ &\le \sum_{i=1}^{m} c_i\Big(\sum_{j=1}^{n}\Big[\frac{\alpha_j}{(f_0:\varphi)}\Big(\frac{f_j(s_i^{-1})}{\sum_{k=1}^{n}\alpha_k f_k(s_i^{-1})}\Big)\Big] + \frac{\varepsilon}{(f_0:\varphi)}\Big) \\ &= \sum_{i=1}^{m} c_i\Big(\frac{1+\varepsilon}{(f_0:\varphi)}\Big) \end{aligned}$$

Now $\dfrac{\sum_{i=1}^{m} c_i}{(f_0:\varphi)}$ may be chosen to be arbitrarily close to $I_\varphi(\sum_{i=1}^{n}(\alpha_i f_i) + \lambda g)$

Hence

$$\begin{aligned}\sum_{i=1}^{n} \alpha_i I_\varphi(f) &\le I_\varphi(\sum_{i=1}^{n}(\alpha_i f_i) + \lambda g)(1+\varepsilon)\\ &\le [I_\varphi(\sum_{i=1}^{n} \alpha_i f_i) + \lambda I_\varphi(g)](1+\varepsilon)\\ &\le [I_\varphi(\sum_{i=1}^{n} \alpha_i f_i) + \lambda (g:f_0)](1+\varepsilon)\\ &\qquad [\text{Property (4) of } I_\varphi]\\ &= (I_\varphi(\sum_{i=1}^{n} \alpha_i f_i) + \varepsilon)(1+\varepsilon).\square\end{aligned}$$

<u>Lemma 4.11.</u> Suppose $\varepsilon > 0$ and $f \in C(G)^{0+}$. Choose $\delta > 0$ so that $|f(x) - f(y)| \le \varepsilon$ whenever $\rho(x^{-1}y,e) \le \delta$. If $g \in C(G)^+$ has the property that $g(x) = 0$ whenever $\rho(x,e) > \frac{\delta}{2}$, and if α is any constant greater than ε, then there exist $c_1,\dots,c_k$ in R^{0+} and $s_1,\dots,s_k$ in G such that

$$|f(x) - \sum_{i=1}^{k} c_i\, g\circ Ts_i(x)| \le \alpha$$

for all $x \in G$.

<u>Proof:</u> For all <u>x</u> and <u>s</u> in G,

$$(f(x) - \varepsilon)g(s^{-1}x) \le f(s)g(s^{-1}x) \le (f(x) + \varepsilon)g(s^{-1}x) \qquad (1)$$

$[\rho(s^{-1}x,e) \le \delta \Rightarrow |f(x) - f(y)| \le \varepsilon$, and $\rho(s^{-1}x,e) > \frac{\delta}{2} \Rightarrow g(s^{-1}x) = 0.]$

Define $g^* \in C(G)^+$ by $g^*(x) = g(x^{-1})$, and write $\eta = \dfrac{\alpha - \varepsilon}{2(f:g^*)}$. Choose ν so that $|g(x) - g(y)| \le \eta$ whenever $\rho(x^{-1}y,e) \le \nu$.

Suppose K is a compact support for f. Then there exist $s_1^{-1},\dots,s_k^{-1}$ in K such that, for any $x \in K$, there is an $s_i^{-1} \in K$ $(1 \le i \le k)$, with

$\rho(s_i x,e) < \frac{\nu}{2}$. Choose $h_1,\ldots,h_k$ in $C(G)^+$ with

(a) $\sum_{i=1}^{k} h_i(x) = 1$ for every $x \in K$

(b) $h_i(x) = 0$ if $\rho(s_i x,e) \geq \nu$.

