STRESS IN THE VICINITY OF A CRACK IN A THICK ELASTIC PLATE

by

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ABSTRACT

The problem of a thick plate containing a penny shaped crack lying in a plane parallel to the surfaces of the plate is considered. It is assumed that shear stresses and normal displacements are specified on both surfaces. Using Hankel transforms the problem is represented as a set of simultaneous dual integral equations and a method contingent on the small value of the ratio of the crack radius to the plate thickness is presented to reduce the problem to a set of simultaneous Fredholm equations. The special case of normally clamped shear free surfaces and uniform pressure applied to the crack surfaces is considered and the Fredholm equations are solved approximately leading to the evaluation of the stress intensity factors for the problem. The case of a uniform displacement of one of the surfaces is also considered.

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TABLE OF CONTENTS

LIST OF GRAPHS

§ 1 Introduction

The problem of determining, within the classical limits of the theory of elasticity, the stress field in the vicinity of a crack in an infinite medium was first discussed by Sneddon [6]. This problem has also been discussed by Green [12], Payne [13] and Sack [14] but much of the later studies of problems concerned with penny-shaped cracks have been based on the work of Sneddon and of methods due to Lebedev and Ufliand [8].

An investigation of the state of stress in a plate containing a penny-shaped crack has been conducted by Lowengrub [1] based on Sneddon's original paper. The crack was taken to lie in the central plane of the plate with its surfaces parallel to those of the plate. Furthermore, the method of solution required that the thickness of the plate be large compared to the radius of the crack.

Lowengrub considered two sets of boundary conditions corresponding to the case of a small uniform normal displacement of the surfaces of the plate or of the case of an uniform tension applied to both surfaces. In both cases the surfaces of the plate were specified as being shear stress free and the surfaces of the crack were specified as being completely stress free.

The symmetry of each problem was used to reduce it to that of an elastic layer and the resulting problem ultimately reduced to the solution of a pair of dual integral equations. These integral equations were in turn reduced to Fredholm equations of the second kind by the method of Lebedev and Ufliand [8]. The Fredholm equations were solved by standard methods and the paper concluded by employing the solution in the determination of the shape of the crack after deformation.

The determination of the state of stress in the vicinity of a crack in a semi-infinite solid has been discussed by Kuz'min and Ufliand [15] and by Srivastava and Singh [2]. The Kuz'min-Ufliand paper is mathematically less pertinent to the present paper and is, in the view of Srivastava and Singh, in error and will not be discussed here.

In their paper Srivastava and Singh discussed two problems with respect to the halfspace. In the first case the bounding plane was assumed to be stress free and in the second it was taken to be rigidly clamped. Stresses were taken to be prescribed on the surfaces of the crack for both problems. Another set of boundary conditions was obtained by considering the medium to be composed of a region above the plane containing the crack and of a region below it. The problem was thus made to resemble a layer problem and as is standard with such problems conditions of continuity were imposed in the region of the plane unoccupied by the crack.

Hankel transforms were applied to the field equations and all the quantities involved and ultimately both problems were represented as sets of simultaneous dual integral equations. With slight modification a method due largely to Sneddon [4] was used to reduce the dual integral equations to simultaneous Fredholm equations of the second kind. For the case in which the distance from the crack to the bounding plane is large compared to the radius of the crack solutions to the Fredholm equations were obtained in power series form. From this solution were obtained representations of the stress intensity factors and the energy required to open out the crack and consequently the critical stress necessary to cause the crack to extend. Finally, for the case in which the distance from the crack to the bounding plane differs little from the radius of the crack, the method of Fox and Goodwin [11] was used to solve the Fredholm equations numerically.

The concern of this thesis is to consider the more general problem of the plate containing a crack, i.e., the case for which the plane containing the crack is not the central one. The problem wherein the normal component of the stress tensor and the normal component of the displacement vector are both prescribed on the surfaces of the plate and the surfaces of the crack are prescribed to be stress free is discussed and the methods used and developed by Srivastava and Singh in their paper on the halfspace are adapted and used throughout. As in the Srivastava-Singh paper a set of simultaneous Fredholm integral equations is obtained and these are solved approximately and used to give power series representations of the two stress intensity factors for the problem. The thesis concludes with a discussion of other possible boundary conditions and proposes that the one discussed here is the only one that is mathematically tractible while still being of physical interest.

§ 2 Solution to the Equilibrium Equations for the Plate

A penny-shaped crack is a disc-shaped discontinuity in a medium and a convenient coordinate system in which to represent such a phenomenon is the polar cylindrical coordinate system (r,θ,z). With the centre of the crack taken to be the origin (Figure 1) the crack is seen to lie in the plane defined by taking $z = 0$. If, furthermore, the medium is considered to be contained between the planes $z = a$ and $z = b$ $(a > 0 > b)$ then the plate containing a penny-shaped crack is completely defined (Figure B).

In general the deformation vector $\overline{u}$ has three components u,v,w representing deformations in the radial or r direction, the angular or θ direction, and the normal or z direction respectively. Each component is a function of position in the medium, i.e.,

$$u = u(r,\theta,z),$$

$$v = v(r,\theta,z),$$

$$w = w(r,\theta,z).$$

Taking body forces to be zero the equations of equilibrium in terms of the components of the deformation vector are

$$(\lambda+2\mu)(u,_{rr}+\frac{1}{r}u,_{r}-\frac{1}{r^2}u) - (\lambda+3\mu)\frac{1}{r^2}v,_{\theta}$$

$$+ (\lambda+\mu)(\frac{1}{r}v,_{r\theta}+w,_{zr})$$

$$+ \mu(\frac{1}{r^2}u,_{\theta\theta}+u,_{zz}) = 0$$

$$(\lambda+\mu)\frac{1}{r}(u,_{r\theta}+w,_{z\theta}) + (\lambda+3\mu)\frac{1}{r^2}u,_{\theta} + (\lambda+2\mu)\frac{1}{r^2}v,_{\theta\theta}$$

$$+ \mu(v,_{rr}+\frac{1}{r}v,_{r}-\frac{1}{r^2}v+v,_{zz}) = 0$$

$$(\lambda+\mu)(u,_{rz}+\frac{1}{r}u,_{z}+\frac{1}{r}v,_{\theta z})$$

$$+ (\lambda+2\mu)w,_{zz}+\mu(w,_{rr}+\frac{1}{r}w,_{r}+\frac{1}{r^2}w,_{\theta\theta}) = 0 \tag{1}$$

where λ and μ are the Lamé constants, and a subscript following a comma represents a partial derivative

i.e.,
$$u,_{r} \equiv \frac{\partial u}{\partial r}.$$

If the boundaries of the plate and crack are initially deformed or stressed in a manner independent of the θ coordinate, i.e., the boundary conditions are axially symmetric, then the angular component v of the deformation vector will vanish, i.e.,

$$v(r,\theta,z) \equiv 0,$$

and furthermore the components u and w will be independent of θ, i.e.,

$$u = u(r,z),$$

$$w = w(r,z).$$

The equations of equilibrium simplify in the case of axial symmetry to become

$$(\lambda+2\mu)(u_{,rr} + \frac{1}{r} u_{,r} - \frac{1}{r^2} u) + \mu\, u_{,zz} + (\lambda+\mu)\, w_{,rz} = 0, \tag{2a}$$

$$(\lambda+\mu)(u_{,rz} + \frac{1}{r} u_{,z}) + \mu(w_{,rr} + \frac{1}{r} w_{,r}) + (\lambda+2\mu)\, w_{,zz} = 0\ . \tag{2b}$$

The Hankel transform of order n is defined as

$$H_n\{f(r)\} = \int_0^\infty J_n(\xi r)\, r f(r)\, dr.$$

$J_n(\xi)$ is the Bessel function of the first kind of order n. As is standard when dealing with the type of problem under consideration the Hankel transform of order 1 is applied to equation (2a) and the Hankel transform of order 0 is applied to equation (2b) to get

$$-\xi^2(\lambda+2\mu)H_1\{u\} + \mu \frac{d^2}{dz^2} H_1\{u\} - (\lambda+\mu)\xi \frac{d}{dz} H_0\{w\} = 0, \tag{3}$$

$$\xi(\lambda+\mu) \frac{d}{dz} H_1\{u\} - \mu\xi^2 H_0\{w\} + (\lambda+2\mu) \frac{d^2}{dz^2} H_0\{w\} = 0. \tag{4}$$

Solutions to equations (3) and (4) are of the type

$$\bar{u} = H_1\{u\} = (A(\xi) + z\xi C(\xi)) \cosh \xi z + (B(\xi) + z\xi D(\xi)) \sinh \xi z \tag{5}$$

$$\bar{w} = H_0\{w\} = (E(\xi) + z\xi G(\xi)) \sinh \xi z + (F(\xi) + z\xi H(\xi)) \cosh \xi z. \tag{6}$$

Substituting these solutions back into equations (3) and (4) we find

$$E = -(A + \frac{\lambda+3\mu}{\lambda+\mu} H)$$

$$F = -(B + \frac{\lambda+3\mu}{\lambda+\mu} G)$$

$$C = -G, \quad D = -H . \tag{7}$$

Thus (5) and (6) become

$$\bar{u} = (A - z\xi G) \cosh \xi z + (B - z\xi H) \sinh \xi z \qquad (8)$$

$$\bar{w} = \left(-A - \frac{\lambda+3\mu}{\lambda+\mu} H + z\xi G\right) \sinh \xi z + \left(-B - \frac{\lambda+3\mu}{\lambda+\mu} G + z\xi H\right) \cosh \xi z. \qquad (9)$$

For the purpose of the problem of a plate with a penny-shaped co-planar crack, the solution above the crack is written

$$\bar{u} = (A_a - (z-a)\xi G_a) \cosh \xi(z-a) + (B_a - (z-a)\xi H_a) \sinh \xi(z-a) \qquad (10)$$

$$\bar{w} = -A_a \sinh \xi(z-a) - B_a \cosh \xi(z-a)$$

$$+ G_a\left[-\frac{\lambda+3\mu}{\lambda+\mu} \cosh \xi(z-a) + (z-a)\xi \sinh \xi(z-a)\right]$$

$$+ H_a\left[-\frac{\lambda+3\mu}{\lambda+\mu} \sinh \xi(z-a) + (z-a)\xi \cosh \xi(z-a)\right]. \qquad (11)$$

The solution for the portion of the medium below the crack is written similarly with the subscript a replaced by b. The solution for the entire medium is thus expressed in terms of the eight unknown functions A_a, A_b, B_a, B_b, G_a, G_b, H_a, H_b.

For the case of axial symmetry the components σ_{zz}, σ_{rr} and σ_{zr} are the only nonvanishing members of the stress tensor. The expressions relating displacements to the stresses are

$$\sigma_{zz} = (\lambda+2\mu)\, w_{,z} + \lambda(u_{,r} + \frac{1}{r} u), \tag{12a}$$

$$\sigma_{zr} = \mu(u_{,z} + w_{,r}). \tag{12b}$$

σ_{rr} is unimportant to the solution of the problem.

Applying the Hankel transform of order 0 to equation (12a) and order 1 to (12b) gives

$$\bar{\sigma}_{zz} = H_0\{\sigma_{zz}\} = (\lambda+2\mu)\overline{w}' + \lambda\overline{u}\xi$$

$$\bar{\sigma}_{zr} = H_1\{\sigma_{zr}\} = \mu(\overline{u}' - \xi\overline{w}). \tag{13}$$

Thus for $z > 0$,

$$\bar{\sigma}_{zr} = 2\mu\xi\{A_a \sinh \xi(z-a) + B_a \cosh \xi(z-a)$$

$$- G_a[(z-a)\xi \sinh \xi(z-a) - \frac{\mu}{\lambda+\mu} \cosh \xi(z-a)]$$

$$- H_a[(z-a)\xi \cosh \xi(z-a) - \frac{\mu}{\lambda+\mu} \sinh \xi(z-a)]\} \tag{14}$$

$$\bar{\sigma}_{zz} = -2\mu\xi\{A_a \cosh \xi(z-a) + B_a \sinh \xi(z-a)$$

$$+ G_a[\frac{\lambda+2\mu}{\lambda+\mu} \sinh \xi(z-a) - (z-a)\xi \cosh \xi(z-a)]$$

$$+ H_a[\frac{\lambda+2\mu}{\lambda+\mu} \cosh \xi(z-a) - (z-a)\xi \sinh \xi(z-a)]\} . \qquad (15)$$

And similarly for $z < 0$.

§ 3 The Boundary Conditions

A physically interesting problem, and the one most often considered for problems of this type, is the one for which the appropriate components of the stress tensor are completely prescribed on the surface of the crack. For the case of axisymmetric boundary conditions it is sufficient to specify σ_{zz} and σ_{zr} and consequently for $r < r_1$, only the stress conditions

$$\sigma_{zz}(r,0^+) = p_+(r)$$

$$\sigma_{zr}(r,0^+) = q_+(r)$$

$$\sigma_{zz}(r,0^-) = p_-(r)$$

$$\sigma_{zr}(r,0^-) = q_-(r) \tag{16}$$

will be considered. $\sigma_{zz}(r,0^+)$, $\sigma_{zr}(r,0^+)$ are the stresses on the upper surface $(z > 0)$ of the crack, and $\sigma_{zz}(r,0^-)$, $\sigma_{zr}(r,0^-)$ are the stresses on the lower surface $(z < 0)$ of the crack.

The conditions on the outer surfaces of the plate will, for the purposes of this paper, be those wherein the normal deformation and the shear stress is prescribed on each plate surface. That is,

$$w(r,a) = g_a(r)$$

$$\sigma_{zr}(r,a) = k_a(r)$$

$$w(r,b) = g_b(r)$$

$$\sigma_{zr}(r,b) = k_b(r) \tag{17}$$

are assumed to be the known surface conditions.

That (16) and (17) represent a well-defined problem; i.e., that the existence and uniqueness of the solution are guaranteed is well documented and will not be discussed here.

