

**THE ROLE OF EXAMPLE-GENERATION TASKS IN  
STUDENTS' UNDERSTANDING OF LINEAR ALGEBRA**

by

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## ABSTRACT

There is little doubt that linear algebra is a fundamentally important course in undergraduate studies. It is required for students majoring not only in mathematics, but also in engineering, physics, and economics, to name just a few. However, to date, research on students' understanding of linear algebra is rather slim.

This study is a contribution to the ongoing research in undergraduate mathematics education, focusing on linear algebra. It is guided by the belief that better understanding of students' difficulties leads to improved instructional methods.

The questions posed in this study are: What is students' understanding of the key concepts of linear algebra? What difficulties do students experience when engaged in these tasks? What can example-generation tasks reveal about students' understanding of linear algebra? Are these tasks effective and useful as a data collection tool for research in mathematics education?

This study identifies some of the difficulties experienced by students with learning several key concepts of linear algebra: vectors and vector spaces, linear dependence and independence, linear transformations, and basis, and also isolates some possible obstacles to such learning. In addition, this study introduces learner-generated examples as a pedagogical tool that helps learners partly overcome these obstacles.

There are several contributions of this study to the field of undergraduate mathematics education. Firstly, focusing on specific mathematical content, it provides a

finer and deeper analysis of students' understanding of linear algebra. Secondly, focusing on methodology, it introduces an effective data collection tool to investigate students' learning of mathematical concepts. Thirdly, focusing on pedagogy, it enhances the teaching of linear algebra by developing a set of example-generation tasks that are a valuable addition to the undergraduate mathematics education. The tasks I have designed can be used in assignments, tutorials, and other educational settings serve not only as an assessment tool but also as an instructional tool that provides learners with an opportunity to engage in mathematical activity.

**Keywords:** linear algebra, mathematics education, learner-generated examples, example-generation tasks, postsecondary

# **DEDICATION**

To Myself.

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## CHAPTER 1: INTRODUCTION

There is little doubt that linear algebra is a fundamentally important course in undergraduate studies. It is required for students majoring not only in mathematics, but also in engineering, physics, and economics, to name just a few. And yet, there has not been much research into students' understanding and learning of linear algebra. It is time to give this problem the attention it deserves.

### 1.1 Personal motivation

In the Fall 2003, I started working in the Algebra Workshop at Simon Fraser University. One of the courses serviced by the workshop was an elementary linear algebra course. Right away, I was struck by the organization and presentation of the material in this course. It was very different from the linear algebra course that I took as a freshman. The course I remember was axiomatic and theoretical, where an abstract vector space over a field was defined in the first lecture. There were no applications discussed and very few examples presented. In contrast, the linear algebra course I encountered in the workshop began with the introduction of systems of linear equations, followed by the concepts of linear algebra in the context of  $\mathbf{R}^n$ , and only later, the formal definition of a general vector space was introduced. It seemed to me that this approach was more conducive to students' learning. Nonetheless, despite this concrete and gradual development of the theory, students were having difficulties grasping the material.

As I observed students' struggles with linear algebra concepts, I tried to probe the depth of their understanding. Unfortunately, in many cases, it boiled down to reproducing the procedures or worked examples from the textbook without any real comprehension. While helping students to interpret the questions, I noticed that using examples in my explanations improved students' understanding. Not only did examples help the students, coming up with my own examples provided a different view of the concepts for me as well. Generating examples made me think of the connections between the objects in linear algebra that had not occurred to me before. Reflecting on my own experiences in the workshop, I encouraged students to use examples in their learning.

In one particular episode, one of the students, Ted, came to me with the questions:

“Is the intersection of the two subspaces of a vector space  $V$  a subspace of  $V$ ? Is the union of the two subspaces of a vector space  $V$  a subspace of  $V$ ?”

Ted had to either give a proof of a positive answer or provide a counter-example. He started to prove the first statement, but was not convinced by his proof. He could not see why the statements should be true or false. I suggested we consider specific subspaces of  $\mathbf{R}^3$ , such as planes and lines, to investigate the above questions. Once we analyzed the questions with several examples in a context familiar to Ted, he had a clearer idea of what the subspace of a vector space was, and what the union and intersection of the different subspaces were.

There were many such episodes that demonstrated how helpful examples were in students' learning. It seemed it would be a useful addition to teaching the course.

Linear algebra is not an easy subject. The symbolic representations and formal definitions can leave many students with little or no understanding of the concepts. The

learner-generated examples seemed to be an important tool to help this understanding. I have always been interested in undergraduate mathematics education, and my experience in the workshop helped me narrow down and shape my research questions.

## **1.2 Rationale, purpose of the study, and research questions**

Many mathematicians and mathematics educators note that linear algebra is one of the most difficult subjects to teach and for students to understand. In order to address this difficulty, the didactic study, that is, an analysis of students' understanding of linear algebra, together with an epistemological analysis of the subject is necessary. To my knowledge, the previous research in mathematics education focused mostly on the *teaching* of linear algebra, and there is not significant work done on students' learning. The common findings expressed in the literature show that many students, at best, could recreate memorized algorithms in a familiar situation or task. Teaching and learning are closely related. In order to design effective instructional strategies and to improve students' learning, it is essential to understand better how people learn, how they create their knowledge, and what factors impede it. For those engaged in the teaching of linear algebra, it is beneficial to know the learning processes involved. These issues have not been addressed in the research on linear algebra in detail. The lack of attention to students' understanding and learning of linear algebra concepts constitutes a gap in the research. The current study attempts to fill the gap by exploring students' understanding of several of the central concepts of linear algebra: vectors and vector spaces, linear dependence and independence, linear transformations, and basis. In addition, this research identifies the difficulties that students experience learning the concepts of linear algebra and possible sources of these difficulties. Having understanding of students'

understanding of linear algebra will guide the design of pedagogical strategies that allow students' to construct an understanding of the mathematical concepts as well as to apply them in different situations.

Working in the Algebra Workshop I noticed that examples helped students make sense of the linear algebra concepts, theorems, and problems. Examples have been used extensively in mathematics education as a pedagogical tool. It is known that employing examples for explanation is beneficial to students' understanding of mathematical situations. The question arises whether and in what way learner-generated examples would affect students' understanding. Research indicates that the construction of examples by students contributes to their learning. Watson and Mason (2002, 2004), in their analysis of learning by involving students directly in construction activities, showed that such activities led students to reorganize their knowledge to fit the kind of examples the teacher was seeking. It helped students to move away from routine algorithms and limited perceptions of concepts, and towards wider ranges of objects.

The purpose and focus of this study is two-fold. On one hand, it explores students' understanding of linear algebra through the lens of learner-generated examples. For this study, I chose to look at several key concepts of linear algebra: vectors and vector spaces, linear dependence and independence, linear transformations, and basis. I designed tasks, requiring participants to generate examples related to the above concepts, to explore what students' responses, examples and construction processes, may reveal about their understanding. Furthermore, this study not only identifies some of the difficulties that students experience but also highlights the sources of these difficulties and opens possible avenues for dealing with them. On the other hand, the study



investigates how effective example-generation tasks are as research tools for extending the knowledge about (mis)conceptions and obstacles in students' thinking, in the context of linear algebra. The role attributed to learner-generated examples in the past focused on the pedagogical purposes. In this research, I examine whether example-generation tasks may provide a finer granularity to, and reveal weaknesses and gaps in, students' understanding of the subject.

In summary, there are two research questions addressed in this study:

1. What is students' understanding of the key concepts of linear algebra? What difficulties do students experience when engaged in the example-generation tasks?
2. What can example-generation tasks reveal about students' understanding of mathematics? Are these tasks effective and useful as a data collection tool for research in mathematics education?

### **1.3 Thesis organization**

Chapter 2 presents an overview of research on linear algebra, while chapter 3 acquaints the reader with the literature on the use of examples in mathematics education. The latter chapter discusses various types of examples such as reference examples, generic examples, counter-examples, examples provided in the instruction, and learner-generated examples.

Research on learning linear algebra falls within research on learning undergraduate mathematics, a field that only in the past two decades received attention in mathematics education research. Several theoretical frameworks have been developed to

interpret students' learning in other undergraduate mathematics courses. Two frameworks that I found more appropriate for this study, to interpret learner-generated examples, are APOS (Action-Process-Object-Schema) (Asiala, Brown, DeVries, Dubinsky, Mathews, and Thomas, 1996) and concept image /concept definition (Tall and Vinner, 1981). These frameworks are discussed in Chapter 4.

The main focus of Chapter 5 is the setting of the study. This chapter describes the participants in this research, the course, and the methodology used for gathering data. Furthermore, it presents the example-generation tasks used in the study. It offers the rationale for considering each task as an appropriate instrument for the study, and suggests anticipated participants' responses.

Chapter 6 is devoted to the results and analysis of participants' responses that are interpreted through the lens of genetic decomposition and the framework of concept image/concept definition. It further identifies the connections between linear algebra concepts that are present in students' schemas.

The summary of the findings and the major outcomes of this research are discussed in Chapter 7. This chapter also presents the contributions of this study. Firstly, the study provides a better understanding of students' learning of specific topics in linear algebra. Secondly, it introduces a methodology for investigating students' understanding of mathematics. Thirdly, the study presents pedagogical tools for engaging students in mathematical activity.

## **CHAPTER 2: RESEARCH ON LINEAR ALGEBRA**

In the past two decades, there has been a growing body of research on curriculum development, students' learning, and the place of technology in the teaching of calculus. Linear algebra is the second most popular course, after calculus, in the undergraduate mathematics curriculum. Despite its popularity, linear algebra has received disproportionate attention in mathematics education research. This chapter reviews the research on linear algebra relevant to this study.

### **2.1 Concerns expressed in the literature**

There is a common concern expressed in the literature that students leaving a linear algebra course have very little understanding of the basic concepts, mostly knowing how to manipulate different algorithms. Carlson (1993) stated that solving systems of linear equations and calculating products of matrices is easy for the students. However, when they get to subspaces, spanning, and linear independence students become confused and disoriented: "it is as if the heavy fog has rolled in over them." Carlson further identified some reasons why certain topics in linear algebra are so difficult for students. Presently linear algebra is taught far earlier and to less sophisticated students than before. The topics that create difficulties for students are concepts, not computational algorithms. Also, different algorithms are required to work with these ideas in different settings.

Dubinsky (1997) pointed out different sources of students' difficulties in learning linear algebra. First, the overall pedagogical approach in linear algebra is that of telling students about mathematics and showing how it works. The strength, and at the same time the pedagogical weakness, of linear situations is that the algorithms and procedures work even if their meaning is not understood. Thus, students just learn to apply certain well-used algorithms on a large number of exercises, for example, computing echelon forms of matrices using the Gaussian row elimination method. Secondly, students lack the understanding of background concepts that are not part of linear algebra but important to learning it. Dorier, Robert, Robinet, and Rogalski (2000) identified students' lack of knowledge of set theory, logic needed for proofs, and interpretation of formal mathematical language as being obstacles to their learning of linear algebra. Thirdly, there is a lack of pedagogical strategies that give students a chance to construct their own ideas about concepts in linear algebra.

Another concern has to do with the way linear algebra is taught in the undergraduate programs. As the applications of linear algebra range over a broad range of disciplines such as engineering, computer science, economics, and statistics, the majority of students taking the course come from non-mathematics majors. Therefore, a course in linear algebra should accommodate these students. To address this concern the Linear Algebra Curriculum Study Group (LACSG), consisting of mathematicians, representatives of client disciplines, and mathematics educators, was formed in January 1990. This group generated a set of recommendations for the first linear algebra course:

1. The syllabus and presentation of the first course in linear algebra must respond to the needs of client disciplines. Students should see the course as one of the most potentially useful mathematics courses they will take as an undergraduate.

2. Mathematics departments should seriously consider making their first course in linear algebra a matrix-oriented course. It should proceed from concrete and practical examples to the development of general concepts, principles and theory. At the same time, representatives from client disciplines have stressed the need for a solid, intellectually challenging course with careful definitions and statements of theorems, and proofs that show relationships between various concepts and enhance understanding.
3. Faculty should consider the needs and interests of their students as learners.
4. Faculty should be encouraged to utilize technology in the first linear algebra course.
5. At least one “second course” in matrix theory / linear algebra should be a high priority for every mathematics curriculum (Carlson, Johnson, Lay, Porter, Watkins, and Watkins, 1997).

## **2.2 Teaching and learning linear algebra**

In France, a group of researchers developed a program on the teaching and learning of linear algebra in the first year of science university (Dorier et al, 2000). This work included the design and evaluation of experimental teaching built upon an analysis of the historical development, didactics and students' difficulties. They identified the source of students' difficulties as the obstacle of formalism that relates to the nature of linear algebra itself. Historically, linear algebra evolved over several centuries into a unifying and generalizing theory. This subject is presented to students as a completed package. Therefore, according to Dorier et al, one main issue in the teaching of linear algebra is to give students better ways of connecting the formal objects of the theory with their previous conceptions, in order to promote more intuitively-based learning. In their research, Dorier et al (2000) built teaching situations leading students to reflect on the nature of the concepts with explicit reference to students' previous knowledge, what researchers called meta-level. The evaluation of the research proved that students that

had followed an experimental teaching based on this approach were more efficient in the use of the definitions, even in formal contexts.

Students construct their own knowledge and it is important for them to connect new ideas with prior experience or knowledge to develop an understanding. As geometry is intrinsically linked to visual perception, it is a potential source for intuitive thinking. Harel (1989, 1990, 1997, 2000) proposed exposing students to some linear algebra methods and concepts at the secondary level, as a foundation for the abstraction and precision of college linear algebra courses. He noted that the incorporation of geometric thinking in teaching the first course in linear algebra makes a significant contribution to students' understanding. Accessing students' knowledge of geometry can provide important background support for language and meaning in linear algebra as well as mental images for a large number of concepts. However, the incorporation of geometry in linear algebra courses must be sequenced in such a way that students understand the context of investigation. For example, thinking about vectors and transformations in a geometric context links these concepts with more familiar ones. However, many students do not see it as a bridge to abstract algebraic concepts. They do not see diagrams as representations of the abstract setting but as the actual object of inquiry, and seem to take such illustrations literally. New findings, reported in Harel (2000), indicated that when geometry is introduced before the algebraic concepts of linear algebra have been formed, students remain in the restricted domain of geometric vectors and do not move up to the general case.

Since linear algebra emerged as a union of various disciplines or areas of mathematics, it brought together different representations or descriptions as well. Hillel

(2000) distinguished three modes of description in linear algebra: geometric, algebraic, and abstract. Abstract mode refers to using the language and concepts of the general formalized theory, including vector spaces, subspaces, linear span, dimension, operators, and kernels. Algebraic mode refers to using the language and concepts of a more specific theory of  $\mathbf{R}^n$  including  $n$ -tuples, matrices, rank, solutions of systems of linear equations, and row space. Geometric mode refers to using the language and concept of 2-dimensional and 3-dimensional space including directed line segments, points, lines, planes, and geometric transformations. Representations allow one to go back and forth between the modes. However, different representations are the source of errors and confusion for many students. For example, representing vectors with arrows and points, and shifting back and forth between the two representations, while it may be unconscious for teachers, creates problems for students. The relationship between arrows and points on a line (plane) is not so clear, particularly if one moves away from the origin.

Sierpinska (2000) considered three modes of thinking in linear algebra: synthetic geometric, analytic-arithmetic, and analytic-structural, that correspond to its three interacting 'languages': visual geometric, arithmetic, and structural. For example, thinking of the possible solutions to the system of three linear equations in three variables as an intersection of planes in  $\mathbf{R}^3$  corresponds to the synthetic-geometric mode. If one thinks about the same problem in terms of row reduction of a matrix, one is in the analytic-arithmetic mode. Finally, thinking about the solutions in terms of singular or invertible matrices would correspond to analytic-structural mode. Each of the three modes of thinking leads to different meanings of the notions involved. However, these meanings are not equally accessible to beginning students. As a result, students have

trouble transferring from one mode to another, and seeing which mode is more appropriate to use in a given situation. For example, students attempted to solve the tasks on transformations presented in purely geometric form in an analytic way, as if the standard coordinate system was in place. They implicitly referred to the Cartesian coordinate system. Consequently, introducing basic linear algebra concepts with the aid of dynamic geometry software using a coordinate-free approach did not facilitate the notion of general basis. A computer screen as well as paper implicitly provides a preferred coordinate system.

Haddad (1999) reflected on two teaching experiences in linear algebra: one traditional, involving an algebraic-only approach, and one experimental, involving a geometric-only approach using the geometry software Cabri. He concluded that neither of these approaches in isolation resulted in a deep understanding. However, there are problems with multiple representations of the same concept as well. Pimm (1986) pointed out that the carry-over of the terminology of a domain to a new setting carries with it the expectations that the same (or at least, similar) relationships are going to hold. One problem is that some borrowed images or relations may be inappropriate to the expanded or new situation. For example, going back and forth between  $\mathbf{R}^n$  and any finite-dimensional vector space creates difficulty and confusion between a vector in  $V$  and its representation or isomorphic coordinate vector in  $\mathbf{R}^n$ . In addition, linking notions to the properties specific to  $\mathbf{R}^n$  in solving systems of linear equations becomes an obstacle to understanding the general theory and to the acceptance of other kinds of objects such as functions, matrices, or polynomials as vectors (Hillel, 2000).



## 2.3 Summary

This chapter presented an overview of research in learning and teaching linear algebra. Several views can be noted in reference to the course and teaching of the subject. On one hand, the course should proceed from concrete and practical examples to formal theory (Carlson et al, 1997). On the other hand, this creates obstacles for students since they tend to stick with concrete objects and not see the abstract structures and properties (Sierpinska, 2000). For example, thinking of vectors as  $n$ -tuples prevents them from conceiving of functions or matrices as vectors. Some research points out the need for multiple representations and visualizations in teaching linear algebra (Haddad, 1999), while other research indicates that it creates confusion for students (Hillel, 2000). Most students tend to think in practical rather than theoretical ways, and this affects their reasoning, and consequently, understanding in linear algebra.

The different methods of teaching linear algebra have been investigated from a variety of perspectives. However, students' understanding of the subject has not been examined through student-generated examples. This study discusses students' difficulties with constructing examples in linear algebra, and explores possible correlations of students' understanding with the generated examples. In the following chapter, I will discuss the role of examples and their use in mathematics education.

## CHAPTER 3: EXAMPLES IN MATHEMATICS EDUCATION

Examples play an important role in mathematics education. Whether provided by an instructor or textbook as an illustration or constructed by students, they have been used in teaching throughout the ages. This chapter discusses various uses of examples in mathematics education.

### 3.1 Types of examples in mathematics education

What is an example? Watson and Mason (2004) used *example* to refer to: “illustrations of concepts and principles, such as a specific equation which illustrates linear equations, or two fractions which demonstrate the equivalence of fractions; placeholders used instead of general definitions and theorems, such as using a dynamic image of an angle whose vertex is moving around the circumference of a circle to indicate that angles in the same segment are equal; questions worked through in textbooks or by teachers as a means of demonstrating the use of specific techniques - commonly called ‘worked examples’; questions to be worked on by students as a means of learning to use, apply, and gain fluency with specific techniques - usually called ‘exercises’; representatives of classes used as raw material for inductive mathematical reasoning, such as numbers generated by special cases of a situation and then examined for patterns; specific contextual situations which can be treated as cases to motivate mathematics” (p.17).

There are several types of examples identified in literature. For example, Michener (1978) described four types of examples distinguished by their use in teaching and learning, which can motivate concepts and results: start-up examples, reference examples, model examples, and counter-examples. Start-up examples help one get started in a new subject by motivating basic definitions and results and setting up useful intuitions. They motivate fundamental concepts, can be understood by themselves, provide a simple and suggestive picture, and their specific situation can be lifted up to general case. Reference examples are examples that one refers to repeatedly. They are basic, widely applicable and provide a common point of contact through which many results and concepts are linked together. In linear algebra, a very useful reference example is the collection of  $2 \times 2$  matrices whose entries are 0's and 1's. Model examples are generic examples. They suggest and summarize expectations and default assumptions about results and concepts. Counter-examples show that a statement is false. They sharpen distinction between concepts.

Some examples are too extreme to be representative of entire classes, but they do show what happens at the 'edges', referred to as boundary examples. They may prevent the misunderstanding of the scope of a concept. There are also non-examples, examples that demonstrate the boundaries or necessary conditions of a concept (Watson and Mason, 2004). It is common experience that learners apply a theorem without checking all the conditions. Examining non-examples or constructing their own examples can force learners to consider the importance of the theorem and the significance and necessity of the conditions.

Mason and Pimm (1984) pointed out that the role of example is to help students see the generality, which is represented by the particular. In order for examples to fulfil this role, students need to see examples as representing some more general statement and appreciate the generality that is being articulated (Mason, 2002). A generic example is an actual example, but one presented in such a way as to bring out its intended role as the carrier of the general. The study by Rowland (1998) suggested that generic examples can be used for proving theorems in number theory and provide a better understanding of the topic as opposed to formal proof. In the teaching context, the purpose of proof is to explain, to illuminate why something is the case rather than be assured that it is the case. The generic example serves to provide insight as to why the proposition holds true. Although given in terms of a particular number, the generic proof nowhere relies on any specific properties of that number.

Counter-examples are used frequently in mathematics instruction. Learners can be asked to construct counter-examples in order to explore the limitations of a concept or relationship, as well as to challenge conjectures. A common task involving construction of counter-examples is the 'True / False task' where students have to identify the validity of the mathematical statements and provide a counter-example if the statement is false. However, counter-example construction turns out to be deeply problematic, especially where learners have not had a history of personal construction (Watson and Mason, 2004).

The status of a counter-example is very powerful compared to the status of other examples. One counter-example is enough to draw very definite conclusions, while several supporting and verifying examples do not suffice. Many students are not

convinced by a counter-example and view it as an exception that does not contradict the statement in question. Zaslavsky and Ron (1998) examined to what extent students understand the special role of counter-examples in refuting false mathematical statements; how far they succeed in generating correct counter-examples; and what difficulties they encounter. Many students did not consider a counter-example as sufficient evidence to prove the statement false. Students were more persistent in their view that a counter-example is sufficient for refuting a geometric statement than an algebraic one. The findings also suggest that many students who accepted a counter-example as sufficient evidence for refuting a mathematical statement were not able to distinguish between an example that satisfies the conditions of a counter-example and one that does not satisfy them.

Peled and Zaslavsky (1997) investigated different types of counter-examples and the extent to which counter-examples generated by in-service and pre-service teachers have explanatory nature. Their findings suggest that there are several categories of counter-examples according to the example's explanatory power and ranging from specific to general. Specific counter-examples do not include any reference to or explanation of an underlying mechanism of a more general case. General counter-examples provide the mechanism that explains a claim being refuted and shows how the counter-examples can be generated.

Many students rely on worked examples in the textbook as they provide illustrations of the principles that the text aims to introduce. Chi and Bassok (1989) found that studying examples in textbooks for learning did not give students understanding of the material, just syntactic representation of procedures to apply. As a result, they could

not adjust and modify the examples to solve other problems. Learning from examples is diminished if the statements in the solution are not explicit about the conditions under which the actions apply. However, if justifications are made too explicit, the examples still may be unclear and not understandable by some students. Students need to complement the procedures with self-explanation consistent with their understanding of the text.

Rowland, Thwaites, and Huckstep (2003) looked at how pre-service elementary teachers use examples in the classroom and how it is related to their content knowledge of mathematics. They identified two different uses of examples in teaching: inductive, i.e. providing or motivating students to provide examples of something, and illustrative and practice-oriented. The examples used for pedagogical purposes should be the outcome of a reflective process of choice on the part of a teacher, an informed selection from the available options. However, the researchers found that this was not the case for some teachers. In particular, teachers' poor choice of examples included calculations to illustrate a procedure when another procedure was more appropriate, and randomly generated examples when more carefully chosen examples were more suitable.

### **3.2 Learner-generated examples**

The above discussion looked at the types of examples, which were in most situations provided for learners rather than constructed by learners. In fact, learners are rarely asked to construct examples for mathematical concepts explicitly, especially in the postsecondary level mathematics courses. Watson and Mason (2004) claimed that 'the examples learners produce arise from a small pool of ideas that just appear in response to particular tasks in particular situations' (p.3). The authors referred to these pools as

*example spaces*. Watson and Mason (2004) further advocated two principles for the use of learner-generated examples. First, learning mathematics consists of exploring, rearranging and extending example spaces, and the relationships between and within them. Through developing familiarity with those spaces, learners can gain fluency and facility in associated techniques and discourse. Second, experiencing extensions of learner's example spaces (if sensitively guided) contributes to flexibility of thinking not just within mathematics but perhaps even more generally, and empowers the appreciation and adoption of new concepts. Learners will model their methods of mathematical enquiry on those presented, used and expected of them by the teacher. The teacher therefore has a role in providing a model of mathematical questioning and example creation as well as organizing learning in such a way as to encourage it.

Constructing examples of objects promotes and contributes to learning, when viewed as becoming better at constructing and reconstructing generalities. By actively working on examples, one learns about classes of objects; by constructing objects with specified constraints, one learns about the structure of the objects, and comes to appreciate the concepts they exemplify. Furthermore, when a learner generates an example under a given constraint, a mental construction is created in his/her mind in parallel. When one constructs an example for a particular concept, which satisfies certain properties, s/he also constructs a link between two (or more) concepts (Hazzan and Zazkis, 1996).

The task of generating an example is considered powerful in terms of revealing strengths and weaknesses. A limited set of early examples has been shown to cause a wide variety of misconceptions, because it is very common for learners to identify

concepts with one or two early examples a teacher has showed them. Since these early examples are often simple ones, the learner is left with an incomplete and restricted sense of the concept, as was found by Zaslavsky and Peled (1996). Their study was designed to identify difficulties associated with the concept of binary operation regarding the associative and commutative properties and to investigate possible sources of these difficulties. The findings of Zaslavsky and Peled's study show that the concept image of binary operation held by teachers and student teachers is primitive, influenced by past experiences of both groups. This is evident in their similar distributions of types of difficulties as well as in their similar tendencies to suggest examples from a limited, familiar and basic content scope. As the study also showed, experienced teachers were more willing to generate unconventional examples that could have been the result of constantly having to generate examples in the classroom.