[E.g. $h_i(x) = \dfrac{[\nu - \rho(s_i x,e)]^+}{\sum_{j=1}^{k}[\nu - \rho(s_j x,e)]^+}$] Then for each h_i and each $s \in G$,

$$h_i(s)f(s)[g(s^{-1}x) - \eta] \leq h_i(s)f(s)g(s_i x) \leq h_i(s)f(s)[g(s^{-1}x) + \eta] \tag{2}$$

for all $x \in G$. [If $h_i(s) > 0$ then $\rho(s_i s,e) = \rho(s_i x x^{-1} s,e) < \nu$ and $|g(s_i x) - g(s^{-1}x)| \leq \eta$. If $h_i(s) = 0$, then (2) is clearly valid. Hence by continuity, (2) holds for each $s \in G$.]

From (1) and (2) we have

$$\begin{aligned}(f(x) - \varepsilon)g(s^{-1}x) - \eta f(s) &\leq f(s)g(s^{-1}x) - \eta f(s) \\ &= \sum_{i=1}^{k} h_i(s)f(s)(g(s^{-1}x) - \eta) \\ &\qquad [\Sigma h_i(s) \neq 1 \Rightarrow s \in -K \Rightarrow f(s) = 0] \\ &\leq \sum_{i=1}^{k} h_i(s)f(s)g(s_i x) \\ &\leq \Sigma h_i(s)f(s)(g(s^{-1}x) + \eta) \\ &\leq (f(x) + \varepsilon)g(s^{-1}x) + \eta f(s).\end{aligned} \tag{3}$$

Now fix $x \in G$ and consider all the functions in (3) as functions of $\underline{s}$. Let $\varphi \in C(G)^+$. Then since $g(s^{-1}x) = g^*(x^{-1}s)$ and $I_\varphi(g^* \circ Tx^{-1}) = I_\varphi(g^*)$ for each fixed $\underline{x}$, we have

$$\begin{aligned}(f(x) - \varepsilon)I_\varphi(g^*) - \eta I_\varphi(f) &\leq I_\varphi\left(\sum_{i=1}^{k} g(s_i x)h_i f\right) \\ &\leq (f(x) + \varepsilon)I_\varphi(g^*) + \eta I_\varphi(f).\end{aligned}$$

Dividing by $I_\varphi(g^*)$, and noting that $\frac{I_\varphi(f)}{I_\varphi(g^*)} = \frac{(f:\varphi)}{(g^*:\varphi)} \leq (f:g^*) = \frac{\alpha - \varepsilon}{2\eta}$, we obtain

$$f(x) - \frac{\alpha+\varepsilon}{2} \leq \frac{I_\varphi(\sum_{i=1}^{k} g(s_i x) h_i f)}{I_\varphi(g^*)} \leq f(x) + \frac{\alpha+\varepsilon}{2} \tag{4}$$

Let $f_i = h_i f$ $(1 \leq i \leq k)$, and choose an $m \in Z^+$ so that $\frac{1}{m} < \frac{\alpha - \varepsilon}{2}$ and $m \geq g(s_i x)(f_0 : g^*)$ for every $x \in G$ and $i = 1, \ldots, k$. Write

$$g_i(x) = \frac{g(s_i x)}{I_\varphi(g^*)} \qquad (1 \leq i \leq k)$$

Then $0 \leq g_i(x) = g(s_i x) \cdot \frac{(f_0:\varphi)}{(g^*:\varphi)} \leq g(s_i x)(f_0 : g^*) \leq m$. Hence, by Lemma 4.10, we can restrict φ so that

$$\sum_{i=1}^{k} g_i(x) I_\varphi(f_i) \leq I_\varphi(\sum_{i=1}^{k} g_i(x) f_i) + \frac{1}{m}$$

Then let $c_i = \frac{I_\varphi(f_i)}{I_\varphi(g^*)}$, $(1 \leq i \leq k)$, and consider any $x \in G$.

