CONTROLLABLE INFINITESIMAL DEFORMATIONS IN HOMOGENEOUSLY DEFORMED COMPRESSIBLE ELASTIC DIELECTRICS

by

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ABSTRACT

In the first part of this presentation, the theory governing infinitesimal deformations superposed on finite deformations of a compressible, homogeneous, isotropic, elastic dielectric is developed. The latter part of this paper concerns finding all possible infinitesimal deformations that can be superposed on any finite homogeneous deformations with a prescribed electric field such that the combined deformation and field can be supported without the body force and distributed charge in every homogeneous, isotropic, compressible elastic dielectric.

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1. INTRODUCTION

In finite elasticity theory, there are certain static deformations that are called universal solutions. The deformation is prescribed, and then it is shown that such a deformation can be supported without the body forces in every homogeneous, isotropic, elastic solid [1]. While there is a large class of such solutions when the elastic solid is incompressible [2], [3], Ericksen [4] has shown that only homogeneous deformations can be and are universal for compressible elastic solids.

The theory governing finite deformations of continuous elastic dielectrics subjected to the simultaneous application of mechanical forces and an electric field has been derived in recent years by Toupin [5], and Singh and Pipkin [6]. In the latter paper, the authors define also what is called a controllable state. In such a state, the deformation and the electric field are both specified to begin with and then the field equations are shown to be satisfied without the distributed charge or body forces irrespective of the form of the stored energy function. The usefulness of such a controllable state lies in the determination of elastic response functions by the comparison of theoretical results with those observed experimentally. Whereas this important feature of controllable deformations has been extensively utilized in finite elasticity theory [7], the experimentation of similar nature has not been attempted for elastic dielectrics so far. Singh and Pipkin [6] find a complete class of controllable states for homogeneous, isotropic, incompressible elastic dielectrics. Furthermore, Singh [8] has proved that when the dielectric is compressible, only homogeneous deformations with uniform electric fields are controllable.

The theory of infinitesimal deformations superposed on finite deformations was developed by Shield and Green [9] in which the authors consider the infinitesimal twist superposed on finite extension of a right circular cylinder. The same problem has been solved for an incompressible elastic dielectric by Verma and Chaudhry [10]. In this paper, the authors take it for granted that the electric field of the basic state is not affected by the superposed additional deformation.

Recently, Currie [11] has attempted to find all possible infinitesimal deformations that can be superposed on any finite homogeneous deformation without the body forces and without the knowledge of the form of the elastic response coefficients.

In this presentation, we follow the procedure laid out by Currie [11] to develop the constitutive equations for an infinitesimal deformation superposed on a finite homogeneous deformation of a compressible elastic dielectric with the electric field which is initially uniform remaining unchanged during the superposed deformation. Section 2 is devoted to such a development. Then we seek all such solutions that are possible in every homogeneous, isotropic, compressible, elastic dielectric without the body forces and distributed charge. The necessary and sufficient conditions governing these controllable states are derived in Section 3. The next three sections are devoted to finding all possible controllable states. Explicit solutions are obtained for the case when the principal stretches of the initial large static homogeneous deformation are not all equal. However, when the principal stretches of the basic deformation are all equal, it is shown that an infinite number of controllable states are permissible.

It is instructive as well as interesting to compare the results in this presentation with those of Currie [11]. Since our equations reduce to those of Currie when the electric field is set equal to zero, it is obvious that we cannot expect more controllable deformations than those listed in Currie's paper. However, because of the presence of the electric field in the constitutive equations and additional field equations, the number of elastic response coefficients is eight rather than three. Consequently, the relations which govern the controllability of states are much more restrictive here. These additional constraints allow us to determine solutions explicitly when the two principal stretches of the basic homogeneous deformation are equal, and somewhat more information about the solutions when all the principal stretches are equal.

It may be pointed out that our analysis is confined to the situation when the electric field prescribed for the basic deformation remains unchanged during the superposed infinitesimal deformation. The analytical convenience of this assumption is motivated by the fact that the electric field is uniform for controllable deformations of a compressible elastic dielectric [8], and also for an infinitesimal controllable state [12].

The technique developed in this presentation can now be employed to find all possible infinitesimal deformations superposed on finite controllable states of incompressible elastic dielectrics.

2. CONTINUUM ELECTROELASTOSTATICS

I ELECTROSTATICS

Consider a deformable elastic dielectric continuum occupying a region D and bounded by the surface ∂D. Let the body be deformed and polarized by applied mechanical forces and applied electric field. Referred to a fixed Cartesian system, let the particle at the point $\vec{X}$ in the undeformed state occupy the position $\vec{x}$ in the deformed state.

We assume that the inertial forces are negligible i.e. the deformation takes place so slowly that at any instant of time, the external forces are in equilibrium with the mechanical and electrical forces inside the medium.

According to Maxwell-Faraday electrostatic theory of dielectrics, there exist two vector fields in space: the macroscopic electric field E_i $(i = 1, 2, 3)$ with the dimensions of the force per unit charge and a macroscopic field of flux or dielectric displacement D_i $(i = 1, 2, 3)$ with the dimensions of charge per unit area. These fields satisfy the following integral equations:

$$\int_C E_i \, dx_i = 0 \tag{2.1}$$

and

$$\int_S D_i \, n_i \, dS = Q \tag{2.2}$$

where C is an arbitrary closed space curve, S the boundary of an arbitrary regular region R of D, Q the total free charge contained inside R and n_i the unit outward normal to S.

Let V_k denote the regions enclosed by the charge bearing surfaces B_k and V_0 denote the remainder of the space D occupied by the dielectric. We assume that the electric field E_i and the displacement D_i are continuously differentiable functions of position in each of the regions D, V_k and V_0. Then applying Stokes' theorem to (2.1) and noting that C is arbitrary, we obtain

$$E_{i,j} = E_{j,i} \tag{2.3}$$

where $(\ )_{,i}$ denotes the partial differentiation of $(\ )$ w. r. t. coordinate x_i.

From (2.1) it also follows that across the surface ∂D of the dielectric, the tangential component of E_i is continuous:

$$e_{ijk}(E_j^+ - E_j^-)\ n_k = 0 \tag{2.4}$$

where n_i is the unit outward normal to the surface ∂D , E_j^+ and E_j^- respectively denote the values of the electric field outside and inside the dielectric surface , and e_{ijk} is the usual permutation symbol.

We restrict our considerations to the case in which the dielectric body and its surface are free of electric charge. With this restriction, applying Gauss' divergence theorem to (2.2) and noting that S is arbitrary, we get

$$D_{i,i} = 0\ . \tag{2.5}$$

Furthermore, applying (2.2) to a cylindrical "pill box" that contains the boundary of the dielectric, we can show that the normal component of D_i is continuous across the surface ∂D of the dielectric:

$$D_i^+ n_i = D_i^- n_i \tag{2.6}$$

where as before D_i^+ and D_i^- denote respectively the values of D_i outside and inside the dielectric surface.

II MECHANICAL EQUILIBRIUM

We further assume that the resultant force F_i and the moment about the origin G_i (excluding gravitational or inertial forces and moments) exerted on any arbitrary volume V of the dielectric, can entirely be expressed in terms of the stress vector T_i acting on the surface S of V. That is

$$F_i = \int_S T_i \, dS$$

and

$$G_i = \int_S e_{ijk} \, x_j \, T_k \, dS \; .$$

The stress vector T_i accounts for all electro-mechanical forces except gravitational and inertial forces which we have excluded.

We also assume that there are no surface couples or body couples.

If V is an arbitrary volume of the dielectric bounded by the surface S, then for equilibrium we have

$$\int_S T_i \, dS + \int_V \rho \, F_i \, dV = 0$$

and (2.8)

$$\int_S e_{ijk} \, x_j \, T_k \, dS + \int_V \rho \, e_{ijk} \, x_j \, F_k \, dV = 0$$

where ρ denotes the mass density of the body and F_i represents the body force per unit mass.

By applying the first of (2.8) to a tetrahedron, it is easy to show that the stress vector T_i on a surface with outward unit normal n_i is completely determined by

$$T_i = t_{ji} n_j \tag{2.9}$$

where t_{ij} represents the stress matrix. Using (2.9) in (2.8), applying the divergence theorem and noting that the region V is arbitrary, we obtain

$$t_{ij,j} + \rho F_i = 0 \tag{2.10}$$

and

$$t_{ij} = t_{ji} . \tag{2.11}$$

If mechanical surface tractions T_i per unit area of the deformed body are applied to the surface of the dielectric, it also follows from (2.8) that:

$$T_i = (t^-_{ij} - t^+_{ij}) n_j \tag{2.12}$$

at the boundary, where our conventions on the superscripts are as in (2.4) and (2.6).

In the situation where electrical effects are absent, the stress t^+_{ij} in the medium surrounding the dielectric is taken to be zero. In this paper, due to the electrical field outside the dielectric, there is a stress t^+_{ij} called the Maxwell stress (denoted by M_{ij}) present outside the dielectric.

III CONSTITUTIVE EQUATIONS

The Maxwell equations (2.3) through (2.5) and the equilibrium equations (2.10) and (2.11), alone are insufficient to determine the behaviour of the medium. For a determinate system, apart from these equations, we also need the constitutive equations which relate the material response with the forces applied to the medium.

We assume that in free space surrounding the dielectric, the flux D_i is directly proportional to the electric field strength E_i :

$$D_i = \varepsilon E_i \tag{2.13}$$

where ε is a dielectric constant.

We also assume that in free space, the stress t_{ij} is the Maxwell stress M_{ij} defined by

$$M_{ij} = \varepsilon E_i E_j - \varepsilon/_2 E_k E_k \delta_{ij} . \tag{2.14}$$

Clearly the Maxwell stress tensor M_{ij} satisfies the equilibrium equations (2.10) and (2.11) identically when the body forces are not present.

We describe the deformation of the dielectric media by the mapping

$$x_i = x_i(X_1, X_2, X_3) \tag{2.15}$$

referred to a single fixed Cartesian coordinate system. The deformation gradients $\partial x_i/_{\partial X_j}$ then provide a measure for deformation.

