

SPHERICALLY-SYMMETRIC MONOPOLE SOLUTIONS
IN SU(2) and SU(3) GAUGE THEORIES

by

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ABSTRACT

A numerical study is made of the 't Hooft-Polyakov monopole and its spherically symmetric generalization to an SU(3) gauge theory. Also, dyon solutions are investigated for an SU(2) gauge theory. The nonlinear differential equations are solved by a collocation method. The masses (and electric charges of the dyons) are plotted as a function of $\beta = \frac{\lambda}{e^2}$ and it is found that the masses (and electric charges) approach an upper (lower) bound as β becomes asymptotically large. We call this phenomena saturation. An explanation of this phenomena is proposed which in principle should predict the saturated values of monopole (dyon) masses and electric charges. We obtain good agreement for the SU(2) dyons.

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Chapter 1

Introduction

Years ago, Dirac¹ introduced into the theory of electrodynamics point particles with magnetic charge, the Dirac monopole. It was an uncomfortable fit. To make the Dirac monopole field part of a dynamical theory one must introduce non-electromagnetic dynamical degrees of freedom, the magnetic monopole, whose mass and spin are free parameters. Further, it is necessary to introduce into the vector potential a line of singularities extending from the monopole to infinity, the Dirac string. There has been extensive investigation extending the ideas of Dirac, but these unpleasant features remain².

To put things in perspective we shall take a short but important digression from the relatively narrow subject of monopoles.

Recently there has been a great deal of interest in non-Abelian gauge theories of the Yang-Mills type as possible candidates for unifying the theories of strong, weak, and electromagnetic interactions³. These theories are not yet completely understood on a quantum level. However, the

¹The original articles are by Dirac (1931, 1948).

²For a nice review of the subject up to 1968 see Amaldi (1968).

³See the articles of Abers and Lee (1973) or S. Weinberg (1974).

classical solutions to the equations of motion have important implications about the structure of the full quantal theory⁵. Of particular interest are stable, finite energy, non-dissipative solutions to the classical field equations. In other words, we are interested in a stable lump of finite field energy that does not diffuse into a constant zero-energy density over all space. These objects, called solitons, are artifacts of non-linear field theories. The soliton may achieve its stability⁶ via one of two mechanisms. Thus, solitons are classified as either topological or non-topological solitons. Non-topological solitons achieve their stability dynamically via time varying field amplitudes and conserved Noether currents. We will not be concerned with such objects in this thesis. The stability of topological solitons is a result of the existence of a non-trivial mapping from the manifold of the internal field space onto the manifold of the real d-dimensional space of the theory. The study of such continuous mappings is called homotopy theory⁷. It turns out that each map is associated with an

⁵I have stolen a number of nice phrases from Marciano and Pagels (1978).

⁶The soliton solutions are solutions of the Euler-Lagrange equations which follows from Hamilton's principle, $\delta S=0$. A solution is stable if it minimizes the action, $\delta S \geq 0$.

⁷Appendix E contains an introduction to homotopy theory where terms like "non-trivial mapping" are defined.

integer which is in turn associated with a topological charge⁸. This means that the soliton cannot diffuse into a vacuum with zero topological charge. The topology of the fields acts as an infinite potential barrier making the static topological soliton stable. There is speculation that the only static and stable lumps of field energy are topological solitons. These classical solutions provide the first approximations for the full quantum theory from which one can calculate the quantum fluctuations. This is important because soliton solutions cannot be generated by standard perturbation theory⁹. Thus, we gain a new class of quantum solutions in our theory. An example of a soliton and a potentially spectacular confirmation of these ideas is the possible existence of magnetic monopoles.

't Hooft (1974) and Polyakov (1974) discovered independently a static, finite-energy and stable classical solution in a spontaneously broken Yang-Mills theory. Specifically, their theory contains a triplet of scalar fields along with a triplet of gauge fields whose interactions are invariant under an $SU(2)$ symmetry¹⁰. With an appropriate

⁸For each soliton solution we can define a conserved current that is not a result of a Lagrangian symmetry (i.e. this is not a Noether current). The corresponding charge is the topological charge.

⁹Soliton amplitudes go as the inverse of the coupling constant.

¹⁰This barrage of nomenclature is explained in detail in following chapters.

identification of the electromagnetic fields, this soliton solution can be interpreted as having the properties of a magnetic monopole.

I should emphasize at this point that the particles we shall consider are completely different from the Dirac monopole. The 't Hooft-Polyakov monopoles are by-products in theories of non-Abelian gauge fields interacting with scalar fields. Unlike the Dirac monopole, the fields are non-singular with finite energy, there are no new dynamical degrees of freedom, and all the properties of these monopoles are determined by the parameters of the original theory of interacting scalar fields and gauge fields.

Julia and Zee (1974) generalized the 't Hooft and Polyakov monopole so that the solution could be interpreted as having both magnetic and electric charges. Such objects are called dyons, following Schwinger (1969). Since that time there has appeared in the literature several papers that generalize the solution of 't Hooft and Polyakov in an $SU(2)$ theory to larger symmetry groups. The fundamental idea behind these papers is that to produce a spherically-symmetric ansatz for the fields which reduces the Euler-Lagrange equations to a set of coupled nonlinear ordinary differential equations¹¹. There seems to be no complete analytic solution to these problems, although Prasad and

¹¹Spherical-symmetry is defined and its use is explained in Chapter 3.

Sommerfield (1975) and Czechowski (1977) have succeeded in obtaining analytic solutions in special cases of SU(2) and SU(3) theories respectively. Any extensive analysis of these problems requires the use of numerical techniques.

The equations to be solved are boundary valued problems, thus numerical solutions cannot be obtained trivially. Previous numerical studies on monopoles in an SU(2) theory have been done by Bais and Primack (1976) and Cutler and Wyld (1976)¹². Bais and Primack investigated solutions for values of the parameter $\beta = \lambda/e^2$ ¹³ ranging from zero to 10 while Cutler and Wyld extended this range from zero to 100 (while we have solved the case of the SU(2) monopole for β 's up to $10^{11.5}$). Until this time there has been no publication of numerical results involving monopoles in an SU(3) gauge theory.

Chapter 2 is devoted to a brief review of non-Abelian Yang-Mills theories. In chapter 3 the crucial concept of spherical-symmetry is defined and then used to produce simple ansatz for finding monopole solutions in a theory with a general compact semisimple gauge group G. Using these principles the equations of motion for a dyon/monopole solution in an SU(2) gauge theory are derived in chapter 4.

¹²A brief description of their methods is given in chapter 6.

¹³These parameters are explained in the following chapters.

An integral expression is given for the energy (mass) which is later calculated numerically. Further, we identify the electromagnetic fields and thereby show that the solution may be interpreted as having a point magnetic charge of $\frac{4\pi}{e}$. The electric charge appears to be "extended" and the total charge is obtained by an integral expression. The electric charge may vary depending upon the boundary conditions applied to the fields. The fifth chapter is devoted to monopole solutions in an SU(3) gauge theory. The process is more involved than the SU(2) case of chapter 4 because of the more complex nature of the SU(3) group. There are two types of monopoles corresponding to the two distinct embeddings of SU(2) as a subgroup of SU(3). Further, there are two distinct types of vacuum symmetries, SU(2)xU(1) and U(1)xU(1), that lead to corresponding identifications of the electromagnetic fields and their associated magnetic charges. Again, an integral expression is given for the masses of the various monopoles which is calculated numerically once the field equations are solved. Chapter 6 contains an explanation of the numerical method used in this thesis to solve the differential equations, collocation. A brief description of the methods of Bais and Primack and Cutler and Wyld is given. In practice the differential equations are solved on the interval $[0,1)$ rather than $[0,\infty)$ on which the problem is originally cast. The corresponding mapping and its restrictions are described. The results of the calculations are given in chapter 7. The fields of dyons in an SU(2)

theory are calculated for various values of β and η ¹⁴. The corresponding charges and masses are calculated from these solutions. The fields of the SU(3) monopoles are solved for a number of values of β . Unfortunately, the case of the SO(3) embedding could be solved only for the case of $\beta=0$. For the monopole/dyon cases that were solved there appears an interesting phenomena. The mass of monopole/dyon monotonically approaches an upper bound as β becomes asymptotically large. This upper bound is different for each case considered. An explanation of this "saturation" is given for the SU(2) monopole/dyon by considering the analytic behavior of the solutions as β goes to infinity. We find there is a good agreement between the infinite β model and the large β limit of the numerical results. Unfortunately, a similar explanation could not be made for the other cases of mass "saturation" of the SU(3) monopole. This presents the intriguing idea that all monopole solutions may saturate in this manner.

¹⁴ η is a parameter that determines one of the boundary conditions on the fields. Increasing η has the effect of increasing the electric charge on the dyon.

Chapter 2

The Lagrangian Density¹

The fundamental object of any field theory is the Lagrangian density \mathcal{L} which is a function of all the fields of the theory and their derivatives $\phi, \partial_\mu \phi$. The Lagrangian L is the integral of \mathcal{L} over all Euclidean three-space.

$$L(t) = \int d^3x \mathcal{L}(\phi^a(x), \partial_\mu \phi^a(x)). \quad (2.1)$$

The equations of motion follow from Hamilton's principle,

$$\delta \int_{t_1}^{t_2} L(t) dt = 0 \quad (2.2)$$

for any t_1 and t_2 , where the variations of the fields are assumed to vanish at t_1 and t_2 . The condition (2.2) implies the fields of the theory satisfy the Euler-Lagrange equations:

$$\frac{\delta \mathcal{L}}{\delta \phi^a} = \partial_\mu \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi^a)}. \quad (2.3)$$

In classical field theory \mathcal{L} must be real and Lorentz invariant, so that one obtains covariant equations of motion. These conditions are generally obvious by the way \mathcal{L} is constructed.