Dahlberg and Housman (1997) suggested that a student's explanation of his/her understanding of the concept definition could reveal the student's concept image, discussed in detail in Chapter 4, and also cause further development in the student's concept image. In their study, upper level undergraduate mathematics students were given a definition of a fine function, were asked to study the definition and write down what comes to mind, and then answer a set of questions. The authors found that students who consistently employed example generation were able to encapsulate more examples into their concept image of fine function, and were more able to use these examples than those who primarily used other learning strategies. They suggested that it might be beneficial to introduce students to new concepts by requiring them to generate their own examples or have them verify and work with instances of a concept before providing



them with examples or commentary. This coincides with the suggestion by Watson and Mason (2002) that learner-generated examples are useful for learning new concepts. By constructing examples learners construct and extend their example spaces. As learners repeatedly construct example spaces associated with a concept, they are building a concept image by relating things that come to mind with a definition or instructions.

### **3.3 Summary and conclusion**

The purpose of this chapter was twofold: to introduce the reader to the different types of examples and to describe a range of use of examples in mathematics education. Several types of examples are noted in the literature: start-up examples, reference examples, model/generic examples, counter-examples, boundary examples, and non-examples. Examples offered by teachers or textbooks are used to provide, or to motivate students to provide, examples of something, or to illustrate mathematical situations. For example, the generic example can have explanatory power and can offer insight as to why the propositions hold true. Watson and Mason (2004) advocate the pedagogical value of using learner-generated examples. For research purposes, learner-generated examples can give an indication about students' understanding of the subject. What types of examples do student produce when presented with the task? Do their examples conform to the given conditions and constraints? What difficulties do they encounter when faced with "give an example" tasks?

Tasks prompting for learner-generated examples can be used to gain information about students' concept image. If learners are asked to come up with an object that has very little, if any, constraint but which is reasonably familiar, then all they do is select the first thing that comes to mind. It may be the prototypical representative of their concept

image. As constraints are added, learners may be forced to search for less obvious and unconventional examples, realizing that there are possibilities other than the ones that had come immediately to mind. Watson and Mason (2004) suggested that since exemplification involves copying given examples, or constructing, manipulating, and transforming the knowledge of the concept, or modifying familiar objects, it possibly leads to reorganization of the concept image. It can be a major influence in concept development. In the next chapter, I will explore the theories of learning and discuss theoretical frameworks applicable to this study to interpret students' example construction process.

## **CHAPTER 4: THEORETICAL CONSIDERATIONS**

The issue of understanding how students learn has arguably that made the greatest impact on current mathematics education research. Using the lens of constructivism to interpret human cognition, a number of learning theories have been developed that focused research on learning mathematics by undergraduate students. Among them were Sfard's theory of Reification (1991), Tall and Vinner's Concept Image / Concept Definition theory (1981), Dubinsky's Action-Process-Object-Schema theory (Asiala et al, 1996), and Gray and Tall's framework of Procept (1994). All the theories listed above share a common underlying assumption that learning mathematics entails construction of knowledge. Having examined the different theories, I will briefly describe the two frameworks that appeared to be most appropriate for this study. APOS theory offers a set of mental constructions, actions, processes, and objects, necessary to learn mathematical concepts, and has been proven effective in designing undergraduate mathematics curriculum. The theory of concept image / concept definition provides an additional insight into learners' existing understanding of the mathematical concepts.

### **4.1 APOS theoretical framework**

The APOS (Action-Process-Object-Schema) theoretical framework for modelling mathematical mental constructions was developed for research and curriculum development in undergraduate mathematics education (Asiala et al, 1996; Dubinsky and McDonald, 2001). The theoretical component of the framework is based on a constructivist theory rooted in Piaget's work on reflective abstraction. The theory

maintains that an individual's mathematical knowledge is developed through the construction of mental Actions, Processes, and Objects, which are organized and linked through Schemas.

An *action* is a transformation of objects, perceived by an individual as being at least somewhat external. A *process* is an action that has been interiorized to the extent that the individual responds to internal rather than external cues. An *object* is an encapsulation of a process. It is determined by an individual's ability to reflect on operations applied to a particular process, to view the process as a totality, to realize that transformations (either actions or processes) can act on it, and to construct such transformations. An individual operating with an object conception of a transformation can also de-encapsulate the object into the component processes. An additional indication of an individual object conception is when s/he refers to the properties of a mathematical concept (Hazzan and Zazkis, 2003; Hazzan, 1996). Tall, D., Tomas, M., Davis, G., Gray, E., and Simpson, A. (2000), in their discussion of the nature of an object being encapsulated, stated that objects are described by their properties, their relationships with other objects, and the ways in which they can be used. Therefore, "it is the use of language in a way that intimates properties, relationships, usage of a concept which indicates that the individual is, in fact, conceiving" algebraic concepts as objects (Tall et al, 2000, p.230). A *schema* consists of a structured collection of processes and objects. It represents the totality of knowledge that is connected (consciously or subconsciously) to a particular mathematical topic. A schema can itself be treated as an object and included in the organization of other schemas.

In what follows, I will illustrate the application of the APOS theoretical

framework to the concept of a system of linear equations. An individual operating with an action conception of a system of linear equations requires external cues that give precise details of the necessary steps. That is, in order to decide if a particular linear system has a solution, such a person has to find this solution explicitly or see that it does not exist. An individual operating with a process conception of a system of linear equations can decide whether this system is consistent or not and how many solutions it has by analyzing an echelon form of an augmented matrix of this linear system. An individual operating with an object conception of a linear system is able to construct linear systems with one, infinitely many or no solutions. S/he should be able to answer conceptual questions such as explaining why a linear system  $A\mathbf{x} = \mathbf{b}$ , where  $A$  is an  $m \times n$  matrix with more rows than columns, cannot be consistent for all  $\mathbf{b}$  in  $\mathbf{R}^m$ .

The APOS theoretical framework has been used to analyze and interpret students' constructions of mathematical knowledge in various undergraduate subjects such as calculus, abstract algebra, and differential equations. Based on the data analysis, the theory makes predictions that if a particular collection of actions, processes, objects, and schemas is constructed in a certain manner by a student, then this student will likely succeed in using certain mathematical concepts in certain problem situations. The detailed descriptions of schemas, called genetic decompositions, allow the researchers to hypothesize about how a learning process may occur. In Chapter 5, I will introduce the genetic decomposition of the following concepts of linear algebra: linear dependence and independence, column and null spaces, and linear transformation.

## **4.2 Concept image / concept definition theoretical framework**

A second theory that has been used in the research on students' understanding in undergraduate mathematics as well is the theory of concept image and concept definition. It was introduced by Tall and Vinner (1981). They defined the concept image to describe the total cognitive structure that is associated with the concept including examples and non-examples, representations (symbolic, graphical, pictorial, verbal, etc.), definitions and alternative characterizations, properties, results, processes and objects, contexts, and the relationships among them. According to the researchers, the concept image is built up over the years through experiences of all kinds, changing as an individual meets new stimuli and matures. The concept image is distinguished from the evoked concept image, which is the portion of the concept image activated at a particular time. Depending on the situation or context, conflicting images may be evoked. When conflicting aspects are evoked simultaneously this creates confusion and can lead to erroneous results.

Concept definition refers to a form of words used to specify that concept. There are two types of concept definition: a personal concept definition (a personal reconstruction by the student of a definition), and a formal concept definition (a concept definition accepted by the mathematical community at large). According to Tall and Vinner (1981), a student, when asked to define a concept, may respond with a personal concept definition, which may not agree with a mathematically acceptable formal concept definition but which instead might be described as an ad hoc description of his or her concept image. Thus, some parts of the concept image function as definitions.

The concept image is built on the experiences of the students. Solving a mathematical problem (or engaging in any mathematical activity) involves recalling or

reconstructing examples, representations, objects, or processes and establishing connections to other examples, representations, objects, or processes. By presenting mathematics to a student in a simplified or restricted context, these simplified features become part of the individual concept image. Later these cognitive structures can cause serious cognitive conflict and act as obstacles to learning (Tall, 1989). For example, the tangent to a circle touches the circle at one point only and does not cross the circle. Vinner (1983) observed that many students believed that a tangent to a more general curve touches it, but may not cross it. When students were asked to draw the tangent to the curve  $y = x^3$  at the origin, many drew a line a little to one side which did not pass through the curve.

The distinction between the concept image and concept definition brings out two ideas about students' learning of mathematics. First, around any mathematical concept, students' thinking is strongly influenced by the examples, non-examples, representations, and contexts in which they have previously experienced the concept. Second, students do not typically refer to a formal definition in response to mathematical tasks but rather rely entirely on their concept image. Tall (1988) observed that when students meet an old concept in a new context, it is the concept image, with all the implicit assumptions abstracted from earlier contexts, that responds to the task. Referring to the concept definition of a function, Vinner (1983) claimed that in order to handle concepts one needs a concept image and not a concept definition and when the concept is introduced by means of a definition, it will remain inactive or even be forgotten. Harel (2000) noted that even though students were exposed to the same formal concept definition of a vector space, their performances in solving problems that could be solved directly by applying

the definition were different. This was because the students were exposed to different experiences applying the formal definition that resulted in formation of different concept images. In addition, even when students can recall a concept definition, the concept definition and the concept image might conflict with or contradict one another. Students may be able to reproduce the formal concept definition without any understanding of it or without having a connection to their concept image. Vinner (1997) referred to this phenomenon as pseudo-conceptual understanding.

A potential conflict factor occurs when part of the concept image conflicts with another part of the concept image or concept definition (Tall and Vinner, 1981). Such factors can seriously impede the learning of a formal theory. For instance, students often form a concept image of " $s_n \rightarrow s$ " to imply  $s_n$  approaches  $s$ , but never actually reaches there. Thus, students infer that  $0.\bar{9}$  is not equal to 1 because the process of getting closer to 1 goes on for ever without ever being completed. Gaining insight into the variety of concept images may suggest ways in the teaching and learning mathematics that reduce the potential conflict factors, for instance, by offering a broad range of examples and non-examples necessary to gain a coherent image (Tall, 1988). When a teacher is aware of possible conflict images it may help to bring incorrect images to the surface and explore the conflict productively.

### **4.3 Summary**

The APOS theory and the theory of concept image and concept definition have been used extensively in the research of students' understanding in mathematics. The APOS theory is a tool that can be used objectively to explain students' difficulties with a



broad range of mathematical concepts and to suggest ways that students learn these concepts (Dubinsky and McDonald, 2001). Thus, this theory was selected for this study as a lens for analyzing students' construction of understanding of the concepts of linear algebra. The idea of schema is very similar to the concept image introduced by Tall and Vinner (1981). The latter framework provides an additional perspective on students' understanding and is used in this study as well. Analyzing students' concept images can provide insight into learners' existing understanding of the fundamental concepts in linear algebra. Investigating potential conflict factors can further inform teaching practice.

## CHAPTER 5: RESEARCH SETTING

### 5.1 The course

Math 232, 'Elementary Linear Algebra', is a standard one-semester introductory linear algebra course at Simon Fraser University. It is a required course not only for mathematics majors but also for students majoring in computing science, physics, statistics, etc. The course is conducted through three one-hour lectures per week, for 13 weeks. In addition to regularly scheduled lectures, students registered in the course are entitled and encouraged to come to the open workshop for assistance. At the workshop students meet with teaching assistants and other students, and work together to understand mathematics in a friendly and helpful environment. Assessment of the students is based on the weekly homework assignments, two midterms and a final exam.

The content of the course includes vectors, systems of linear equations, matrices, linear transformations, vector spaces, eigenvalues and eigenvectors, orthogonality, distance and approximation. The required textbook for the course is 'Linear Algebra and Its Applications' by David C. Lay. Since the author was a member of the LACSG (Linear Algebra Curriculum Study Group), the textbook is strongly influenced by the LACSG syllabus. One of the LACSG recommendations was that geometric interpretation of concepts should always be included in linear algebra. Thus, the textbook offers a geometric perspective for concepts such as linear dependence and linear transformations. To give a better idea of the subject content included in the course and the chronological order in which it is taught, a summary of the syllabus is presented in Table 1:

Table 1: Summary of syllabus

Chapter 1: Linear Equations in Linear Algebra	Systems of Linear Equations; Row Reduction and Echelon Form; Vector Equations; The Matrix Equation $A\mathbf{x} = \mathbf{b}$ ; Solution Sets of Linear Systems; Linear Independence; Linear Transformations
Chapter 2: Matrix Algebra	Matrix Operations; The Inverse of a Matrix; Characterization of Invertible Matrices
Chapter 3: Determinants	Introduction to Determinants; Properties of Determinants
Chapter 4: Vector Spaces	Vector Spaces and Subspaces; Null Spaces, Column Spaces, and Linear Transformations; Linearly Independent Sets, Bases; Coordinate Systems; Dimension; Rank; Change of Basis
Chapter 5: Eigenvalues and Eigenvectors	Eigenvectors and Eigenvalues; The Characteristic Equation; Diagonalization; Eigenvectors and Linear Transformations
Chapter 6: Orthogonality and Least Squares	Inner Product; Length and Orthogonality; Orthogonal Sets; Orthogonal Projections; The Gram-Schmidt Process; Least-Squares Problems

## 5.2 Participants

The participants of the study were students enrolled in Math 232 in the Spring and Summer 2005 semesters. 68 students agreed to participate in the study in the Spring 2005 semester offering of the course. Later in the course the students were asked to participate in individual, clinical interviews. A total of six students volunteered to participate in the interviews from both classes. These students represented different levels of achievement and sophistication.

## 5.3 Data collection

The methodology used for gathering data includes both written and oral student responses. Thus, the data for this study comes from the following sources: students' written responses to the questions designed for this study and posed in three homework assignments described in detail below, and clinical interviews. There were informal

observations of students' work in the class and open workshop throughout the course. These observations informed the design of questions for data collection.

The written responses were collected from the three assignments, with later tasks designed in light of students' responses to the previous tasks. The students had one week to work on each assignment. The interviews were conducted during the last part of the course after the second midterm in both Spring and Summer 2005 semesters. In the Spring 2005 offering, the example-generation was emphasized during the course instruction. In particular, the distinction between "good examples" and "bad examples" was explained and illustrated with several concept definitions. For instance, a 2x3 matrix

$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$  is an example of a matrix in echelon form, but the properties of an

echelon form of a matrix are not visible in this example. A matrix  $A =$

$\begin{bmatrix} 0 & 3 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$  offers more information about an example space of echelon forms of

a matrix, and also puts limitations on what can be claimed. The lectures were presented with emphasis on understanding and promoting the building of mental models for the linear algebra concepts. The students were always encouraged to come up with the 'most testing' examples for the definitions and theorems they encountered. In addition, the textbook used in the course had a number of questions requiring students to construct examples for mathematical statements and objects. In the Summer 2005 offering, the course focused mainly on applications of linear algebra. Nonetheless, the textbook for the course remained the same. Therefore, students were still familiar with example-generation tasks.

### 5.3.1 Interviews

The emphasis of the interview was on the methods of constructing examples and thinking processes rather than evaluation of the final answer. The interviews were conducted in a private office and students were provided with pencil and paper. The students were informed that the interviews were not designed to judge or evaluate their knowledge but to follow their thinking process. The students were encouraged to attempt every task and were told that all answers (right or wrong) were equally important for the study.

The students were asked to 'think aloud', that is, to describe everything they were doing and thinking while working on the problem. The interview questions were semi-structured. That is, the initial tasks were predetermined, but the students were asked for clarification or explanation during the interviews to justify their examples and the choice of construction method. Occasional prompts were given by the interviewer to direct the students to correct their minor mistakes, to clarify the wording of the question, or to help students advance with a question. All written work produced by the students was kept at the end of the interviews to complement their verbal work. The interviews were audio taped and carefully transcribed, cross-referencing with students' written work.

The students were mostly confident during the interviews. The first set of questions was designed to familiarize them with the process of generating examples before attempting the main tasks, to make the students comfortable with the questions and to put them at ease.

## 5.4 Tasks

This section addresses the following questions: how the selection of tasks helps to answer research questions asked, the reasons the tasks I have designed were posed, and how the data were analyzed. As was mentioned in Chapter 1, the purpose of this study was to investigate what role example-generation tasks play in students' understanding of linear algebra and how they can inform researchers about students' understanding of the subject. For this study, a set of linear algebra concepts was selected to investigate the research questions. The concepts of vector, linear dependence and independence, linear transformations, vector spaces and bases are some of the central concepts of linear algebra. Research has shown that students have difficulty with these concepts (Dorier, 2000; Carlson et al, 1997). Therefore, the tasks concentrate on these problematic areas.

A summary of the tasks posed in the study is presented in Table 2. To follow the example-generation process, Task 1 was included in the interview questions as well. Having students generate examples and justify their choices through written tasks and in an interview setting provided an opportunity not only to observe the final product of a student's thinking process but also to follow it through interaction with a student during his/her example-generation. The detailed analysis of the tasks is presented in Sections 5.5.1 and 5.5.2.

Table 2: Summary of tasks

Tasks	Topics covered	HW / Interview
<p><b>Task 1. Linear (in)dependence</b>            a). (1). Give an example of a <math>3 \times 3</math> matrix <math>A</math> with real nonzero entries whose columns <math>\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3</math> are linearly dependent.            (2). Now change as few entries of <math>A</math> as possible to produce a matrix <math>B</math> whose columns <math>\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3</math> are linearly independent, explaining your reasoning.            (3). Interpret the span of the columns of <math>A</math> geometrically            b). Repeat part a (involving <math>A</math> and <math>B</math>), but this time choose your example so that the number of changed entries in going from <math>A</math> to <math>B</math> takes a different value from before.</p>	<p>Systems of linear equations, vector and matrix equations, and linear dependence and independence in <math>\mathbf{R}^n</math></p>	<p>HW / Interview</p>
<p><b>Task 2. Column space / Null space</b>            Find an example of a matrix <math>A</math> with real entries for which <math>\text{Nul } A</math> and <math>\text{Col } A</math> have at least one nonzero vector in common. For this matrix <math>A</math>, find <u>all</u> vectors common to <math>\text{Nul } A</math> and <math>\text{Col } A</math>. If <math>T</math> is the linear transformation whose standard matrix is <math>A</math>, determine the kernel and range of <math>T</math>.</p>	<p>Matrix operations, Invertible Matrix Theorem, subspaces associated with matrices, general vector spaces, and linear transformations</p>	<p>HW</p>
<p><b>Task 3: Linear transformations</b>            Let <math>T</math> be a linear transformation from a vector space <math>V</math> to a vector space <math>W</math> and let <math>\mathbf{u}, \mathbf{v}</math> be vectors in <math>V</math>. State whether the following is true or false, giving either a proof or a counter-example: if <math>\mathbf{u}</math> and <math>\mathbf{v}</math> are linearly independent then <math>T(\mathbf{u})</math> and <math>T(\mathbf{v})</math> are linearly independent.</p>		<p>HW</p>

<p><b>Task 4: Basis</b></p> <p>Let <math>M_{2 \times 2}</math> be the space of real-valued matrices. Let <math>H</math> be the subspace of <math>M_{2 \times 2}</math> consisting of all matrices of the form <math>\begin{bmatrix} a &amp; b \\ -b &amp; c \end{bmatrix}</math>, where <math>a, b, c</math> are real.</p> <p>(1). Determine <math>\dim H</math>.</p> <p>(2). Give a basis for <math>H</math>.</p> <p>(3). Expand it to a basis for <math>M_{2 \times 2}</math>.</p>	Subspaces, bases	HW
<p><b>Task 5: Linear transformations (revisited)</b></p> <p>(a). Give an example of a linear transformation.</p> <p>(b). Give an example of a linear transformation from <math>\mathbf{R}^2</math> to <math>\mathbf{R}^2</math> that maps the vector <math>\begin{bmatrix} 0 \\ 1 \end{bmatrix}</math> to the vector <math>\begin{bmatrix} 3 \\ 5 \end{bmatrix}</math>.</p> <p>(c). Give an example of a linear transformation from <math>\mathbf{R}^3</math> to <math>\mathbf{R}^2</math> that maps the vector <math>\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}</math> to the vector <math>\begin{bmatrix} 3 \\ 1 \end{bmatrix}</math>.</p> <p>(d). Give an example of a transformation that is not linear.</p>	All above topics	Interview
<p><b>Task 6: Vectors</b></p> <p>Give an example of a vector.</p> <p>Give an example of a vector from a different vector space.</p> <p>Give another example of a vector from a vector space which is fundamentally different from the first two.</p>	All above topics	Interview

The example-generation tasks may reveal students' (mis)understanding of the mathematical concepts. Furthermore, the written responses to the tasks also provide a description of collective example spaces related to the concepts involved, that is, example spaces local to a classroom or other group at a particular time, which act as a local



conventional space (Watson and Mason, 2004). The tasks I have designed are non-standard questions that require understanding of the concept rather than merely demonstrating a learned algorithm or technique. In particular, Tasks 1 and 2 are highly original and, I believe, very instructive. In addition to analysis of the tasks from the research point of view presented in Section 5.5, the pedagogical value of the tasks is discussed in Chapter 7.

## 5.5 Tasks analysis

This section presents an analysis of the tasks used for data collection. For each task, I describe the purpose of the task, possible responses to the task, and how the theoretical frameworks of APOS and concept image/concept definition have been used to interpret students' responses.

### 5.5.1 Tasks used for written responses

#### *Task 1. Linear (in)dependence*

- a). (1). *Give an example of a  $3 \times 3$  matrix  $A$  with real nonzero entries whose columns  $\mathbf{a}_1$ ,  $\mathbf{a}_2$ ,  $\mathbf{a}_3$  are linearly dependent.*
- (2). *Now change as few entries of  $A$  as possible to produce a matrix  $B$  whose columns  $\mathbf{b}_1$ ,  $\mathbf{b}_2$ ,  $\mathbf{b}_3$  are linearly independent, explaining your reasoning.*
- (3). *Interpret the span of the columns of  $A$  geometrically*
- b). *Repeat part a (involving  $A$  and  $B$ ), but this time choose your example so that the number of changed entries in going from  $A$  to  $B$  takes a different value from before.*

The prerequisite knowledge for many concepts in linear algebra is the linear dependence relation between vectors. The purpose of the task was to investigate students' understanding of the concept of linear dependence and linear independence of vectors, in

particular, in  $\mathbf{R}^3$ . Many concepts of linear algebra are connected, and students should be able to use all these terms freely and with understanding. On one hand, this task connects the number of linearly independent columns in a matrix  $A$ , the number of pivots in an echelon form of  $A$ , and the dimension of the vector space spanned by the column vectors of  $A$ . On the other hand, it connects the minimum number of entries required to be changed in  $A$  to make its columns linearly independent, and the number of free variables in the matrix equation  $Ax = \mathbf{0}$ . This task also explores the possible proper subspaces of a vector space  $\mathbf{R}^3$  (excluding the subspace spanned by the zero vector,  $\text{Span}\{\mathbf{0}\}$ ). It can be further extended to a  $4 \times 4$  case, and then to the general case of  $n \times n$  matrices.

This is an open-ended task with no learnt procedures to accomplish it. The routine tasks ask students to determine if a set of vectors is linearly dependent or independent by applying the definition or theorems presented in the course. In part (a1) of the task, the given and the question are reversed. Zazkis and Hazzan claim that ‘such “inversion” usually presents a greater challenge for students than a standard situation’ (1999, p.433). To complete Task 1(a1) students have to adjust their prior experiences in order to construct a set of three linearly dependent vectors in  $\mathbf{R}^3$ , viewed as columns of a  $3 \times 3$  matrix  $A$ .

#### *Constructing a 3x3 matrix with linearly dependent columns*

There are several approaches for constructing an example for the Task 1(a1). One can try a guess-and-test strategy – starting with an arbitrary  $3 \times 3$  matrix with nonzero real entries and then checking whether its columns are linearly dependent; if not, change some entries and try again.

The procedure to determine if a set of vectors is linearly dependent involves row reducing a matrix  $A$  with vectors as columns to see if the associated homogeneous linear system has a nontrivial solution, or equivalently if there are free variables. In the case of a square matrix  $A$ , this implies that an echelon form of  $A$  has a zero row. Thus, to construct a  $3 \times 3$  matrix  $A$  with nonzero real entries whose columns are linearly dependent, one can explicitly start with such echelon form  $U$  of  $A$  and perform elementary row operations to eliminate the zero entries in  $U$ . Alternatively, one can perform elementary row operations mentally to construct:

- a matrix  $A$  with two identical rows;
- a matrix  $A$  with one row being a multiple of another row.

The procedure works since a sequence of elementary row operations is reversible and transforms a matrix into a row equivalent matrix, and row equivalent matrices have the same linear dependence relations between the columns.

A more sophisticated approach calls for the use of the properties of a linearly dependent set of vectors to construct a matrix  $A$ . One such property is the characterization of linearly dependent sets theorem: a set of two or more vectors is linearly dependent if and only if at least one of the vectors in the set is a linear combination of the other vectors. Applying this property to the set of three vectors  $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$  in  $\mathbf{R}^3$ , one can construct:

- a matrix  $[\mathbf{a}_1 \ c\mathbf{a}_1 \ d\mathbf{a}_1]$  – with two columns being multiples of the first one where  $\mathbf{a}_1$  has nonzero real entries and  $c$  and  $d$  are both nonzero real numbers. Then  $\mathbf{a}_2 = c\mathbf{a}_1 + d\mathbf{a}_3$ ;

· a matrix  $[\mathbf{a}_1 \ \mathbf{a}_2 \ c\mathbf{a}_1+d\mathbf{a}_2]$  – with any two columns,  $\mathbf{a}_1$  and  $\mathbf{a}_2$ , having nonzero real entries where  $\mathbf{a}_2$  is not a multiple of  $\mathbf{a}_1$ , and  $\mathbf{a}_3 = c\mathbf{a}_1 + d\mathbf{a}_2$ .