If we assume that the displacement field D_i and the stress tensor t_{ij} are functions of the deformation gradients $\partial x_i/_{\partial X_j}$ and the electric field E_i , then for an isotropic, homogeneous elastic dielectric body, the constitutive equations as derived by Singh and Pipkin [6] are:

$$D_i = (\phi_0 \delta_{ij} + \phi_1 B_{ij} + \phi_2 B_{ik} B_{kj}) E_j \tag{2.16}$$

and

$$t_{ij} = N_0 \delta_{ij} + N_1 B_{ij} + N_2 B_{ik} B_{kj} + 2N_3 E_i E_j + N_4(E_i B_{jk} + E_j B_{ik}) E_k + N_5(E_i B_{jn} B_{nk} + E_j B_{in} B_{nk}) E_k \tag{2.17}$$

where B_{pq} denotes the Finger Strain tensor

$$B_{pq} = \frac{\partial x_p}{\partial X_k} \frac{\partial x_q}{\partial X_k} \tag{2.18}$$

and where ϕ_0, ϕ_1, ϕ_2; N_0, N_1, ..., N_5 are the response functions which depend upon the invariants

$$I_1 = B_{rr}, \quad I_2 = B_{rs} B_{rs}, \quad I_3 = E_r E_r, \quad I_4 = E_r B_{rs} E_s, \quad I_5 = E_p B_{pq} B_{qr} E_r, \quad I_6 = \det. B_{pq} . \tag{2.19}$$

From (2.12) and (2.14), it follows that the surface tractions T_i, per unit area of the deformed body, required to maintain the deformation are

$$T_i = (t_{ij} - M_{ij}) n_j$$

where t_{ij} is the stress tensor given by (2.17); M_{ij} is the Maxwell stress tensor given by (2.14) and n_i the unit exterior normal to the surface of the dielectric body.

An infinitesimal deformation in electro-elastostatics, called the uncoupled theory of electrostriction, is defined as the one in which the displacement gradients $\frac{\partial u_i}{\partial x_j} << 1$ and the electric field E_i is sufficiently weak so that only terms linear in $\frac{\partial u_i}{\partial x_j}$ and quadratic in E_i may be retained in the constitutive equations. Furthermore, in the uncoupled theory, the terms involving the products of E_i and $\frac{\partial u_i}{\partial x_j}$ are also neglected. Equations (2.16) and (2.17) then reduce to [13]:

$$D_i = \varepsilon_0 E_i$$

and (2.20)

$$t_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij} + a E_i E_j + b E_k E_k \delta_{ij}$$

where ε_0 is the dielectric constant; λ, μ are Lame's constants; a, b are the scalar constants which characterize the dielectric properties of the material, and e_{ij} is the strain tensor of infinitesimal theory:

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{2.21}$$

3. CONTROLLABLE INFINITESIMAL DEFORMATIONS IN HOMOGENEOUSLY DEFORMED COMPRESSIBLE DIELECTRICS

A controllable static deformation is a deformation which may be maintained in all materials of a given class without the body forces or the distributed charge.

In the present problem, we are interested in finding all possible controllable infinitesimal deformations which can be superposed on all possible controllable finite deformations in homogeneous, isotropic, compressible, elastic dielectrics. Singh [8] has proved that the only finite controllable deformations in compressible isotropic dielectrics are homogeneous ones, in which the strain tensor is constant and the electric field is uniform.

Before we proceed with our problem, we summarize below the set of equations from the previous sections.

For a finite deformation of the dielectric, the dielectric displacement field D_i and the stress tensor t_{ij} are given by

$$D_i = (\phi_0 \delta_{ij} + \phi_1 B_{ij} + \phi_2 B_{ik} B_{kj}) E_j \tag{3.1}$$

and

$$\begin{aligned} t_{ij} = {} & N_0 \delta_{ij} + N_1 B_{ij} + N_2 B_{ik} B_{kj} + 2N_3 E_i E_j \\ & + N_4(E_i B_{jk} + E_j B_{ik}) E_k + N_5(E_i B_{jn} B_{nk} + E_j B_{in} B_{nk}) E_k \end{aligned} \tag{3.2}$$

where B_{ij} is the Finger-strain tensor given by

$$B_{ij} = \frac{\partial x_i}{\partial X_k} \frac{\partial x_j}{\partial X_k} \quad . \tag{3.3}$$

The coefficients ϕ_0, ϕ_1, ϕ_2 ; $N_0, N_1, \ldots, N_5$ are functions of the following invariants:

$$I_1 = B_{rr} \ , \quad I_2 = B_{rs} B_{rs} \ , \ I_3 = E_r E_r \ , \quad I_4 = E_r B_{rs} E_s$$

$$I_5 = E_p B_{pq} B_{qr} E_r \ , \quad I_6 = \det . \ B_{pq} \ . \tag{3.4}$$

In the assumed absence of distributed charge, the flux D_i is solenoidal:

$$D_{i,i} = 0 \ . \tag{3.5}$$

The equations of equilibrium in the absence of body forces and any volume charge distribution are

$$\frac{\partial t_{ij}}{\partial x_j} = 0 \ . \tag{3.6}$$

The surface tractions T_i required to maintain the deformation are given by

$$T_i = (t_{ij} - M_{ij}) \ n_j \tag{3.7}$$

where

$$M_{ij} = \varepsilon \ E_i \ E_j - \varepsilon/2 \ E_k \ E_k \ \delta_{ij} \tag{3.8}$$

is the Maxwell stress tensor; n_i is the unit outward normal to the surface of the deformed body.

If we choose the coordinate axes along the principal directions of the strain tensor B_{ij} , then every homogeneous deformation can be described by the mapping:

$$x_1 = \lambda_1 X_1 , \quad x_2 = \lambda_2 X_2 , \quad x_3 = \lambda_3 X_3 \tag{3.9}$$

where $\lambda_1, \lambda_2, \lambda_3$ are the principal stretches.

For a homogeneous deformation (3.9), the strain tensor (3.3) is given by

$$B_{ij} = \lambda_1^2 \delta_{1i} \delta_{1j} + \lambda_2^2 \delta_{2i} \delta_{2j} + \lambda_3^2 \delta_{3i} \delta_{3j} . \tag{3.10}$$

Corresponding to this deformation field (3.9), the stress field given by (3.2), which we shall denote by $\overset{0}{t}_{ij}$ and the dielectric displacement field given by (3.1), which we shall denote by $\overset{0}{D}_i$, are:

$$\begin{aligned}
\overset{0}{t}_{11} &= N_0 + N_1 \lambda_1^2 + N_2 \lambda_1^4 + 2E_1^2(N_3 + N_4 \lambda_1^2 + N_5 \lambda_1^4) \\
\overset{0}{t}_{22} &= N_0 + N_1 \lambda_2^2 + N_2 \lambda_2^4 + 2E_2^2(N_3 + N_4 \lambda_2^2 + N_5 \lambda_2^4) \\
\overset{0}{t}_{33} &= N_0 + N_1 \lambda_3^2 + N_2 \lambda_3^4 + 2E_3^2(N_3 + N_4 \lambda_3^2 + N_5 \lambda_3^4) \\
\overset{0}{t}_{12} &= E_1 E_2 \left[2N_3 + N_4(\lambda_1^2 + \lambda_2^2) + N_5(\lambda_1^4 + \lambda_2^4)\right] \\
\overset{0}{t}_{23} &= E_2 E_3 \left[2N_3 + N_4(\lambda_2^2 + \lambda_3^2) + N_5(\lambda_2^4 + \lambda_3^4)\right] \\
\overset{0}{t}_{31} &= E_3 E_1 \left[2N_3 + N_4(\lambda_3^2 + \lambda_1^2) + N_5(\lambda_3^4 + \lambda_1^4)\right]
\end{aligned} \tag{3.11}$$

$$\begin{aligned}
\overset{0}{D}_1 &= (\phi_0 + \lambda_1^2 \phi_1 + \lambda_1^4 \phi_2) E_1 \\
\overset{0}{D}_2 &= (\phi_0 + \lambda_2^2 \phi_1 + \lambda_2^4 \phi_2) E_2 \\
\overset{0}{D}_3 &= (\phi_0 + \lambda_3^2 \phi_1 + \lambda_3^4 \phi_2) E_3 .
\end{aligned} \tag{3.12}$$

Consider now an infinitesimal deformation $\vec{u}(\vec{x})$ superposed on the basic homogeneous deformation.

The dielectric displacement and the stress matrix are functions of the strain matrix B_{ij} and the electric field E_i :

$$D_i = D_i(B_{pq}, E_p)$$

$$t_{ij} = t_{ij}(B_{pq}, E_p)$$

where E_p are constants and B_{pq} are now functions of $\vec{y} = \vec{x} + \vec{u}$.

It has been claimed [10] that when an infinitesimal deformation is superposed on a finite homogeneous deformation, then the electric field if uniform during the basic deformation remains unchanged during the superposed infinitesimal deformation. Singh[12] has also shown that if an infinitesimal deformation is controllable, then the field must be uniform. In view of these results, the electric field E_i that we take during the superposed infinitesimal deformation is identical to the constant field $E_i^{(0)}$ of the basic homogeneous deformation.

With the help of Taylor's Theorem, we can then write

$$\begin{aligned} D_i &= D_i\big|_0 + \frac{\partial D_i}{\partial B_{pq}}\bigg|_0 (B_{pq} - B_{pq}\big|_0) + \ldots \\ t_{ij} &= t_{ij}\big|_0 + \frac{\partial t_{ij}}{\partial B_{pq}}\bigg|_0 (B_{pq} - B_{pq}\big|_0) + \ldots \end{aligned} \qquad (3.13)$$

where $(\ )\big|_0$ denotes the quantity $(\ )$ in the homogeneously deformed state. Now

$$B_{pq} = \frac{\partial y_p}{\partial X_k} \frac{\partial y_q}{\partial X_k} = \frac{\partial (x_p + u_p)}{\partial X_k} \cdot \frac{\partial (x_q + u_q)}{\partial X_k}$$

$$= \frac{\partial x_p}{\partial X_k} \frac{\partial x_q}{\partial X_k} + \frac{\partial x_p}{\partial X_k} \frac{\partial x_m}{\partial X_k} \frac{\partial u_q}{\partial x_m} + \frac{\partial x_q}{\partial X_k} \frac{\partial x_m}{\partial X_k} \frac{\partial u_p}{\partial x_m} \quad \text{on neglecting}$$

second order terms in $u_{p,q}$

$$= B_{pq}|_0 + B_{pm}|_0\, u_{q,m} + B_{qm}|_0\, u_{p,m} \; .$$

Thus,

$$\left.\frac{\partial D_i}{\partial B_{pq}}\right|_0 (B_{pq} - B_{pq}|_0) = \left.\frac{\partial D_i}{\partial B_{pq}}\right|_0 (B_{pm}|_0\, u_{q,m} + B_{qm}|_0\, u_{p,m})$$

$$= 2\left.\frac{\partial D_i}{\partial B_{pq}}\right|_0 B_{pm}|_0\, u_{q,m}$$

$$= 2\left.\frac{\partial D_i}{\partial B_{pr}}\right|_0 B_{rq}|_0\, u_{p,q}$$

$$= A_{ipq}\, u_{p,q} \; ,$$

where $A_{ipq} = 2\left(\frac{\partial D_i}{\partial B_{pr}} B_{rq}\right)_0$.