¹This chapter is a shameless plagiarism of the first section of Abers and Lee (1973).

A very powerful observation is that to every continuous symmetry of \mathcal{L} there corresponds a conserved current. It is well known that in a Lorentz invariant theory, the energy, momentum and angular momentum can be defined and are conserved (Roman (1969), Barut (1964)). Now we shall extend this idea to non-classical space-time symmetries, called internal symmetries. Internal symmetry transformations consist of linear transformations among the fields of the theory,

$$\phi(x) \rightarrow \phi'(x) = \text{EXP} [-i\theta^a L^a] \phi(x) \equiv U(\theta) \phi(x) \quad (2.4)$$

where $\phi(x)$ is a column vector and L^a is a matrix representation of the generators of a group G . We say the fields come in multiplets which form a basis for representations of G . If the transformation parameters θ^a are independent of space-time, (2.4) is called a global gauge transformation or a gauge transformation of the first kind. Under a global transformation terms of the form, $\phi^\dagger \phi$ and $\partial_\mu \phi^\dagger \partial^\mu \phi$, are invariant and a Lagrangian density can be constructed using these basic units².

Assuming we start with a Lagrangian density with a global symmetry, how do we construct a theory that is invariant under a local gauge transformation where the transformation parameters θ^a become functions of space and time $\theta^a(x)$?

²It will be assumed for simplicity that the fields $\phi = \begin{pmatrix} \phi^1 \\ \vdots \\ \phi^n \end{pmatrix}$ are Lorentz scalars.

The answer is that we are forced to introduce a set of vector fields $A_\mu^a(x)$ called gauge fields that transform in such a way as to keep \mathcal{L} invariant under gauge transformations.

Under a local gauge transformation

$$\phi(x) \longrightarrow \phi'(x) = U(\theta(x)) \phi(x), \quad (2.5)$$

the derivatives of the fields transform as,

$$\partial_\mu \phi(x) \longrightarrow U(\theta) \partial_\mu \phi(x) + (\partial_\mu U(\theta)) \phi(x). \quad (2.6)$$

Consequently, $\phi^\dagger \phi$ remains invariant while $\partial_\mu \phi^\dagger \partial^\mu \phi$ does not.

To resolve this difficulty we introduce a covariant derivative

$D_\mu \phi(x)$ which transforms like $\phi(x)$:

$$D_\mu \phi(x) \longrightarrow D'_\mu \phi'(x) = U(\theta) D_\mu \phi(x). \quad (2.7)$$

Thus, terms of the form $(D_\mu \phi)^\dagger (D^\mu \phi)$ are invariant and so they replace the role $\partial_\mu \phi^\dagger \partial^\mu \phi$ had in the global theory.

The covariant derivative $D_\mu \phi(x)$ is defined by introducing a vector field³ $A_\mu^a(x)$ for each dimension of the Lie algebra,

$$D_\mu \phi(x) \equiv \left(\partial_\mu - ig L^a A_\mu^a(x) \right) \phi(x), \quad (2.8)$$

³We shall use the terms vector field and gauge field interchangeably.

where the coupling constant g is arbitrary.

In order to satisfy (2.7) $A_\mu^a(x)$ must transform in the following manner:⁴

$$\vec{A}_\mu \cdot \vec{L} \longrightarrow \vec{A}'_\mu \cdot \vec{L} = U(\theta) \vec{A}_\mu \cdot \vec{L} U^{-1}(\theta) - \frac{i}{g} (\partial_\mu U(\theta)) U^{-1}(\theta) \quad (2.9)$$

where $\vec{A}_\mu \cdot \vec{L} \equiv A_\mu^a L^a$. (2.10)

Now we must add a kinetic energy term \mathcal{L}_0 which contains only the fields A_μ^a and their derivatives. We shall propose that \mathcal{L}_0 be constructed out of tensors $F_{\mu\nu}^a$ according to

$$\mathcal{L}_0 = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a}, \quad (2.11)$$

such that it is gauge invariant. We define

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g C_{abc} A_\mu^b A_\nu^c, \quad (2.12)$$

where C_{abc} are the structure constants of the Lie group G .⁵

⁴The transformation rule appears to depend upon the particular representation of G , but it only depends upon the commutators $[L^a, L^b]$ whose form is independent of representation.

⁵The generators of G satisfy $[T^a, T^b] = i C_{abc} T^c$ as do any representation of the generators, $[L^a, L^b] = i C_{abc} L^c$. This set of commutation relations defines the Lie algebra of G .

A convenient notation employs the regular representation matrices

$$(T^a)_{bc} \equiv -i C_{abc}, \quad (2.13)$$

where we define

$$A_\mu(x) \equiv A_\mu^a(x) T^a, \quad (2.14)$$

and
$$F_{\mu\nu} \equiv F_{\mu\nu}^a T^a. \quad (2.15)$$

Our Yang-Mills⁶ tensor can now be defined by,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig [A_\mu, A_\nu], \quad (2.16)$$

and transforms as,

$$F_{\mu\nu} \longrightarrow F'_{\mu\nu} = U_R(\theta) F_{\mu\nu} U_R^{-1}(\theta), \quad (2.17)$$

where $U_R(\theta)$ is the transformation matrix under the regular representation,

$$U_R(\theta) \equiv \text{EXP}[-i\vec{\theta}(\alpha) \cdot \vec{T}]. \quad (2.18)$$

⁶Yang and Mills (1954) first studied local gauge invariance under non-Abelian groups, for the case of isotopic spin, SU(2).

Using this notation we may express \mathcal{L}_0 by,

$$\mathcal{L}_0 = -\frac{1}{4N} \text{Tr} (F_{\mu\nu} F^{\mu\nu}) \quad (2.19)$$

where $\text{Tr}(T^a T^b) = N \delta_{ab}$.⁷ (2.20)

Gauge invariance is now obvious because under gauge transformations,

$$F_{\mu\nu} F^{\mu\nu} \longrightarrow F'_{\mu\nu} F'^{\mu\nu} = U_r(\theta) F_{\mu\nu} F^{\mu\nu} U_r^{-1}(\theta) \quad (2.21)$$

and

$$\text{Tr} (F_{\mu\nu} F^{\mu\nu}) = \text{Tr} (F'_{\mu\nu} F'^{\mu\nu}). \quad (2.22)$$

Gathering everything together, we may write down the form of the Lagrangian density for a Yang-Mills theory invariant under a gauge group G,

$$\mathcal{L} = -\frac{1}{4N} \text{Tr} (F_{\mu\nu} F^{\mu\nu}) + \frac{1}{2} (D_\mu \phi)^\dagger (D^\mu \phi) - V(\phi) \quad (2.23)$$

where $V(\phi)$ is a gauge invariant potential term.

The form of the potential chosen is extremely important because it partially determines the configuration of the fields $\phi^a(x)$ in the vacuum and provides the cause for

⁷This property is not obvious, but for the groups we will consider it will be true. For those interested in the gory details, see Gilmore (1974).

spontaneous symmetry breaking and the Higgs mechanism⁸.

⁸For those unfamiliar with these phenomena, a crude explanation is provided in appendix A.

Chapter 3

Monopoles, Spherical Symmetry and Point Monopoles

Recently there has been several papers that generalize the 't Hooft-Polyakov monopole solution of an $SU(2)$ gauge theory ('t Hooft (1974), Polyakov (1974) to larger gauge groups (Bais and Primack (1977), Michel and O'Raighfeartaigh (1977) Horvath and Palla (1976), Wilkinson and Goldhaber (1977)). I will show a crude method for producing an ansatz that generates static, spherically-symmetric monopole solutions in a theory of a general compact semisimple¹ gauge group G . The approach used is based on Bais and Primack (1977) and Wilkinson and Goldhaber (1977) because they use the appealing idea of spherical symmetry.

The original monopole solution of 't Hooft and Polyakov has the distinctive property that the fields are invariant under combined spatial and isospin rotations, we shall call this property spherical symmetry. This property allows one to reduce the equations of motion to ordinary differential equations in the radial variable. By considering distinct embeddings of $SU(2)$ in a compact semisimple group G we can generalize this notion.

