44892

National Library of Canada | Bibliothèque nationale du Canada

CANADIAN THESES ON MICROFICHE | THÈSES CANADIENNES SUR MICROFICHE

NAME OF AUTHOR/*NOM DE L'AUTEUR* JAMES J. DUKARM

TITLE OF THESIS/*TITRE DE LA THÈSE* ALGEBRAIC AND LOCALLY ALGEBRAIC FUNCTORS

UNIVERSITY/*UNIVERSITÉ* SIMON FRASER UNIVERSITY

DEGREE FOR WHICH THESIS WAS PRESENTED/ *GRADE POUR LEQUEL CETTE THÈSE FUT PRÉSENTÉE* PH. D.

YEAR THIS DEGREE CONFERRED/*ANNÉE D'OBTENTION DE CE DEGRÉ* 1980

NAME OF SUPERVISOR/*NOM DU DIRECTEUR DE THÈSE* A. H. LACHLAN

Permission is hereby granted to the NATIONAL LIBRARY OF CANADA to microfilm this thesis and to lend or sell copies of the film.

The author reserves other publication rights, and neither the thesis nor extensive extracts from it may be printed or otherwise reproduced without the author's written permission.

L'autorisation est, par la présente, accordée à la BIBLIOTHÈQUE NATIONALE DU CANADA de microfilmer cette thèse et de prêter ou de vendre des exemplaires du film.

L'auteur se réserve les autres droits de publication; ni la thèse ni de longs extraits de celle-ci ne doivent être imprimés ou autrement reproduits sans l'autorisation écrite de l'auteur.

DATED/*DATÉ* 1980 04 16 SIGNED/*SIGNÉ*

PERMANENT ADDRESS/*RÉSIDENCE FIXÉ* c/o MATHEMATICS DEPT.
SIMON FRASER UNIVERSITY
BURNABY, B.C., CANADA V5A 1S6

National Library of Canada
Collections Development Branch

Canadian Theses on
Microfiche Service

Bibliothèque nationale du Canada
Direction du développement des collections

Service des thèses canadiennes
sur microfiche

NOTICE

The quality of this microfiche is heavily dependent upon the quality of the original thesis submitted for microfilming. Every effort has been made to ensure the highest quality of reproduction possible.

If pages are missing, contact the university which granted the degree.

Some pages may have indistinct print especially if the original pages were typed with a poor typewriter ribbon or if the university sent us a poor photocopy.

Previously copyrighted materials (journal articles, published tests, etc.) are not filmed.

Reproduction in full or in part of this film is governed by the Canadian Copyright Act, R.S.C. 1970, c. C-30. Please read the authorization forms which accompany this thesis.

THIS DISSERTATION
HAS BEEN MICROFILMED
EXACTLY AS RECEIVED

AVIS

La qualité de cette microfiche dépend grandement de la qualité de la thèse soumise au microfilmage. Nous avons tout fait pour assurer une qualité supérieure de reproduction.

S'il manque des pages, veuillez communiquer avec l'université qui a conféré le grade.

La qualité d'impression de certaines pages peut laisser à désirer, surtout si les pages originales ont été dactylographiées à l'aide d'un ruban usé ou si l'université nous a fait parvenir une photocopie de mauvaise qualité.

Les documents qui font déjà l'objet d'un droit d'auteur (articles de revue, examens publiés, etc.) ne sont pas microfilmés.

La reproduction, même partielle, de ce microfilm est soumise à la Loi canadienne sur le droit d'auteur, SRC 1970, c. C-30. Veuillez prendre connaissance des formules d'autorisation qui accompagnent cette thèse.

LA THÈSE A ÉTÉ
MICROFILMÉE TELLE QUE
NOUS L'AVONS REÇUE

Ottawa, Canada
K1A 0N4

NL-339 (Rev. 8/80)

ALGEBRAIC AND LOCALLY ALGEBRAIC FUNCTORS

by

James J. Dukarm

B.A., Michigan State University, 1969

M.S., St. Mary's University (Texas), 1973

A THESIS SUBMITTED IN PARTIAL FULFILLMENT

OF THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

in the Department

of

Mathematics

© James J. Dukarm 1980

SIMON FRASER UNIVERSITY

April 1980

All rights reserved. This thesis may not be
reproduced in whole or in part, by photocopy
or other means, without permission of the author.

PARTIAL COPYRIGHT LICENSE

I hereby grant to Simon Fraser University the right to lend my thesis or dissertation (the title of which is shown below) to users of the Simon Fraser University Library, and to make partial or single copies only for such users or in response to a request from the library of any other university, or other educational institution, on its own behalf or for one of its users. I further agree that permission for multiple copying of this thesis for scholarly purposes may be granted by me or the Dean of Graduate Studies. It is understood that copying or publication of this thesis for financial gain shall not be allowed without my written permission.

Title of Thesis/Dissertation:

ALGEBRAIC AND LOCALLY ALGEBRAIC FUNCTORS

Author:

(signature)

JAMES J. DUKARM

(name)

1980 04 16

(date)

APPROVAL

Name: James J. Dukarm

Degree: Doctor of Philosophy

Title of Thesis: Algebraic and Locally Algebraic Functors

Examining Committee:

Chairman: S. K. Thomason

A. H. Lachlan
Senior Supervisor

B. S. Thomson

H. Gerber

A. R. Freedman

B. Banaschewski
External Examiner
Professor
Department of Mathematics
McMaster University

Date Approved: April 16, 1980

ABSTRACT

Many algebraic constructions can be viewed as algebra-valued functors. Using a category-theoretic formulation of universal algebra and first-order logic originated by F. W. Lawvere, we obtain algebraic and logical results concerning functors which correspond to important kinds of algebraic constructions--in particular, to Boolean powers and bounded Boolean powers.

The notion of an equational interpretation of an equational theory T' in an equational theory T is introduced and shown to be the syntactical counterpart to coalgebras. By means of equational interpretations $T' \longrightarrow T$, the representable functors $\mathrm{Mod}(T) \longrightarrow \mathrm{Mod}(T')$ are shown to be obtainable as T'-algebras defined "within" the underlying-set functor $U_T\colon \mathrm{Mod}(T) \longrightarrow \mathrm{Set}$ when U_T is treated as a T-algebra in

the functor category $\mathrm{Set}^{\mathrm{Mod}(T)}$. The algebraic functors, i.e., the representable functors $G\colon \mathrm{Mod}(T) \longrightarrow \mathrm{Mod}(T')$ whose set-valued component $U_{T'}.G$ is monadic, are characterized similarly. The latter result is associated with a syntactical characterization of all the equational theories T' such that $\mathrm{Mod}(T')$ is equivalent as a category to $\mathrm{Mod}(T)$. Finitary and infinitary versions of the Boolean power construction are described as algebraic functors which correspond to Post algebras in a functor category.

A functor-theoretic characterization of locally equational categories is given which is analogous to F. E. J. Linton's characterization of equational categories. The characterization of locally equational categories leads naturally to the notion of a locally algebraic functor. Bounded Boolean powers are described as locally algebraic functors, and a new proof of T. K. Hu's theorem characterizing the category of Boolean algebras as a locally equational category is sketched.

DEDICATION

This thesis is respectfully dedicated to Helena Rasiowa.

ACKNOWLEDGEMENTS

My supervisor, Prof. Alistair Lachlan, has earned my deep appreciation and thanks for his tireless help and friendship over the last several years. Very special thanks also go to Doc. dr. Antoni Wiweger of the Mathematics Institute of the Polish Academy of Sciences, Warsaw, for his valuable help and encouragement during my stays in Warsaw in 1975-76 and in 1977. Prof. Helena Rasiowa arranged my extended stay in Warsaw in 1975-76 and supervised my work at Warsaw University, where I began to be a mathematician instead of a student largely as a result of opportunities which she provided. Countless other young mathematicians have been helped similarly by Rasiowa's generosity, and the dedication of this thesis to Prof. Rasiowa is a gesture of appreciation for her help and example to all of us.

The content of this thesis has been affected significantly by all three of the individuals named above. Prof. Lachlan convinced me that locally equational classes were worthy of investigation and also encouraged and helped my study of Boolean powers. Docent Wiweger also played an important role in directing my attention to categorially interesting aspects of Boolean powers, and he provided much useful criticism and many very helpful suggestions concerning my work on those functors and other algebraic problems. Prof. Rasiowa suggested the relevance of Cat-Ho Nguyen's generalized Post algebras to my work on Boolean powers.

Thanks are also offered to the many persons who contributed references, advice, and opportunities for discussing my research. Prominent among these people are B. Banaschewski, M. Barr, S. Burris, F. Linton, M. Makkai, E. Nelson, A. Obtułowicz, A. Pixley, K. Prikrý, and K. Sokolnicki.

TABLE OF CONTENTS

CHAPTER 1. INTRODUCTION

The m-valued Post algebras, for finite $m \geqslant 2$, were introduced by P. C. Rosenbloom [38] as "many-valued" analogues to the Boolean algebras in connection with a kind of many-valued propositional logic which was designed to be functionally complete with respect to a semantics of m-valued truth tables. It was natural for logicians and algebraists to try to determine the extent and nature of the similarities between the Post algebras and the Boolean algebras. In search of "Boolean" properties of Post algebras, p-rings, and other analogues to Boolean algebras, A. L. Foster invented the Boolean power construction and the notion of a normal subdirect power. One of his main results was the following theorem.

<u>1.1.</u> <u>Theorem</u> (A. L. Foster [14], [15]). Let A be a primal

algebra. The following classes of algebras are coextensive:

i) The equational class generated by A;

ii) The class of all isomorphic copies of normal subdirect powers of A;

iii) The class of all isomorphic copies of Boolean powers of A.

Here, a <u>primal algebra</u> (in Foster's terminology, a "functionally strictly complete" finite algebra) is a finite universal algebra A, having at least two elements, such that, for each finite n, every function $A^n \longrightarrow A$ is a polynomial of A. Foster explicitly recognized that the m-element m-valued Post algebra was primal, and in his proof he showed that the Boolean algebra involved in constructing any given algebra in a primally-generated equational class was recoverable from that algebra. Thus, Foster's proof almost shows that the category of Boolean algebras is equivalent to the equational category (with algebras as objects and homomorphisms as arrows) generated by any primal algebra, with the Boolean power construction providing the equivalence functor.

In a subsequent paper [16], Foster generalized the Boolean power construction, defining "bounded" Boolean powers of an infinite algebra and showing that the bounded Boolean power construction enjoyed properties which were "localized" versions of corresponding properties of the original Boolean power construction. Much of Foster's work concerning Boolean powers and bounded Boolean powers was tidied up and extended by M. I. Gould and G. Grätzer [18], with most of the material of the latter paper appearing in Grätzer's book [19].

Foster's notions of "local" properties were studied and refined in a succession of papers by T. K. Hu, most notably [22], [23], [24], and [25], in which Hu formulated the definition of a locally equational class of universal algebras. Hu's final definition and characterization of locally equational classes, which appeared in [25], are given at the beginning of Chapter 5 of this thesis. The next theorem, one of Hu's most significant results, seems to have remained somewhat obscure, possibly because it never appeared in print with both a correct proof and the final definition of a locally equational class.

A finitary universal algebra A having at least two elements is locally primal (see [16]) if, for every finite n and every finite subset X of A, each partial function $X^n \longrightarrow A$ is the restriction of a polynomial $A^n \longrightarrow A$ of A.

1.2. Theorem (T. K. Hu [22], [24], [25]). Let K be a locally equational class of finitary universal algebras, regarded as a category whose arrows are all homomorphisms between algebras of K. Then K is equivalent to the category of Boolean algebras if and only if K is generated as a locally equational class by a locally primal algebra.

1.3. Corollary. Let K be an equational class of finitary universal algebras, regarded as a category whose arrows are all homomorphisms between algebras of K. Then K is equivalent to the category of Boolean algebras if and only if K is generated as an equational class by a primal algebra.

Hu's proof of (1.2) in [22] shows that a locally equational class K is dually equivalent to the category of Boolean spaces if and only if K is generated by a locally primal algebra A; then K is an equational class if and only if A is finite, hence primal. The functor which provides the dual equivalence assigns to each Boolean space X an algebra of continuous functions $C(X, A)$, where A has the discrete topology. It is now well known that, if X is the Stone space of a Boolean algebra B, then $C(X, A)$ is isomorphic to the bounded Boolean power $A[B]$, which is an ordinary Boolean power if A is finite (see Banaschewski and Nelson [3] for details).

Investigations by numerous authors (see Burris [5] and Banaschewski and Nelson [3] for an extensive bibliography) have shown that the Boolean power and bounded Boolean power constructions have extremely nice logical and algebraic properties. Most of the work done on Boolean powers to date makes use of Foster's original definition of Boolean powers, Foster's characterization of Boolean powers as normal subdirect powers, the algebra-of-continuous-functions characterization, or a characterization of Boolean powers as a simple type of sheaf construction.

The important work on Post algebras as lattices done by Epstein [13], Traczyk [41], and others, has no apparent connection with the Boolean power construction. The lattice-theoretic studies of Post algebras express the correspondence between Post algebras and Boolean algebras by describing the m-valued Post algebras as coproducts of an m-element chain with Boolean algebras in the category of bounded distributive lattices, or as chain-based distributive lattices, or as algebras of nonincreasing (m-1)-element chains in Boolean algebras. These constructions are discussed in Balbes and Dwinger [1] and in Rasiowa [37].

The results presented in this thesis have their origins in a study of the functorial properties of the Boolean power construction; some preliminary results of that study, such as a construction of

"Post algebras" as the Eilenberg-Moore algebras for a Boolean power monad in the category of sets, were presented in seminars at Warsaw University and at the Mathematics Institute of the Polish Academy of Sciences in Warsaw in early 1976 and 1977. The writer's paper [9] on coalgebra-representable Boolean power functors shows that much of the niceness of Boolean powers is attributable to a special relationship between their representing algebras and free algebras in a category of infinitary Boolean algebras.

Chapter 3 of this thesis, on Morita equivalence and algebraic functors, contains improved versions of results originally presented in [10] which generalize the methods and results of [9]. Equational theories T and T' are said to be <u>Morita equivalent</u> if the respective equational categories Mod(T) and Mod(T') of algebras are equivalent as categories. A functor $G: \mathrm{Mod}(T) \longrightarrow \mathrm{Mod}(T')$ is <u>algebraic</u> if its composite $U_{T'}G$ with the underlying-set functor $U_{T'}: \mathrm{Mod}(T') \longrightarrow \mathrm{Set}$ is monadic; every algebraic functor is coalgebra-representable. In Chapter 3 we define an <u>equational interpretation</u> of one equational theory in another and show that such interpretations are the syntactical counterparts to coalgebras. Morita equivalence of equational theories is characterized syntactically, and a syntactical characterization of algebraic functors is derived.

The construction of m-valued Post algebras as chain-based lattices provides an example of an equational interpretation of the equational theory BA of Boolean algebras in the equational theory P_m of m-valued Post algebras which witnesses the Morita equivalence of those theories. Likewise, the construction of m-valued Post algebras as lattices of nonincreasing (m-1)-element chains in Boolean algebras is directly related to an equational interpretation of P_m in BA which not only witnesses the Morita equivalence of the two theories but actually gives rise to a representable Boolean power functor.

Much of the material of [9] appears in Chapter 4, which is a discussion of representable Boolean power functors. It is shown that the representable Boolean power functors are algebraic, and the equational theory of generalized Post algebras associated with a representable Boolean power functor is examined in some detail.

The connection between bounded Boolean powers and locally equational classes as demonstrated by Hu's theorem (1.2) suggests that there might be a "local" generalization of algebraic functors corresponding to bounded Boolean powers. The required "locally equational" counterpart to monadicity (also appearing in [11]) is given in Chapter 5 in the form of a functor-theoretic characterization of locally equational categories. Bounded Boolean power functors do turn out to be locally algebraic, and a new proof of the Hu theorem (1.2) is outlined using the results of Chapter 5.

One contribution of this study which is not expressible as a theorem is the demonstration that a consistent category-theoretic approach to algebraic constructions, based on Lawvere's analysis in [29] of algebra-valued functors, can be a practical way of obtaining algebraically meaningful results.

CHAPTER 2. FUNDAMENTALS OF CATEGORY-THEORETIC ALGEBRA

The reader is assumed to be familiar with basic notions of category theory as presented in Mac Lane [32] and with standard universal algebra as in Grätzer [19]. The treatment of universal algebra described in this chapter is similar in spirit to the model-theoretic approach exemplified by Grätzer [19], but is more suitable for dealing with the interactions of algebraic and category-theoretic phenomena. In Chapter 3, for example, we find it useful to treat certain set-valued functors as algebras in a functor category; the structure and properties of such algebras cannot conveniently be explained in terms of elements and mappings of elements. This particular style of category-theoretic universal algebra originated with F. W. Lawvere's Ph.D. thesis [29] and was adapted by F. E. J. Linton [30] to cover infinitary universal algebra. A good exposition of finitary category-theoretic algebra is given in Pareigis [36], while Wraith [42] provides a detailed account of the basics of infinitary category-theoretic algebra. The paper [17] by P. Freyd must be included in the canon of the Lawvere-Linton approach to algebra, since it has motivated much of the subsequent research in category-theoretic algebra.

We assume no particular set-theoretic foundation for category theory; since we only discuss functor categories of finite "depth," the type of category theory which we use is no more hazardous than Zermelo-Fraenkel set theory with a few Grothendieck universes added on. For our purposes, then, all the categories (except Cat) which are mentioned below are considered as being objects in a very big category Cat whose arrows are functors.

The category Set of sets and functions is assumed to satisfy some form of the Axiom of Choice, and cardinals in Set are identified with initial ordinals. The finite cardinals are $0, 1, 2, \ldots,$ while ω is the smallest infinite cardinal. Note that, for category-theoretic purposes, the domain and codomain are part of the data which define a function; thus, for example, the identity function $X \longrightarrow X$ is distinct from the inclusion map $X \longrightarrow Y$, if X is a proper subset of Y.

If A and B are objects in a category M, then the family of all arrows $A \longrightarrow B$ in M is normally denoted by $M(A, B)$; however, this usage is abandoned below in certain cases where the notation would be confusing. The category M is <u>small</u> if the family of all arrows in M is a set, i.e., an object in the category Set; M is <u>locally small</u> if $M(A, B)$ is a set, for all M-objects A and B. The identity arrow $A \longrightarrow A$ is id_A, or sometimes just id, and the composite arrow $A \xrightarrow{g} B \xrightarrow{f} C$ is $f.g$, or simply fg. If f and g are functions, then the value of $f.g$ at the point $a \in A$ may be denoted by $[f.g](a)$. A <u>subobject</u> of A in M is an object B with a monomorphism $B \xrightarrow{b} A$; we sometimes write $B \hookrightarrow A$ to indicate that B is a subobject of A. If $B \xhookrightarrow{b} A$ and $C \xhookrightarrow{c} A$ are subobjects of A, then $B \leqslant C$ is defined to mean that there is an arrow $B \xrightarrow{f} C$ with $c.f = b$; in that case, f is a uniquely determined monomorphism. If f is an isomorphism, the subobjects B and C are equivalent. The category M° is the opposite of M; the objects of M° are the same as those of M, and the arrows $A \longrightarrow B$ in M° are in bijective correspondence with the arrows $B \longrightarrow A$ in M; usually, no confusion will result if the same name is used for corresponding arrows in M and in M°.

All functors are considered to be covariant, but frequently a functor $M^\circ \longrightarrow K$ will be described as though it were an arrow-reversing transformation defined on M. Note that each functor $G: M \longrightarrow K$ determines a functor $G^\circ: M^\circ \longrightarrow K^\circ$ in an obvious way. The functor category K^M has as its objects all functors $M \longrightarrow K$, and as its arrows all natural transformations between such functors. A natural transformation $G \xrightarrow{f} H$ in K^M is given as a family (not necessarily a set) of arrows $f_A: G(A) \longrightarrow H(A)$ in K indexed by the objects A of M. In general, K^M may be a rather large category, but most of the functor categories which we use are of the form Set^M, where M is locally small.

If M is locally small, then each object A determines a functor $\underline{A}: M \longrightarrow \mathrm{Set}$ defined by the following:

i) For each object B, $\underline{A}(B)$ is $M(A, B)$;

ii) For each arrow $B \xrightarrow{f} C$, $\underline{A}(f)$ is the function $M(A, B) \longrightarrow M(A, C)$ which sends each $A \xrightarrow{g} B$ to $A \xrightarrow{fg} C$.

Any functor $U: M \longrightarrow \mathrm{Set}$ for which there is an object A such that U is naturally isomorphic to $\underline{A}$ is called a <u>representable functor</u>, and the object A is said to <u>represent</u> U. Each arrow $A \xrightarrow{f} B$ in M determines a natural transformation $\underline{f}: \underline{B} \longrightarrow \underline{A}$ which acts by composition with f on the right: for each object C and each arrow $B \xrightarrow{g} C$, we have $\underline{f}_C(g) = g.f.$

The Yoneda Lemma says that the natural transformations $\underline{A} \longrightarrow U$, for any M-object A and any set-valued functor $U: M \longrightarrow \mathrm{Set}$, are in bijective correspondence with the elements of the set $U(A)$. For each natural transformation $\underline{A} \xrightarrow{f} U$, the corresponding element of $U(A)$ is

$f_A(id_A)$; for each element a of $U(A)$, the corresponding natural transformation $\underline{a}$ is defined by $\underline{a}_B(g) = [U(g)](a)$, for each object B and each arrow $A \xrightarrow{g} B$. An important consequence of the Yoneda Lemma is that the Yoneda embedding $M^\circ \longrightarrow \mathrm{Set}^M$, which takes each arrow $A \xrightarrow{f} B$ of M to $\underline{B} \xrightarrow{f} \underline{A}$ in Set^M, is a full embedding. The Yoneda embedding preserves limits, i.e., takes colimits of diagrams in M to limits of diagrams in Set^M.

An <u>equational theory</u> is a locally small skeletal category T with all products, such that every object of T is a power of one particular object $\mathcal{T}$. An arrow of the form $\mathcal{T}^n \longrightarrow \mathcal{T}^m$ in T is called an <u>(m, n)-ary operation</u>. Any (m, n)-ary operation is simply a product arrow induced by an m-sequence of $(1, n)$-ary operations. For any function $m \xrightarrow{f} n$, there is a corresponding operation $\mathcal{T}^n \xrightarrow{f^*} \mathcal{T}^m$ whose composite with the i-th projection $\mathcal{T}^m \longrightarrow \mathcal{T}$ is the $f(i)$-th projection $\mathcal{T}^n \longrightarrow \mathcal{T}$, for each $i < m$. The assignment of $m \xrightarrow{f} n$ in Set to $\mathcal{T}^n \xrightarrow{f^*} \mathcal{T}^m$ in T determines a functor

j_T: Card° $\longrightarrow$ T, where Card is the full subcategory in Set of all cardinals. There are (up to isomorphism) two exceptional equational theories T, for which the functor j_T is not faithful. The first exception is the theory T which has only one object and one arrow. The second exception is the theory T for which $T^0 \neq T$, but $T^n = T$ for all $n \neq 0$, and where there is exactly one (1, 0)-ary operation.

A (1, n)-ary operation f is said to be trivial when there is a function g such that $f = g^*$, or when there is an $m < n$ such that that f factors as $f = g^*.h$ for some function $m \xrightarrow{g} n$ and some (1, m)-ary operation h. Given any cardinal n, an equational theory T is said to have rank(T) $\leqslant$ n provided that, for all $m > n$, there are no nontrivial (1, m)-ary operations. In particular, T is finitary if rank(T) $\leqslant \omega$, and rank(T) is defined to be ∞ if there is no cardinal n such that rank(T) $\leqslant$ n. For any equational theory T and any cardinal k, the k-ary part of T is the equational theory T' obtained by deleting all the nontrivial (1, n)-ary operations from T, for all $n \geqslant k$.

Let T be an equational theory, and let M be any category. A T-algebra in M is a product-preserving functor $A: T \longrightarrow M$, and a T-homomorphism in M is a natural transformation between T-algebras. We sometimes informally use the same letter to denote both an algebra (or homomorphism) and its underlying M-object (or M-arrow). This generalizes the common practice in universal algebra of ignoring the distinction between an algebra (or homomorphism) and its underlying set (or function). When we refer to an M-object A as being a T-algebra, it is to be understood that there is a T-algebra $A': T \longrightarrow M$ such that $A'(T) = A$ which is being referred to. Similarly, we might refer to an M-arrow $A \longrightarrow B$ as being a homomorphism.