Similarly,

$$\left.\frac{\partial t_{ij}}{\partial B_{pq}}\right|_0 (B_{pq} - B_{pq}|_0) = 2\left.\frac{\partial t_{ij}}{\partial B_{pr}}\right|_0 B_{rq}|_0\, u_{p,q}$$

$$= C_{ijpq}\, u_{p,q}$$

where $C_{ijpq} = 2\left(\frac{\partial t_{ij}}{\partial B_{pr}} B_{rq}\right)_0$.

Using these results, (3.13) finally takes the form

$$D_i = D_i^0 + A_{ipq}\, u_{p,q} \tag{3.14}$$

$$t_{ij} = t_{ij}^0 + C_{ijpq}\, u_{p,q} \tag{3.15}$$

where

$$A_{ipq} = 2\left(\frac{\partial D_i}{\partial B_{pr}} B_{rq}\right)_0$$
$$C_{ijpq} = 2\left(\frac{\partial t_{ij}}{\partial B_{pr}} B_{rq}\right)_0 \quad . \tag{3.16}$$

In view of (3.14), (3.15), equations (3.5) and (3.6) imply

$$A_{ipq}\, u_{p,qi} = 0 \tag{3.17}$$

and

$$C_{ijpq}\, u_{p,qj} = 0 \ . \tag{3.18}$$

In the remainder of the analysis $\underset{\sim}{B}$ *will denote the tensor having the value given by* (3.10).

Now, from (3.16)

$$\begin{aligned} A_{ipq} &= 2\,\frac{\partial D_i}{\partial B_{pr}} B_{rq} \\ &= \phi_1[\delta_{ip} B_{jq} + \delta_{jp} B_{iq}]\, E_j \\ &\quad + \phi_2[\delta_{ip} B_{jr} B_{rq} + \delta_{jp} B_{ir} B_{rq} + B_{iq} B_{jp} + B_{ip} B_{jq}]\, E_j \\ &\quad + (G^0_{pq}\,\delta_{ij} + G^1_{pq} B_{ij} + G^2_{pq} B_{ik} B_{kj})\, E_j \end{aligned} \tag{3.19}$$

where

$$\begin{aligned} \frac{1}{2} G^{\alpha}_{pq} = \frac{\partial\phi_\alpha}{\partial I_1} B_{pq} + 2\,\frac{\partial\phi_\alpha}{\partial I_2} B_{pr} B_{rq} + \frac{\partial\phi_\alpha}{\partial I_4} E_p E_r B_{rq} \\ + \frac{\partial\phi_\alpha}{\partial I_5}(E_p E_k B_{kr} B_{rq} + E_k E_r B_{kp} B_{rq}) + I_6 \frac{\partial\phi_\alpha}{\partial I_6}\delta_{pq} \ , \end{aligned}$$

for $\alpha = 0, 1, 2.$

Also ,

$$C_{ijpq} = 2\frac{\partial t_{ij}}{\partial B_{pr}} B_{rq}$$

$$\begin{aligned}
&= N_1(\delta_{ip}B_{jq}+\delta_{jp}B_{iq}) + N_2(\delta_{ip}B_{jr}B_{rq}+\delta_{jp}B_{ir}B_{rq}+B_{ip}B_{jq}+B_{jp}B_{iq}) \\
&+ N_4[\delta_{ip}E_jE_rB_{rq}+\delta_{jp}E_iE_rB_{rq}+E_iE_pB_{jq}+E_jE_pB_{iq}] \\
&+ N_5[\delta_{ip}E_jE_kB_{kr}B_{rq}+\delta_{jp}E_iE_kB_{kr}B_{rq}+E_iE_rB_{rp}B_{jq}+E_jE_rB_{rp}B_{iq} \\
&\qquad + E_iE_pB_{jr}B_{rq}+E_jE_pB_{ir}B_{rq}+E_iE_rB_{jp}B_{rq}+E_jE_rB_{ip}B_{rq}] \\
&+ H^0_{pq}\delta_{ij} + H^1_{pq}B_{ij} + B_{it}B_{tj}H^2_{pq} + 2E_iE_jH^3_{pq} \\
&+ (E_iB_{jt}+E_jB_{it})\, E_tH^4_{pq} + (E_iB_{jt}B_{ts}+E_jB_{it}B_{ts})\, E_sH^5_{pq} \quad ,
\end{aligned} \tag{3.20}$$

where

$$\begin{aligned}
\frac{1}{2}H^\alpha_{pq} &= \frac{\partial N_\alpha}{\partial I_1}B_{pq} + 2\frac{\partial N_\alpha}{\partial I_2}B_{pr}B_{rq} + \frac{\partial N_\alpha}{\partial I_4}E_pE_rB_{rq} \\
&+ \frac{\partial N_\alpha}{\partial I_5}(E_pE_kB_{kr} + E_kE_rB_{kp})B_{rq} + I_6\frac{\partial N_\alpha}{\partial I_6}\delta_{pq}
\end{aligned}$$

for $\alpha = 0, 1, 2, \ldots, 5$.

These expressions for A_{ipq} and C_{ijpq} are evaluated in the homogeneously finite deformed state described by (3.9).

Since we want to find all controllable infinitesimal deformations that can be superposed, we therefore intend to seek the solutions $\vec{u}(\vec{x})$ of (3.17) and (3.18) which hold irrespective of the response functions ϕ_0, ϕ_1, ϕ_2 ; $N_0, N_1, \ldots, N_5$. If such a solution is to exist, it is necessary and sufficient that in (3.17) and (3.18) the coefficient of each response function and its derivative be zero.

Substituting (3.19) into (3.17) and equating to zero the coefficients of each response function and their derivatives, we have the

the following set of equations to be satisfied by $\vec{u}$:

$$
\begin{aligned}
E_i\, u_{p,pi} &= E_i\, B_{pq}\, u_{p,qi} = E_i\, B_{pr}\, B_{rq}\, u_{p,qi} = E_i\, E_p\, E_r\, B_{rq}\, u_{p,qi} \\
&= E_i (E_p\, E_k\, B_{kr} + E_k\, E_r\, B_{kp})\, B_{rq}\, u_{p,qi} \\
&= E_j\, B_{jq}\, u_{p,pq} + E_p\, B_{iq}\, u_{p,qi} \\
&= E_j\, B_{jr}\, B_{rq}\, u_{p,pq} + E_p\, B_{ir}\, B_{rq}\, u_{p,qi} + E_j\, B_{iq}\, B_{jp}\, u_{p,qi} \\
&\quad + E_j\, B_{ip}\, B_{jq}\, u_{p,qi} = 0\ .
\end{aligned}
\tag{3.21}
$$

Substituting (3.20) into (3.18) and equating to zero the coefficients of each response functions and their derivatives; and on removing the redundancies, we find that $\vec{u}$ must also satisfy the following set of equations:

$$
\begin{aligned}
u_{p,pi} &= B_{pq}\, u_{p,qi} = B_{pq}\, u_{i,pq} = B_{pr}\, B_{rq}\, u_{p,qi} \\
&= B_{pr}\, B_{rq}\, u_{i,pq} = E_p\, E_r\, B_{rq}\, u_{p,qi} \\
&= (E_p\, E_k\, B_{kr}\, B_{rq} + E_k\, E_r\, B_{kp}\, B_{rq})\, u_{p,qi} \\
&= E_j (E_r\, B_{rq}\, u_{i,qj} + E_p\, B_{iq}\, u_{p,qj}) \\
&= E_j (E_k\, B_{kr}\, B_{rq}\, u_{i,qj} + E_r\, B_{rp}\, B_{iq}\, u_{p,qj}) = 0\ .
\end{aligned}
\tag{3.22}
$$

Thus $\vec{u}$ has to satisfy (3.21) and (3.22) if (3.17) and (3.18) hold irrespective of the response functions. We show now that the set of equations (3.21) are identically satisfied in view of (3.22) and are thus redundant.

$E_i\, u_{p,pi} \equiv 0$ in view of (3.22) (i)

$E_i\, B_{pq}\, u_{p,qi} \equiv 0$ in view of (3.22) (ii)

$E_i\, B_{pr}\, B_{rq}\, u_{p,qi} \equiv 0$ in view of (3.22) (iv)

$E_i\, E_p\, E_r\, B_{rq}\, u_{p,qi} \equiv 0$ in view of (3.22) (vi)

$E_i (E_p\, E_k\, B_{kr}\, B_{rq}\, u_{p,qi} + E_k\, E_r\, B_{kp}\, B_{rq}\, u_{p,qi}) \equiv 0$ in view of (3.22) (vii)

$E_j\, B_{jq}\, u_{p,pq} + E_p\, B_{iq}\, u_{p,qi} \equiv 0$ in view of (3.22) (i) and (iii)

$E_j\, B_{jr}\, B_{rq}\, u_{p,pq} + E_p\, B_{ir}\, B_{rq}\, u_{p,qi} + E_j\, B_{iq}\, B_{jp}\, u_{p,qi} + E_j\, B_{ip}\, B_{jq}\, u_{p,qi} \equiv 0$

on using (3.22) (i), (v), (iii) and (ii) in the first, second, third and fourth terms respectively.

Hence conditions (3.22) are necessary and sufficient that $\vec{u}(\vec{x})$ generates a controllable superposed infinitesimal deformation.

Three cases arise now depending upon whether the principal stretches are equal or unequal. We discuss these cases in the next sections.

4. GENERAL CASE - WHEN NO TWO PRINCIPAL STRETCHES ARE EQUAL

Here we discuss the most general case when $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$. We use (3.10) to simplify (3.22).