To obtain a concrete definition of spherical symmetry consider first the effects of a rotation in Euclidean

¹Gilmore (1974) provides exhaustive definitions of these terms, however for the purposes of this thesis we may assume these conditions to hold, since we consider only $SU(2)$ and $SU(3)$.

three-space². A position will transform as a vector,

$$\underline{x} \rightarrow \underline{x}' = R(\underline{x}) \approx \underline{x} + \delta\theta \times \underline{x}, \quad (3.1)$$

while a scalar field, such as the Higgs field will transform as,

$$\phi'(\underline{x}') = \phi(\underline{x}), \quad (3.2)$$

and the gauge field $A_\mu(x)$ will transform as a vector field,

$$\underline{A}'(\underline{x}') = R(\underline{A}(\underline{x})). \quad (3.3)$$

Defining the change in a field by,

$$\delta\psi(\underline{x}) \equiv \psi'(\underline{x}) - \psi(\underline{x}), \quad (3.4)$$

we obtain for an infinitesimal rotation $\delta\theta$,

$$\delta\phi(\underline{x}) = -\delta\theta \cdot (\underline{x} \times \nabla\phi(\underline{x})) \quad (3.5)$$

$$\delta A_i(\underline{x}) = -\delta\theta \cdot (\underline{x} \times \nabla A_i(\underline{x})) + (\delta\theta \times \underline{A}(\underline{x}))_i. \quad (3.6)$$

²We will normally call such operations space rotations which are a subgroup of the Lorentz transformations whereas the isospin rotations form an SU(2) subgroup of the gauge transformations.

In order to conform with the literature³ we shall use the rather loose notation,

$$(\underline{x} \times \nabla \psi(\underline{x}))_i \equiv [iL^i, \psi(\underline{x})], \quad (3.7)$$

where
$$\underline{L} \equiv \underline{x} \times (-i\nabla). \quad (3.8)$$

So we may write,

$$\delta\phi(\underline{x}) = -i \delta\theta^i [L^i, \phi(\underline{x})], \quad (3.9)$$

and
$$\delta A_j(\underline{x}) = -i \delta\theta^i [L^i, A_j(\underline{x})] + \epsilon_{jik} \delta\theta^i A_k(\underline{x}) \quad (3.10)$$

Now consider the generators of an SU(2) subgroup of our general group G, which we will denote by $\{T^i\}$ $i = 1, 2, 3$. An isospin rotation on the Higgs field can be written,

$$\Phi \longrightarrow \Phi' = \text{EXP}[-i\vec{\theta} \cdot \vec{T}] \Phi \text{ EXP}[+i\vec{\theta} \cdot \vec{T}], \quad (3.11)$$

or
$$\Phi'(\underline{x}) = U(\theta) \Phi(\underline{x}) U^{-1}(\theta). \quad (3.12)$$

³Bais and Primack (1977), Wilkinson and Golhaber (1977) and many other authors have adopted this notation.

⁴For notational simplicity we shall use the regular representation of the Higgs field, defined by $\Phi \equiv \phi^a T^a$, $a = 1, 2, \dots, \dim G$. It is not hard to show that under gauge transformations Φ transforms as in (3.11) if one considers infinitesimal transformations.

where
$$\vec{\theta} \cdot \vec{T} \equiv \theta^1 T^1 + \theta^2 T^2 + \theta^3 T^3, \quad (3.13)$$

$$U(\theta) = \text{EXP} [-i \vec{\theta} \cdot \vec{T}]. \quad (3.14)$$

Under the same transformation the gauge field becomes,

$$A_\mu \rightarrow A'_\mu = U(\theta) A_\mu U^{-1}(\theta) - \frac{i}{g} (\partial_\mu U(\theta)) U^{-1}(\theta). \quad (3.15)$$

we will consider only global transformations so that the second term in (3.15) vanishes. Under an infinitesimal isospin rotation we see,

$$\delta \Phi = -i \delta \theta^i [\tau^i, \Phi], \quad (3.16)$$

and
$$\delta A_j = -i \delta \theta^i [\tau^i, A_j]. \quad (3.17)$$

Now we shall define fields as spherically symmetric if they are invariant under a combined space and isospin rotation. By considering (3.9), (3.10), (3.16) and (3.17) we define spherically symmetric fields as those that satisfy,

$$[L^i + T^i, A_j] = i \epsilon_{ijk} A_k, \quad (3.18)$$

$$[L^i + T^i, \Phi] = 0. \quad (3.19)$$

To construct spherically symmetric monopole solutions, one uses the most general ansatz satisfying (3.18) and (3.19)

for each embedding of an SU(2) subgroup in G. The specific form of these ansatz must be such that they conform to the boundary conditions of the theory.

As the radius becomes asymptotically large the Higgs field should approach its lowest possible energy configuration, the vacuum. These conditions can be expressed by,

$$D^\mu \Phi \xrightarrow{r \rightarrow \infty} 0, \quad (3.20)$$

$$\frac{\delta V}{\delta \Phi} \xrightarrow{r \rightarrow \infty} 0. \quad (3.21)$$

A configuration of fields which satisfies these conditions everywhere in space except at the origin is known as point monopole. Obviously, a knowledge of all the possible point monopoles of a theory would be useful in determining the boundary conditions of our fields. In fact, Wilkinson and Goldhaber (1977) have shown a method for obtaining all the point monopoles of a theory using group theory techniques. They use these point monopoles as the starting point for defining the ansatz for a general finite-energy, static and spherically symmetric monopole solutions.

In this thesis we use the opposite approach; where we start with the most general spherically symmetric ansatz and then force it to conform to a monopole solution as the radius goes to infinity.

Chapter 4

Dyons in an SU(2) Gauge Theory

The dyon is a particle with both magnetic and electric charge and has been discussed previously by Schwinger (1969) and Zwanzinger (1968). Julia and Zee (1974) have extended the arguments of 't Hooft (1974) and Polyakov (1974) by showing the possibility of constructing classical solutions having both magnetic and electric charges.

We shall start with a Yang-Mills theory (under an SU(2) gauge group) with symmetry breaking. Then, we choose a spherically symmetric ansatz for our fields thereby reducing the equations of motion to ordinary differential equations in the radial variable. A suitable definition of the electromagnetic field is given so that the electric and magnetic charges become clearly defined. The 't Hooft-Polyakov monopole may then be identified as the special case of such a dyon with zero electric charge.

The Lagrangian density for an SU(2) gauge theory with a triplet of scalar fields is given by,

$$\mathcal{L} = -\frac{1}{2} \text{Tr} (F_{\mu\nu} F^{\mu\nu}) + \text{Tr} (D_\mu \Phi D^\mu \Phi) - V(\Phi), \quad (4.1)$$

where

$$\Phi \equiv \phi^a T^a, \quad (4.2)$$

$$A_\mu \equiv A_\mu^a T^a, \quad (4.3)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ie [A_\mu, A_\nu], \quad (4.4)$$

and
$$T^a \equiv \frac{1}{2} \tau^a, \quad (4.5)$$

$a = 1, 2, 3.$

The matrices τ^1, τ^2, τ^3 are the 2x2 Pauli spin matrices so that the representation matrices T^a have the following properties:

$$[T^a, T^b] = i\epsilon_{abc} T^c, \quad (4.6)$$

(i.e. they form a representation of the generators of SU(2))

and
$$\text{Tr} (T^a T^b) = \frac{1}{2} \delta_{ab}. \quad (4.7)$$

Further, one may show that the covariant derivative (2.8) may be expressed as

$$D_\mu \Phi = \partial_\mu \Phi - ie [A_\mu, \Phi]. \quad (4.8)$$

From the discussion of appendix A we choose a symmetry breaking potential of the form ($|\phi| = v$ minimizes V),

$$\begin{aligned} V(\Phi) &= \frac{\lambda}{4} (2 \text{Tr} \Phi^2 - v^2)^2 \\ &= \frac{\lambda}{4} (\phi^a \phi^a - v^2)^2. \end{aligned} \quad (4.9)$$

At this point we propose a spherically symmetric ansatz for the Higgs field Φ and the gauge field A_μ ;

$$A_i(\underline{x}) = \epsilon_{ija} n^j \frac{(1-H(r))}{er} T^a, \quad (4.10)$$

$$\Phi(\underline{x}) = \frac{F(r)}{er} n^a T^a, \quad (4.11)$$

$$i = 1, 2, 3 \quad a = 1, 2, 3; \quad n^i \equiv \frac{x^i}{|\underline{x}|},$$

Under rotation A^0 has the same Lorentz transformation properties as Φ so,

$$A_0(\underline{x}) = \frac{J(r)}{er} n^a T^a. \quad (4.12)$$

It is a straightforward computation to show that $A_i(\underline{x})$ satisfies (3.18) while $\Phi(\underline{x})$ and $A_0(\underline{x})$ satisfy (3.19), meaning that the fields are indeed spherically symmetric. In fact, Weinberg and Guth (1976) prove that this is the only finite energy and spherically symmetric ansatz in an SU(2) gauge theory.

It should be noted that in (4.10), (4.11) and (4.12) the angular dependence of the fields is displayed explicitly while the unknowns are functions of the radius only. Additionally there is no time dependence because we are assuming static solutions.