Substituting the above equations into (10), (11), (14) and (15) we find

$$G_a = -\frac{\lambda+\mu}{\lambda+2\mu}\left(\overline{g}_a + \frac{\overline{k}_a}{2\mu\xi}\right)$$

$$B_a = \frac{\mu}{\lambda+2\mu}\,\overline{g}_a + \frac{\overline{k}_a}{2\mu\xi}\,\frac{\lambda+3\mu}{\lambda+2\mu}$$

where

$$\overline{g}_a = H_0\{g_a\},\ \overline{k}_a = H_1\{k_a\}\ . \tag{18}$$

B_b and G_b are determined similarly. Thus B_a, B_b, G_a, G_b are determined in terms of the known functions.

§ 4 Development of the Integral Equations

It has been assumed, up to this point, that the solution to the problem is represented differently above the plane containing the crack than below it. However, outside the region of the crack the plane $z = 0$ is a quite artificial boundary and it is therefore necessary that all the quantities pertinent to the state of stress in the plate be required to be continuous across this boundary. For example, in the case of the radial deformation the relationship

$$u(r,0^+) = u(r,0^-), \quad r > r_1 \tag{19}$$

must hold. It is useful at this point to introduce a new function $M(\xi)$ such that

$$M = A_a \cosh \xi a - A_b \cosh \xi b - a\xi\, H_a \sinh \xi a + b\xi\, H_b \sinh \xi b + U,$$

$$U = -B_a \sinh \xi a + B_b \sinh \xi b + a\xi\, G_a \cosh \xi a - b\xi\, G_b \cosh \xi b\ . \tag{20}$$

Then (19) can be expressed as

$$\int_0^\infty \xi\, M(\xi)\, J_1(\xi r)\, d\xi = 0, \quad r > r_1. \tag{21}$$

For the normal deformation the requirement of continuity is expressed as

$$w(r,0^+) = w(r,0^-), \quad r > r_1. \tag{22}$$

Here a new function $N(\xi)$ is introduced such that

$$N = A_a \sinh \xi a - A_b \sinh \xi b + H_a(\frac{\lambda+3\mu}{\lambda+\mu} \sinh \xi a$$

$$- a\xi \cosh \xi a) - H_b(\frac{\lambda+3\mu}{\lambda+\mu} \sinh \xi b - b\xi \cosh \xi b) + W.$$

$$W = -B_a \cosh \xi a + B_b \cosh \xi b + G_a(- \frac{\lambda+3\mu}{\lambda+\mu} \cosh \xi a$$

$$+ a\xi \sinh \xi a) - G_b(- \frac{\lambda+3\mu}{\lambda+\mu} \cosh \xi b + b\xi \sinh \xi b). \tag{23}$$

Thus (22) becomes

$$\int_0^\infty \xi\, N(\xi)\, J_0(\xi r)\, d\xi = 0, \quad r > r_1. \tag{24}$$

Note that U and W are defined in terms of the known functions.

Relationships similar to (19) and (22) exist for the stresses σ_{zz} and σ_{zr} also in the region $r > r_1$. However, due to the nature of the boundary conditions on the surface of the crack relationships can be obtained for the entire region $r > 0$. Define

$$d_1(r) = \begin{cases} p_+(r) - p_-(r), & r < r_1 \\ 0 & , \quad r > r_1 \end{cases}$$

$$d_2(r) = \begin{cases} q_+(r) - q_-(r), & r < r_1 \\ 0 & , \quad r > r_1 \end{cases} \tag{25}$$

From (16) the following can be seen to hold

$$\sigma_{zz}(r,0^+) = \sigma_{zz}(r,0^-) + d_1(r), \quad r > 0,$$

$$\sigma_{zr}(r,0^+) = \sigma_{zr}(r,0^-) + d_2(r), \quad r > 0.$$

Further let

$$Z_+ = -\frac{1}{2\mu\xi}\,\bar{\sigma}_{zz}(\xi,0^+)$$

$$Z_- = -\frac{1}{2\mu\xi}\,\bar{\sigma}_{zz}(\xi,0^-)$$

$$R_+ = \frac{1}{2\mu\xi}\,\bar{\sigma}_{zr}(\xi,0^+)$$

$$R_- = \frac{1}{2\mu\xi}\,\bar{\sigma}_{zr}(\xi,0^-) \ . \tag{26}$$

These quantities can alternatively be written as

$$Z_+ = A_a \cosh \xi a + H_a(\frac{\lambda+2\mu}{\lambda+\mu} \cosh \xi a - a\xi \sinh \xi a) + P_+ \quad \ldots a$$

$$Z_- = A_b \cosh \xi b + H_b(\frac{\lambda+2\mu}{\lambda+\mu} \cosh \xi b - b\xi \sinh \xi b) + P_- \quad \ldots b$$

$$R_+ = -A_a \sinh \xi a - H_a(-a\xi \cosh \xi a + \frac{\mu}{\lambda+\mu} \sinh \xi a) + Q_+ \quad \ldots c$$

$$R_- = -A_b \sinh \xi b - H_b(-b\xi \cosh \xi b + \frac{\mu}{\lambda+\mu} \sinh \xi b) + Q_- \quad \ldots d \qquad (27)$$

where

$$P_+ = -B_a \sinh \xi a + G_a(- \frac{\lambda+2\mu}{\lambda+\mu} \sinh \xi a + \xi a \cosh \xi a) \quad \ldots a$$

$$P_- = -B_b \sinh \xi b + G_b(- \frac{\lambda+2\mu}{\lambda+\mu} \sinh \xi b + \xi b \cosh \xi b) \quad \ldots b$$

$$Q_+ = B_a \cosh \xi a - G_a(a\xi \sinh a\xi - \frac{\mu}{\lambda+\mu} \cosh a\xi) \quad \ldots c$$

$$Q_- = B_b \cosh \xi b - G_b(b\xi \sinh b\xi - \frac{\mu}{\lambda+\mu} \cosh b\xi) \quad \ldots d \qquad (28)$$

Note that P_+, P_-, Q_+, Q_- are expressed in terms of known functions. Then

$$-2\mu \int_0^\infty Z_+(\xi)\ \xi^2 J_0(\xi r)\, d\xi = p_+(r), \quad r < r_1 \quad \ldots a$$

$$-2\mu \int_0^\infty Z_-(\xi)\ \xi^2 J_0(\xi r)\, d\xi = p_-(r), \quad r < r_1 \quad \ldots b$$

$$2\mu \int_0^\infty R_+(\xi)\ \xi^2 J_1(\xi r)\, d\xi = q_+(r), \quad r < r_1 \quad \ldots c$$

$$2\mu \int_0^\infty R_-(\xi)\ \xi^2 J_1(\xi r)\, d\xi = q_-(r), \quad r < r_1 \quad \ldots d \qquad (29)$$

We see that

$$Z_+ - Z_- = -d_1^* \tag{30}$$

$$R_+ - R_- = d_2^* \tag{31}$$

where

$$d_1^* = \frac{1}{2\mu\xi} \int_0^\infty d_1(r)\ rJ_0(\xi r)\ dr$$

$$d_2^* = \frac{1}{2\mu\xi} \int_0^\infty d_2(r)\ rJ_1(\xi r)\ dr\ . \tag{32}$$

Now substitute equation (27a) into equation (20) to eliminate A_a, A_b to get

$$(M-M_1)\ \frac{\lambda+\mu}{\lambda+2\mu} = -\ H_a \cosh a\xi + H_b \cosh b\xi$$

$$M_1 = U - P_+ + P_- - d_1^*\ , \tag{33}$$

and substitute equation (27c) into equation (23) to get

$$(N-N_1)\ \frac{\lambda+\mu}{\lambda+2\mu} = H_a \sinh a\xi - H_b \sinh b\xi$$

$$N_1 = Q_+ - Q_- - d_2^* + W \tag{34}$$

Solving for H_a and H_b yields

$$H_a = \frac{\lambda+\mu}{(\lambda+2\mu)} \frac{1}{\sinh \xi(a-b)} [(N-N_1) \cosh \xi b + (M-M_1) \sinh \xi b] \tag{35}$$

$$H_b = \frac{\lambda+\mu}{(\lambda+2\mu)} \frac{1}{\sinh \xi(a-b)} [(N-N_1) \cosh \xi a + (M-M_1) \sinh \xi a] . \tag{36}$$

Substitute equations (35), (36) into equation (20) to get

$$M(1 + \frac{(\lambda+\mu)\xi}{(\lambda+2\mu)\sinh \xi(a-b)} \cdot (a-b) \sinh \xi a \sinh \xi b)$$

$$+ N \frac{(\lambda+\mu)\xi}{(\lambda+2\mu)\sinh \xi(a-b)} (-b \sinh \xi b \cosh \xi a$$

$$+ a \sinh \xi a \cosh \xi b) - M_1 \frac{(\lambda+\mu)\xi}{(\lambda+2\mu)\sinh \xi(a-b)} \cdot (a-b)$$

$$\sinh \xi a \sinh \xi b$$

$$- N_1 \frac{(\lambda+\mu)\xi}{(\lambda+2\mu)\sinh \xi(a-b)} (-b \sinh \xi b \cosh \xi a + a \sinh \xi a \cosh \xi b)$$

$$= A_a \cosh \xi a - A_b \cosh \xi b + U . \tag{37}$$

Substitute equations (35), (36) into equation (23) to get

$$N\left(-\frac{\mu}{\lambda+2\mu}+\frac{(\lambda+\mu)\xi}{(\lambda+2\mu)\sinh\xi(a-b)}\cdot(a-b)\cosh\xi b\cosh\xi a\right)$$

$$+M\frac{(\lambda+\mu)\xi}{(\lambda+2\mu)\sinh\xi(a-b)}(-b\sinh\xi a\cosh\xi b+a\sinh\xi b\cosh\xi a)$$

$$+N_1\left(-\frac{\lambda+3\mu}{\lambda+2\mu}+\frac{(\lambda+\mu)}{(\lambda+2\mu)}\xi\frac{\cosh\xi b\cosh\xi a}{\sinh\xi(a-b)}(a-b)\right)$$

$$-M_1\frac{(\lambda+\mu)\xi}{(\lambda+2\mu)\sinh\xi(a-b)}(-b\sinh\xi a\cosh\xi b+a\sinh\xi b\cosh\xi a)$$

$$=A_a\sinh\xi a-A_b\sinh\xi b+W\ . \tag{38}$$

Solving for A_a yields

$$A_a=\frac{(\lambda+\mu)\xi}{(\lambda+2\mu)\sinh^2\xi(a-b)}[(M-M_1)(a\sinh\xi b\cosh\xi(a-b)-b\sinh\xi a)$$

$$+(N-N_1)(a\cosh\xi b\cosh\xi(a-b)-b\cosh\xi a)]$$

$$+\left[-M\sinh\xi b-\frac{\mu}{\lambda+2\mu}N\cosh\xi b+\frac{\lambda+3\mu}{\lambda+2\mu}N_1\cosh\xi b\right.$$

$$\left.+U\sinh\xi b-W\cosh\xi b\right]/\sinh\xi(a-b)\ . \tag{39}$$

Similarly the result for A_b is

$$A_b = \frac{(\lambda+\mu)\xi}{(\lambda+2\mu)\sinh^2 \xi(a-b)} [(M-M_1)(a \sinh \xi b - b \sinh \xi a \cosh \xi(a-b))$$

$$+ (N-N_1)(a \cosh \xi b - b \cosh \xi a \cosh \xi(a-b))]$$

$$+ [- M \sinh \xi a - \frac{\mu}{\lambda+2\mu} N \cosh \xi a + \frac{\lambda+3\mu}{\lambda+2\mu} N_1 \cosh \xi a$$

$$+ U \sinh \xi a - W \cosh \xi a]/\sinh \xi(a-b) \ . \qquad (40)$$

Henceforth only results valid for $z > 0$ need be given. Substitution of equations (35) and (39) into equation (27a) results in

$$Z_+ = \frac{(\lambda+\mu)\xi}{(\lambda+2\mu)\sinh^2 \xi(a-b)} [(M-M_1)(a \sinh \xi b \cosh \xi b$$

$$- b \sinh \xi a \cosh \xi a) + (N-N_1)(a \cosh^2 \xi b - b \cosh^2 \xi a)]$$

$$- M_1 \frac{\cosh \xi a \sinh \xi b}{\sinh \xi(a-b)}$$

$$+ \frac{\cosh \xi a \cosh \xi b}{(\lambda+2\mu)\sinh \xi(a-b)} [(\lambda+\mu)N + \mu N_1]$$

$$+ \frac{U \sinh \xi b - W \cosh \xi b}{\sinh \xi(a-b)} \cosh \xi a + P_+ \ . \qquad (41)$$

Equations (35) and (39) into equation (27c) gives

$$R_+ = \frac{(\lambda+\mu)\xi}{(\lambda+2\mu)\sinh^2 \xi(a-b)} - [(M-M_1)(-a \sinh^2 \xi b + b \sinh^2 \xi a)$$

$$+ (N-N_1)(-a \sinh \xi b \cosh \xi b + b \sinh \xi a \cosh \xi a)]$$

$$+ \frac{\sinh \xi a \sinh \xi b}{(\lambda+2\mu)\sinh \xi(a-b)} [(\lambda+\mu)M + \mu M_1] - N_1 \frac{\sinh \xi a \cosh \xi b}{\sinh \xi(a-b)}$$

$$- \frac{U \sinh \xi b - W \cosh \xi b}{\sinh \xi(a-b)} \sinh \xi a + Q_+ \qquad (42)$$