$$\begin{aligned}
I_\varphi(\sum_{i=1}^{k} \frac{g(s_i x)}{I_\varphi(g^*)} h_i f) &= I_\varphi(\sum_{i=1}^{k} g_i(x) f_i) \\
&\leq \Sigma\, g_i(x) I_\varphi(f_i) \\
&\qquad [\text{Property (3) of } I_\varphi] \\
&= \Sigma\, g(s_i x) \cdot \frac{I_\varphi(f_i)}{I_\varphi(g^*)} \\
&= \Sigma\, c_i g(s_i x) \\
&\leq I_\varphi(\Sigma\, g_i(x) f_i) + \frac{1}{m} \\
&\leq I_\varphi(\sum_{i=1}^{k} \frac{g(s_i x)}{I_\varphi(g^*)} h_i f) + \frac{\alpha - \varepsilon}{2}
\end{aligned}$$

Together with (4), this gives

$$f(x) - \frac{\alpha+\varepsilon}{2} \leq \sum_{i=1}^{k} c_i g(s_i x) \leq f(x) + \alpha \qquad (x \in G)$$

and so

$$\left|f(x) - \sum_{i=1}^{k} c_i g\circ Ts_i(x)\right| \leq \alpha$$

for every $x \in G$.□

Now to construct the Haar integral, we consider a sequence of functionals $\{I_{\varphi_n}\}_{n=1}^{\infty}$, where $\varphi_n \in C(G)^+$ has the property that $\varphi_n(x) = 0$ if $\rho(e,x) \geq \frac{1}{n}$. It will be shown that, for each $f \in C(G)^+$, $\{I_{\varphi_n}(f)\}_{n=1}^{\infty}$ is a Cauchy sequence, converging to a number $I(f)$. The functional I, so defined, is positive and left-invariant. It can be extended to $C(G)$ by defining, for each $f \in C(G)$, $I(f) = I(f^+ + \varphi) - I(f^- + \varphi)$ for some $\varphi \in C(G)^+$.

<u>Theorem 4.12.</u> Let G be a locally compact group. Then

(i) there exists a positive left-invariant integral I on $C(G)$, with the property that $I(f) > 0$ if $f > 0$.

(ii) if J is any positive left-invariant integral on $C(G)$, then $J = cI$ for some constant $c \in R^+$.

<u>Proof</u>: (i) Let $f \in C(G)^+$ and let $\{\varphi_n\}_{n=1}^{\infty}$ be as above. To show that $\{I_{\varphi_n}(f)\}_1^{\infty}$ converges, it is sufficient to show that for any $\varepsilon > 0$, there exists an $N \in Z^+$ such that

$$|I_{\varphi_n}(f) - I_{\varphi_k}(f)| \leq \varepsilon$$

if $n,k \geq N$.

Suppose $0 < \varepsilon < 1$. Choose $\lambda > 0$ so that $\rho(x,s^{-1}) \leq \varepsilon$ whenever

$\rho(sx,e) \le \lambda$. [Since $(x,y) \to xy$ is continuous, there exists a $\lambda > 0$ such that $\rho^*((s^{-1},sx), (s^{-1},e)) = \rho(sx,e) \le \lambda$ implies $\rho(x,s^{-1}) \le \varepsilon$.] Let K be a compact support for f and K_0 be a compact support for f_0. Then $(K \cup K_0)_1 = \{x \in G : \rho(x, K \cup K_0) \le 1\}$. Let ω be any function in $C(G)^+$ with $\omega(x) = 1$ for all x in $(K \cup K_0)_1$.

Write $\gamma = \dfrac{\varepsilon}{4[1 + (\omega:f_0)][1 + (f:f_0)]}$ and pick $\delta \in R$ so that $0 < \delta < \lambda$ and

$$|f(x) - f(y)| \le \frac{\gamma}{2} \text{ if } \rho(y^{-1}x,e) \le \delta$$

$$|f_0(x) - f_0(y)| \le \frac{\gamma}{2} \text{ if } \rho(y^{-1}x,e) \le \delta$$