*Constructing a 3x3 matrix with linearly independent columns*

The part (a2) of Task 1 asks students to construct a matrix  $B$  with linearly independent columns by changing as few entries in a matrix  $A$  as possible. Construction of a matrix  $B$  depends completely on the original choice of a matrix  $A$ . There are the only two cases of importance. If the columns of  $A$  are multiples of one column, that is,  $A = [\mathbf{a}_1 \ c\mathbf{a}_1 \ d\mathbf{a}_1]$  and so the rank of  $A$  is 1, then two entries have to be changed in  $A$  to construct  $B$ . Otherwise, two column vectors in a matrix  $A$  are linearly independent with the third vector being a linear combination of the other two, and so the rank of  $A$  is 2. Then it is sufficient to change only one entry in  $A$  to make the columns linearly independent. This is a crucial step in solving the second part of the task.

There are several approaches possible for this part leading to both correct and incorrect results and conclusions. One may construct a matrix  $B$  correctly, that is, satisfying the condition of having linearly independent columns, but by changing more than the minimum number of entries of  $A$ . A construction leading to the incorrect matrix  $B$ , not having linearly independent columns, results from changing the correct number of entries in a matrix  $A$  but in incorrect positions.

Examples alone cannot provide sufficient evidence for students' level of understanding. It is justifications of the constructions that indicate the possible conflicts and misconceptions. Thus, the interpretation and analysis of correct responses depend on and require investigating the explanations given for the correct constructions.

*Geometric interpretation of the span of columns of a matrix  $A$*

There are two possible geometric interpretations of the span of the columns of  $A$ . If the columns of  $A$  are multiples of one column, the span is a line through the origin in  $\mathbf{R}^3$ . Otherwise, the span of columns of  $A$  is a plane through the origin in  $\mathbf{R}^3$ .

Correct responses may be expressed either graphically, representing column vectors appropriately and indicating the linear dependence relation, or with words: a plane or a line in  $\mathbf{R}^3$  through the origin. In the latter case, analysis of the justification is required to investigate a student's level of understanding.

For this part, a student may give a geometric representation of the generic span of linearly dependent vectors in  $\mathbf{R}^3$ . Usually, it's a three dimensional space with three linearly dependent vectors located in the  $xy$ -plane. This type of example is referred to as a figural example, a figural image which stands for and constitutes the associated concept (Watson and Mason, 2004). In this case, the concept is the linear dependence of vectors in  $\mathbf{R}^3$ . This representation may be completely disconnected from the actual matrix  $A$ , not only because the vectors are drawn in the incorrect location but also because it may show a vector space of erroneous dimension, for instance, a plane instead of a line.

Another incorrect response may come from giving a geometric representation of the solution set of a matrix equation  $A\mathbf{x} = \mathbf{0}$ , or  $\text{Nul } A$ . In this case, a student is concentrating on a process of finding  $\text{Nul } A$  rather than  $\text{Col } A$ . Instead of providing a geometric interpretation of the span of the columns of  $A$ , students may identify the solution set of a homogeneous linear system  $A\mathbf{x} = \mathbf{0}$  as a span of one or two vectors - a line or a plane, respectively. However, the source of the mistake does not lie in the geometric interpretation but in giving a geometric interpretation of an incorrect object.

### *Second round of Linear (in)dependence task*

In order to complete Task 1(b) students have to create an example of a matrix  $A$  using a different type of linear dependence relation from the one they employed previously. The second part of the linear (in)dependence task pushes the student to think of another similar example with a change in one constraint. For this part, a student might associate the different number of entries required to be changed with a different linear dependence relation between vectors.

Using the APOS theoretical framework for analyzing students' responses to Task 1, one can identify different levels of students' understanding of the linear dependence concept. When students construct examples using random guess-and-test strategy, they may be operating with an action conception of linear dependence. They have to perform row reduction on a matrix to find out if its columns are linearly dependent. Students that construct examples of matrices with the same rows or rows being multiples of each other, i.e. inverting the row reduction procedure mentally, may understand linear dependence as a process. Students that emphasize relations between column vectors may have encapsulated linear dependence as an object, and consequently might be able to construct any set of linearly dependent vectors.

The case of constructing a matrix  $A$  starting explicitly with an echelon form deserves special attention. A student may not be operating just with the process concept of linear dependence. To construct a matrix in this way requires more sophisticated knowledge and understanding. In the course, a linearly dependent set of vectors is defined in terms of solutions to a vector equation. A linear dependence relation for the columns of a matrix  $A$  corresponds to a nontrivial solution of  $A\mathbf{x} = \mathbf{0}$ . A student may

realize that a matrix having linearly dependent columns has a certain echelon form. So the object, linear dependence, is de-encapsulated to construct an appropriate echelon form of a matrix corresponding to  $A\mathbf{x} = \mathbf{0}$  having a nontrivial solution. Then the elementary row operations are performed on this echelon form to generate an example of  $A$ .

The above is just one of the ways of interpreting students' understanding. However, one has to look at the construction of examples for both parts of Task 1 to seek a more complete picture. A student may answer Task 1(a) of the question correctly but construct a matrix  $A$  for the second part with the same linear dependence relation and yet change a different number of entries from Task 1(a). In this case, the answer will be incorrect. Seeing just one possibility for a linear dependence relation and not changing it for the other case may indicate the process conception of linear dependence, namely that a student is not able to analyze the linear dependence relation in a set of vectors without referring to specific elements. A student may be at the action level of linear dependence if s/he can construct a set of linearly dependent vectors but is having difficulty altering the vectors to create a linearly independent set.

### ***Task 2. Column space / Null space***

*Find an example of a matrix  $A$  with real entries for which  $\text{Nul } A$  and  $\text{Col } A$  have at least one nonzero vector in common. For this matrix  $A$ , find all vectors common to  $\text{Nul } A$  and  $\text{Col } A$ . If  $T$  is the linear transformation whose standard matrix is  $A$ , determine the kernel and range of  $T$ .*

Every matrix has associated with it two intrinsic and complementary subspaces: the column space and null space. The column space of a matrix  $A$  is the set of all linear combinations of the column vectors of  $A$ , or the span of the columns of  $A$ . The null space of the matrix  $A$  is the set of solutions to a homogeneous linear system  $A\mathbf{x} = \mathbf{0}$ . Both types

of subspaces are related to the linear transformation  $\mathbf{x} \rightarrow A\mathbf{x}$  in that the column space of  $A$  is the range of the linear transformation and the null space of  $A$  is the kernel of the linear transformation. In the case where  $A$  is a square  $n \times n$  matrix, the column space and null space of  $A$  are both subspaces of  $\mathbf{R}^n$ , and have the zero vector in common. However, there are cases when these subspaces share nonzero vectors, and so we can ask how large their intersection might be.

The purpose of Task 2 was to explore how students treat these special cases, or non-examples. Non-examples are examples which demonstrate the boundaries or necessary conditions of a concept (Watson and Mason, 2004). They can simultaneously be counter-examples to an implicit conjecture. In this task, a matrix  $A$  is a non-example of a square matrix with  $\text{Col } A \cap \text{Nul } A = \{\mathbf{0}\}$ . At the same time, it is a counter-example to the conjecture that for any square matrix  $A$ ,  $\text{Col } A$  and  $\text{Nul } A$  have only the zero vector in common.

In general, for a linear transformation  $T: V \rightarrow W$ , the kernel and range of  $T$  lie in different vector spaces. Students get used to treating them as nonintersecting by definition. This task probes whether students realize that, when  $V = W$ , part of the range of  $T$  may be in the kernel. Overall, Task 2 can help researchers uncover what connection students build between the fundamental subspaces associated with matrices and linear transformations.

A key realization in starting Task 2 is that a matrix  $A$  must be square in order for  $\text{Nul } A \cap \text{Col } A$  to be non-empty. Further, a student might try  $2 \times 2$  or  $3 \times 3$  matrices first, in the search for an easy example. If  $A$  is not a zero matrix,  $\text{Col } A$  contains at least one nonzero vector – one of the nonzero columns of  $A$ . However,  $\text{Nul } A$  contains at least one



nonzero vector if the homogeneous system  $A\mathbf{x} = \mathbf{0}$  has a non-trivial solution, which happens if the columns of  $A$  are linearly dependent, or, equivalently, if  $A$  is not invertible.

Not all singular matrices can serve as examples for this task. However, limiting the choice of examples to singular matrices reduces the potential search space. Again one can use a guess-and-test strategy to find a matrix satisfying given constraints. In this case, the level of understanding is not the same as in the guess-and-test approach above as there is an initial observation of the restricted example space.

Another approach is to start with a general matrix, for instance, a 2x2 matrix

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  and a vector  $\begin{bmatrix} x \\ y \end{bmatrix}$  in  $\text{Nul } A$ , and set up the required system of equations:

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$  for  $\text{Nul } A$ , and  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$  where  $s, t \in \mathbf{R}$  for  $\text{Col } A$ . The next

step is to assign values to the variables to satisfy the equations. As  $\begin{bmatrix} x \\ y \end{bmatrix}$  has to be a nonzero vector, it forces the matrix to be singular. However, unless explicitly stated in the solution, it may not be clear if a student is conscious of this fact. If the student is not, then the approach is just a symbolic version of the first guess-and-test strategy.

One may start with another general situation, but less general than the previous one, by combining the requirement of the task that  $\text{Col } A$  and  $\text{Nul } A$  have a nonzero vector in common with the observation that a column of a matrix is a vector in  $\text{Col } A$ . Then one may construct a general matrix but let one of the columns be a vector in  $\text{Nul } A$ , or a solution to the homogeneous system,  $A\mathbf{x} = \mathbf{0}$ . Symbolically, let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  and  $\mathbf{x}$

$= \begin{bmatrix} a \\ c \end{bmatrix}$  with  $a$  and  $c$  not both 0. Then the corresponding homogeneous system is

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ , and one can easily find an example that satisfies this equation.

Possible responses and approaches to this task, resulting in both correct and incorrect responses, may indicate students' level of understanding. Giving an invertible  $2 \times 2$  matrix as an example may show students' lack of connection between the existence of a nonzero vector in  $\text{Nul } A$  and a matrix  $A$  not being invertible, even though it is one of the implications of the invertible matrix theorem. Students constructing a square matrix  $A$  with the given constraints using a guess-and-test strategy and choosing  $A$  from the set of all  $n \times n$  matrices,  $M_{n \times n}$  ( $n = 2, 3$ ), or offering an invertible matrix as an example may indicate an action conception of the null and column spaces. The link between the process of finding nontrivial solution to  $A\mathbf{x} = \mathbf{0}$  (this matrix equation having free variables) and a matrix not being invertible may not be present for these students.

Starting with a singular matrix may indicate that a student is treating  $\text{Nul } A$  as an object but the two vector spaces are not coordinated to meet the requirement of the task.  $\text{Nul } A$  and  $\text{Col } A$  are disconnected even though they are associated with the same object, a matrix  $A$ . This may indicate that the vector spaces associated with a matrix are still at the process level of understanding for these students. These notions are disjoint which can lead to the incorrect examples of singular matrices with  $\text{Nul } A \cap \text{Col } A = \{\mathbf{0}\}$ .

$\text{Col } A$  is generated by the columns of a matrix  $A$ , and  $\text{Nul } A$  is generated by solution vectors to the homogeneous matrix equation  $A\mathbf{x} = \mathbf{0}$ . If a student is able to think of the two vector spaces without resorting to the specific elements, by noticing the

relation between the vector spaces, s/he is operating with the object conception of  $\text{Nul } A$  and  $\text{Col } A$ . In Task 2 [Column space / Null space], the last general approach to constructing such a matrix may correspond to treating the associated vector spaces as objects. The condition of the task is incorporated in the example generation by combining the properties of the vector spaces associated with a matrix  $A$ , namely, a nonzero column of  $A$  is set to be a solution to  $A\mathbf{x} = \mathbf{0}$ .

Students' choices of matrices and explanations of their constructions of the matrices may reveal gaps in students' understanding of the nature of subspaces associated with a matrix, and the (dis)connection of the concept image formed by a student and the formal concept definition of the column and null spaces. Task 2 can also indicate the difficulties that students have with representations of the objects in linear algebra.

### ***Task 3: Linear transformations***

*Let  $T$  be a linear transformation from a vector space  $V$  to a vector space  $W$  and let  $\mathbf{u}, \mathbf{v}$  be vectors in  $V$ . State whether the following is true or false, giving either a proof or a counter-example: if  $\mathbf{u}$  and  $\mathbf{v}$  are linearly independent then  $T(\mathbf{u})$  and  $T(\mathbf{v})$  are linearly independent.*

It is a property of any linear transformation  $T$  from a vector space  $V$  to a vector space  $W$ ,  $T:V \rightarrow W$ , that if  $\{T(\mathbf{u}), T(\mathbf{v})\}$  is a linearly independent set of vectors in  $W$ , then  $\{\mathbf{u}, \mathbf{v}\}$  is a linearly independent set in  $V$ . Equivalently, if  $\{\mathbf{u}, \mathbf{v}\}$  is a linearly dependent set of vectors in  $V$ , then  $\{T(\mathbf{u}), T(\mathbf{v})\}$  is linearly dependent in  $W$ . However, other variations require certain conditions on a linear transformation to hold and are not true in general. Task 3 asks students to identify the validity of the mathematical statement and provide a counter-example if the statement is false.

One role of example-generation tasks is to explore the limitations of a concept or relationship, as well as to challenge conjectures. In this task, students are asked to explore when and why some properties do not hold for any linear transformation, and how linear transformations, abstract vector spaces and linear (in)dependence of vectors are related. On one hand, Task 3 helps researchers to investigate students' understanding of the properties of linear transformations. On the other hand, it helps students to better understand what is preserved by a linear transformation and what is not, and may preview for them the one-to-one property of a linear transformation.

When the vector spaces  $V$  and  $W$  are  $\mathbf{R}^n$  and  $\mathbf{R}^m$  respectively, this task also emphasizes the connection between a linear transformation and its standard matrix. A common problem in establishing this connection is that many students do not realize that the columns of the standard matrix of a linear transformation are exactly the images of the standard basis vectors. With this understanding, it is very easy to construct a counter-example. One can take an  $m \times n$  matrix  $A$  with two linearly dependent columns,  $\mathbf{a}_i$  and  $\mathbf{a}_j$  with  $1 \leq i < j \leq n$ , for example, at least one column being a multiple of another. These two columns are the images of the standard basis vectors  $\mathbf{e}_i$  and  $\mathbf{e}_j$  under the linear transformation  $T: \mathbf{R}^n \rightarrow \mathbf{R}^m$  whose standard matrix is  $A$ . Since the basis vectors are linearly independent, this linear transformation is a counter-example.

The linearly independent vectors in a vector space  $\mathbf{R}^n$  don't have to be restricted to the standard basis vectors. A linear transformation mapping  $\mathbf{R}^n$  to  $\mathbf{R}^m$  is uniquely determined by its action on any set of basis vectors of  $\mathbf{R}^n$ . Also, the coordinates of the images of the basis vectors form the columns of an  $m \times n$  matrix  $A$  of a linear transformation with respect to this basis. Thus, to construct a counter-example one can

take an  $m \times n$  matrix  $A$  with two linearly dependent columns so that these columns are the coordinates of the images of any two linearly independent vectors in  $\mathbf{R}^n$  under the linear transformation  $T(\mathbf{x}) = A\mathbf{x}$  represented by the matrix  $A$  with respect to a basis  $\beta$ . Since students are more comfortable with square matrices, they may give an example of a linear transformation with a square matrix, either standard or with respect to a given basis.

A linear transformation  $T: V \rightarrow W$  such that  $T(\mathbf{x}) = \mathbf{0}$  for all  $\mathbf{x} \in V$  is an easy counter-example for this task. It works for any vector spaces  $V$  and  $W$  since any two linearly independent vectors are mapped to the zero vector in  $W$ , and any set of vectors is linearly dependent if it contains the zero vector.

Different approaches to constructing examples for this task may indicate varying levels of students' understanding of the concept of linear transformation. Some students may use properties of linear transformations to try to prove the statement of the task, incorrectly using quantifiers and implications:  $\mathbf{u}, \mathbf{v}$  linearly independent  $\rightarrow c\mathbf{u} + d\mathbf{v} = \mathbf{0} \rightarrow c = d = 0 \rightarrow T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v}) = \mathbf{0} \rightarrow T(\mathbf{u}), T(\mathbf{v})$  linearly independent since  $c = d = 0$ . Others may construct a transformation with an invertible standard matrix, which is not a counter-example. In both cases, students may be operating with an action conception of linear transformation. If a student constructs a matrix of a linear transformation that is singular and uses the standard unit vectors to provide a counter-example but still checks that the conditions are met and that the images of the unit vectors form a linearly dependent set, s/he may understand linear transformation as a process. A student is thinking of linear transformation as an object when the student is able to evaluate how and why the properties of a linear transformation can be applied to construct a counter-example, such as giving a zero transformation as an example.

#### **Task 4: Basis**

Let  $M_{2 \times 2}$  be the space of real-valued matrices. Let  $H$  be the subspace of  $M_{2 \times 2}$  consisting of all matrices of the form  $\begin{bmatrix} a & b \\ -b & c \end{bmatrix}$ , where  $a, b, c$  are real. (1). Determine  $\dim H$ . (2). Give a basis for  $H$ . (3). Expand it to a basis for  $M_{2 \times 2}$ .

This task provides researchers with tools to gauge students' understanding of the concept of a basis. The task offers a further glimpse into students' understanding of basis and dimension, in particular, how a basis of a subspace of a vector space can be extended to a basis of the vector space itself, and how "big" the subspace can be inside a space.

There is no explicit request to construct an example in this task. However, 'give a basis for  $H$ ' requires a student 'to give an example' for a possible basis for  $H$ .

In response to this task, a student may present an obvious basis of this subspace:

$$\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}, \text{ verifying its properties by showing that it is a subset of}$$

$H$ , spans  $H$ , and is a linearly independent set. For part (3), one may complete the basis

with  $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ , or with  $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ . The dimension of  $H$  is 3, which is the cardinality of any

basis of  $H$ , and dimension of  $M_{2 \times 2}$  is 4.

This task might allow us to capture students' concept image of a basis. One of the problems that students encounter in a linear algebra course is not knowing how to work with definitions in a mathematically precise way. The unfamiliar definitions are usually encountered one after another and many definitions depend on previous ones. For example, the definition of a basis depends on the ideas of linear independence and spanning set. Another requirement for a set of vectors to form a basis for a given vector

space is that it has to be a subset of that vector space first. For a nontrivial vector space spanned by a finite set of vectors, the number of elements in a basis is invariant and is referred to as the dimension of this vector space. Thus, the concept image of a basis should include the concepts: linear independence, spanning set, subset of a vector space, dimension. Students' responses may indicate what components are missing from their concept image.

### 5.5.2 Tasks used for clinical interview

#### *Task 5: Linear transformations (revisited)*

(a). *Give an example of a linear transformation.*

(b). *Give an example of a linear transformation from  $\mathbf{R}^2$  to  $\mathbf{R}^2$  that maps a vector  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$  to the vector  $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$ .*

(c). *Give an example of a linear transformation from  $\mathbf{R}^3$  to  $\mathbf{R}^2$  that maps a vector  $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$  to the vector  $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ .*

(d). *Give an example of a transformation that is not linear.*

The purpose of this task is to investigate students' concept image of a linear transformation. The exercises and problems dealing with linear transformations usually ask students to check if a given transformation is linear, or one-to-one, or onto, or has other properties. These tasks can be successfully completed without necessarily understanding the concepts. This is not the case when finding or constructing an example of a linear transformation.

As definitions provide the foundation for every subject, it is necessary to understand why all the conditions are needed and what class of objects is being defined. To be able to reproduce a definition does not guarantee understanding of the concept.

Every linear transformation from  $\mathbf{R}^n$  to  $\mathbf{R}^m$  can be represented as a matrix transformation. This property can be used to provide an example for the first part of the task. One can choose any  $m \times n$  matrix  $A$  and indicate the corresponding vector spaces involved to define a linear transformation. However, examples don't have to be restricted to these vector spaces. One can give an example of a linear transformation of the space of polynomials, for example,  $T(\mathbf{p}) = D\mathbf{p}$  from  $\mathbf{P}_n$  to  $\mathbf{P}_{n-1}$  where  $\mathbf{P}_n$  is the space of polynomials of degree at most  $n$  and  $D\mathbf{p}$  is the derivative of  $\mathbf{p}$ .

To generate examples for parts (b) and (c), one can find a matrix of a linear transformation by setting up a system of linear equations with unknowns being the entries in a matrix, and then try to find a possible solution to this system. Then in both questions the entries  $a_{21}$  and  $a_{22}$  are determined by the coordinates of the image vectors. The remaining entries can have any values.

Alternatively, one may remember that an image of a vector under a matrix transformation is the linear combination of the columns of a matrix with weights corresponding to the entries in the vector, for example for part (c),  $T(\mathbf{x}) = A\mathbf{x} = [\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3]\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3$ . Then one can find the entries in  $A$  by solving the vector equation:

$$0\mathbf{a}_1 + 1\mathbf{a}_2 + 0\mathbf{a}_3 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}. \text{ Again, this shows that the second column of } A \text{ has to be } \begin{bmatrix} 3 \\ 1 \end{bmatrix} \text{ and}$$

the other two columns can have any values.



Another approach is to use the property that the standard matrix of a linear transformation is uniquely determined by the action of the transformation on the standard basis of  $\mathbf{R}^n$ . This property can be used to construct infinitely many matrices representing a linear transformation with the given requirements. Again for part (c), one may choose any  $2 \times 3$  matrix  $A$  so that the second column of  $A$  is  $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ .

These three approaches may correspond to three levels of understanding in action-process-object theoretical framework. In the first approach, a student may be thinking of the matrix of the linear transformation as an action. To find it s/he has to set up a system of linear equations and solve it explicitly. In the second case, a student has interiorized one of the actions of finding an image of a vector as a linear combination of the columns of a matrix. S/he is able to mentally perform this procedure to find numerical values for the matrix. However, a student may not understand the defining property of the standard matrix of a linear transformation. When no computations are performed to construct a matrix, one may have encapsulated the concept of a matrix transformation as an object.

The vector  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$  or  $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$  is immediately recognized as one of the basis vectors of  $\mathbf{R}^2$  or  $\mathbf{R}^3$ ,

respectively. Thus, its image will be the second column of the required matrix. With this approach a student can construct other matrices with the given property which will be evident from the response to a request for another example of a matrix with the same constraints. In addition, viewing the matrix of a linear transformation in terms of the actions on the basis vectors will help him/her later to work with the matrix of a linear transformation relative to any basis.

A nonlinear transformation is a non-example. As was pointed out in the description of Task 2, non-examples are examples which demonstrate the boundaries or necessary conditions of a concept (Watson and Mason, 2004). Non-examples also contribute to and are part of the concept image of a linear transformation. This question explores what are the defining properties for students of a linear transformation and how these properties are modified to generate a transformation that is not linear. Thus, it complements the first question of the task and provides further indication of the students' concept image.

### ***Task 6: Vectors***

*Give an example of a vector.*

*Give an example of a vector from a different vector space.*

*Give another example of a vector from a vector space which is fundamentally different from the other two.*

This was the first task that students were asked during the clinical interviews. The purpose of this task was two-fold: to make the students comfortable with the questions, as was mentioned above, and to see what objects constitute their concept image of the central concept of linear algebra.

Students first encounter vectors in  $\mathbf{R}^n$ , and then move on to abstract vector spaces. Usually, the definition of an abstract vector space is followed by a set of examples of different vector spaces including the vector space,  $\mathbf{P}_n$ , of polynomials of degree at most  $n$ ;  $M_{m \times n}$ , the vector space of  $m \times n$  matrices; the space of continuous functions; etc.

When a student is asked for an example of a vector, the assumption and expectation is that a student will produce an example of a vector from his/her prototypical

vector space: a vector space that dominates the concept image and is easily accessible. A possible response may include a vector in  $\mathbf{R}^2$  or  $\mathbf{R}^3$ . The second prompt may elicit a response of an element from  $\mathbf{R}^n$  for a value of  $n$  other than 2 or 3. The third request was added to push students to search for other possibilities that haven't come up in their examples so far by imposing a constraint that vectors have to be from a vector space fundamentally different from the ones mentioned in the previous two questions. In case the vectors in the first two examples belonged to a vector space  $\mathbf{R}^n$  for some values of  $n$ , the third example of a vector would have to belong to a vector space other than  $\mathbf{R}^n$ .

Overall, students' responses to this task may provide some indication about students' understanding of the concepts of vector and vector space and about students' personal/local example space of a vector, the example space triggered by current task as well as by recent experience (Watson and Mason, 2004); the characteristics of students' concept images of vectors; what objects are conceivable as vectors by students; and what modes of representation are used. When one is asked for three examples of the same concept s/he may start to look in the personal potential example space. Personal potential example spaces, from which personal/local space is drawn, consists of one person's past experience (even though not explicitly remembered or recalled), and may not be structured in ways which afford easy access (Watson and Mason, 2004). If a student is unable to go beyond  $\mathbf{R}^n$ , the concept image of a vector and consequently of a vector space of this student is limited. This task helps identify these limitations.

## 5.6 Summary

The intention of this chapter was to introduce the reader to the participants of the study, the research setting, and the data collection process. Furthermore, the chapter

presented the discussion of the tasks included in the study. Firstly, the rationale for considering each task as an appropriate instrument for the study was offered. Secondly, anticipated participants' responses to the task were suggested, and interpreted through the lens of genetic decomposition and the framework of concept image/concept definition.