Let

$$\alpha(\xi) \doteq -b \cosh \xi a \sinh \xi a + a \cosh \xi b \sinh \xi b$$

$$\beta_1(\xi) = -b \cosh^2 \xi a + a \cosh^2 \xi b$$

$$\beta_2(\xi) = -b \sinh^2 \xi a + a \sinh^2 \xi b$$

$$\overline{F}_1(\xi) = H_0\{F_1(r)\} = \frac{-2\mu(\lambda+\mu)\xi^2}{(\lambda+2\mu)\sinh^2 \xi(a-b)} [-M_1\alpha(\xi) - N_1\beta_1(\xi)]$$

$$+ \frac{\mu}{\lambda+2\mu} \frac{\cosh \xi a \cosh \xi b}{\sinh \xi(a-b)} N_1 - \frac{\cosh \xi a \sinh \xi b}{\sinh \xi(a-b)} M_1$$

$$+ P_+ + \frac{U \sinh \xi b - W \cosh \xi b}{\sinh \xi(a-b)} \cosh \xi a \ .$$

$$\overline{F}_2(\xi) = H_1\{F_2(r)\} = \frac{2\mu(\lambda+\mu)\xi^2}{(\lambda+2\mu)\sinh^2 \xi(a-b)} [-M_1\beta_2(\xi) - N_1\alpha(\xi)]$$

$$+ \frac{\mu}{\lambda+2\mu} \frac{\sinh \xi a \sinh \xi b}{\sinh \xi(a-b)} M_1 - \frac{\cosh \xi b \sinh \xi a}{\sinh \xi(a-b)} N_1$$

$$+ Q_+ - \frac{U \sinh \xi b - W \cosh \xi b}{\sinh \xi(a-b)} \sinh \xi a \ . \qquad (43)$$

Combining equations (26), (41), (42) and (43), we obtain

$$- \frac{2\mu(\lambda+\mu)}{(\lambda+2\mu)} \int_0^\infty \frac{\xi^2}{\sinh^2 \xi(a-b)} [M(\xi)\xi\alpha(\xi) + N(\xi)(\xi\beta_1(\xi)$$

$$+ \cosh \xi a \cosh \xi b \sinh \xi(a-b))] J_0(\xi r)\, d\xi$$

$$= - F_1(r) + p_+(r), \quad r < r_1 ; \tag{44}$$

$$\frac{2\mu(\lambda+\mu)}{(\lambda+2\mu)} \int_0^\infty \frac{\xi^2}{\sinh^2 \xi(a-b)} [M(\xi)(\xi\beta_2(\xi)$$

$$- \sinh \xi b \sinh \xi a \sinh \xi(a-b)) + N(\xi)\, \xi\alpha(\xi)] J_1(\xi r)\, d\xi$$

$$= - F_2(r) + q_+(r), \quad r < r_1 . \tag{45}$$

Thus equations (21), (24), (44) and (45) represent a pair of simultaneous dual integral equations. It should be noted at this point that were the expressions for the functions Z_- and R_- derived expressions identical to (41) and (42) would have resulted.

§ 5 Significance of $M(\xi)$ and $N(\xi)$

The determination of the solution, originally expressed in terms of eight unknown functions, has now been reduced to the problem of finding two unknown functions $M(\xi)$ and $N(\xi)$. If M and N can be resolved all quantities pertaining to the solution above the crack, i.e., $z > 0$, can be determined from them and the known functions M_1, N_1, B_a and G_a defined in (33),(34),(18). The solution below the crack, i.e., $z < 0$, is found in an identical fashion and will not be given. For example, defining the component of A_a that can be expressed in terms of the functions M_1 and N_1 as

$$\phi = -\frac{\lambda+\mu}{\lambda+2\mu}\frac{\xi}{\sinh^2 \xi(a-b)}[M_1(-b \sinh \xi a + a \sinh \xi b \cosh \xi(a-b))$$

$$+ N_1(-b \cosh \xi a + a \cosh \xi b \cosh \xi(a-b))]$$

$$+ \frac{\lambda+3\mu}{\lambda+2\mu} N_1 \frac{\cosh \xi b}{\sinh \xi(a-b)} + \frac{U \sinh \xi b - W \cosh \xi b}{\sinh \xi(a-b)} \tag{46}$$

and the component of H_a that can be defined in terms of M_1 and N_1 as

$$\psi = -\frac{\lambda+\mu}{\lambda+2\mu}\frac{1}{\sinh \xi(a-b)}[N_1 \cosh \xi b + M_1 \sinh \xi b] \tag{47}$$

and furthermore, defining functions

$$L_1(\xi,z) = -b \sinh \xi a \cosh \xi(z-a) + a \sinh \xi b \cosh \xi(z-a) \cosh \xi(a-b)$$

$$- (z-a) \sinh \xi b \sinh \xi(z-a) \sinh \xi(a-b) \tag{48}$$

$$L_2(\xi,z) = -b\cosh\xi a\cosh\xi(z-a) + a\cosh\xi b\cosh\xi(z-a)\cosh\xi(a-b)$$
$$+ (-(z-a)\sinh\xi(z-a)\cosh\xi b$$
$$+ \frac{1}{\xi}\cosh\xi b\cosh\xi(z-a))\sinh\xi(a-b) . \qquad (49)$$

$$L_3(\xi,z) = -b\sinh\xi a\sinh\xi(z-a) + a\sinh\xi b\cosh\xi(a-b)\sinh\xi(z-a)$$
$$- ((z-a)\sinh\xi b\cosh\xi(z-a)$$
$$+ \frac{1}{\xi}\sinh\xi b\sinh\xi(z-a))\sinh\xi(a-b) . \qquad (50)$$

$$L_4(\xi,z) = -b\cosh\xi a\sinh\xi(z-a) + a\cosh\xi b\cosh\xi(a-b)\sinh\xi(z-a)$$
$$- (z-a)\cosh\xi b\cosh\xi(z-a)\sinh\xi(a-b) . \qquad (51)$$

$$L_5(\xi,z) = -b\sinh\xi a\cosh\xi(z-a) + a\sinh\xi b\cosh\xi(z-a)\cosh\xi(a-b)$$
$$- ((z-a)\sinh\xi b\sinh\xi(z-a)$$
$$+ \frac{\lambda+2\mu}{\lambda+\mu}\frac{1}{\xi}\sinh\xi b\cosh\xi(z-a))\sinh\xi(a-b) . \qquad (52)$$

$$L_6(\xi,z) = -b\cosh\xi a\cosh\xi(z-a) + a\cosh\xi b\cosh\xi(z-a)\cosh\xi(a-b)$$
$$- ((z-a)\sinh\xi(z-a)\cosh\xi b$$
$$+ \frac{\mu}{\lambda+\mu}\frac{1}{\xi}\cosh\xi b\cosh\xi(z-a))\sinh\xi(a-b) . \qquad (53)$$

$$L_7(\xi,z) = b \sinh \xi a \sinh \xi(z-a) - a \sinh \xi b \sinh \xi(z-a) \cosh \xi(a-b)$$
$$+ ((z-a) \sinh \xi b \cosh \xi(z-a)$$
$$- \frac{\mu}{\lambda+\mu} \frac{1}{\xi} \sinh \xi b \sinh \xi(z-a)) \sinh \xi(a-b) \quad . \tag{54}$$

$$L_8(\xi,z) = b \cosh \xi a \sinh \xi(z-a) - a \cosh \xi b \sinh \xi(z-a) \cosh \xi(a-b)$$
$$+ ((z-a) \cosh \xi(z-a) \cosh \xi b$$
$$- \frac{\lambda+2\mu}{\lambda+\mu} \frac{1}{\xi} \cosh \xi b \sinh \xi(z-a)) \sinh \xi(a-b) \; . \tag{55}$$

The expression representing the normal stress is

$$\sigma_{zz}(r,z) = - 2\mu \, H_0\{\frac{\lambda+\mu}{\lambda+2\mu} \frac{\xi^2}{\sinh^2 \xi(a-b)} [ML_1 + NL_2]$$
$$+ \xi[\phi \cosh \xi(z-a) + B_a \sinh \xi(z-a)$$
$$+ G_a(\frac{\lambda+2\mu}{\lambda+\mu} \sinh \xi(z-a) - (z-a)\xi \cosh \xi(z-a))$$
$$+ \psi(\frac{\lambda+2\mu}{\lambda+\mu} \cosh \xi(z-a) - (z-a)\xi \sinh \xi(z-a))]\} \tag{56}$$

and the expression for the shear component is

$$\sigma_{zr}(r,z) = 2\mu \, H_1\{\frac{\lambda+\mu}{\lambda+2\mu} \frac{\xi^2}{\sinh^2 \xi(a-b)} [ML_3 + NL_4]$$
$$+ \xi[\phi \sinh \xi(z-a) + B_a \cosh \xi(z-a)$$
$$- G_a((z-a)\xi \sinh \xi(z-a) - \frac{\mu}{\lambda+\mu} \cosh \xi(z-a))$$
$$- \psi((z-a)\xi \cosh \xi(z-a) - \frac{\mu}{\lambda+\mu} \sinh \xi(z-a))]\}. \tag{57}$$

The expression for the normal component of the displacement vector is

$$w(r,z) = H_0\{\frac{1}{\lambda+2\mu} \frac{\xi}{\sinh^2 \xi(a-b)} [ML_7 + NL_8]$$

$$- \phi \sinh \xi(z-a) - B_a \cosh \xi(z-a)$$

$$+ G_a(- \frac{\lambda+3\mu}{\lambda+\mu} \cosh \xi(z-a) + (z-a)\xi \sinh \xi(z-a))$$

$$+ \psi(- \frac{\lambda+3\mu}{\lambda+\mu} \sinh \xi(z-a) + (z-a)\xi \cosh \xi(z-a))\} \qquad (58)$$

and that for the radial component is

$$u(r,z) = H_1\{\frac{1}{\lambda+2\mu} \frac{\xi}{\sinh^2 \xi(a-b)} [ML_5 + NL_6]$$

$$+ \phi \cosh \xi(z-a) + B_a \sinh \xi(z-a)$$

$$- (z-a)\xi \psi\sinh \xi(z-a) - (z-a)\xi G_a \cosh \xi(z-a)\} . \qquad (59)$$

For the case of a constant pressure opening the crack, an expression for the energy, T, required to open out the crack can be determined from

$$T = -2\pi \int_0^1 r\sigma_{zz}(r,0)\ w(r,0)\ dr , \qquad (60)$$

where, without loss of generality r_1 has been set to 1. Since $\sigma_{zz}(r,0) = -p_0$, $r < 1$ we have

$$T = 2\pi p_0 \int_0^1 r \int_0^\infty \xi \, \overline{w}(r,0) \, J_0(\xi r) \, d\xi dr$$

$$= 2\pi p_0 \int_0^\infty \overline{w}(\xi,0) \, J_1(\xi) \, d\xi$$

$$= \frac{2\pi p_0}{\lambda+2\mu} \int_0^\infty \{[ML_7+NL_8]/\sinh^2 \xi h - \phi \sinh \xi(z-a) - B_a \cosh \xi(z-a)$$

$$+ G_a(- \frac{\lambda+3\mu}{\lambda+\mu} \cosh \xi(z-a) + (z-a)\xi \sinh \xi(z-a))$$

$$+ \psi(- \frac{\lambda+3\mu}{\lambda+\mu} \sinh \xi(z-a) + (z-a)\xi \cosh \xi(z-a))\} \, J_1(\xi) \, d\xi \; . \quad (61)$$

§ 6 A Method of Solution of the Integral Equations

Equations (44) and (45), while successfully representing the problem as a set of integral equations, do not appear to lend themselves to easy solution. Most equations of this type are reduced by some variation of Sneddon's method for equations with a weight function to equations of Fredholm type of the second kind. The particular variation which will be employed here is the one developed by Srivastava and Singh.

However, in order to proceed it will be helpful to introduce three new parameters: h, p, q. The plate thickness will be represented by h and the position of the plane containing the crack will be indicated by p and q as follows:

$$h = (a-b)$$

$$p = \frac{a}{h} \quad , \quad p > 0$$

$$q = -\frac{b}{h} \, , \quad q > 0 \tag{62}$$

and of course it automatically holds that

$$p + q = 1 \, . \tag{63}$$

It is now possible to indicate the position of the crack plane within the plate without reference to the coordinate system.

The next step in accomplishing the reduction of equations (44) and (45) to Fredholm equations is the modification of the form of the coefficient of N in equation (44) and M in equation (45). First note that

$$\frac{1}{\sinh^2 h\xi} = 4 \sum_{k=1}^{\infty} ke^{-2kh\xi} \tag{64}$$

is a uniformly convergent series if $\xi > 0$. Next make the substitutions (62), (63) and expand the coefficient of N in (44) as

$$\beta_1(\xi) + \cosh \xi hp \cosh \xi hq \frac{\sinh \xi h}{\xi}$$

$$= qh \cosh^2 ph\xi + ph \cosh^2 qh\xi$$

$$+ \frac{1}{8\xi} (e^{2ph\xi} - e^{-2ph\xi} - e^{-2qh\xi} + e^{2qh\xi} + e^{2\xi h} - e^{-2\xi h}) . \tag{65}$$

Multiply through by $\xi^2/\sinh^2 h\xi$:

$$\xi^2/\sinh^2 \xi h(\beta_1(\xi) + \cosh \xi hp \cosh \xi hq \frac{\sinh \xi h}{\xi})$$

$$= 4\xi \sum_{k=1}^{\infty} ke^{-2hk\xi}[qh\xi \frac{1}{4}(2 + e^{2ph\xi} + e^{-2ph\xi})$$

$$+ ph\xi \frac{1}{4}(2 + e^{-2qh\xi} + e^{2qh\xi}) + \frac{1}{8}(e^{2ph\xi} - e^{-2ph\xi}$$

$$- e^{-2qh\xi} + e^{2qh\xi} + e^{2h\xi} - e^{-2h\xi})]$$

$$= 4\xi \sum_{k=1}^{\infty} k[qh\xi \frac{1}{4}(2e^{-2kh\xi} + e^{-2h\xi(k-p)}$$

$$- e^{-2h\xi(k+p)}) + ph\xi \frac{1}{4}(2e^{-2kh\xi} + e^{-2h\xi(k+q)}$$

$$+ e^{-2h\xi(k-q)}) + \frac{1}{8}(e^{-2\xi h(k-p)} - e^{-2\xi h(k+p)}$$

$$- e^{-2\xi h(k+q)} + e^{-2\xi h(k-q)} + e^{-2\xi h(k-1)} - e^{-2\xi h(k+1)}] . \quad (66)$$