Substituting (4.10), (4.11) and (4.12) into (4.1), then integrating over all space, we obtain for the Lagrangian

$$L = -\frac{1}{\alpha} \int_0^{\infty} dr \left(\left(\frac{dH}{dr} \right)^2 + \frac{(H^2-1)^2}{2r^2} - \frac{J^2 H^2}{r^2} - \frac{\left(r \frac{dJ}{dr} - J \right)^2}{2r^2} \right. \\ \left. + \frac{F^2 H^2}{r^2} + \frac{\left(r \frac{dF}{dr} - F \right)^2}{2r^2} + \frac{\beta}{4r^2} \left(F^2 - e^2 v^2 r^2 \right)^2 \right), \quad (4.13)$$

where $\beta \equiv \frac{\lambda}{e^2}$, $\alpha \equiv \frac{e^2}{4\pi} \simeq \frac{1}{137}$. (4.14)

The equations of motion result from extremizing this integral. However it is advantageous to convert to dimensionless variables first. Let,

$$z \equiv evr, \quad (4.15)$$

so that $L = -\frac{M_w}{\alpha} l(\beta, \eta)$, (4.16)

where $M_w = ev^1$, (4.17)

and 2 $l(\beta, \eta) = \int_0^{\infty} dz \left(H'^2 + \frac{(H^2-1)^2}{2z^2} - \frac{J^2 H^2}{z^2} \right. \\ \left. - \frac{(\eta J' - J)^2}{2z^2} + \frac{F^2 H^2}{z^2} + \frac{(\eta F' - F)^2}{2z^2} + \frac{\beta}{4z^2} \left(F^2 - z^2 \right)^2 \right), \quad (4.18)$

¹ M_w is the mass of the vector boson obtained by way of the Higgs mechanism. For a crude explanation see appendix A.

² η is a parameter that determines the asymptotic behavior of $J(r)$. See appendix B for details.

where $f' \equiv \frac{df}{dz}$.

Extremizing \mathcal{L} by use of the Euler-Lagrange equations gives the following ordinary differential equations in z ,

$$z^2 H'' = H(H^2 - J^2 + F^2 - 1), \quad (4.19)$$

$$z^2 J'' = 2JH^2, \quad (4.20)$$

$$z^2 F'' = 2FH^2 + \beta F(F^2 - z^2). \quad (4.21)$$

The boundary conditions on H, J and F are discussed in detail in appendix B. The criteria we use for deciding these conditions are that the energy be finite and that the fields approach the point monopole solutions in the asymptotic limit.

I shall simply state the result here:

$$\begin{aligned} H(0) &= 1, & H(\infty) &= 0, \\ J(0) &= 0, & J \underset{z \rightarrow \infty}{\sim} \eta z, & (0 \leq \eta < 1), \\ F(0) &= 0, & F \underset{z \rightarrow \infty}{\sim} z. & \end{aligned} \quad (4.22)$$

The gauge invariant Hamiltonian density is given by,

$$\mathcal{H} = F_{0i}^a F_{0i}^a + D_0 \phi^a D_0 \phi^a - \mathcal{L}. \quad (4.23)$$

³This formula is not obvious but it does reduce to our normal definition in the case of an Abelian theory. For further details see Julia and Zee (1974), Prasad and Sommerfield (1975) or Coleman (1975).

Since all time derivatives vanish, the total energy is interpreted as the mass of the dyon. Making the appropriate substitutions we obtain:

$$M = \int \mathcal{H}(\underline{x}) d^3x = \frac{M_w}{\alpha} C(\beta, \eta), \quad (4.24)$$

where

$$C(\beta, \eta) = \int_0^\infty dz \left(H'^2 + \frac{(H^2-1)^2}{2z^2} + \frac{H^2(J^2+F^2)}{z^2} + \frac{(zJ'-J)^2}{2z^2} + \frac{(zF'-F)^2}{2z^2} + \frac{\beta}{4z^2} (F^2-z^2)^2 \right). \quad (4.25)$$

Having solved (4.19), (4.20) and (4.21) this can be calculated.

In order to determine the electric and magnetic charge of the object we have constructed, we must identify the electromagnetic fields $\mathcal{F}^{\mu\nu}$. Any choice of $\mathcal{F}^{\mu\nu}$ must satisfy the following conditions:

1. $\mathcal{F}^{\mu\nu}$ transforms as a 2-contravariant tensor.
2. $\mathcal{F}^{\mu\nu}$ must be gauge invariant.
3. If we make a gauge transformation on the Higgs field Φ , in some region, such that it points only in one direction in isospin space, T^3 say, the fields should reduce to the usual definition,

$$\mathcal{F}^{\mu\nu} \longrightarrow \partial_\mu A_\nu^3 - \partial_\nu A_\mu^3. \quad (4.26)$$

The last condition requires an explanation. Ordinary electrodynamics is a gauge theory, a special case of the

general formalism presented in the second chapter. The symmetry group of electrodynamics is $G=U(1)$. Under the $U(1)$ group, a complex field will transform as,

$$\phi(x) \rightarrow \phi'(x) = e^{-iq\theta(x)} \phi(x), \quad (4.27)$$

while a real (Hermitian) field is invariant,

$$\Phi(x) = \Phi'(x) = e^{-iq\theta(x)} \Phi(x) e^{+iq\theta(x)}. \quad (4.28)$$

The corresponding transformation of the single gauge field is given by,

$$A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) - \frac{1}{e} \partial_\mu \theta(x), \quad (4.29)$$

which should look familiar to those acquainted with the covariant formulation of electromagnetism. Since all $U(1)$ transformation commute (i.e. $U(1)$ is an Abelian group) the structure constants C_{ijk} vanish and thus the Yang-Mills tensor (2.12) becomes the ordinary gauge invariant definition of the electromagnetic field tensor,

$$F_{\mu\nu} = \mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (4.30)$$

In this light, the third criterion for the generalized definition of $\mathcal{F}_{\mu\nu}$ should make more sense. The gauge transformation in some region will pictorially do something like this:

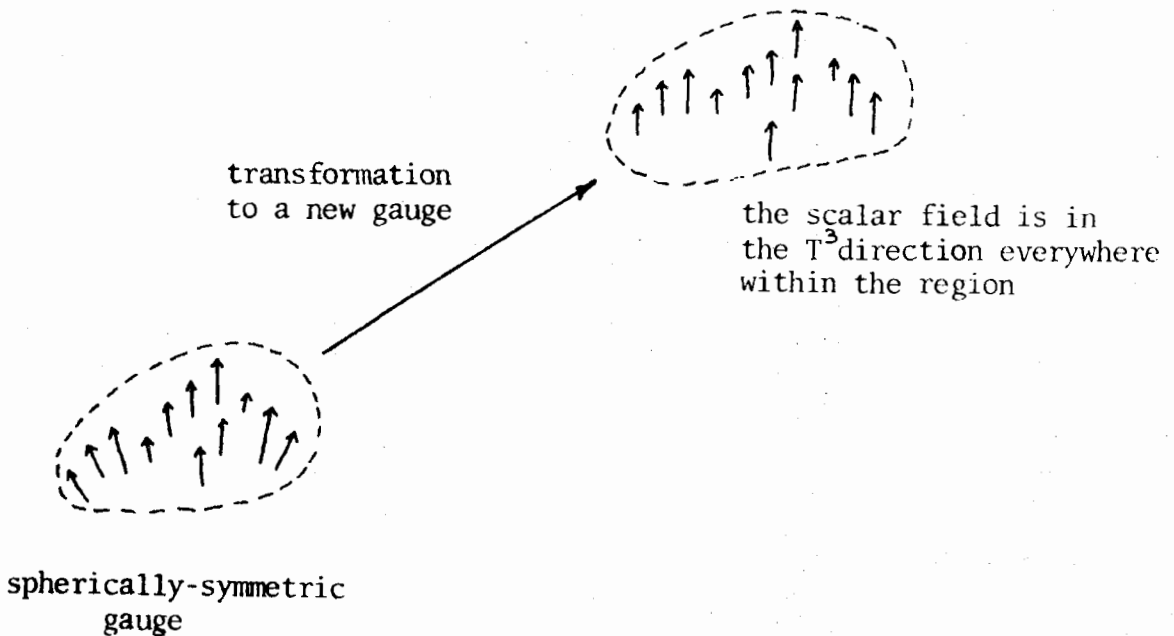


Fig.4.1 A gauge transformation to a U(1) symmetry

In this new gauge the scalar fields may be expressed as,

$$\Phi(x) = \phi(x) T^3, \tag{4.31}$$

and are invariant under the U(1) subgroup of transformations,

$$U(\theta) = \text{EXP}[-i\theta(x) T^3]. \tag{4.32}$$

If we consider only these transformations the gauge field $A_\mu^3(x)$ has the same transformation properties as $A_\mu(x)$ did

in (4.29), and we have the corresponding gauge invariant tensor

$$\mathcal{F}_{\mu\nu} = \partial_\mu A_\nu^3(x) - \partial_\nu A_\mu^3(x). \quad (4.33)$$

Finally, the vacuum of this theory has a U(1) symmetry⁴. Thus the Higgs mechanism leaves one vector field massless which we identify as the electromagnetic field⁵. The gauge transformation shown in Fig. 4.1 makes the U(1) symmetry and the corresponding massless field obvious (T^3 and A^3), thus we demand the third criterion.

The electromagnetic field tensor suggested by 't Hooft (1974) is,

$$\mathcal{F}_{\mu\nu} = \hat{\phi}^a F_{\mu\nu}^a - \frac{1}{e} \epsilon_{abc} \hat{\phi}^a D_\mu \hat{\phi}^b D_\nu \hat{\phi}^c, \quad (4.34)$$

where
$$\hat{\phi}^a = \frac{\phi^a}{\sqrt{\phi^b \phi^b}}, \quad (4.35)$$

or in matrix language,

$$\mathcal{F}_{\mu\nu} = 2 \text{Tr}(\hat{\Phi} F_{\mu\nu}) + \frac{2i}{e} \text{Tr}(\hat{\Phi} [D_\mu \hat{\Phi}, D_\nu \hat{\Phi}]). \quad (4.36)$$

⁴This U(1) symmetry is made obvious in Chapter 5.