Choose $g \in C(G)^+$ so that $g(x) = 0$ on $\{x \in G : \rho(x,e) \ge \frac{\delta}{2}\}$. By Lemma 4.11, there exist $c_1,\ldots,c_m$ in R^{0+} and $t_1^{-1},\ldots,t_m^{-1}$ in K such that

$$|f(x) - \sum_{i=1}^{m} c_i g(t_i x)| \le \gamma \qquad (x \in G)$$

We then have

$$|f(x) - \sum_{1}^{m} c_i g(t_i x)| \le \gamma\omega(x) \qquad (x \in G)$$

[If $|f(x) - \Sigma c_i g(t_i x)| > \gamma\omega(x)$ for some $x \in G$, then $\omega(x) < 1$ and $|f(x) - \Sigma c_i g(t_i x)| > 0$. If $f(x) > 0$, $x \in K$ and so $\omega(x) = 1$. Hence $f(x) = 0$. If $\Sigma c_i g(t_i x) > 0$ then $g(t_j x) > 0$ for some $\underline{j}$, which implies that $\rho(t_j x,e) \le \delta$. But then, since $\delta < \lambda$, $\rho(x,t_j^{-1}) \le \varepsilon < 1$, and so $x \in (K \cup K_0)_1$ and $\omega(x) = 1$.]

So

$$\Sigma c_i g(t_i x) - \gamma\omega(x) \le f(x) \le \Sigma c_i g(t_i x) + \gamma\omega(x)$$

and

$$I_\varphi(\Sigma\ c_i g\circ Tt_i) - \gamma I_\varphi(\omega) \le I_\varphi(f) \le I_\varphi(\Sigma\ c_i g\circ Tt_i) + \gamma I_\varphi(\omega)$$

for any $\varphi \in C(G)^+$. Hence

$$|I_\varphi(f) - I_\varphi(\Sigma\ c_i g\circ Tt_i)| \le \gamma I_\varphi(\omega) = \gamma \frac{(\omega:\varphi)}{(f_0:\varphi)} \le \gamma(\omega:f_0) \quad (1)$$

From the proof of Lemma 4.11, we know that $c_j = \dfrac{I_\varphi(h_j f)}{I_\varphi(g^*)}$ for $j = 1,\ldots,m$.

Now $h_j f \le f$, hence

$$c_j \le \frac{I_\varphi(f)}{I_\varphi(g^*)} = \frac{(f:\varphi)}{(g^*:\varphi)} \le (f:g^*) \qquad (1 \le j \le m)$$

We can now apply Lemma 4.10 to obtain an $N \in Z^+$ such that

$$|I_{\varphi_k}(\sum_{i=1}^{m} c_i g\circ Tt_i) - \sum_{i=1}^{m} c_i I_{\varphi_k}(g\circ Tt_i)| \le \gamma$$

and hence

$$|I_{\varphi_k}(\sum_{i=1}^{m} c_i g\circ Tt_i) - \sum_{i=1}^{m} c_i I_{\varphi_k}(g)| \le \gamma \quad (2)$$

for every φ_k with $k \ge N$.

Combining (1) and (2), we have

$$|I_{\varphi_k}(f) - cI_{\varphi_k}(g)| \le \gamma[1 + (\omega:f_0)] \quad (3)$$

where $c = \sum_{i=1}^{m} c_i > 0$ and $k \ge N$.

We can substitute f_0 for f in all of the above inequalities, and, since $I_{\varphi_k}(f_0) = 1$, we obtain

$$|1 - dI_{\varphi_k}(g)| \le \gamma[1 + (\omega:f_0)]$$

for some $d > 0$, or, equivalently,

$$\left|\frac{c}{d} - cI_{\varphi_k}(g)\right| \le \frac{c}{d}\,\gamma[1 + (\omega:f_0)] \tag{4}$$