The equations $u_{p,pi} = 0$, $B_{pq}\, u_{p,qi} = 0$, $B_{pr}\, B_{rq}\, u_{p,qi} = 0$ become

$$u_{1,1i} + u_{2,2i} + u_{3,3i} = 0$$

$$\lambda_1^2\, u_{1,1i} + \lambda_2^2\, u_{2,2i} + \lambda_3^2\, u_{3,3i} = 0$$

$$\lambda_1^4\, u_{1,1i} + \lambda_2^4\, u_{2,2i} + \lambda_3^4\, u_{3,3i} = 0 \,.$$

Since the determinant of coefficients $(\lambda_1^2 - \lambda_2^2)(\lambda_2^2 - \lambda_3^2)(\lambda_3^2 - \lambda_1^2) \neq 0$, therefore

$$u_{1,1i} = u_{2,2i} = u_{3,3i} = 0 \,, \tag{4.1}$$

Again the equations $B_{pq}\, u_{i,pq} = 0$, $B_{pr}\, B_{rq}\, u_{i,pq} = 0$ yield

$$u_{i,11} = u_{i,22} = u_{i,33} = 0 \,. \tag{4.2}$$

Also the remaining equations in (3.22), with the use of (3.10), (4.1) and (4.2) simplify to:

$$E_1E_2(\lambda_1^2u_{2,1i}+\lambda_2^2u_{1,2i})+E_2E_3(\lambda_2^2u_{3,2i}+\lambda_3^2u_{2,3i})+E_3E_1(\lambda_3^2u_{1,3i}+\lambda_1^2u_{3,1i}) = 0$$

$$E_1E_2(\lambda_1^2+ {}_2^2)(\lambda_1^2u_{2,1i}+\lambda_2^2u_{1,2i})+E_2E_3(\lambda_2^2+\lambda_3^2)(\lambda_2^2u_{3,2i}+\lambda_3^2u_{2,3i})$$
$$+E_3E_1(\lambda_3^2+\lambda_1^2)(\lambda_3^2u_{1,3i}+\lambda_1^2u_{3,1i}) = 0$$

$$E_1E_2[(\lambda_1^2+\lambda_2^2)u_{i,12}+B_{i3}(u_{2,31}+u_{1,23})]+E_2E_3[(\lambda_2^2+\lambda_3^2)u_{i,23}+B_{i1}(u_{3,12}+u_{2,13})]$$
$$+E_3E_1[(\lambda_1^2+\lambda_3^2)u_{i,13}+B_{i2}(u_{3,21}+u_{1,23})] = 0,$$

and

$$E_1E_2[(\lambda_1^4+\lambda_2^4)u_{i,12}+B_{i3}(\lambda_1^2u_{1,32}+\lambda_2^2u_{2,31})]+E_2E_3[(\lambda_2^4+\lambda_3^4)u_{i,23} \tag{4.3}$$

$$+B_{i1}(\lambda_2^2u_{2,31}+\lambda_3^2u_{3,12})] + E_3E_1[(\lambda_3^4+\lambda_1^4)u_{i,31}+B_{i2}(\lambda_3^2u_{3,21}+\lambda_1^2u_{1,23})] = 0 .$$

We now consider the following three subcases of this case.

(A) *Let the electric field vector be parallel to one of the coordinate planes, say* x_2x_3 *-plane* $(E_1 = 0 , \; E_2 \neq 0 , \; E_3 \neq 0)$.

Then in this case, the equations (4.1), (4.2), (4.3) further simplify to

$$\begin{gathered} u_{1,1i} = u_{2,2i} = u_{3,3i} = u_{i,11} = u_{i,22} = u_{i,33} = 0 \\ \lambda_2^2 \, u_{3,21} + \lambda_3^2 \, u_{2,31} = 0 \\ (\lambda_2^2 + \lambda_3^2) \, u_{1,23} + \lambda_1^2(u_{3,12} + u_{2,13}) = 0 \\ (\lambda_2^4 + \lambda_3^4) \, u_{1,23} + \lambda_1^2(\lambda_2^2 \, u_{2,31} + \lambda_3^2 \, u_{3,21}) = 0 . \end{gathered} \tag{4.4}$$

The general solution $\vec{u}$ satisfying

$$u_{1,1i} = u_{2,2i} = u_{3,3i} = u_{i,11} = u_{i,22} = u_{i,33} = 0$$

is

$$\begin{aligned} u_1 &= c_1 \, x_2 \, x_3 + d_1 \, x_1 + e_1 \, x_2 + f_1 \, x_3 + \text{const.} \\ u_2 &= c_2 \, x_3 \, x_1 + d_2 \, x_1 + e_2 \, x_2 + f_2 \, x_3 + \text{const.} \\ u_3 &= c_3 \, x_1 \, x_2 + d_3 \, x_1 + e_3 \, x_2 + f_3 \, x_3 + \text{const.} \end{aligned} \tag{4.5}$$

where c_r, d_r, e_r, f_r $(r = 1, 2, 3)$ are arbitrary constants.

If $\vec{u}$ has also to satisfy the other three equations in (4.4) then, using (4.5), we have

$$c_2 \lambda_3^2 + c_3 \lambda_2^2 = 0$$

$$c_1(\lambda_2^2 + \lambda_3^2) + c_2 \lambda_1^2 + c_3 \lambda_1^2 = 0$$

$$c_1(\lambda_2^4 + \lambda_3^4) + c_2 \lambda_1^2 \lambda_2^2 + c_3 \lambda_1^2 \lambda_3^2 = 0 .$$

Since the determinant $2\lambda_1^2 \lambda_2^2 \lambda_3^2(\lambda_2^2 - \lambda_3^2) \neq 0$, therefore

$$c_1 = c_2 = c_3 = 0 .$$

Hence it follows that the general solution of (4.4) is a linear displacement field so that in this case the only controllable infinitesimal deformations which can be superposed are the homogeneous ones.

(B) *Let the electric field be along one of the coordinate axes - say along the* x-*axis* $(E_1 \neq 0 , E_2 = E_3 = 0)$.

In this case, all the equations in (4.3) are satisfied identically in view of (4.1) and (4.2), and the only equations left to be satisfied by $\vec{u}$ are:

$$u_{1,1i} = u_{2,2i} = u_{3,3i} = 0$$
$$u_{i,11} = u_{i,22} = u_{i,33} = 0 . \qquad (4.6)$$

Apart from homogeneous deformations, the general solution of (4.6) is

$$u_1 = k_1 x_2 x_3 , \quad u_2 = k_2 x_1 x_3 , \quad u_3 = k_3 x_1 x_2 \qquad (4.7)$$

where k_1, k_2, k_3 are arbitrary constants. This deformation can be given a physical interpretation. The displacement u_1 represents a shear in the x_2-direction, the magnitude of the shear varying linearly with x_3. Thus the total deformation (4.7) represents the superposition of three such shear displacements.

If $k_1 + k_2 + k_3 = 0$, the deformation may be thought of as a pure torsional deformation about two of the axes, say a twist k_3 about the x_1-axis and a twist $-k_1$ about the x_3-axis.

Calculating the stresses produced by the finite homogeneous deformation (3.9) together with infinitesimal deformation (4.7), we have, on using (3.11), (3.15) and (3.2),

$$\begin{aligned}
t_{11} &= t^0_{11} = N_0 + N_1 \lambda_1^2 + N_2 \lambda_1^4 + 2E_1^2(N_3 + N_4 \lambda_1^2 + N_5 \lambda_1^4) \\
t_{22} &= t^0_{22} = N_0 + N_1 \lambda_2^2 + N_2 \lambda_2^4 \\
t_{33} &= t^0_{33} = N_0 + N_1 \lambda_3^2 + N_2 \lambda_3^4 \\
t_{12} &= (\lambda_1^2 k_2 + \lambda_2^2 k_3)\, x_3\, [N_1 + N_4 E_1^2 + (\lambda_1^2 + \lambda_2^2)(N_2 + N_5 E_1^2)] \\
t_{23} &= (\lambda_2^2 k_3 + \lambda_3^2 k_2)[N_1 + N_2(\lambda_2^2 + \lambda_3^2)]\, x_1 \\
t_{31} &= (\lambda_1^2 k_3 + \lambda_3^2 k_1)\, [N_1 + N_4 E_1^2 + (\lambda_1^2 + \lambda_3^2)(N_2 + N_5 E_1^2)]\, x_2 .
\end{aligned} \tag{4.8}$$

Suppose that in its initial homogeneously deformed state, the material is in the form of a rectangular block with faces $x_i = \pm a_i$. The infinitesimal deformation (4.7) deforms the point (a_1, x_2, x_3) on the plane $x_1 = a_1$ into the point (y_1, y_2, y_3) where

$$y_1 = a_1 + k_1 x_2 x_3 , \quad y_2 = x_2 + k_2 a_1 x_3 , \quad y_3 = x_3 + k_3 a_1 x_2 .$$

Eliminating x_2 , x_3 from these equations, we get

$$f(y_1, y_2, y_3) \equiv (1 - k_2 k_3 a_1^2) y_1 - a_1(1 - k_2 k_3 a_1^2)^2$$

$$- k_1(y_2 y_3 - k_2 a_1 y_3^2)(1 - k_2 k_3 a_1^2) + k_1 k_3 a_1(y_2 - k_2 a_1 y_3^2) = 0$$

which is the equation of the surface to which the plane $x_1 = a_1$ deforms after infinitesimal deformation. The normal to this surface in the deformed state is proportional to

$$\left(\frac{\partial f}{\partial y_1}, \frac{\partial f}{\partial y_2}, \frac{\partial f}{\partial y_3}\right) = [(1 - k_2 k_3 a_1^2)^2 , k_1(1 - k_2 k_3 a_1^2)(k_3 a_1 x_2 - x_3),$$

$$k_1(1 - k_2 k_3 a_1^2)(k_2 a_1 x_3 - x_2)] .$$

Thus in the final deformed state, the normal to the bounding surface which was originally in the homogeneous deformed state as $x_1 = a_1$ is parallel to

$$\vec{n} = [1 - k_2 k_3 a_1^2 , k_1(k_3 a_1 x_2 - x_3) , k_1(k_2 a_1 x_3 - x_2)] \quad . \quad (4.9)$$

Thus, by (3.7), the components of surface traction T_1, T_2, T_3 on this face necessary to maintain the deformation are given by

$$T_i = (t_{ij} - M_{ij}) \frac{n_j}{|\vec{n}|} \quad (4.10)$$

where t_{ij} is given by (4.8), M_{ij} by

$$M_{ij} = \varepsilon E_i E_j - \varepsilon/2 \, E_k E_k \delta_{ij} , \quad (4.11)$$

and $\vec{n}$ by (4.9).

Substituting (4.8), (4.11) and (4.9) into (4.10), and linearizing with respect to k_1, k_2, k_3, we find that to the first order in the infinitesimal displacements, the surface tractions on the face $x_1 = a_1$, required to maintain this deformation, are:

$$
\begin{aligned}
T_1 &= N_0 + N_1 \lambda_1^2 + N_2 \lambda_1^4 + 2E_1^2(N_3 + N_4 \lambda_1^2 + N_5 \lambda_1^4 - \varepsilon/4) \\
T_2 &= \Big[\{N_1 + N_4 E_1^2 + (\lambda_1^2 + \lambda_2^2)(N_2 + N_5 E_1^2)\}(k_1 \lambda_2^2 + k_2 \lambda_1^2) \\
&\qquad - k_1(N_0 + N_1 \lambda_2^2 + N_2 \lambda_2^4 + \varepsilon/2\, E_1^2)\Big] x_3 \qquad (4.12) \\
T_3 &= \Big[[(N_1 + N_4 E_1^2) + (N_2 + N_5 E_1^2)(\lambda_1^2 + \lambda_3^2)](k_1 \lambda_3^2 + k_3 \lambda_1^2) \\
&\qquad - k_1(N_0 + N_1 \lambda_3^2 + N_2 \lambda_3^4 + \varepsilon/2\, E_1^2)\Big] x_2 .
\end{aligned}
$$

The surface tractions required on the other faces to maintain the deformation to the first order of infinitesimal displacements can be calculated in a similar manner.