The category Mod(T) is the full subcategory in Set^T of all functors $T \longrightarrow Set$ which are T-algebras. A category is said to be equational if it is of the form Mod(T), for some equational theory. The set of all homomorphisms $A \longrightarrow B$ in Mod(T) is $hom_T(A, B)$. Equational theories, equational categories, and algebras (in Set) correspond closely to their counterparts in model-theoretic universal algebra. There are some "size differences" which arise because model-theoretic universal algebra does not normally deal with theories having a proper class of nontrivial operations, and the language-free category-theoretic approach to algebra leads naturally to the consideration and use of entities such as coalgebras (coproduct-preserving functors $T^{\circ} \longrightarrow M$) which are somewhat odd from a model-theoretic viewpoint. The notion of a T-algebra in Set differs slightly from the model-theoretic notion of an algebra in that the latter requires that, for any algebra A, the operations $A^n \longrightarrow A$ should be functions from the n-th Cartesian power of the set A into A. The category version of a T-algebra allows the n-th power A^n of the underlying set to be any set $X = \mathbf{A}(T^n)$ which, relative to some n-sequence of projections $X \longrightarrow A$, is an n-fold product of A with itself in Set in the category-theoretic sense. Clearly, every T-algebra A in Set is canonically isomorphic to a T-algebra A' in Set which is constructed from the Cartesian powers of the set A.

The category $Mod(T)$ is equipped with a faithful underlying-set functor U_T: Mod(T) $\longrightarrow$ Set, where for each T-algebra A, we have $U_T(A) = A(T)$, while for each homomorphism h we have $U_T(h)$ being the T-component of h. In accordance with the informal usage mentioned above, we shall not ordinarily distinguish between A and $U_T(A)$, or between h and $U_T(h)$ notationally. The Yoneda embedding $T° \longrightarrow Set^T$ factors through Mod(T), so it determines a full embedding Y_T: $T° \longrightarrow$ Mod(T). Since Card is a skeleton of Set, there is an equivalence functor E: Set $\longrightarrow$ Card; let $J_T = j_T.E°$. The composite functor $F_T = Y_T.J_T$ is a left adjoint for U_T, i.e., F_T is a free algebra functor for Mod(T). The functor Y_T determines an equivalence of categories between T° and the full subcategory in Mod(T) of free T-algebras.

Every equational category Mod(T) is small-complete and small-cocomplete, i.e., has limits and colimits for all small diagrams. If A is a T-algebra, then $n \otimes A$ is a T-algebra which is an n-th copower, i.e., a coproduct of n copies of A, in Mod(T). It will be clear from the context whether A^n is an n-th power of the algebra

A in Mod(T) or the set $A(T^n)$.

A <u>regular epimorphism</u> in any category M is an arrow h such that, for some pair of arrows (u, v), h is a coequalizer of (u, v). If h has a kernel pair in M, then h is a regular epimorphism if and only if h is a coequalizer of its kernel pair. In Mod(T), but not in all categories, any composite of regular epimorphisms is a regular epimorphism, and a homomorphism h is a regular epimorphism if, for some homomorphism g, the composite hg is a regular epimorphism. This is so because a homomorphism h is a regular epimorphism in Mod(T) if and only if h is surjective as a function; a function is a regular epimorphism in Set if and only if it is a surjection. This fact can be summarized by the statement that U_T preserves and reflects regular epimorphisms. The functor U_T also preserves and reflects monomorphisms, i.e., h is a monomorphism in Mod(T) if and only if h is injective as a function.

A <u>binary relation</u> on an object A in a finitely complete category $\underline{M}$ is a subobject $R \xhookrightarrow{r} A \times A$. An <u>equivalence relation</u> on A is a binary relation on A which is reflexive, symmetric, and transitive in an appropriate sense. One way of characterizing equivalence relations which is adequate for our purposes is the approach taken by Pareigis (see [36], p. 99, on monomorphic equivalence relations). A binary relation $R \xhookrightarrow{r} A \times A$ is an equivalence relation in $\underline{M}$ if and only if $\underline{B}(R) \longrightarrow \underline{B}(A \times A)$ is an equivalence relation in Set (i.e., equivalent as a subobject of $\underline{B}(A) \times \underline{B}(A)$, via the canonical isomorphism $\underline{B}(A \times A) \longrightarrow \underline{B}(A) \times \underline{B}(A)$, to a "real" equivalence relation on $\underline{B}(A)$), for every $\underline{M}$-object B.

Let p, q be the projections $A \times A \longrightarrow A$; then $R \xhookrightarrow{r} A \times A$ is a <u>congruence relation</u> on A if and only if there is an arrow $A \xrightarrow{h} B$ such that $(p.r,\ q.r)$ is a kernel pair of h; here, h may be taken to be a coequalizer of $(p.r,\ q.r)$, if one exists. It is easy to show that every congruence relation in $\underline{M}$ is an equivalence relation in $\underline{M}$, but the converse is not true in general.

In Set and in Mod(T), however, the congruence relations and the equivalence relations coincide; furthermore, the category-theoretic notion of congruence relation agrees with the usual one in universal algebra. If $A \xrightarrow{h} B$ is a homomorphism with kernel pair (u, v), where u and v are homomorphisms $R \longrightarrow A$, then the induced homomorphism $R \xrightarrow{r} A \times A$ is a monomorphism which embeds R in $A \times A$ as a congruence relation, the kernel congruence $\ker(h)$ of h. If $A \xrightarrow{g} C$ is a coequalizer of (u, v), then there is a canonical isomorphism $C \longrightarrow A/\ker(h)$ such that $A \xrightarrow{g} C \longrightarrow A/\ker(h)$ is the canonical projection. The functor U_T preserves and reflects congruence relations, i.e., $R \xhookrightarrow{r} A \times A$ is a congruence in Mod(T) if and only if it is a congruence relation in Set.

2.1. Theorem (Linton [30]). A category M is equivalent to Mod(T), for some equational theory T, if and only if M has all kernel pairs and coequalizers, and there is a functor $U\colon M \longrightarrow \mathrm{Set}$ such that:

i) U has a left adjoint;

ii) U preserves and reflects congruence relations and regular epimorphisms.

An object A in a category M is tractable if all powers of A exist in M and, for all cardinals m and n, there is only a set of arrows $A^n \longrightarrow A^m$. If A is a tractable object of M, the equational structure of A is an equational theory T_A which is a skeleton of the full subcategory in M of all powers of A. The dual notions, cotractability and equational costructure, are also important.

The two-element set 2 is a tractable object in Set; its equational structure T_2 is an equational theory of rank ∞, and $\mathrm{Mod}(T_2)$ is equivalent to the category of all complete atomic Boolean algebras, with complete homomorphisms. The finitary part of T_2 is the equational theory BA of finitary Boolean algebras; Mod(BA) is equivalent to the category of all Boolean algebras, with Boolean homomorphisms.

For any equational theory T, the free T-algebra $F_T(1)$ on one free generator is cotractable in $\mathrm{Mod}(T)$; for each n, $F_T(n)$ is an n-th copower of $F_T(1)$. The equational costructure of $F_T(1)$ is T. Since the Yoneda embedding $\mathrm{Mod}(T)^\circ \longrightarrow \mathrm{Set}^{\mathrm{Mod}(T)}$ takes coproducts in $\mathrm{Mod}(T)$ to products in $\mathrm{Set}^{\mathrm{Mod}(T)}$, and since, for each n, the functor ${U_T}^n$ is represented by $F_T(n)$, it follows that the functor U_T is a tractable object in $\mathrm{Set}^{\mathrm{Mod}(T)}$ whose equational structure is T; thus, U_T is, in the informal sense, a T-algebra in $\mathrm{Set}^{\mathrm{Mod}(T)}$. The latter observation is fundamental to our results in Chapter 3.

P. Freyd's approach to algebra-valued functors in [17] is based on the notion of a <u>coalgebra</u>, or a coproduct-preserving functor $T^\circ \xrightarrow{A} M$, where T is an equational theory and M is a category. The M-object $A(T)$ is the <u>underlying M-object</u> of the coalgebra, and for each T-operation f the arrow $A(f)$ is a <u>co-operation</u> of the coalgebra. The same kind of informal usage as used for algebras applies as well to coalgebras; thus, for example, $F_T(1)$ might be said to be a T-coalgebra in Mod(T), although technically the coalgebra which is referred to is $Y_T\colon T^{\circ} \longrightarrow \mathrm{Mod}(T)$, which we obtained above by factoring the Yoneda embedding.

Given a coalgebra $A\colon T^{\circ} \longrightarrow M$, there is a corresponding functor $\underline{A}\colon M \longrightarrow \mathrm{Mod}(T)$, which is said to be <u>represented</u> by A. For each M-object B, the T-algebra $\underline{A}(B)$ is the functor $T \longrightarrow M^{\circ} \longrightarrow \mathrm{Set}$ obtained by composing $M^{\circ}(B, -)$ with A°. Freyd proved the following two useful theorems.

<u>2.2.</u> <u>Theorem</u> (Freyd [17]). Let M be a small-complete category, and let T be an equational theory. A functor $G\colon M \longrightarrow \mathrm{Mod}(T)$ has a left adjoint if and only if G is represented by a T-coalgebra in M.

Freyd originally stated the theorem above only for finitary T, but his proof works for all equational theories T.

Let k be an infinite cardinal; a <u>k-directed set</u> is a small partial order (regarded as a category in the usual way) in which every set of fewer than k elements has an upper bound. A monomorphic k-directed system in M is a functor $D: I \longrightarrow M$ such that I is a k-directed set and, for every $i \longrightarrow j$ in I, the arrow $D(i) \longrightarrow D(j)$ in M is a monomorphism. A colimit of a monomorphic k-directed system is called a k-directed union. If $k = \omega$, then the prefix "ω-" on all these terms is omitted.

2.3. Theorem (Freyd [17]). Let T and T' be finitary equational theories. Then a functor $G: \mathrm{Mod}(T) \longrightarrow \mathrm{Mod}(T')$ is represented by a T'-coalgebra in $\mathrm{Mod}(T)$ whose underlying T-algebra is generated by a set of fewer than k elements if and only if G preserves products, equalizers, and k-directed unions.

Note that the identity functor $\mathrm{Mod}(T) \longrightarrow \mathrm{Mod}(T)$ is represented by the coalgebra Y_T. Much interesting material on coalgebras and coalgebra-representable functors is presented in Wraith's monograph [42]. We shall return to the subject of coalgebra-representable functors after taking a look at Linton's adaptation [30] of Lawvere's fundamental results concerning algebra-valued functors [29].

The equational theories are the objects in a category ET in which an arrow $T \longrightarrow T'$, called a <u>mapping of theories</u>, is a product-preserving functor $g\colon T \longrightarrow T'$ such that $g(\mathcal{T}) = \mathcal{T}'$. The induced functor $g^*\colon \mathrm{Mod}(T') \longrightarrow \mathrm{Mod}(T)$, which acts by composing T'-algebras with g, is called a <u>reduct functor</u>. A functor $G\colon \mathrm{Mod}(T') \longrightarrow \mathrm{Mod}(T)$ has the property that $U_T G = U_{T'}$ ("preserves underlying sets") if and only if there is a mapping of theories g such that $G = g^*$. Since the category Set itself is equivalent to $\mathrm{Mod}(\mathrm{Card}^\circ)$, where Card° is the opposite of the full subcategory of all cardinals in Set, regarded as an equational theory, the functor $U_T\colon \mathrm{Mod}(T) \longrightarrow \mathrm{Set}$ may be identified with the reduct functor $j_T{}^*$, where $j_T\colon \mathrm{Card}^\circ \longrightarrow T$ is the functor defined earlier in this chapter. Every reduct functor is faithful and has a left adjoint.

The tractable set-valued functors form a category TF in which an arrow $U \longrightarrow V$ is a commutative triangle $V.G = U$. We define a "structure" functor STR: $TF \longrightarrow ET°$ which assigns to each tractable set-valued functor U its equational structure T_U. To see what STR does to arrows, let $V.G = U$ be an arrow $U \longrightarrow V$ in TF. Note that for every n we have $(VG)^n = V^n.G = U^n$. If $V^n \xrightarrow{f} V^m$ is an operation in T_V, then $V^n.G \xrightarrow{fG} V^m.G$ is an (m, n)-ary operation in T_U. The assignment of fG to each T_V-operation f defines a mapping of theories $T_V \longrightarrow T_U$ which is the image under STR of the arrow $V.G = U$ from U to V in TF.

Now define a "semantics" functor SEM: $ET° \longrightarrow TF$ which sends each equational theory T to the corresponding underlying-set functor U_T, and each mapping of theories $g: T \longrightarrow T'$ to the commutative triangle $U_T.g^* = U_{T'}$ (which is an arrow $U_{T'} \longrightarrow U_T$ in the category TF).

It is a remarkable fact that STR is left adjoint to SEM; this "structure-semantics adjointness" is the category-theoretic counterpart to the Galois connection between sets of identities and classes of algebras which is known in universal algebra. The unit

$Id_{TF} \longrightarrow$ SEM.STR is a natural transformation whose U-component, for every tractable functor $U: M \longrightarrow Set$, is a commutative triangle $U_T.E_U = U$, where T is the equational structure of U; the functor $E_U: M \longrightarrow Mod(T)$ is called the <u>comparison functor</u> for U. Since every component of the unit of an adjoint pair is a universal arrow, it follows that the comparison functor E_U has the following universal property: for each functor $G: M \longrightarrow Mod(T')$ such that $U_{T'}.G = U$, there is a unique mapping of theories $T' \xrightarrow{g} T$ such that $G = g^*.E_U$. The practical meaning of this is that, to study any algebra-valued functor $G: M \longrightarrow Mod(T')$ whose set-valued component $U_{T'}.G$ is tractable, it is sufficient to investigate $U_{T'}.G$ and its equational structure T. That is, in order to understand G, we should examine the T'-reduct of the T-algebra $U_{T'}.G$, which lives in Set^M.

Suppose that G is represented by a T'-coalgebra A in M. Then $U_{T'}.G$ is isomorphic to <u>$A(T')$</u>: $M \longrightarrow Set$, whose equational structure T is the equational costructure of the M-object $A(T')$. Thus, studying the T'-coalgebra A in M is equivalent to studying the representable functor $\underline{A(T')} = U_{T'}.G$ as a T'-algebra in Set^M. The technique of treating a functor as an algebra in a functor category has the big advantage of being applicable to both representable and non-representable functors.

Although the Lawvere-Linton approach to universal algebra is nominally language-free, every equational theory T is associated with a language L_T which corresponds to the first-order language $L(\tau)$ described in Grätzer [19]. The language L_T is the many-sorted canonical language for the category T, in the terminology of Makkai and Reyes [33]. Our description of L_T and its interpretation follows [33] quite closely, with minor omissions and adaptations appropriate to the special nature of an equational theory T as a category.

The language L_T consists of the following items:

i) For each arrow f of T, there is an <u>operation symbol</u> f; it is assumed that $f \neq g$ whenever $f \neq g$.

ii) For each cardinal n, there is an infinite set of

<u>n-ary free variables</u> and an infinite set of <u>n-ary bound variables</u>; it is assumed that no free variable is a bound variable and that no n-ary variable is an m-ary variable if $m \neq n$.

iii)[1] There are an identity symbol $=$, an infinitary disjunction symbol $\bigvee$, an infinitary conjunction symbol $\bigwedge$, a negation symbol $\neg$, an implication symbol $\rightarrow$, an existential quantifier symbol $\exists$, a universal quantifier symbol $\forall$, and parentheses ( ).

The <u>terms</u> of L_T are defined as follows:

t_1) Every n-ary free variable is an (n, n)-ary term.

t_2) If t is an (m, n)-ary term and f is a (k, m)-ary operation, then $f(t)$ is a (k, n)-ary term.

t_3) A string of symbols of L_T is a term if and only if it can be constructed by finitely many applications of t_1 and t_2.

The formulas of L_T which we require constitute only a small fragment of the language described by Makkai and Reyes. For a more complete discussion of the formulas, the reader is referred to [33]. We shall use only equations and conjunctions and disjunctions of equations. An <u>(m, n)-ary equation</u> is an expression of the form $r = s$, where both r and s are (m, n)-ary terms. If S is any set of equations such that there are only finitely many free variables occurring in the members of S, then $\bigvee S$ and $\bigwedge S$ are formulas.

A <u>sequent</u> is an expression of the form $S \Rightarrow S'$, where both S and S' are finite sets (which may be empty) of formulas, and where $\Rightarrow$ is a new symbol. In practice, we shall allow S and S' to be single formulas instead of sets of formulas.

Let M be any finitely complete category. A <u>many-sorted structure</u> for L_T in M is a morphism of graphs $A: T \longrightarrow M$ (in the sense of Mac Lane [32]). In other words, A sends objects of T to objects of M and arrows of T to arrows of M, preserving domains and codomains of arrows, but not necessarily preserving composition of arrows, and not necessarily sending identity arrows to identity arrows.

Let X be the k-tuple $(x_0, x_1, \ldots, x_k)$, where x_i is an n_i-ary free variable, for $i = 0, 1, \ldots, k$. Let t be a term whose (only) free variable is x_j. If t is x_j, then the interpretation $[t]_{A,X}$ of t in A relative to X is defined to be the j-th projection

$$A(T^{n_0}) \times \ldots \times A(T^{n_k}) \longrightarrow A(T^{n_j}).$$

If t is $f(r)$, where r is an (m, n_j)-ary term whose interpretation $[r]_{A,X}$ is already defined as an arrow $A(T^{n_0}) \times \ldots \times A(T^{n_k}) \longrightarrow A(T^m)$, then $[t]_{A,X}$ is defined to be $A(f).[r]_{A,X}$. Now let $r = s$ be an (m, n_j)-ary equation whose (only) free variable is x_j. The interpretation $[r = s]_{A,X}$ of $r = s$ in A relative to X is an equalizer of the arrows $[r]_{A,X}$ and $[s]_{A,X}$. As we see from the diagram

$$[r = s]_{A,X} \longrightarrow \prod_{i<k} A(T^{n_i}) \rightrightarrows A(T^m),$$

$[r = s]_{A,X}$ is a subobject of $\prod_{i<k} A(T^{n_i})$. If S is a set of

equations such that all the free variables occurring in members of S are among those of X, then $[\bigwedge S]_{A,X}$ is defined to be the intersection, if it exists, of the subobjects of $\prod_{i<k} A(T^{n_i})$ corresponding to the equations belonging to S, while $[\bigvee S]_{A,X}$ is defined to be the supremum, if it exists, of those subobjects. The interpretation of each formula of L_T in the structure A relative to X, if there is such an interpretation, is a subobject of the product $\prod_{i<k} A(T^{n_i})$. Note that the interpretations of terms and formulas as defined above are specified only up to canonical isomorphism in the category M; this fact is discussed in [33].

Now suppose that S and S' are finite sets of formulas such that all the free variables occurring in the members of $S \cup S'$ are among those of the k-tuple X given above. Then the structure A satisfies the sequent $S \Rightarrow S'$ if and only if the following conditions are satisfied:

i) Every formula in $S \cup S'$ has an interpretation in A

relative to X.

ii) The intersection $\bigwedge_{F \in S} [F]_{A,X}$ and the supremum $\bigvee_{G \in S'} [G]_{A,X}$ both exist among the subobjects of $\prod_{i<k} A(T^{n_i})$ in M.

iii) In the natural quasi-order of the subobjects of $\prod_{i<k} A(T^{n_i})$, we have $\bigwedge_{F \in S} [F]_{A,X} \leq \bigvee_{G \in S'} [G]_{A,X}$.

We write $A \models S \Rightarrow S'$ to say that the structure A satisfies the sequent $S \Rightarrow S'$.

A T-algebra is obviously a rather special kind of structure for L_T, in that it satisfies all the identities of T, i.e., all the sequents of the form $\{\ \} \Rightarrow f(g(x)) = h(x)$, where f, g, and h are operations such that $f.g = h$ in T. It is useful to distinguish structures resembling T-algebras in this respect in which the projections and other trivial operations are well-behaved. We shall say that a structure $A: T \longrightarrow M$ for L_T is a many-sorted T-algebra in M if A is a functor and if, for all cardinals m and n and for all (m, n)-ary terms r and s, the structure A satisfies the sequent

$$\bigwedge\{p_i(r) = p_i(s) \ : \ i < m\} \Rightarrow r = s,$$

where for each $i < m$ the symbol p_i corresponds to the i-th projection $T^m \xrightarrow{p_i} T$ in T. These conditions are necessary and sufficient for every (m, n)-ary equation $r = s$ to be equivalent to the conjunction of the corresponding set of (1, n)-ary equations $p_i(r) = p_i(s)$ in every many-sorted T-algebra. We shall write $T \models S \Rightarrow S'$ to say that, for every many-sorted T-algebra A in which the necessary interpretations exist, we have $A \models S \Rightarrow S'$.

It is common practice in universal algebra to specify an equational theory by means of a set of equations in a language over a similarity type (see [19]); in the category-theoretic context, this amounts to providing a presentation of the equational theory. Both Lawvere [29] and Wraith [42] discuss the category-theoretic technicalities of presentations, but we shall say informally that a presentation of an equational theory T is given by specifying a family of distinguished T-operations and describing their behaviour sufficiently that the category T is determined up to isomorphism. For example, a presentation of the finitary equational theory BA of Boolean algebras might name the constant 0, the meet operation $\wedge$, and the complement operation $\neg$, and provide a list of axioms describing the behaviour of these operations in Boolean algebras. Another presentation of BA might list the ring operations 0, 1, +, -, and $\cdot$ and describe how they work in a Boolean ring. The difference between Boolean rings and Boolean algebras is, from the standpoint of category-theoretic algebra, not a difference of algebras at all but rather a matter of distinct presentations of the one theory BA.

At about the same time that the Lawvere-Linton approach to universal algebra was taking shape, category-theoretic investigations of adjointness gave rise to the notion of a monad (or "triple") and of a monadic functor. Both Mac Lane [32] and Pareigis [36] provide detailed expositions of the basic theory of monads, so we shall only say enough about them to establish notation and to state some results which will be referred to in later chapters. Given categories M and K and functors $U: M \longrightarrow K$ and $F: K \longrightarrow M$, with F left adjoint to U, the monad in K determined by U and F will be called H. The category of all Eilenberg-Moore algebras over H in K is K^H, and its "forgetful" functor and free algebra functor are respectively $U^H: K^H \longrightarrow K$ and $F^H: K \longrightarrow K^H$. The canonical comparison functor for the monad H is $C: M \longrightarrow K^H$. We shall say that the functor $U: M \longrightarrow K$ is <u>strictly monadic</u> if C is an isomorphism of categories, and <u>monadic</u> if C is a category equivalence. Note that, whether U is monadic or not, F^H is left adjoint to U^H, and the monad in K determined by U^H and F^H is just H. The canonical comparison functor C is unique such that $U^H.C = U$ and $C.F = F^H$.

With U, F, and H as given above, the Kleisli category for H is K_H, and its associated functors are $U_H: K_H \longrightarrow K$ and

$F_H: K \longrightarrow K_H$. Here, F_H is left adjoint to U_H, and the monad in K determined by U_H and F_H is just H again. The Kleisli functor $C': K_H \longrightarrow M$ is unique with the property that $U.C' = U_H$ and $C'.F_H = F$.

We shall cite two results which illuminate the connection between equational theories and monads. The first result is a theorem adapted from Pareigis [36], p. 135, which shows that when H is the monad in Set determined by U_T and F_T, where T is an equational theory, the Kleisli category Set_H for H is equivalent to the full subcategory in Mod(T) of free T-algebras.

2.4. Theorem [36]. Let T be an equational theory, and let H be the monad in Set determined by U_T and F_T. Then there is an equivalence of categories $Q: Set_H{}^\circ \longrightarrow T$ such that $Q.F_H{}^\circ = J_T$.

The second result is a special case, for monads in Set, of a theorem of Linton (Theorem 9.3, p. 41 of [31]) which explains the connection between Eilenberg-Moore algebras and T-algebras.

2.5. Theorem (Linton [31]). Let $U\colon M \longrightarrow Set$ be a functor with a left adjoint; let T be the equational structure of U, and let $E\colon M \longrightarrow Mod(T)$ be the comparison functor for U. Also let H be the monad in Set determined by U and its adjoint, and let $C\colon M \longrightarrow Set^H$ be the canonical comparison functor for H. Then there is an equivalence of categories $R\colon Mod(T) \longrightarrow Set^H$ such that the following diagram commutes:

$$\begin{tikzcd}
M \arrow[r, "C"] \arrow[d, "E"'] & Set^H \arrow[d, "U^H"] \\
Mod(T) \arrow[ur, "R"] \arrow[r, "U_T"'] & Set
\end{tikzcd}$$

In particular, U is monadic if and only if its comparison functor $E\colon M \longrightarrow Mod(T)$ is a category equivalence.

CHAPTER 3. MORITA EQUIVALENCE AND ALGEBRAIC FUNCTORS

A natural question arises in connection with equational theories: given an equational theory T, for what other equational theories T' is the category $\mathrm{Mod}(T')$ equivalent to $\mathrm{Mod}(T)$? Theories T and T' such that $\mathrm{Mod}(T)$ is equivalent to $\mathrm{Mod}(T')$ are said to be <u>Morita equivalent</u> (see [42], p. 54).