## **CHAPTER 6: ANALYSIS OF RESULTS**

### **6.1 Introduction**

The goal of this chapter is to analyze students' understanding of the concepts of linear algebra through the example-generation tasks described in Chapter 5. To reiterate, for the purpose of this research several topics were selected from the undergraduate course in linear algebra. These topics are: vectors, linear dependence, column and null spaces, linear transformations, and basis. Strategies and tools used in students' responses helped to shed light on their understanding. The data were analyzed through the lens of two frameworks: APOS and concept image / concept definition. The analysis does not follow the order of the numbered tasks, with Tasks 3 and 5 (Linear Transformations) analyzed together, but corresponds to the topics under investigation. To protect students' confidentiality, the students' names have been changed; however, the gender has been preserved. 6 students that have been interviewed are: Nicole, Stan, Leon, Anna, Sarah and Joan.

### **6.2 Vectors**

Vectors are the building blocks of many concepts of linear algebra. They are the elements of any set that satisfies the axioms for a vector space. It is important for students to be able to think of vectors in general to understand and apply the theory of vector spaces. On the other hand, their concept image of vector should include all possible objects that can be treated as vectors.

**Task 6: Vectors**

*Give an example of a vector.*

*Give an example of a vector from a different vector space.*

*Give another example of a vector from a vector space which is fundamentally different from the first two.*

Task 6 was asked during the clinical interviews to acquaint students with example-generation and to gain some understanding of their concept image of a vector and vector space. As students first encounter vectors in  $\mathbf{R}^n$ , these vectors are the prototypical examples for them. Thus, it is not surprising that all but one student gave examples of vectors in  $\mathbf{R}^2$  or  $\mathbf{R}^3$  in response to the first prompt in the task as can be seen in the following interview excerpts:

**I:** Can you give an example of a vector?

**Nicole:** OK (writes  $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ )... in  $\mathbf{R}^3$ .

**I:** Can you give an example of a vector?

**Anna:** It can be (writes:  $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ )... in  $\mathbf{R}^2$ .

**I:** Can you give an example of a vector?

**Joan:** (1,3,5) ... in  $\mathbf{R}^3$ .

In reply to the second request for an example of a vector, the example space of all students again consisted of vectors in  $\mathbf{R}^n$  where  $n = 2, 3$ , or 4.

**Nicole:** OK. (writes:  $\begin{bmatrix} 2 \\ 4 \\ 5 \\ 6 \end{bmatrix}$ ) ... in  $\mathbf{R}^4$ .

**Joan:** (1,0) in  $\mathbf{R}^2$ .

**Stan:** (writes:  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ ) in  $\mathbf{R}^3$ .

**Anna:** Sure. Another one would be a 4 by 1, so it could be (writes:  $\begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ ), and this vector is in  $\mathbf{R}^4$ .

However, only two students were able to provide an example for the third prompt that required them to look beyond  $\mathbf{R}^n$ , as the students had to look for an example of a vector space fundamentally different from the two they used to answer parts (a) and (b). The answers indicated that students either didn't know how to respond to this question or thought of vectors as  $n \times 1$  arrays of numbers.

**I:** Can you think of any other vector space different from  $\mathbf{R}^n$ ?

**Joan:** I don't know how to do that. Maybe an equation.

**I:** Can you think of any other vector space different from  $\mathbf{R}^n$ ?

**Anna:** Different from  $\mathbf{R}^1$ ,  $\mathbf{R}^2$ , and  $\mathbf{R}^4$ , or completely different from the general one?

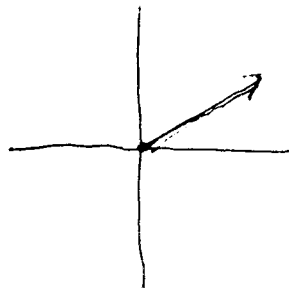
**I:** Different from the general one.

**Anna:** No. I think of vectors as always something  $n$  by 1. So it can be as many rows but only 1 column.

Two students used the geometric representation of vectors. For instance, Leon made the diagrams of vectors in  $\mathbf{R}^2$  and  $\mathbf{R}^3$  to support his explanations.

**I:** Can you give an example of a vector?

**Leon:** A vector? So from the origin and to any point in the space, I will construct a line, and then to another point, somewhere in the space, I would connect it to, and I would call it a vector, in such a direction.

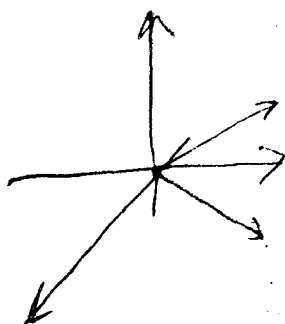


**I:** What vector space does this vector belong to?

**Leon:**  $\mathbf{R}^2$ .

**I:** Can you give me an example of another vector from a different vector space?

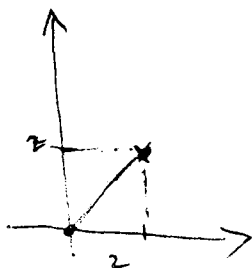
**Leon:** Alright. Then I would try  $\mathbf{R}^3$ . I would again start from the origin, and to some point in this space, I would direct this line, and give it direction.



The other student, Sarah, had difficulty representing her examples symbolically as an array of numbers even though she indicated the coordinates of the vector on the diagram. She didn't indicate that this is a directed line segment. For her, it was a point on the plane.

**I:** Can you represent your example numerically?

**Sarah:** No. I can see it as a diagram... It's a point.

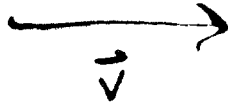


For the second request, she just drew a general vector as an example. She didn't specify the vector space that contains this vector.



**I:** Can you give an example of another vector from a different vector space?

**Sarah:** It could be this.



**I:** What vector space does this vector belong to?

**Sarah:** I don't know... Could be any vector space.

Students once exposed to  $\mathbf{R}^n$  have difficulty considering other elements as vectors. Only two students gave examples of vectors from vector spaces other than  $\mathbf{R}^n$ . Only one of them provided a correct representation for his examples. In response to the first question, this student offered a polynomial as an example of a vector.

**I:** Can you give me an example of a vector?

**Stan:** Sure... (writes:  $p(t) = 1+t+t^2$ ).

The same student offered several more objects that could be treated as vectors. In fact, he claimed that an element of any set that satisfies the axioms of a vector space could be viewed as an example for Task 6 (Vectors).

**I:** Can you think of any other vectors from different vector spaces?

**Stan:** Real-valued functions, for example,  $y = x$ ... I don't know how to write it properly, but I know they can be treated as vectors. I mean, any  $n$ -tuple could be a vector,  $\{\mathbf{0}\}$ . Anything I can find that has the property of scalar multiplication and addition.

Another student, after giving examples of vectors in  $\mathbf{R}^n$  for different values of  $n$ , was confused with the representation of polynomials in a vector space  $\mathbf{P}_2$ . Consider the following interview excerpt.

**I:** Can you give me an example of a vector that doesn't belong to any vector space  $\mathbf{R}^n$ ?

**Nicole:** OK. (writes:  $\begin{bmatrix} 5 \\ 8t \\ 2t^2 \end{bmatrix}$ ).

**I:** What vector space is this an element of?

**Nicole:**  $P_2$ . I think.

**I:** What is  $P_2$ ?

**Nicole:** Polynomials of degree two, I think.

**I:** If  $P_2$  is the space of polynomials of degree at most two, would you represent an element of this space as you did before?

**Nicole:** Maybe...Actually, no. It will not be  $t$ . It will be (writes:  $\begin{bmatrix} 5 \\ 8 \\ 2 \end{bmatrix}$ ). Not sure.

**I:** From your answers, the vectors in  $\mathbf{R}^3$  and  $P_2$  look the same. Do  $\mathbf{R}^3$  and  $P_2$  share the same vectors?

**Nicole:** That what it looks like.

**I:** Are they the same vector space?

**Nicole:** They will be in the same one big vector space but they are different ones... They are in the same big  $V$ , but then they are different subspaces.

**I:** What do you mean by that?

**Nicole:** You know, it's like an apple and orange are in the same space but then they are different things.

**I:** What is this  $V$ ?

**Nicole:** Vector space.

**I:** What are the vectors in this vector space?

**Nicole:** It can be these vectors that have 3 entries.

In the courses under consideration, students first encounter vectors as  $n \times 1$  arrays

of real numbers:  $\begin{bmatrix} x_1 \\ \dots \\ x_n \end{bmatrix}$ , and the concepts of linear algebra are first introduced in the

context of  $\mathbf{R}^n$ . Later in the course, a formal definition of a vector space is presented. Examples of vector spaces other than  $\mathbf{R}^n$  are presented such as the space of polynomials of degree at most  $n$  or the space of  $m \times n$  matrices, and there is a discussion of an isomorphism between a finite dimensional vector space over  $\mathbf{R}$  and  $\mathbf{R}^n$ . At this point, students' concept image of a vector includes arrays of numbers and the new objects. However, since the concept of an isomorphism is not yet internalized this creates a conflict. For example, in the preceding interview excerpt the student represented a

polynomial in  $\mathbb{P}_2$  first as  $\begin{bmatrix} 5 \\ 8t \\ 2t^2 \end{bmatrix}$ , then corrected her answer to  $\begin{bmatrix} 5 \\ 8 \\ 2 \end{bmatrix}$ . Both of these answers

are incorrect representations of vectors in  $\mathbb{P}_2$ . The first one indicates the confusion between the different representations of vectors: as a coordinate vector in  $\mathbb{R}^n$  and as an element of a space of polynomials, in this case. In fact, she was convinced that this is what elements of  $\mathbb{P}_2$  look like, and claimed that  $\mathbb{P}_2$  and  $\mathbb{R}^3$  have exactly the same elements but are different subspaces of “the big vector space  $\mathcal{V}$ ”. While Nicole and Stan could envision polynomials as vectors, for other students the concept image included only vectors in  $\mathbb{R}^n$ : either geometric vectors as directed line segments or  $n \times 1$  arrays.

Interpreting students’ responses to Task 6 with the concept image/concept definition theory reveals that for the majority of students the concept image of a vector at this point in the course is limited to elements of  $\mathbb{R}^n$ . They are only able to view the vectors as  $n \times 1$  arrays or directed line segments or points in  $\mathbb{R}^2$  or  $\mathbb{R}^3$ . This restricts students’ access to conceptual understanding of the general theory of vector spaces as is manifested in their understanding of the concept of a basis discussed in Section 6.6.

### 6.3 Linear dependence and linear independence

The two concepts of linear dependence and independence are closely connected. To have a solid understanding of one of them involves having understanding of the other. I will first present the summary of students’ responses for constructing matrix  $A$  with linearly dependent columns, and then use APOS theoretical framework to analyze students’ understanding of linear (in)dependence.

**Task 1: Linear (in)dependence**

a). (1). Give an example of a 3x3 matrix  $A$  with real nonzero entries whose columns  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$  are linearly dependent.

(2). Now change as few entries of  $A$  as possible to produce a matrix  $B$  whose columns  $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$  are linearly independent, explaining your reasoning.

(3). Interpret the span of the columns of  $A$  geometrically.

b). Repeat part a (involving  $A$  and  $B$ ), but this time choose your example so that the number of changed entries in going from  $A$  to  $B$  takes a different value from before.

**6.3.1 Constructing matrix  $A$  with linearly dependent columns**

In Task 1(a1) the students were required to give an example of three linearly dependent vectors represented as a 3x3 matrix with nonzero real entries. Table 3 presents the summary of different approaches used to complete this part of the task. The total frequencies exceed the number of participants as some students provided two examples for the task. Although all but 6% of the students constructed correct examples, their methods indicate different levels of understanding.

Table 3: Constructing 3x3 matrix  $A$  with linearly dependent columns

Method	Examples	Frequency of occurrence
Guess-and-Check method	$A = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 5 & 1 \\ 3 & 6 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 2 \\ 2 & 5 & 1 \\ 3 & 6 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 2 \\ 2 & 5 & 1 \\ 3 & 6 & 0 \end{bmatrix}$	17%
Rows method: same rows, one row multiple of another row	$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 1 \\ 2 & 5 & 1 \end{bmatrix}; \text{ or } A = \begin{bmatrix} 4 & 6 & 9 \\ 8 & -7 & -5 \\ 2 & 3 & 9/2 \end{bmatrix}$	23%

<p>Echelon method: start with echelon form <math>U</math> of <math>A</math></p>	$U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R3 \rightarrow R3 + R2 + R1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ $\xrightarrow[\begin{matrix} R1 \rightarrow R1 + R3 \\ R2 \rightarrow R2 + R3 \end{matrix}]{ } \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix} = A$	5%
<p>Identical columns method:  <math>[a_1 \ a_1 \ a_1]</math> where <math>a_1</math> has nonzero real entries</p>	$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}; \text{ or } A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$	8%
<p>Multiple columns method: two columns are multiples of the first one - <math>[a_1 \ ca_1 \ da_1]</math> where <math>a_1</math> has nonzero real entries and <math>c</math> and <math>d</math> are both nonzero real numbers</p>	$A = [a_1 \ 3a_1 \ 9a_1] = \begin{bmatrix} 1 & 3 & 9 \\ 1 & 3 & 9 \\ 1 & 3 & 9 \end{bmatrix};$ $A = [a_1 \ 2a_1 \ 3a_1] = \begin{bmatrix} 3 & 6 & 9 \\ 2 & 4 & 6 \\ 4 & 8 & 12 \end{bmatrix}$	19%
<p>Two multiple columns method: two identical columns or two columns multiples of each other and the third column having any nonzero real entries:  <math>[a_1 \ ca_1 \ a_3]</math></p>	$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 4 \end{bmatrix}; \text{ or } A = [a_1 \ 2a_1 \ a_3] = \begin{bmatrix} 2 & 4 & 5 \\ 3 & 6 & 6 \\ 4 & 8 & 1 \end{bmatrix}$	21%
<p>Linear combination method: any two columns, <math>a_1</math> and <math>a_2</math>, having nonzero real entries and <math>a_3 = ca_1 + da_2</math></p>	$A = [a_1 \ a_2 \ a_1 + a_2] = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 4 & 6 \\ 3 & 5 & 8 \end{bmatrix};$ $A = [a_1 \ a_2 \ 2a_1 + (-1/2)a_2] = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -2 & 3 \\ 1/2 & -1 & 3/2 \end{bmatrix}$	18%

The responses to the remaining parts of the task depended on the construction of a matrix  $A$ . The results and analysis of these remaining parts are presented with examples of students' work in Sections 6.2.2 – 6.2.5.

### 6.3.2 Linear dependence as action

For students using a guess-and-check strategy the linear dependence was concluded as an outcome of an action performed on a chosen  $3 \times 3$  matrix  $A$ . To complete Task 1(a1), these students had to pick 9 numbers to perform a set of operations on these numbers getting a certain result, in this case, at least one zero row in the modified form of  $A$ . This is a consequence of the condition for the columns of a matrix to be linearly dependent. These students had to go through calculations explicitly to verify that their example satisfied the requirement of the task.

Many students changed an arbitrary number of entries in a matrix  $A$  to construct a matrix  $B$  with linearly independent columns. 17% of the students changed one entry or two entries in the same row when the columns of  $A$  span a line. This approach always leads to an incorrect answer. For example, Lucy changed two entries in the same row –

going from  $A = \begin{bmatrix} 5 & 10 & 15 \\ 10 & 20 & 30 \\ 20 & 40 & 60 \end{bmatrix}$  to  $B = \begin{bmatrix} 5 & 10 & 15 \\ 10 & 20 & 30 \\ 20 & 39 & 10 \end{bmatrix}$  with the explanation 'I changed one

number in each column that was a multiple of another column except for the first column.' In this case, the entries should be changed in different rows and different columns. Some students incorrectly claimed linear independence since none of the vectors was a multiple of another. They did not check whether the columns were still linearly dependent.

Leon made the same mistake as Lucy during the interview when working on Task 1b. He changed two entries in the same row when the entries should have been changed

in different rows. His matrix  $A$  was  $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 4 & 8 & 12 \end{bmatrix}$ . When prompted for justification of his

claim, Leon checked his result by row reducing the matrix  $B$ . He wasn't satisfied with the result of that approach. For him, it was possible that the vectors were not linearly independent, but it wasn't sufficient evidence.

**Leon:** You see that all three of these are linearly dependent on each other, all multiples of one another, and possibly just change the last entries of each of the vectors to make them no longer multiples of one another. (changes the last entries in the first and second columns of  $A$  to 5 and 9,

respectively; writes:  $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 5 & 9 & 12 \end{bmatrix}$ ).

**I:** If you change two entries in the same row, will the resulting vectors be linearly independent?

**Leon:** ... They may or may not be.

**I:** Will the three vectors in your new matrix: (1,2,5), (2,4,9) and (3,6,12) be linearly independent?

**Leon:** Maybe not necessarily. The method I can think of checking it is by row reducing it. (row

reduces the matrix to get:  $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 5 & 9 & 12 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ ). Yes, they are not necessarily linearly

independent in this way.

To be positive that the resulting vectors were linearly independent using the linear combination method, Leon chose large numbers and different positions in a matrix  $A$  and generated another matrix  $B$ . He then was confident that none of the vectors could be a linear combination of the other two. Leon was applying the property of a linearly dependent set of vectors. He further supported his argument geometrically. Though, he didn't register that the second time he changed the entries in different positions and that

was the reason for vectors to become linearly independent, not the magnitude of the entries.

**Leon:** For instance, let's say by changing a number here to a really large number, 80 (changes  $a_{22}$  from 4 to 80 in the original matrix  $A$  above). Then the first and the third shouldn't come together to form this (i.e. the corresponding entries in the first and third columns will not add up to 80) [...] if another number was very very large, let's say, in the bottom here ( $a_{31}$ ) and this was 90, if then these two (the third and the first column vectors) would come together to form the second vector, and yet they'd have to form a smaller number even though the combination of the first and the third are much larger. I don't think it would be possible. This should be a linearly independent matrix.

**I:** How can you be sure that the vectors  $(1,2,90)$ ,  $(2,80,8)$  and  $(3,6,12)$  are linearly independent?

**Leon:** One way to do is to row reduce, another way for me would be to put it in  $\mathbf{R}^3$  space and see if such relationship exists... So, I'll do that.  $(1, 2, 90)$  is way up here (draws on the diagram the three vectors), and even though my sketch isn't accurate, I can be pretty sure that they are not multiples of one another, and they are not parallel.

The answers to part (a2) revealed students' misconceptions about linearly dependent sets of more than two vectors. One of the theorems that students encounter says that a set of two vectors is linearly dependent if and only if at least one of the vectors is a multiple of another. However, students internalize the connection: linear dependence  $\leftrightarrow$  vectors are multiples of each other, but not the assumption about the number of vectors. As a result, they misgeneralize and use this property to determine if a set of more than two vectors is linearly independent. Namely, the fact that none of the vectors is a multiple of any other was misgeneralized to show that the vectors are linearly independent. For justification of linear independence of the columns of  $B$ , a number of students used the above property. Since this property is not a sufficient condition, some students gave incorrect examples of matrices  $B$ . For example, one student claimed that

the columns of the matrix  $B = \begin{bmatrix} 0 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$  are linearly independent because  $\mathbf{b}_2$  and  $\mathbf{b}_3$  are



not multiples of  $\mathbf{b}_1$ , and another claimed that  $B = \begin{bmatrix} 7 & 4 & 1 \\ 4 & 2 & 1 \\ 16 & 8 & 4 \end{bmatrix}$  has linearly independent

columns because the columns are not multiples of each other.

On the whole, several types of responses can be identified from participants' solutions to Task 1 (Linear (in)dependence). Some students constructed matrix  $A$  correctly but failed to perform the next step either changing an arbitrary number of entries in  $A$ , or generating matrix  $B$  with linearly dependent columns. These students are most likely operating with the action conception of linear dependence.

### 6.3.3 Linear dependence as process

Applying the APOS theoretical framework, students are operating with the process conception of linear dependence when they construct a matrix  $A$  emphasizing the relations between the rows. They may know that in order for the columns of  $A$  to be linearly dependent an echelon form of a matrix has to have a zero row. The row reduction process is an intended action in this case. It is performed mentally, and then reversed to generate a required matrix.

23% of the students constructed their examples with one row being a multiple of another as was indicated in some responses: 'I made one row multiple of another', or 'take R3 to be 2xR1'. This is a possible way to get a zero row in an echelon form since one of the elementary row operations involves replacing a row with a sum of that row and a multiple of another row.

Joan used the Rows method (see Table 3) in her construction of matrix  $A$ . She gave an example of a matrix with 2 rows being multiples of the first row and proceeded

immediately to reduce this matrix to echelon form and represent the variables of a homogeneous equation. After showing that she got 2 free variables, she offered an example of a matrix so that  $A\mathbf{x} = \mathbf{0}$  has one free variable. However, in her explanations she confused vectors with variables, and linear dependence of vectors with representing a variable in terms of other variables. She couldn't interpret the span of the columns geometrically and was always going back to free variables: '... if I reduce the matrix of the 3 equations, then I get a matrix and I have to get a free variable and because it's one of the criteria for the vectors to be linearly dependent.'

**Joan:** So 3, 5, 2 (starts writing the first row), and 6, 10, 4, and 1, 5/3 and 2/3. (writes:  $\begin{bmatrix} 3 & 5 & 2 \\ 6 & 10 & 4 \\ 1 & 5/3 & 2/3 \end{bmatrix}$ ) and this would simplify to (writes:  $\begin{bmatrix} 3 & 5 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ ) where  $x_1$  is  $-5x_2 + (-2x_3)$ ;  $x_2$  is  $x_2$

which is a free variable;  $x_3$  is linearly dependent on  $x_2$ ... Then if  $x_2$  is  $t$ , and  $x_3$  is  $s$ , then  $x_1$  is  $-5t - 2s$ ;  $x_1$  is linearly dependent on  $x_2$  and  $x_3$ ... And if you want two vectors that dependent on one variable, you'd have to have something else here (in the third row) so new matrix is

(writes:  $\begin{bmatrix} 3 & 5 & 2 \\ 6 & 10 & 4 \\ 3 & 4 & 3 \end{bmatrix}$ ). So when I row reduce, I get (writes:  $\begin{bmatrix} 3 & 5 & 2 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ ).

There is an intermediate step that links the linear dependence of columns of a matrix and its echelon form having a zero row. The definition of linear dependence of a set of vectors is given in terms of a solution to a vector equation. That is, a set of vectors  $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  is linearly dependent if the vector equation,  $c_1\mathbf{v}_1 + \dots + c_n\mathbf{v}_n = \mathbf{0}$  has a nontrivial solution. The solution set of this vector equation corresponds to the solution set of a matrix equation  $A\mathbf{x} = \mathbf{0}$  having the  $\mathbf{v}_i$ 's as columns which in turn corresponds to the solution set of the system of linear equations whose augmented matrix is  $[A \ \mathbf{0}]$ . In the prior instruction it was shown that the linear system  $A\mathbf{x} = \mathbf{0}$  has a nontrivial solution if it has free variables, and this can be inferred from an echelon form of  $A$ . Thus, some students formed the connection: linear dependence  $\leftrightarrow$  free variables  $\leftrightarrow$  zero row in

echelon form. As a result some examples were justified with the following statements: ‘a linearly dependent matrix is a matrix with free variables’, ‘columns of  $A$  are linearly dependent since  $x_3$  is free variable which implies  $A\mathbf{x} = \mathbf{0}$  has not only trivial solution’, or ‘when the forms are reduced into reduced echelon form, the linearly dependent matrix has a free variable  $x_3$ ; however, the linearly independent doesn’t – it has a unique solution’.

Anna used the Echelon method to construct her correct example. As her first example she generated a matrix with the zero row, and then changed the zero entries to keep the column vectors identical. She referred to the rows of the matrix for justification.

**Anna:** So, if it’s a 3 by 3 matrix, and linearly dependent, then I’m gonna have free variables in my matrix. Because, when you have linearly independent, it means it’s a pivot column. So it can be

(writes:  $\begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ ).

**I:** How can you modify your matrix to eliminate the zeros?

**Anna:** Maybe replace the 0 with 2. (writes:  $\begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 2 & 2 & 2 \end{bmatrix}$ ). It’s still gonna be linearly dependent,

no matter what it is. There always are some scalar times the multiple of the first one (row). Yeah, I’ll get the second row...

The reference to free variables and echelon forms is observed in responses and explanations for constructing the matrix  $B$  by changing as few entries in matrix  $A$  as possible. 16% of the students modified the link: linear dependence  $\leftrightarrow$  free variables to complete Task 1 (a2). Consider the following response. The student correctly changed  $A$

$= \begin{bmatrix} 2 & 4 & 5 \\ 3 & 6 & 6 \\ 4 & 8 & 1 \end{bmatrix}$  to  $B = \begin{bmatrix} 2 & 4 & 3 \\ 3 & 6 & 6 \\ 4 & 9 & 1 \end{bmatrix}$  claiming that, in matrix  $A$ ,  $\mathbf{a}_2$  is a multiple of  $\mathbf{a}_1$  which

produces a free variable: ‘by definition of linear independence, solution can’t have a free

variable – can't have nontrivial solution'. In this case, students used the link: linear independence  $\leftrightarrow$  no free variables. Since in a linear system corresponding to a 3x3 matrix there are three variables, and each one is either free or basic, students only need to change as many entries in  $A$  as there are free variables. These students constructed one of the connections intended by the task:

number of free variables  $\leftrightarrow$  number of entries needed to be changed,

as can be seen in the explanation ' $A$  has 2 free variables, then changing 2 values of  $A$  will make  $B$  have no free variables.' However, only 3% of the students extended this connection to complete Task 1(b). Many used the same linear dependence relation between the columns of  $A$ , but changed a different number of entries, incorrectly in this case, to get  $B$ .