⁵This Higgs mechanism is discussed in appendix A.

This definition indeed satisfies the three criteria. Upon substituting the ansatz (4.10), (4.11) and (4.12) into (4.34) we obtain

$$\mathcal{F}_{ij} = -\epsilon_{ijk} \frac{n^k}{er^2}, \quad (4.37)$$

$$\mathcal{F}_{oi} = -n^i \frac{d}{dr} \left(\frac{\mathcal{J}}{er} \right). \quad (4.38)$$

The corresponding electric and magnetic fields are given by,

$$\mathbf{B}_i = -\frac{1}{2} \epsilon_{ijk} \mathcal{F}_{jk} = \frac{n^i}{er^2}, \quad (4.39)$$

$$\mathbf{E}_i = \mathcal{F}_{oi} = -n^i \frac{d}{dr} \left(\frac{\mathcal{J}}{er} \right). \quad (4.40)$$

It appears we have a point monopole of strength $(+\frac{1}{e})$ whereas the electric charge is extended. The magnetic and electric charges are obtained using Gauss' law:

$$Q_m = \oint_{S \rightarrow \infty} \underline{\mathbf{B}} \cdot \hat{n} da = \frac{4\pi}{e}, \quad (4.41)$$

$$\begin{aligned} Q_e &= \oint_{S \rightarrow \infty} \underline{\mathbf{E}} \cdot \hat{n} da = \int (\nabla \cdot \underline{\mathbf{E}}) d^3x \\ &= - \int d^3x \partial_i \left[n^i \frac{d}{dr} \left(\frac{\mathcal{J}}{er} \right) \right], \end{aligned}$$

from equation (4.20) and integrating by parts we obtain,

$$Q_e = -\frac{8\pi}{e} \zeta(\beta, \eta), \quad (4.42)$$

where

$$\zeta(\beta, \eta) = \int_0^\infty dz \frac{J(z) H^2(z)}{z}. \quad (4.43)$$

The result of this thesis is the solution F, J and H and the calculation of $C(\beta, \eta)$ and $\zeta(\beta, \eta)$.

The magnetic monopole of 't Hooft and Polyakov correspond to a special case of the dyon where $A_0(x) = 0$, which corresponds to $\eta = 0$.

Chapter 5

Magnetic Monopoles in an SU(3) Gauge Theory

In this chapter we shall deal only with the essential ideas needed to obtain the differential equations of the fields. For further details I suggest the reader to consult Sinha (1976) or Corrigan, Olive, Fairlie and Nuyts (1976)¹ whose papers provide the basis for this chapter.

There are two fundamental differences between SU(2) and SU(3) with respect to the classification of monopole solutions. First, octet vectors of equal length cannot, in general, be SU(3) rotated into each other whereas triplet vectors of equal length can always be SU(2) rotated into each other. Second, there are two distinct ways of embedding SU(2) in SU(3).

Weinberg and Guth (1976) proved that the 't Hooft-Polyakov ansatz is the only spherically symmetric monopole of finite energy in an SU(2) gauge theory, so the classification of these monopoles is a trivial matter. However, SU(3) is a more complicated animal having more than one type of monopole solution.

We shall represent our fields by 3x3 traceless Hermitian matrices:

$$\Phi = \frac{1}{2} \lambda_a \phi^a, \quad (5.1)$$

$$A^\mu = \frac{1}{2} \lambda_a A^{\mu a}, \quad (5.2)$$

¹Hereafter we shall use the abbreviation Corrigan et al.

$$F_{\mu\nu} = \frac{1}{2} \lambda_a F_{\mu\nu}^a = \partial_\mu A_\nu - \partial_\nu A_\mu - ie[A_\mu, A_\nu], \quad (5.3)$$

$$D_\mu \Phi = \partial_\mu \Phi - ie[A_\mu, \Phi], \quad (5.4)$$

$a = 1, 2, \dots, 8.$

The λ -matrices form a convenient representation of the generators of SU(3). Following the convention of Gell-Mann² these matrices have the following properties:

$$\text{Tr}(\lambda_a \lambda_b) = 2 \delta_{ab}, \quad (5.5)$$

$$[\lambda_a, \lambda_b] = 2if_{abc} \lambda_c, \quad (5.6a)$$

$$\{\lambda_a, \lambda_b\} = \frac{4}{3} \delta_{ab} 1 + d_{abc} \lambda_c, \quad (5.6b)$$

(anticommutator)

where the structure constants f_{abc} are real and totally anti-symmetric while d_{abc} are real and totally symmetric.

An SU(3) rotation is a unitary transformation,

$$\Phi(x) \rightarrow \Phi'(x) = U(\theta) \Phi(x) U^\dagger(\theta), \quad (5.7)$$

$$U^\dagger(\theta) U(\theta) = 1, \quad \det U(\theta) = 1, \quad (5.8)$$

and so the eigenvalues of Φ are invariant. Thus, Φ 's with different eigenvalues cannot be SU(3) rotated into each other,

²The eight matrices and their properties are listed in appendix D. The source used was Gell-Mann and Ne'eman (1964).

they lie on distinct SU(3) orbits. Hence, one classification scheme of monopole solutions would be to distinguish the eigenvalues of the Higgs field Φ in the vacuum. Further, these eigenvalues identify the symmetry H of the vacuum.

It is the scalar potential $V(\Phi)$ that determines the eigenvalues of Φ in the vacuum. Consider the most general renormalizable gauge invariant potential (having at most a quartic term in Φ),

$$V(\Phi) = -\mu^2 \text{Tr } \Phi^2 + \lambda (\text{Tr } \Phi^2)^2 + \alpha_3 \text{Tr } \Phi^3 + \text{constant}. \quad (5.9)$$

Since Φ is a traceless Hermitian matrix, it can be diagonalized by a unitary transformation such that

$$\Phi \longrightarrow \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & -(\epsilon_1 + \epsilon_2) & 0 \\ 0 & 0 & \epsilon_2 \end{pmatrix} \quad (5.10)$$

ϵ_1, ϵ_2 are real,

and we obtain

$$\text{Tr } \Phi^2 = 2(\epsilon_1^2 + \epsilon_2^2 + \epsilon_1 \epsilon_2), \quad (5.11)$$

and
$$\text{Tr } \Phi^3 = -3\epsilon_1 \epsilon_2 (\epsilon_1 + \epsilon_2). \quad (5.12)$$

Substituting (5.11) and (5.12) into the potential (5.9) we obtain,

$$V = -2\mu^2(\epsilon_1^2 + \epsilon_2^2 + \epsilon_1\epsilon_2) + 4\lambda(\epsilon_1^2 + \epsilon_2^2 + \epsilon_1\epsilon_2)^2 - 3\alpha_3\epsilon_1\epsilon_2(\epsilon_1 + \epsilon_2) + \text{constant} . \quad (5.13)$$

The vacuum (or point monopole values of the eigenvalues are such that V is minimized,

$$\frac{\partial V}{\partial \epsilon_1} = \frac{\partial V}{\partial \epsilon_2} = 0, \quad (5.14)$$

which yields,

$$0 = (2\epsilon_1 + \epsilon_2)(-2\mu^2 + 8\lambda(\epsilon_1^2 + \epsilon_1\epsilon_2 + \epsilon_2^2) - 3\alpha_3\epsilon_2), \quad (5.15)$$

and

$$0 = (2\epsilon_2 + \epsilon_1)(-2\mu^2 + 8\lambda(\epsilon_1^2 + \epsilon_1\epsilon_2 + \epsilon_2^2) - 3\alpha_3\epsilon_1). \quad (5.16)$$

These equations have several solutions, but many are redundant because they correspond to permuting eigenvalues along the diagonal in (5.10), only one of the permutations need be considered. Suppose,

$$2\epsilon_1 + \epsilon_2 = 0, \quad (5.17)$$

then from (5.16),

$$24\lambda\epsilon_1^2 - 3\alpha_3\epsilon_1 - 2\mu^2 = 0, \quad (5.18)$$

which has the solutions,

$$\epsilon_1 = \frac{\alpha_3 \pm \sqrt{\alpha_3^2 + \frac{64}{3} \mu^2 \lambda}}{16\lambda}, \quad (5.19)$$

where we choose the sign that minimizes V . So the vacuum value of the Higgs field has the diagonal form,

$$\Phi_{\text{vac}} = \epsilon_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad (5.20)$$

which is a multiple of the eighth Gell-Mann matrix,

$$\lambda_8 = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & -\frac{2}{\sqrt{3}} \end{pmatrix}. \quad (5.21)$$