The classical Morita theorem [35] of module theory, which is the motivation for the notion of Morita equivalence of equational theories, provides necessary and sufficient conditions on rings R and S for the categories R-mod and S-mod of left modules to be equivalent (see Pareigis [36] or Cohn [7] for a detailed exposition of the Morita theorem). Since R-mod and S-mod are equivalent to equational categories, the Morita theorem provides a characterization of Morita equivalence for equational theories of the kind which Wraith [42] calls "annular." T. K. Hu's result (1.3) also provides some nontrivial examples of Morita equivalent equational theories. For any finite set X having m elements, where $m > 1$, the finitary part of the equational structure T_X of X is an equational theory P_m such that every m-element P_m-algebra belonging to $\mathrm{Mod}(P_m)$ is primal and $\mathrm{Mod}(P_m)$ is identifiable with the equational class generated by an m-element primal algebra. In keeping with the remarks in Chapter 2 on presentations of the theory BA of Boolean algebras (note that $BA = P_2$), we shall call P_m the <u>equational theory of m-valued Post algebras</u>, and we shall refer to P_m-algebras as <u>m-valued Post algebras</u>. Hu's result (1.3) says that the theories P_m, for all finite $m > 1$, are precisely the finitary equational theories which are Morita equivalent to BA.

It should be noted that there are various category-theoretic generalizations of Morita equivalence which apply to categories which are not necessarily equational theories. The relevance of some recent papers on category-theoretic Morita equivalence to the more strictly algebraic results discussed here or in Wraith [42] is commented upon later in this chapter.

One of our main results is a syntactical characterization (3.8) of all the equational theories T' which are Morita equivalent to a given equational theory T; this result is an improved version of the main result of Dukarm [10]. We also provide syntactical characterizations of coalgebra-representable functors (3.4) and of algebraic functors, i.e., algebra-valued functors with monadic set-valued component (3.14).

When are equational theories T and T' Morita equivalent? Suppose that $E\colon \mathrm{Mod}(T) \longrightarrow \mathrm{Mod}(T')$ is an equivalence functor; let $U = U_{T'}.E$ be the set-valued component of E, and let T'' be the equational structure of U. The composite $L.F_{T'}$ of the left adjoints L of E and $F_{T'}$ of $U_{T'}$ is a left adjoint for U; it follows that U is represented by the T-algebra $A = L(F_{T'}(1))$. Since L is an equivalence functor, it is full and faithful and preserves coproducts, so for every n we have $n \otimes A \cong L(F_{T'}(n))$, and the equational costructure T" of A is isomorphic to the equational costructure T' of $F_{T'}(1)$. ~~It~~ follows that E is actually the comparison functor for U. But then by (2.5) U is monadic, since its comparison functor is a category equivalence.

Now we can reformulate the original problem as follows: find all the monadic functors $\mathrm{Mod}(T) \longrightarrow \mathrm{Set}$, and characterize their equational structure theories.

Let M be an arbitrary nonempty category. An object A of M is <u>regular-projective</u> in M if, for every regular epimorphism $B \xrightarrow{f} C$ and every arrow $A \xrightarrow{h} C$ there is an arrow $A \xrightarrow{g} B$ such that $f.g = h$. A <u>regular generator</u> in M is an object A such that for every object B there is a cardinal n and a regular epimorphism $n \otimes A \longrightarrow B$. A regular-projective regular generator is called a <u>regular progenerator</u>. These definitions are, with minor variations, standard in the literature. We also define two objects A and B of M to be retract-equivalent if each one is a retract of some power of the other, and dually retract-equivalent if each one is a retract of some copower of the other. In any equational category Mod(T), the regular progenerators are precisely the T-algebras which are dually retract-equivalent to the free T-algebra $F_T(1)$.

3.1. Theorem. Let T be an equational theory. For any functor $U\colon \mathrm{Mod}(T) \longrightarrow \mathrm{Set}$, the following are equivalent:

i) U is monadic;

ii) U is represented by a regular progenerator;

iii) U and U_T are retract-equivalent in $\mathrm{Set}^{\mathrm{Mod}(T)}$.

Proof. Suppose that U is monadic; then U has a left adjoint F, and U is represented by the T-algebra $A = F(1)$. It follows that U is tractable, and the equational structure T' of U is the same as the equational costructure of A. Since U is monadic, its comparison functor $E\colon \mathrm{Mod}(T) \longrightarrow \mathrm{Mod}(T')$ is an equivalence functor, by (2.5). The comparison functor E is represented by A, where A is viewed as a T'-coalgebra in $\mathrm{Mod}(T)$, and $E(A)$ is isomorphic to the free T'-algebra $F_{T'}(1)$, which is a regular progenerator in $\mathrm{Mod}(T')$. Since E is an equivalence functor, it is easy to see that A is a regular progenerator in $\mathrm{Mod}(T)$. This proves that i) implies ii). Before showing that ii) implies i), we shall prove that ii) and iii) are equivalent.

If U is represented by a regular progenerator A, then A is dually retract-equivalent to $F_T(1)$, which represents U_T, so U is retract-equivalent to U_T.

Suppose that U is retract-equivalent to U_T; we shall show that U is represented by a regular progenerator. Let $U_T^{\,n} \xrightarrow{r} U$ be a retraction, and let $U \xrightarrow{s} U_T^{\,n}$ be a coretraction, with $r.s = id_U$. Then s is an equalizer of $s.r$ and id. Since the Yoneda embedding is full and faithful, there is a homomorphism $F_T(n) \xrightarrow{h} F_T(n)$ such that $s.r = \underline{h}$. Let $F_T(n) \xrightarrow{t} A$ be a coequalizer of h and id in Mod(T). The Yoneda embedding takes t to an equalizer of $s.r$ and id in $Set^{Mod(T)}$; but then $\underline{A}$ and U are isomorphic, i.e., A represents U. Since $U \cong \underline{A}$ is, by hypothesis, retract-equivalent to U_T, it follows that A is dually retract-equivalent to $F_T(1)$, i.e., A is a regular progenerator. This completes the proof that ii) is equivalent to iii).

To finish the proof of the theorem, we must show that ii) implies i). Assuming that U is represented by a regular progenerator A, we shall use (2.1) in combination with (2.5) to show that U is monadic. Since U is represented by A, U has a left adjoint F, which sends each set X to an X-indexed copower of A in Mod(T). It is immediate from the definition of a regular-projective object that U preserves regular epimorphisms. Now suppose that $B \xrightarrow{f} C$ is given such that $U(f)$ is a regular epimorphism. Let $F.U \xrightarrow{p} Id$ be the co-unit for the adjoint pair (F, U); we claim that, since A

is a regular generator, every component of p is a regular epimorphism. If this is so, then we have $f.p_B = p_C.F(U(f))$, where the right side of the equation is a composite of regular epimorphisms in $\mathrm{Mod}(T)$ and, hence, is a regular epimorphism, so $f.p_B$ is a regular epimorphism, which implies that f is a regular epimorphism, and we have shown that U reflects regular epimorphisms.

Why is p_C a regular epimorphism? The algebra $F(U(C))$ is a copower of A indexed by $U(C)$, i.e., by the set $\hom_T(A, C)$. For each homomorphism $A \xrightarrow{h} C$, let h' be the coproduct injection $A \longrightarrow F(U(C))$ corresponding to $h \in \hom_T(A, C)$. The co-unit p is constructed so that $p_C.h' = h$ for each h. By hypothesis, A is a regular generator, so for some m there is a regular epimorphism $m \otimes A \xrightarrow{g} C$. Let g_i be the composite of g with the i-th coproduct injection $A \longrightarrow m \otimes A$, for each $i < m$. The homomorphisms $g_i{}'\colon A \longrightarrow F(U(C))$ induce a coproduct homomorphism $m \otimes A \xrightarrow{t} F(U(C))$ with $t_i = g_i{}'$ for each $i < m$. Then $p_C.t = g$, as can be checked by composing with coproduct injections, so p_C is a regular epimorphism as claimed.

Now we must show that U preserves and reflects congruence relations. Preservation is obvious, because U preserves kernel pairs, since U is representable. Both in Set and in Mod(T), it happens that every equivalence relation is a congruence relation, so we need only verify that U reflects equivalence relations.

Suppose that we are given two homomorphisms u, v from R to B in Mod(T) which induce a product homomorphism $R \longrightarrow B \times B$ such that $U(h): U(R) \longrightarrow U(B \times B)$ is an equivalence relation in Set. The functor U is faithful, since it is represented by a generator, so it reflects monomorphisms, hence h is a monomorphism in Mod(T). Using the equivalence of ii) and iii) established above, we may assume that, for some n, U_T is a retract of U^n. Let $U^n \xrightarrow{r} U_T$ be a retraction, and let $U_T \longrightarrow U^n$ be a coretraction, with $r.s = id$. It is easy to see that any nontrivial power of an equivalence relation in Set is an equivalence relation, so $U^n(R) \longrightarrow U^n(B \times B)$ is an equivalence relation. Using the natural transformations r and s, we shall show that $U_T(R) \longrightarrow U_T(B \times B)$ is an equivalence relation.

Without loss of generality, we may identify $U^n(R) \longrightarrow U^n(B \times B)$ with an equivalence relation $R' \subseteq B' \times B'$ in the ordinary sense, i.e., a set of ordered pairs which is reflexive, symmetric, and transitive. We also write $U_T(R) \longrightarrow U_T(B \times B)$ as $R \subseteq B \times B$, in keeping with our informal practice of ignoring the distinction between the algebras and their underlying sets, and identifying h with an inclusion map. The natural transformations $U^n \xrightarrow{r} U_T$ and $U_T \xrightarrow{s} U^n$ are here considered to be such that s_B is the inclusion map for $B \subseteq B'$ and such that r_B is a function $B' \longrightarrow B$ with $r_B(b) = b$ for all $b \in B$. Then $s_{B \times B}$ is the inclusion map for $B \times B \subseteq B' \times B'$, and s_R is the inclusion map for $R \subseteq R'$. The retraction $r_{B \times B}$ is just the product function $r_B \times r_B \colon B' \times B' \longrightarrow B \times B$, and r_R is the restriction of $r_B \times r_B$ to $R' \subseteq B' \times B'$.

Suppose $b \in B$. Then since R' is reflexive, $(b, b) \in R'$, so $(r_B(b), r_B(b)) = (b, b) \in R$, hence R is reflexive. If $(a, b) \in R$, then $(a, b) \in R'$, and by symmetry of R' we have $(b, a) \in R'$, so $(r_B(b), r_B(a)) = (b, a) \in R$, hence R is symmetric. If (a, b) and (b, c) belong to ~~R,~~ then they belong to R'; by transitivity of R', (a, c) belongs to R', so we have $(r_B(a), r_B(c)) = (a, c) \in R$, hence R is transitive. This proves that $R \hookrightarrow B \times B$ is an equivalence relation in Set, so it is also a congruence relation in Set. But U_T reflects congruence relations, so $R \hookrightarrow B \times B$ is a congruence relation in Mod(T). This completes the proof that U reflects congruence relations.

We have shown that, if U is represented by a regular progenerator, then U has a left adjoint and preserves and reflects congruence relations and regular epimorphisms. By (2.1) and (2.5), U is monadic. This completes the proof of the theorem.

The equivalence of i) and ii) in (3.1) is apparently part of the "folklore" of category theory. Lawvere's version of the characterization theorem for finitary equational categories (see [29], p. 79) explicitly states that the underlying-set functor is represented by a regular progenerator. Remarks by Wraith in [42], p. 54, and a theorem in Herrlich and Strecker [21], p. 245, also indicate that (3.1) is not a new result. In a conversation with the writer in March 1978, Michael Barr explained that he and several other category theorists had been stimulated by the Morita theorem [35] to investigate Morita equivalence of equational theories and had soon realized that all equivalences between equational categories were comparison functors represented by regular progenerators. From a category theorist's point of view, the general result lacked any novel features or interesting departures from the original Morita theorem, so the matter was dropped.

There is considerable logical interest in Morita equivalence, however, and by shifting our attention from the regular progenerators to the functors they represent, we shall derive a syntactical characterization of Morita equivalent equational theories which is new.

Let M be any nonempty category, and let $U: M \longrightarrow \text{Set}$ be a functor which is tractable in Set^M. If T is the equational structure of U, then U is, in the informal sense, a T-algebra in Set^M. If f and g are (m, n)-ary operations of T, then the interpretation $[f(x) = g(x)]$ of the equation $f(x) = g(x)$ in the T-algebra U relative to the variable x is an equalizer in Set^M of the arrows f and g as shown in the diagram below.

$$[f(x) = g(x)] \longrightarrow U^n \underset{g}{\overset{f}{\rightrightarrows}} U^m$$

Thus, $[f(x) = g(x)]$ is a subfunctor of U^n. For any object B of M, the set $U(B)$ inherits T-operations from U and is the underlying set of the T-algebra $E_U(B)$, where $E_U: M \longrightarrow \text{Mod}(T)$ is the comparison functor for U. The value of the functor $[f(x) = g(x)]$ at B is the solution set of $f(x) = g(x)$ in the n-th power $U^n(B)$ of the underlying set $U(B)$ of the T-algebra $E_U(B)$. Every subfunctor $G \hookrightarrow U^n$ which is an equalizer of a pair of operations in the equational structure of U will be said to be equationally definable.

3.2. Lemma. For any equational theory T and any cardinal n, a functor $G: \mathrm{Mod}(T) \longrightarrow \mathrm{Set}$ is represented by a T-algebra which is generated by n or fewer of its elements if and only if G is an equationally definable subfunctor of U_T^n.

Proof. Let A be a T-algebra which is generated by a set X of cardinality n or less. Then there is a regular epimorphism $F_T(n) \xrightarrow{p} A$ which sends the free generators $\{x_i : i < n\}$ of $F_T(n)$ onto X. For each $i < n$, let $p(x_i) = a_i$. Every element of $F_T(n)$ is of the form $f(x)$, where f is a $(1, n)$-ary T-operation and $x = (x_i : i < n)$; likewise, every element of A is of the form $f(a)$, where f is a $(1, n)$-ary T-operation and $a = (a_i : i < n)$; so we have $p(f(x_i)) = f(a)$ for every $(1, n)$-ary operation f. Let $(f_j : j < m)$ and $(g_j : j < m)$ be sequences of $(1, n)$-ary operations such that, for each $j < m$, we have $f_j(a) = g_j(a)$, i.e., $(f_j(x), g_j(x)) \in \ker(p)$. Also assume that, for each pair $(u(x), v(x))$ in $\ker(p)$, there is a $j < m$ such that $u = f_j$ and $v = g_j$. For each $(1, n)$-ary operation u, let $F_T(1) \xrightarrow{\hat{u}} F_T(n)$ be the (unique) homomorphism which sends the free generator of $F_T(1)$ to $u(x)$ in $F_T(n)$; then $\hat{u}$ is the co-operation $Y_T(u)$ of the T-coalgebra $Y_T\colon T° \longrightarrow \mathrm{Mod}(T)$ which was defined in Chapter 2, and which is informally identified with $F_T(1)$ as noted in Chapter 2. Let f be the (m, n)-ary operation which is induced in T by the sequence $(f_i : i < m)$ of $(1, n)$-ary operations; then $F_T(m) \xrightarrow{\hat{f}} F_T(n)$ is the coproduct homomorphism induced by the sequence $(\hat{f}_i : i < m)$ of homomorphisms; $\hat{f}$ sends the j-th free generator of $F_T(m)$ to $f_j(x)$ in $F_T(n)$. Define g and $\hat{g}$ similarly with respect to the sequence $(g_j : j < m)$. It is easy to see that $F_T(n) \xrightarrow{p} A$ is a coequalizer of $(\hat{f}, \hat{g})$ in $\mathrm{Mod}(T)$. An alternative construction of $\hat{f}$ and $\hat{g}$ which makes this obvious is to let $F_T(m) \xrightarrow{h} \ker(p)$ be a regular epimorphism from a free algebra onto $\ker(p)$, which is a subalgebra of $F_T(n) \times F_T(n)$, and take $\hat{f}$ and $\hat{g}$ to be the composites of h with the projections $\ker(p) \longrightarrow F_T(n)$; those projections are a kernel pair of p, and p coequalizes its kernel pairs, since p is a regular epimorphism.

The Yoneda embedding $\mathrm{Mod}(T)^\circ \longrightarrow \mathrm{Set}^{\mathrm{Mod}(T)}$ takes $F_T(n) \xrightarrow{p} A$ to $\underline{A} \xrightarrow{\underline{p}} U_T^n$, which is an equalizer of f and g in $\mathrm{Set}^{\mathrm{Mod}(T)}$. This proves that every representable functor is equationally definable as stated in (3.2).

If U is an equationally definable subfunctor of U_T^n, then there is an equalizer diagram $U \longrightarrow U_T^n \underset{g}{\overset{f}{\rightrightarrows}} U_T^m$ which witnesses that fact. The arrows f and g correspond, via the Yoneda embedding, to homomorphisms $\hat{f}, \hat{g}\colon F_T(m) \longrightarrow F_T(n)$. Let $F_T(n) \xrightarrow{p} A$ be a coequalizer in ~~Mod(T)~~ of $(\hat{f}, \hat{g})$; then $\underline{A} \xrightarrow{p} U_T{}^n$ is an equalizer of (f, g), so $U \cong \underline{A}$, i.e., U is represented by a T-algebra A which, as a homomorphic image of $F_T(n)$, is generated by n or fewer of its elements. This completes the proof of (3.2).

Note that the proof of (3.2) illustrates several "tricks" which are applied frequently below without being spelled out in detail. Also note that (3.2) depends heavily on the fact that $F_T(1)$ is a regular generator in Mod(T). Now that we know that every representable functor U: Mod(T) $\longrightarrow$ Set is definable in a power of U_T in a nice way, we describe how the equational structure of U relates to the equational structure T of U_T.

Let T and T' be equational theories, and let n be a nonzero cardinal. An <u>equational interpretation of T' in T of rank n</u> is given by a morphism of graphs $t: T' \longrightarrow T$ and an (m, n)-ary equation $f(x) = g(x)$ of L_T, called the <u>universe</u> of the interpretation, satisfying conditions i) - iv) below.

Let $E_1(x)$ be the equation $f(x) = g(x)$, and for each $k > 1$ let $E_k(y)$ be the formula $\bigwedge\{f(p_i(y)) = g(p_i(y)) : i < k\}$ where, for each $i < k$, p_i is the i-th projection $(T^n)^k = T^{n\times k} \longrightarrow T^n$. For each operation u of T', let u^t be the operation symbol of L_T which corresponds to t(u). The conditions which must be satisfied by t and $f(x) = g(x)$ are as follows.

i) For each (j, k)-ary T'-operation u, t(u) is an $(n \times j,\ n \times k)$-ary T-operation.

ii) For each (j, k)-ary T'-operation u, we have

$$T \models E_k(y) \Rightarrow E_j(u^t(y)).$$

iii) For each k we have

$$T \models E_k(y) \Rightarrow \bigwedge\{p_i(y) = q_i^{\,t}(y) : i < k\},$$

where $T'^k \xrightarrow{q_i} T'$ is the i-th projection in T'.

iv) For all T'-operations u, v, w such that u.v = w, where w is (j, k)-ary, we have

$$T \models E_k(y) \Rightarrow u^t(v^t(y)) = w^t(y).$$

3.3. Lemma. Let $A: T \longrightarrow M$ be a T-algebra in a small-complete category M, and let $t: T' \longrightarrow T$ be an equational interpretation of T' in T of rank n with universe $f(x) = g(x)$. Then there is a T'-algebra $B: T' \longrightarrow M$, with $B(T') = [f(x) = g(x)]_{A,x}$, such that, for every T'-operation u, $B(u)$ is the restriction to $[f(x) = g(x)]_{A,x}$ of $A(t(u)) = [u^t(y)]_{A,y}$.

Proof. To simplify notation, we write $A(u): A(T^n) \longrightarrow A(T^m)$ as $A^n \xrightarrow{u} A^m$ for every (m, n)-ary T-operation u. Also let $[f(x) = g(x)]_{A,x}$ be $E \xhookrightarrow{e} A^n$ and let $[E_k(y)]_{A,y}$ be $E_k \xhookrightarrow{e_k} A^{n\times k}$. First we shall show that E_k and E^k are equivalent as subobjects of $A^{n\times k}$. When $k = 0, 1$, there is nothing to prove, so assume $k > 1$. For each $i < k$, let $(T^n)^k = T^{n\times k} \xrightarrow{p_i} T^n$ and $(T^m)^k = T^{m\times k} \xrightarrow{r_i} T^m$ be the respective i-th projection operations. Note that for any (m, n)-ary T-operation u we have $u.p_i = r_i.u^k$, where $T^{n\times k} \xrightarrow{u^k} T^{m\times k}$ is the k-th power of u. Since E_k is the intersection of the subobjects $E_{k,i} = [f(p_i(y)) = g(p_i(y))]_{A,y} \hookrightarrow A^{n\times k}$, we can prove $E^k \leq E_k$ by showing that $E^k \leq E_{k,i}$ for each $i < k$. But for each i, $E_{k,i} \hookrightarrow A^{n\times k}$ is an equalizer of $(f.p_i, g.p_i)$, and $f.p_i.e^k = r_i.f^k.e^k = r_i.(f.e)^k = r_i.(g.e)^k = r_i.g^k.e^k = g.p_i.e^k$, so indeed $E^k \leq E_{k,i}$ for each i, hence $E^k \leq E_k$. Products and equalizers commute, so $E^k \xrightarrow{e^k} A^{n\times k}$ is an equalizer of (f^k, g^k). Also note that $f.p_i.e_k = g.p_i.e_k$ because $E_k \leq E_{k,i}$. Then we have $r_i.f^k.e_k = f.p_i.e_k = g.p_i.e_k = r_i.g^k.e_k$ for each $i < k$, so $f^k.e_k = g^k.e_k$, which implies that $E_k \leq E^k$. This completes the proof that E_k and E^k are equivalent and permits us to identify B^k and E_k with E^k.

Condition ii) of the definition says that, for any (j, k)-ary T'-operation u, the restriction of $[u^t(y)]_{A,y}$ to $B^k \hookrightarrow A^{n\times k}$ factors through $B^j \hookrightarrow A^{n\times j}$ as in the diagram below.

$$\begin{tikzcd}
A^{n\times k} \arrow[r, "A(t(u))"] & A^{n\times j} \\
B^k \arrow[u, hook, "e^k"] \arrow[r, dashed] & B^j \arrow[u, hook, "e^j"']
\end{tikzcd}$$

We define $B(u)$ to be the arrow $B^k \longrightarrow B^j$ which completes the diagram; since e^j is a monomorphism, there is only one such arrow. Condition iv) guarantees that B preserves composition of T-operations, even though perhaps t does not. Thus, B is a functor $T' \longrightarrow M$. According to condition iii), for each projection operation $T'^k \xrightarrow{q_i} T'$ the interpretation $A(t(q_i))$ coincides with the projection $A^{n\times k} \xrightarrow{p_i} A^n$ on $B^k \hookrightarrow A^{n\times k}$. But then in the commutative diagram

$$\begin{tikzcd} A^{n\times k} \arrow[r, "p_i"] & A^n \\ B^k \arrow[u, hook, "e^k"] \arrow[r, "B(q_i)"'] & B \arrow[u, hook', "e"'] \end{tikzcd}$$

the bottom arrow $B(q_i)$ has to coincide with the "real" i-th projection $B^k \longrightarrow B$, since the composite of the latter with the monomorphism e is equal to $p_i \cdot e^k$. This shows that B is a product-preserving functor $T' \longrightarrow M$, i.e., a T-algebra in M. The proof of (3.3) is now complete.

If $t\colon T' \longrightarrow T$ is an equational interpretation of T' in T of rank n with universe $f(x) = g(x)$, then applying (3.3) to U_T, viewed as a T-algebra in $\mathrm{Set}^{\mathrm{Mod}(T)}$, we see that the interpretation defines a subfunctor U of $U_T^{\,n}$ which is a T'-algebra in $\mathrm{Set}^{\mathrm{Mod}(T)}$; this means that U is the set-valued component of a functor $\mathrm{Mod}(T) \longrightarrow \mathrm{Mod}(T')$ which we shall denote (a little ambiguously) by t^*. Note that any mapping of theories $t\colon T' \longrightarrow T$ is an equational interpretation of rank 1 with universe $x = x$; in this case, $t^*\colon \mathrm{Mod}(T) \longrightarrow \mathrm{Mod}(T')$ is the reduct functor.

3.4. Theorem. A functor $G\colon \mathrm{Mod}(T) \longrightarrow \mathrm{Mod}(T')$ is coalgebra-representable if and only if there is an equational interpretation $t\colon T' \longrightarrow T$ such that $G \cong t^*$.