(C) *Let the electric field be a general one* (i.e. $E_1 \neq 0$, $E_2 \neq 0$, $E_3 \neq 0$).

Using (4.1) and (4.2), the first equation in (4.3) yields

$$\lambda_2^2 u_{3,21} + \lambda_3^2 u_{2,31} = 0$$

$$\lambda_1^2 u_{3,21} + \lambda_3^2 u_{1,23} = 0$$

$$\lambda_1^2 u_{2,13} + \lambda_2^2 u_{1,23} = 0 .$$

Since the determinant of coefficients $2\lambda_1^2 \lambda_2^2 \lambda_3^2 \neq 0$, therefore

$$u_{1,23} = u_{2,31} = u_{3,12} = 0 .$$

In view of this, (4.1) and (4.2), the remaining equations in (4.3) are identically satisfied.

Hence in this case, the equations to be satisfied by $\vec{u}$ reduce to

$$\begin{aligned} u_{1,1i} &= u_{2,2i} = u_{3,3i} = 0 , \\ u_{i,11} &= u_{i,22} = u_{i,33} = 0 , \\ u_{1,23} &= u_{2,13} = u_{3,12} = 0 . \end{aligned} \tag{4.13}$$

Apart from a homogeneous deformation, the general solution $\vec{u}$ of

$$u_{i,11} = u_{i,22} = u_{i,33} = u_{1,1i} = u_{2,2i} = u_{3,3i} = 0$$

is

$$u_1 = k_1 x_2 x_3 , \quad u_2 = k_2 x_1 x_3 , \quad u_3 = k_3 x_1 x_2 . \tag{4.14}$$

This solution has also to satisfy the equations

$$u_{1,23} = u_{2,13} = u_{3,12} = 0$$

which gives $k_1 = k_2 = k_3 = 0$.

Hence this case also does not furnish any non-homogeneous controllable infinitesimal deformation that can be superposed.

5. BIAXIAL CASE - WHEN TWO PRINCIPAL STRETCHES ARE EQUAL

Here we discuss the case, when the basic homogeneous deformation has two equal principal stretches, say $\lambda = \lambda_1 = \lambda_2 \neq \lambda_3$.

Then using (3.10), the equations (3.22) reduce to

$$
\begin{aligned}
& u_{1,1i} + u_{2,2i} = u_{3,3i} = 0 \\
& u_{i,11} + u_{i,22} = u_{i,33} = 0 \\
& (E_1^2 - E_2^2)\, u_{1,1i} + E_1 E_2 (u_{1,2i} + u_{2,1i}) = 0 \\
& E_2 E_3 (\lambda^2 u_{3,2i} + \lambda_3^2 u_{2,3i}) + E_3 E_1 (\lambda^2 u_{3,1i} + \lambda_3^2 u_{1,3i}) = 0 \\
& \lambda^2 [(E_1^2 - E_2^2)\, u_{1,11} + 2E_1 E_2 u_{1,12}] + E_2 E_3 [(\lambda^2 + \lambda_3^2)\, u_{1,23} \\
& \qquad + (\lambda^2 - \lambda_3^2)\, u_{2,13}] + 2\lambda^2 E_1 E_3 u_{1,13} = 0 \\
& (E_1^2 - E_2^2)\, u_{2,11} + 2E_1 E_2 u_{2,12} + 2E_2 E_3 u_{2,23} + E_1 E_3 (u_{1,23} + u_{2,13}) = 0 \\
& (E_1^2 - E_2^2)\, u_{3,11} + 2E_1 E_2 u_{3,12} = 0 \\
& E_2 E_3 (u_{1,23} - u_{2,13}) = 0 \\
& E_1 E_3 (u_{2,13} - u_{1,23}) = 0 \, .
\end{aligned}
\tag{5.1}
$$

Once again, the following subcases may be considered.

(A) *When the electric field is parallel to one of the coordinate planes.*

(i) Let $\vec{E}$ *be parallel to* $x_2 x_3$ *-plane*, i.e. $E_1 = 0$, $E_2 \neq 0$, $E_3 \neq 0$.

Then from (5.1), $\vec{u}$ satisfies the following set of equations:

$$u_{1,1i} + u_{2,2i} = u_{3,3i} = u_{i,11} + u_{i,22} = u_{i,33} = 0$$

$$u_{1,1i} = 0 , \quad \lambda^2 u_{3,2i} + \lambda_3^2 u_{2,3i} = 0$$

$$(\lambda^2 + \lambda_3^2) u_{1,23} + (\lambda^2 - \lambda_3^2) u_{2,13} = 0$$

$$-E_2 u_{2,11} + 2E_3 u_{2,23} = 0 , \quad u_{3,12} = 0 , \quad u_{1,23} - u_{2,13} = 0 .$$

These equations can be further simplified to

$$u_{1,1i} = u_{2,2i} = u_{3,3i} = u_{i,11} = u_{i,22} = u_{i,33} = 0$$

and (5.2)

$$u_{1,23} = u_{2,13} = u_{3,12} = 0$$

which is the same set of equations as (4.13). Thus in this case also, apart from homogeneous deformations, there is no other infinitesimal deformation which can be superposed.

This result is also true when the electric field is parallel to the $X_3 X_1$ plane, i.e. $E_2 = 0$, $E_3 \neq 0$, $E_1 \neq 0$.

(ii) Let $\vec{E}$ *be parallel to the* X_1X_2 *-plane,* i.e. $E_3 = 0$, $E_1 \neq 0$, $E_2 \neq 0$.

Here two subcases arise, according as $E_1 = E_2$ or $E_1 \neq E_2$.

(α) Let $E_1 = E_2$.

Then the equations (5.1) simplify to

$$u_{1,1i} + u_{2,2i} = u_{3,3i} = u_{i,11} + u_{i,22} = u_{i,33} = 0 \qquad \text{(a)}$$

$$u_{1,2i} + u_{2,1i} = 0 \qquad \text{(b)} \quad (5.3)$$

$$u_{3,12} = u_{1,12} = u_{2,12} = 0 \ . \qquad \text{(c)}$$

Apart from homogeneous deformations, the general solution of (a) is

$$u_1 = \phi_{,2} + x_3\,\psi_{,2}\ , \quad u_2 = -\phi_{,1} - x_3\,\psi_{,1}\ , \quad u_3 = \chi \qquad (5.4)$$

where ϕ, ψ, χ are functions of x_1 and x_2 only, and

$$\nabla_1^2\,\phi = 0\ , \quad \nabla_1^2\,\psi = \text{const.} = c \quad \text{(say)}\ , \quad \nabla_1^2\,\chi = 0\ . \qquad (5.5)$$

Here

$$\nabla_1^2 \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\ .$$

Equation (5.3) (b) implies

$$u_{1,2} + u_{2,1} = \text{const.} = 2K \quad \text{(say)}$$

which on using (5.4) and (5.5) yields

$$\phi = \frac{K}{2}(x_2^2 - x_1^2) + B'\,x_1\,x_2 + C'\,x_1 + D'\,x_2 + \text{const.}$$

or $\quad \phi(x_1, x_2) = A_1(x_1^2 - x_2^2) + B_1\,x_1\,x_2 + C_1\,x_1 + D_1\,x_2 + \text{const.}$

and

$$\psi_{,11} = \psi_{,22} = c/2\ . \qquad (5.6)$$

Using (5.4) - (5.6) into (5.3 (c)), χ and ψ have the forms:

$$\chi(x_1, x_2) = p(x_1^2 - x_2^2) + q\, x_1 + r\, x_2 + \text{const.}\ ,$$

and (5.7)

$$\psi(x_1, x_2) = \frac{c}{4}(x_1^2 + x_2^2) + B_2\, x_1\, x_2 + C_2\, x_1 + D_2\, x_2 + \text{const.}$$

In view of (5.6) and (5.7), the equations (5.4) yield

$$u_1 = \frac{c}{2}\, x_2\, x_3 + B_2\, x_1\, x_3 - B_1\, x_1 - 2A_1\, x_2 + D_2\, x_3 + D_1\ ,$$

$$u_2 = -\frac{c}{2}\, x_1\, x_3 - B_2\, x_2\, x_3 - 2A_1\, x_1 - B_1\, x_2 - C_2\, x_3 - C_1\ ,$$

and

$$u_3 = \chi = p(x_1^2 - x_2^2) + q\, x_1 + r\, x_2 + \text{const.}$$

Thus apart from homogeneous deformation, the general solution of (5.3) is:

$$\begin{aligned} u_1 &= A\, x_2\, x_3 + B\, x_1\, x_3 \\ u_2 &= -B\, x_2\, x_3 - A\, x_1\, x_3 \qquad (5.8) \\ u_3 &= p(x_1^2 - x_2^2) \end{aligned}$$

where A, B, p are arbitrary constants.

For infinitesimal deformation (5.8) the stress field can be obtained from (3.15):

$$t_{11} = N_0 + \lambda^2 N_1 + \lambda^4 N_2 + 2E_1^2(N_3 + \lambda^2 N_4 + \lambda^4 N_5)$$
$$+ 2\lambda^2[N_1 + 2\lambda^2 N_2 + 2E_1^2(N_4 + 2\lambda^2 N_5)] B x_3$$
$$t_{22} = N_0 + \lambda^2 N_1 + \lambda^4 N_2 + 2E_1^2(N_3 + \lambda^2 N_4 + \lambda^4 N_5)$$
$$- 2\lambda^2[N_1 + 2\lambda^2 N_2 + 2E_1^2(N_4 + 2\lambda^2 N_5)] B x_3$$
$$t_{33} = t_{33}^0 = N_0 + N_1 \lambda_3^2 + N_2 \lambda_3^4 \tag{5.9}$$
$$t_{12} = t_{12}^0 = 2E_1^2(N_3 + \lambda^2 N_4 + \lambda^4 N_5)$$
$$t_{23} = -[N_1 + (\lambda^2 + \lambda_3^2) N_2][\lambda_3^2(A x_1 + B x_2) + 2p \lambda^2 x_2]$$
$$+ E_1^2[N_4 + (\lambda^2 + \lambda_3^2) N_5][2p \lambda^2 + \lambda_3^2(B - A)](x_1 - x_2)$$
$$t_{31} = [N_1 + (\lambda^2 + \lambda_3^2) N_2][2p \lambda^2 x_1 + \lambda_3^2(A x_2 + B x_1)]$$
$$+ [N_4 + (\lambda^2 + \lambda_3^2) N_5] E_1^2 [2p \lambda^2 + \lambda_3^2(B - A)](x_1 - x_2) .$$