Consider the last two terms in the above expression:

$$\sum_{k=1}^{\infty} ke^{-2h\xi(k-1)} - \sum_{k=1}^{\infty} ke^{-2h\xi(k+1)}$$

$$= \sum_{k=1}^{\infty} ke^{-2h\xi(k-1)} + 1 - \sum_{k=1}^{\infty} ke^{2h\xi(k+1)} .$$

Let $k = n + 1$ in the first sum, $k = n - 1$ in the second, so that the above is

$$= 1 + 2e^{-2h\xi} + 2 \sum_{n=2}^{\infty} e^{-2h\xi n}$$

$$= 1 + 2 \sum_{k=1}^{\infty} e^{-2h\xi k} \ .$$

Thus

$$\xi^2/\sinh^2 \xi h(\beta_1(\xi) + \cosh ph\xi \cosh qh\xi \frac{\sinh h\xi}{\xi}$$

$$= \frac{1}{2}\xi(1+k_0(\xi)) \ . \qquad (67)$$

where

$$k_0(\xi) = \sum_{k=1}^{\infty} [4kh\xi\ e^{-2hk\xi} + 2qkh\xi\ e^{-2h\xi(k-p)}$$

$$+ 2qkh\xi\ e^{-2h\xi(k+p)} + 2pkh\xi\ e^{-2h\xi(k-q)}$$

$$+ 2pkh\xi\ e^{-2h\xi(k+q)} + k(e^{-2h\xi(k-p)} - e^{-2h\xi(k+p)}$$

$$- e^{-2h\xi(k+q)} + e^{-2h\xi(k-q)}) + 2e^{-2kh\xi}] \ . \qquad (68)$$

Further, note the relationship (uniformly convergent for $\xi > 0$)

$$\frac{e^{-2h\xi}}{1 - e^{-2h\xi}} = \sum_{k=1}^{\infty} e^{-2kh\xi}$$

Consequently, $k_0(\xi)$ can also be expressed as

$$k_0(\xi) = (2\xi\beta_1(\xi) + \frac{1}{2}\sinh 2ph\xi + \frac{1}{2}\sinh 2qh\xi)/\sinh^2 \xi h + \frac{2e^{-2\xi h}}{1 - e^{-2\xi h}} . \qquad (69)$$

Similarly,

$$\xi^2/\sinh^2 \xi h(\beta_2(\xi) + \sinh ph\xi \sinh qh\xi \frac{\sinh h\xi}{\xi}) = \frac{1}{2}\xi(1+k_1(\xi)) \qquad (70)$$

where

$$k_1(\xi) = \sum_{k=1}^{\infty} [-4kh\xi\, e^{-2h\xi k} + 2qkh\xi\, e^{-2h\xi(k-p)} + 2qkh\xi\, e^{-2h\xi(k+p)} + 2pkh\xi\, e^{-2h\xi(k-q)} + 2pkh\xi\, e^{-2h\xi(k+q)} - k(e^{-2h\xi(k-p)} - e^{-2h\xi(k+p)} + e^{-2h\xi(k-q)} - e^{-2h\xi(k+q)}) + 2e^{-2kh\xi}] . \qquad (71)$$

Or

$$k_1(\xi) = (2\xi\beta_2(\xi) - \frac{1}{2}\sinh 2ph\xi - \frac{1}{2}\sinh 2qh\xi)/\sinh^2 h\xi + \frac{2e^{-2\xi h}}{1 - e^{-2\xi h}} \qquad (72)$$

Also

$$\frac{\alpha(\xi)}{\sinh^2 h\xi} = \sum_{k=1}^{\infty} hk[-p(e^{-2h\xi(k-q)} - e^{-2h\xi(k+q)})$$

$$- q(e^{-2\xi h(k+p)} - e^{-2\xi h(k-p)})] . \qquad (73)$$

Thus the set of simultaneous dual integral equations (21), (24), (44) and (45) may be written

$$\int_0^\infty \xi M(\xi)\, J_1(\xi r)\, d\xi = 0, \quad r > r_1 ,$$

$$\int_0^\infty \xi N(\xi)\, J_0(\xi r)\, d\xi = 0, \quad r > r_1 ,$$

$$\frac{2\mu(\lambda+\mu)}{\lambda+2\mu} \int_0^\infty \xi(\xi M(\xi)(1+k_1(\xi)) + 2\xi^2 N(\xi)\frac{\alpha(\xi)}{\sinh^2 h\xi})\, J_1(\xi r)\, d\xi$$

$$= -2F_2(r) + 2q_+(r)$$

$$\frac{2\mu(\lambda+\mu)}{\lambda+2\mu} \int_0^\infty \xi(2\xi^2 M(\xi)\frac{\alpha(\xi)}{\sinh^2 h\xi} + \xi N(\xi)(1+k_0(\xi)))\, J_0(\xi r)\, d\xi$$

$$= 2F_1(r) - 2p_+(r), \quad r < r_1 . \qquad (74)$$

It is convenient now to point out that so long as the ratio of the plate thickness, h, to the crack radius, r_1, is known it is unnecessary to leave r_1 unspecified. That is, no loss of generality occurs if the problem is normalized by letting $r_1 = 1$. Now the following substitutions for $M(\xi)$ and $N(\xi)$ can be made:

$$\xi M(\xi) = \xi^{1/2} \int_0^1 m(t)\, J_{3/2}(\xi t)\, dt$$

$$= \frac{1}{\xi}\left(\frac{2}{\pi}\right)^{1/2} \left[-m(1) \sin \xi + \int_0^1 \frac{1}{t} \frac{d}{dt}(t^{1/2} m(t)) \sin \xi t dt\right] , \tag{75}$$

$$\xi N(\xi) = \int_0^1 n(t) \sin \xi t dt$$

$$= - \frac{n(1)}{\xi} \cos \xi + 1/\xi \int_0^1 \frac{d}{dt} n(t) \cos \xi t dt ,$$

$$n(0) = 0 . \tag{76}$$

Observe that (Bateman [10])

$$\int_0^\infty \sin t\xi\, J_1(\xi r)\, d\xi = \begin{cases} 0 & 0 < r < t \\ t/r(r^2-t^2)^{-1/2} & t < r < \infty , \end{cases}$$

$$\int_0^\infty \cos \xi t \, J_0(\xi r) \, d\xi = \begin{cases} 0 & 0 < r < t \\ (r^2-t^2)^{-1/2} & t < r < \infty \, . \end{cases} \tag{77}$$

the last two equations of (74) become (r<1)

$$\left(\frac{2}{\pi}\right)^{1/2} \int_0^r \frac{\frac{d}{dt}(t^{1/2}m(t))}{(r^2-t^2)^{1/2}} dt + r \int_0^\infty \left\{ \xi^{3/2} \left(\int_0^1 m(t) \, J_{3/2}(\xi t) \right. \right.$$

$$\left. + 2\xi \left(\int_0^1 n(t) \sin \xi t dt\right) \frac{\alpha(\xi)}{\sinh^2 \delta\xi} \right\} J_1(\xi r) \, d\xi$$

$$= \frac{\lambda+2\mu}{\mu(\lambda+\mu)} (q_+(r) - F_\alpha(r)) \tag{78}$$

$$\int_0^r \frac{\frac{d}{dt} n(t)}{(r^2-t^2)^{1/2}} dt + \int_0^\infty \left\{ 2\xi^{3/2} \left(\int_0^1 m(t) \, J_{3/2}(\xi t) \, dt\right) \frac{\alpha(\xi)}{\sinh^2 \delta\xi} \right.$$

$$\left. + \xi \left(\int_0^1 n(t) \sin \xi t dt\right) k_0(\xi) \right\} J_0(\xi r) \, d\xi$$

$$= \frac{\lambda+2\mu}{\mu(\lambda+\mu)} (F_1(r) - P_+(r)) \tag{79}$$

Now employ Abel's integral equation, i.e., if

$$\int_0^r \frac{f(t) \, dt}{(r^2-t^2)^{1/2}} = g(r)$$

then $$f(t) = \frac{2}{\pi}\frac{d}{dt}\int_0^t \frac{rg(r)\ dr}{(t^2-r^2)^{1/2}}\ , \tag{80}$$

and two results obtainable from Bateman [10]

$$\left(\frac{2}{\pi}\right)^{1/2} \int_0^t \frac{r^2 J_1(\xi r)\ dr}{(t^2-r^2)^{1/2}} = t^{3/2}\ \xi^{-1/2}\ J_{3/2}(t\xi)\ ,$$

$$\int_0^t \frac{\xi r J_0(\xi r)\ dr}{(t^2-r^2)^{1/2}} = \sin t\xi\ , \tag{81}$$

to get the set of simultaneous Fredholm equations of the second kind:

$$m(t) + t\int_0^1 m(u) \int_0^\infty \xi J_{3/2}(\xi u)\ J_{3/2}(\xi t)\ k_1(\xi)\ d\xi du$$

$$+ 2t\int_0^1 n(u)\int_0^\infty \xi^{1/2}\ J_{3/2}(\xi t)\ \sin \xi u\ \frac{\alpha(\xi)}{\sinh^2 h\xi}\ d\xi du = E_2(t) \tag{82}$$

$$n(t) + \frac{4}{\pi}\int_0^1 m(u)\int^\infty \xi^{1/2}\ J_{3/2}(\xi u)\ \sin \xi t\ \frac{\alpha(\xi)}{\sinh^2 h\xi}\ d\xi du$$

$$+ \frac{2}{\pi}\int_0^1 n(u)\int_0^\infty \sin \xi t\ \sin \xi u\ k_0(\xi)\ d\xi du = E_1(t)\ , \tag{83}$$

where

$$E_1(t) = \frac{2}{\pi}\frac{\lambda+2\mu}{\mu(\lambda+\mu)}\int_0^t \frac{(F_1(r)-P_+(r))}{(t^2-r^2)^{1/2}}\,dr$$

$$E_2(t) = \frac{2}{\pi}\frac{\lambda+2\mu}{\mu(\lambda+\mu)}\int_0^t \frac{(q_+(r)-F_2(r))}{(t^2-r^2)^{1/2}}\,dr \ . \tag{84}$$

The analysis which follows is based on that used in the Lowengrub and Srivastava-Singh papers and demands that the plate thickness be large compared to the radius of the crack. In the case of both the plate and the half space when the crack is in the vicinity of one of the surfaces the method fails due largely to the fact that the definition of the problem becomes ambiguous. The method is straightforward, however, and for the purposes of this thesis instructive, and will be employed here with the restriction that the crack plane remain close to the central plane. This restriction is consistent with the restriction imposed in the Srivastava-Singh paper inasmuch as, with respect to that paper, it is a result of the restriction that the distance from the crack to the single bounding plane be large.

Expand all known quantities in series form to get

$$\xi J_{3/2}(\xi t)\ J_{3/2}(\xi u) = \frac{2}{9\pi}(tu)^{3/2}\ (\xi^4 - \frac{1}{10}\,\xi^6(u^2+t^2)+\dots) \ ,$$

$$\xi^{1/2}\, J_{3/2}(\xi u)\, \sin \xi t = \left(\frac{2}{\pi}\right)^{1/2} tu^{3/2}\left(\frac{1}{3}\,\xi^3 - \frac{8}{6!}\,(3u^2+5t^2)\,\xi^5\right.$$

$$\left. + \frac{16}{8!}\,(3u^4+14u^2t^2+7t^4)\,\xi^7 - \ldots\right) ,$$

$$\sin \xi t \sin \xi u = ut\left(\xi^2 - \frac{1}{3!}(u^2+t^2)\,\xi^4 + \frac{2}{6!}(3u^4+10u^2t^2+3t^4)\,\xi^6 - \ldots\right) . \qquad (85)$$

Thus, integrating term by term,

$$\int_0^\infty \xi J_{3/2}(\xi u)\, J_{3/2}(\xi t)\, k_1(\xi)\, d\xi = (tu)^{3/2}(k_{15}h^{-5}$$

$$+ (u^2+t^2)\, k_{17}h^{-7} + O(h^{-9})) ,$$

$$\int_0^\infty \xi^{1/2}\, J_{3/2}(\xi u)\, \sin \xi t\, \frac{\alpha(\xi)}{\sinh^2 h\xi}\, d\xi = tu^{3/2}(\alpha_4 h^{-4}$$

$$- \alpha_6 h^{-6}(3u^2+5t^2) + \alpha_8(3u^4+14u^2t^2+7t^4)\, h^{-8} + O(h-^9)) ,$$

$$\int_0^\infty \sin \xi t \sin \xi u\, k_0(\xi)\, d\xi = ut(k_{03}\delta^{-3} + k_{05}(u^2+t^2)\, h^{-5}$$

$$+ k_{07}(3u^4+10u^2t^2+3t^4)\, h^{-7} + O(h^{-9})) . \qquad (86)$$

Where k_0, k_1 and $\alpha(\xi)$ are defined in equations (68), (71), and (73), and

$$k_{03} = \frac{1}{2^3} \sum_{k=1}^{\infty} \left(\frac{8}{k^3} - \frac{2k}{(k-p)^3} + \frac{2k}{(k-q)^3} + \frac{2k}{(k+q)^3} - \frac{2k}{(k+p)^3} \right.$$

$$\left. + \frac{3!kq}{(k-p)^4} + \frac{3!kq}{(k+p)^4} + \frac{3!kp}{(k+q)^4} + \frac{3!kp}{(k-q)^4} \right) ,$$

$$k_{05} = \frac{1}{3!} \frac{1}{2^5} \sum_{k=1}^{\infty} \left(\frac{8 \cdot 4!}{k^5} + 4! \left(\frac{k}{(k-p)^5} + \frac{k}{(k+p)^5} - \frac{kp}{(k+q)^5} \right. \right.$$

$$\left. \left. - \frac{k}{(k-q)^5} \right) - 5! \left(\frac{kq}{(k-p)^6} + \frac{kq}{(k+p)^6} + \frac{kp}{(k+q)^6} + \frac{kp}{(k-q)^6} \right) \right) ,$$