In the limit of the cubic term vanishing ($\alpha_3 \rightarrow 0$),

$$\epsilon_1 \xrightarrow{\alpha_3 \rightarrow 0} \pm \left(\frac{\mu}{\sqrt{\lambda}} \right) \cdot \frac{1}{2\sqrt{3}} = \pm \frac{v}{2\sqrt{3}}, \quad (5.22)$$

and
$$\Phi_{\text{vac}} = \pm \frac{v}{2} \lambda_8, \quad (5.23)$$

where
$$v \equiv \mu/\sqrt{\lambda}. \quad (5.24)$$

(We shall conventionally choose the plus sign.) Using the general potential (5.9) we obtain a vacuum value of the Higgs

field that is called λ_8 -like, for obvious reasons. We may now determine the symmetry of such a vacuum. From appendix D we see that λ_8 commutes only with $\lambda_1, \lambda_2, \lambda_3$ and λ_8 . Thus Φ_{vac} is invariant under transformations involving an exponentiation of these four matrices. From the commutation relations,

$$[\lambda_8, \lambda_i] = 0, \quad (5.25)$$

$$\left[\frac{\lambda_i}{2}, \frac{\lambda_j}{2} \right] = i \epsilon_{ijk} \frac{\lambda_k}{2}, \quad (5.26)$$

$$i, j, k = 1, 2, 3$$

one concludes λ_8 generates a U(1) subgroup of SU(3) while $(\lambda_{1/2}, \lambda_{2/2}, \lambda_{3/2})$ generates an SU(2) subgroup. We say that the unbroken symmetry group or the little group of Φ is SU(2)XU(1) (i.e. isomorphic with U(2)),

$$H = U(2) \simeq SU(2) \times U(1). \quad (5.27)$$

Corrigan et al. use the general potential (5.9) and thus the vacuum in their theory has a U(2) symmetry. However, Sinha chose a less general potential of the form³

$$V(\Phi) = \frac{\lambda}{4} (2 \text{Tr} \Phi^2 - v^2)^2 \quad (5.28)$$

³This is a special case of the general potential (5.9) where the cubic term is removed, i.e. $\alpha_3 = 0$.

The vacuum value of the Higgs field has the diagonal form,

$$\Phi_{\text{vac}} = \pm \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & (-\frac{\epsilon}{2} + \frac{1}{2}\sqrt{v^2 - 3\epsilon^2}) & 0 \\ 0 & 0 & (-\frac{\epsilon}{2} - \frac{1}{2}\sqrt{v^2 - 3\epsilon^2}) \end{pmatrix}, \quad (5.29)$$

$$0 \leq \epsilon \leq \frac{v}{\sqrt{3}},$$

plus permutations along the diagonal. A neat way of expressing (5.29) is,

$$\Phi_{\text{vac}} = \frac{v}{2} (a\lambda_3 + b\lambda_8), \quad (5.30)$$

where $a^2 + b^2 = 1.$ (5.31)

For most choices of a and b , Φ_{vac} of (5.30) will commute only with λ_3 and λ_8 and thus the unbroken symmetry group is generally⁴,

$$H = U(1) \times U(1).$$

It is of interest to note that for $SU(2)$ the vacuum has the form,

$$\Phi_{\text{vac}} = \begin{pmatrix} \epsilon & 0 \\ 0 & -\epsilon \end{pmatrix} \quad (5.32)$$

⁴There are choices of a and b that make Φ_{vac} λ_8 -like and $H = U(2)$; eg. $(a,b) = (0,1), (\frac{\sqrt{3}}{2}, -\frac{1}{2})$, but these are exceptions to the rule.

which commutes only with the single SU(2) generator

$$T^3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (5.33)$$

so the vacuum of an SU(2) theory has a U(1) symmetry.

The main objective of this thesis is to compute and compare the monopole solutions of SU(2) and SU(3). In order to do this we must use an equivalent Higgs potential in all cases, namely

$$V(\Phi) = \frac{\lambda}{4} (\phi^a \phi^a - v^2)^2 = \frac{\lambda}{4} (2 \text{Tr} \Phi^2 - v^2)^2 \quad (5.35)$$

(SU(2): $a = 1, 2, 3$: SU(3): $a = 1, 2, \dots, 8$).

As we have shown, this potential is consistent with two types of vacuum symmetries in SU(3). Consequently, we must be clear in every case as to the properties of the vacuum because it is important in the identification of the electromagnetic field.

It has been shown by Bais and Primack (1977) that there are two distinct ways of embedding an SU(2) subgroup in SU(3). The generators for each of these embeddings are given by:

$$\begin{aligned} \text{SU}(2): \quad (T^1, T^2, T^3) &= \left(\frac{1}{2} \lambda_1, \frac{1}{2} \lambda_2, \frac{1}{2} \lambda_3 \right) \\ \text{SO}(3): \quad (T^1, T^2, T^3) &= (\lambda_7, -\lambda_5, \lambda_2). \end{aligned}$$

As the label indicates, the second case is really an SO(3)

embedding in $SU(3)$ ⁵. However, no harm is done, because the generators obey the commutation rules for $SU(2)$.

At this stage we are ready to construct a spherically symmetric ansatz for the scalar fields. There are eight independent scalar fields ϕ^a which transform as an octet representation of $SU(3)$. The question we must answer is how do these fields transform under the embedded subgroups?⁶ The fundamental representation of $SU(3)$ is a three-by-three matrix acting on a three dimensional vector space,

$$\begin{pmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} q'_1 \\ q'_2 \\ q'_3 \end{pmatrix} . \quad (5.36)$$

The action of one of the subgroups on such a vector space can be written

$$\text{EXP}[-i\vec{\theta}\cdot\vec{\lambda}] \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \equiv U(\theta) \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} . \quad (5.37)$$

⁵Properly speaking $SO(3)$ is a $2 \rightarrow 1$ homomorphic image of $SU(2)$ (Gilmore (1974)).

⁶The argument presented here is my "hand-wavy" version of Corrigan et al. (1976). More sophisticated group theoretical approaches are presented in Bais and Primack (1977) and Wilkinson and Goldhaber (1977).

For the SU(2) embedding the transformation takes the form,⁷

$$U_{SU(2)}(\theta) \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} q'_1 \\ q'_2 \\ q_3 \end{pmatrix}, \quad (5.38)$$

where $\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$ transforms as a two component isospinor under this SU(2) rotation while (q_3) is unaffected. The three component vector may be regarded as the sum of an isospin- $\frac{1}{2}$ and isospin-0 state, symbolically

$$(3) = (2) + (1). \quad (5.39)$$

The standard method of constructing an octet representation of SU(3) is well known⁸ to be the direct product of the fundamental representation with the contragradient representation,

$$(3) \otimes (\bar{3}) = (8) \oplus (1). \quad (5.40)$$

But, if we restrict ourselves to the SU(2) subgroup of transformations, (5.40) becomes our familiar Clebsch-Gordan

⁷See the form of $\lambda_1, \lambda_2, \lambda_3$ in appendix D.

⁸For those as ignorant as I was, see Fonda and Ghirardi (1970) for details.

direct product⁹. Now we have,

$$\begin{aligned}
 (3) \otimes (\bar{3}) &= ((2)+(1)) \otimes ((2)+(1)) \\
 &= (2) \otimes (2) \oplus (2) \otimes (1) \oplus (1) \otimes (2) \oplus (1) \otimes (1) \\
 &= (3) \oplus (1) \oplus (2) \oplus (2) \oplus (1),
 \end{aligned}
 \tag{5.41}$$

so we may write for the SU(2) embedding,

$$(8) = (3) \oplus (2) \oplus (2) \oplus (1).
 \tag{5.42}$$

Which means that we may construct the Higgs field Φ with components that transform under the SU(2) subgroup as isospin-1, $\frac{1}{2}, \frac{1}{2}$ and isospin-0 states respectively. All of these components cannot be used because we must require to satisfy the spherical symmetry criteria.

In the fundamental representation the SO(3) subgroup transforms the three-dimensional vector as

$$U_{SO(3)} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} q'_1 \\ q'_2 \\ q'_3 \end{pmatrix},
 \tag{5.43}$$

where (q'_1, q'_2, q'_3) transforms as an isovector (isospin-1 state)

⁹See Gottfried (1966).

under an $SO(3)$ rotation. So in this case our vector decomposes as,

$$(3) = (3). \quad (5.44)$$

The corresponding decomposition for the octet of fields is,

$$(8) = (5) \oplus (3), \quad (5.45)$$

providing a clue as to the form of the ansatz for Φ .

Corrigan et al. use the following ansatz for the Higgs field:

SU(2) Embedding:¹⁰

$$\Phi(\underline{x}) = af(r)\psi_1 + bg(r)\psi_2, \quad (5.46)$$

where $f(r)$ and $g(r)$ are real functions of the radius while a and b are real constants chosen such that,

$$a^2 + b^2 = 1, \quad (5.47)$$

and

$$n^i \equiv x^i/|\underline{x}|,$$

$$\psi_1 \equiv \sum_{i=1}^3 \frac{\lambda_i}{2} n^i, \quad (5.48)$$

$$\psi_2 \equiv \frac{\lambda_8}{2}. \quad (5.49)$$

¹⁰There is no time dependence because we are assuming static solutions.

SO(3) Embedding:

$$\Phi(x) = A(r)\phi_1 + B(r)\phi_2, \quad (5.50)$$

$$A(r), B(r) \in \mathbb{R},$$

where

$$(\phi_1)_{ij} = n^i n^j - \frac{1}{3} \delta_{ij}, \quad (5.51)$$

$$(\phi_2)_{ij} = i \epsilon_{ijk} n^k, \quad (5.52)$$

$$i, j, k = 1, 2, 3.$$

One may check that Φ satisfies the spherical symmetry requirement,

$$[L^i + T^i, \Phi] = 0, \quad (5.53)$$

where it is understood that T^i are the appropriate generators of the SU(2) or SO(3) subgroups of SU(3).