6 students referred to the number of pivot positions in a matrix  $A$  when

constructing a matrix  $B$ . For example, starting with  $A = \begin{bmatrix} 2 & -1 & -1 \\ 2 & 2 & 1 \\ 4 & 4 & 2 \end{bmatrix}$ , one student

correctly observed that the last row of  $A$  is a multiple of the second row and has no pivot, but incorrectly concluded that changing the last entry of the last row of  $A$  from 2 to 1 would make it a pivot row: 'in order to make it a pivot row the last entry in this row should be nonmultiplication of the second row, that is why I changed that entry'. When the columns of  $A$  were the same, i.e.  $A = [a_1 \ a_1 \ a_1]$ , to construct  $B$  another student correctly changed two entries, 'because if you change only one, it is still going to be linearly dependent – there should be three basic variables'. These students connected the linear independence of columns to a homogeneous linear system having three basic

variables and therefore three pivot positions: linear independence  $\leftrightarrow$  pivot positions / basic variables.

Similarly, Anna approached this part of Task 1 (Linear (in)dependence) in terms of the pivot positions. First she wanted to change 3 entries in her original matrix  $A$

$= \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 2 & 2 & 2 \end{bmatrix}$  so that the pivot positions would be clearly visible in a matrix  $B$ .

**Anna:** I am gonna make  $-1$  here ( $a_{21}$ ) to be  $0$ . Because automatically when I set this to be  $0$ , then I have a leading entry right here ( $a_{11}$ ). But if I wanna do the minimum, then I wanna do one more [...] I can set these two ( $a_{31}$  and  $a_{32}$ ) to be  $0$ . So it's linearly independent.

She then realized that one of the changes was unnecessary and corrected her mistake. However, Anna wasn't confident in her result and had to use row reduction to check that her matrix  $B$  has linearly independent columns.

**Anna:** I can just set this ( $a_{32}$ ) to be  $0$ , and this ( $a_{31}$ ) could be a  $2$  still (writes:  $\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 2 & 0 & 2 \end{bmatrix}$ ).

because when I row reduce... No, that's not good. I'll just set it ( $a_{31}$ ) to  $0$ , just to be safe... Oh, actually it will work out. So I can replace this  $-1$  here ( $a_{21}$  in original matrix  $A$ ) with  $0$ , and this ( $a_{32}$ ) with  $0$ , and this ( $a_{31}$ ) can stay a  $2$ .

Anna didn't connect the dependence relation to the number of entries she had to change. She was concerned whether the row reduction process gives the right echelon form. When she thought that her calculations wouldn't produce the desired outcome she wanted to go back to her original intention of changing 3 entries: '... no, that's not good. I'll just set it ( $a_{31}$ ) to  $0$ , just to be safe.'

An explicit use of the link: linear independence  $\leftrightarrow$  pivot positions / basic variables can be observed in the following response. One student found that the matrix  $A$

in the first part of the task is row equivalent to  $\begin{bmatrix} 1 & 3 & 2 \\ 0 & -3 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ ; then she assigned a non-zero value to  $a_{33}$  in this echelon form of  $A$ ,  $a_{33} = 1$ , to get an echelon form of  $B$ , and used reversed elementary row operations to get  $B$ . 'If this,  $a_{33}$ , is nonzero, then the augmented matrix has only trivial solution, and column  $\mathbf{a}_3$  is not a linear combination of  $\mathbf{a}_1$  and  $\mathbf{a}_2$ , so the system is linearly independent.' This student is making every column in an echelon form of  $A$  a pivot column. Since elementary row operations preserve the linear dependence relations, and because the elementary row operations were reversed, the resulting matrix  $B$  has linearly independent columns and differs from the original matrix  $A$  in only one entry.

Leon was using the same strategy to generate matrix  $B$ . He reduced his matrix  $A =$

$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 6 & 7 \end{bmatrix}$  to an echelon form  $U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 5 \end{bmatrix}$  to see what entries he should change.

Leon then correctly changed  $a_{22}$  from 4 to 3, so that a new echelon form  $U' = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 5 \end{bmatrix}$

corresponded to a possible echelon form of a matrix with linearly independent columns.

**Leon:** So then what I would do is to try and row reduce this matrix, and see what values would make this matrix consistent, from reduced echelon form. (reduces the matrix to get  $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 5 \end{bmatrix}$ ).

And then I would look and see what I can change here... If I make this ( $a_{22}$ ) 3, then I'll get 1 ( $a_{22}$ ) instead of 0 in echelon form. And then this would be a linearly independent matrix.

Only 50% of the students completed both parts of the task, with 63% of incorrect responses to Task 1(b). In the majority of incorrect responses students ignored the

different structures of linear dependence relations between vectors. They either used the same matrix  $A$  in both parts of the task or a matrix  $A$  having the same linear dependence relations between columns. Then if the students changed the correct number of entries in Task 1(a), their response to Task 1(b) was incorrect. For instance, one student constructed matrix  $A$  for both parts with the same dependence relation between columns,  $\mathbf{a}_3$  in  $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2\}$ , and changed 1 entry in the first part but 3 entries in the second part. Similarly, another student used the same construction: ‘columns of  $A$  are linearly dependent, so  $\mathbf{a}_3 = c\mathbf{a}_1 + d\mathbf{a}_2$  with  $c$  and  $d$  scalars’, but then used the same vectors  $\mathbf{a}_1$  and  $\mathbf{a}_2$  and different scalars for parts (a) and (b) of the task. Even though this student indicated one of the general construction methods for 3 linearly dependent vectors, he didn’t perceive the purpose of the task.

Some students used the same matrix in both parts of the task, but changed a different number of entries in each part, changing the correct number of entries in part

(a), but making more changes than needed in part (b). For example,  $A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 2 & 4 \\ 1 & 1 & 2 \end{bmatrix}$ ,  $B =$

$\begin{bmatrix} 1 & 3 & 4 \\ 2 & 2 & 4 \\ 1 & 1 & 3 \end{bmatrix}$  in part (a), and  $B = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 2 & 3 \\ 1 & 1 & 3 \end{bmatrix}$  in part (b): ‘this time I changed 2 entries of  $A$

to make  $B$ .’

Interviews revealed the analogous tendency to generate a matrix  $A$  for part (b) with the same linear dependence relations. Anna first offered a matrix with identical columns again and changed an incorrect number of entries. After being reminded that she had to change as few entries as possible, she knew that to change ‘two would be just

enough'. Anna then tried a second strategy and said that changing entries in other positions would affect the minimum number of entries required to change. But she had not yet linked this requirement to the linear dependence relations:

**I:** If you compare your two matrices, what is similar about them?

**Anna:** The columns are multiples of each other, they are the same. My  $v_1, v_2, v_3$  are all the same.

Her third attempt used vectors that were multiples of each other, and so also failed:

**I:** What is a different way to construct 3 linearly dependent vectors in  $\mathbf{R}^3$ ?

**Anna:** They don't necessarily have to be the same, the vectors. So, I could have (writes:  $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ )

and then this one can be multiple of the first one, so it could be (writes:  $\begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}$ ). Or I could have a

completely different one. I could have (writes:  $\begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 2 & 4 & 6 \end{bmatrix}$ ).

Anna as well as many other students correctly completed Task 1(a), but used the same matrix  $A$  or a matrix  $A$  having the same linear dependence relations between columns for part (b). She then changed a different number of entries of  $A$  from part (a), but in this case this number wasn't minimal. In both cases, Anna and the students that made the same mistake as her didn't relate the number of entries needed to be changed with the type of dependence relation between columns of  $A$ . They were able to check the mechanics of the task but didn't make a connection between different parts of it. These students may be understanding linear dependence as a process. They don't think of linear



dependence globally, considering different structures of linear dependence relations, but revert to specific vectors.

#### 6.3.4 Linear dependence as object

The row reduction process is central to linear algebra. It is an essential tool, an algorithm that allows students to compute concrete solutions to elementary linear algebra problems. However, encapsulation of linear dependence as an object requires a movement beyond the outcome of actual or intended procedures of row reduction toward a conceptual understanding of the structure of linear dependence relations in a set of vectors.

An indication of the construction of linear dependence as an object is demonstrated when students emphasize the relation between vectors, when they use the linear combination method to construct their examples of three linearly dependent vectors. In the linear combination method, there could be recognized different levels of generality for constructing an example. Either students gave a specific example of a matrix with a linear dependence relations between columns easily identified, as can be seen in Table 3, or they identified a general strategy for constructing a class of 3x3 matrices with linearly dependent columns. For example, Amy wrote: ‘to be linearly dependent, at least one of the columns of a matrix  $A$  has to be a linear combination of the others ...  $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{0}$  with weights not all zero. Pick  $\mathbf{a}_1$  and  $\mathbf{a}_2$ . Then for  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$  to be linearly independent,  $\mathbf{a}_3$  has to be a linear combination of  $\mathbf{a}_1$  and  $\mathbf{a}_2$ . So, let  $\mathbf{a}_3 = \mathbf{a}_1 + \mathbf{a}_2$ ’. In the latter case, students applied the property that if  $\mathbf{u}$  and  $\mathbf{v}$  are linearly independent vectors in  $\mathbf{R}^n$ , then the set of three vectors  $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$  is linearly dependent if and only if  $\mathbf{w}$  is in  $\text{Span}\{\mathbf{u}, \mathbf{v}\}$  (i.e.  $\mathbf{w}$  is a linear combination of  $\mathbf{u}$  and  $\mathbf{v}$ ).

Sarah generated her matrix with the Linear combination method making the third

column the sum of the first two:  $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 1 & 2 & 3 \end{bmatrix}$ . Leon used the Two multiple columns

method for his construction. He explained his reasoning by pointing out the linear dependence relations between columns.

**Leon:** Right away if I put two vectors that are linearly dependent to start,  $(1, 2, 4)$  and  $(2, 4, 8)$ , and  $\mathbf{v}_3$  that is not dependent on the first two,  $(3, 5, 7)$ . Then I can simply put:  $\mathbf{v}_2 = 2\mathbf{v}_1 + 0\mathbf{v}_3$ . And I think this will suffice.

There were some representational errors in the responses as well. Carrie

constructed  $A = \begin{bmatrix} 1 & 4 & -5 \\ 2 & 1 & -3 \\ 1 & 4 & -5 \end{bmatrix}$  of the form  $[\mathbf{a}_1 \ \mathbf{a}_2 \ -\mathbf{a}_1 - \mathbf{a}_2]$  and justified that the columns

were linearly dependent because  $\mathbf{a}_3$  was a linear combination of  $\mathbf{a}_1$  and  $\mathbf{a}_2$ :  $a_1 \begin{bmatrix} \phantom{0} \\ \phantom{0} \\ \phantom{0} \end{bmatrix} +$

$a_2 \begin{bmatrix} \phantom{0} \\ \phantom{0} \\ \phantom{0} \end{bmatrix} = a_3 \begin{bmatrix} \phantom{0} \\ \phantom{0} \\ \phantom{0} \end{bmatrix}$ . The first part of the response is correct, however, the last claim indicates

the gap in symbolic representation of the statement. The confusion between the weights/coordinates and vectors may create an obstacle in applying these concepts in other situations.

Several students that correctly produced a matrix  $B$  from matrix  $A$  explicitly referenced the linear dependence relations in their explanations. For example, one student expressed the second column of a matrix  $A$  as a linear combination  $\mathbf{a}_2 = 2\mathbf{a}_1 + 0\mathbf{a}_3$  and, by changing an entry in the second column, concluded that in a matrix  $B$  the columns 'can

not be put in a linear combination except  $0\mathbf{a}_1 + 0\mathbf{a}_2 + 0\mathbf{a}_3 = \mathbf{0}$ , while another student wrote that 'since  $\mathbf{a}_3 = \mathbf{a}_1 + \mathbf{a}_2$ , it is in  $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2\}$ , so  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$  are linearly dependent. However,  $\mathbf{b}_3$  is not in  $\text{Span}\{\mathbf{b}_1, \mathbf{b}_2\}$ , there is only the trivial solution for  $B\mathbf{x} = \mathbf{0}$ , so  $\mathbf{b}_1, \mathbf{b}_2,$

$\mathbf{b}_3$  are linearly independent' where  $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & 7 \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 4 & 8 \end{bmatrix}$ . These students

are using the definition of linear dependence in reference to a solution of a homogeneous linear system. They are referring to the defining properties of linear dependence which may indicate that these students are treating the concept as object.

Sarah changed correctly one entry,  $a_{13}$ , in her matrix  $A = [\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_1+\mathbf{a}_2]$  to construct a matrix  $B$ , claiming that the columns of  $B$  formed a linearly independent set, explaining that 'because it's not the multiple again, and ... other two, any two that adds together with scalar that times with each vector, column vector cannot make the third, cannot make the other ... any two of the column space in the matrix...adding them together and timesing them with any scalar cannot make the sum of the third in the matrix'. It appears she was trying to explain that now it is not possible to form the third vector as a linear combination of the other two. However, when she attempted to justify this symbolically, she used different scalars in the linear combination  $\mathbf{a}_3 = c\mathbf{a}_1+d\mathbf{a}_2$ , and couldn't show this in general.

Sarah identified the span of the columns of her matrix  $A = [\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_1+\mathbf{a}_2]$  correctly, but initially wasn't confident in her statement: '... I can try to guess, but I don't think I'll do it right though...it will be just a plane'. However, her reasoning for this answer illustrates she wasn't guessing but understood the concepts she was dealing with.

**I:** A plane through which points?

**Sarah:** This is tricky. Through the first two columns, through the first two points, because the third one is dependent on the other two.

**I:** So you just need the first two vectors to figure out what the span is of these three vectors?

**Sarah:** It's more like the dependence factor, it doesn't necessarily have to be the last one, but it's not the linear dependence column vector. Because the span of the first two will include the dependence of the last vector.

During clinical interviews, students were able to move from the process understanding of linear dependence to the object level. Initially, both Anna and Leon used matrix  $A$  with the same linear dependence relation between columns to complete Task 1(b). They were changing different number of entries but knew that some changes were unnecessary. To allow Anna to move forward with the task the interviewer suggested she looked at the geometric interpretation of the column space of a matrix.

**I:** In all of these cases, the column span of your matrix is a line in  $\mathbf{R}^3$ . What is the other possible geometric interpretation of the column space of a 3 by 3 matrix in  $\mathbf{R}^3$  (with the prescribed properties)?

**Anna:** Is it just a flat plane? That's like all possible vectors in that span, I mean, in that space. Like all the possible combinations, linear combinations that I can have of these vectors.

**I:** What is an example of a 3 by 3 matrix whose column space is a plane?

**Anna:** It will be a linearly independent set.

It seems Anna knew the alternative geometric interpretation but she thought it was formed by a linearly independent set of 3 vectors. She struggled with this question: '... with 3 by 3 matrix, I don't know, I can't see it as a plane. I should, I don't know why... It will be the span of these three vectors that make a plane. But that doesn't make any sense'. Finally, Anna was able to complete the task when she started with two linearly independent vectors and found the third vector to make a set linearly dependent:

**Anna:** Let's say I have  $\mathbf{e}_1$  and  $\mathbf{e}_2$  again as my two vectors. If I wanna add the third vector to make it linearly dependent, I would add, for example,  $2\mathbf{e}_1$ . And this  $2\mathbf{e}_1$ , it's just gonna look like  $(2, 0, 0)$ , 2 times the vector  $\mathbf{e}_1$ , which is  $(1, 0, 0)$ .

**I:** What would be the span of these three vectors,  $\{\mathbf{e}_1, \mathbf{e}_2, 2\mathbf{e}_1\}$ ?

**Anna:** It's linearly dependent... A plane.

She then asserted that to make this set linearly independent only one change is necessary:

**Anna:** Just change this one ( $a_{33}$ , changes 0 to 1), because you want them (the vectors) to be different from each other. So I will need to change something in  $2\mathbf{e}_1$  to make it different from the other two. So if I write it out (writes:  $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ ), and I want to make it linearly independent; all I have to do is add a 2 or, probably 1 (in  $a_{33}$ ) is the best way to go (to make a change).

Leon used the same matrix in his first attempt at solving Task 1(b). He drew a diagram this time to demonstrate his reasoning. He still changed one entry and, after his error was pointed out, Leon suggested that he can change two entries. In his second attempt to generate an example for this part, Leon's reasoning shows that he had a clear idea of how to approach the problem: '... what I am really trying to do right now is find a relationship that will relate two vectors to one and still be linearly dependent after the first change and then make it linearly independent... what I was originally thinking is that I could change one of the entries and still make it linearly dependent in another way, and then change an entry to make it linearly independent'. He finally constructed a matrix

with 3 columns being multiples of each other:  $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 4 & 8 & 12 \end{bmatrix}$ .

Working through this task helped students understand the connection between the linear dependence relations, the geometric interpretation, and the minimum number of entries needed to change:

**Leon:** ...actually, I think to make two changes is minimum, because all three vectors are linearly dependent to one another. Changing one will not change the relationship overall. You will still have at least two linearly dependent vectors. So can I draw from that with three linearly dependent vectors you need two changes and with only two linearly dependent vectors you only need one change.

Anna even attempted to generalize her strategy for an  $nxn$  case:

**Anna:** First, if my vectors are the same it's going to take more than one step to make them linearly independent. But if two of the vectors are different and the last one is the same as one of the other ones, I just need to change the leading entry number in that matrix; so when I row reduce it, I have an identity matrix...If I have an  $n$  by  $n$  matrix, and I have  $\{v_1, \dots, v_{n-1}\}$  and then I have  $2v_1$ . This one vector is twice  $v_1$ , or three times  $v_1$ , just to keep it general, as my  $v_n$ . So if I make  $\{v_1, \dots, v_{n-1}\}$  linearly independent, and the very last one is  $cv_1$  then I need to change only one entry.

Object conception of linear dependence relation includes mastery of all possible characterizations of a linearly dependent set of vectors, in particular, the ability to recognize the possible ways to alter a set in order to obtain a linearly independent set. In Task 1 (Linear (in)dependence), encapsulation of linear dependence as an object includes viewing a matrix as a set of column vectors, not as discrete entries that have certain values after performing algebraic manipulations. The latter perspective inhibits students' geometric interpretation of the span of columns, because the structure of linear dependence relations is not visible. Thus, the students that correctly completed both parts of the task might be operating with the object conception of linear dependence. An interesting note is that in the majority of the correct responses the students constructed matrix  $A$  of rank 2 in Task 1(a), and of rank 1 in Task 1(b).

### 6.3.5 Geometric interpretation of span

Many students had problems with the geometric interpretation part of Task 1 (Linear (in)dependence). It appears that for these students, linear dependence and span are just algebraic manipulations of symbols. To interpret the span of the columns of  $A$  geometrically students used a visual representation and/or a verbal description. 14% of the students used a visual representation of the span. However, 88% of these students didn't correctly represent linear dependence relations. Nonetheless, of these, 64% still

correctly identified the geometric object. For example, a geometric representation of a matrix  $A$  with two identical columns, that is,  $A = [a_1 \ a_2 \ a_3=a_2]$ , is reproduced in Fig. 1:

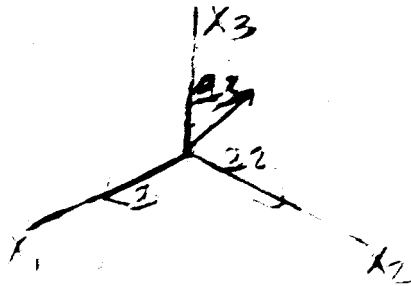


Figure 1: Geometric representation of the column span of  $A = [a_1 \ a_2 \ a_2]$

This diagram does not correctly represent the linear dependence relations between the columns of  $A$ . Moreover, only three vectors are indicated, but not the actual span of the vectors. One student drew three vectors with  $a_2 = a_1 + a_3$  in  $x_1x_3$ -plane when  $A = [a_1 \ a_1 \ a_2]$ . These students did not form a mental model of linear dependence and span. As another example, to express the geometric interpretation for  $A = [a_1 \ 2a_1 \ 3a_1]$  the student drew a plane in  $\mathbb{R}^3$  with 3 vectors not related to each other, as in Fig. 2.

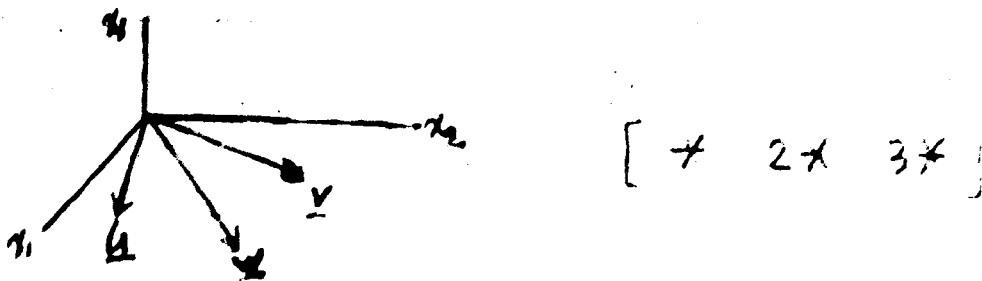


Figure 2: Geometric representation of the column span of  $A = [a_1 \ 2a_1 \ 3a_1]$

In all of these examples the span of the columns of  $A$  is disconnected from the actual vectors. Also in the latter example the object that these vectors span is misrepresented: vectors lying on a plane and not on a line through the origin.

Several other misconceptions could be noted in relation to the geometric interpretation of the span of vectors. One of the misconceptions comes from using the

row vectors in an echelon form of a matrix to represent the span of the columns of  $A$ . For

instance, one student correctly constructed matrix  $A = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \\ 5 & 7 & 9 \end{bmatrix}$ , showed that it is row

equivalent to  $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ , and then used the rows of this echelon form of  $A$  to give a

geometric interpretation of the span of the columns of  $A$ . It seems the student confused the column space of  $A$  with the row space of an echelon form of  $A$ . He might have realized that to construct the span of the columns, he had to consider vectors in a matrix, but wasn't certain what vectors he had to look at for this task.

The same confusion is noted in Anna's initial response to a request for a geometric interpretation of the column space of  $A$ .

**I:** How can you represent the span of the column of the original matrix geometrically,

$$\left( \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 2 & 2 & 2 \end{bmatrix} \right)?$$

**Anna:** So  $(1, 1, 1)$  will be here;  $(-1, -1, -1)$  here, and  $(2, 2, 2)$ ... So this is  $\mathbf{v}_1$ , this is  $\mathbf{v}_3$ , and this is  $\mathbf{v}_2$ ....

She started working with row vectors, but then remembered that in her matrix all vectors were the same and identified the span as a straight line.

Another student started with a correct justification of why the span of the columns of  $A$  is not  $\mathbf{R}^3$ , however, it was poorly phrased: 'the span of the columns of  $A$  have only two pivot positions and therefore only two pivot columns of three columns (making it linearly dependent) so  $A$  will not span  $\mathbf{R}^3$ '. But the student didn't carry through this



explanation and instead switched to interpreting the solution set of  $A\mathbf{x} = \mathbf{0}$ , ' $A$  has one free variable making it a line through the origin.'

Leon first claimed the column space of his matrix  $A$  cannot exist '...there cannot be any... the space would not be  $\mathbf{R}^3$  by linearly dependent set.' After it was suggested that the column space of  $A$  may be a subspace of  $\mathbf{R}^3$  he stated that it would be  $\mathbf{R}^2$  then, and explained why  $\mathbf{R}^2$  is a subspace of  $\mathbf{R}^3$ .

**Leon:** It would be a subspace of  $\mathbf{R}^3$ ,  $\mathbf{R}^2$ .

**I:** Is  $\mathbf{R}^2$  a subspace of  $\mathbf{R}^3$ ?

**Leon:** Yes.  $\mathbf{R}^2$  has... there is  $x$ , there is  $y$  and there is  $0$ , then it would be a subspace of  $\mathbf{R}^3$ . Or at least in  $\mathbf{R}^3$ ... Not, actually  $(x, y)$  which is in  $\mathbf{R}^2$ .

It appears that some students incorrectly transferred the statement that the linear system  $A\mathbf{x} = \mathbf{0}$  has infinitely many solutions to describe the span of the columns of  $A$  as being an infinite number of planes. For example, one student stated that for the columns of  $A$  to be linearly dependent there must be at least one free variable, and constructed the third row,  $R_3$ , of  $A$  to be  $2xR_1$ . He then claimed that  $x_3$ , free variable, can be any point so the span of columns of  $A$  'would be infinite planes'.

Several students claimed the same geometric interpretation of the span of the columns of  $A$  in Task 1(a) and 1(b), even though the echelon forms of  $A$  and the number of entries changed were different: one in the first case and two in the second. These students didn't connect the structure of the linear dependence relation to the object spanned by the vectors. They might have had difficulty understanding the dimension of a vector space later in the course.

A common confusion appeared in students' responses of the span of the columns of  $A$  with the solution set of  $A\mathbf{x} = \mathbf{0}$ . Instead of providing a geometric interpretation of the

span of the columns of  $A$ , some students gave a geometric interpretation of the solution set of the homogeneous system  $A\mathbf{x} = \mathbf{0}$ . The nature of the above confusion is explored in Section 6.3.

#### 6.4 Column space / Null space

Recasting some of the analysis and conclusions of Task 1 in terms of column space and null space of a matrix, we saw previously that students commonly confused the span of the columns of a matrix  $A$  (namely  $\text{Col } A$ ) with the solution set of the homogeneous equation  $A\mathbf{x} = \mathbf{0}$  (namely  $\text{Nul } A$ ). Having been exposed to systems of linear equations, when students see the words ‘matrix’ and ‘span’ in the same problem, many of the students may rush into solving the homogeneous system  $A\mathbf{x} = \mathbf{0}$  without comprehending what the question asks. Particular words in the problem trigger a certain reaction in some students without reflection on the meaning. These students develop only symbolic meaning of the words and this becomes an obstacle. Some examples of such confusion in students’ responses to Task 1 are: ‘The span of the columns of  $A$  will geometrically be a line through the origin since the equation  $A\mathbf{x} = \mathbf{0}$  has only 1 free variable  $x_3$ ’, in this case, the span of the columns of  $A$  is a plane, or similarly, ‘ $x_1$  and  $x_2$  are dependent on  $x_3$ ; this is a line through the origin’.