$$k_{07} = \frac{2}{6!} \frac{1}{2^7} \sum_{k=1}^{\infty} \left(\frac{12 \cdot 6!}{k^7} - 6! \left(- \frac{k}{(k-p)^7} + \frac{k}{(k+p)^7} \right. \right.$$

$$\left. + \frac{k}{(k+q)^7} - \frac{k}{(k-q)^7} \right) + 7! \left(\frac{kq}{(k-p)^8} + \frac{kq}{(k+p)^8} + \frac{kp}{(k+q)^8} + \frac{kp}{(k-q)^8} \right) ,$$

$$\alpha_4 = \left(\frac{2}{\pi}\right)^{1/2} \frac{1}{2^3} \sum_{k=1}^{\infty} \left[q\left(\frac{k}{(k+p)^5} - \frac{k}{(k-p)^5}\right) + p\left(\frac{k}{(k-q)^5} - \frac{k}{(k+q)^5}\right)\right] ,$$

$$\alpha_6 = \left(\frac{2}{\pi}\right)^{1/2} \frac{1}{2^4} \sum_{k=1}^{\infty} \left[q\left(\frac{k}{(k+p)^7} - \frac{k}{(k-p)^7}\right) + p\left(\frac{k}{(k-q)^7} - \frac{k}{(k+q)^7}\right)\right] ,$$

$$\alpha_8 = \left(\frac{2}{\pi}\right)^{1/2} \frac{1}{2^5} \sum_{k=1}^{\infty} \left[q\left(\frac{k}{(k+p)^9} - \frac{k}{(k-p)^9}\right) + p\left(\frac{k}{(k-q)^9} - \frac{k}{(k+q)^9}\right)\right] ,$$

$$k_{15} = \frac{4}{9\pi} \frac{1}{2^5} \sum_{k=1}^{\infty} \left(\frac{-22 \cdot 4!}{k^5} + 4! \left(\frac{k}{(k-p)^5} + \frac{k}{(k-q)^5}\right.\right.$$

$$\left.\left. - \frac{k}{(k+q)^5} \quad \frac{k}{(k+p)^5}\right) + 5! \left(\frac{kq}{(k+p)^6} + \frac{kq}{(k-p)^6} + \frac{kp}{(k+q)^6} + \frac{kp}{(k-q)^6}\right)\right) ,$$

$$k_{17} = \frac{4}{90\pi} \frac{1}{2^7} \sum_{k=1}^{\infty} \left(\frac{16 \cdot 6!}{k^7} + 6! \left(\frac{k}{(k-p)^7} - \frac{k}{(k+p)^7}\right.\right.$$

$$\left.\left. - \frac{k}{(k+q)^7} + \frac{k}{(k-q)^7}\right) + 7! \left(\frac{kq}{(k-p)^8} + \frac{kq}{(k+p)^8} + \frac{kp}{(k-q)^8} + \frac{kp}{(k+q)^8}\right)\right) . \tag{87}$$

The result is

$$m(t) + t^{5/2} \int_0^1 m(u)\, u^{3/2}(k_{15}h^{-5} + k_{17}h^{-7} + O(h^{-9}))\, du$$

$$+ 2t^{5/2} \int_0^1 un(u)\,(\alpha_4 h^{-4} - \alpha_6 h^{-6}(3t^2+5u^2)$$

$$+ \alpha_8 h^{-8}(3t^4 + 14u^2t^2 + 7u^4) + O(h^{-9}))\, du = E_2(t) . \tag{88}$$

$$n(t) + \frac{4}{\pi} t \int_0^1 u^{3/2} m(u)(\alpha_4 h^{-4} - \alpha_6 h^{-6}(3u^2+5t^2)$$

$$+ \alpha_8 h^{-8}(3u^4 + 14u^2t^2 + 7t^4) + O(h^{-9}))\ du$$

$$+ \frac{2}{\pi} t \int_0^1 un(u)(k_{03}h^{-3} + k_{05}h^{-5}(u^2+t^2)$$

$$+ k_{07}h^{-7}(3u^4 + 10u^2t^2 + 3t^4) + O(h^{-9}))\ du = E_1(t) \ . \qquad (89)$$

Graphs I through VI illustrate the behaviour of k_{03}, k_{05}, k_{07}, α_4, α_6, α_8 with respect to the position of the plane containing the crack to the plate surfaces. It can be seen that these quantities are meaningful only if the crack plane is restricted to lie not too far from the centre of the plate.

§ 7 The Case of Constant Normal Pressure on the Surfaces of the Crack

If the kernels of equations (88) and (89) are truncated at terms of order h^{-8} they will represent the approximation of equations (88) and (89) by equations with a so-called degenerate kernel (c.f. Tricomi [9]). If, further, the assumption is made that each of $m(t)$ and $n(t)$ can be expressed as a Taylor series in h then equations (88) and (89) can be solved easily simply by substituting the series representations and comparing coefficients.

Now consider the circumstance wherein the points on the surfaces of the plate are fixed so that they cannot move in a direction perpendicular to the surfaces, i.e., the direction of the z-axis, but are allowed freedom to move radially, i.e., along an r-axis. Angular displacement will not occur because the initial disturbance of the plate will be symmetric (§ 2). Thus it is required that the normal component of the displacement vector be zero on the surfaces, or

$$w(r,a) = w(r,b) = 0 , \qquad (90)$$

and that there is no restricting shear stress to inhibit movement in the radial direction, or

$$\sigma_{zr}(r,a) = \sigma_{zr}(r,b) = 0 . \qquad (91)$$

Further, suppose that some substance of the nature of a fluid or a gas is present between the surfaces of the crack. Such a phenomenon would incur almost no shearing stresses on the crack surfaces and so, referring to equation (16), we set

$$q_+(r) = q_-(r) = 0 \ . \tag{92}$$

Since the deformations are constrained to be small the pressure, $-p_0$, exerted by the gas acts approximately normally to the crack surfaces and thus, using equations (16), we set

$$p_+(r) = p_-(r) = p_0 \ . \tag{93}$$

The known functions M_1, N_1, G_a, B_a, P_+, Q_+ defined in equations (33), (34), (18), (28) are easily shown to be identically zero in this case and consequently, from equation (43)

$$F_1(r) = F_2(r) = 0 \ .$$

Thus from equation (84)

$$E_1(t) = \frac{2}{\pi}\frac{\lambda+2\mu}{\mu(\lambda+\mu)}\,p_0 \ ,$$

$$E_2(t) = 0 \ . \tag{94}$$

Now make the representation

$$m(t) = m_0(t) + m_1(t)\, h^{-1} + m_2(t)\, h^{-2} + \dots$$

$$n(t) = n_0(t) + n_1(t)\, h^{-1} + n_2(t)\, h^{-2} + \dots \qquad (95)$$

Substituting the above into equations (88) and (89) yields

$$n_0(t) = \frac{2}{\pi} \frac{\lambda+2\mu}{\mu(\lambda+\mu)} p_0\, t ,$$

$$m_0(t) = m_1(t) = m_2(t) = m_3(t) = 0 ,$$

$$n_1(t) = n_2(t) = 0 ,$$

$$n_3(t) = - \frac{4}{3\pi^2} \frac{\lambda+2\mu}{\mu(\lambda+\mu)} p_0\, k_{03}\, t ,$$

$$m_4(t) = - \frac{4}{3\pi} \frac{\lambda+2\mu}{\mu(\lambda+\mu)} p_0\, \alpha_4\, t^{5/2} ,$$

$$m_6(t) = \frac{4}{\pi} \frac{\lambda+2\mu}{\mu(\lambda+\mu)} p_0\, \alpha_6'\, t^{5/2}(t^2+1) ,$$

$$n_4(t) = 0$$

$$n_5(t) = - \frac{4}{\pi^2} \frac{\lambda+2\mu}{\mu(\lambda+\mu)} p_0\, k_{05}\, t(1/5 + \frac{t^2}{3}) ,$$

$$n_6(t) = \frac{8}{9\pi^3} \frac{\lambda+2\mu}{\mu(\lambda+\mu)} p_0\, k_{03}^{\,2}\, t ,$$

$$m_7(t) = \frac{8}{9\pi^2} \frac{\lambda+2\mu}{\mu(\lambda+\mu)} p_0 k_{03} \alpha_4 t^{5/2} ,$$

$$n_7(t) = -\frac{4}{7\pi^2} \frac{\lambda+2\mu}{\mu(\lambda+\mu)} p_0 k_{07} t(3+14t^2+7t^4) ,$$

$$m_8(t) = -\frac{4}{5\pi} \frac{\lambda+2\mu}{\mu(\lambda+\mu)} p_0 \alpha_8 t^{5/2}(5t^4+14t^2+5) ,$$

$$n_8(t) = \frac{2}{\pi} \frac{\lambda+2\mu}{\mu(\lambda+\mu)} p_0 t\left(\frac{8}{15} \alpha_4^2 + \frac{k_{03}k_{05}}{\pi^2}\left(\frac{8}{15} + \frac{4}{3}\left(\frac{1}{5} + \frac{t^2}{3}\right)\right)\right) . \qquad (96)$$

The stress intensity factors, as defined in Sneddon [5], are

$$K_{zz} = \lim_{r\to1^+} \sqrt{2(r-1)} \left\{\sigma_{zz}(r,0), \quad r > 1\right\} \qquad (97)$$

$$k_{rz} = \lim_{r\to1^+} \sqrt{2(r-1)} \left\{\sigma_{rz}(r,0), \quad r > 1\right\} . \qquad (98)$$

where K_{zz} and K_{rz} are the factors associated with the normal stress and the shear stress respectively at the tip of the crack.

We have

$$K_{zz} = -\frac{2\mu(\lambda+\mu)}{\lambda+2\mu} \lim_{r\to1^+} \sqrt{2(r-1)} \int_0^\infty \xi \left\{\frac{\xi M(\xi)}{\sinh^2 \xi h} + \frac{1}{2} \xi N(\xi)(1+k_0(\xi))\right\} J_0(\xi r)\, d\xi \qquad (99)$$

It is not difficult to show that the term

$$\int_0^\infty \xi \left\{ \frac{M(\xi)}{\sinh^2 \xi h} + \frac{1}{2} \xi N(\xi) \ k_0(\xi) \right\} J_0(\xi r) \ d\xi$$

is bounded near $r = 1$. Employing equation (76) to express $N(\xi)$ in terms of $n(t)$ yields

$$\int_0^\infty \xi N(\xi) \ J_0(\xi r) \ d\xi =$$

$$= \int_0^\infty \xi(-n(1) \cos\xi + \int_0^1 \frac{d}{dt} n(t) \cos \xi t dt) \ J_0(\xi r) \ d\xi \ . \qquad (100)$$

The last term can also be shown to be bounded near $r = 1$ and so

$$K_{zz} = - \frac{2\mu(\lambda+\mu)}{\lambda+2\mu} \lim_{r \to 1^+} \sqrt{2(r-1)} \left[\int_0^\infty - \frac{1}{2} n(1) \cos\xi \ J_0(\xi r) \ d\xi + O(1) \right]$$

$$= \frac{\mu(\lambda+\mu)}{\lambda+2\mu} n(1) \qquad (101)$$

$$K_{zz} = \frac{2}{\pi} p_0 \left\{ 1 - \frac{2}{3\pi} k_{p3} h^{-3} - \frac{16}{15\pi} k_{05} h^{-5} + \frac{4}{9\pi^2} k_{03}{}^2 h^{-6} - \frac{48}{7\pi} k_{07} h^{-7} \right.$$

$$\left. + \left(\frac{8}{15\pi} \alpha_4{}^2 + \frac{112}{45\pi^2} k_{03} k_{05}\right) h^{-8} + O(h^{-9}) \right\} . \qquad (102)$$

Similarly,

$$K_{rz} = \left(\frac{2}{\pi}\right)^{1/2} \frac{\mu(\lambda+\mu)}{\lambda+2\mu} m(1)$$

$$= \left(\frac{2}{\pi}\right)^{1/2} p_0 \left\{ - \frac{4}{3\pi} \alpha_4 h^{-4} + \frac{8}{\pi} \alpha_6 h^{-6} \right.$$

$$\left. + \frac{8}{9\pi^2} k_{03}\alpha_4 h^{-7} - \frac{96}{5\pi} \alpha_8 h^{-8} + O(h^{-9}) \right\} . \qquad (103)$$

For the problem of an infinite medium the stress intensity factor is (Sneddon [6])

$$K_{zz}^{\infty} = \frac{2}{\pi} p_0 . \qquad (104)$$

Due to the finite distance of the crack from the normally clamped boundaries there is a decrease in stress intensity at the tip of the crack compared to the case in which the medium is unbounded. The percent decrease with respect to the infinite medium of the normal stress intensity factor can be expressed as

$$K = \frac{K_{zz}^{\infty} - K_{zz}}{K_{zz}^{\infty}} 100 . \qquad (105)$$

The value of K when the crack is located a distance of $.4h$ from the upper plate surface; i.e., $a = .4h$,

$$K = 403.590h^{-3} - 1852.45h^{-5} - 1628.85h^{-6} + 9138.31h^{-7}$$

$$+ 13993.6h^{-8} + O(h^{-9}) . \qquad (106)$$

The value of the sheer stress intensity at this point is

$$K_{rz} = -.572140h^{-4} + 10.7097h^{-6} + 2.30910h^{-7} - 72.4282h^{-8}$$

$$+ O(h^{-9}) \ . \qquad (107)$$

§ 8 The Case of an Uniform Surface Displacement

Suppose that the problem of Section 7 is varied so that the upper surface of the plate is no longer fixed but rather is given an uniform displacement in the positive direction along the z-axis. That is, the normal component of the displacement vector is non-zero on the surface $z = a$, so that from equation (17),

$$w(r,a) = g_a = \varepsilon \; . \tag{108}$$

Let, however, the remaining conditions of Section 7 be unchanged. That is,

$$w(r,b) = g_b(r) = 0$$

$$\sigma_{zr}(r,a) = k_a(r) = 0$$

$$\sigma_{zr}(r,b) = k_b(r) = 0 \; . \tag{109}$$

Modify the problem further so that the crack surfaces are no longer stressed. That is

$$\sigma_{zz}(r,0^+) = p_+(r) = 0$$

$$\sigma_{zz}(r,0^-) = p_-(r) = 0$$

$$\sigma_{zr}(r,0^+) = q_+(r) = 0$$

$$\sigma_{zr}(r,0^-) = q_-(r) = 0 \ . \qquad (110)$$

It is assumed here that ε is sufficiently small that the resultant warping, due to the presence of the crack, of the plane containing the crack, will be several orders of magnitude smaller than ε. In other words the crack-plane will remain approximately plane.