Proof. Let $G\colon \mathrm{Mod}(T) \longrightarrow \mathrm{Mod}(T')$ be represented by a coalgebra $A\colon T'^{\circ} \longrightarrow \mathrm{Mod}(T)$. In accordance with our informal usage, we identify A with its underlying T-algebra, equipped with T'-co-operations. By (3.2), there is, for some m and n, an (m, n)-ary equation $f(x) = g(x)$ of L_T such that the set-valued ~~functor~~ $\cancel{U_{T'}}.G \cong \underline{A}$: Mod(T) $\longrightarrow$ Set can be identified with $[f(x) = g(x)] \hookrightarrow U_T{}^n$. This means that in Mod(T) there is a coequalizer diagram $F_T(m) \underset{\hat{g}}{\overset{\hat{f}}{\rightrightarrows}} F_T(n) \xrightarrow{p} A$, so that p is a regular epimorphism. Because copowers commute with coequalizers, every copower $F_T(n \cancel{\times k}) \xrightarrow{k \otimes p} k \otimes A$ of p is a regular epimorphism. Now let u be any (j, k)-ary T'-operation, and let $j \otimes A \xrightarrow{\hat{u}} k \otimes A$ be the corresponding co-operation. Consider the following diagram:

$$\begin{tikzcd}
F_T(n \times j) \arrow[r, dashed, "\hat{h}"] \arrow[d, "j \otimes p"'] & F_T(n \times k) \arrow[d, "k \otimes p"] \\
j \otimes A \arrow[r, "\hat{u}"'] & k \otimes A
\end{tikzcd}$$

If $j > 0$, then $F_T(n \times j)$ is regular-projective, so there is a homomorphism $\hat{h}$ as shown which completes the diagram. If $j = 0$, then both $F_T(n \times j)$ and $j \otimes A$ are isomorphic copies of the algebra $F_T(0)$ of constants, which is an initial object in $\mathrm{Mod}(T)$, so the diagram commutes with $\hat{h}$ being the embedding $F_T(0) \hookrightarrow F_T(n \times k)$. Define a morphism of graphs $t\colon T' \longrightarrow T$ by setting $t(u)$ equal to h for each (j, k)-ary operation u of T', where h is any $(n \times j, n \times k)$-ary operation of T such that $(k \otimes p).\hat{h} = \hat{u}.(j \otimes p)$. This morphism of graphs defines an equational interpretation of T' in T with universe $f(x) = g(x)$.

Given an equational interpretation $t\colon T' \longrightarrow T$ with (m, n)-ary universe $f(x) = g(x)$, let A be a coequalizer in $\mathrm{Mod}(T)$ of the T-co-operations $\hat{f}, \hat{g}\colon F_T(m) \longrightarrow F_T(n)$. By pulling the T'-algebra operations of $U_{T'}.t^* \cong \underline{A} \hookrightarrow U_T{}^n$ back through the Yoneda embedding, we endow A with T'-co-operations which make A a T'-coalgebra in $\mathrm{Mod}(T)$. It is obvious that t^* is represented by this coalgebra A. This concludes the proof of the theorem.

Theorem (3.4) shows that equational interpretations are the syntactical counterparts of coalgebras. In this connection it is interesting to note Wraith's observation in [42], p. 62, that a coalgebra is a kind of generalized mapping of theories. The syntactical description of a coalgebra-representable functor in terms of an equational interpretation of theories enables us to investigate the logical properties of the functor without recourse to ultraproducts or modifications of the Feferman-Vaught Theorem as employed, for example, in Burris [5] or Banaschewski and Nelson [3]. In particular, it is immediately obvious that the representable functor t^* corresponding to an equational interpretation $t\colon T' \longrightarrow T$ will preserve any logical properties which the definable substructures provided by the interpretation (see (3.3)) inherit from the "parent" T-algebras.

3.5. Lemma. Let A and B be tractable objects in a category M, and let T be the equational structure of A. Then B is a retract of A^n if and only if there is an idempotent (n, n)-ary

T-operation $A^n \xrightarrow{u} A^n$ such that B is isomorphic to the subobject $[u(x) = x] \hookrightarrow A^n$.

Proof. This lemma is a thinly-disguised version of a well-known fact about retracts. First, suppose that B is a retract of A^n. Then we have arrows $B \xrightarrow{s} A^n$ and $A^n \xrightarrow{r} B$ with $r.s = id_B$. The composite $s.r$ is an idempotent (n, n)-ary T-operation, since $s.r.s.r = s.id_B.r = s.r$. Furthermore, it is easy to verify that s is an equalizer of s.r and id. Letting $u = s.r$, we have $B \cong [u(x) = x]$ as required.

Now suppose $A^n \xrightarrow{u} A^n$ is given with $u.u = u$ and with $B = [u(x) = x] \overset{s}{\hookrightarrow} A^n$. The arrow s is an equalizer of u and id, but u itself satisfies $u.u = id.u$, so there is a unique arrow $A^n \xrightarrow{r} B$ such that $s.r = u$. But s, being an equalizer, is a monomorphism, and $s.r.s = u.s = id.s = s.id_B$, so it follows that $r.s = id_B$, i.e., B is a retract of A^n.

We shall say that an equational interpretation of T' in T is strong if the morphism of graphs $t: T' \longrightarrow T$ preserves composition of arrows and if the universe of the interpretation is $u(x) = x$, where u is the interpretation under t of the $(1, 1)$-ary identity operation of T'. Let f' be a (j, k)-ary T'-operation, and let $f = t(f')$. Then we have $u^j.f.u^k = f$. Let f', g', and h' be T'-operations whose interpretations in T are f, g, and h respectively. Then because the morphism of graphs t preserves composition of arrows, any identity $f'.g' = h'$ in T' will correspond to an identity $f.g = h$ in T.

Given an equational theory T with an idempotent (n, n)-ary operation u, we define a new equational theory $T|u$, the restriction of T to u, as follows. Say that an $(n \times j, n \times k)$-ary operation g of T is a u-operation if $u^j.g.u^k = g$. Note that a composite of u-operations is a u-operation. The equational theory $T|u$ is the category whose arrows are the u-operations of T and whose identity arrows are the powers (which may be computed in T) of the operation u. The verification that $T|u$ is actually an equational theory is contained in the proof of lemma (3.6) below. The inclusion $T|u \longrightarrow T$, together with the equation $u(x) = x$, defines a strong equational interpretation of $T|u$ in T of rank n; obviously, a strong equational interpretation of any equational theory T' in T with universe $u(x) = x$ will always take the form of a morphism of graphs $T' \longrightarrow T$ which factors as a mapping of theories $T' \longrightarrow T|u$ followed by the inclusion morphism $T|u \longrightarrow T$.

<u>3.6</u>. <u>Lemma</u>. Let A be a tractable object with equational structure T in a small-complete category M. Let u be an idempotent (n, n)-ary operation in T. Then $[u(x) = x]$ is a tractable object whose equational structure is isomorphic to $T|u$.

Proof. Let $B = [u(x) = x]$; then by (3.5) B is a retract of A^n. Let $A^n \xrightarrow{r} B$ and $B \xrightarrow{s} A^n$ be arrows such that $r.s = id_B$

and $s.r = u$. Let T' be a skeleton of the full subcategory in M of powers of B. Define a morphism of graphs $t\colon T' \longrightarrow T|u$ as follows. For each arrow $B^k \xrightarrow{g} B^j$ of T', let $t(g)$ be the $(n \times j,\ n \times k)$-ary T-operation $s^j.g.r^k$; it is easy to see that $t(g)$ is a u-operation of T, and it is also easy to see that t preserves composition of arrows and takes the identity arrow $B^k \longrightarrow B^k$ to u^k, for each k. Hence, $t\colon T' \longrightarrow T|u$ is a functor and is surjective on objects. Because powers preserve retractions and coretractions, every power of r is an epimorphism and every power of s is a monomorphism, so it follows that t is faithful. Finally, t is full, since for any $(n \times j,\ n \times k)$-ary u-operation f of T we have $f = u^j.f.u^k = s^j.r^j.f.s^k.r^k = t(r^j.f.s^k)$. We have shown that t is an isomorphism of categories, so $B = [u(x) = x]$ is a tractable object, since $T' \cong T|u$ is locally small. This proves the lemma.

3.7. __Lemma.__ Let A and B be tractable objects in a small-complete category M, and let T be the equational structure of A. Then A and B are retract-equivalent, with B being a retract of A^n and A being a retract of B^m, if and only if T satisfies the following conditions:

i) There is an idempotent (n, n)-ary T-operation u such that $B \cong [u(x) = x] \hookrightarrow A^n$;

ii) There is an $(n \times m, 1)$-ary T-operation d such that $u^m.d = d$;

iii) There is a $(1, n \times m)$-ary T-operation p such that $p.d = id$.

Proof. By (3.5), condition i) is necessary and sufficient for B to be a retract of A^n. Suppose that A and B are retract-equivalent as described above. Then we have $B \xrightarrow{s} A^n$ and $A^n \xrightarrow{r} B$ with $r.s = id_B$, and we also have $A \xrightarrow{h} B^m$ and $B^m \xrightarrow{g} A$ with $g.h = id_A$. The operations called for by the theorem are $u = s.r$, $d = s^m.h$, and $p = g.r^m$. On the other hand, now suppose that conditions i), ii), and iii) are true. We may suppose that $u = s.r$, where s and r are as above. Consider the arrows $A \xrightarrow{r^m.d} B^m$ and $B^m \xrightarrow{p.s^m} A$; using ii) and iii), we compute

$$p.s^m.r^m.d = p.u^m.d = p.d = id_A,$$

i.e., A is a retract of B^m. This completes the proof.

Combining (3.1), (3.6), and (3.7), we now have a syntactical characterization of all the equational theories T' which are Morita equivalent to a given equational theory T.

<u>3.8</u>. <u>Theorem</u>. Equational theories T and T' are Morita equivalent if and only if, for some cardinals m and n, the following conditions are satisfied:

i) There is an idempotent (n, n)-ary T-operation u such that T' is isomorphic to $T|u$;

ii) There is an $(n \times m, 1)$-ary T-operation d such that $u^m.d = d$;

iii) There is a $(1, n \times m)$-ary T-operation p such that $p.d = id$.

A strong equational interpretation $t: T' \longrightarrow T$ with (n, n)-ary universe $u(x) = x$ will be called a spanning equational interpretation if t determines an isomorphism of T' and $T|u$, and if conditions ii) and iii) of (3.8) are satisfied. It is evident that $G: Mod(T) \longrightarrow Mod(T')$ is an equivalence functor if and only if there is a spanning equational interpretation $t: T' \longrightarrow T$ such that $G \cong t^*$.

There is a category-theoretic generalization of Morita equivalence which has been studied in various forms and for which some partial characterizations have been published. Given a fixed base category M, two small categories A and B are Morita equivalent over M if the functor categories M^A and M^B are equivalent.

U. Knauer [27] has characterized Morita equivalence of monoids over Set; his Theorem 6.1 is actually a special case of (3.8), since each monoid A is identifiable with an equational theory A' which

features a monoid of unary operations isomorphic to A, so that the category Set^A of "left A-acts" is identifiable with Mod(A').
According to Knauer, Banaschewski [2] contains similar results on monoids.

A related paper is Elkins and Zilber [12], in which Morita equivalence of arbitrary small categories over Set is characterized in a way which is obviously strongly related to our (3.8). In the terminology of [12], a weak functor is a composition-preserving morphism of graphs. A weak equivalence A ---→ B is a weak functor satisfying conditions analogous to those which define a category equivalence. Theorem 4.4 of [12] states that, for small categories A and B, the functor categories Set^A and Set^B are equivalent if and only if there is a weak equivalence A ---→ B.

It should also be noted that Freyd [17] contains a syntactical characterization of auto-equivalences Mod(T) ⟶ Mod(T), for finitary equational theories T.

Next we present a few corollaries to (3.8).

<u>3.9</u>. <u>Corollary</u>. Retract-equivalent tractable objects in a category M have Morita equivalent equational structures.

<u>3.10</u>. <u>Corollary</u>. If A is a tractable object in a small-complete category M, and if the equational structure of A is Morita equivalent to T, then A is retract-equivalent to some object B whose equational structure is isomorphic to T.

The significance of the next lemma is that, if we are interested only in finitary equational theories, the cardinals m and n mentioned in (3.8) may be taken to be finite. Part i) of (3.11) is certainly not new, but part ii) might be.

<u>3.11</u>. <u>Lemma</u>. Let T be a finitary equational theory, and let A be a T-algebra belonging to Mod(T). Then:

i) If A is finitely generated, then the equational costructure of A is finitary;

ii) If A is a regular progenerator whose equational costructure is finitary, then A is finitely generated.

Proof. If A is finitely generated, then it is evident from the proof of (3.4) that there is an equational interpretation of finite rank of the equational costructure T' of A in T, so it follows that T' is finitary. On the other hand, suppose that A is a regular progenerator whose equational costructure T' is finitary. Then by (3.1) the comparison functor $E: \mathrm{Mod}(T) \longrightarrow \mathrm{Mod}(T')$ for $\underline{A}$ is an equivalence functor, so by (2.3) A is finitely generated. This completes the proof of the lemma.

Note that the converse of condition i) above is not valid. If A has no proper endomorphisms, then the equational costructure of A is the trivial theory, i.e., the theory Card° whose operations are all trivial, which is certainly a finitary theory. An example of an algebra with no proper endomorphisms but which is not finitely generated is constructed by equipping a countably infinite set X with a sequence $(u_i : i < \omega)$ of unary operations, each of which is a permutation of X with a single fixed point, with each point of X being fixed by at least one of the operations, and with the operations chosen so that, for each point a of X, the set $X - \{u_i(a) : i < \omega\}$ is infinite.

Applied to finitary theories, (3.8) obviously has great potential for transferring logical properties such as categoricity, stability, decidability, and so on. For the moment we shall content ourselves with two logical corollaries to (3.8). A finitary equational theory T is <u>countably presentable</u> if there is a presentation of T in terms of countably many equations and countably many distinguished operations. An equational theory T is <u>locally finite</u> if, for each m and n, there are only finitely many nontrivial (m, n)-ary T-operations.

3.12. Corollary. A countably presentable finitary equational theory is Morita equivalent to exactly ω different finitary equational theories, and they are all countably presentable.

3.13. Corollary. Every finitary equational theory which is Morita equivalent to a finitary and locally finite equational theory is locally finite.

The canonical language L_T of an equational theory T, while adequate for technical purposes, is inconvenient for informal discourse because all of its operation symbols are formally unary, which makes it necessary to employ a multitude of operation symbols for projections and argument-shuffling operations. We shall frequently resort to a self-explanatory notation, employing variables in the familiar way, when discussing the operations of an equational theory T. In the terminology of Makkai and Reyes [33], we are using the extended canonical language of T. The following examples, using the extended language, may help to provide an intuitive understanding of how the operations u, d, and p mentioned in (3.8) determine an equivalence of categories.

Recall that, for finite $m > 1$, P_m is the finitary equational theory of m-valued Post algebras, which was pointed out at the beginning of this chapter as being Morita equivalent to the finitary equational theory BA of Boolean algebras. A survey of basic lattice-theoretic results concerning Post algebras is given in Balbes and Dwinger [1] and in Rasiowa [37]. Rasiowa provides a presentation of P_m in terms of:

i) Constants $e_0, e_1, \ldots, e_{m-1}$

ii) Unary operations $\neg$, $D_1, D_2, \ldots, D_{m-1}$

iii) Binary operations $\wedge$, $\vee$, $\rightarrow$

iv) A list of equational axioms $(p_0), (p_1), \ldots, (p_8)$.

The equational axioms ensure that every m-valued Post algebra is a Heyting algebra with respect to e_0, e_{m-1}, $\neg$, $\wedge$, $\vee$, and $\rightarrow$, with e_0 being the "zero" and e_{m-1} being the "one." The operation D_1 can be proved to coincide with "double negation," i.e., $D_1(x) = \neg\neg x$ (see [37], p. 137).

The operations u, d, and p required for a spanning equational interpretation of rank 1 of BA in P_m are as follows:

i) u is the (1, 1)-ary "double negation" operation

$$u(x) = \neg\neg x$$

ii) d is the (m-1, 1)-ary operation

$$d(x) = (D_1(x), D_2(x), \ldots, D_{m-1}(x))$$

iii) p is the (1, m-1)-ary operation

$$p(x_1, \ldots, x_{m-1}) = (e_1 \wedge x_1) \vee (e_2 \wedge x_2) \vee \ldots \vee (e_{m-1} \wedge x_{m-1}).$$

The identities which u, d, and p are required to satisfy are given in [37] as (p_5), which takes care of both i) and ii) of (3.8), and (p_7), which corresponds to condition iii) of (3.8). The operation u picks out the subset of all complemented, or "Boolean," elements of any m-valued Post algebra. The operation d decomposes each element of the Post algebra into a chain of Boolean elements which can be re-assembled by p to recover the original element. The u-operations of P_m are precisely the operations which preserve Boolean elements; it is easily seen (see [37], p. 136) that the Boolean elements of any Post algebra form a Boolean algebra with respect to the operations which preserve them. Thus, the analysis of Post algebras as chain-based distributive lattices by Epstein [13] and Traczyk [41] corresponds precisely to the syntactical conditions posed by (3.8) for P_m to be Morita equivalent to the theory BA of Boolean algebras.

The representation of the m-valued Post algebras as lattices of nonincreasing (m-1)-element chains in Boolean algebras (see Rasiowa [37], pp. 143-144 for details) provides a spanning equational interpretation of rank m-1 of P_m in BA. In this case, the three BA-operations which determine the spanning interpretation are:

i) An (m-1, m-1)-ary idempotent operation

$$u(x_1, \ldots, x_{m-1}) = (x_1, x_1 \wedge x_2, \ldots, x_1 \wedge \ldots \wedge x_{m-1})$$

ii) The (m-1, 1)-ary diagonal operation

$$d(x) = (x, x, \ldots, x)$$

iii) The (1, m-1)-ary projection operation

$$p(x_1, \ldots, x_{m-1}) = x_1.$$

For the next example, let T be any equational theory, and let ${}^kT$ be the full subcategory in T of powers of T^k; then ${}^kT$ is an equational theory, called the k-th <u>matrix theory</u> of T for reasons explained in Wraith [42]. Since ${}^kT$ is the equational structure of T^k, which is retract-equivalent in T to T, it follows by (3.9) that ${}^kT$ is Morita equivalent to T. A spanning equational interpretation of rank k of ${}^kT$ in T is given by the inclusion functor ${}^kt\colon {}^kT \hookrightarrow T$, which is *not* a mapping of theories unless k = 1. The operation u in this case is the identity arrow of T^k, while d is the diagonal arrow $T \longrightarrow T^k$ and p is any projection $T^k \longrightarrow T$.

Another significant example relevant to (3.8) is provided by the role of idempotents in the endomorphism ring, i.e., equational costructure, of a free module in defining Morita contexts (see Cohn [7], pp. 46-47); the counterparts to d and p do not seem to be pointed out in Cohn's discussion, however.

Lawvere [29] defines an algebraic functor to be any functor of the form $g^*: \mathrm{Mod}(T) \longrightarrow \mathrm{Mod}(T')$, where T and T' are equational and $g: T' \longrightarrow T$ is any product-preserving functor. The degree of such an algebraic functor g^* is that cardinal n such that

$U_{T'}.g^* = U_T{}^n$.

Let $G: \mathrm{Mod}(T) \longrightarrow \mathrm{Mod}(T')$ be any functor such that $U_{T'}.G = U_T{}^n$. Then $U_{T'}.G$ is represented by $F_T(n)$, and the equational structure of $U_{T'}.G$ is isomorphic to ${}^nT$. Consider the following commutative diagram.

$$\begin{tikzcd} \mathrm{Mod}(T) \arrow[r, "E"] \arrow[d, "U_T^{\,n}"'] \arrow[dr, "G"] & \mathrm{Mod}({}^nT) \arrow[d, "g^*"] & {}^nT \\ \mathrm{Set} & \mathrm{Mod}(T') \arrow[l, "U_{T'}"] & T' \arrow[u, "g"'] \end{tikzcd}$$

Here, g is a uniquely-determined mapping of theories, and the comparison functor E for $U_T^{\,n}$ is the equivalence functor ${}^nt^*$, where ${}^nt\colon {}^nT \hookrightarrow T$ is the spanning equational interpretation of ${}^nT$ in T discussed above. Thus, we have $G = g^*.{}^nt^* = ({}^nt.g)^*$, so G is an algebraic functor of degree n, in Lawvere's terminology.

Now let $G\colon \mathrm{Mod}(T) \longrightarrow \mathrm{Mod}(T')$ be any functor whose set-valued component $U = U_{T'}.G$ is monadic; then U is represented by a regular progenerator, and the equational structure T'' of U is Morita equivalent to T. Consider the following commutative diagram.

$$\begin{tikzcd}
\mathrm{Mod}(T) \arrow[r, "E"] \arrow[d, "U"'] \arrow[dr, "G"] & \mathrm{Mod}(T'') \arrow[d, "g^*"] & T'' \\
\mathrm{Set} & \mathrm{Mod}(T') \arrow[l, "U_{T'}"] & T' \arrow[u, "g"']
\end{tikzcd}$$

Here again the comparison functor E is an equivalence functor, and g is a uniquely determined mapping of theories. In this case, $E = t^*$, where $t\colon T'' \longrightarrow T$ provides a spanning equational interpretation of T'' in T as demanded by (3.8). Note that t factors as $T'' \longrightarrow {}^nT \hookrightarrow T$, where n is the rank of the interpretation. Thus, we have $G = g^*.t^* = (t.g)^*$, where g is a mapping of theories and t is a spanning equational interpretation of rank n.

The foregoing considerations suggest a broadening of Lawvere's definition. Say that a functor $G\colon \mathrm{Mod}(T) \longrightarrow \mathrm{Mod}(T')$ is <u>algebraic</u> if G is represented by a regular progenerator.

3.14. Theorem. For any equational theories T and T' and any functor $G: \mathrm{Mod}(T) \longrightarrow \mathrm{Mod}(T')$, the following are equivalent:

i) G is algebraic;

ii) G has a left adjoint and preserves and reflects regular epimorphisms;

iii) G factors as an equivalence functor followed by a reduct functor;

iv) The set-valued component $U_{T'}.G$ of G is monadic;

v) $G \cong (t.g)^*$, where $g: T' \longrightarrow T"$ is a mapping of equational theories and $t: T" \longrightarrow T$ is a spanning equational interpretation.

Note that ii) is used as the definition of the term "algebraic functor" in Herrlich and Strecker [21], except that the domain of an algebraic functor as defined by them need not be an equational category. The rôle of regular progenerators in representing algebraic functors is explicitly pointed out in [21], Theorem 32.21.

The equivalence of conditions iii) and iv) above is implied by Linton's result (2.5). The syntactical characterization v) is new.

Define the rank of an algebraic functor G: $\mathrm{Mod}(T) \longrightarrow \mathrm{Mod}(T')$ to be the smallest cardinal n such that the regular progenerator which represents G is generated by n elements. If G has rank n, then n is the smallest cardinal k for which $U_{T'}.G$ is a retract of $U_T{}^k$ and for which the spanning interpretation t mentioned in condition v) above factors through ${}^k t$. Note that, by (3.11), the equational structure of $U_{T'}.G$ is finitary if and only if G has finite rank.

CHAPTER 4. REPRESENTABLE BOOLEAN POWER FUNCTORS

The notion of the Boolean extension A[B] of a finitary universal algebra A by a Boolean algebra B was introduced by A. L. Foster [14] as a device for making structural comparisons between Boolean algebras and other kinds of algebras such as p-rings. One of Foster's principal results was that, when A is primal, i.e., finite and nontrivial, having all possible finitary operations, the equational class generated by A is the class of all isomorphic copies of Boolean extensions of A (see [14], [15]). From our point of view, Foster, without the help of category theory, defined and studied a class of functors, showing that certain of those functors were category equivalences. Our goal in this chapter is to prove some results about Foster's functors which would be difficult to formulate without using category-theoretic concepts, but which have algebraic import. In showing that these results carry over to infinitary algebra, we prove an infinitary version of Foster's theorem cited above which could not be proved by directly extending Foster's original method. Our analysis of Boolean power functors as algebraic functors provides a useful alternative to the topological or sheaf-theoretic approach as exemplified by Burris [5] and Banaschewski and Nelson [3].

We begin with a naive search for all the monadic functors $\mathrm{Mod(BA)} \longrightarrow \mathrm{Set}$ whose equational structure is finitary. Such a search would be motivated, if we did not already have other reasons for it, by the fact that Mod(BA) is an important and comparatively well-understood equational category which enjoys an abundance of interesting and easily-studied coalgebras. As (3.11) informs us, our search is equivalent to the problem of identifying all the finitely-generated regular progenerators in Mod(BA). Since BA is a locally finite theory, all finitely generated Boolean algebras are finite. The finite regular progenerators in Mod(BA) are rather easy to find.

<u>4.1</u>. <u>Lemma</u>. Every finite Boolean algebra having more than two elements is a regular progenerator.

Proof. Every finite Boolean algebra is a power of a two-element Boolean algebra; the free algebras are the ones of the form 2^{2^n}, where n is the number of free generators. Obviously, for $k > 2$ the free algebra $F_{BA}(1) \cong 2^2$ is a retract of the algebra 2^k; furthermore, for sufficiently large finite n, 2^k is a retract of $2^{2^n} \cong F_{BA}(n)$. Thus, every finite Boolean algebra having more than two elements is dually retract-equivalent to $F_{BA}(1)$, hence is a regular progenerator.