It seems that for students such as these, the procedure of solving systems of linear equations, and especially the expression of solutions of a homogeneous linear system in parametric form after row reducing a coefficient matrix or an augmented matrix  $[A \ \mathbf{0}]$ , has become a routine exercise so that the mechanics creates a barrier in answering the question about the column span of  $A$ . Up to that point in the course students were exposed to questions of whether or not columns of an  $m \times n$  matrix  $A$  span  $\mathbf{R}^m$ , but were

not required to find the span of the columns of  $A$  in the case of a negative answer. So many students reverted to procedures when incorrectly giving the geometric interpretation of a solution set to  $Ax = \mathbf{0}$  instead of the span of the columns of  $A$ .

### Task 2: Column space / Null space

Find an example of a matrix  $A$  with real entries for which  $\text{Nul } A$  and  $\text{Col } A$  have at least one nonzero vector in common. For this matrix  $A$ , find all vectors common to  $\text{Nul } A$  and  $\text{Col } A$ . If  $T$  is the linear transformation whose standard matrix is  $A$ , determine the kernel and range of  $T$ .

Task 2 was designed to further investigate the students' understanding of the fundamental subspaces  $\text{Col } A$  and  $\text{Nul } A$ , and how they can be related. This task calls for generating a non-example, an example that tests the boundaries of a concept. The summary of responses to this task is presented in Table 4.

Table 4: Constructing matrix  $A$  with  $\mathbf{v}$  in  $\text{Nul } A \cap \text{Col } A$ ,  $\mathbf{v} \neq \mathbf{0}$

Type of Response	Example	Frequency
Invertible matrix (necessarily incorrect)	$A = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	12%
Singular matrix with $\text{Nul } A \cap \text{Col } A = \{\mathbf{0}\}$	$A = \begin{bmatrix} 1 & 5 \\ 2 & 10 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$	23%
Any other specific singular matrix (necessarily correct)	$A = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$ or $\begin{bmatrix} 3 & -1 \\ 9 & -3 \end{bmatrix}$	48%
Semi-general construction	<p><math>\text{Nul } A: Ax = \mathbf{0}</math>; <math>\text{Col } A</math>: can be any column in <math>A</math>. Pick one column and determine the other column.</p> <p>Let <math>\mathbf{x}</math> equal to first column. <math>\begin{bmatrix} 2 &amp; a \\ 1 &amp; b \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}</math>;</p> <p><math>a = -4</math> and <math>b = -2</math> and <math>A = \begin{bmatrix} 2 &amp; -4 \\ 1 &amp; -2 \end{bmatrix}</math>.</p>	11%
General construction	$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ; $A = \begin{bmatrix} 1 & -1/2 \\ 2 & -1 \end{bmatrix}$ .	6%

#### 6.4.1 Nul $A$ / Col $A$ as action

Students that constructed an invertible matrix as an example may be operating with an action conception of the vector spaces associated with a matrix  $A$ . They need to work with specific numbers and perform calculations explicitly to find the solution set of a homogeneous equation  $A\mathbf{x} = \mathbf{0}$ , and to compute the column space of  $A$ . As a result, computational mistakes lead them to incorrect conclusions.

12 % of the students constructed an invertible matrix as an example for this task such as the 2x2 identity matrix  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  or the 2x2 invertible matrix  $\begin{bmatrix} 3 & 6 \\ 6 & 2 \end{bmatrix}$ . This may show students' lack of making the connection between the existence of a nonzero vector in Nul  $A$  and a matrix  $A$  not being invertible, even though it is one of the implications of the invertible matrix theorem.

#### 6.4.2 Nul $A$ / Col $A$ as process

The process conception of column space and null space may entail realizing that one has to work with a singular matrix since its null space contains a nonzero vector. The action of solving a matrix equation is performed mentally to satisfy one of the requirements of the task. However, an individual has to compute the null space to show that null space and column space share at least one nonzero vector, which doesn't follow from the construction of an example. Students acting at a process level of understanding are unlikely to connect the vector spaces Col  $A$  and Nul  $A$ .

In this task 88% of the students chose a matrix from the set of 2x2 singular matrices to satisfy constraints of the task. 48% of the students provided a specific example for Task 2 (Null space / Column space). However, the example space for these

students was very limited. The examples are summarized in Table 5. The scalar multiples of representative matrices are included in the same group. As can be seen from the table, 95% of the matrices come from a set of 6 groups of matrices, which can be identified as reference examples (Michener, 1979; see also Chapter 3).

Table 5: Examples of singular matrices with  $\mathbf{v}$  in  $\text{Nul } A \cap \text{Col } A$ ,  $\mathbf{v} \neq \mathbf{0}$

Example	Frequency
$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}; 2A$	26%
$A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}; 2A; 5A$	30%
$A = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$	6%
$A = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}; 2A; 3A$	10%
$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; 2A; -1A$	16%
$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$	7%
Misc	5%

Some examples of students' correct explanations include: ' $A = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$  where

$\text{Col } A$  and  $\text{Nul } A$  contain  $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$  ... Therefore,  $\begin{bmatrix} -x \\ x \end{bmatrix}$  where  $x =$  all real #s will be all

vectors common to  $\text{Nul } A$  and  $\text{Col } A$ '; ' $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$  ... All vectors common to both

subspaces can be expressed as  $\mathbf{w} = \begin{bmatrix} -a \\ a \end{bmatrix}$  where  $a$  is all real numbers'; or ' $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$  ...

$\text{Col } A = \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix}\right\}$ , but  $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$  is a linear combination of  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ , so it does not add to the spanning set.  $\text{Col } A = \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ .  $\text{Nul } A$  and  $\text{Col } A$  are equal and both have infinitely many common vectors.’

Not every singular matrix can serve as an example for Task 2 (Column space / Null space). Some students accounted for the requirement of Task 2 regarding null space, but ignored the intersection condition. These students may not link the null space to the column space, and may be operating with the process conception of two vector spaces. 23% of the students generated incorrect examples. One of the reasons that led to incorrect examples such as  $A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$  was overgeneralization: ‘any 2x2 matrix with one free variable will do’.

For students that generated incorrect examples, the computational and interpretational errors prevented them from realizing their mistake. Several incorrect responses included matrices  $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$  or  $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ . In both cases, students incorrectly claimed that  $\text{Nul } A$  and  $\text{Col } A$  have the vector  $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$  in common. However, in the first case,  $\mathbf{v}$  is an element of  $\text{Col } A$  but not of  $\text{Nul } A$ , while, in the second case,  $\mathbf{v}$  is an element of  $\text{Nul } A$  but not of  $\text{Col } A$ . Some of the possible explanations included finding a nonzero vector in one of the two vector spaces,  $\text{Nul } A$  and  $\text{Col } A$ , and concluding that this vector belongs to both; or computational mistakes that prevented students from reaching the

correct conclusion. Starting from the incorrect example,  $A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ , Tina attempted to show that  $\mathbf{v} \in \text{Col } A$  by row reducing the augmented matrix  $[A \ \mathbf{v}]$ . However, she failed to apply the same row reduction operations to the vector  $\mathbf{v}$  as to the matrix  $A$  resulting the matrix  $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ . At this point her combined errors led her to the conclusion that the vector  $\mathbf{v}$  is in  $\text{Col } A$  since the rightmost column is not a pivot column. Gary did get a pivot in the rightmost column of the augmented matrix  $\begin{bmatrix} 2 & 3 & -3/2 \\ 4 & 6 & 1 \end{bmatrix}$ , however, he still claimed that vector  $\begin{bmatrix} -3/2 \\ 1 \end{bmatrix}$  is common to both vector spaces. For both solutions students were using an algorithm for verifying that a vector lies in the column space of a matrix. Lack of interpretation of the results created an obstacle for these students.

Representational mistakes are very common in linear algebra. Even in the correct responses to this task, many students represented  $\text{Col } A$  as one vector or a set of two vectors, for example,  $\text{Col } A = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$  or  $\text{Col } A = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ,  $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$ , or the null space of  $A$  as a matrix, for example,  $\text{Nul } A = \left\{ \begin{bmatrix} 2 & -1 \\ 4 & 2 \end{bmatrix} \right\}$ . These students treated  $\text{Col } A$  as a set of columns of  $A$  rather than a set of all linear combinations of the columns of  $A$ . Showing that vectors lie in both  $\text{Col } A$  and  $\text{Nul } A$ , some students checked in turn whether each column  $\mathbf{x}$  of a matrix  $A$  satisfied the homogeneous equation  $A\mathbf{x} = \mathbf{0}$ . However, then they failed to find all vectors common to these two vector spaces. A common mistake was that

students indicated only 2 vectors in  $\text{Nul } A$  and  $\text{Col } A$ , for example,  $\text{Nul } A = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \text{Col } A$ . Similarly, students said that  $\text{Nul } A = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$  and  $\text{Col } A = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\}$  so they had only the vector  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$  in common. This may indicate that the students interpreted the column space of a matrix as the set not the span of column vectors, even though the distinction between a set of vectors and their span was emphasized in class and several problems in the textbook that were assigned to students. Many students failed to recall and apply this distinction. Perhaps they found it trivial and so did not attach any importance to it.

Regardless of the computational and representational mistakes, students that incorporated the requirement that the null space and column space of a matrix have at least one nonzero vector and limited their potential example space to the set of singular matrices may be working with the concepts of  $\text{Nul } A$  and  $\text{Col } A$  as a process. They internalized the procedures for finding these two vector spaces but were still unable to connect them.

### 6.4.3 $\text{Nul } A / \text{Col } A$ as object

An indication that students have encapsulated  $\text{Nul } A$  and  $\text{Col } A$  as objects is demonstrated when the students are able to analyze a new situation and recognize how and why to apply the properties of  $\text{Nul } A$  and  $\text{Col } A$ . In this task, an individual might recognize that  $\text{Nul } A$  having at least one nonzero vector implies  $A$  is singular. Further, the analysis of  $\text{Col } A$  having an element in common with  $\text{Nul } A$  should lead to construction of



a matrix such that one of its columns,  $\mathbf{a}_i$ , also satisfies the matrix equation  $A\mathbf{x} = \mathbf{0}$ , so that,  $A\mathbf{a}_i = \mathbf{0}$ .

17% of the students indicated a general construction method for this task. All conditions of the task were incorporated in their responses. Several strategies could be identified within these responses.

Some students used the echelon form of a 2x2 singular nonzero matrix  $A$ . In this case, the last row of  $A$  has to be zero: ' $A = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ , then  $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \mathbf{0}$  with condition that  $x \neq 0$ . Then  $a + b = x$  and  $ax + b0 = 0$ ; so  $a = 0$  (because if  $x = 0$  we have the trivial solution) and  $b = x$ . Then setting  $x = 1$ , the matrix  $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$  satisfies the given constraints.' This is a subcase of the general method of Table 4.

Some students picked one specific column of a 2x2 matrix and set it to be a solution to the homogeneous equation  $A\mathbf{x} = \mathbf{0}$ . This determines uniquely the entries in the other column. Another strategy is the generalization of the previous approach. Students started with a general 2x2 matrix  $A$ , and set the first column of  $A$  to be a vector in  $\text{Nul } A$ :

'Let  $A$  be 2x2 matrix; let  $\mathbf{v}$  be a vector in  $\text{Nul } A$  and  $\text{Col } A$ . Let  $\mathbf{a}_1 = \mathbf{v}$ . Let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

and  $\mathbf{v} = \begin{bmatrix} a \\ c \end{bmatrix}$ . Then  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$  and  $\begin{matrix} a^2 + bc = 0 \\ ca + cd = 0 \end{matrix}$ ; e.g.  $A = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}$ ,

Not all general constructions led to correct results. For example, Peter started with a general 2x2 matrix  $A$  and a vector  $\mathbf{v}$  in  $\mathbf{R}^2$ : 'Let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  and  $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ . Then  $\text{Nul } A$

$= A\mathbf{x} = \mathbf{0}$ ;  $\text{Col } A = s \begin{bmatrix} a \\ c \end{bmatrix} + t \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ . Then  $ax_1 + bx_2 = 0$ ; if  $a = 1$  and  $x_1 = -1$ , then  $bx_2$

$= 1$  and  $b = 1$  and  $x_2 = 1$ . Then  $cx_1 + dx_2 = 0$ ;  $-c + d = 0$ ;  $c = d = 1$ .' This leads to the

incorrect matrix  $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ . However, he still claimed that  $\text{Nul } A$  and  $\text{Col } A$  have the

vector  $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$  in common. Even though Peter started with a general matrix and vector and

included all the requirements of the task in the set-up, he didn't ensure that the resulting matrix is valid. A source of his error is choosing arbitrary values for  $a$  and  $x_1$ .

Students that used the general construction methods of Table 4 included all the conditions of Task 2 (Column space / Null space) in their example-generation. There is strong evidence that they understand the concepts of  $\text{Nul } A$  and  $\text{Col } A$  as object.

## 6.5 Linear transformations

Linear transformation is another central concept in linear algebra. To determine students' understanding of this concept, they were presented with several linear transformations tasks. Students' examples provide significant data on their understanding of linear transformation. The responses are very diverse and are interpreted using a combination of both frameworks: APOS and concept image / concept definition.

### ***Task 3: Linear Transformations***

*Let  $T$  be a linear transformation from a vector space  $V$  to a vector space  $W$  and let  $\mathbf{u}, \mathbf{v}$  be vectors in  $V$ . State whether the following is true or false, giving either a proof or a counter-example: if  $\mathbf{u}$  and  $\mathbf{v}$  are linearly independent then  $T(\mathbf{u})$  and  $T(\mathbf{v})$  are linearly independent.*

**Task 5: Linear Transformations (revisited)**

(a). Give an example of a linear transformation.

(b). Give an example of a linear transformation from  $\mathbf{R}^2$  to  $\mathbf{R}^2$  that maps the vector  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$  to the vector  $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$ .

(c). Give an example of a linear transformation from  $\mathbf{R}^3$  to  $\mathbf{R}^2$  that maps the vector  $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$  to the vector  $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ .

(d). Give an example of a non-linear transformation.

In Task 3 students were required to construct a counter-example for a mathematical statement. However, they first had to identify whether the statement is true or false. The summary of students' responses is shown in Table 6. Both correct and incorrect responses were included.

### 6.5.1 Concept Image of a linear transformation

Task 5(a) asks students to give an example of a linear transformation. Students were free to choose the representation and vector spaces to describe their examples. To construct an example of a linear transformation, students either started by recalling the definition and then constructed their examples to fit the definition, or they first came up with an example and then used the properties of a linear transformation to justify that their transformations were indeed linear. Nonetheless, students that used the second approach had difficulty with the verification part. For these students the link linear transformation  $\leftrightarrow$  defining properties is a part of their concept image, but they had difficulty applying the definition to the problem. The definition did not appear to carry much meaning for the students. They were still operating at the level of symbolic

manipulations. The students that provided examples of a matrix transformation had incorporated the link linear transformation  $\leftrightarrow$  matrix transformation in their concept image. These links are explored in detail below.

Table 6: Summary of students' responses to Task 3 (Linear transformations)

Type of Response	Example	Frequency
Statement is true	Since $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, so the transformation is one-to-one. Therefore, the transformation is linearly independent, so the statement is true	22%
Incorrect Examples	<p>(1) <math>\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}</math>, <math>\mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}</math>, <math>T = \begin{bmatrix} 1 &amp; 0 \\ 0 &amp; 1 \end{bmatrix}</math>.</p> <p><math>T(\mathbf{u}) = A\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}</math>, <math>T(\mathbf{v}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}</math>. Thus this is linearly dependent;</p> <p>(2) <math>\mathbf{u} = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}</math> and <math>\mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}</math> are linearly independent because they are not multiples of each other; and</p> <p><math>T(\mathbf{u}) = \begin{bmatrix} 3 &amp; 1 &amp; 5 \\ 2 &amp; 1 &amp; 6 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}</math> and <math>T(\mathbf{v}) = \begin{bmatrix} 2 &amp; 1 &amp; 2 \\ 3 &amp; 4 &amp; 7 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}</math>.</p> <p><math>T(\mathbf{u})</math> and <math>T(\mathbf{v})</math> are linearly dependent since the number of vectors in each matrix is greater than the number of entries in each vector.</p>	10%
Unit vectors / singular matrix	$\mathbf{u}$ and $\mathbf{v}$ are unit vectors in $\mathbf{R}^2$ and $A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$	21%
2 linearly independent vectors / singular matrix	$\mathbf{u} = \begin{bmatrix} 10 \\ 20 \\ 40 \end{bmatrix}$ , $\mathbf{v} = \begin{bmatrix} 10 \\ 50 \\ 40 \end{bmatrix}$ , $A = \begin{bmatrix} 1 & 0 & 4 \\ 2 & 0 & 5 \\ 3 & 0 & 6 \end{bmatrix}$ .	16%
Zero transformation	Let $T: \mathbf{R}^n \rightarrow \mathbf{R}^m$ be a linear transformation whose standard matrix $A$ is an $m \times n$ zero matrix. Then $T(\mathbf{u}) = A\mathbf{u} = \mathbf{0}$ and $T(\mathbf{v}) = A\mathbf{v} = \mathbf{0}$ , so $T(\mathbf{u})$ and $T(\mathbf{v})$ are linearly dependent even though $\mathbf{u}$ and $\mathbf{v}$ are linearly independent: $cT(\mathbf{u}) + dT(\mathbf{v}) = \mathbf{0}$ for all $c, d$ in $\mathbf{R}$ .	21%
Verbal / geometric / other representations	Projection in $\mathbf{R}^2$ ; If $T$ maps $\mathbf{u}$ to $\mathbf{0}$ , and $T$ maps $\mathbf{v}$ to $T(\mathbf{v})$ then $T(\mathbf{u})$ and $T(\mathbf{v})$ are not linearly independent.	10%

### 6.5.1.1 Linear transformations ↔ defining properties

For two students, Sarah and Leon, an immediate reaction to the request for an example of a linear transformation was to recall its definition:

**Sarah:** If it is a linear transformation, it means it must qualify that... (writes:  $cT(\mathbf{u})=T(c\mathbf{u})$  and  $T(\mathbf{u}+\mathbf{v})=T(\mathbf{u})+T(\mathbf{v})$ ) for any  $\mathbf{u}, \mathbf{v}$ . So if it's a linear transformation, it must satisfy that a scalar times that transformation equals to the scalar times a vector and then transformation. And then it has to satisfy that a transformation of the sum of two vectors must satisfy... equals to the transformation of the first vector plus the transformation of the second vector.

**Leon:** A linear transformation is required, has to satisfy two requirements...  $T(\mathbf{x}+\mathbf{y})$  has to equal  $T(\mathbf{x}) + T(\mathbf{y})$  and  $T(c\mathbf{x})$  equals  $cT(\mathbf{x})$ .

Only after recalling the definition did they start to look for examples satisfying the requirements:

**I:** Can you give me an example of a transformation that satisfies these properties?

**Sarah:** Let's say our transformation would be... from  $\mathbf{R}^2$  to  $\mathbf{R}^2$ . And then it would be just  $(x, y)$  to  $(y, x)$ : so changing the two... vector... you change the coordinates ... I don't know what it's called... Interchange it, so  $x$  becomes  $y$  and  $y$  becomes  $x$ , the value, not exactly the point.

**I:** Can you give me an example of a transformation that satisfies these properties?

**Leon:** ... Trying to satisfy these two requirements... I am thinking, for instance, just multiplication perhaps, where  $T(\mathbf{x}) = 2\mathbf{x}$  and  $T(2\mathbf{x}) = 4\mathbf{x}$ . So just trying to satisfy these properties. Then  $T(\mathbf{x}+\mathbf{y}) = 2(\mathbf{x}+\mathbf{y}) = 2\mathbf{x} + 2\mathbf{y} = T(\mathbf{x}) + T(\mathbf{y})$ . So I would say that the closest transformation I can think of is  $T(\mathbf{x})=2\mathbf{x}$ .

To present an example of a linear transformation one has to specify the vector spaces involved, namely, domain and co-domain of the linear transformation, and the action of that transformation. While Sarah correctly indicated the vector spaces,  $\mathbf{R}^2$  to  $\mathbf{R}^2$ , and defined the action of the transformation symbolically,  $(x, y)$  to  $(y, x)$ , her explanation showed the confusion between the vectors and the coordinates. The same mistake appeared in her solution to part (b) of Task 5 when she was trying to show that the non-linear transformation  $(x, y) \rightarrow (x+3, y+4)$  is linear.

**I:** Is this a linear transformation?

**Sarah:** I am hoping it is.  $T(x+y) = T(x) + T(y)$ , and this gives me  $T(x+y) = (x+3) + (y+4)$  and  $T(x) = (x+3)$  and  $T(y) = (y+4)$ ... As long as you can write it, as long as it satisfies the condition, then it will be a linear transformation.

**I:** In the equation:  $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ ,  $\mathbf{u}$  and  $\mathbf{v}$  are vectors, but  $x$  and  $y$  in  $T(x+y) = T(x) + T(y)$  are the coordinates of one vector.

**Sarah:** It looks like I am mixing my vectors and coordinates.

Sarah didn't have an appropriate language to support her reasoning. This was observed in students' responses to other tasks as well. Another point to notice is that she described the transformation on the coordinates of a vector, which is still correct, and not as an action on a vector space as a whole. In this case,  $T(x, y) = (y, x)$  is a linear transformation of  $\mathbf{R}^2$  to  $\mathbf{R}^2$ , which is a reflection through the line  $y = x$ . A similar response was given by Leon. He defined his transformation as an action on an element of a vector space, namely multiplication by a scalar. He then made sure the properties of a linear transformation were satisfied, checking the scalar multiplication property for a particular scalar 2:  $T(2\mathbf{x}) = 4\mathbf{x}$ . For both of these students, even though they referred to the properties of linear transformation, it doesn't seem that they understood what these properties meant and why they were required.

In addition, Leon didn't identify the vector spaces in his example for which the transformation was defined and was prompted by the interviewer to complete his example:

**I:** If your transformation is  $T(\mathbf{x}) = 2\mathbf{x}$ , what vector space does  $\mathbf{x}$  belong to and where would  $2\mathbf{x}$  be?

**Leon:** It would be from  $\mathbf{R}^1$  to  $\mathbf{R}^1$ . But if there is  $x$  and  $y$  coordinates then from  $\mathbf{R}^2$  to  $\mathbf{R}^2$ .

To justify that the transformations in their examples were linear, students proceeded to check the properties of the linear transformations. However Joan, as well as

Leon, concluded that the transformation she defined was linear because it satisfied just one of the properties:

**I:** Why is this transformation linear?

**Joan:** Because if I transform this using this property, the property of the linear transformation, and if I tried to determine if the transformation of vector  $\mathbf{v}$  and vector  $\mathbf{u}$  equals the transformation of vector  $\mathbf{u}$  plus the transformation of vector  $\mathbf{v}$  separately (writes:  $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u})+T(\mathbf{v})$ ), they would be equivalent. Then they would prove it's a linear transformation.

**I:** You showed that the transformation of  $\mathbf{u}+\mathbf{v}$  is the same as transformation of  $\mathbf{u}$  plus the transformation of  $\mathbf{v}$ . Is it enough to show that  $T$  is a linear transformation?

**Joan:** Yes.

Anna verified the above property of linear transformation for the two specific vectors and not in the general case. Only after questioning did she complete her verification of both properties of linear transformation for any two vectors in a vector space:

**I:** Why is this transformation linear?

**Anna:** It has to satisfy the property... If I take  $T(\mathbf{u}) = 2\mathbf{u}$  and  $T(\mathbf{v}) = 2\mathbf{v}$ ... Let's say (writes:  $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ ,  $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ .  $T(\mathbf{u} + \mathbf{v}) = T\left(\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 4 \\ -2 \end{bmatrix}$ ). And if I compute (writes:  $T(\mathbf{u}) = \begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix}$ ,  $T(\mathbf{v}) = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$ ), and when I add them together I get (writes:  $T(\mathbf{u}) + T(\mathbf{v}) = \begin{bmatrix} 4 \\ 4 \\ -2 \end{bmatrix}$ ). So they are equal.

**I:** Does one example prove the general statement?

**Anna:** You mean ... I cannot use the numbers... Then I think I can write  $T(\mathbf{u}+\mathbf{v}) = 2(\mathbf{u}+\mathbf{v})$ , and ... I said before  $T(\mathbf{u}) + T(\mathbf{v}) = 2\mathbf{u} + 2\mathbf{v} = 2(\mathbf{u}+\mathbf{v})$ ... Yes, they are still equal.

**I:** Is this sufficient condition for a transformation to be linear?

**Anna:** Yeah... Oh, wait. There is another property... Scalar multiplication? For some scalar  $c$  in  $\mathbf{R}$  I will have  $T(c\mathbf{u}) = 2(c\mathbf{u})$ ... and it has to be the same as  $cT(\mathbf{u}) = c(2\mathbf{u})$ ... Yes. They are the same.

The reference to the properties of a linear transformation was reiterated in students' written responses to Task 3 that asked for a counter-example to the statement. 22% of the students claimed incorrectly that the statement is true. Several types of



explanation can be noted in this case. Some students used the property of a one-to-one linear transformation incorrectly, by transferring the linear independence of the vectors  $\mathbf{u}$  and  $\mathbf{v}$  in the domain of  $T$  to the linear independence of the columns of the standard matrix of  $T$ : ‘since  $\mathbf{u}$  and  $\mathbf{v}$  are linearly independent, then  $A\mathbf{u} = \mathbf{0}$  and  $A\mathbf{v} = \mathbf{0}$  have only the trivial solution, and by definition,  $A\mathbf{u} = T(\mathbf{u})$  and  $A\mathbf{v} = T(\mathbf{v})$  and they are linearly independent’. Others referred to the IMT (invertible matrix theorem) in their justifications: ‘since  $\mathbf{u}$  and  $\mathbf{v}$  are linearly independent, by IMT  $T(\mathbf{v})$  and  $T(\mathbf{u})$  will be one-to-one transformations and it will be linearly independent.’ These responses show that students were using the right words but applying them to the wrong concepts. These students made no distinction between vectors, a linear transformation and a matrix. The IMT applies to a matrix of a linear transformation from  $\mathbf{R}^n$  to  $\mathbf{R}^m$  in the case  $m = n$ , and a linear transformation from  $\mathbf{R}^n$  to  $\mathbf{R}^m$  is one-to-one if and only if the column vectors of the matrix are linearly independent.