From equation (18) we have

$$G_a = -\frac{\lambda+\mu}{\lambda+2\mu}\,\varepsilon\,\frac{\delta(\xi)}{\xi} \ ,$$

$$B = \frac{\mu}{\lambda+\mu}\,\varepsilon\,\delta(\xi)/\xi \ ,$$

$$G_a = G_b = 0 \ ,$$

and from equations (20), (23), (28), (33), (34)

$$U = -\frac{\varepsilon}{\lambda+2\mu}\frac{\delta(\xi)}{\xi}(\mu \sinh \xi a + (\lambda+\mu)a\xi \cosh \xi a) ,$$

$$W = \frac{\varepsilon}{\lambda+2\mu}\frac{\delta(\xi)}{\xi}((\lambda+2\mu) \cosh \xi a - (\lambda+\mu)a\xi \sinh \xi a) ,$$

$$P_+ = \varepsilon \frac{\delta(\xi)}{\xi}\frac{\lambda+\mu}{\lambda+2\mu}(\sinh \xi a + a\xi \cosh \xi a) ,$$

$$P_- = 0 ,$$

$$\overline{d}_1 = 0 ,$$

$$Q_+ = \varepsilon\, \delta(\xi) \frac{\lambda+\mu}{\lambda+2\mu} a \sinh \xi a ,$$

$$Q_- = 0 ,$$

$$\overline{d}_2 = 0 ,$$

$$M_1 = -\varepsilon \frac{\delta(\xi)}{\xi} \sinh \xi a ,$$

$$N_1 = \varepsilon \frac{\delta(\xi)}{\xi} \cosh \xi a \ , \tag{111}$$

where $\delta(\xi)$ is Dirac's delta function.

Thus

$$F_1(r) = 2\mu\varepsilon \int_0^\infty \xi \ \delta(\xi) \left\{ \frac{\lambda+\mu}{\lambda+2\mu} \frac{\xi \ \delta(\xi)}{\sinh^2 \xi h} [\sinh \xi a \ \alpha(\xi) - \cosh \xi a \ \beta_1(\xi)] \right.$$

$$- \frac{\mu}{\lambda+2\mu} \sinh \xi a - \frac{\lambda+\mu}{\lambda+2\mu}(- \frac{\lambda+2\mu}{\lambda+\mu} \sinh \xi a + \xi a \cosh \xi a)$$

$$+ \frac{\lambda+\mu}{\lambda+2\mu}(\sinh \xi b \sinh \xi a \cosh \xi a - \cosh^2 \xi a \cosh \xi b)/\sinh \xi h$$

$$\left. + a\xi \cosh \xi \right\} J_0(\xi r) \ d\xi \quad . \tag{112}$$

It is not difficult to show that the above reduces to

$$F_1(r) = \frac{4\mu(\lambda+\mu)}{\lambda+2\mu} \frac{\varepsilon}{h} \ . \tag{113}$$

Similarly we have

$$F_2(r) = 0 \ . \tag{114}$$

Equations (44) and (45) become

$$\int_0^\infty \frac{\xi^2}{\sinh^2 \xi h} [M(\xi)\xi\, \alpha(\xi) + N(\xi)(\xi\, \beta_1(\xi) + \cosh \xi a \cosh \xi b \sinh \xi h]$$

$$J_0(\xi r)\, d\xi = \frac{2\varepsilon}{h}, \qquad r < 1, \tag{115}$$

$$\int_0^\infty \frac{\xi^2}{\sinh^2 \xi h} [M(\xi)(\xi\, \beta_2(\xi) - \sinh \xi b \sinh \xi a \sinh \xi h) + N(\xi)\xi\, \alpha(\xi)]$$

$$J_1(\xi r)\, d\xi = 0, \qquad r < 1, \tag{116}$$

The stress intensity factors for this problem can be obtained from the problem of a constant pressure exerted on the crack surfaces by taking

$$p_+(r) = -\frac{4\mu(\lambda+\mu)}{\lambda+2\mu} \frac{\varepsilon}{h}.$$

§ 9 Some Other Boundary Conditions

The following results can be obtained by following the procedure of Section 4. The first set of equations, (118) and (119), obtains if the outer surfaces of the plate are specified to be normal stress free, i.e., there is no tension applied along the z-axis, and if the points of the surface are fixed with respect to the z-axis, i.e., the radial component of the displacement vector is zero. The second set of equations, (121) and (122), is the result of applying the above conditions to one surface of the plate and the conditions of Section 3 to the other. In both cases the surfaces of the crack are assumed to have a constant pressure, $-p_0$, applied to them.

Thus if the surface conditions are

$$u(r,a) = u(r,b) = 0$$

$$\sigma_{zz}(r,a) = \sigma_{zz}(r,b) = 0 \ . \tag{117}$$

Then equations (44) and (45) take the form

$$\frac{2\mu(\lambda+\mu)}{\lambda+2\mu} \int_0^\infty \frac{\xi^2}{\sinh^2 \xi h} [M'(\xi)\ \alpha(\xi) + N'(\xi)(\xi\ \beta_2(\xi)$$

$$- \sinh \xi a \sinh \xi b \sinh \xi h)]\ J_0(\xi r)\ d\xi$$

$$= p_0 \ , \qquad r < 1 \ . \tag{118}$$

$$\frac{2\mu(\lambda+\mu)}{\lambda+2\mu} \int_0^\infty \frac{\xi^2}{\sinh^2 \xi h} [M'(\xi)(\xi\ \beta_1(\xi) + \cosh \xi b \cosh \xi a \sinh \xi h)$$

$$+ N'(\xi)\xi\ \alpha(\xi)]\ J_1(\xi r)\ d\xi$$

$$= 0\ , \qquad r < 1\ . \tag{119}$$

And if the surface conditions are such that

$$w(r,a) = 0$$

$$u(r,b) = 0$$

$$\sigma_{zr}(r,a) = 0$$

$$\sigma_{zz}(r,a) = 0 \tag{120}$$

then equations (44) and (45) become

$$\frac{2\mu(\lambda+\mu)}{\lambda+2\mu} \int_0^\infty \frac{\xi^2}{\cosh^2 \xi h} [M''(\xi)\xi\ \alpha(\xi) + N''(\xi)(\xi\ \beta_1(\xi)$$

$$+ \sinh \xi b \cosh \xi a \cosh \xi h)]\ J_0(\xi r)\ d\xi$$

$$= p_0\ , \qquad r < 1\ . \tag{121}$$

$$\frac{2\mu(\lambda+\mu)}{\lambda+2\mu} \int_0^\infty \frac{\xi^2}{\cosh^2 \xi h} [M''(\xi)(\xi\ \beta_2(\xi) - \cosh \xi b \sinh \xi a \cosh \xi h)$$

$$+ N''(\xi)\xi\ \alpha(\xi)]\ J_1(\xi r)\ d\xi$$

$$= 0\ , \qquad r < 1\ . \tag{122}$$

Although the above surface conditions seem unlikely to occur naturally a method of solution similar to that employed in Section 6 can be used to reduce the problems described by conditions (117) and (120) to Fredholm equations. Only the expressions on the right of the equations (118), (119) and (121), (122) change if the surface conditions are non-zero.

The expressions analogous to (44) and (45) for the boundary conditions representing the plate that is clamped on both sides and the plate that is stress free both sides are quite complicated. Consider the surface conditions for the clamped plate:

$$u(r,a) = 0$$

$$w(r,a) = 0$$

$$u(r,b) = 0$$

$$w(r,b) = 0\ , \tag{123}$$

with constant pressure prescribed on the crack surface. Furthermore, consider the surface conditions for the stress free plate:

$$\sigma_{zr}(r,a) = 0$$

$$\sigma_{zz}(r,a) = 0$$

$$\sigma_{zr}(r,b) = 0$$

$$\sigma_{zz}(r,b) = 0 \ , \tag{124}$$

with constant pressure prescribed on the crack surface.

Define the matrices

$$A = \begin{pmatrix} S_3^{+}(a\xi) & -S_3^{+}(b\xi) & -\xi a \sinh \xi a & \xi b \sinh \xi b \\ \xi a \sinh \xi a & -\xi b \sinh \xi b & S_3^{-}(a\xi) & -S_3^{-}(b\xi) \\ S_2^{+}(\xi a) & -S_2^{+}(\xi b) & C_2^{-}(a\xi) & -C_2^{-}(b\xi) \\ C_2^{+}(a\xi) & C_2^{+}(b\xi) & -S_2^{-}(a\xi) & S_2^{-}(b\xi) \end{pmatrix} \tag{125}$$

$$B = \begin{pmatrix} S_2^+(a\xi) & -S_2^+(b\xi) & -C_2^-(a\xi) & C_2^-(b\xi) \\ -C_2^-(a\xi) & C_2^-(b\xi) & -S_2^-(a\xi) & S_2^-(b\xi) \\ a\xi \sinh a\xi & -b\xi \sinh b\xi & -S_1^+(a\xi) & S_1^+(b\xi) \\ S_1^-(a\xi) & -S_1^-(b\xi) & -a\xi \sinh \xi a & b\xi \sinh \xi b \end{pmatrix} \qquad (126)$$

where

$$S_1^-(z) = \sinh z - z \cosh z$$

$$S_1^+(z) = \sinh z + z \cosh z$$

$$S_2^-(z) = \frac{\mu}{\lambda+\mu} \sinh z - z \cosh z$$

$$S_2^+(z) = \frac{\mu}{\lambda+\mu} \sinh z + z \cosh z$$

$$S_3^-(z) = \frac{\lambda+3\mu}{\lambda+\mu} \sinh z - z \cosh z$$

$$S_3^+(z) = \frac{\lambda+3\mu}{\lambda+\mu} \sinh z + z \cosh z$$

$$C_2^-(z) = \frac{\lambda+2\mu}{\lambda+\mu} \cosh z - z \sinh z$$

$$C_2^+(z) = \frac{\lambda+2\mu}{\lambda+\mu} \cosh z + z \sinh z \qquad (127)$$

Furthermore, define A_{ij} to be the determinant of the matrix obtained by deleting the ith row and the jth column of A, and B_{ij} to be the determinant of the matrix obtained in the same way from B. Then for the clamped plate the expressions analogous to (44) and (45) are

$$\frac{2\mu(\lambda+\mu)}{\lambda+2\mu} \int_0^\infty \frac{\xi^2}{\det A} [M'''(S_2^+(\xi a)\ A_{11} + C_2^-(\xi a)\ A_{13})$$

$$- N'''(S_2^+(\xi a)\ A_{21} + C_2^-(\xi a)\ A_{23})]\ J_0(\xi r)\ d\xi$$

$$= p_0\ , \qquad r < 1\ . \qquad (128)$$

$$\frac{2\mu(\lambda+\mu)}{\lambda+2\mu} \int_0^\infty \frac{\xi^2}{\det A} [M'''(C_2^+(\xi a)\ A_{11} + S_2^-(a\xi)\ A_{13})$$

$$- N'''(C_2^+(a\xi)\ A_{21} + S_2^-(a\xi)\ A_{23})]\ J_1(\xi r)\ d\xi$$

$$= 0\ , \qquad r < 1\ . \tag{129}$$

For the stress-free plate the expressions (44) and (45) become

$$\frac{2\mu(\lambda+\mu)}{\lambda+2\mu} \int_0^\infty \frac{\xi^2}{\det B} [M''''(S_1^-(a\xi)\ B_{11} - a\xi \sinh \xi a\ B_{13})$$