In fact, every countable Boolean algebra having more than two elements is a regular progenerator in Mod(BA); not much is known about the uncountably infinite Boolean algebras which are regular progenerators (see Balbes and Dwinger [1], p. 107).

4.2. Lemma. For any finite $m > 1$, the equational costructure of the Boolean algebra 2^m in Mod(BA) is isomorphic to the finitary part of the equational structure of an m-element set, i.e., to the equational theory P_m of m-valued Post algebras.

Proof. Since 2^m is finitely generated, its equational costructure is finitary, by (3.11). Thus, it is sufficient to verify that the finitary parts of the two theories cited are isomorphic. For all finite n, the Boolean algebra 2^{m^n} is an n-th copower of 2^m in Mod(BA), and the assignment of $2^m \xrightarrow{f^*} 2^{m^n}$ to each finitary operation $m^n \xrightarrow{f} m$ establishes the required isomorphism.

Thus, if the set A has $m > 1$ elements, then the representable functor $\underline{2}^A\colon \mathrm{Mod(BA)} \longrightarrow \mathrm{Set}$ is an m-valued Post algebra in $\mathrm{Set}^{\mathrm{Mod(BA)}}$; the corresponding comparison functor $\mathrm{Mod(BA)} \longrightarrow \mathrm{Mod}(P_m)$ is a category equivalence, since 2^A is a regular progenerator. By combining (3.1), (3.11), (4.1), and (4.2), we have a proof of T. K. Hu's result (1.3).

If A is a finite set having at least two elements and B is a Boolean algebra, the Boolean power $A[B]$ is the set of all A-indexed partitions of unity in B, i.e., the set of all functions u from A into the underlying set of B such that:

i) For $a \neq b$ in A, we have $u(a) \wedge u(b) = 0$ in B;

ii) $\bigvee_{a \in A} u(a) = 1$ in B.

The definition above is A. L. Foster's (see [14]). For each homomorphism $B \xrightarrow{h} C$, let $A[h]$ be the function which sends each partition u in $A[B]$ to the partition $h.u$ in $A[C]$; we have now defined a set-valued Boolean power functor $A[-]: \mathrm{Mod}(BA) \longrightarrow \mathrm{Set}$.

4.3. Lemma. The functor $A[-]$ is represented by the Boolean algebra 2^A.

Proof. It is easy to see that the restriction of any homomorphism $2^A \longrightarrow B$ to the atoms of 2^A determines an A-indexed partition of unity in B, and that each partition u belonging to A[B], regarded as a function from the atoms of 2^A into B, has a unique extension to a homomorphism $2^A \longrightarrow B$. More formally, the isomorphism $\underline{2}^A \longrightarrow A[-]$ in the functor category $\mathrm{Set}^{\mathrm{Mod(BA)}}$ corresponds, by the Yoneda Lemma, to an element of $A[2^A]$ or, in Mac Lane's terminology, to a universal element of A[-]. That element is the partition $A \longrightarrow 2^A$ defined by the insertion of the atoms.

The representability of A[-] does not seem to have been directly exploited before to any significant extent, although Banaschewski and Nelson [3] treat A[-] as a contravariant algebra-of-continuous-functions functor C(-, A) from topological spaces into **ISP**(A), and it is pointed out that C(-, A) has an "adjoint on the right."

Foster's definition of $A[B]$ in [14] includes a formula which shows how each finitary operation $A^n \xrightarrow{f} A$ can be transmogrified into an operation $A[B]^n \xrightarrow{f'} A[B]$. If A is a finite algebra in some finitary equational category $\mathrm{Mod}(T')$, then the Boolean algebra 2^A is a T'-coalgebra in $\mathrm{Mod}(BA)$; each finitary operation $A^n \xrightarrow{f} A$ determines a co-operation $2^A \xrightarrow{f^*} 2^{A^n}$. By exploiting the fact that every Boolean algebra B has an embedding into a power of a two-element Boolean algebra, we can show that both Foster's formula for f' and the co-operation f^* yield the same natural transformation $A[-]^n \longrightarrow A[-]$. (Note that from now on we shall often identify $A[-]$ with $\underline{2^A}$, in light of (4.3)).

4.4. Lemma. Let $A^n \xrightarrow{f} A$ be a finitary operation on a finite set A having at least two elements, and let $f' = \underline{f^*}$ be the corresponding operation on the set-valued functor $A[-]$. Then for any Boolean algebra B, the action of $f'_B : A[B]^n \longrightarrow A[B]$ is described as follows. For any $u = (u_0, u_1, \ldots, u_{n-1})$ in $A[B]^n$ and any $c \in A$, we have

$$[f'_B(u)](c) = \bigvee_{f(b)=c} \bigwedge_{j<n} u_j(b_j).$$

Proof. Each of the partitions u_j corresponds to a homomorphism $v_j^*: 2^A \longrightarrow 2^I$ which factors through $B \hookrightarrow 2^I$; these homomorphisms collectively induce a coproduct homomorphism $2^{A^n} \xrightarrow{v^*} 2^I$ which factors through B and corresponds to $u \in A[B]^n$. To simplify the notation, we identify elements of B with subsets of I. Then

$$\begin{aligned} [f'_B(u)](c) &= [\underline{f^*}_B(v^*)](\{c\}) \\ &= [v^*.f^*](\{c\}) \\ &= [(f.v)^*](\{c\}) \end{aligned}$$

$$= \{i \in I : [f.v](i) = c\}$$

$$= \{i \in I : f(v_0(i), \ldots, v_{n-1}(i)) = c\}$$

$$= \bigcup_{f(b)=c} \bigcap_{j<n} v_j{}^*(\{b_j\})$$

$$= \bigcup_{f(b)=c} \bigcap_{j<n} u_j(b_j).$$

Note that (4.4) only serves to show that our viewing $A[-]$ as a representable functor does not conflict in any respect with Foster's original definition of the Boolean extension construction; the computation upon which the proof is based is, with minor variations depending on context, widely known (see [3], p. 5, for example), and was probably used by Foster himself to derive the formula given in (4.4). Indeed, since $A[-]$ is representable, it preserves products and monomorphisms, so if $B \hookrightarrow 2^I$ we have $A[B] \hookrightarrow A[2^I] \cong A^I$. The function $I \xrightarrow{v} A^n$ in the proof above is equivalent to an n-tuple of elements of A^I, and the composite $I \xrightarrow{v} A^n \xrightarrow{f} A$ corresponds to the result of applying f "coordinatewise" to that n-tuple in the familiar way. This is how one shows that Boolean powers of an algebra A are isomorphic to certain subdirect powers of A which Foster defined in [14] and called "normal" subdirect powers.

There is one fact which has apparently never been pointed out before that can be derived from (4.4) together with (4.3) and (4.2), namely, that the Boolean power functors A[-], where A is finite, have finitary equational structure, i.e., they are incapable of carrying nontrivial infinitary operations, so that in fact every nontrivial operation which A[-] admits is of the kind described by the formula in (4.4).

The set-of-continuous-functions version of A[-], featured in [3], is recoverable from our representable functor version quite easily. Let X be the Stone space of B; then B is the subalgebra of clopen sets in 2^X (identifying 2^X with the power set algebra of X).

Informally, we have a chain of correspondences

$$
\begin{aligned}
A[B] &\leftrightarrow \text{homomorphisms } 2^A \longrightarrow B \\
&\leftrightarrow \text{homomorphisms } 2^A \longrightarrow 2^X \text{ which factor} \\
&\qquad \text{through } B \hookrightarrow 2^X \\
&\leftrightarrow \text{functions } X \xrightarrow{f} A \text{ such that } f^*(\{a\}) \in B \\
&\qquad \text{for all } a \in A \\
&\leftrightarrow \text{continuous functions from } X \text{ to the} \\
&\qquad \text{discrete space } A.
\end{aligned}
$$

We shall refer to any functor $\mathrm{Mod(BA)} \longrightarrow \mathrm{Mod(T)}$, where T is equational, as a <u>representable Boolean power functor</u> if it is represented by a finite Boolean algebra with more than two elements. The next result is really just a restatement of (4.1) in more impressive language.

<u>4.5.</u> <u>Theorem.</u> For any equational theory T, the algebraic functors $\mathrm{Mod(BA)} \longrightarrow \mathrm{Mod(T)}$ of finite rank are precisely the representable Boolean power functors.

The definition of the functor $A[B]$ given above depends upon the finiteness of the set A to guarantee that the join in condition ii) of the definition is an operation in the Boolean algebra B, rather than just an order-theoretic supremum, so that partitions will be preserved by composition with homomorphisms. For infinite A, there are two versions of the Boolean power $A[B]$ to choose from. One version, which remains faithful to the set-of-continuous-functions approach, is called a bounded Boolean power and uses only partitions $u: A \longrightarrow B$ for which $u(a)$ is nonzero for only finitely many $a \in A$; bounded Boolean powers are discussed in Chapter 5. The other way of defining $A[B]$ when A is infinite is to require that B be a complete Boolean algebra, so that the definition of $A[B]$ given above still works, with the join in condition ii) being an infinitary one. This approach has a serious disadvantage, from a category-theoretic point of view: the category of complete Boolean algebras with complete homomorphisms (needed to preserve partitions) is not an equational category and even lacks some desirable features such as coproducts of infinite families of algebras. Rather than struggle with functors defined on such an unpleasant domain category, we may restrict the domain of $A[-]$ to an equational category of Boolean algebras which are sufficiently complete to have A-indexed joins. This approach is not unknown in the literature (see Karatay [26], for example), but is not common, since many investigators like to treat the Boolean power construction as a bifunctor, with both the set A and the Boolean algebra B being variable; the Boolean algebras are required to be complete in order to accomodate arbitrarily large sets A. It seems likely that a serious functor-theoretic investigation of Boolean powers as bifunctors would best proceed by restricting to sets of bounded cardinality (a well-behaved category) and to Boolean algebras belonging to some equational category of infinitary Boolean algebras as defined below.

Let k be an infinite regular cardinal. A finitary Boolean algebra B is <u>k-complete</u> if every family of fewer than k elements has a supremum and an infimum in B relative to the customary partial order. A <u>k-complete Boolean homomorphism</u> is a Boolean homomorphism which preserves the suprema and infima of families of fewer than k elements. We deviate from Sikorski's usage in [40] to avoid having to refer continually to algebras which are "m-complete for all $m < k$." It follows from Sikorski [40], p. 131 ff., that the category k-Bool of all k-complete Boolean algebras with k-complete homomorphisms has a representable underlying-set functor U with a left adjoint F, i.e., a free algebra functor.

<u>4.6.</u> <u>Lemma</u>. The category k-Bool is monadic over Set, i.e., there is an underlying-set-preserving equivalence of categories k-Bool $\longrightarrow$ Mod(k-BA), where k-BA is the equational structure of the underlying-set functor $U: \text{k-Bool} \longrightarrow \text{Set}$.

Proof. We shall use (2.1) to prove the theorem. It follows from Sikorski [40], p. 131 ff. that U has a left adjoint (the free algebra functor). According to R. Lagrange [28], a k-complete homomorphism $B \xrightarrow{f} C$ is an epimorphism in k-Bool if and only if $U(f)$ is surjective. In particular, if f is a regular epimorphism, then it is an epimorphism, so $U(f)$ is surjective. Thus, U preserves regular epimorphisms. To show that U reflects regular epimorphisms, suppose that $U(f)$ is surjective. Form the kernel congruence $D \subseteq U(B) \times U(B)$ of $U(f)$ in Set, i.e., $D = \{(a, b) : f(a) = f(b)\}$. Then the two projections $D \longrightarrow U(B)$ are a kernel pair for $U(f)$, and $U(f)$ is a coequalizer for them in Set. By routine calculations, it can be verified that D is a k-complete subalgebra of $B \times B$, that the kernel pair in Set lifts into k-Bool as a kernel pair of f, and that f coequalizes its kernel pair. Since U is representable (by the free algebra $F(1)$), it preserves congruence relations. To show that U reflects congruence relations, let $C \hookrightarrow B \times B$ be a k-complete subalgebra such that $U(C)$ is an equivalence relation on $U(B)$. Then by routine calculations check that $U(B)/U(C)$ is a k-complete Boolean algebra with the obvious operations, and that the projections $C \longrightarrow B$ are a kernel pair for the projection $B \longrightarrow B/C$. There are no surprises in any of these computations, because the fact that the filter $f^*(\{1\})$ for any k-complete homomorphism f is a k-filter and that the quotient of any k-complete Boolean algebra by a k-filter is a k-complete Boolean algebra ([40], §21) guarantees that everything works as it should.

Lemma (4.6) should be regarded as a "folklore" result; see Manes [34] for related examples.

It is intuitively helpful to have a convenient presentation of the equational theory k-BA of k-complete Boolean algebras. As one would hope, it suffices to distinguish the constants 0 and 1, the complement operation $\neg$, and the m-ary join operations, for every $m < k$. That this works follows from the fact that a Boolean algebra is k-complete if and only if it admits these operations, and that a Boolean homomorphism is k-complete if and only if it preserves these operations. From now on, a k-complete Boolean algebra is identified with a k-BA-algebra in Set, i.e., a Boolean algebra with infinitary operations which pick out the suprema and infima of families of fewer than k elements.

Note that there are obvious mappings of theories $BA \longrightarrow k\text{-}BA \longrightarrow T_2$, where T_2 is the equational structure of a two-element set, otherwise known as the equational theory of complete atomic Boolean algebras. Define an equational theory T to be an <u>equational theory of k-complete Boolean algebras</u> if T is a quotient theory of k-BA, i.e., if there is a full mapping of theories $k\text{-}BA \longrightarrow T$. The corresponding equational category Mod(T) will be called an <u>equational category of k-complete Boolean algebras</u>. Note that, if T is a quotient of k-BA, then Mod(T) is a full subcategory of Mod(k-BA) which is closed under **H**, **S**, and **P**.

Intuitively, an equational theory of k-complete Boolean algebras is the result of adding distributivity axioms to k-BA.

For the rest of this chapter, let BA* be a fixed equational theory of k-complete Boolean algebras.

4.7. Lemma. If A is a set of cardinality less than k, but greater than 1, then 2^A and all of its copowers in Mod(BA*) are regular progenerators in Mod(BA*).

Proof. Every nontrivial copower of a regular-projective object is regular-projective, and every BA*-algebra with more than two elements has $F_{BA*}(1) \cong 2^{2}$ as a retract, so to prove the lemma it is sufficient to show that 2^A itself is regular-projective. Let $B \xrightarrow{f} C$ be a regular epimorphism, and let $2^A \xrightarrow{h} C$ be any homomorphism. We shall construct a homomorphism $2^A \xrightarrow{g} B$ such that $f.g = h$. Since $|A| < k$, the BA*-algebra 2^A is generated by its atoms, so it suffices to describe what g does to the atoms.

Let $(a_i : i \leqslant m)$ be a well-ordering of A, and let $g(\{a_0\})$ be any element b_0 of B such that $f(b_0) = h(\{a_0\})$; b_0 exists, because $U_{BA*}(f)$ is surjective. (Note that we are identifying the elements of 2^A with the subsets of A). If $0 < i < m$, and $g(\{a_j\})$ is defined for each $j < i$ so that $f(g(\{a_j\})) = h(\{a_j\})$ and so that for all $j' < j$ we have $g(\{a_j\}) \wedge g(\{a_{j'}\}) = 0$, then let $g(\{a_i\}) = b_i - \bigvee_{j<i} g(\{a_j\})$, where b_i is an element of B such that $f(b_i) = h(\{a_i\})$. The foregoing suffices to define $g(\{a_i\})$ for all $i < m$. Finally, let $g(\{a_m\}) = \neg \bigvee_{i<m} g(\{a_i\})$.

Lemma (4.7) provides an important example of how an equational category of k-complete Boolean algebras, for uncountable k, differs from the category of finitary Boolean algebras. In the latter category, no infinite power of 2 is regular-projective. Indeed, no infinite complete finitary Boolean algebra is even embeddable in a free finitary Boolean algebra (see Sikorski [40], p. 67).

4.8. Lemma. When $|A^n| < k$, the BA*-algebra 2^{A^n} is a retract of the n-th copower $n \otimes 2^A$ of 2^A in Mod(BA*). If n is finite, then the retraction is an isomorphism.

Proof. Each projection $A^n \xrightarrow{p} A$ induces a (complete) homomorphism $2^A \xrightarrow{p^*} 2^{A^n}$, which we shall call a coprojection; the n coprojections induce a coproduct homomorphism $n \otimes 2^A \xrightarrow{r_n} 2^{A^n}$. By hypothesis, $|A^n| < k$, so 2^{A^n} is generated by its atoms and is a regular progenerator in Mod(BA*), by (4.7). For each $i < n$, the i-th coprojection $2^A \longrightarrow 2^{A^n}$ sends $\{c\}$ to $\{a \in A^n : a_i = c\}$. For any given $b \in A^n$, we have $\{b\} = \bigcap_{i<n} \{a \in A^n : a_i = b_i\}$, so clearly each atom $\{b\}$ of 2^{A^n} is in the image of r_n, but then r_n must be surjective, hence it is a retraction since 2^{A^n} is regular-projective.

If n is finite, then $n \otimes 2^A$ is atomic, and r_n is bijective on atoms. To see this, let $(s_i : i < n)$ be the coproduct injections $2^A \longrightarrow n \otimes 2^A$. Each homomorphism $n \otimes 2^A \longrightarrow 2$ is induced by an n-sequence of homomorphisms $2^A \longrightarrow 2$, each of which sends a single atom of 2^A to 1. If the i-th homomorphism in the sequence sends $\{a_i\}$ to 1, then the induced homomorphism $n \otimes 2^A \longrightarrow 2$ sends $\bigcap_{i<n} s_i(\{a_i\})$ to 1. On the other hand, any homomorphism $n \otimes 2^A \xrightarrow{h} B$ which sends $\bigcap_{i<n} s_i(\{a_i\})$ to 1 must be such that $[h.s_i](\{a_i\}) = 1$ for all $i < n$. This means that $h.s_i$ factors through $2 \hookrightarrow B$, for all $i < n$, so h does too. But then it follows that $\bigcap_{i<n} s_i(\{a_i\})$ is an atom in $n \otimes 2^A$. It is easy to see that the homomorphism r_n defined above establishes a bijective correspondence between the atoms of $n \otimes 2^A$ and of 2^{A^n}; if we show that $n \otimes 2^A$ is complete and atomic, then it follows that r_n is an isomorphism. (Here, a complete BA*-algebra is one which has a supremum and an infimum for every family of elements; obviously, the BA-reduct of a complete atomic BA*-algebra is complete and atomic in the ordinary sense, hence is isomorphic to a power of 2).

First, note that $n \otimes 2^A$ is complete because it is a k-complete Boolean algebra which is generated by fewer than k elements.

It is atomic because the join of its atoms is 1 (see [40], p. 59).

We prove this by induction on n, using the fact that $(n \otimes 2^A) \sqcup 2^A$ is isomorphic to $(n+1) \otimes 2^A$. First, note that $0 \otimes 2^A \cong 2$ is atomic. Suppose that $n \otimes 2^A$ is atomic; to show that $(n+1) \otimes 2^A$ is atomic, we compute

$$\begin{aligned}
\bigvee_{a \in A^{n+1}} \bigwedge_{i<n+1} s_i(\{a_i\}) &= \bigvee_{b \in A} \bigvee_{a \in A^n} \left(s_n(\{b\}) \wedge \bigwedge_{i<n} s_i(\{a_i\}) \right) \\
&= \bigvee_{b \in A} \left(s_n(\{b\}) \wedge \bigvee_{a \in A^n} \bigwedge_{i<n} s_i(\{a_i\}) \right) \\
&= 1 \wedge \bigvee_{a \in A^n} \bigwedge_{i<n} s_i(\{a_i\}) \\
&= 1 \wedge 1 = 1.
\end{aligned}$$

It should be noted that, if n is infinite, r_n is not generally an isomorphism. For example, if $k > 2^{\omega}$, the free k-complete Boolean algebra $\omega \otimes 2^2$ is not isomorphic to $2^{2^{\omega}}$, by [40], Proposition 31.3. If the equational theory BA^* imposes sufficient infinitary distributivity conditions on its algebras, then for some infinite values of n the algebra $n \otimes 2^A$ is atomic, and r_n is an isomorphism (see [40], Proposition 24.5).

Let A be a set with $1 < |A| < k$. The set-valued Boolean power functor $A[-]$: $\mathrm{Mod}(BA^*) \longrightarrow \mathrm{Set}$ is defined as follows.

i) For each BA^*-algebra B, $A[B]$ is the set of all A-indexed partitions of unity in B;

ii) For each homomorphism $B \xrightarrow{h} C$, $A[h]$ is the function $A[B] \longrightarrow A[C]$ which sends each partition u in $A[B]$ to $h.u$ in $A[C]$.

<u>4.9.</u> <u>Lemma</u>. The functor $A[-]$ is represented by the BA^*-algebra 2^A.

Proof. Identical to the proof of (4.3).

On the basis of (4.9), we shall refer to all algebraic functors of the form G: $\mathrm{Mod}(BA^*) \longrightarrow \mathrm{Mod}(T)$, where $U_T.G \cong \underline{2}^A$ for some A, as <u>representable Boolean power functors</u>.

At this point we can begin to appreciate an important difference between the infinitary Boolean powers $A[-]: \mathrm{Mod}(BA^*) \longrightarrow \mathrm{Set}$ which we are now discussing and the finitary ones $A[-]: \mathrm{Mod}(BA) \longrightarrow \mathrm{Set}$. In the finitary case, taking $A = 2$, the Boolean power functor $2[-]$ is represented by the finitary free Boolean algebra $2^2 \cong F_{BA}(1)$, i.e., $2[-]$ is isomorphic to the underlying-set functor U_{BA}, and its equational structure is just the finitary theory BA, which is the finitary part of the equational structure of a two-element set. In the infinitary case, it is similarly true that $2[-]: \mathrm{Mod}(BA^*) \longrightarrow \mathrm{Set}$ is represented by the BA^*-algebra $2^2 \cong F_{BA^*}(1)$, so that $2[-]$ is isomorphic to the underlying-set functor U_{BA^*} and has equational structure BA^*. When BA^* is an equational theory of k-complete Boolean algebras, where $k \geq \omega^+$, it is not generally true that BA^* is isomorphic to a theory of operations on a two-element set, i.e., to a subtheory of T_2, since BA^* may lack necessary distributivity properties. The "image" of the mapping of theories $BA^* \longrightarrow T_2$ is the equational theory k-RBA of k-representable k-complete Boolean algebras; Mod(k-RBA) is identifiable with the full subcategory **HSP**({2}) of Mod(BA*). Thus, we cannot count on being able to extrapolate from operations on the set 2 to describe the equational structure of 2[-], and yet it is evident that the infinitary operations admitted by 2[-] are an important feature of the functor and should not be ignored.

Such considerations indicate that the infinitary Boolean power functors A[-] will generally have infinitary equational structure which is somehow not fully expressible in terms of operations on the set A; the Foster formula (4.4) cannot be relied upon unless we resort to draconian measures: either throw away the infinitary structure of the functors, which so far has been the usual practice in studies of Boolean powers, or drastically restrict the domain of the functors. Karatay [26] does both in order to describe

$A[-]$ as a "normal subdirect power" construction.

In the remainder of this Chapter we shall show how, for any equational theory BA^* of k-complete Boolean algebras, the representable Boolean power functors with domain $\mathrm{Mod}(BA^*)$ can be analyzed by means of the results of Chapter 3.

The functor $A[-]$ is sometimes called a Boolean *extension* functor because each Boolean power $A[B]$ contains a copy of the set A, namely $A[2]$. An element a of A is represented in $A[B]$ by the unique homomorphism $2^A \longrightarrow B$ which sends $\{a\}$ to 1. Evidently, each natural transformation $A[-]^n \xrightarrow{f} A[-]$ induces a function $A^n \xrightarrow{f'} A$ corresponding to the 2-component $f_2\colon A[2]^n \longrightarrow A[2]$ of f. The assignment of f' to f defines a mapping of theories $P \longrightarrow T_A$, where P is the equational structure of $A[-]$, i.e., the equational costructure of 2^A in $\mathrm{Mod}(BA^*)$, while T_A is the equational structure of the set A. In the case where $|A| = 2$, this gives us the standard mapping $BA^* \longrightarrow T_2$.

<u>4.10.</u> <u>Lemma</u>. A function $A^n \xrightarrow{g} A$ is induced on A by a P-operation $A[-]^n \xrightarrow{f} A[-]$ if and only if, for each $c \in A$, $g^*(\{c\})$ belongs to the BA^*-subalgebra of 2^{A^n} which is generated by the family of all subsets of A^n of the form $\{a \in A : a_i = c\}$, $i < n$.

Proof. The BA^*-subalgebra of 2^{A^n} described in the lemma is the image of the homomorphism r_n defined in the proof of (4.8); for sufficiently large n, the algebra 2^{A^n} is not generated by its atoms, and then r_n fails to be surjective.