From (3) and (4), we conclude

$$\left|I_{\varphi_k}(f) - \frac{c}{d}\right| \le \gamma[1 + (\omega:f_0)](1 + \frac{c}{d}) \tag{5}$$

for every $k \ge N$. Then

$$\frac{c}{d}\,(1 - \gamma[1 + (\omega:f_0)]) \le I_{\varphi_k}(f) + \gamma[1 + (\omega:f_0)]$$

and so

$$\frac{c}{d} + 1 \le \frac{I_{\varphi_k}(f) + 1}{1 - \gamma[1 + (\omega:f_0)]} \le \frac{(f:f_0) + 1}{1 - \dfrac{\varepsilon}{4[(f:f_0) + 1]}} \le 2[(f:f_0) + 1]$$

since $0 < \varepsilon < 1$. Combined with (5), this gives

$$\left|I_{\varphi_k}(f) - \frac{c}{d}\right| \le \gamma[1 + (\omega:f_0)]2[(f:f_0) + 1] = \frac{\varepsilon}{2}$$

for every $k \ge N$. Hence if $k,n \ge N$,

$$\left|I_{\varphi_k}(f) - I_{\varphi_n}(f)\right| \le \varepsilon$$

and so $\lim_{n\to\infty} I_{\varphi_n}(f) = I(f)$ exists for each $f \in C(G)^+$.

I(f) has the following properties:

(a) If $f > 0$, $I(f) > 0$, because $0 < \dfrac{1}{(f_0:f)} \le I_{\varphi_n}(f)$ for each $n \in Z^+$.

(b) $I(\alpha f + g) = \alpha I(f) + I(g)$. [By Lemma 4.10, for each $\varepsilon > 0$ there is an $N \in Z^+$ such that $\alpha I_{\varphi_k}(f) + I_{\varphi_k}(g) \le I_{\varphi_k}(\alpha f + g) + \varepsilon$ for every

$k \geq N$. Hence $\alpha I(f) + I(g) \leq I(\alpha f + g)$. $I(\alpha f + g) \leq \alpha I(f) + I(g)$ by Properties (2) and (3) of I_φ.

(c) $I(f \circ Ts) = I(f)$ for every $s \in G$.

Now if $f \in C(G)$, then f^+ and f^- are in $C(G)^{0+}$. Therefore we can choose any $\varphi \in C(G)^+$, and have $f = (f^+ + \varphi) - (f^- + \varphi)$. Define $I(f) = I(f^+ + \varphi) - I(f^- + \varphi)$. [If we also have $f = f_1 - f_2$, $(f_1, f_2 \in C(G)^+)$, then $f_1 + f^- + \varphi = f_2 + f^+ + \varphi$, and so $I(f_1) + I(f^- + \varphi)$ $= I(f_2) + I(f^+ + \varphi)$. Hence $I(f)$ does not depend on the particular choice of f_1 and f_2.] Clearly (a), (b), and (c) hold for the extended functional I.

(ii) Let J be a left-invariant positive integral on C(G). If $f \in C(G)^+$ and $J(f_1) > 0$, then there exist $d_1, \ldots d_n$ in R^{0+} and $t_1, \ldots, t_n$ in G with

$$f_1 \leq \sum_{i=1}^{n} d_i f \circ Tt_i$$

and

$$J(f_1) \leq \sum_{1}^{n} d_i J(f)$$

where $\sum_{1}^{n} d_i > 0$. Consequently, $J(f) > 0$ whenever $f \in C(G)^+$.

Now let f_1 and φ_n be in $C(G)^+$ with

$$f_1 \leq \sum_{j=1}^{m} c_j \varphi_n \circ Ts_j$$

Then

$$J(f_1) \le \sum_{j=1}^{m} c_j J(\varphi_n)$$

and it follows that

$$(f_1:\varphi_n) \ge \frac{J(f_1)}{J(\varphi_n)} \tag{6}$$

for every $f_1, \varphi_n \in C(G)^+$.