We must now choose an ansatz for the gauge fields¹¹ $A_\mu(x)$ which satisfy the corresponding spherical symmetry requirement,

$$[L^i + T^i, A_j] = i \epsilon_{ijk} A_k. \quad (5.54)$$

¹¹ Corrigan et al. show that these are the most general ansatz for the gauge fields.

I shall attempt to provide a motivation for the choice of the various ansatz used for the scalar and gauge fields. The basic guide is that the transformation properties under simultaneous gauge and space rotations, where the scalar fields should be invariant while the gauge field transforms as a vector. The SU(2) embedding is very simple because the scalar field has components of the form (5.48) and (5.49)

$$\vec{\lambda} \cdot \hat{n}, \quad \lambda_8$$

which are obviously invariant. The gauge field (5.56) has the form

$$\vec{\lambda} \times \hat{n}$$

which will transform in the proper manner.

For the SO(3) embedding the generators $\lambda_1, \lambda_3, \lambda_4, \lambda_6, \lambda_8$ or more appropriately some linear combination transforms as a quadrupole moment tensor under the SO(3) subgroup of SU(3). To make an invariant quantity under simultaneous rotation we simply contract it with a real space quadrupole moment tensor, thus (5.51). The form (5.52) is simply $\hat{n} \cdot \vec{\lambda}$ where $\vec{\lambda} = (\lambda_7, \lambda_5, \lambda_2)$ which is invariant. The gauge fields (5.57) are combinations of contractions and derivatives to give a quantity which transforms as a vector under simultaneous rotations.

SU(2) Embedding:

$$A_\mu(x) = (1-h(r)) B_\mu(x), \quad (5.55)$$

where
$$B_\mu(x) = -\sum_{i=1}^3 \frac{\lambda_i}{2} \epsilon_{\mu ij} \frac{n^j}{er}, \quad (5.56)$$

($\epsilon_{\mu ij}$ is the usual ϵ symbol for $\mu=1,2,3$ and $\epsilon_{0ij} = 0$);

SO(3) Embedding:

$$eA_\mu(x) = i(1-D_1(r))[\phi_1, \partial_\mu \phi_1] - iD_2(r)[\phi_2, \partial_\mu \phi_2]. \quad (5.57)$$

With some algebraic manipulation one may show that these ansatz for the gauge fields do indeed satisfy the criterion of spherical symmetry.

The covariant derivative of the Higgs field (5.4) takes the form ¹²,

SU(2) Embedding:

$$D_k \Phi = \frac{a}{2r} [\lambda_k - n^k (n^j \lambda_j)] h(r) f(r) + a n^k f'(r) \frac{(n^j \lambda_j)}{2} + b \frac{\lambda_3}{2} n^k g'(r), \quad (5.58)$$

SO(3) Embedding:

$$D_k \Phi = (\partial_k \phi_1)(A D_1 + B D_2) + (\partial_k \phi_2)(A D_2 + B D_1) + n^k \phi_1 A' + n^k \phi_2 B'. \quad (5.59)$$

¹²Many of the algebraic manipulations involved in producing these terms are very tedious. For those interested in the details see Corrigan et al. (1976).

The Higgs potential (5.35) takes the form:

SU(2) Embedding:

$$V(\Phi) = \frac{\lambda}{4} (a^2 f(r)^2 + b^2 g^2(r) - v^2)^2, \quad (5.60)$$

SO(3) Embedding:

$$V(\Phi) = \frac{\lambda}{4} \left(\frac{4}{3} A^2(r) + 4B^2(r) - v^2 \right)^2. \quad (5.61)$$

We shall assume all time derivatives vanish and that the gauge is such that $A_0 = 0$ ¹³. In this case the Hamiltonian density is simply the negative of the Lagrangian density,

$$\mathcal{H}(\underline{x}) = -\mathcal{L}(\underline{x}) \quad (5.62)$$

and the equations of motion follow from minimizing the total energy. Applying the various ansätze to the Lagrangian density,

$$\mathcal{L} = -\frac{1}{2} \text{Tr} (F_{\mu\nu} F^{\mu\nu}) + \text{Tr} (D_\mu \Phi D^\mu \Phi) - V(\Phi), \quad (5.63)$$

we obtain:

SU(2) Embedding:

$$\begin{aligned} \mathcal{H}(r) = -\mathcal{L}(r) = & \frac{1}{e^2 r^2} \left[\frac{(h(r)-1)^2}{2r^2} + h'(r)^2 \right] \\ & + \frac{1}{r^2} \left[a^2 f(r)^2 h(r) + \frac{r^2}{2} (a^2 f'(r)^2 + b^2 g'(r)^2) \right] + \frac{\lambda}{4} (a^2 f(r)^2 + b^2 g^2(r) - v^2)^2. \end{aligned} \quad (5.64)$$

¹³ It is always possible to choose such a gauge; see Coleman (1975).

SO(3) Embedding:

$$\begin{aligned} \mathcal{H}(r) = -\mathcal{L}(r) = & \frac{4}{e^2 r^2} \left[D_1'^2(r) + D_2'^2(r) + \frac{(D_1^2(r) + D_2^2(r) - 1)^2 + 12 D_1^2(r) D_2^2(r)}{2r^2} \right] \\ & + \left(\frac{A'(r)^2}{3} + B'(r)^2 \right) \\ & + \frac{4}{r^2} \left[(A(r) D_1(r) + B(r) D_2(r))^2 + (A(r) D_2(r) + B(r) D_1(r))^2 \right] \\ & + \frac{\lambda}{4} \left(\frac{4}{3} A(r)^2 + 4 B(r)^2 - v^2 \right)^2. \end{aligned} \quad (5.65)$$

The total energy or mass of the monopole is then,

$$M = \int_0^\infty \mathcal{H}(z) dz = 4\pi \int_0^\infty r^2 \mathcal{H}(r) dr. \quad (5.66)$$

The mass integrals are simplified by the following substitutions:

$$z \equiv e v r, \quad (5.67)$$

SU(2) Embedding:

$$f \equiv v F, \quad g \equiv v G, \quad h \equiv H, \quad (5.68)$$

SO(3) Embedding:

$$D_1 \equiv \mathcal{H}_1, \quad D_2 \equiv \mathcal{H}_2, \quad A \equiv v \mathcal{J}, \quad B \equiv v \mathcal{D}, \quad (5.69)$$

together with the convention that a prime indicates a derivative with respect to z we obtain:

SU(2) Embedding:

$$M = \frac{M_w}{\alpha} C(\beta, a, b), \quad (5.70)$$

$$C(\beta, a, b) = \int_0^\infty dz \left[H'^2 + \frac{(1-H^2)^2}{2z^2} + a^2 F^2 H^2 + \frac{z^2}{2} (a^2 F'^2 + b^2 G'^2) + \frac{\beta z^2}{4} (a^2 F^2 + b^2 G^2 - 1)^2 \right], \quad (5.71)$$

SO(3) Embedding:

$$M = \frac{M_w}{\alpha} C(\beta), \quad (5.72)$$

$$C(\beta) = \int_0^\infty dz \left[4 \left(\rho_1'^2 + \rho_2'^2 + \frac{(\rho_1^2 + \rho_2^2 - 1)^2 + 12 \rho_1^2 \rho_2^2}{2z^2} \right) + 2z^2 \left(\frac{\mathcal{J}'^2}{3} + \mathcal{Y}'^2 \right) + 4 \left((\mathcal{J} \rho_1 + \mathcal{Y} \rho_2)^2 + (\mathcal{J} \rho_2 + \mathcal{Y} \rho_1)^2 \right) + \frac{\beta z^2}{4} \left(\frac{4}{3} \mathcal{J}^2 + 4 \mathcal{Y}^2 - 1 \right)^2 \right], \quad (5.73)$$

where $M_w = ev$, $\alpha = \frac{e^2}{4\pi}$, $\beta = \frac{\lambda}{e^2}$. (5.74)

The equations of motion follow from minimizing the C's by way of the Euler-Lagrange equations.

SU(2) Embedding:

$$z^2 H'' = H(H^2 + a^2 z^2 F^2 - 1), \quad (5.75)$$

$$(z^2 F')' = F(2H^2 + \beta z^2 (a^2 F^2 + b^2 G^2 - 1)), \quad (5.76)$$

$$(z^2 G')' = \beta z^2 (a^2 F^2 + b^2 G^2 - 1), \quad (5.77)$$

SO(3) Embedding:

$$z^2 H_1'' = H_1 (H_1^2 + 7H_2^2 - 1) + z^2 (H_1 (z^2 + y^2) + 2zy H_2), \quad (5.78)$$

$$z^2 H_2'' = H_2 (H_2^2 + 7H_1^2 - 1) + z^2 (H_1 (z^2 + y^2) + 2zy H_1), \quad (5.79)$$

$$(z^2 z')' = 6(z(H_1^2 + H_2^2) + 2H_1 H_2 y) + \beta z^2 z' (\frac{4}{3} z^2 + 4y^2 - 1), \quad (5.80)$$

$$(z^2 y')' = 2(y(H_1^2 + H_2^2) + 2H_1 H_2 z) + \beta z^2 y' (\frac{4}{3} z^2 + 4y^2 - 1). \quad (5.81)$$

The boundary conditions that we use for these functions are:

SU(2) Embedding:

$$\begin{aligned} H(0) &= 1, & H(\infty) &= 0, \\ G'(0) &= 0, & G(\infty) &= 1, \\ F(0) &= 0, & F(\infty) &= 0, \end{aligned} \quad (5.82)$$

SO(3) Embedding:

$$\begin{aligned} H_1(0) &= 1, & H_1(\infty) &= 1/\sqrt{8}, \\ H_2(0) &= 0, & H_2(\infty) &= -1/\sqrt{8}, \\ z(0) &= 0, & z(\infty) &= \sqrt{3}/4, \\ y(0) &= 0, & y(\infty) &= \sqrt{3}/4. \end{aligned} \quad (5.83)$$

The explanation of these boundary conditions is given in appendix C. The criteria used for deciding these boundary conditions were that:

1. The mass integrals must be finite.
2. The radial functions $F(z)$, $G(z)$, ...etc. must be bounded

everywhere including $z=0$.