Suppose that g is induced by $A[-]^n \xrightarrow{f} A[-]$, i.e., g is obtained in the following way. Let $2^A \xrightarrow{\hat{f}} n \otimes 2^A$ be the co-operation representing f; for each $a \in A^n$, let $n \otimes 2^A \xrightarrow{h_a} 2$ be the homomorphism which sends each of the elements $s_i(\{a_i\})$ to 1 (if n is too large, the atom $\bigwedge_{i<n} s_i(\{a_i\})$ might not exist, since BA^* might not have an n-ary meet operation). Then $g(a)$ is that element c of A such that $h_a . \hat{f}$ sends $\{c\}$ to 1.

For each $a \in A^n$, let p_a be the a-th projection $2^{A^n} \longrightarrow 2$, which sends $\{a\}$ to 1. It is easy to see that g is induced by f as explained above if and only if, for each $a \in A^n$, we have $p_a . g^* = h_a . \hat{f}$. Furthermore, we have $p_a . r_n = h_a$ for all $a \in A^n$; this can be checked by verifying that $p_a . r_n$ sends $s_i(\{a_i\})$ to 1, for each $i < n$.

We claim that g is induced by f if and only if $g^* = r_n . \hat{f}$. This claim implies the lemma.

If g is induced by f, then for each $a \in A^n$ we have $p_a . r_n . \hat{f} = h_a . \hat{f} = p_a . g^*$ as noted above, so it follows that $r_n . \hat{f} = g^*$. On the other hand, if $r_n . \hat{f} = g^*$, then we have $p_a . g^* = p_a . r_n . \hat{f} = h_a . \hat{f}$, so g is induced by f. This completes the proof of the lemma.

In the last proof we were essentially finding the image of the free-generator-preserving mapping $A[r_n]$ from the free P-algebra

$A[n \otimes 2^A]$ to the P-algebra $A[2^{A^n}] \cong A^{A^n}$, the latter being free when viewed as a T_A-algebra. The image of $A[r_n]$ is a free algebra of rank n in the full subcategory **HSP**$(\{A[2]\})$ of Mod(P).

The fact that the infinitary P-co-operations on the BA*-algebra 2^A are not generally inverse-image mappings between powers of 2 in Mod(BA*) means that the Foster formula of (4.4), which fully explains the equational structure of a finitary Boolean power functor, does not work for the kind of Boolean powers which we are presently discussing, except for isolated cases as described in the next result.

<u>4.11</u>. <u>Lemma</u>. Let f be an n-ary operation in the equational structure P of $A[-]$, where $n < k$. For any BA*-algebra B which is isomorphic to an m-complete field of sets, where $n < m < k$, the action of $f_B : A[B]^n \longrightarrow A[B]$ is described as follows. For all $u = (u_j : j < n)$ in $A[B]^n$ and all $c \in A$,

$$[f_B(u)](\{c\}) = \mathrm{Sup}\left\{ \bigwedge_{j<n} u_j(\{a_j\}) \; : \; f'(a) = c \right\},$$

where the supremum may be computed by an infinitary join operation of BA* if $|A^n| < k$.

Proof. Identify B with an m-complete field of subsets of some set I. Also let the coproduct injections $2^A \longrightarrow n \otimes 2^A$ be $(s_j : j < n)$. Then for each $j < n$ we have $u_j = u.s_j$, if the n-sequence $u \in A[B]^n$ is identified with a homomorphism $n \otimes 2^A \longrightarrow B$ (note that $A[-]^n$ is represented by $n \otimes 2^A$). For each $i \in I$ and $j < n$, there is exactly one element $a_{ij} \in A$ such that $i \in u_j(\{a_{ij}\})$. Then we have

$$i \in \bigcap_{j<n} u_j(\{a_{ij}\}) = u\left(\bigwedge_{j<n} s_j(\{a_{ij}\})\right),$$

for all $i \in I$, so it follows that

$$\mathrm{Sup}\left\{u\left(\bigwedge_{j<n} s_j(\{a_j\})\right) : a \in A^n\right\} = 1$$

in B. Now

$$[f_B(u)](\{c\}) = [u.\hat{f}](\{c\})$$

$$= \operatorname{Sup}\left\{[u.\hat{f}](\{c\}) \cap u\left(\bigwedge_{j<n} s_j(\{a_j\})\right) : a \in A^n\right\}$$

$$= \operatorname{Sup}\left\{u\left(\hat{f}(\{c\}) \wedge \bigwedge_{j<n} s_j(\{a_j\})\right) : a \in A^n\right\}.$$

Note that the meet of $\hat{f}(\{c\})$ with $\bigwedge_{j<n} s_j(\{a_j\})$ is nonzero if and only if $\hat{f}(\{c\}) \geqslant \bigwedge_{j<n} s_j(\{a_j\})$, since the latter is an atom in $n \otimes 2^A$. But for any $a \in A^n$, $\hat{f}(\{c\})$ contains the atom $\bigwedge_{j<n} s_j(\{a_j\})$ if and only if $h_a.\hat{f}$ sends $\{c\}$ to 1, i.e., if and only if $f'(a) = c$, where f' is the operation on A induced by f. Thus,

$$[f_B(u)](\{c\}) = \operatorname{Sup}\left\{u\left(\bigwedge_{j<n} s_j(\{a_j\})\right) : f'(a) = c\right\}$$

$$= \operatorname{Sup}\left\{\bigcap_{j<n} u_j(\{a_j\}) : f'(a) = c\right\},$$

and the proof is complete.

4.12. Corollary. Every finitary P-operation f is determined by its corresponding T_A-operation as shown in (4.11).

A. J. Foster's device in [14] of using a two-element "subframe" of the "kernel" algebra A to recover the "core" Boolean algebra B from A[B] was one of the original inspirations for the results in Chapter 3. Before any of those results had been discovered by the writer, Foster's method was adapted in Dukarm [9] for a brute-force proof that the infinitary representable Boolean power functors are monadic. In [9], the results which appear above as lemmas (4.7) through (4.12) were the basis for a description of P-operations which were strongly reminiscent of Post algebra operations. We now present a modified version of that description, showing that the P-algebras are actually generalized Post algebras as defined by Cat-Ho Nguyen [6]. This approach, which is an extension of Foster's method of analyzing Boolean powers, will then be contrasted with an analysis based on the results of Chapter 3.

Recall that BA* is an equational theory of k-complete Boolean algebras, A is a set with $1 < |A| < k$, and P is the equational structure of $\underline{2}^A = A[-]\colon \mathrm{Mod}(BA^*) \longrightarrow \mathrm{Set}$.

Let $(a_i : i \leqslant m)$ be a well-ordering of A, where m is an ordinal of the same cardinality as A. This ordering of A determines lattice operations $\wedge$ and $\vee$ on A, relative to which A is a complete linearly-ordered lattice with a least element a_0 and a greatest element a_m. This lattice admits a pseudocomplement operation $\neg$, where

$$\neg a_i = \begin{cases} a_m, & \text{if } i = 0 \\ a_0, & \text{if } i \neq 0. \end{cases}$$

A relative pseudocomplement operation $\rightarrow$ is defined by

$$a_i \rightarrow a_j = \begin{cases} a_m, & \text{if } i \leqslant j \\ a_j, & \text{if } i > j. \end{cases}$$

For each *successor* ordinal $i < m$, define a unary operation D_i by

$$D_i(a_j) = \begin{cases} a_m, & \text{if } i \leqslant j \\ a_0, & \text{if } i > j. \end{cases}$$

By (4.12), the operations defined above correspond to uniquely determined P-operations which we shall denote by the same symbols as above. Each element a_i of A determines a constant ((1, 0)-ary) P-operation e_i; in particular, we write e_0 as 0 and e_m as 1. It is easy to see from (4.8) and (4.10) that the finitary part of P is actually isomorphic to the finitary part of T_A (see [9]). It follows that each P-algebra is, relative to the finitary operations defined above, a relatively pseudocomplemented bounded distributive lattice, i.e., a Heyting algebra, in which there is a complete linearly-ordered sublattice of constants, which we shall identify with A, and some extra unary operations D_i. Note that the chain of constants A is actually a copy of the initial P-algebra $F_P(0)$. By examining the corresponding operations on A, we see that the finitary operations cited above satisfy the same kinds of identities as their counterparts in a finitary equational theory $P_{m'}$ of m'-valued Post algebras.

In order for the P-algebras to really look like Post algebras, there must be a join operation so that

$$x = \bigvee_{i<m} (D_{i+1}(x) \wedge e_{i+1})$$

is an identity of P (expressed in the extended canonical language). If A is finite, the join is finitary and easily found. If A is infinite, we would like to be able to "lift" the m-ary join from the complete lattice A and use it as a P-operation, as we did with the finitary "Post algebra" operations. This can be done, since the m-ary join in A satisfies the condition of (4.10). To see this, note that, for any $i \leqslant m$, we have

$$\{b \in A^m : \bigvee b > a_i\} = \bigcup_{j<m} \bigcup_{i<n} \{b \in A^m : b_j = a_n\}$$

belonging to the image of r_m, and we can write

$$\{b \in A^m : \bigvee b = a_i\} = \bigcap_{j<i} \{b \in A^m : \bigvee b > a_j\} - \{b \in A^m : \bigvee b > a_i\}.$$

Thus, the m-ary join on A is induced by an m-ary P-operation $\sum$. There is no guarantee that $\sum$ is a true join operation in P, if

$\bar{m}$ is infinite, but $\sum$ has all the same finitary properties as a true join operation: the result of restricting $\sum$ to finitely many distinct arguments is a finitary join operation. Most importantly, we do have the identity

$$x = \sum_{i<m} (D_{i+1}(x) \wedge e_{i+1}),$$

since the *unary* composite P-operation on the right side of the equation induces the operation $\bigvee_{i<m} (D_{i+1}(x) \wedge e_{i+1})$ on A, and it is easy to see that the latter operation is the identity operation on A.

In a study of many-valued infinitary propositional logics [6], Cat-Ho Nguyen defined and investigated a class of generalized Post algebras. As we shall see, our P-algebras are generalized Post algebras of the kind which Cat-Ho Nguyen studied, and in fact the P-algebras correspond to the Lindenbaum-Tarski algebras of the propositional logics which Cat-Ho Nguyen worked with. The following is a version of his characterization ([6], Theorem 1.6) of his generalized Post algebras;

we have specialized it to the case where the constants form a well-ordered chain, and we are using the result as a definition, since the original definition given in [6] is much too broad for our purposes.

Let m be a nonzero ordinal. A <u>generalized Post algebra of type m</u> is a universal algebra C such that:

i) C is a Heyting algebra relative to operations $0, 1, \neg,$ $\wedge, \vee,$ and $\rightarrow$;

ii) The constants in C form a chain $(e_i : i \leq m)$ of order type $m+1$, where $e_0 = 0$ and $e_m = 1$, and where each family of constants has a supremum in C which is a constant;

iii) There is a family $(D_{i+1} : i < m)$ of (1, 1)-ary operations such that, for every element c of C, we have $c = \operatorname{Sup}_{i<m} \{D_{i+1}(c) \wedge e_{i+1}\}$;

iv) The following identities hold in C:

$$D_i(x \wedge y) = D_i(x) \wedge D_i(y)$$

$$D_i(x \vee y) = D_i(x) \vee D_i(y)$$

$$D_i(x) \vee \neg D_i(x) = 1$$

$$D_i(D_j(x)) = D_j(x)$$

$$D_i(e_j) = \begin{cases} 1, & \text{if } i \leqslant j \\ 0, & \text{if } i > j. \end{cases}$$

4.13. Theorem. Let BA^* be an equational theory of k-complete Boolean algebras, and let P be the equational structure of the Boolean power functor $A[-]$, where A is a set with $1 < |A| < k$. Then for every ordinal m with $|m+1| = |A|$ there is a presentation of P relative to which every P-algebra is a generalized Post algebra of type m.

Proof. It is clear from the foregoing discussion that we need only verify the parts of conditions ii) and iii) of (4.13) pertaining to suprema. The pseudojoin $\sum$ is constructed so that it induces a true join operation on the subalgebra A of constants. Then in A we have the identity $y \wedge \sum_{j<m} e_{i_j} = \sum_{j<m} (y \wedge e_{i_j})$, which has only one free variable and thus also holds in P; this identity says

that $\sum_{j<m} e_{i_j}$ is less than or equal to each upper bound of $\{e_{i_j} : j<m\}$. It follows that condition ii) is satisfied in every P-algebra, since $\sum_{j<m} e_{i_j}$ is an upper bound of $\{e_{i_j} : j<m\}$ and belongs to the algebra of constants.

A similar trick works for condition iii). The following finitary equations hold in A, and therefore are identities of P.

$$(D_{j+1}(x)\wedge e_{j+1}) \wedge \sum_{i<m} (D_{i+1}(x)\wedge e_{i+1}) = D_{j+1}(x)\wedge e_{j+1}$$

$$y \wedge \sum_{i<m} (D_{i+1}(x) \wedge e_{i+1}) = \sum_{i<m} (y \wedge D_{i+1}(x) \wedge e_{i+1})$$

The equations of the first kind collectively assert that $\sum_{i<m} (D_{i+1}(x)\wedge e_{i+1})$ is an upper bound for the set $\{D_{i+1}(x)\wedge e_{i+1} : i<m\}$, while the equation in the second line says that it is a *least* upper bound. This concludes the proof of the theorem.

A slightly more laborious version of the method of the proof above can be used to show that the pseudojoin operation $\sum$ induces a true join operation on the Boolean elements of any P-algebra; the same can be shown by diagram-chasing in Mod(BA*); however, this approach to describing P is rather inefficient for treating in any detail the infinitary properties of the infinitary P-operations. In particular, it seems to be difficult to show that the P-algebras can be presented as generalized Post algebras with k-complete lattice operations.

Using the results of Chapter 3, we can find a familiar-looking presentation of the P-algebras as generalized Post algebras. Taking a cue from (3.8), we seek a spanning equational interpretation of P in BA*; such an interpretation is provided by the retraction $F_{BA^*}(m) \xrightarrow{r} 2^A$ and coretraction $2^A \xrightarrow{s} F_{BA^*}(m)$ given as follows.

For each $i < m$, the retraction r sends the i-th free generator x_i of $F_{BA^*}(m)$ to the set $A - \{a_j : j \leqslant i\}$ (viewed as an element of the BA*-algebra 2^A), while the coretraction s sends that set to $\bigwedge_{j \leqslant i} x_j$ in $F_{BA^*}(m)$. Here, we are assuming that A is well-ordered as $(a_i : i \leqslant m)$. The idempotent (m, m)-ary BA*-operation $\underline{r.s} = u: U_{BA^*}{}^m \longrightarrow U_{BA^*}{}^m$ determines a spanning equational interpretation $p: P \longrightarrow BA^*$ analogous to the "nonincreasing chains" interpretation of the finitary theory of Post algebras in BA which was given as an example in Chapter 3. Because p is a spanning interpretation, the P-operations are the BA*-operations which are u-operations, i.e., the ones which preserve nonincreasing m-indexed chains in every BA*-algebra.

The presentation of BA* in terms of 0, 1, $\neg$, $\rightarrow$, and the n-ary join and meet operations, for all $n < k$, thus provides a presentation of P in terms of those basic Boolean operations together with u. In particular, the finitary and infinitary lattice operations of P are given by m-sequences of copies of the corresponding operations of BA*; for example, the binary join operation of P is interpreted in BA* as an m-sequence of binary joins; in symbols,

$$(x_i : i < m) \vee (y_i : i < m) = (x_i \vee y_i : i < m).$$

Thus, it is evident that the "nonincreasing chain" spanning interpretation of P in BA^* yields a presentation of P as an equational theory of generalized Post algebras which are k-complete as lattices.

The interpretations of the other "Post operations" of P in BA^* are as follows. For each $i \leqslant m$, the constant e_i is the chain $(c_j : j < m)$ in which $c_j = 1$ for all $j < i$, and $c_j = 0$ for all $j \geqslant i$. For each $i < m$, the operation D_{i+1} is given by

$$D_{i+1}(x_j : j < m) = (x_i, x_i, \ldots, x_i, \ldots).$$

The pseudocomplement is

$$\neg(x_j : j < m) = (\neg x_0, \neg x_0, \ldots, \neg x_0, \ldots),$$

while the relative pseudocomplement is

$$(x_j : j < m) \rightarrow (y_j : j < m) = \left(\bigwedge_{j \leqslant i} (x_j \rightarrow y_j) : i < m\right).$$

The possibility of representing generalized Post algebras as lattices of nonincreasing chains of elements in Boolean algebras is discussed in [6]; the representation mentioned there is essentially the one we have obtained by means of the spanning equational interpretation $p: P \longrightarrow BA^*$, except that there is an error in [6] which we have corrected here.

For any equational theory T, the action of a representable Boolean power functor $A[-]: \mathrm{Mod}(BA^*) \longrightarrow \mathrm{Mod}(T)$ can be described conveniently in terms of the spanning interpretation $p: P \longrightarrow BA^*$ defined above, since by (3.14) we have $A[-] \cong (p.t)^*$, where $t: T \longrightarrow P$ is an ordinary mapping of theories. To compute $A[B]$, just construct the Post algebra of nonincreasing m-indexed chains in B and forget some of the Post operations, as directed by t.

To conclude this Chapter, we note that the infinitary counterpart to Hu's result (1.3) would be to find all of the regular progenerators in $\mathrm{Mod}(BA^*)$ which are generated by fewer than k

elements. This problem seems to be a rather difficult one, since for infinitary equational theories BA* of k-complete Boolean algebras it is possible for Mod(BA*) to contain regular progenerators, generated by fewer than k elements, which are not powers of 2; for example, some free algebras of rank $n < k$ will usually fail to be powers of 2. In many cases, the precise solution to the problem seems to depend upon properties of the category Set corresponding to the Generalized Continuum Hypothesis, the existence or nonexistence of various kinds of exotic large cardinals, and so on.

CHAPTER 5. LOCALLY EQUATIONAL CATEGORIES AND LOCALLY ALGEBRAIC FUNCTORS

The notion of a locally equational class of finitary universal algebras originated in the work of A. L. Foster [16] and was further developed in a series of investigations by T. K. Hu culminating in a 1973 paper [25] in which a characterization of locally equational classes is given in the style of Birkhoff's "**HSP**" characterization of equational classes (see [19] for an exposition of Birkhoff's result). In this Chapter we provide necessary and sufficient conditions for a category to be equivalent to some locally equational category (i.e., to a full subcategory of a finitary equational category, where the objects of the subcategory form a locally equational class). Our category-theoretic characterization of locally equational categories is analogous to Linton's result (2.1) for equational categories, but the proof of our result is very different from the proof of (2.1) sketched in Linton's paper [30]. The characterization of locally equational categories leads naturally to the notion of a locally algebraic functor, and we show that bounded Boolean powers can be regarded as locally algebraic functors. Finally, we sketch a new proof of Hu's result (1.2) characterizing Mod(BA) among all locally equational categories. The characterization theorem (5.3) and the claim that bounded Boolean powers are "locally monadic" appear in Dukarm [11]. Definitions and results cited from Hu [25] are adapted to our category-theoretic frame of reference.

Let T be a finitary equational theory, let A be an algebra belonging to Mod(T), and let X be a subset of A. For any n, we say that a (1, n)-ary equation $f(x) = g(x)$ in the language L_T is an <u>identity of X</u> provided that X^n is a subset of $[f(x) = g(x)]_{A,x}$.

For each n, the set of all $(1, n)$-ary equations which are identities of X is called $\mathrm{Id}_n(X)$.

The locally equational closure of a class K of algebras belonging to $\mathrm{Mod}(T)$ is the full subcategory $L(K)$ of $\mathrm{Mod}(T)$ whose objects are all the algebras A having the following property. For each finite subset X of A, there are a finite sequence $B_1, \ldots, B_n$ of algebras in K and a finite subset Y of the algebra $\prod_i B_i$ such that $\mathrm{Id}_\omega(Y) \subseteq \mathrm{Id}_\omega(X)$. A full subcategory K of $\mathrm{Mod}(T)$ is said to be a locally equational category if K is the locally equational closure of the class of all algebras which are objects of K. These definitions are adapted from Hu [25].

As evidence that the concept of a locally equational category is of algebraic interest, we cite a few examples. Hu points out in [25] that, if T is finitary, the category $\mathrm{Mod}(T)$ itself is locally equational, as are its full subcategories of locally finite algebras or locally simple algebras. It should be interesting to logicians that, for every infinite k, the category of all locally finite-dimensional cylindric algebras of dimension k (see Henkin, Monk, and Tarski [20], p. 231) is locally equational, but not equational; these cylindric algebras are the Lindenbaum-Tarski algebras of first-order theories.

5.1. <u>Theorem</u> (T. K. Hu [25]). Let T be a finitary equational theory, and let M be a full subcategory of $\mathrm{Mod}(T)$. Then M is locally equational if and only if M is closed under the formation of directed unions, homomorphic images, subalgebras, and finite products.

In his paper [25], Hu points out that certain algebras, called homogeneously generated algebras, play an important role in locally equational categories which is similar in some respects to the role of the free algebras in an equational category. Let T be a finitary equational theory, and let X be a subset of an algebra A

which belongs to Mod(T). Then X is homogeneous if it has the following property: for each finitary equation $f(x) = g(x)$ in L_T, if $X^n \cap [f(x) = g(x)]_{A,x}$ is nonempty, then $f(x) = g(x)$ is an identity of X. The algebra A is said to be homogeneously generated of rank n if there is an n-element homogeneous subset of A which generates A.

5.2. Theorem (T. K. Hu [25]). Let X be a set of generators for an algebra A in Mod(T). Then the following are equivalent:

i) X is homogeneous;

ii) Every function $X \longrightarrow X$ has a unique extension to an endomorphism of A;

iii) If Y is a subset of an algebra B which belongs to Mod(T), and $Id_n(X) \subseteq Id_n(Y)$ for every $n \leq |X|$, then every function $X \longrightarrow Y$ has a unique extension to a homomorphism $A \longrightarrow B$.

Let X be a subset of A, and let m be any cardinal. Then define $H_X(m)$ to be the subalgebra of A^{X^m} which is generated by the projections $X^m \longrightarrow A$. It is not difficult to verify that $H_X(m)$ is homogeneously generated by those projections and that, for any subset Y of an algebra B with $Id_\omega(Y) = Id_\omega(X)$, we have $H_X(m) \cong H_Y(m)$. Furthermore, if X is a homogeneous generating set for A, then $A \cong H_X(m)$, where m is the cardinal of X. The proof of the characterization theorem (5.3) will show in precisely what sense the homogeneously generated algebras in a locally equational category are analogous to the free algebras in an equational category.

An <u>inverse system</u> in a category M is a diagram of the form $D\colon I^\circ \longrightarrow M$, where I is a directed set; the inverse system D is <u>epimorphic</u> if, for each arrow f of I, the arrow $D(f)$ is an epimorphism. A limit of an inverse system is called an <u>inverse limit</u>. Let $D\colon I^\circ \longrightarrow M$ be an inverse system. Application of the Yoneda embedding $M^\circ \longrightarrow Set^M$ yields a directed system $\underline{D}\colon I \longrightarrow M^\circ \longrightarrow Set^M$

of representable functors in Set^{M}. If D is an epimorphic inverse system, then $\underline{D}$ is a monomorphic directed system, and we say that the functor $U = \mathrm{Colim}\,\underline{D}$ is <u>locally represented</u> by the epimorphic inverse system D. A functor $U: M \longrightarrow \mathrm{Set}$ is said to be <u>locally representable</u> if there is an epimorphic inverse system $D: I^{\circ} \longrightarrow M$ such that $U \cong \mathrm{Colim}\,\underline{D}$; by the Yoneda Lemma, it follows that U is locally representable if and only if U is a directed union of representable subfunctors.

If U is represented by an object A, i.e., $U \cong \underline{A}$, then the Yoneda Lemma says that the assignment $a \longmapsto \underline{a}$ defines a bijective correspondence between the elements of $U(A)$ and the natural transformations $U \longrightarrow U$. Correspondingly, it is not hard to see that there is a bijective correspondence between the "global elements" $1 \longrightarrow U.D$ (where $1: I^{\circ} \longrightarrow \mathrm{Set}$ has the constant value "one") and the natural transformations $U \longrightarrow U$ when $U: M \longrightarrow \mathrm{Set}$ is locally represented by D. We say that D is <u>coherent</u>, and that $U \cong \mathrm{Colim}\,\underline{D}$ is coherently locally represented by D, if the following condition is satisfied.

Coherence condition. For every $i \in I$ and every element a of $U(D_i)$, there is a global element $t\colon 1 \longrightarrow U.D$ such that $t_i\colon 1 \longrightarrow U(D_i)$ is the constant function which picks out the element a.

5.3. Theorem. A category M is equivalent to a locally equational category if and only if M is finitely complete, with directed unions and coequalizers of kernel pairs, and there is a functor $U\colon M \longrightarrow \mathrm{Set}$ such that:

i) For every finite n, the functor $U^n\colon M \longrightarrow \mathrm{Set}$ is coherently locally representable;

ii) U preserves and reflects congruence relations and regular epimorphisms;

iii) U preserves directed unions.