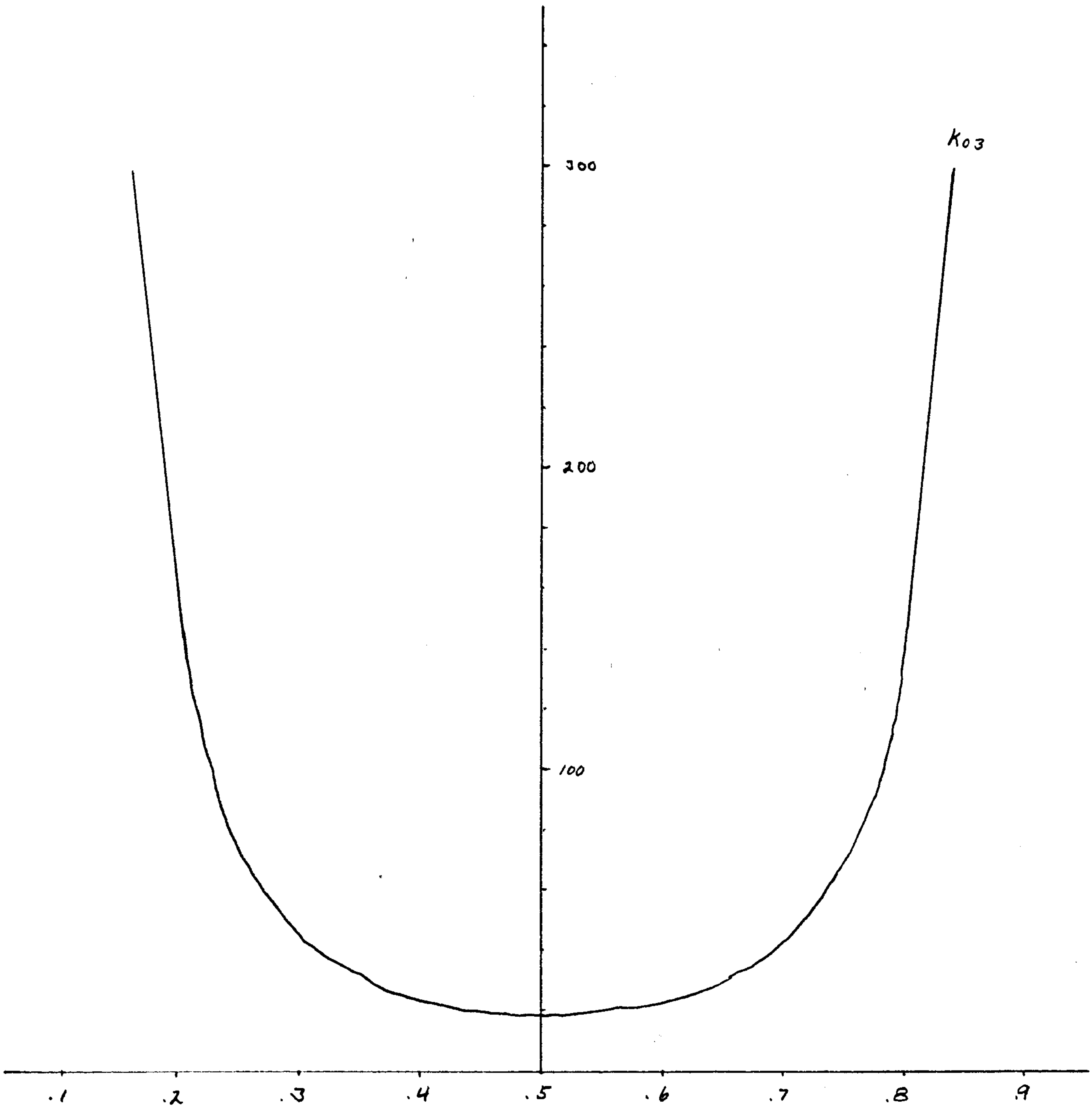
$$- N''''(S_1^-(a\xi)\ B_{21} - a\xi \sinh \xi a\ B_{23})]\ J_0(\xi r)\ d\xi$$

$$= p_0\ , \tag{130}$$

$$\frac{2\mu(\lambda+\mu)}{\lambda+2\mu} \int_0^\infty \frac{\xi^2}{\det B} [M''''(\xi a \sinh \xi a\ B_{11} + S_1^+(a\xi)\ B_{13})$$

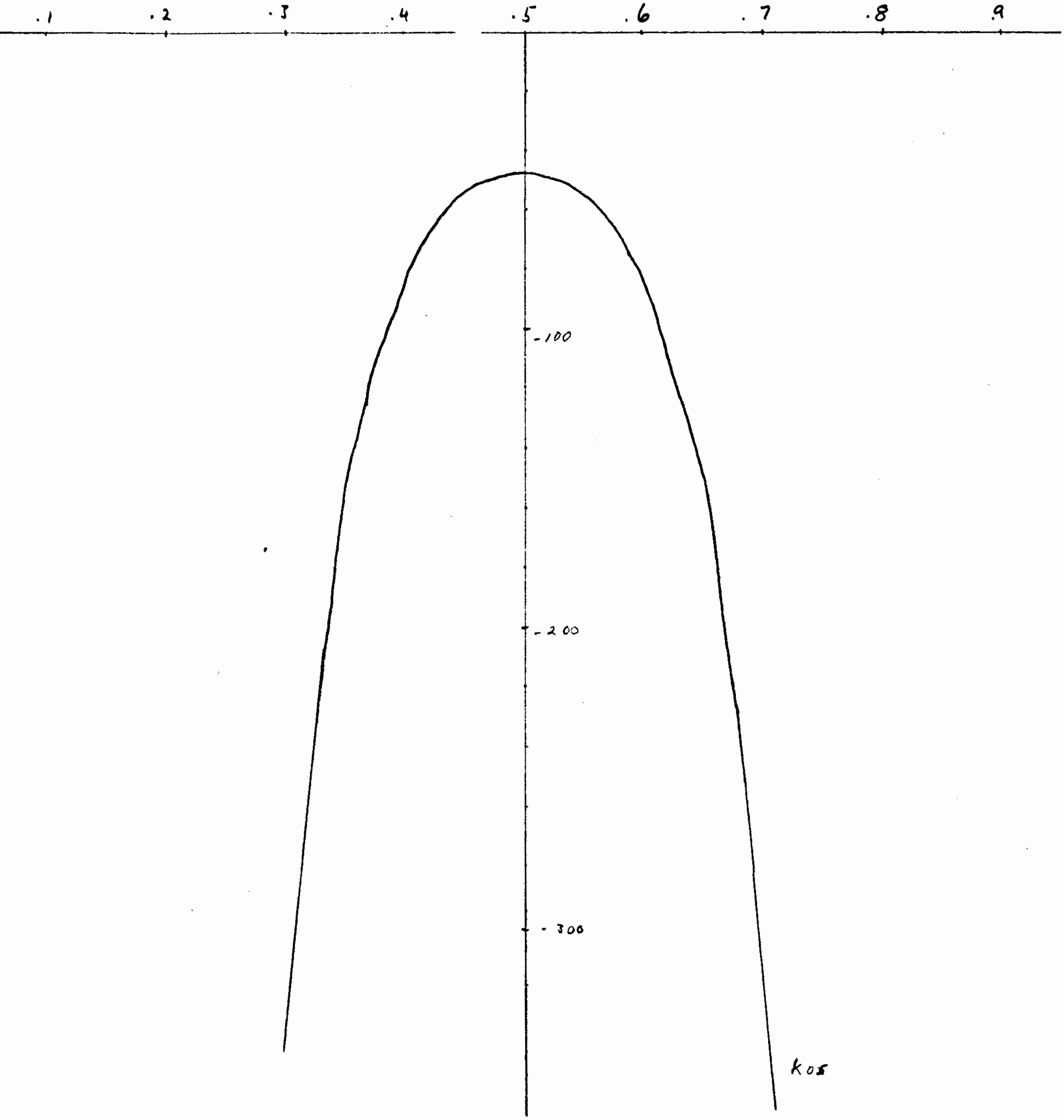
$$- N''''(\xi a \sinh \xi a\ B_{21} + S_1^+(a\xi)\ B_{23})]\ J_1(\xi r)\ d\xi$$

$$= 0\ . \tag{131}$$



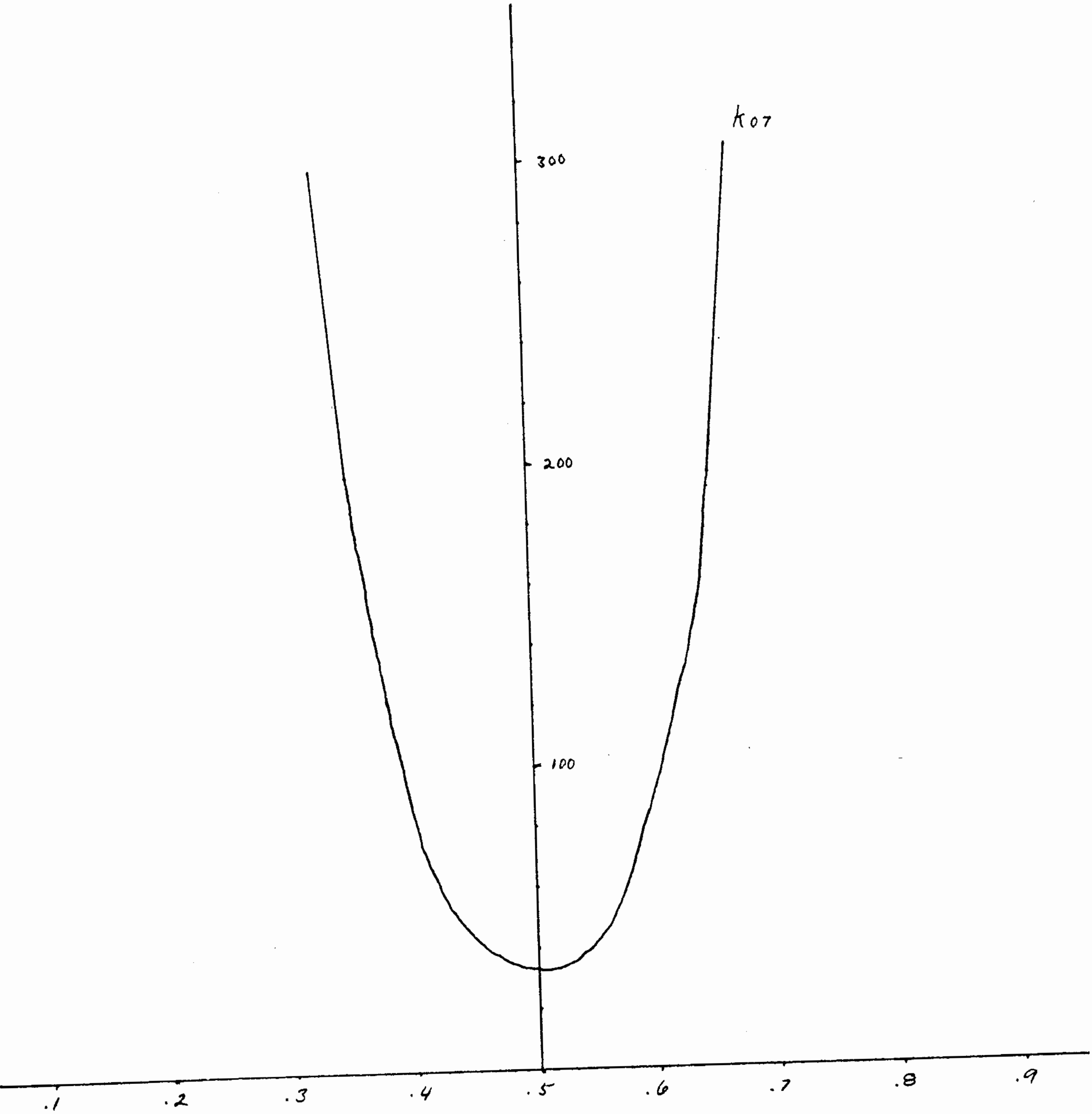
The variation of k_{03} with the value of p; i.e., the position of the crack with respect to the plate surfaces.

GRAPH II



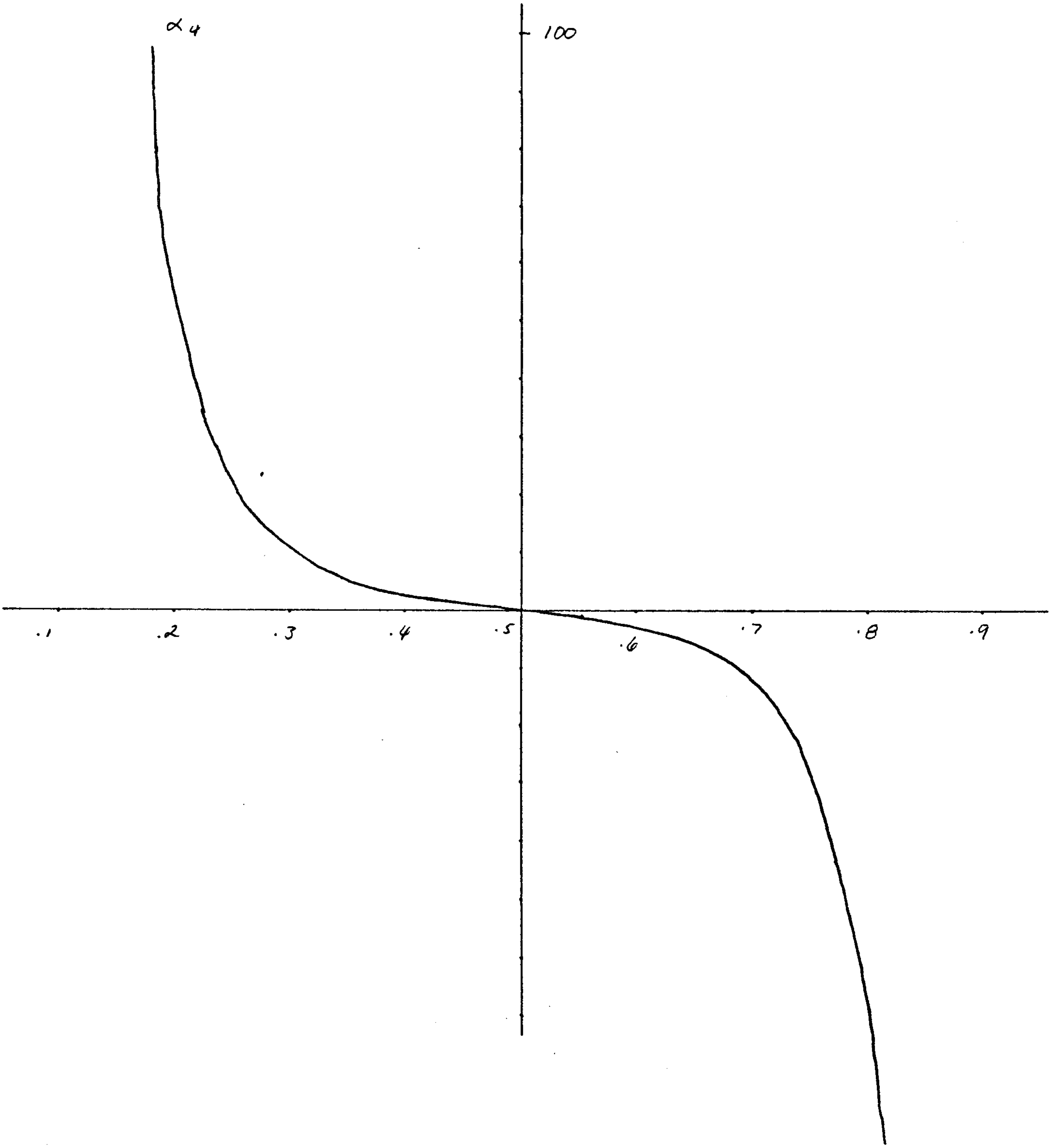
The variation of k_{05} with the position of the crack.