Some students tried to impose constraints on  $T$  to make the statement true rather than look for a counter-example. For example, several students incorrectly assumed that the linear transformation is invertible (which means it is also one-to-one, even though that was not stated explicitly in their solution), so the statement of the task is true, and the images of  $\mathbf{u}$  and  $\mathbf{v}$  are linearly independent: ‘ $c_1T(\mathbf{u}) + c_2T(\mathbf{v}) = \mathbf{0}$ ;  $T^{-1}(c_1T(\mathbf{u}) + c_2T(\mathbf{v})) = T^{-1}(c_1T(\mathbf{u})) + T^{-1}(c_2T(\mathbf{v})) = T^{-1}(\mathbf{0})$ ;  $c_1\mathbf{u} + c_2\mathbf{v} = \mathbf{0}$  and  $c_1 = c_2 = 0$  because  $\mathbf{u}$  and  $\mathbf{v}$  are linearly independent’.

Other students correctly inferred that since  $\mathbf{u}$  and  $\mathbf{v}$  are linearly independent then the vector equation  $c_1\mathbf{u} + c_2\mathbf{v} = \mathbf{0}$  implies that  $c_1 = c_2 = 0$ . Then they used the same symbols for the scalars in the equation  $c_1T(\mathbf{u}) + c_2T(\mathbf{v}) = \mathbf{0}$ , concluding incorrectly that

this equation also implies  $c_1 = c_2 = 0$  so that  $\{T(\mathbf{u}), T(\mathbf{v})\}$  is a linearly independent set. Yet others incorrectly started with the assumption that  $T(\mathbf{u})$  and  $T(\mathbf{v})$  are linearly independent and proved the correct conclusion that  $\mathbf{u}$  and  $\mathbf{v}$  are linearly independent, which is always true. Both situations may be due to the problems with the logical reasoning. It was indicated in the previous research that such problems are an important prerequisite for understanding linear algebra, and could be an obstacle in students' learning (Dorier et al, 2000).

It seems that in most of these responses, students were using the properties of a linear transformation without attaching meaning to them. They were manipulating the symbols to fit the requirements of the tasks.

#### 6.5.1.2 Linear transformations $\leftrightarrow$ matrix transformation

Two students represented their transformations as a matrix transformation:  $T(\mathbf{x}) = A\mathbf{x}$ . They first gave a general description of a linear transformation, specifying the vector spaces, and when the question was repeated, the second request gave a correct example of such a transformation:

**I:** Can you give an example of a linear transformation?

**Stan:** Sure. I have  $\mathbf{x}$  goes to  $A\mathbf{x}$  (writes:  $\mathbf{x} \rightarrow A\mathbf{x}$ ). For example, a transformation from  $\mathbf{R}^2$  to  $\mathbf{R}^2$  ( $T: \mathbf{R}^2 \rightarrow \mathbf{R}^2$ ).

**I:** What is an example of such transformation?

**Stan:** So, any value in  $\mathbf{R}^2$ , any vector maps (writes:  $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \leftarrow A\mathbf{x}$ ) to that ( $A\mathbf{x}$ ).

**I:** Can you give an example of a linear transformation?

**Nicole:** I start with  $\mathbf{R}^3$  and end up with  $\mathbf{R}^3$  and there is a matrix,  $3 \times 3$ .

**I:** What is an example of such transformation?

**Nicole:** It can be anything. Let's say (writes:  $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ ). And (writes:  $A \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ ) equals some vector in  $\mathbf{R}^3$ .

Another indication of the link linear transformation  $\leftrightarrow$  matrix transformation is seen in the construction of the examples of a nonlinear transformation. Leon tried to find a transformation that could not be represented as a matrix transformation. However, his example of scaling a vector by a factor of 2, even though not represented as a matrix multiplication explicitly, can still be written as  $T(\mathbf{x}) = A\mathbf{x}$  where  $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ :

**Leon:** OK, transformation that is not linear. So you want something that just doesn't meet the criteria... So according to the definition  $T(\mathbf{x}) = A\mathbf{x}$ ... So, for instance, we have a vector with two entries:  $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} c \\ d \end{bmatrix}$  and  $T(2a, 2b)$  should get to  $(2c, 2d)$ , (writes:  $T\left(\begin{bmatrix} 2a \\ 2b \end{bmatrix}\right) = \begin{bmatrix} 2c \\ 2d \end{bmatrix}$ ).

## 6.5.2 Linear transformations with APOS

### 6.5.2.1 Linear transformations as action

Students may be operating with the action conception of a linear transformation when they concentrate on the numbers/coordinates of a specific vector and not the properties of linear transformations. They perform an action by calculating the image of a vector under a linear transformation, or they recall this action for constructing their examples of a linear transformation. For instance, students offered an example of a linear transformation as an action on a specific vector:

**Joan:** Say from  $\mathbf{R}^2$  to  $\mathbf{R}^4$ ...so  $3x^3+x^2+5x-1$  would go to  $3x^5+x^4+5x^3-x^2$ ... This would be... from  $\mathbf{R}^3$  to  $\mathbf{R}^5$ .

**Anna:** It could be (writes:  $T \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix}$ ).

The same approach was used by Sarah for Task 5(b) when she was looking for an example of a linear transformation that maps  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$  to  $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$ . She found a transformation satisfying this particular condition but it wasn't linear.

**I:** Can you give an example of a linear transformation that sends the vector (0,1) to the vector (3,5)?

**Sarah:** I don't know if you can write it this way (writes:  $(x,y) \rightarrow (x+3,y+4)$ )? It satisfies only this condition: (0,1) goes to (3,5).

Sarah's example of a transformation is a translation. Every vector  $\mathbf{v}$  in the domain of  $T$  is translated by the vector  $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$  to a vector  $\mathbf{v} + \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ . In this case the zero vector is mapped to a vector  $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ , violating the property of a linear transformation requiring  $T(\mathbf{0}) = \mathbf{0}$ . In fact, non-trivial translations are a class of transformations that are nonlinear. Although Sarah's example fulfils the condition on mapping (0, 1) to (3, 5), she did not realize that it is an example of a non-linear transformation.

Considering Joan's response, not only did she construct her example for a specific vector but she also indicated the vector spaces incorrectly, apparently recalling an isomorphism of the space of polynomials  $\mathbf{P}_n$  with  $\mathbf{R}^{n+1}$  incorrectly. In fact this student wasn't comfortable with the task; she said she was never asked to write her own transformation. In the routine exercises students are asked to find an image of a vector given a linear transformation. It seems that they adopted this process in generating examples of a linear transformation acting on a specific vector.

While providing examples of a linear transformation with imposed constraints as in parts (b) and (c) of Task 5, some students concentrated on the coordinates of vectors

and not the properties of linear transformations. These students either guessed the answer or tried to solve the problem by trial and error, looking for numbers that would work.

After struggling with Task 5(b), Sarah was asked explicitly for a matrix to represent a linear transformation. As can be seen, she still didn't know how to approach construction of an example, and was guessing the position of the numbers to put in a matrix.

**I:** How else can you describe a linear transformation from  $\mathbf{R}^n$  to  $\mathbf{R}^m$ ?

**Sarah:** I don't know.

**I:** If you have a vector  $\mathbf{x}$  in  $\mathbf{R}^n$ , and you want to write your transformation as an action of some object on your vector, how can you write it?

**Sarah:** With something in  $\mathbf{R}^n$ ?

**I:** You are starting with vector in  $\mathbf{R}^n$  and you want to get a vector in  $\mathbf{R}^m$ .

**Sarah:** It's a vector space that has  $m$  vectors in it. You can take some matrix that has...you want  $m \times n$  matrix... you multiply it by  $n \times 1$  vector...so you get an  $m \times 1$  vector.

**I:** So if you describe your linear transformation as an action of a matrix on a vector, what does the dimension of your matrix have to be?

**Sarah:** It will be 2 by 2.

**I:** Then how can you find the matrix of a linear transformation that maps vector  $(0,1)$  to  $(3,5)$ ?

**Sarah:** I don't know... I am timesing something, I am timesing  $A$  with what,  $(0,1)$  to get  $(3,5)$ , so that (writes:  $[A] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ )... Something like (writes:  $\begin{bmatrix} 0 & 1 \\ 3 & 5 \end{bmatrix}$ )... No... I don't know how to justify it. Maybe (writes:  $\begin{bmatrix} 0 & 3 \\ 1 & 5 \end{bmatrix}$ ).

Even after other constraints were added to the task to limit the potential example space to one matrix, Sarah couldn't find a strategy to solve the problem. She continued guessing the matrix:

**I:** I'll add another condition. Can you give an example of a linear transformation from  $\mathbf{R}^2$  to  $\mathbf{R}^2$  that sends the vector  $(0,1)$  to the vector  $(3,5)$  and sends the vector  $(1,0)$  to the vector  $(-4,7)$ ?

**Sarah:** (writes:  $T \left( \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) \rightarrow \left( \begin{bmatrix} 3 & -4 \\ 5 & 7 \end{bmatrix} \right)$ ).  $(0,1)$ ,  $(1,0)$  goes to  $(3,5)$ ,  $(-4,7)$ . Yes, that would be my standard matrix.

**I:** If you write the transformation as  $T(\mathbf{x})=A\mathbf{x}$ , what is the matrix  $A$ ?

**Sarah:** It would be this one (writes:  $\begin{bmatrix} 3 & -4 \\ 5 & 7 \end{bmatrix}$ ).  $T$  of this guy (writes:  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ ) will be this guy (writes:  $\begin{bmatrix} 3 & -4 \\ 5 & 7 \end{bmatrix}$ ).

**I:** But your transformation is from  $\mathbf{R}^2$ . What would be the image of the vector  $(0,1)$  under this linear transformation?

**Sarah:** It will be  $(-4,7)$ . That's not right (laughs).

**I:** And if you apply your transformation to the vector  $(1,0)$ ...

**Sarah:** It will be  $(3,5)$ , so...which means the standard (matrix) would be (writes:  $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ).

**I:** So, what is your matrix  $A$ ?

**Sarah:** If I multiply it by  $(1,0)$ , I get  $(1,0)$ . No, I don't get  $(-4,7)$ ... That has nothing to do with this matrix... So matrix  $A$  would be... I'll do a wild guess... (writes:  $\begin{bmatrix} 3 & -4 \\ 5 & 7 \end{bmatrix}$ ).

Every linear transformation from  $\mathbf{R}^n$  to  $\mathbf{R}^m$  can be represented as a matrix transformation. The standard matrix of a linear transformation is uniquely determined by the action of the transformation on the standard basis of  $\mathbf{R}^n$ , namely the columns of the  $n \times n$  identity matrix,  $I_n$ . This property can be used to solve Task 3, and to construct infinitely many matrices in Task 5(b) and (c).

Students may be aware of this property but not many of them were able to apply it in the problem-solving situation. Anna didn't know how to describe a linear transformation. She restated the question: 'if I call  $(0,1,0)$   $\mathbf{u}$  and I call this,  $(3,1)$ ,  $\mathbf{v}$ , then I want to say that some transformation of  $\mathbf{u}$  will give me  $\mathbf{v}$ ', but was stuck after that. Even after being prompted to find a matrix of a linear transformation, she tried to recall the procedures she knew to see which one might work in this case.

**I:** How can you find a matrix of a linear transformation that will map the vector  $(0,1,0)$  to the vector  $(3,1)$ ?

**Anna:** It's just a vector, so I can't see how I can put it into a matrix. ...If I multiply it by a matrix, it has to be something by 3 to give me 2 by 1. If my matrix is something by 2 and I multiply it by  $\mathbf{u}$ , which is 3 by 1 to get a 2 by 1, it has to be a 2 by 3 matrix.

(writes:  $A \quad \mathbf{u} \quad = \quad \mathbf{v}$   
 $2 \times 3 \quad 3 \times 1 \quad \quad \quad 2 \times 1$ ).

... If I row reduce  $\mathbf{u}$  and  $\mathbf{v}$ , I should end up with  $A$ . No. It's not invertible, it's not going to be inverse of  $\mathbf{v}$  and  $\mathbf{u}$ . But, all I can think of is row reducing... But one of them is in  $\mathbf{R}^3$  and another one is in  $\mathbf{R}^2$ , so they can't be put in the same matrix... I know it has to be multiplied by some matrix, the vector  $(0,1,0)$  to give me  $(3,1)$ ... So I have to figure out what that matrix is.

From this excerpt it can be seen that Anna was familiar with the row reduction procedure as it is associated with a matrix, and she was prompted to use a matrix in some way: 'all I can think of is row reducing', but she didn't know where and when it can be applied. She couldn't put the vectors in the same matrix since they were elements of different vector spaces. Organizing vectors in a matrix is one of the first steps in solving linear algebra problems involving several vectors from the same vector space, for example to check if a set of vectors is linearly dependent. Then Anna realized that one of the vectors has to be multiplied by a matrix to get the second vector: '...it has to be multiplied by some matrix...' This suggests that some students first search for some known algorithm to apply right away rather than considering the statement of the question and its meaning.

In her search for a matrix, Anna was looking for specific numbers to be sure that matrix vector multiplication works out, without comprehending what the vectors and matrix represent. The multiplication process is detached from the action of a matrix on a vector, and from a linear transformation. However, Anna's first attempt at generating a matrix,  $\begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ , was incorrect: 'if I don't want the third row then it has to be all 0's'

After checking her matrix, Anna corrected her mistake:

**Anna:** Oh, no. It can't be all 0's, because I have a 1 in (3,1). One of the 0's is a 1. It has to be the middle one (changes  $a_{22}$  from 0 to 1). And this one ( $a_{11}$ ) has to be a 3. Oh, no. It has to be this one ( $a_{12}$ ). Because...(checks the matrix vector multiplication)... When I multiply my matrix  $\begin{pmatrix} 0 & 3 & 0 \\ 0 & 1 & 0 \end{pmatrix}$  with vector  $\mathbf{u}$ , I am going to end up with vector  $\mathbf{v}$ .

When faced with a request for another linear transformation that had the same property, namely transforming the vector  $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$  to the vector  $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ , Anna was surprised and didn't think it was possible.

**I:** Can you find an example of another matrix associated with a linear transformation that maps the vector (0,1,0) to the vector (3,1)?

**Anna:** Another matrix?... I was thinking it's a unique one, because you need to have a 3 and 1 to multiply with 1 in  $\mathbf{u}$  to give me (3,1).

The theorem stating that a linear transformation between  $\mathbf{R}^n$  and  $\mathbf{R}^m$  can be represented as a matrix transformation also states that matrix involved is unique and is determined by the action of the linear transformation on the standard basis of  $\mathbf{R}^n$ . It seems Anna was referring to the uniqueness part of this theorem, but ignoring the other conditions. Even though Anna was able to generate examples of other linear transformations, the relationship of the position of the image vector in a matrix to the fact that the vectors in the domain were unit vectors didn't guide her example-generation. She wasn't able to connect the theorem, part of which she recalled, to this task. She concentrated on the vectors not the properties of linear transformations. This suggests that Anna is operating with the action conception of a linear transformation.

The need to refer to specific numbers or symbol manipulation is seen again in students' counter-examples. Some students claimed that the statement presented in Task 3 is true based on a single example. For example, Tara constructed the linear



transformation:  $T \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a+b \\ a-b \end{bmatrix}$ , but showed only that the images of the unit vectors are

linearly independent:  $T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$  and  $T(\mathbf{e}_2) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ , while Kim defined  $T(\mathbf{x}) = A\mathbf{x}$  with

$A$  being an invertible matrix:  $T(\mathbf{x}) = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 2 & 3 \\ 5 & 3 & 1 \end{bmatrix} \mathbf{x}$ .

The students that claim the statement of Task 3 is true after imposing additional constraints on the linear transformation or making assumptions that are not specified in the given task, and students that used invertible matrices in their examples may have an action conception of a linear transformation. These students appeared unable to interpret the problem without computing values.

#### 6.5.2.2 Linear transformations as process

Students understand linear transformations as a process if they internalize the action of finding the image of a vector in  $\mathbf{R}^n$  under a matrix transformation as a linear combination of the columns of this matrix. They are able to mentally perform this procedure to find the appropriate entries in the matrix. However, at this stage, the defining property of the standard matrix of a linear transformation may not be understood by the students. For example, Nicole represented a linear transformation,  $T(\mathbf{x}) = A\mathbf{x}$  as

$[\mathbf{a}_1 \ \mathbf{a}_2]\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2$  and then found the vectors that solved this vector equation:

I: Give an example of a linear transformation that takes the vector (0,1) to the vector (3, 5) in  $\mathbf{R}^2$ ?

Nicole: ... I have  $A$  times  $(0, 1)$  has to equal  $(3,5)$ . So now I can do (writes:  $0 \begin{bmatrix} \phantom{0} \\ \phantom{0} \end{bmatrix} + 1 \begin{bmatrix} \phantom{0} \\ \phantom{0} \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ ).

Then  $A$  can be (writes:  $\begin{bmatrix} 3 & 3 \\ 4 & 5 \end{bmatrix}$ ).

It was common to take the unit vectors in  $\mathbf{R}^2$  or  $\mathbf{R}^3$  as the two linearly independent vectors in the domain of a linear transformation to construct a counter-example for Task 3. 21% of the students offered correct counter-examples using the unit vectors and appropriate matrices to represent a linear transformation. However, less than half of these students gave coherent explanations for their examples. In fact, some explanations revealed students' lack of understanding or pseudo-conceptual understanding (Vinner, 1997).

Vinner introduces the notion of pseudo-conceptual understanding, and further categorizes it into two types. One is when students don't understand the topic but want to appear as if they do. In this case, they would put down as many mathematical statements and as much terminology as they can remember in their response to a question, for example, to make it look as if they understand it. The other type of pseudo-conceptual understanding is when students think they understand the topic but in reality they don't. In this case, the answer may be partially correct. For instance, Nick started with a correct counter-example, defining  $T$  as a matrix transformation with  $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ . If he had stopped at this point, his answer would be considered correct. However, his subsequent remarks reveal misconceptions in Nick's understanding. Namely, he computed  $T(\mathbf{u}) = A\mathbf{u}$  as the 2x2 matrix  $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ , similarly, for  $A\mathbf{v}$ , and then calculated that  $A\mathbf{u} + A\mathbf{v} = A$ . The

conclusion that  $T(\mathbf{u})$  and  $T(\mathbf{v})$  were linearly dependent was based on the fact that the columns of the matrix  $A$  were linearly dependent. This is a correct conclusion, since the columns of  $A$  are the images of the unit vectors,  $\mathbf{u}$  and  $\mathbf{v}$ . But it is difficult to conclude that this was Nick's understanding as well.

An example of the second type of pseudo-conceptual understanding is seen in Lola's response. She defined the standard matrix of  $T$  to be  $A = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}$  with  $T(\mathbf{u}) = \begin{bmatrix} a \\ b \end{bmatrix}$  and  $T(\mathbf{v}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ , however, she never indicated what the vectors  $\mathbf{u}$  and  $\mathbf{v}$  were. Also, the first part of her explanation using the properties of a linear transformation:  $c_1\mathbf{u} + c_2\mathbf{v} = \mathbf{0}$  with  $c_1 = c_2 = 0$ , then  $c_1T(\mathbf{u}) + c_2T(\mathbf{v}) = \mathbf{0}$ , indicates that Lola may not know what the standard matrix of a linear transformation represents and how it is defined. Even though students were using unit vectors and the standard matrix of a linear transformation to generate a counter-example, they still proceeded to calculate  $T(\mathbf{u}) = A\mathbf{u}$  and  $T(\mathbf{v}) = A\mathbf{v}$ . These students didn't associate the columns of the standard matrix  $A$  with the images of the unit vectors.

In response to Task 3, 16% of the students selected specific linearly independent vectors in  $\mathbf{R}^2$  or  $\mathbf{R}^3$  and a singular matrix to represent a linear transformation. 45% of these responses offered linear transformations from  $\mathbf{R}^2$  to  $\mathbf{R}^2$ . For instance, Sam correctly stated that if  $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$  and  $T$  has the standard matrix  $\begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix}$ , then  $T(\mathbf{u}) = \mathbf{0}$  making the set  $\{T(\mathbf{u}), T(\mathbf{v})\}$  linearly dependent. 30% of counter-examples were of linear transformations from  $\mathbf{R}^3$  to  $\mathbf{R}^3$ . Some of the linear transformations were of the form  $T(\mathbf{x}) = A\mathbf{x}$  where  $A = [\mathbf{a}_1 \ \mathbf{0} \ \mathbf{a}_3]$  or  $A = [\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{0}]$ . In the former case,  $\mathbf{u}$  and  $\mathbf{v}$  can have the

first and third coordinates in common; in the latter case,  $\mathbf{u}$  and  $\mathbf{v}$  can have the first and second coordinates in common. As a result,  $\mathbf{u}$  and  $\mathbf{v}$  are mapped to the same vector. For

example,  $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & 0 \\ 3 & 4 & 0 \end{bmatrix}$  with  $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ , and  $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$ . This is a special case of the general

construction offered by one student with  $T(\mathbf{u}) = T(\mathbf{v}) = \mathbf{b}$  that leads to  $T(\mathbf{u})$  and  $T(\mathbf{v})$  being linearly dependent.

Several students either supported or represented their examples visually indicating that  $\mathbf{u}$  and  $\mathbf{v}$  were linearly independent vectors while  $T(\mathbf{u})$  and  $T(\mathbf{v})$  were linearly dependent. For example, one student correctly chose linearly independent vectors  $\mathbf{u} =$

$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$  and  $\mathbf{v} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$  with a linear transformation defined as  $\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} 2x \\ 0 \end{bmatrix}$ . Her explanation

included the geometric interpretation: the images of the vectors lie on the same line, so since the vectors are multiples of each other, they are linearly dependent.

Students using particular vectors and matrices in their examples may be operating with a linear transformation as a process. They internalized the mechanics of finding the images of vectors under the action of a linear transformation and looked for a combination of matrix / vectors that would give the desired outcome.

### 6.5.2.3 Linear transformations as object

Students may conceive of a mathematical notion as an object, when they talk about its properties (Hazzan and Zazkis, 2003). Thus, students may be operating with an object conception of a linear transformation when they emphasize the properties of a linear transformation in their solution. They do not perform any computations to

construct a matrix for their example of a linear transformation for Task 5. This approach may suggest that the students have encapsulated the concept of a matrix transformation as an object.

Stan and Leon emphasized the columns of a matrix in their example-generation in Task 5(b) and (c). They correctly placed the coordinates of the image vectors in the appropriate column of a matrix.

**I:** Can you find a linear transformation from  $\mathbf{R}^2$  to  $\mathbf{R}^2$  that maps (0,1) to (3,5)?

**Stan:** OK, I saw that the column, one of the columns of the matrix, the second column has to be (3,5); it's the image of the unit vector (0,1). And then I can choose any values to satisfy that property, so, that would be (writes:  $\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ ).

Stan immediately recognized that  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$  was one of the standard basis vectors of  $\mathbf{R}^2$ . Thus,

its image would be the second column of the required matrix. Leon, on the other hand, first determined the dimensions of a matrix that would satisfy the requirements of the question and then found a class of matrices that would work as examples for a linear transformation:

**I:** Can you give me an example of a linear transformation from  $\mathbf{R}^3$  to  $\mathbf{R}^2$  that sends the vector (writes:  $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ ) to the vector (writes:  $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ )?

**Leon:** So we are looking at the linear transformation matrix... So we are going from  $\mathbf{R}^3$  to  $\mathbf{R}^2$ , and then we know that the transformation of (writes:  $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ ) is to (writes:  $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ ). Then I would multiply... What would I multiply by to get a 2 by 1 vector? This is a 3 by 1 and I want it to become 2 by 1, so this (matrix) has to be 2 by 3... And afterwards, we have (writes:  $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ ), so the second column has to be (3,1) and the first and the third columns can be anything... So I will call the 2 entries here ( $a_{11}$  and  $a_{13}$ )  $a$  and  $b$ , where  $a$  and  $b$  can be any number, and then  $c$  and  $d$  here

( $a_{21}$  and  $a_{23}$ , writes:  $\begin{bmatrix} a & 3 & b \\ c & 1 & d \end{bmatrix}$ ). And this would be a possible linear transformation matrix and it should get you to (3,1).

The same strategy was used by the students to generate another example of a linear transformation with the given image of the one of the standard basis vectors. While Stan again offered a specific example, Leon presented a general form of a matrix that could serve as an example:

**I:** Give an example of a linear transformation from  $\mathbf{R}^3$  to  $\mathbf{R}^2$  that maps (0,1,0) to (3, 1).

**Stan:** This could be ... (writes:  $\begin{bmatrix} 1 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ ), so (1, 3) will be in the second column.

**I:** Can you give an example of a linear transformation again from  $\mathbf{R}^3$  to  $\mathbf{R}^2$  that sends the vector  $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$  to  $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ ?

**Leon:** ... Then I just call these  $a$ ,  $b$ ,  $c$ , and  $d$  [the first and second columns] and this has to be 1 and 4 (writes:  $\begin{bmatrix} a & b & 1 \\ c & d & 4 \end{bmatrix}$ ).