If in the initially homogeneously deformed state, the body is in the shape of a rectangular block bounded by surfaces $x_i = \pm a_i$, then the infinitesimal deformation (5.8) deforms the point (a_1, x_2, x_3) on the plane $x_1 = a_1$ into the point

$$[a_1 + A x_2 x_3 + B a_1 x_3 , x_2 - B x_2 x_3 - A a_1 x_3 , x_3 + p(a_1^2 - x_2^2)] .$$

In the final deformed state the normal to the boundary surface which was originally in the homogeneous state $x_1 = a_1$ is parallel to

$$\vec{n} = [1-Bx_3-pB(a_1^2-x_2^2), -Ax_3-pA(a_1^2-x_2^2), -Ba_1-Ax_2+(A^2-B^2)a_1x_3] . \tag{5.10}$$

Hence, by (3.7), the components of surface tractions T_1, T_2, T_3 on this face necessary to maintain the deformation are given by

$$T_i = (t_{ij} - M_{ij})\, n_j / |\vec{n}| \tag{5.11}$$

where t_{ij} is given by (5.9), M_{ij} by (3.8) and $\vec{n}$ by (5.10). Substituting (5.9), (3.8), (5.10) into (5.11), noting that $E_3 = 0$, $E_2 = E_1 \neq 0$, and linearizing with respect to the constants A, B, p, we find that, to the first order in displacements,

$$\begin{aligned}
T_1 &= N_0 + \lambda^2 N_1 + \lambda^4 N_2 + 2E_1^2(N_3 + \lambda^2 N_4 + \lambda^4 N_5)(1 - A x_3) \\
&\quad + 2\lambda^2 B x_3[N_1 + 2\lambda^2 N_2 + 2E_1^2(N_4 + 2\lambda^2 N_5)] - \varepsilon A E_1^2 x_3 \\
T_2 &= E_1^2[2(1 - A x_3)(N_3 + \lambda^2 N_4 + \lambda^4 N_5) - \varepsilon] + A(N_0 + \lambda^2 N_1 + \lambda^4 N_2) x_3 \\
T_3 &= E_1^2[N_0 + (\lambda^2 + \lambda_3^2) N_5][2p \lambda^2 + \lambda_3^2(B - A)](x_1 - x_2) \\
&\quad - [N_1 + (\lambda^2 + \lambda_3^2) N_2][\lambda_3^2(B x_2 + A x_1) + 2p \lambda^2 x_2] \\
&\quad - [N_0 + \lambda_3^2 N_1 + \lambda_3^4 N_2 + \varepsilon E_1^2](B a_1 + A x_2) .
\end{aligned} \tag{5.12}$$

(β) Now let $E_3 = 0$, $E_1 \neq E_2 \neq 0$.

In this case, the equations (5.1) simplify to

$$\begin{aligned}
&u_{1,1i} + u_{2,2i} = u_{3,3i} = u_{i,11} + u_{i,22} = u_{i,33} = 0 && \text{(a)} \\
&(E_1^2 - E_2^2)\, u_{1,1i} + E_1 E_2(u_{1,2i} + u_{2,1i}) = 0 && \text{(b)} \\
&(E_1^2 - E_2^2)\, u_{1,11} + 2E_1 E_2\, u_{1,12} = 0 && \text{(c)} \\
&(E_1^2 - E_2^2)\, u_{2,11} + 2E_1 E_2\, u_{2,12} = 0 && \text{(d)}
\end{aligned} \tag{5.13}$$

$$(E_1^2 - E_2^2)\, u_{3,11} + 2E_1 E_2\, u_{3,12} = 0 \;. \qquad \text{(e)}$$

Apart from homogeneous deformations, the general solution of (5.13 (a)) is

$$u_1 = \frac{\partial\alpha}{\partial x_2}\,, \quad u_2 = -\frac{\partial\alpha}{\partial x_1}\,, \quad u_3 = \chi(x_1, x_2)$$

where $\alpha = \phi(x_1, x_2) + x_3\, \psi(x_1, x_2)$,

$$\nabla_1^2\, \alpha = c\, x_3\,, \quad \nabla_1^2\, \chi = 0\,, \quad \nabla_1^2 \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \;.$$

These may also be written as

$$u_1 = \phi_{,2} + x_3\, \psi_{,2}\,, \quad u_2 = -\phi_{,1} - x_3\, \psi_{,1}\,, \quad u_3 = \chi \qquad (5.14)$$

where ϕ, ψ, χ are functions of x_1, x_2 only, and

$$\nabla_1^2\, \phi = 0\,, \quad \nabla_1^2\, \chi = 0\,, \quad \nabla_1^2\, \psi = \text{const.} = c \;. \qquad (5.15)$$

Now (5.13 (b)) implies

$$(E_1^2 - E_2^2)\, u_{1,1} + E_1 E_2 (u_{1,2} + u_{2,1}) = \text{const.} = K \quad \text{(say)}$$

which, on using (5.14) and (5.15), gives

$$(E_1^2 - E_2^2)\, \psi_{,21} + 2E_1 E_2\, \psi_{,22} = c\, E_1 E_2$$

and (5.16)

$$(E_1^2 - E_2^2)\, \phi_{,1} + 2E_1 E_2\, \phi_{,2} = K\, x_2 + f(x_1)$$

where f is an arbitrary function. The equation (5.13 (c)) is identically satisfied in view of (5.14) - (5.16).

Using (5.14) - (5.16) into (5.13 (d)) and solving the resulting partial differential equations in ϕ and ψ by Langrangian Method, we find that ψ and ϕ are of the forms

$$\begin{aligned} \psi &= A_1 x_1^2 + B_1 x_2^2 + C_1 x_1 x_2 + D_1 x_1 + F_1 x_2 + \text{const.} \\ \phi &= A_2 x_1^2 + B_2 x_2^2 + C_2 x_1 x_2 + D_2 x_1 + F_2 x_2 + \text{const.} \end{aligned} \tag{5.17}$$

where $A_1, \ldots, F_1$ and $A_2, \ldots, F_2$ are arbitrary constants such that

$$2(A_1 + B_1) = c \, , \quad A_2 + B_2 = 0$$

and (5.18)

$$(E_1^2 - E_2^2) C_1 + 4 B_1 E_1 E_2 = c E_1 E_2$$

using (5.14) and (5.15) into (5.13 (e)), and solving the resulting partial differential equation for χ, we find that χ assumes the form

$$\chi = A_3 x_1^2 + B_3 x_2^2 + C_3 x_1 x_2 + D_3 x_1 + F_3 x_2 + \text{const.}$$

where

$$A_3 + B_3 = 0 \, , \tag{5.19}$$

and

$$A_3(E_1^2 - E_2^2) + C_3 E_1 E_2 = 0 \, .$$

Thus, with the explicit forms of ϕ, ψ and χ, when used in (5.14), we get

$$u_1 = A\,x_2\,x_3 + B\,x_1\,x_3 + C\,x_1 + D\,x_2 + F\,x_3 + \text{const.}$$

$$u_2 = -B\,x_2\,x_3 + (A - c)\,x_1\,x_3 + D\,x_1 - C\,x_2 + G\,x_3 + \text{const.}$$

$$u_3 = p(x_1^2 - x_2^2) + q\,x_1\,x_2 + r\,x_1 + s\,x_2 + \text{const.}$$

where we have put

$$2B_1 = A\,,\; C_1 = B\,,\; C_2 = C\,,\; 2B_2 = D\,,\; F_1 = F\,,\; 2A_1 = c - 2B_1 = c - A$$

and (5.20)

$$A_3 = p\,,\; C_3 = q\,,\; D_3 = r\,,\; F_3 = s\,.$$

Hence apart from homogeneous deformations, the solution of (5.13) is:

$$\begin{aligned} u_1 &= A\,x_2\,x_3 + B\,x_1\,x_3 \\ u_2 &= -B\,x_2\,x_3 + (A - c)\,x_1\,x_3 \\ u_3 &= p(x_1^2 - x_2^2) + q\,x_1\,x_2\,, \end{aligned} \tag{5.21}$$

where in view of (5.18), (5.19), and (5.20),

$$B(E_1^2 - E_2^2) + 2A\,E_1\,E_2 = c\,E_1\,E_2$$

$$p(E_1^2 - E_2^2) + q\,E_1\,E_2 = 0\,. \tag{5.22}$$

Calculating the stresses produced by the homogeneous deformation combined with the infinitesimal deformation (5.21), we find, on using (3.10), (3.15) and (3.20), that

$$t_{11} = N_0 + \lambda^2 N_1 + \lambda^4 N_2 + 2E_1^2(N_3 + \lambda^2 N_4 + \lambda^4 N_5)$$
$$+ 2\lambda^2[N_1 + 2\lambda^2 N_2 + (E_1^2 + E_2^2)(N_4 + 2\lambda^2 N_5)] B x_3$$

$$t_{22} = N_0 + \lambda^2 N_1 + \lambda^4 N_2 + 2E_2^2(N_3 + \lambda^2 N_4 + \lambda^4 N_5)$$
$$- 2\lambda^2[N_1 + 2\lambda^2 N_2 + (E_1^2 + E_2^2)(N_4 + 2\lambda^2 N_5)] B x_3$$

$$t_{33} = t_{33}^0 = N_0 + \lambda_3^2 N_1 + \lambda_3^4 N_2 \tag{5.23}$$

$$t_{12} = 2E_1 E_2(N_3 + \lambda^2 N_4 + \lambda^4 N_5) + \lambda^2[N_1 + 2\lambda^2 N_2$$
$$+ (E_1^2 + E_2^2)(N_4 + 2\lambda^2 N_5)](2A - c) x_3$$

$$t_{23} = [N_1 + (\lambda^2 + \lambda_3^2) N_2][\{\lambda^2 q + \lambda^2(A - c)\} x_1 - (2p \lambda^2 + B \lambda_3^2) x_2]$$
$$+ [N_4 + (\lambda^2 + \lambda_3^2) N_5][E_1 E_2\{(2\lambda^2 p + B \lambda_3^2) x_1 + (\lambda^2 q + \lambda_3^2 A) x_2\}$$
$$+ E_2^2\{(\lambda^2 q + \lambda_3^2(A - c)) x_1 - (2\lambda^2 p + B \lambda_3^2) x_2\}]$$

$$t_{31} = [N_1 + (\lambda^2 + \lambda_3^2) N_2][(2\lambda^2 p + B \lambda_3^2) x_1 + (\lambda^2 q + A \lambda_3^2) x_2]$$
$$+ [N_4 + (\lambda^2 + \lambda_3^2) N_5][E_1^2\{(2p \lambda^2 + B \lambda_3^2) x_1 + (q \lambda^2 + A \lambda_3^2) x_2\}$$
$$+ E_1 E_2\{(q \lambda^2 + \lambda_3^2(A - c)) x_1 - (2p \lambda^2 + B \lambda_3^2) x_2\}]$$

where the constants B, A, c, p, q are subject to conditions (5.22).