Let $f \in C(G)^+$ and let $\omega \in C(G)^+$ be defined as in part (i). For each $\varepsilon > 0$ there is an $N \in Z^+$ such that for all $n \ge N$, [and $\varphi_n(x) = 0$ for all x in $\{x \in G: \rho(e,x) \ge \frac{1}{n}\}$], there exist $c_1,\ldots,c_m$ in R^{0+} and $s_1,\ldots,s_m$ in G with

$$f(x) \le \varepsilon\omega(x) + \sum_{i=1}^{m} c_i\varphi_n(s_i x)$$

and

$$\sum_{i=1}^{m} c_i\varphi_n(s_i x) \le f(x) + \varepsilon\omega(x) \qquad (x \in G)$$

From this we can conclude that

$$(f:\varphi_n) \le \varepsilon(\omega:\varphi_n) + \sum_1^m c_i$$

and

$$\sum_{i=1}^{m} c_i J(\varphi_n) \le J(f) + \varepsilon J(\omega) \qquad (n \ge N)$$

Hence

$$J(f) + \varepsilon J(\omega) \ge (f:\varphi_n)[1 - \varepsilon \frac{(\omega:\varphi_n)}{(f:\varphi_n)}]J(\varphi_n) \ge [1 - \varepsilon(\omega:f)](f:\varphi_n)J(\varphi_n)$$

for each $n \geq N$.

Now let f_1 be any fixed function in $C(G)^+$. From (6) we have

$$\frac{J(f)}{J(f_1)} + \varepsilon \frac{J(\omega)}{J(f_1)} \geq [1 - \varepsilon(\omega{:}f)] \frac{(f{:}\varphi_n)}{(f_1{:}\varphi_n)} = [1 - \varepsilon(\omega{:}f)] \frac{I_{\varphi_n}(f)}{I_{\varphi_n}(f_1)}$$

for every $n \geq N$. Therefore

$$\frac{J(f)}{J(f_1)} + \varepsilon \frac{J(\omega)}{J(f_1)} \geq [1 - \varepsilon(\omega{:}f)] \frac{I(f)}{I(f_1)}$$

and since this holds for each $\varepsilon > 0$,

$$\frac{J(f)}{J(f_1)} \geq \frac{I(f)}{I(f_1)}$$

or,

$$J(f) \geq \frac{J(f_1)}{I(f_1)} I(f)$$

for every $f \in C(G)^+$. Holding f_1 fixed, we can repeat the above argument with f and f_1 interchanged and obtain

$$J(f) \leq \frac{J(f_1)}{I(f_1)} I(f)$$

for any given $f \in C(G)^+$. Hence $J(f) = \frac{J(f_1)}{I(f_1)} I(f)$ for every $f \in C(G)^+$, and consequently, $J(f) = \frac{J(f_1)}{I(f_1)} I(f)$ for every $f \in C(G)$.□

We can now use the Daniell extension method of Definition 2.4 to extend I to $C_1(G)$.

Corollary 4.13. Let G be a locally compact group. Then there exists a left Haar integral on $C_1(G)$.

Proof: Let I be a left-invariant positive integral on C(G). Clearly I is also a positive integral on $C_1(G)$. Suppose $g \in C_1(G)$ and $\{f_n\}_{n=1}^{\infty}$ is a representation for g. Then

$$g\circ Ts(x) = g(sx) = \sum_{n=1}^{\infty} f_n(sx) = \sum_1^{\infty} f_n\circ Ts(x)$$

whenever $\sum_1^{\infty} |f_n\circ Ts(x)|$ converges, and

$$\sum_1^{\infty} I(|f_n\circ Ts|) = \sum_1^{\infty} I(|f_n|\circ Ts) = \sum_1^{\infty} I(|f_n|) < \infty$$

since $\{f_n\}$ is a representation for g. Hence $g\circ Ts$ is integrable, and

$$I(g\circ Ts) = \sum_1^{\infty} I(f_n\circ Ts) = \sum_1^{\infty} I(f_n) = I(g).\square$$

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