Proof. First we shall prove that M as specified above, with a functor $U: M \longrightarrow \mathrm{Set}$ satisfying i), ii), and iii), is equivalent to a locally equational category, To do this we show, by a series of lemmas, that the finitary comparison functor $E: M \longrightarrow \mathrm{Mod}(T)$, where T is the finitary part of the equational structure of U, is exact, full, and faithful; then, using Hu's result (5.1), we show that the closure of the image category $E(M)$ in $\mathrm{Mod}(T)$ under the formation of isomorphic copies is a locally equational category.

According to condition i), we may assume that for each finite n there is a coherent epimorphic inverse system $H(n): I(n)^{\circ} \longrightarrow M$ which locally represents U^n. Let $H(n)$ send each $i \longrightarrow j$ in $I(n)$ to $H_j(n) \xrightarrow{h_{ij}} H_i(n)$ in M. For each $i \in I(n)$, the colimit injection $\underline{H_i(n)} \hookrightarrow U^n$ is $\underline{h_i}$, where h_i belongs to $U^n(H_i(n))$, i.e., $h_i = (h_{i,0}, h_{i,1}, \ldots, h_{i,n-1})$ is an n-tuple of elements of $U(H_i(n))$.

5.4. Lemma. The functor U is faithful and preserves and reflects finite limits, monomorphisms, and isomorphisms.

Proof. Each of the representable functors $\underline{H_i(1)}$ preserves limits. Because directed unions in Set commute with finite limits, it follows that U preserves finite limits. Every functor preserves isomorphisms, and every functor which preserves finite limits also preserves monomorphisms, so U preserves isomorphisms and monomorphisms.

To show that U is faithful, let f and g be arrows $A \longrightarrow B$ in M with $U(f) = U(g)$, and let $C \xrightarrow{s} A$ be an equalizer of (f, g). Then s is a monomorphism, and $U(s)$ is an equalizer of $(U(f), U(g))$, since U preserves finite limits. But then $U(s)$ is an isomorphism, hence a regular epimorphism, since $U(f) = U(g)$. But U reflects regular epimorphisms, so s is a regular epimorphism. Then s is an isomorphism, so $f = g$; this proves that U is faithful.

Faithful functors reflect monomorphisms; since U reflects regular epimorphisms, it reflects monomorphisms which are regular epimorphisms, i.e., U reflects isomorphisms.

Now we shall show that U reflects limits of finite diagrams. Suppose that $D: J \longrightarrow M$ is a finite diagram in M, and that $U(B)$, with projections $U(f_j): U(B) \longrightarrow U(D_j)$, is a limit of $U.D$ in Set. Also suppose that A, with projections $d_j: A \longrightarrow D_j$, is a limit of D in M; since U preserves limits of finite diagrams, $U(A)$ with projections $U(d_j)$, $j \in J$, is a limit of $U.D$ in Set. Because U is faithful, and because the functions $U(f_j)$ constitute a cone in Set from $U(B)$ to $U.D$, it follows that the arrows f_j are a cone in M from B to D. Let $B \xrightarrow{f} A$ be the induced arrow; f is unique with the property that $d_j.f = f_j$ for all $j \in J$. Since $U(d_j).U(f) = U(f_j)$ for all $j \in J$, the arrow $U(f): U(B) \longrightarrow U(A)$ is a limit arrow in Set; but, since both $U(B)$ and $U(A)$, with their respective projections, are limits of $U.D$, the limit arrow $U(f)$ is an isomorphism. It follows that f is an isomorphism in M, which means that B is a limit of D in M, so we conclude that U reflects limits of finite diagrams. This concludes the proof of (5.4).

Recall that in any category, if (u, v) is a kernel pair of $A \xrightarrow{f} B$, then f is a regular epimorphism if and only if f is a coequalizer of (u, v). A <u>regular factorization</u> of an arrow h (see Grillet [4]) is a factorization of the form $h = s.f$, where f is a regular epimorphism and s is a monomorphism. The factorization $h = s.f$ is <u>unique</u> (up to canonical isomorphism) if, for every regular factorization $h = s'.f'$, there is an isomorphism g such that $f' = g.f$ and $s'.g = s$. If $h = s.f$ is a unique regular factorization of h, then the subobject $B \xhookrightarrow{s} C$ is an <u>image</u> of h, written $\mathrm{im}(h)$, and all images of h are equivalent as subobjects of $C = \mathrm{codom}(h)$.

<u>5.5.</u> <u>Lemma</u>. The category M has unique regular factorizations, and U preserves and reflects them.

Proof. Given $A \xrightarrow{f} B$ in M, let (u, v) be a kernel pair of f, and let r be a coequalizer of (u, v). Since f.u = f.v, there is a unique arrow s such that f = s.r. In Set we have U(f) = U(s).U(r), where U(r) is a coequalizer of a kernel pair (U(u), U(v)) of U(f). This guarantees that U(f) = U(s).U(r) is a regular factorization of U(f) in Set. In particular, U(s) is a monomorphism, so s is too, so f = s.r is a regular factorization of f in M. Preservation and reflection of regular factorizations by U is obvious, since U preserves and reflects both regular epimorphisms and monomorphisms. Uniqueness of regular factorizations in M is easily seen from the uniqueness of regular factorizations in Set, together with the preservation and reflection of regular factorizations and isomorphisms by U. This concludes the proof of (5.5).

5.6. Lemma. Let u and v be arrows $A \longrightarrow B$ in M, and let $B \xrightarrow{f} C$. Then f.u = f.v is exact if and only if U(f).U(u) = U(f).U(v) is exact.

Proof. This one is obvious, since by definition $f.u = f.v$ is exact if and only if (u, v) is a kernel pair of f and f is a regular epimorphism, but U preserves and reflects kernel pairs and regular epimorphisms.

The next result is a crucial one. It belongs to the genre of results known to category theorists as "fill-in" lemmas, and is adapted from Herrlich and Strecker [21], Proposition 32.7.

5.7. Lemma. Let r and g be arrows in M, with r being a regular epimorphism. If there is a function f such that $f.U(r) = U(g)$, then there is a unique arrow f' in M such that $U(f') = f$.

Proof. Let (u, v) be a kernel pair of r; then $r.u = r.v$ is exact. Since $U(g) = f.U(r)$, it follows that we have $U(g).U(u) = U(g).U(v)$, so by faithfulness of U we have $g.u = g.v$. But r is a coequalizer of (u, v), so there is a unique arrow f'

such that $f'.r = g$. Then $U(f').U(r) = U(g) = f.U(r)$, so $U(f') = f$, since $U(r)$ is an epimorphism. This completes the proof of (5.7).

5.8. Lemma. The functor U is finitely tractable, i.e., the full subcategory in Set^{M} of finite powers of U is locally small.

Proof. Let m and n be finite; there is a bijective correspondence between arrows $U^n \longrightarrow U^m$ and cones from $\underline{H(n)}$ to U^m, since $U^n = \mathrm{Colim}\,\underline{H(n)}$. Since, by the Yoneda Lemma, each arrow $\underline{H_i(n)} \longrightarrow U^m$ corresponds to a uniquely determined element of $U^m(H_i(n))$, it follows that each arrow $U^n \longrightarrow U^m$ can be matched with an element of the set $\prod_i U^m(H_i(n))$.

On the basis of (5.8) we know that the functor U has a finitary comparison functor $E: M \longrightarrow \mathrm{Mod}(T)$, where T is a finitary equational theory whose finitary operations are the arrows in a skeleton of the full subcategory in Set^{M} of all finite powers of U. This comparison functor is the one described in Lawvere's thesis [29], which dealt exclusively with finitary theories. In keeping with the viewpoint of our Chapter 3, we can say that U is a T-algebra in Set^M, and that for each object B in M the T-algebra $E(B)$ is the one whose underlying set is $U(B)$ and which inherits its T-operations from U in the obvious way. Lawvere's structure-semantics adjointness works in the setting of finitary theories, so (as in the infinitary case) $E\colon M \longrightarrow \mathrm{Mod}(T)$ has a universal property similar to that ascribed to infinitary comparison functors in Chapter 2. Using the preceding lemmas, together with $U = U_T.E$ and the fact that U_T has all the properties claimed for U, it is easy to prove the following result.

5.9. Lemma. The finitary comparison functor E for U is faithful and preserves and reflects finite limits, directed unions, monomorphisms, regular epimorphisms, isomorphisms, congruence relations, regular factorizations, and exact diagrams.

5.10. Lemma. The inverse limit of $E.H(n)$ is $F_T(n)$.

Proof. First, note that the inverse limit $\mathrm{Lim}\ E.H(n)$ in $\mathrm{Mod}(T)$ is constructed on the underlying-set level as the set of all "compatible families" of elements of the algebras $E(H_i(n))$, $i \in I(n)$, with the limit algebra $\mathrm{Lim}\ E.H(n)$ being a subalgebra of the product $\prod_i E(H_i(n))$. The compatible families of elements are identifiable with the global elements $1 \longrightarrow U.H(n)$, as noted in Mac Lane [32], p. 106.

To prove the lemma, we define a function p which maps elements of $F_T(n)$ into elements of $\mathrm{Lim}\ E.H(n)$; it is evident that p is bijective and preserves T-operations, hence may be regarded as an isomorphism in $\mathrm{Mod}(T)$. Note the following chain of bijective correspondences.

Elements $f(x_0, \ldots, x_{n-1})$ of $F_T(n)$

$\leftrightarrow$ T-operations $U^n \xrightarrow{f} U$

$\leftrightarrow$ cones $(f.h_i : i \in I(n))$ from $\underline{H(n)}$ to U

$\leftrightarrow$ compatible families $(f(h_{i,0}, \ldots, h_{i,n-1}) : i \in I(n))$

of elements of the algebras $E(H_i(n))$

$\leftrightarrow$ elements $f((h_{i,j} : i \in I(n)) : j < n)$ of $\mathrm{Lim}\, E.H(n)$.

Define $p(f(x_0, \ldots, x_{n-1}))$ to be $f((h_{i,j} : i \in I(n)) : j < n)$. Note that the composite of p with the i-th projection $\mathrm{Lim}\, E.H(n) \longrightarrow E(H_i(n))$ is a homomorphism p_i which takes the j-th free generator x_j to $h_{i,j}$, for all $j < n$. Relative to the family $(p_i : i \in I(n))$ of projections, $F_T(n)$ is an inverse limit of $E.H(n)$.

5.11. Lemma. For each $n < \omega$ and each $i \in I(n)$, the algebra $E(H_i(n))$ is generated by the set $\{h_{i,0}, \ldots, h_{i,n-1}\}$.

Proof. We shall show that the projection p_i is surjective. Let a be any element of $U_T(E(H_i(n))) = U(H_i(n))$, and let $a' = (a, a, \ldots, a)$ belong to $U^n(H_i(n))$. By the Coherence Condition, there is a global element $t: 1 \longrightarrow U^n.H(n)$ such that $t_i: 1 \longrightarrow U^n(H_i(n))$ picks out a'. But t corresponds to a compatible family of n-tuples of elements of the algebras $E(H_j(n))$, $j \in I(n)$, which in turn corresponds to some n-tuple of elements of $F_T(n)$ which is sent to a' by p_i. This suffices to show that p_i is surjective.

5.12. Lemma. For every homomorphism $E(H_i(n)) \xrightarrow{f} E(A)$, there is an M-arrow $H_i(n) \xrightarrow{f'} A$ such that $E(f') = f$.

Proof. Let f be as shown above, and let $a \in U^n(A)$ be the n-tuple of elements onto which f sends the generating n-tuple $h_i \in U^n(H_i(n))$; i.e., $f(h_{i,j}) = a_j$, for all $j < n$. Since U^n is a directed union of the representable functors $H_j(n)$, $j \in I(n)$, it follows that there is a $j \geq i$ in $I(n)$ and an M-arrow $H_j(n) \xrightarrow{g} A$ such that g corresponds to a in $H_i(n)(A) \subseteq U^n(A)$. Then $U(g)$ sends h_j onto a. Since both $f.p_i$ and $E(g).p_j$ are homomorphisms $F_T(n) \longrightarrow E(A)$ which send the free generators $(x_0, . . . , x_{n-1})$ onto $(a_0, . . . , a_{n-1})$, we have $E(g).p_j = f.p_i = f.E(h_{ij}).p_j$; but

p_j is an epimorphism, so $E(g) = f.E(h_{ij})$. But since $p_i = E(h_{ij}).p_j$ is a regular epimorphism, $E(h_{ij})$ is a regular epimorphism. Thus, by the fill-in lemma (5.7) there is an M-arrow f' such that $U_T(f) = U(f')$, from which it follows that $f = E(f')$. This completes the proof of (5.12).

5.13. Lemma. For every M-object A and every finitely-generated subalgebra B of E(A), there is an M-object B' such that $E(B') \cong B$.

Proof. Suppose that $B \subseteq E(A)$ is generated by the set $\{b_0, \ldots, b_{n-1}\}$. Since $U^n(A)$ is a directed union of the sets $H_i(n)(A)$, $i \in I(n)$, there are $j \in I(n)$ and an M-arrow $H_j(n) \xrightarrow{g} A$ such that $U(g)$ sends h_i onto the sequence $b = (b_0, \ldots, b_{n-1})$. Since $\{b_0, \ldots, b_{n-1}\}$ generates B, evidently B is an image of $E(g)$, and by (5.5) it follows that $B \cong E(B')$, where B' is an image in M of g. This proves the lemma.

<u>5.14</u>. <u>Lemma</u>. For any M-objects A and B, if $E(A)$ is finitely generated, then for each homomorphism $E(A) \xrightarrow{f} E(B)$ there is an M-arrow $A \xrightarrow{f'} B$ such that $E(f') = f$.

Proof. Suppose that $E(A)$ is generated by a finite number n of its elements, and that $E(A) \xrightarrow{f} E(B)$ is a homomorphism. It is clear from the proof of (5.13) that there is a $j \in I(n)$ and a regular epimorphism $H_j(n) \xrightarrow{g} A$. By (5.12), there is an M-arrow $H_j(n) \xrightarrow{g'} B$ such that $E(g') = f.E(g)$. Thus, we have $U(g') = U_T(f).U(g)$, where g is a regular epimorphism. By the fill-in lemma (5.7), there is an M-arrow $A \xrightarrow{f'} B$ such that $U(f') = U_T(f)$, from which it follows that $E(f') = f$. This completes the proof.

<u>5.15</u>. <u>Lemma</u>. The comparison functor E is full.

Proof. Let A and B be arbitrary objects of M, and let $E(A) \xrightarrow{f} E(B)$ be a homomorphism. Since T is finitary, every

T-algebra is a directed union of its finitely-generated subalgebras; by (5.13) it follows that $E(A)$ is a directed union of finitely-generated algebras of the form $E(A_j) \hookrightarrow E(A)$, indexed by the elements j of a directed set J, where $A_j \overset{g_j}{\hookrightarrow} A$ is a subobject of A in M which is an image of an arrow $H_i(n) \longrightarrow A$, for some finite n and some $i \in I(n)$. By (5.9), A is a directed union in M of the subobjects $A_j \overset{g_j}{\hookrightarrow} A$. By (5.14), each of the homomorphisms $f_j = f.E(g_j)$ is of the form $f_j = E(f_j')$, for some M-arrow $A_j \xrightarrow{f_j'} B$. These arrows in M constitute a cone from the directed system $(A_j : j \in J)$ to B; the cone determines a unique arrow $A \xrightarrow{f'} B$ such that $f'.g_j = f_j'$ for each $j \in J$. Then we have $E(f').E(g_j) = E(f_j') = f_j.E(g_j)$, for each $j \in J$, which proves that $E(f') = f$. Thus, E is full.

Lemmas (5.9) and (5.15) jointly imply that $E: M \longrightarrow Mod(T)$ determines a category equivalence between M and a full subcategory $E(M)$ of $Mod(T)$. Let M' be the closure of $E(M)$ in $Mod(T)$ under the formation of isomorphic copies. Clearly, M is equivalent to M'.

5.16. Lemma. The category M' is locally equational.

Proof. We shall show that M' is closed under the formation of directed unions, homomorphic images, subalgebras, and finite products, and then apply Hu's characterization theorem (5.1).

Closure under the formation of directed unions and finite products is obvious from the preceding lemmas. Let B be a subalgebra of $E(A)$. Then B is a directed union of its finitely-generated subalgebras, which are also finitely-generated subalgebras of $E(A)$ and, thus, by (5.13), belong to M'. Closure of M' under directed unions enables us to conclude that B belongs to M; this is sufficient to show that M' is closed under the formation of subalgebras.

Now let $E(A) \xrightarrow{f} B$ be a surjective homomorphism. The kernel congruence $\ker(f)$ is an object of M', since it is a subalgebra of $E(A) \times E(A)$. We may assume without loss of generality that $\ker(f) = E(C)$, for some object C of M; since E reflects congruence relations, C is a congruence relation on A, i.e., there is an exact diagram

$$\begin{tikzcd} C \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & A \arrow[r, "g"] & B' \end{tikzcd}$$

in M, where $f.E(u) = f.E(v)$ is exact in $\mathrm{Mod}(T)$. But E preserves exact diagrams, so $E(g).E(u) = E(g).E(v)$ is exact in $\mathrm{Mod}(T)$. It follows that $E(B')$ is canonically isomorphic in $\mathrm{Mod}(T)$ to B, so B belongs to M'. This shows that M' is closed under homomorphic images and concludes the proof of (5.16).

So far, we have proved that the conditions given in (5.3) are sufficient to guarantee that M is equivalent, via the finitary comparison functor E of U, to a locally equational category. It remains to be shown that the conditions are also necessary.

If M is a locally equational full subcategory of a finitary equational category Mod(T), then the functor $U: M \longrightarrow \mathrm{Set}$ required by (5.3) is the restriction to M of U_T. For each finite n, a set of representatives of all the isomorphism types of the algebras in M which are homogeneously generated by an n-element set forms an epimorphic inverse system in M in a natural way, and it is easy to check that $F_{T'}(n)$ is an inverse limit of the system, where T' is the finitary equational structure of U. (In other words, Mod(T') is the closure of M in Mod(T) under H, S, and P). The fact that the limit projections from $F_{T'}(n)$ to the homogeneously generated algebras of rank n are surjective guarantees that the inverse system coherently locally represents U^n. The regular epimorphisms, congruence relations, and directed unions in M are the same as in Mod(T), and U preserves and reflects them, since U_T does. This completes the proof of (5.3).

There is an obvious way of generalizing the notions of locally equational categories and locally representable functors by allowing an infinite regular cardinal k to play the role of ω in the appropriate definitions; the proof above could be adapted easily to provide an infinitary counterpart to (5.3).

According to (3.14), a functor $G: \mathrm{Mod}(T) \longrightarrow \mathrm{Mod}(T')$ is algebraic if and only if its set-valued component $U_{T'}.G$ satisfies conditions i) and ii) of Linton's characterization theorem (2.1) for equational categories. Let T and T' be finitary theories. We shall say that $G: \mathrm{Mod}(T) \longrightarrow \mathrm{Mod}(T')$ is <u>locally algebraic</u> if its set-valued component $U_{T'}.G$ satisfies conditions i), ii), and iii) of the characterization theorem (5.3) for locally equational categories.

After some remarks concerning locally algebraic functors in general, we identify a class of locally algebraic functors, the elementary locally algebraic functors, which offer some promise of being analyzable syntactically. We prove a theorem giving sufficient conditions for a functor to be an elementary locally algebraic functor, and then show that bounded Boolean powers can be regarded as elementary locally algebraic functors. The chapter concludes with a sketch of a proof of Hu's theorem (1.2) employing our results on bounded Boolean powers.

A syntactical analysis of locally representable functors by the methods of Chapter 3 may be impossible, since it is not clear that a locally representable functor $U\colon \mathrm{Mod}(T) \longrightarrow \mathrm{Set}$ is necessarily a subfunctor of a power of U_T. Although U itself is a directed union of representable functors $\underline{D_i}$, $i \in I$, which are equationally definable subfunctors of some sufficiently large power $U_T{}^m$ of U_T, the natural monomorphisms $\underline{D_i} \hookrightarrow U_T{}^m$ might not constitute a *cone* from $\underline{D}\colon I \longrightarrow \mathrm{Set}^{\mathrm{Mod}(T)}$ to $U_T{}^m$. Let $A = \mathrm{Lim}\, D$; if the projections $A \longrightarrow D_i$ are surjective, then for some cardinal m we can obtain a cone p of surjective homomorphisms from $F_T(m)$ to D by composing a surjection $F_T(m) \longrightarrow A$ with the cone of projections $A \longrightarrow D$. In this case (and only in this case) the monomorphisms $\underline{D_i} \xhookrightarrow{p_i} U_T^{\,m}$ are a cone from $\underline{D} \to U_T^{\,m}$, so that U is embedded in $U_T^{\,m}$ as a union of equationally definable subfunctors of $U_T^{\,m}$. For each $i \in I$, we then have $\underline{D_i} \cong [E_i] \hookrightarrow U_T^{\,m}$, where E_i is an (n_i, m)-ary equation in the language L_T or, equivalently, a conjunction of $(1, m)$-ary equations. Since $\underline{D_i} \leqslant \underline{D_k}$ whenever $i \leqslant k$ in I, it follows that $T \models E_i \Rightarrow E_k$; thus, the formulas E_i, $i \in I$, form a directed system with respect to implication. Since U is the union of the functors $[E_i]$, $i \in I$, we have $U = [\bigvee_{i \in I} E_i]$. A locally representable functor $U\colon \mathrm{Mod}(T) \longrightarrow \mathrm{Set}$ which is embeddable in a power of U_T as shown above will be called an <u>elementary</u> locally representable functor.

Finite products commute with directed unions in Set, and limits and colimits are computed "pointwise" in $\mathrm{Set}^{\mathrm{Mod}(T)}$, so it follows that finite products commute with directed unions in $\mathrm{Set}^{\mathrm{Mod}(T)}$; in particular, U^n is a directed union of the functors $\underline{D_i}^{\,n}$, $i \in I$, for all finite n. But $\underline{D_i}^n$ is represented by the algebra $n \otimes D_i$. By taking the n-th copower in $Mod(T)$ of each of the algebras and connecting homomorphisms in the inverse system D, we obtain an inverse system $n \otimes D$ which locally represents U^n, and the n-th copower of the cone $F_T(m) \xrightarrow{p} D$ is a cone $F_T(m \times n) \xrightarrow{n \otimes p} n \otimes D$ of surjective homomorphisms which corresponds to the cone of product monomorphisms $\underline{D_i}^n \hookrightarrow U_T^{m \times n}$. With U and its finite powers thus firmly planted in U_T^m and its powers, we would like to be able to obtain the finitary equational structure of U by restricting T-operations, as we did with the representable subfunctors of U_T^m in (3.3) and (3.4).

Let $U^n \xrightarrow{f} U$ be given; the composites

$$\underline{n \otimes D_i} \hookrightarrow U^n \xrightarrow{f} U \hookrightarrow U_T^m$$

constitute a cone from $n \otimes D$ to U_T^m corresponding to a cone $g\colon F_T(m) \longrightarrow n \otimes D$ in $Mod(T)$. If g could be "lifted" through

$n \otimes p$ to get a homomorphism h as shown in the diagram,

$$\begin{tikzcd} F_T(m \times n) \arrow[r, "n \otimes p"] & n \otimes D \\ F_T(m) \arrow[u, dashed, "h"] \arrow[ur, "g"'] & \end{tikzcd}$$

then $U_T^{m \times n} \xrightarrow{\underline{h}} U_T^{m}$ would be a T-operation whose restriction to $U^n \hookrightarrow U_T^{m \times n}$ would induce f. Indeed, for each $i \in I$ this program can be carried out as shown below, since $F_T(m)$ is regular-projective and $n \otimes p_i$ is a regular epimorphism.

$$\begin{tikzcd} F_T(m \times n) \arrow[r, "n \otimes p_i"] & n \otimes D_i \\ F_T(m) \arrow[u, dashed, "h_i"] \arrow[ur, "g_i"'] & \end{tikzcd}$$

There seems to be no guarantee, however, that there exists a single homomorphism h such that, for all $i \in I$, $(n \otimes p_i).h = g_i$.

We arrive at the conclusion that each finitary operation

$U^n \xrightarrow{f} U$ is given as a cone of partial operations $\underline{D_i}^n \xrightarrow{f_i} U$, where each partial U-operation f_i is the restriction to $\underline{D_i}^n$ of some T-operation $U_T^{m \times n} \xrightarrow{h_i} U_T^m$.

A more detailed syntactical analysis of elementary locally representable and locally algebraic functors must wait until the algebraic and category-theoretic properties of these functors are better understood. At the time of writing, it is unclear to what extent locally algebraic functors as defined above are syntactically analogous to algebraic functors.