The infinitesimal deformation (5.21) deforms the point (a_1, x_2, x_3) on the plane $x_1 = a_1$ in the initially homogeneously deformed state into the point

$$(a_1+Ax_2x_3+Ba_1x_3 \ , \quad x_2-Bx_2x_3+(A-c)a_1x_3 \ , \quad x_3+p(a_1^2-x_2^2)+qa_1x_2) \ .$$

The normal to this bounding surface is parallel to the vector $\vec{n}$, where

$$\vec{n} = [1-Bx_3 - Bp(a_1^2-x_2^2) - Bqa_1x_2 \, , \, - Ax_3 - Ap(a_1^2-x_2^2) - Aqa_1x_2 \, ,$$
$$- Ba_1 - Ax_2 - B^2a_1x_3 - A(A-c)a_1x_3] \, . \qquad (5.24)$$

The components of surface traction T_i on this surface necessary to maintain this deformation are given by (5.11), where now t_{ij} are given by (5.23), M_{ij} by (3.8) and $\vec{n}$ by (5.24).

Substituting (5.23), (3.8) and (5.24) into (5.11), noting that $E_3 = 0$, and linearizing w. r. t. A, B, p, q, A' = A - c , we have to the first order of displacements:

$$T_1 = t_{11} - \varepsilon/_2 \, (E_1^2 - E_2^2) - (t_{12}^0 - \varepsilon \, E_1 \, E_2) \, A \, x_3$$
$$T_2 = t_{21} - \varepsilon \, E_1 \, E_2 - A[t_{22}^0 - \varepsilon/_2(E_2^2 - E_1^2)] \, x_3 \qquad (5.25)$$
$$T_3 = t_{13} + t_{33}(-B \, a_1 - A \, x_2)$$

where t_{ij} are given by (5.23) and t_{ij}^0 by (3.9) [with $\lambda_1 = \lambda_2 = \lambda$, $E_3 = 0$].

(B) *When the electric field is parallel to one of the coordinate axes.*

(i) Let $\vec{E}$ *be parallel to* x_1 *-axis,* i.e. $E_1 \neq 0$, $E_2 = E_3 = 0$.

In this case the equations (5.1) simplify to

$$u_{1,1i} + u_{2,2i} = u_{3,3i} = u_{i,11} + u_{i,22} = u_{i,33} = 0$$

$$u_{1,1i} = 0 \, , \quad u_{1,11} = u_{2,11} = u_{3,11} = 0$$

or

$$u_{1,1i} = u_{2,2i} = u_{3,3i} = u_{i,11} = u_{i,22} = u_{i,33} = 0 \ . \tag{5.26}$$

The general solution of (5.26), apart from homogeneous deformations, is

$$u_1 = k_1 x_2 x_3 \ , \quad u_2 = k_2 x_1 x_3 \ , \quad u_3 = k_3 x_1 x_2 \tag{5.27}$$

which is the same as (4.7).

Thus the stresses t_{ij} in the final deformed state and the surface tractions T_i required on the face $x_1 = a_1$ to maintain the deformations (5.26), to the first order of displacements are given by changing $\lambda_1 = \lambda_2 = \lambda$ in (4.8) and (4.12) respectively.

The case when $E_2 \neq 0$, $E_3 = E_1 = 0$ also yields the deformation (5.27) and hence gives a similar result as above.

(ii) Let $\vec{E}$ *be parallel to* x_3 *-axis,* i.e. $E_3 \neq 0$, $E_1 = {-}_2 = 0$.

Then the equations (5.1) reduce to

$$u_{1,1i} + u_{2,2i} = u_{3,3i} = u_{i,11} + u_{i,22} = u_{i,33} = 0 \ . \tag{5.28}$$

Apart from homogeneous deformations, the general solution of (5.28) is

$$u_i = \phi_{,2} + x_3 \psi_{,2} \ , \quad u_2 = -\phi_{,1} - x_3 \psi_{,1} \ , \quad u_3 = \chi \tag{5.29}$$

where ϕ, ψ, χ are functions of x_1, x_2 only and $\nabla_1^2 \phi = \nabla_1^2 \chi = 0$, $\nabla_1^2 \psi = \text{const.} = c$ (say), and where

$$\nabla_1^2 \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \ . \tag{5.30}$$

Substituting (5.29), (3.10) and (3.11) into (3.15), the stresses in the final deformed state are:

$$
\begin{aligned}
t_{11} &= N_0 + \lambda^2 N_1 + \lambda^4 N_2 + 2\lambda^2(N_1 + 2\lambda^2 N_2)(\phi_{,21} + x_3 \psi_{,21}) \\
t_{22} &= N_0 + \lambda^2 N_1 + \lambda^4 N_2 - 2\lambda^2(N_1 + 2\lambda^2 N_2)(\phi_{,21} + x_3 \psi_{,21}) \\
t_{33} &= t^0_{33} = N_0 + \lambda_3^2 N_1 + \lambda_3^4 N_2 + 2E_3^2(N_3 + \lambda_3^2 N_4 + \lambda_3^4 N_5) \\
t_{12} &= \lambda^2(N_1 + 2\lambda^2 N_2)[(\phi_{,22} - \phi_{,11}) + x_3(\psi_{,22} - \psi_{,11})] \\
t_{23} &= [N_1 + (\lambda^2 + \lambda_3^2) N_2 + E_3^2\{N_4 + (\lambda^2 + \lambda_3^2) N_5\}](\lambda^2 \chi_{,2} - \lambda_3^2 \psi_{,1}) \\
t_{31} &= [N_1 + (\lambda^2 + \lambda_3^2) N_2 + E_3^2\{N_4 + (\lambda^2 + \lambda_3^2) N_5\}](\lambda^2 \chi_{,1} + \lambda_3^2 \psi_{,2})
\end{aligned}
\tag{5.31}
$$

We consider a particular case of the general infinitesimal deformation (5.29), when $\phi = \chi = 0$. This may be thought of as the indentation of the sides of the rectangular block. In the homogeneously deformed state, the block is assumed to be contained within the planes $x_i = \pm a_i$.

Then a superposed deformation (5.29) with $\phi = \chi = 0$ implies that there is no displacement in the x_3-direction. In this particular case, (5.29) and (5.31) reduce to

$$u_1 = x_3 \psi_{,2}, \quad u_2 = -x_3 \psi_{,1}, \quad u_3 = 0 \tag{5.32}$$

where

$$\nabla_1^2 \psi = \frac{\partial^2 \psi}{\partial x_1^2} + \frac{\partial^2 \psi}{\partial x_2^2} = \text{const.} = c$$

and

$$t_{11} = N_0 + \lambda^2 N_1 + \lambda^4 N_2 + 2\lambda^2(N_1 + 2\lambda^2 N_2)\ x_3\ \psi_{,12}$$

$$t_{22} = N_0 + \lambda^2 N_1 + \lambda^4 N_2 - 2\lambda^2(N_1 + 2\lambda^2 N_2)\ x_3\ \psi_{,12}$$

$$t_{33} = t^0_{33} = N_0 + \lambda_3^2 N_1 + \lambda_3^4 N_2 + 2E_3^2(N_3 + \lambda_3^2 N_4 + \lambda_3^4 N_5)$$

$$t_{12} = \lambda^2(N_1 + 2\lambda^2 N_2)(\psi_{,22} - \psi_{,11})\ x_3 \tag{5.33}$$

$$t_{23} = -[N_1 + (\lambda^2 + \lambda_3^2)\ N_2 + E_3^2\{N_4 + (\lambda^2 + \lambda_3^2)\ N_5\}]\ \lambda_3^2\ \psi_{,1}$$

$$t_{31} = [N_1 + (\lambda^2 + \lambda_3^2)\ N_2 + E_3^2\{N_4 + (\lambda^2 + \lambda_3^2)\ N_5\}]\ \lambda_3^2\ \psi_{,2}\ .$$

The infinitesimal deformation (5.32) deforms the point (a_1, x_2, x_3) on the plane $x_1 = a_1$ in the homogeneous state into the point

$$\left(a_1 + x_3 \left.\frac{\partial\psi}{\partial x_2}\right|_{x_1=a_1},\ \ x_2 - x_3 \left.\frac{\partial\psi}{\partial x_1}\right|_{x_1=a_1},\ \ x_3\right).$$

Denoting this new point by (y_1, y_2, y_3), we see that

$$y_1 = a_1 + x_3 \left(\left.\frac{\partial\psi}{\partial x_2}\right|_{x_1=a_1}\right) = a_1 + x_3\ g(x_2) \quad \text{say}$$

$$y_2 = x_2 - x_3 \left(\left.\frac{\partial\psi}{\partial x_1}\right|_{x_1=a_1}\right) = x_2 - x_3\ f(x_2) \quad \text{say}$$

$$y_3 = x_3\ .$$

The equation of the bounding surface, which was the plane $x_1 = a_1$ in the initially homogeneously deformed state is:

$$F(y_1, y_2, y_3) \equiv y_2 - g^{-1}(z) + y_3\ f[g^{-1}(z)] = 0 \tag{5.34}$$

where $z = \dfrac{y_1 - a_1}{y_3}$.

The normal to this bounding surface is parallel to the vector $\left(\frac{\partial F}{\partial y_1}, \frac{\partial F}{\partial y_2}, \frac{\partial F}{\partial y_3}\right)$ or is parallel to the vector $\vec{n}$, where

$$\vec{n} = \Big[\psi_{,2}\big|_{x_1=a_1} - x_3(\psi_{,2}\ \psi_{,12})_{x_1=a_1}\ ,\ -x_3(\psi_{,2}\ \psi_{,22})_{x_1=a_1}\ ,$$
$$-1 + x_3(\psi_{,12})_{x_1=a_1} - x_3(\psi_{,1}\ \psi_{,2}\ \psi_{,12})_{x_1=a_1}\Big]\ . \qquad (5.35)$$

The components of surface tractions T_i on this bounding surface (5.34), which shall maintain the infinitesimal deformation (5.32), are given by (5.11), where t_{ij} to be used are furnished by (5.33), M_{ij} by (3.8), and $\vec{n}$ by (5.35).