GRAPH III



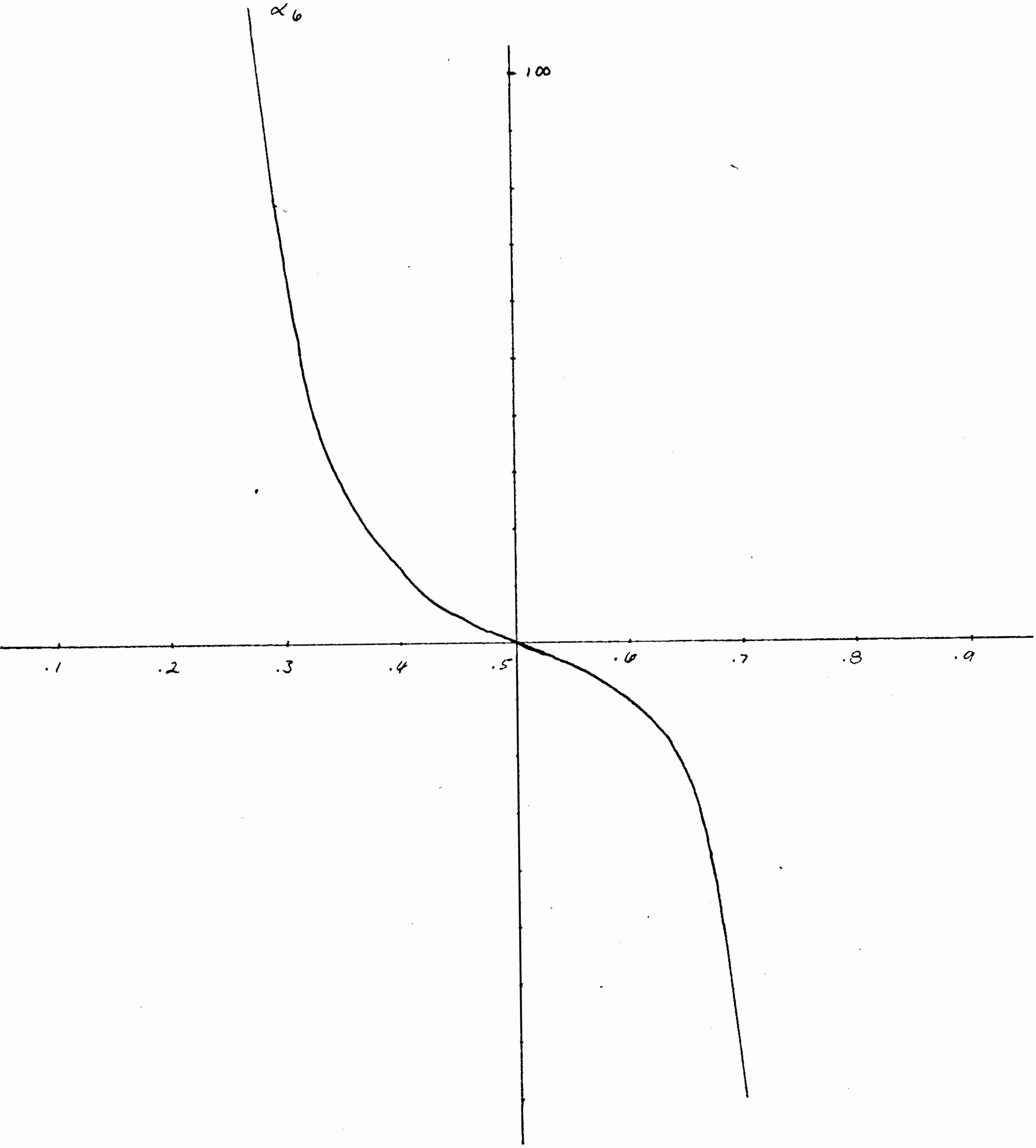
The variation of k_{07} with the position of the crack.

GRAPH IV

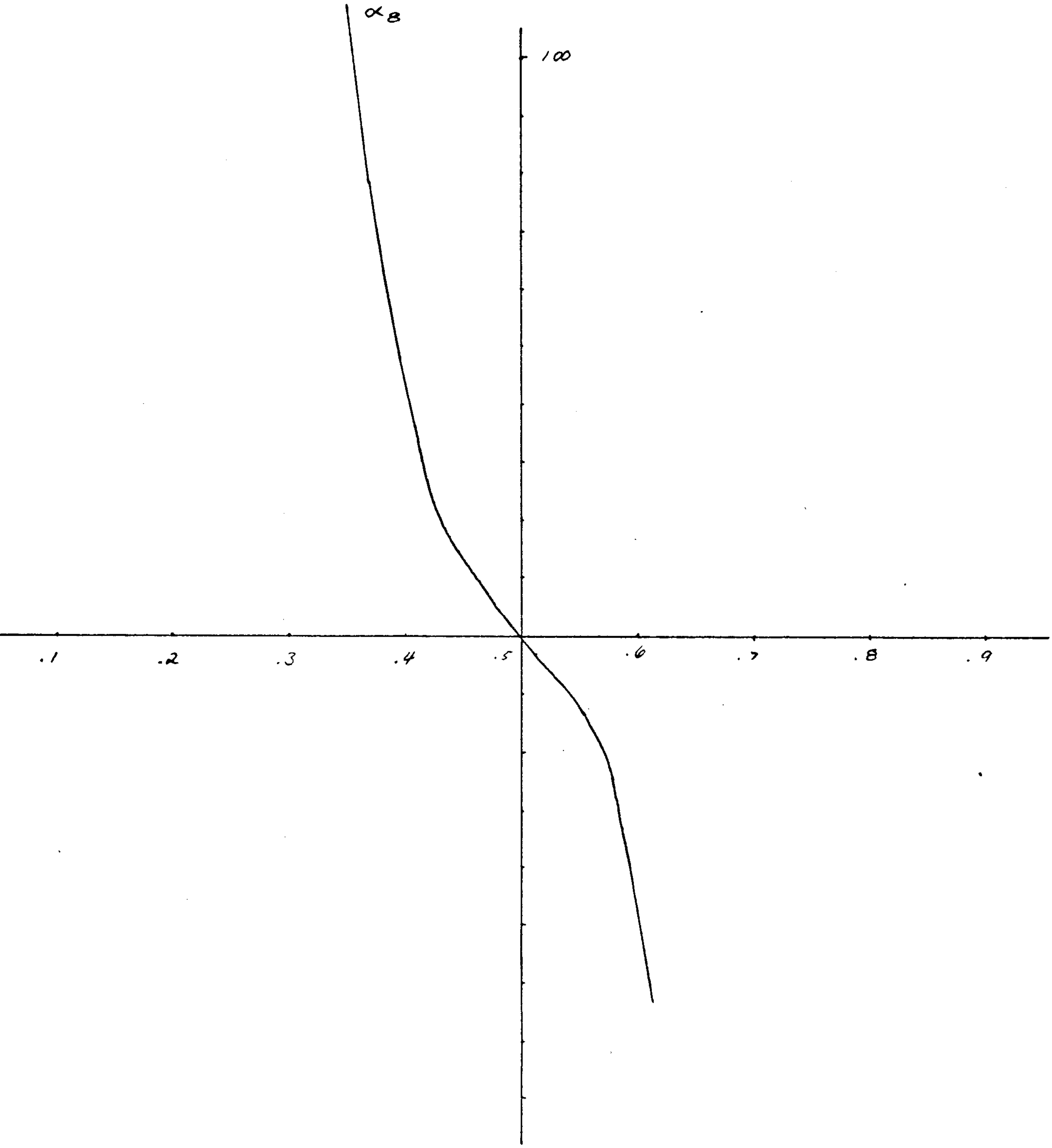


The variation of α_4 with the position of the crack.

GRAPH V



The variation of α_6 with the position of the crack.



The variation of α_8 with the position of the crack.

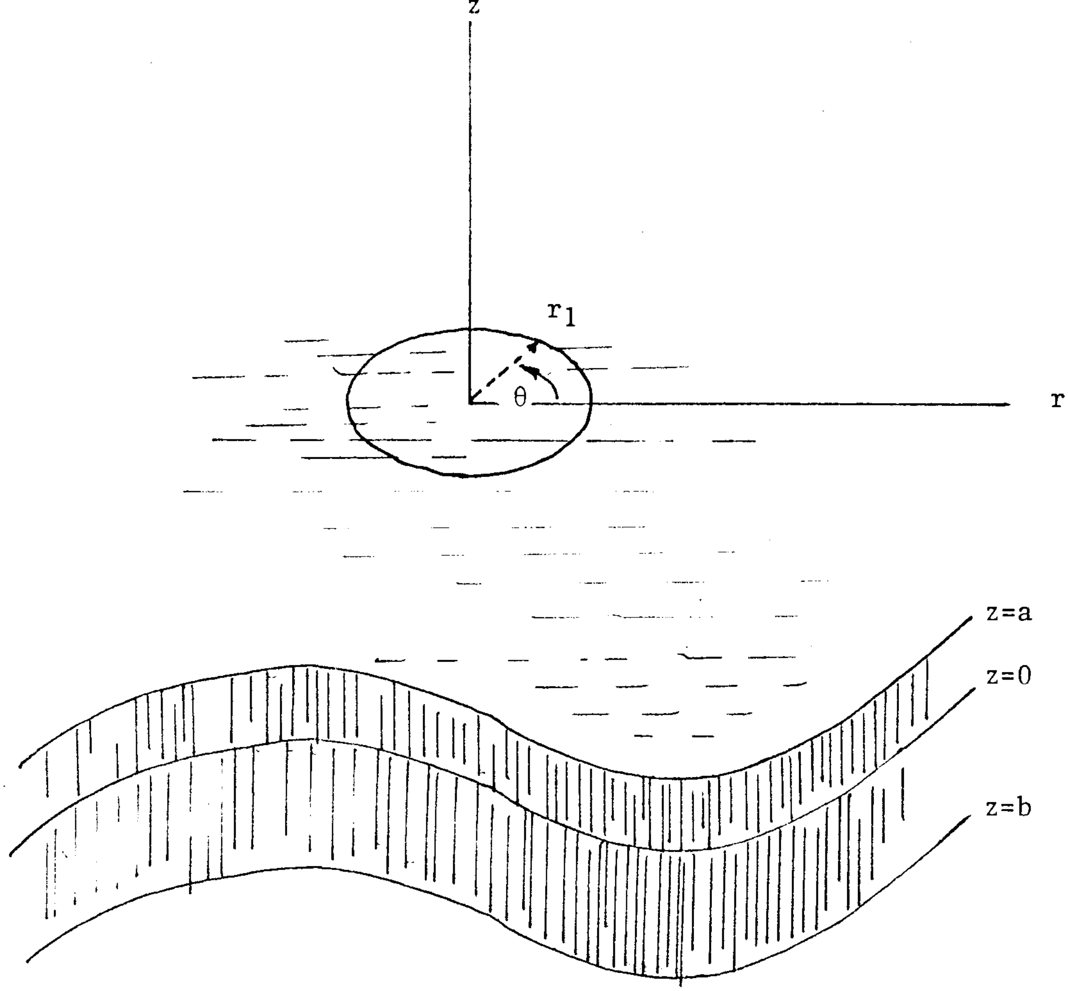


FIGURE A

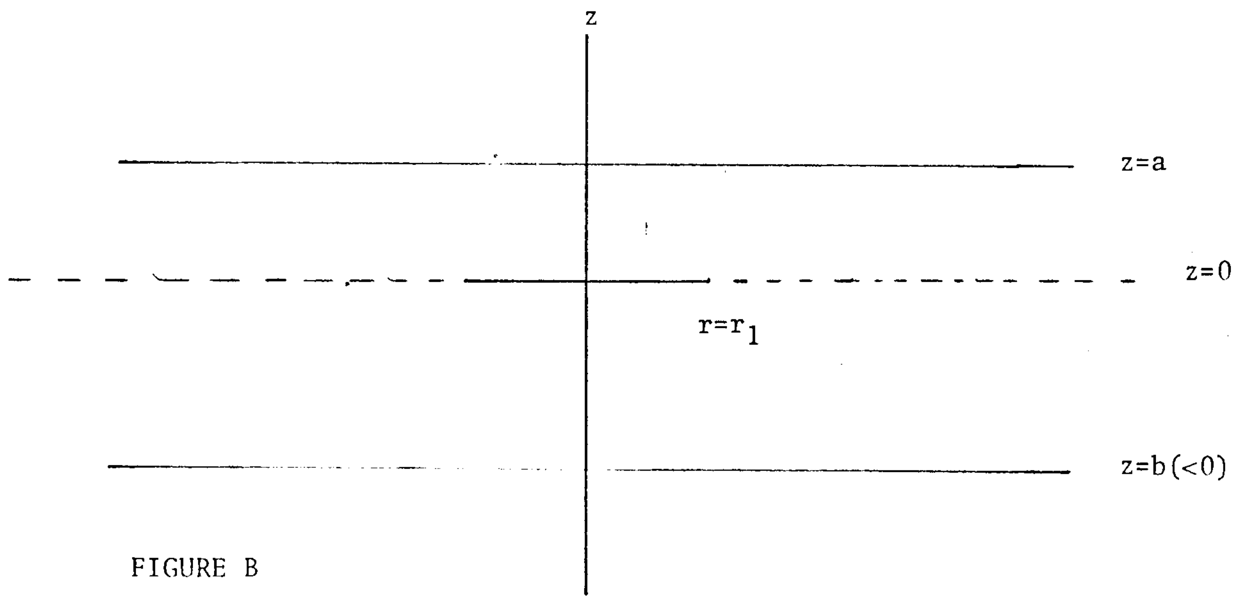


FIGURE B

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