We shall say that a locally algebraic functor G is an <u>elementary locally algebraic functor</u> if its set-valued component is an elementary locally representable functor. Just as a representable functor whose representing algebra is a regular progenerator is algebraic, the next result tells us that an elementary locally representable functor which is locally represented by an inverse system of regular progenerators is an elementary locally algebraic functor.

5.17. Theorem. Let T be a finitary equational theory, and let $D\colon I^{\circ} \longrightarrow \mathrm{Mod}(T)$ be an inverse system of finitely generated regular progenerators such that there is a cone of surjective homomorphisms $A \xrightarrow{p} D$, for some T-algebra A. Then the functor $U\colon \mathrm{Mod}(T) \longrightarrow \mathrm{Set}$ which is locally represented by D is an elementary locally algebraic functor.

Proof. It is obvious from the preceding discussion that U is elementary, and that the cone of surjections $A \longrightarrow D$ presents $\underline{D}$ as a directed system of equationally definable subfunctors of $\underline{A}$, with $U \hookrightarrow \underline{A}$ as the union of that system. For each finite n, U^n is locally represented by $n \otimes D$.

Since D_i is finitely generated, $\underline{D_i}$ preserves directed unions, for all $i \in I$, so U preserves directed unions. Each of the functors $\underline{D_i}$, $i \in I$, preserves limits, so it follows that U preserves finite limits. In particular, U preserves kernel pairs, so U preserves congruence relations. It remains to be shown that U

preserves regular epimorphisms and reflects regular epimorphisms and congruence relations, and that, for each finite n, $n \otimes D$ is coherent.

Whenever it is convenient, we shall treat the colimit injections $\underline{D}_i \hookrightarrow U$ as though, for each T-algebra B, the B-component $\underline{D}_i(B) \hookrightarrow U(B)$ were an inclusion map.

Suppose that $B \xrightarrow{h} C$ is a regular epimorphism. If $u \in U(C)$, we may identify it with an element of $\underline{D}_i(C) \subseteq U(C)$, for some $i \in I$. But D_i is regular-projective, so $\underline{D}_i$ preserves regular epimorphisms. Then $\underline{D}_i(h) : \underline{D}_i(B) \longrightarrow \underline{D}_i(C)$ is surjective, so there is some $v \in \underline{D}_i(B) \subseteq U(B)$ such that $h.v = u$. This suffices to show that $U(h)$ is surjective, so U preserves regular epimorphisms.

To prove that U reflects regular epimorphisms, suppose that $B \xrightarrow{h} C$ is given with $U(h)$ surjective. Then every element of $\underline{D}_i(C) \subseteq U(C)$ is a $U(h)$-image of some element of $U(B)$, i.e., for each $D_i \xrightarrow{u} C$ there is $j \geq i$ in I and an arrow $D_j \xrightarrow{v} B$ such that $h.v = u.d_{ij}$, where $D_j \xrightarrow{d_{ij}} D_i$ is the connecting arrow. (This amounts to viewing u as an element of a bigger piece $\underline{D_j}(C)$ of $U(C)$, where j is chosen so that the pre-image v of u lives in $\underline{D_j}(B) \subseteq U(B)$). Since $d_{ij}.p_j = p_i$, where $A \xrightarrow{p_i} D_i$ is surjective, it follows that d_{ij} is surjective. But D_i is regular-projective, so there is $D_i \xrightarrow{s} D_j$ with $d_{ij}.s = id_{D_i}$. But then $u = u.d_{ij}.s = h.v.s$, which shows that the function $\underline{D_i}(h)$ is surjective; since D_i is a regular generator, it follows that h is surjective.

Now we must show that U reflects congruence relations. Suppose that $C \hookrightarrow B \times B$ is given such that in Set $U(C) \hookrightarrow U(B \times B) = U(B) \times U(B)$ is isomorphic to a congruence relation (i.e., an equivalence relation in the ordinary sense). For any $i \in I$, let $(u, v): D_i \longrightarrow B \times B$ be the homomorphism whose composites with the projections $B \times B \longrightarrow B$ are respectively $u, v: D_i \longrightarrow B$. The Elements of $\underline{D_i}(C)$ are in bijective correspondence with the arrows

$(u, v): D_i \longrightarrow B \times B$ which factor through $C \hookrightarrow B$. Let $i \in I$ be fixed; we shall show that $\underline{D_i}(C) \hookrightarrow \underline{D_i}(B \times B) = \underline{D_i}(B) \times \underline{D_i}(B)$ is an equivalence relation.

Let $D_i \xrightarrow{u} B$ be given. Since $U(C)$ is reflexive, there is $j \geqslant i$ in I such that (u', u') factors through C, where $u' = u.d_{ij}$. As we have pointed out, d_{ij} has a coretraction s, so $(u', u').s = (u'.s, u'.s) = (u, u)$ factors through C, so $\underline{D_i}(C)$ is reflexive.

Now suppose that $(u, v): D_i \longrightarrow B \times B$ is given which factors through C. Since $U(C)$ is symmetric, for some $j \geqslant i$ in I we have $(v', u'): D_j \longrightarrow B \times B$ which factors through C, where $v' = v.d_{ij}$ and $u' = u.d_{ij}$. Then $(v', u').s = (v'.s, u'.s) = (v, u)$ factors through C, so $\underline{D_i}(C)$ is symmetric.

The same trick shows that $\underline{D_i}(C)$ is transitive, so it is an equivalence relation, i.e., a congruence relation in Set. Since D_i is a regular progenerator, $\underline{D_i}$ reflects congruence relations, by

(3.1), so C is a congruence relation in Mod(T). This proves that U reflects congruence relations.

Finally, we shall show that $n \otimes D$ is coherent, for all finite n. Note that $n \otimes D$ is coherent if and only if, for each $i \in I$, the projection $(\mathrm{Lim}\ U^n.n\otimes D) \xrightarrow{d_i} U^n(n\otimes D_i)$ is surjective. By hypothesis we have a cone $n\otimes A \xrightarrow{n\otimes p} n\otimes D$ of surjections; since U preserves surjections, U^n does too, and $U^n(n\otimes p)$ is a cone of surjections from $U^n(A)$ to $U^n.n\otimes D$. Let $U^n(A) \xrightarrow{f} \mathrm{Lim}\ U^n.n\otimes D$ be the induced arrow. For each $i \in I$, we have that $d_i.f = U^n(n\otimes p_i)$ is surjective, so d_i is surjective. This completes the proof that U is an elementary locally algebraic functor.

Let A be any set having at least two elements. The <u>set-valued bounded Boolean power functor</u> $A[-]: \mathrm{Mod}(BA) \longrightarrow \mathrm{Set}$ is defined as follows.

i) For each Boolean algebra B, $A[B]$ is the set of all

A-indexed partitions of unity $u: A \longrightarrow B$ such that the set $\{a \in A : u(a) \neq 0\}$ is finite.

ii) For each Boolean homomorphism $B \xrightarrow{h} C$, let $A[h]: A[B] \longrightarrow A[C]$ be the function which sends each $u \in A[B]$ to $h.u \in A[C]$.

An <u>algebra-valued bounded Boolean power functor</u> is any functor of the form $G: \mathrm{Mod}(BA) \longrightarrow \mathrm{Mod}(T)$, where $U_T.G \cong A[-]$, for some set A.

When A is finite, the bounded Boolean power functor $A[-]$ coincides with the finitary representable Boolean power functor defined in Chapter 4. The bounded Boolean power construction was introduced by A. L. Foster [16] and has been studied extensively in conjunction with Boolean powers. Major references on bounded Boolean powers are Grätzer [19], Burris [5], and Banaschewski and Nelson [3]. Bounded Boolean powers admit convenient representations as "bounded normal subdirect powers" (see [18]) and as algebras of continuous functions (see [3]). Our treatment of bounded Boolean powers as elementary locally algebraic functors seems to be new.

5.18. Theorem. Every bounded Boolean power functor is an elementary locally algebraic functor; furthermore, every locally algebraic functor of the form $G\colon \mathrm{Mod}(BA) \longrightarrow \mathrm{Mod}(T)$ is a bounded Boolean power functor.

Proof. Since a functor is or is not an elementary locally algebraic functor by virtue of the properties of its set-valued component, it is sufficient to consider only set-valued functors. First, we shall prove that $A[-]\colon \mathrm{Mod}(BA) \longrightarrow \mathrm{Set}$ is an elementary locally algebraic functor. Let I be the directed system of all finite subsets of the set A, with inclusion maps as connecting arrows. Assigning to each $X \xhookrightarrow{f} Y$ the corresponding Boolean homomorphism $2^Y \xrightarrow{f^*} 2^X$, we obtain an inverse system $P\colon I^o \longrightarrow \mathrm{Mod}(BA)$ such that:

i) All connecting homomorphisms $2^Y \longrightarrow 2^X$ are surjective;

ii) The surjections $2^A \xrightarrow{p_X} 2^X$, corresponding to $X \hookrightarrow A$ for all $X \in I$, constitute a cone p from 2^A to P.

For each finite n, we have an inverse system $n \otimes P: I^\circ \longrightarrow \mathrm{Mod(BA)}$ of quotients of $n \otimes 2^A$; $n \otimes P$ sends each $X \hookrightarrow Y$ of I to $2^{Y^n} \longrightarrow 2^{X^n}$ in $\mathrm{Mod(BA)}$. The conditions i) and ii) above are satisfied, mutatis mutandis, by $n \otimes P$.

We claim that P locally represents $A[-]$. For each $X \in I$, a natural transformation $\underline{2}^X \xrightarrow{g_X} A[-]$ is defined as follows. For each homomorphism $2^X \xrightarrow{h} B$, let $g_{X,B}(h)$ be the partition $u: A \longrightarrow B$ defined by $u(a) = h.p_X(\{a\})$ for all $a \in A$. Obviously, for each Boolean algebra B, the arrow $g_{X,B}$ is an injective function which embeds $\underline{2}^X(B)$ in $A[B]$. In fact, $g = (g_X : X \in I)$ is a cone of monomorphisms from $\underline{P}$ to $A[-]$. To show that $A[-]$ is a directed union of the functors $\underline{2}^X$, $X \in I$, let B be any Boolean algebra, and let u be any element of $A[B]$. Let $X = \{a \in A : u(a) \neq 0\}$, and let $2^X \xrightarrow{h} B$ be the Boolean homomorphism which sends $\{a\}$ to $u(a)$, for each $a \in X$. Then $g_{X,B}(h) = u$. This shows that $A[B] = \operatorname{Colim}_{X \in I} \underline{2}^X(B)$ for each B, so $A[-] = \operatorname{Colim}_{X \in I} \underline{2}^X = \operatorname{Colim} \underline{P}$. It is immediate that, for each finite n, $A[-]^n$ is locally represented by $n \otimes P$. By (5.17) $A[-]$ is an elementary locally algebraic functor.

Now suppose that U: Mod(BA) $\longrightarrow$ Set is locally representable. Let $D: I^\circ \longrightarrow$ Mod(BA) be an epimorphic inverse system which locally represents U. Epimorphisms in Mod(BA) are surjective, so all the connective arrows $D_j \xrightarrow{d_{ij}} D_i$ are surjective. Suppose that D_i is not finitely generated; then neither is D_j, for any $j \geq i$ in I, so we may as well suppose that none of the algebras in the inverse system D is finitely generated. Let $i \in I$ be fixed, and let the finitely generated subalgebras of D_i be $B_k \hookrightarrow D_i$, where those subalgebras are indexed by an appropriate directed set K so that we can write the directed system of finitely generated subalgebras of D_i as $B: K \longrightarrow$ Mod(BA). Of course, $\operatorname{Colim} B = D_i$ since BA is a finitary equational theory. We shall show that U does not preserve the directed union $\mathrm{Colim}\ B = D_i$.

Application of U to the cone of inclusions $B \longrightarrow D_i$ produces a cone of monomorphisms $U.B \longrightarrow U(D_i)$, so obviously the directed union $\mathrm{Colim}\ U.B$ is embedded in $U(D_i)$ by the induced arrow $\mathrm{Colim}\ U.B \longrightarrow U(D_i)$. The functor U preserves the directed union $\mathrm{Colim}\ B = D_i$ if and only if the embedding $\mathrm{Colim}\ U.B \longrightarrow U(D_i)$ is an isomorphism. If D_i is not finitely generated, however, the element of $U(D)$ represented by the identity arrow in $\underline{D_i}(D_i) \subseteq U(D_i)$ has no counterpart in $\mathrm{Colim}\ U.B$, so the embedding of $\mathrm{Colim}\ U.B$ in $U(D_i)$ is not an isomorphism, and U does not preserve directed unions.

The preceding argument shows that if the functor $U\colon \mathrm{Mod}(BA) \longrightarrow \mathrm{Set}$ is locally algebraic then all of the algebras D_i, $i \in I$, must be finitely generated, since U preserves directed unions. But then each algebra D_i is a finite power 2^{X_i} of a two-element Boolean algebra. Each of the surjective connecting homomorphisms $2^{X_j} \xrightarrow{d_{ij}} 2^{X_i}$ is induced by an injection $X_i \hookrightarrow X_j$, and it is easy to see that $U \cong A[-]$, where A is the directed union in Set of the directed system of finite sets X_i, $i \in I$. This completes the proof of (5.18).

5.19. Theorem. Let A be a set having at least two elements. The finitary equational structure of $A[-]\colon \mathrm{Mod(BA)} \longrightarrow \mathrm{Set}$ is isomorphic to the finitary equational structure of the set A.

Proof. If A is finite, this result is a consequence of (4.2) and (4.3). Suppose that A is infinite, and let $P\colon I^{\circ} \longrightarrow \mathrm{Mod(BA)}$ be the inverse system, defined in the proof of (5.18), which locally represents $A[-]$. Recall that $p\colon 2^A \longrightarrow P$ is the obvious cone of surjections, and that $g\colon \underline{P} \longrightarrow A[-]$ is the cone of colimit injections. Given a finitary operation $A[-]^n \xrightarrow{f} A[-]$, consider the composite

$$n \otimes P \xrightarrow{n \otimes g} A[-]^n \xrightarrow{f} A[-] \xrightarrow{r} \underline{2}^A,$$

where r is the colimit arrow induced by the cone $\underline{p}: \underline{P} \longrightarrow \underline{2^A}$.

This composite corresponds to a cone $2^A \longrightarrow n \otimes P$ in Mod(BA) and, hence, to a homomorphism $2^A \xrightarrow{f'} 2^{A^n}$, since $2^{A^n} = \operatorname{Lim} n \otimes P$. To verify that f' really does induce f on $A[-]$ in some reasonable sense, observe that there is a natural transformation $A[-]^n \longrightarrow \underline{2^{A^n}}$ which, for each Boolean algebra B, embeds $A[B]^n$ in $\hom_{BA}(2^{A^n}, B)$ as the set of all homomorphisms u which factor, for some $X \in I$, as $2^{A^n} \xrightarrow{q_X} 2^{X^n} \xrightarrow{u'} B$, where q_X is the projection induced by $X^n \hookrightarrow A^n$. The action of f on an element u of $A[B]^n$ (viewed as an arrow $2^{A^n} \longrightarrow B$) is as follows.

$$[r_B.f_B](u) = [r_B.f_B.n \otimes q_{X,B}](u') = \underline{[q_X.f']}_B(u') = u'.q_X.f' = u.f'$$

Since $u.f'$ is an element of $\hom_{BA}(2^A, B)$ which represents an element of $A[B]$, it must factor as $2^A \xrightarrow{p_Y} 2^Y \xrightarrow{v} B$, for some $Y \in I$ and some homomorphism v. This means that $u.f'$ is a complete homomorphism.

We claim that f' is a complete homomorphism, i.e., that $f' = h^*$, for some function $A^n \xrightarrow{h} A$. The claim is true because, if f' is not complete, the original operation f is not well-defined on $A[2]$. To see this, let $A_0 \supsetneq A_1 \supsetneq \ldots \supsetneq A_m \supsetneq \ldots$ be a descending chain of subsets of A such that $\bigcap_i A_i = 0$ while $\bigcap_i f'(A_i) \neq 0$; such a chain must exist if f' is not complete. Let $b \in \bigcap_i f'(A_i)$, and let $u\colon 2^{A^n} \longrightarrow 2$ be the homomorphism which sends $\{b\}$ to 1. Clearly, u belongs to $A[2]^n \subseteq \hom_{BA}(2^{A^n}, 2)$. For each of the sets A_i we have $[u.f'](A_i) = u(f'(A_i)) \geq u(\{b\}) = 1$. Thus, $\bigcap_i [u.f'](A_i) = 1 \neq 0$, so $u.f'$ is not a complete homomorphism and, hence, not an element of $A[2] \subseteq \hom_{BA}(2^A, 2)$.

So far, we have established that each finitary operation $A[-]^n \xrightarrow{f} A[-]$ is induced by a finitary operation $A^n \longrightarrow A$. But the Foster formula for bounded Boolean powers

$$[f_B(u)](c) = \operatorname{Sup}\left\{\bigwedge_{j<n} u_j(b_j) \ : \ f(b) = c\right\}$$

for all $c \in A$ and $u \in A[B]^n$ (derived as in (4.4)), shows that every finitary operation on A induces an operation on $A[-]$, and that no two distinct A-operations induce the same $A[-]$-operation. It follows that $A[-]$ and A have isomorphic finitary equational structure. This completes the proof of (5.19).

The new content of (5.19) is simply that Foster's formula is sufficient for bounded Boolean powers; $A[-]$ is incapable of carrying any finitary operations which are not inherited from A. The proof of (5.19) is interesting, in that it shows that, while (as noted earlier in this Chapter) the operations on $A[-]$ are not directly representable as restrictions to $A[-]^n \subseteq (\underline{2}^A)^n$ of operations in the equational structure of the functor $\underline{2}^A$, the $A[-]$-operations *are* obtainable as restrictions of operations in a ***many-sorted*** T_A-algebra in $\mathrm{Set}^{\mathrm{Mod(BA)}}$, namely, the functor which sends each function $A^m \xrightarrow{h} A^n$ to $\underline{h}^*: \underline{2}^{A^m} \longrightarrow \underline{2}^{A^n}$. This suggests a possible means of avoiding the difficulties in the syntactical analysis of locally representable functors, as pointed out in the discussion preceding (5.17).

Finally, note that (5.18) and (5.19) together can be used to prove Hu's theorem (1.2). The finitary equational structure of a set A is the equational theory T of a locally primal algebra A^+ whose underlying set is A. Theorems (5.18) and (5.19) together show that $\mathrm{Mod}(BA)$ is equivalent, via the comparison functor for the bounded Boolean power functor $A[-]$, to the locally equational subcategory $L(\{A^+\})$ of $\mathrm{Mod}(T)$ generated by the locally primal algebra A^+. Given any algebra A' whose underlying set is A and which belongs to an equational category $\mathrm{Mod}(T'')$, let T' be the finitary equational theory of the algebra A' (i.e., look at A' itself as an equational theory; $\mathrm{Mod}(T')$ is essentially just $\mathbf{HSP}(\{A'\}) \subseteq \mathrm{Mod}(T'')$). The algebra A' itself determines a mapping of theories $T'' \longrightarrow T'$, and the inclusion $T' \hookrightarrow T$ is a mapping of theories. The composite $T'' \longrightarrow T' \longrightarrow T$ determines a reduct functor $U\colon \mathrm{Mod}(T) \longrightarrow \mathrm{Mod}(T'')$

which establishes a category equivalence between $L(\{A^+\})$ and $L(\{A'\})$ exactly when A' is locally primal.

LIST OF REFERENCES

[1] R. Balbes and P. Dwinger, Distributive Lattices, University of Missouri Press, Columbia, Missouri 1974.

[2] B. Banaschewski, Functors into categories of M-sets, Abh. Math. Sem. Univ. Hamburg 38 (1972), 49-64.

[3] B. Banaschewski and E. Nelson, Boolean powers as algebras of continuous functions, Manuscript, McMaster University 1976.

[4] M. Barr, P. A. Grillet, and D. H. Van Osdol, Exact Categories and Categories of Sheaves, Lecture Notes in Mathematics 236, Springer-Verlag, Berlin-Heidelberg 1971.

[5] S. Burris, Boolean powers, Algebra Universalis 5 (1975), 341-360.

[6] Cat-Ho Nguyen, Generalized Post algebras and their application to some infinitary many-valued logics, Dissertationes Mathematicae 107 (1973).

[7] P. M. Cohn, Morita equivalence and duality, Queen Mary College Mathematics Notes, London 1968.

[8] A. Day, Injectivity in equational classes of algebras, Canadian Journal of Mathematics 24, No. 2 (1972), 209-220.

[9] J. J. Dukarm, Weakly monadic Boolean extension functors, Fundamenta Mathematicae (to appear).

[10] J. J. Dukarm, Morita equivalence of varietal theories, Manuscript, Simon Fraser University 1978.

[11] J. J. Dukarm, A characterization of locally equational categories of algebras, Manuscript, Simon Fraser University 1978.

[12] B. Elkins and J. A. Zilber, Categories of actions and Morita equivalence, Rocky Mountain Journal of Mathematics 6, No. 2 (1976), 199-225.

[13] G. Epstein, The lattice theory of Post algebras, Transactions of the American Mathematical Society 95 (1960), 300-317.

[14] A. L. Foster, Generalized "Boolean" theory of universal algebras (I), Mathematische Zeitschrift 58 (1953), 306-336.

[15] A. L. Foster, Generalized "Boolean" theory of universal algebras (II), Mathematische Zeitschrift 59 (1953), 191-199.

[16] A. L. Foster, Functional completeness in the small, Mathematische Annalen 143 (1961), 29-58.

[17] P. Freyd, Algebra-valued functors in general and tensor products in particular, Colloquium Mathematicum 14 (1966), 89-106.

[18] M. Gould and G. Grätzer, Boolean extensions and normal subdirect powers of finite universal algebras, Mathematische Zeitschrift 99 (1967), 16-25.

[19] G. Grätzer, Universal Algebra, D. Van Nostrand Co., Inc., Princeton 1968.

[20] L. Henkin, J. D. Monk, and A. Tarski, Cylindric Algebras (I), North-Holland Publishing Co., 1971.

[21] H. Herrlich and G. E. Strecker, Category Theory, Allyn and Bacon, Boston 1973.

[22] T. K. Hu, Stone duality for primal algebra theory, Mathematische Zeitschrift 110 (1969), 180-198.

[23] T. K. Hu, On the fundamental subdirect factorization theorems of primal algebra theory, Mathematische Zeitschrift 112 (1969), 154-162.

[24] T. K. Hu, On the topological duality for primal algebra theory, Algebra Universalis 1 (1971), 152-154.

[25] T. K. Hu, Locally equational classes of universal algebras, Chinese Journal of Mathematics (Taiwan) 1, No. 2 (1973), 143-165.

[26] A. B. Karatay, A generalization of the characterization theorem of Boolean extensions, Algebra Universalis 4 (1974), 185-189.

[27] U. Knauer, Projectivity of acts and Morita equivalence of monoids, Semigroup Forum 3 (1971-72), 359-370.

[28] R. Lagrange, Amalgamations and epimorphisms in m-complete Boolean algebras, Algebra Universalis 4 (1974), 277-279.

[29] F. W. Lawvere, Functorial semantics of algebraic theories, Ph.D. Dissertation, Columbia University 1963.

[30] F. E. J. Linton, Some aspects of equational categories, Proceedings of the La Jolla conference on categorical algebra, Springer-Verlag, Berlin, 1963, pp. 84-94.

[31] F. E. J. Linton, An outline of functorial semantics, Lecture Notes in Mathematics 80, Springer-Verlag, Berlin-Heidelberg 1969, pp. 7-52.

[32] S. Mac Lane, Categories for the Working Mathematician, Graduate Texts in Mathematics 5, Springer-Verlag, New York 1971.

[33] M. Makkai and G. E. Reyes, First Order Categorical Logic, Lecture Notes in Mathematics 611, Springer-Verlag, Berlin-Heidelberg 1977.

[34] E. G. Manes, Algebraic Theories, Graduate Texts in Mathematics 26, Springer-Verlag, New York 1976.

[35] K. Morita, Scientific Reports of Tokyo Tyoiku, Daigaku, Sect A 9 (1965), 40-71.

[36] B. Pareigis, Categories and Functors, Academic Press, New York 1970.

[37] H. Rasiowa, An Algebraic Approach to Nonclassical Logics, Studies in Logic and the Foundations of Mathematics, Vol. 78, North-Holland Publishing Co., Amsterdam, and Pańdstwowe Wydawnictwo Naukowe, Warsaw 1974.

[38] P. C. Rosenbloom, Post algebras I. Postulates and general theory, American Journal of Mathematics 64 (1942), 167-188.

[39] J. R. Shoenfield, Mathematical Logic, Addison-Wesley Publishing Co., Reading, Massachusetts 1967.

[40] R. Sikorski, Boolean Algebras (third edition), Springer-Verlag, New York 1969.

[41] T. Traczyk, Axioms and some properties of Post algebras, Colloquium Mathematicum 10, No. 2 (1963), 193-209.

[42] G. C. Wraith, Algebraic Theories, Lecture Notes Series No. 22, Matematisk Institut, Aarhus Universitet, February 1975.