Substituting (5.33), (3.8), (5.35) into (5.11) and noting that $E_3 \neq 0$, $E_1 = E_2 = 0$ and further linearizing with respect to the derivatives of ψ, to the first order in displacements, we have

$$T_1 = (N_0 + \lambda^2 N_1 + \lambda^4 N_2 + \varepsilon/2\ E_3^2)\ \psi_{,2}\big|_{x_1=a_1}$$
$$- [N_1 + (\lambda^2 + \lambda_3^2)\ N_2 + E_3^2\{N_4 + (\lambda^2 + \lambda_3^2)\ N_5\}]\ \lambda_3^2\ \psi_{,2}$$
$$T_2 = -t_{23} = [N_1 + (\lambda^2 + \lambda_3^2)\ N_2 + E_3^2\{N_4 + (\lambda^2 + \lambda_3^2)\ N_5\}]\ \lambda_3^2\ \psi_{,2}$$
$$T_3 = (t_{33} + \varepsilon/2\ E_3^2)(-1 + x_3\ \psi_{,12}\big|_{x_1=a_1} \qquad (5.36)$$
$$= [N_0 + \lambda_3^2 N_1 + \lambda_3^4 N_2 + 2E_3^2(N_3 + \lambda_3^2 N_4 + \lambda_3^4 N_5) + \varepsilon/2\ E_3^2]$$
$$\times (-1 + x_3\ \psi_{,12}\big|_{x_1 = a_1})\ .$$

(C) *When the electric field is a general one* $(E_1 \neq 0,\ E_2 \neq 0,\ E_3 \neq 0)$

Two subcases arise according as $E_1 = E_2$ or $E_1 \neq E_2$.

(i) Let $E_1 \neq E_2$.

In this case the set of equations (5.1), to be satisfied by $\vec{u}$, simplify to

$$u_{1,1i} + u_{2,2i} = u_{3,3i} = u_{i,11} + u_{i,22} = u_{i,33} = 0 \qquad \text{(a)}$$

$$(E_1^2 - E_2^2)\ u_{1,1} + E_1\ E_2(u_{1,2} + u_{2,1}) = \text{const.} \qquad \text{(b)}$$

$$E_1(\lambda^2\ u_{3,1} + \lambda_3^2\ u_{1,3}) + E_2(\lambda^2\ u_{3,2} + \lambda_3^2\ u_{2,3}) = \text{const.} \qquad \text{(c)}$$

$$\lambda^2[(E_1^2 - E_2^2)\ u_{1,11} + 2E_1\ E_2\ u_{1,12}] + E_2\ E_3[(\lambda^2 + \lambda_3^2)\ u_{1,23} \qquad (5.37)$$

$$+ (\lambda^2 - \lambda_3^2)\ u_{2,13}] + 2\lambda^2\ E_1\ E_3\ u_{1,13} = 0 \qquad \text{(d)}$$

$$(E_1^2 - E_2^2)\ u_{2,11} + 2E_1\ E_2\ u_{2,12} + E_1\ E_3(u_{1,23} + u_{2,13}) + 2E_2\ E_3\ u_{2,23} = 0 \qquad \text{(e)}$$

$$(E_1^2 - E_2^2)\ u_{3,11} + 2E_1\ E_2\ u_{3,12} = 0 \qquad \text{(f)}$$

$$u_{1,23} - u_{2,13} = 0\ . \qquad \text{(g)}$$

As before the general solution of (a), apart from homogeneous deformations, is

$$u_1 = \phi_{,2} + x_3\ \psi_{,2}\ ,\quad u_2 = -\phi_{,1} - x_3\ \psi_{,1}\ ,\quad u_3 = \chi \qquad (5.38)$$

where ϕ, ψ, χ are functions of x_1, x_2 only and

$$\nabla_1^2\ \phi = \nabla_1^2\ \chi = \nabla_1^2\ \psi - c = 0\ ,\quad \nabla_1^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\ ,\quad c = \text{const.}$$

Using this, (g) implies $\nabla_1^2\ \psi = 0$ or $c = 0$. Again, using (5.38) in (f), solving the resulting P. D. E. by Lagrangian Method and using the fact that $\nabla_1^2\ \chi = 0$, we find that χ is of the form

$$\chi(x_1, x_2) = p(x_1^2 - x_2^2) + q\, x_1\, x_2 + r\, x_1 + s\, x_2 + \text{const.}$$

where p, q, r, s are arbitrary constants. Using (5.38) in (5.37 (b)) and solving the resulting P. D. E. for ψ with $\nabla_1^2 \psi = 0$, we obtain

$$\psi(x_1, x_2) = A_1(x_1^2 - x_2^2) + C_1\, x_1\, x_2 + D_1\, x_1 + F_1\, x_2 + \text{const.} \tag{5.39}$$

and

$$(E_1^2 - E_2^2)\, \phi_{,21} + 2E_1\, E_2\, \phi_{,22} = \text{const.} \tag{5.40}$$

where $A_1, C_1, \ldots, F_1$ are arbitrary constants such that

$$(E_1^2 - E_2^2)\, C_1 - 2A_1\, E_1\, E_2 = 0\ .$$

In view of (5.38) - (5.40), equation (5.37 (e)) yields

$$\phi(x_1, x_2) = A_2(x_1^2 - x_2^2) + C_2\, x_1\, x_2 + D_2\, x_1 + F_2\, x_2 + \text{const.}$$

where $A_2, \ldots, F_2$ are arbitrary constants.

Using (5.38) and the expressions for ϕ, ψ, χ in the remaining two equations (5.37 (c), (d)), we obtain further restrictions on the constants involved in ϕ, ψ, χ :

$$E_1(2p\, \lambda^2 + C_1\, \lambda_3^2) + E_2(q\, \lambda^2 - 2A_1\, \lambda_3^2) = 0$$

$$E_1(q\, \lambda^2 - 2A_1\, \lambda_3^2) + E_2(-2p\, \lambda^2 - \lambda_3^2\, C_1) = 0$$

and

$$E_1\, C_1 - 2E_2\, A_1 = 0\ .$$

These equations for A_1, C_1, p, q yield

$$A_1 = C_1 = p = q = 0 \ .$$

Thus, finally, we have

$$\phi(x_1, x_2) = D_2 \ x_1 + F_2 \ x_2 + \text{const.} \ ,$$

$$\psi(x_1, x_2) = D_1 \ x_1 + F_1 \ x_2 + \text{const.} \ ,$$

and

$$\chi(x_1, x_2) = r \ x_1 + s \ x_2 + \text{const.}$$

Upon substitution in (5.38), we conclude that the infinitesimal deformations u_i obtained are also homogeneous ones only.

(ii) Let $E_1 = E_2$.

In this case the equations (5.37) or (5.1) simplify to

$$\begin{gathered}
u_{1,1i} + u_{2,2i} = u_{3,3i} = u_{i,11} + u_{i,22} = u_{i,33} = 0 \\
u_{1,2} + u_{2,1} = \text{const.} \\
\lambda^2(u_{3,1} + u_{3,2}) + \lambda_3^2(u_{1,3} + u_{2,3}) = \text{const.} \\
2\lambda^2[E_1 \ u_{1,12} + E_3 \ u_{1,13}] + E_3[(\lambda^2 + \lambda_3^2) \ u_{1,23} + (\lambda^2 - \lambda_3^2) \ u_{2,13}] = 0 \\
2E_1 \ u_{2,12} + 2E_3 \ u_{2,23} + E_3(u_{1,23} + u_{2,13}) = 0 \\
u_{3,12} = 0 \\
u_{1,23} - u_{2,13} = 0 \ .
\end{gathered} \tag{5.41}$$

Solving the equations as in subcase (C) (i) above, we again find that u_i

are linear in x_1, x_2 and x_3 ; so that the only controllable deformation furnished is a homogeneous one.

6. EQUAL PRINCIPAL STRETCHES

Finally we consider the case, when in the basic homogeneous deformation, the three principal stretches are equal:

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda \ .$$

Since the material is isotropic, the basic homogeneous state is one of hydrostatic pressure. The dielectric body will have the same shape as in its reference state, either uniformly expanded or uniformly compressed.

In this case (3.10) implies

$$B_{11} = B_{22} = B_{33} = \lambda^2 \ , \quad B_{12} = B_{23} = B_{31} = 0 \ .$$

Also in this case, every direction is a principal direction of $\underset{\sim}{B}$. Thus, without loss of generality, we can choose our coordinate system in such a way that our X-axis is parallel to the electric field vector, so that

$$E_1 = |\vec{E}| \neq 0 \ , \quad E_2 = E_3 = 0 \ .$$

The equations (3.22) in this case reduce to

$$u_{1,1i} + u_{2,2i} + u_{3,3i} = 0$$

$$u_{i,11} + u_{i,22} + u_{i,33} = 0$$

$$u_{1,1i} = 0 \ , \qquad u_{i,11} = 0$$

which may be rewritten as

$$u_{1,1} = \text{const.} = A \quad (\text{say}) \qquad \text{(a)}$$

$$u_{2,2} + u_{3,3} = \text{const.} = B \quad (\text{say}) \qquad \text{(b)} \qquad (6.1)$$

$$u_{i,11} = 0 \qquad \text{(c)}$$

$$u_{i,22} + u_{i,33} = 0 \qquad \text{(d)}$$

where A, B are arbitrary.

The general solution of (6.1(c)) is

$$u_i = x_1\, f_i(x_2, x_3) + g_i(x_2, x_3)\ , \quad i = 1, 2, 3 \qquad (6.2)$$

where f_i and g_i are twice differentiable arbitrary functions of x_2 and x_3.

Using (6.2) in (6.1(a)), we have

$$f_1(x_2, x_3) = A\ .$$

In view of (6.2), (6.1(d)) implies

$$f_{i,22} + f_{i,33} = 0$$

and $\qquad (6.3)$

$$g_{i,22} + g_{i,33} = 0\ .$$

This means that f_i and g_i $(i = i, 2, 3)$ are harmonic functions of x_2 and x_3.

Using (6.2) in (6.1(b)), we have

$$f_{2,2} + f_{3,3} = 0$$

and

$$g_{2,2} + g_{3,3} = B \ . \tag{6.4}$$

The solutions to (6.3) and (6.4) are infinite. In fact, if we choose

$$f_2 = \text{Re. } F(x_2 + ix_3)$$

$$f_3 = -\text{Im. } F(x_2 + ix_3)$$

$$g_2 = \text{Re. } G(x_2 + ix_3) + B\, x_2 \quad \text{and} \quad g_3 = -\text{Im. } G(x_2 + ix_3)$$

or $$g_2 = \text{Re. } G(x_2 + ix_3) \quad \text{and} \quad g_3 = -\text{Im. } G(x_2 + ix_3) + B\, x_3$$

where F and G are any two analytic functions of $x_2 + ix_3$, then equations (6.3) for $i = 2, 3$ and (6.4) are identically satisfied. Again in view of (6.3), $g_1(x_2, x_3)$ is any harmonic function of x_2, x_3.

Thus the solutions u_i given by (6.2) are infinite. Once we know the explicit form of a solution (6.2), then as before we can calculate the surface tractions required to maintain this deformation.

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