When students do not perform any computations to construct a matrix of a linear transformation but rely on the properties, it may indicate that they have encapsulated the concept of a matrix transformation as an object. Thus, both Stan and Leon may operate with an object conception of a linear transformation. With this approach they can construct other linear transformations with the given constraints as can be seen in their responses to the requests for another example. In addition, viewing the matrix of a transformation in terms of the actions on the standard basis vectors will help them later to work with a matrix of a linear transformation relative to any basis.

In response to Task 3, three students went beyond the requirements of the task and identified a general class of situations that refute the statement of the task. One can take any two linearly independent vectors  $\mathbf{u}$  and  $\mathbf{v}$  in the kernel of a linear transformation  $T: V \rightarrow W$ , or two linearly independent vectors with the same image. Then  $T(\mathbf{u}) = \mathbf{0}$  and  $T(\mathbf{v}) = \mathbf{0}$ , or  $T(\mathbf{u}) = \mathbf{b}$  and  $T(\mathbf{v}) = \mathbf{b}$ , respectively, so the set  $\{T(\mathbf{u}), T(\mathbf{v})\}$  is linearly dependent. The third student correctly stated that if  $T(\mathbf{u}) = \mathbf{0}$  and  $T(\mathbf{v})$  is any vector, the set  $\{T(\mathbf{u}), T(\mathbf{v})\}$  is again linearly dependent.

Projection is another class of linear transformations that can be used as a counter-example for Task 3. Alana represented one such projection geometrically; however, her representation of vectors in the domain was incorrect:

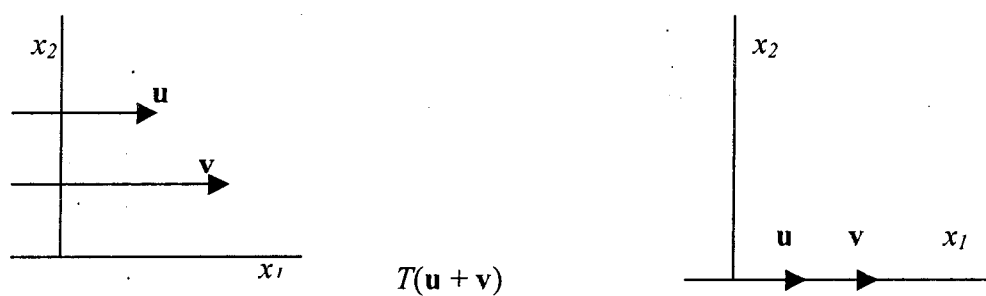


Figure 3: Counter-example for Task 3

Students that offered general strategies for constructing counter-examples for Task 3 or general scenarios that refute the statement may understand a linear transformation as an object. They were able to operate on a set of linear transformations without reference to individual elements.

## 6.6 Basis

The concept of a basis includes and is related to the concepts of linear independence, spanning set, dimension and subset of a vector space (Fig. 4).

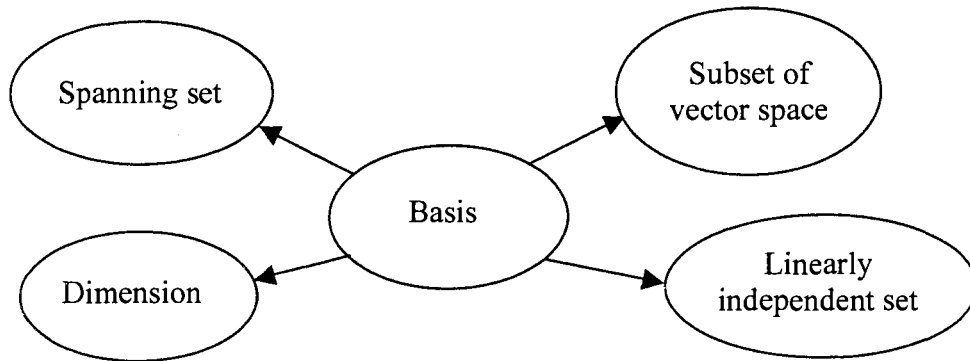


Figure 4: Concept map of a basis

**Task 4: Basis**

Let  $M_{2 \times 2}$  be the space of real-valued matrices. Let  $H$  be the subspace of  $M_{2 \times 2}$  consisting of all matrices of the form  $\begin{bmatrix} a & b \\ -b & c \end{bmatrix}$ , where  $a, b, c$  are real. Determine  $\dim H$ , give a basis for  $H$ , and expand it to a basis for  $M_{2 \times 2}$ .

In Task 4, the students were asked to give an example of a basis of a vector space, a subspace  $H$  of  $M_{2 \times 2}$ , consisting of all matrices of the form  $\begin{bmatrix} a & b \\ -b & c \end{bmatrix}$ , where  $a, b$ , and  $c$  are real numbers. To be able to construct an appropriate example for this task students need to possess all the above links in their concept image of a basis. However, the results indicate that for many students one or more connections either have not been formed yet or have not been invoked in this task. The summary of students' responses is presented in Table 7 below. The responses are categorized based on the number and type of connections identified in students' answers.



Table 7: Summary of students' responses to Task 4 (Basis)

Connections present	Example	Frequency
Dimension, Linear Independence	$\mathbf{v}_1 = \begin{bmatrix} a \\ -b \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} b \\ c \end{bmatrix}$ . It is possible for these two vectors to be linearly independent. So $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent and hence is a basis for $H$ . Thus, $\dim H = 2$ .	26%
Subset	$\begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}$ is a basis for $H$ ; $\dim H = 2$ since its columns are linearly independent.	23%
Linear Independence, Spanning Set, Dimension	By isomorphism with $\mathbf{R}^4$ , $\begin{bmatrix} a \\ b \\ -b \\ c \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$ 3 linearly independent vectors, so $\dim H = 3$ . A basis for $H$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ .	8%
Subset, Spanning Set, Dimension	$\begin{bmatrix} a & b \\ -b & c \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$ So basis for $H$ is $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ There are 3 vectors, $\dim H = 3$ .	10%
Subset, Dimension	Basis of $H = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ ; $\dim H = 3$ because there are 3 free variables	6%

All	$H = \left\{ a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$ $= \text{Span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$ $a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 0 \text{ only when } a = b = c = 0, \text{ so linearly independent. Basis for } H \text{ is } \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}. \text{Dim } H = 3.$	24%
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As can be seen in the table, 24% of responses illustrated all connections associated with the concept of a basis. Students represented an element of  $H$  as a linear combination of 3 matrices and showed that the set of these matrices is linearly independent and spans  $H$ . For example, a student correctly stated that  $\begin{bmatrix} a & b \\ -b & c \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ . The three matrices are linearly independent because they are not multiples of each other and not linear combination of the preceding ones, and  $H = \text{Span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ . As another explanation, some students used the definition of a linearly dependent set of vectors to show that the three vectors are linearly independent: 'the three matrices are linearly independent because for  $a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$  the weights  $a = b = c = 0$ '. Jeff considered the entries of vectors to justify linear independence: 'since each of the vectors does not have the nonzero entries in the corresponding position of the other two.' To expand the basis of  $H$

to a basis of  $M_{2 \times 2}$ , Nora first combined the above basis of  $H$  and the standard basis of  $M_{2 \times 2}$  to get a spanning set of  $M_{2 \times 2}$ ,  $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$ , and then removed the vector that was expressed as the linear combination of the others,  $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = (-1) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ , to get a linearly independent spanning set. In above responses, all students indicated that a basis has to be a subset of  $H$ , a spanning set, linearly independent, and the number of elements in the basis gives the dimension of a vector space.

26% of the students used the columns of a specific or general element in  $H$  to form a basis for  $H$ . For these students the connection basis  $\leftrightarrow$  subset may be absent. The first part of the concept definition of a basis of a vector (sub)space  $H$  says that ‘an indexed set of vectors  $\{v_1, \dots, v_n\}$  in  $V$  is a basis for  $H$  if...’ (Lay, 2003; p238), that is, the basis vectors are the elements of a vector space in the first place. The students, however, don’t treat this clause as part of the defining property of a basis. They focus on the second part of the definition that emphasizes the conditions that make this subset a basis, namely, linear independence and spanning set. Furthermore, many students still couldn’t apply these conditions correctly in the context of the vector space of matrices.

The vector space of matrices is a particularly problematic concept for students. It requires students to treat matrices as objects, as elements of a vector space. (As was seen in Section 6.2, for many students their concept image of a vector is restricted to  $\mathbf{R}^n$ , to  $n \times 1$  array of numbers). At the same time, each matrix has associated with it other vector spaces such as column space, row space, and null space. This creates an obstacle for

many students when working with the vector space of matrices as can be seen in the following response:

'dim  $H = 2$ . A basis for  $H \rightarrow \begin{bmatrix} 2 & 1 \\ -1 & 1/2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}$ . Expand to basis for  $M_{2 \times 2}$ ,

$2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -2 & 3/2 \end{bmatrix} \rightarrow \begin{bmatrix} 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3/2 \end{bmatrix}$ '. In fact, these students do not distinguish

between matrices and vectors in  $\mathbf{R}^n$ . As a result, the students combine the properties and definitions associated with the column space and null space of a matrix with the definitions referring to the general vector spaces. For example, linearly independent columns of a matrix  $A$  form a basis for the column space of  $A$ . This property was transferred to Task 4 (Basis) when students claimed that 'since columns of  $H$  are linearly independent and there are two of them,  $\dim H = 2$  and basis of  $H$  is 2 linearly

independent vectors in  $H$ , e.g. all the linear combinations of  $c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 5 \end{bmatrix}$  where  $c_1 =$

$c_2$ .' In the explanation there was an attempt to include the condition that the basis vectors form a spanning set for the vector space, but the student was not applying it correctly.

Some students that used the columns of a specific matrix in  $H$  as a basis realized that the vector space  $M_{2 \times 2}$  is bigger than  $H$ , so a basis for it has to include more elements. For

example, 'basis for  $H$  has at most 2 vectors. Assume basis for  $H = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$  is

linearly independent because the vectors are not multiples, then basis for  $M_{2 \times 2} =$

$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right\}$ '. However, even though they claimed that a basis for  $H$  had at most 2

elements because more vectors would result in linearly dependent set, they didn't check

that the basis for  $M_{2 \times 2}$  they provided was a linearly dependent set. Another student attempted to show that the expanded basis formed a linearly independent set: ‘a basis for

$H$  is  $B = \{\mathbf{b}_1, \mathbf{b}_2\}$  where  $\mathbf{b}_1 = \begin{bmatrix} a \\ -b \end{bmatrix}$  and  $\mathbf{b}_2 = \begin{bmatrix} b \\ c \end{bmatrix}$ , and a basis for  $M_{2 \times 2}$  is  $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$

where  $\mathbf{b}_3 = \begin{bmatrix} a \\ d \end{bmatrix}$  which is not a linear combination of the other two’.

8% of the students used an isomorphism with  $\mathbf{R}^4$  to represent a general matrix in

$H$ : a matrix  $\begin{bmatrix} a & b \\ -b & c \end{bmatrix}$  could be expressed as  $\begin{bmatrix} a \\ b \\ -b \\ c \end{bmatrix}$ , and then found a basis for a subspace

of  $\mathbf{R}^4$  isomorphic to  $H$  and expanded it to a basis of  $\mathbf{R}^4$ . However, the students didn’t convert the vectors in  $\mathbf{R}^4$  back to the elements of  $H$ . The answers were left as basis of  $H =$

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ . These students may be lacking the connection basis  $\leftrightarrow$  subset of a

vector space. A similar approach is seen in Beth response. She used an isomorphism between  $H$  and  $\mathbf{R}^n$ , but incorrectly used  $\mathbf{R}^3$  instead of  $\mathbf{R}^4$  collapsing two entries  $b$  and  $-b$

into one: ‘ $H = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a, b, c \in \mathbf{R} \right\} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ , so  $\dim H = 3$ . A basis for  $H$  and a

basis for  $M_{2 \times 2}$  are the same thing, the columns of  $I_3$ .’ As a result, she identified the dimension of  $H$  correctly, but didn’t complete the rest of the task. Several students used the number of variables in the general form of an element of  $H$  to justify their conclusion

about  $\dim H$  such as ‘Span  $\left\{ a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, b \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$  is a basis for  $H$ , there are 3 free variables and  $\dim H = 3$ ’.

To find an obvious basis for  $H$ , one can represent a general element in  $H$  as a linear combination of the matrices:  $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ ,  $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ , and  $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ . The following response indicates such an attempt. However, a student again collapsed the matrices into 2x1 arrays which led to the erroneous conclusion: ‘ $H = \left\{ \begin{bmatrix} a & b \\ -b & c \end{bmatrix} : a, b, c \in R \right\} \Rightarrow a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ’. Observing that  $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$  is a linear combination of  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ , he concluded that a basis for  $H$  consists of the standard unit vectors in  $\mathbf{R}^2$ . This student may understand how to construct a basis for a subspace. However, the difficulty of viewing a matrix as an element of a vector space and encapsulating a matrix as an object on which other transformations and operations can be performed creates an obstacle for this student.

Examples constructed for Task 6 indicate that for many students their concept image of a basis is incomplete. When students list the elements of a basis that are not elements of a vector space, the necessary component of a basis being a subset of this vector space is not present or inactive in students’ concept image. If the number of elements in a basis does not correspond to the dimension of a vector space, the link basis  $\leftrightarrow$  dimension may be lacking in the concept image. Some participants of this study did not perceive the defining properties of a basis: linear independence and the spanning set as the necessary components for their examples.

Tall and Vinner (1981) characterize a potential conflict factor as part of the concept image or concept definition that may conflict with another part of the concept image or concept definition. Such factors can seriously impede the learning of a formal theory. In students' responses several potential conflict factors were identified: basis vectors form a subset of a vector space; a basis is a linearly independent set; dimension of a vector space equals the cardinality of a basis. These conflict factors create obstacles for the conceptual understanding of a basis.

## 6.7 Summary

In general, example-generation tasks provide a view of an individual's schema of basic linear algebra concepts. Through the construction process and students' examples we see the relationships between the different concepts.

Task 1 (Linear (in)dependence) revealed that the connections linear dependence  $\leftrightarrow$  free variables / pivot positions / zero row in echelon form, and linear independence  $\leftrightarrow$  no free variables / vectors not multiples of each other are strong in students' schema. This task also showed that students tend to confuse the null space and column space of a matrix, and this confusion was further demonstrated in students' responses to Task 2 (Column space / Null space). In addition, Task 2 identified that some students' schema of null space doesn't have the connection between the existence of a nontrivial solution to the homogeneous equation  $A\mathbf{x} = \mathbf{0}$  and the singularity of  $A$ .

Students' responses to Tasks 3 and 5 (Linear transformations) suggested that some students connected a linear transformation to its defining properties, while others formed the link linear transformation  $\leftrightarrow$  matrix transformation in their schema. At the

same time, the connection of a linear transformation to the properties of its standard matrix is weak or nonexistent for some students.

Vectors were generally associated with the elements of  $\mathbf{R}^n$  as was shown by Task 6 (Vectors). The links to other spaces such as matrices and polynomials have not been formed yet by many students. Task 4 (Basis) showed that the relation basis  $\leftrightarrow$  subset of a vector space was ignored by many students, while some students misinterpreted the link basis  $\leftrightarrow$  dimension.



## CHAPTER 7: CONCLUSION

*"Researchers in mathematics education have suggested that the mistakes students make can provide windows through which we can observe the inner workings of a student's mind as s/he engages in the learning process" (Dubinsky, Dautermann, Leron, and Zazkis, 1994, p.295).*

This study is a contribution to the ongoing research in undergraduate mathematics education, focusing on linear algebra. Despite the centrality of the subject in the undergraduate curriculum for students in mathematics, sciences and engineering, research on learning and understanding of linear algebra is rather slim. In particular, students' learning and understanding of the specific concepts of linear algebra has not been addressed in the research in detail. This study is an attempt to fill this gap and it is guided by the belief that better understanding of student's difficulties leads to improved instructional methods.

Two research questions were posed in this study:

1. What is students' understanding of the key concepts of linear algebra?  
What difficulties do students experience when engaged in the example-generation tasks?
2. What can example-generation tasks reveal about students' understanding of mathematics? Are these tasks effective and useful as a data collection tool for research in mathematics education?

Below I will summarize and discuss the findings addressing each of the above questions. I will highlight the contributions of this study from a pedagogical and from a

methodological perspective. In addition, I will offer recommendations for teaching linear algebra and directions for future studies.

## 7.1 Main findings and contributions of the study

One of the goals of this thesis was to investigate students' understanding of linear algebra. Several fundamental topics were selected for this investigation. My study has identified some of the difficulties experienced by students with learning several key concepts of linear algebra, and has also isolated the possible obstacles to such learning. It was shown that many students have limited example spaces to support their concept formation/understanding. In the following paragraphs, I outline the main findings of my research as related to the concepts under investigation.

Learners' responses to Task 1 (Linear (in)dependence) show that many students treat linear dependence as a process. They think of linear dependence in reference to the row reduction procedure. Some students connected linear dependence to the homogeneous linear system  $A\mathbf{x} = \mathbf{0}$  having free variables that in turn corresponds to the  $n \times n$  matrix  $A$  having a zero row in an echelon form. Other students linked the linear independence of vectors to a homogeneous linear system having only basic variables and therefore  $n$  pivot positions. However, few students considered the different structures of the linear dependence relations.

Even though geometric representation helps in visualizing the concepts, for some students geometric and algebraic representations seem completely detached. This can be seen in students' attempts to provide a geometric interpretation of the span of the columns of a matrix. There was a common confusion of the span of the columns of  $A$

with the solution set of  $A\mathbf{x} = \mathbf{0}$ . Instead of providing a geometric interpretation of the span of the columns of  $A$ , some students gave a geometric interpretation of the solution set of the homogeneous system  $A\mathbf{x} = \mathbf{0}$ . Further analysis of students' understanding of the column space and null space showed that some students did not connect a matrix  $A$  being singular with  $\text{Nul } A$  having at least one nonzero vector. Other students internalized the procedures for finding these two vector spaces but were still unable to connect them.

Learners' concept image of a linear transformation included the links: linear transformation  $\leftrightarrow$  defining properties, or linear transformation  $\leftrightarrow$  matrix transformation. However, students had difficulty applying the definition to the problem. Some students appeared to operate at the level of symbolic manipulations without attaching meaning to definitions.

The study reveals that for the majority of students the concept image of a vector is limited to elements of  $\mathbf{R}^n$ . They are only able to view the vectors as  $n \times 1$  arrays or directed line segments or points in  $\mathbf{R}^2$  or  $\mathbf{R}^3$ . For example, as we saw in the interview with Nicole (Chapter 6, Section 6.2), she was confused between the elements of  $\mathbf{R}^n$  and  $\mathbf{P}_{n-1}$ . Thinking of polynomials as arrays of numbers creates a conflict when a student is asked to check if a set of polynomials in  $\mathbf{P}_n$  satisfying the equation  $\mathbf{p}(0) = 0$  is a subspace of  $\mathbf{P}_n$ , or when considering the action of a linear transformation  $T: \mathbf{P}_3 \rightarrow \mathbf{R}^4$  given by  $T(\mathbf{p}) =$

$$\begin{bmatrix} p(-3) \\ p(-1) \\ p(1) \\ p(3) \end{bmatrix}.$$

The concept image of a basis is fragmented or incomplete for many learners. When constructing an example of a basis, many students failed to check if the elements in

the set are elements of the vector space in question. They jumped to verifying that a set is linearly independent and spans the vector space without considering the above condition.

This study showed that example-generation tasks are a useful tool to discuss students' understanding of the mathematical concepts. In particular, students' examples reveal their appreciation of the structure of the concepts involved, the connections students make between the different concepts, students' level of understanding according to the APOS theoretical framework, and students' existing concept image. All these are components of the complex notion of understanding.

In summary, there are several contributions of this study to the field of undergraduate mathematics education. Firstly, focusing on specific mathematical content, it provides a finer and deeper analysis of students' understanding of linear algebra. Secondly, focusing on methodology, it introduces an effective data collection tool to investigate students' learning of mathematical concepts. Learner-generated examples showed that students' concept images conflicted at times with formal mathematical definitions. The tasks developed in this study provide researchers with useful tools to investigate the scope of students' understanding.

Moreover, focusing on pedagogy, the study enhances the teaching of linear algebra by developing a set of example-generation tasks. As was pointed out in Chapter 2, one of the reasons for students' difficulties in learning linear algebra is the lack of pedagogical practices that allow students to construct their own knowledge (Dubinsky, 1997). Therefore, example-generation tasks are a valuable addition to undergraduate mathematics education since they serve not only as an assessment tool but also as an

instructional tool that provides learners with an opportunity to engage in mathematical activity.

## 7.2 Pedagogical considerations

As mentioned above, the tasks soliciting learner-generated examples were developed by the author in this research for the purpose of data collection. However, these tasks are also effective pedagogical tools for assessment and construction of mathematical knowledge, and can contribute to the learning process.

Part of the power of Task 1 (Linear (in)dependence) is that it anticipates the concept of rank, long before students are exposed to it. In playing with the examples (assigned after only two weeks of classes), students develop their intuition about what linear (in)dependence “really means”. The students may not be able to articulate why the second example works differently from the first, but they are starting to develop a “feel” for the difference. This task can be further extended to higher dimensional vector spaces.

Task 2 (Null space / Column space) explores the connection between the fundamental vector spaces associated with a matrix as well as their relation to a linear transformation represented by this matrix. In general, for a linear transformation  $T:V\rightarrow W$ , the kernel and range of  $T$  lie in different vector spaces. This task helps students realize that, when  $V = W$ , part of the image of  $T$  may be in the kernel. To further guide students' thinking, Task 2 can be extended in several directions, for example, by applying the transformation again, so that some of the vectors in the range of  $T$  are sent to  $\mathbf{0}$ . If  $A$  is the standard matrix of a linear transformation  $T$ , this shows that  $\text{Ker } A^2$  contains  $\text{Ker } A$  and in

this case is larger. This can be restated in terms of the solution sets to homogeneous systems: the solution set of  $A^2\mathbf{x} = \mathbf{0}$  contains the solution set of  $A\mathbf{x} = \mathbf{0}$ .

Task 5 (Linear transformations (revisited)) helps learners explore the properties of linear transformations. This task can be modified to introduce and investigate various classes of linear transformations such as one-to-one, onto, or invertible. A possible variation of Task 3 (Linear transformations) that was not explored in this study is to impose other constraints, such as requiring that any two linearly independent vectors necessarily map into linearly dependent vectors, or that  $\dim V > \dim W$ . Addition of this last constraint would require students to produce a non-square matrix transformation as a counter-example, if the transformation they choose can be represented as a matrix transformation. Task 3 can also be formulated in the form of a question: Is it always possible to find a linear transformation  $T$  from a vector space  $V$  to a vector space  $W$  such that for linearly independent vectors  $\mathbf{u}$  and  $\mathbf{v}$  in  $V$ , the images  $T(\mathbf{u})$  and  $T(\mathbf{v})$  are linearly independent? This wording of the task may trigger different responses from students. It is not only asking for a specific counter-example, but may also provoke students to consider classes of linear transformations for which the possibility always exists.

Task 4 (Basis) helps students to better understand what a basis is, and to realize in a hands-on setting that a basis of a subspace can be extended in more than one way to a basis of a vector space. Furthermore the reasoning must take place outside the familiar comfortable setting of  $\mathbf{R}^n$ . In the subsequent class discussion of the problem, different answers can be investigated to emphasize that a basis is not unique: Why are all correct? Are some more natural than others? When is a standard basis not the most natural?

As was seen in the interviews, example-generation tasks created situations where students were stuck. Their examples did not fit the required constraints. Furthermore, faced with the contradictions, students were led to identify their misconceptions and eventually correct them. Learning happens when there is a challenge, when students are confronted with a problem for which previous solution methods are inadequate. Example-generation tasks present such challenges for students. Working on example constructions, students were able to progress from the process understanding to the object level (interviews with Leon and Anna, Chapter 6, Section 6.3.4). However, this may not be possible in the limited time frame of lectures to a large class, since individual attention is required and the instructor/tutor must avoid the tendency to just tell students the intended answer.

Furthermore, example-generation tasks give students an opportunity to reflect on their knowledge. Joan commented that she found it difficult to construct examples for mathematical concepts. She could perform the calculations, but could not make sense of them. In the interview, she could not produce an example of a linear transformation because she was never asked to write her own transformation. As she acknowledged afterwards, "... now I can understand why I am doing the calculations".

### **7.3 Limitations of the study and suggestions for further explorations**

There are several limitations of this study. Firstly, some conclusions are based only on written data and are not corroborated by interviews. The written responses did not always provide enough information about the genesis of the examples. Secondly, as there was less than full participation from students enrolled in a course, their responses might have not been representative. Thirdly, the data were collected using one type of

tasks, and there were no verifications from the different tasks to further support the findings.

There are several directions that can be suggested for further explorations focusing on the methodology, content and teaching.

The use of example-generation tasks as research tools does not have to be limited to specific concepts discussed in this study, or even to linear algebra. The study can be expanded to other undergraduate level courses such as abstract algebra or real analysis to explore and improve students' learning.

This study examined students' understanding of several key concepts in linear algebra through learner-generated examples. It would be interesting to investigate students' understanding of the same concepts using different tools or mathematically equivalent tasks, and to compare the analysis of the results. Another extension might focus on different concepts of linear algebra such as eigenvectors/eigenvalues, diagonalization of matrices, or orthogonality.

Another extension of this study that could be of interest is the expert-novice study that explores the example-construction by the different groups: mathematicians in general, mathematicians that teach linear algebra, and students. In particular, the study could investigate the diverse approaches used for example-generation by these groups, and how these approaches correlate with the understanding of mathematical concepts.

It is hoped that by examining students' learning, the data collected can lead to teaching strategies, which will help students expand their example spaces of mathematical concepts and broaden their concept images/schemas. It is proposed that



further studies could discuss the design and implementation of example-generation tasks intended specifically as instructional strategies and evaluate their effectiveness. There is also a need for further investigation of the relationship between learner-generated examples and learner's knowledge and understanding.

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