3. The scalar field Φ must approach a form such that its eigenvalues are the same as (5.30),

$$\Phi_{\text{vac}} = \frac{v}{2} (a\lambda_3 + b\lambda_8), \quad a^2 + b^2 = 1.$$

The fields will approach point monopole solutions as a result of the criteria above.

One of the results of this thesis is the numerical solution of the differential equations and the computation of the corresponding mass integral.

Identifying the electromagnetic field is a more complicated matter in SU(3) than it was in SU(2). The vacuum of SU(2) has a U(1) symmetry and so the Higgs mechanism permits only one massless vector field which is identified with the electromagnetic field. However, we have shown that an SU(3) theory may have either a U(2) or U(1)xU(1) vacuum symmetry, depending on the form of the Higgs potential.

For U(1)xU(1) there are two massless vector fields A_μ^3 and A_μ^8 . Thus Sinha ¹⁴(1976) identified two corresponding electromagnetic fields.

As in SU(2), $\mathcal{F}_{\mu\nu}$ must be gauge invariant and reduce to the usual definitions (i.e. to $\partial_\mu A_\nu^3 - \partial_\nu A_\mu^3$ and $\partial_\mu A_\nu^8 - \partial_\nu A_\mu^8$) when the scalar field is transformed to point in one direction (λ_3 say, i.e. $\hat{\phi}^a = \delta_{2a}$) in some region. The electromagnetic

¹⁴The identification of the electromagnetic fields are taken from Sinha's paper.

fields are identified by Sinha as:

$$\begin{aligned} \mathcal{F}_{\mu\nu}^{(3)} = & \hat{\phi}^a F_{\mu\nu}^a - \frac{5}{e} f_{abc} \hat{\phi}^a D_\mu \hat{\phi}^b D_\nu \hat{\phi}^c \\ & + \frac{4}{e} f_{abc} \hat{\phi}^a (f_{bgm} \hat{\phi}^g D_\mu \hat{\phi}^m) (f_{cnp} \hat{\phi}^n D_\nu \hat{\phi}^p), \end{aligned} \quad (5.84)$$

$$\mathcal{F}_{\mu\nu}^{(8)} = \sqrt{3} d_{abc} \hat{\phi}^b \hat{\phi}^c (F_{\mu\nu}^a + \frac{4}{e} f_{amn} D_\mu \hat{\phi}^m D_\nu \hat{\phi}^n), \quad (5.85)$$

where $\hat{\phi}^a \equiv \phi^a / \sqrt{\phi^b \phi^b}$.

The only cases we shall consider ¹⁵ with a U(1)xU(1) vacuum are from the SU(2) embedding (5.46),

$$\Phi(\underline{x}) = a f(r) \psi_1 + b g(r) \psi_2$$

for the cases (a,b) = (1,0), ($\frac{1}{2}, \frac{\sqrt{3}}{2}$). Substituting in the ansatz for (a,b) = (1,0) into (5.84) and (5.85) we obtain,

$$\underline{(a,b) = (1,0)}: \quad \mathcal{F}_{\mu\nu}^{(3)} = - \frac{\epsilon_{\mu\nu k} n^k}{er^2}, \quad (5.86)$$

and $\mathcal{F}_{\mu\nu}^{(8)} = 0, \quad (5.87)$

¹⁵This thesis was set up to calculate the monopole solutions proposed by Sinha and Corrigan et al.. Corrigan et al. started with the general potential (5.9) and so their vacuum symmetry is U(2). On the other hand Sinha used a symmetric potential (5.28) allowing both U(2) and U(1)xU(1) vacuum symmetries. Sinha considered only the SU(2) embedding (5.46) for the cases (a,b) = (1,0), ($\frac{1}{2}, \frac{\sqrt{3}}{2}$), ($-\frac{\sqrt{3}}{2}, -\frac{1}{2}$); the latter represents U(2) which was also produced by Corrigan et al. while the former two correspond to U(1)xU(1) vacuum.

where $\epsilon_{\mu\nu k}$ is the usual ϵ symbol for $\mu, \nu, k = 1, 2, 3$, and zero for $\mu, \nu = 0$. Thus we have a "point" magnetic monopole with magnetic charges $Q_m^{(3)} = 4\pi/e$ and $Q_m^{(8)} = 0$. For the case $(a, b) = (\frac{1}{2}, \frac{\sqrt{3}}{2})$ we have,

$$\mathcal{F}_{\mu\nu}^{(3)} = -\frac{\epsilon_{\mu\nu k} n^k}{er^2} \left[\frac{af(r)}{Q(r)} (1-h^2) + 5h^2 \left(\frac{af(r)}{Q(r)} \right)^3 - 4h^2 \left(\frac{af(r)}{Q(r)} \right)^5 \right], \quad (5.88)$$

and

$$\mathcal{F}_{\mu\nu}^{(8)} = \frac{\epsilon_{\mu\nu k} n^k}{er^2} \left[\frac{2abf(r)g(r)}{Q^2(r)} (1-h^2) + \frac{8(af(r))^3 bg(r)}{Q^4(r)} h^2 \right], \quad (5.89)$$

where $Q(r) = (a^2 f^2(r) + b^2 g^2(r))^{\frac{1}{2}}$. The fields, in this case, correspond to an "extended" spherically-symmetric magnetic charge density. The total magnetic charges are obtained from the asymptotic values of the magnetic fields along with Gauss' law. We find $Q_m^{(3)} = \frac{1}{2} \left(\frac{4\pi}{e} \right)$ and $Q_m^{(8)} = \frac{\sqrt{3}}{2} \left(\frac{4\pi}{e} \right)$.

When the symmetry of the vacuum is $U(2)$ (generated by $\lambda_1, \lambda_2, \lambda_3$ and λ_8) there are four massless vector fields. It is assumed that the $U(1)$ subgroup generated by λ_8 corresponds to the gauge group of the electromagnetic fields while the $SU(2)$ subgroup corresponds to a triplet of $SU(2)$ Yang-Mills fields¹⁶. The gauge invariant electromagnetic fields are

¹⁶Sinha defines the corresponding Yang-Mills tensor and non-Abelian magnetic charges. We shall not deal with these objects in this thesis.

defined such that they reduce to the usual definition

$\partial_\mu A_\nu^{(3)} - \partial_\nu A_\mu^{(3)}$ in some region when the scalar field $\hat{\phi}^a$ is transformed to $\hat{\phi}^a = \delta_{ab}$ in that region. The electromagnetic fields are then given by,

$$\mathcal{F}_{\mu\nu} = \hat{\phi}^a F_{\mu\nu}^a - \frac{4}{3e} f_{abc} \hat{\phi}^a D_\mu \hat{\phi}^b D_\nu \hat{\phi}^c, \quad (5.90)$$

with the usual definition of electric and magnetic charge.

Two cases with a U(2) vacuum are investigated in this thesis (corresponding to the ansatz presented by Corrigan et al.).

SU(2) Embedding: From (5.46)

$$\Phi(x) = af(r)\psi_1 + bg(r)\psi_2,$$

where $(a,b) = (\frac{\sqrt{3}}{2}, -\frac{1}{2})$. The electromagnetic fields are then given by:

$$\mathcal{F}_{\mu\nu} = -\frac{\epsilon_{\mu\nu k} n^k}{er^2} \left[\frac{af(r)}{Q} (1-h^2(r)) + h^2(r) \left(\frac{af(r)}{Q} \right)^2 \right], \quad (5.90)$$

where $Q = (a^2 f^2(r) + b^2 g^2(r))^{\frac{1}{2}}$.

SO(3) Embedding:

From (5.50) the scalar field is given by,

$$\Phi(x) = A(r)\phi_1 + B(r)\phi_2.$$

The corresponding electromagnetic fields are given by,

$$F_{\mu\nu} = -\frac{\epsilon_{\mu\nu k} \pi^k}{er^2} \left\{ \frac{4}{Q} (2AD_1D_2 + B(D_1^2 + D_2^2)) \right.$$

$$\left. -\frac{16}{3Q^3} (2A(AD_1 + BD_2)(AD_2 + BD_1) \right.$$

$$\left. + B(AD_2 + BD_1)^2 \right),$$

(5.91)

where $Q = \left(\frac{4}{3} A^2(r) + 4B^2(r) \right)^{1/2}$.

Both (5.90) and (5.91) represent extended spherically-symmetric magnetic charge densities. The total magnetic charges are given by:

SU(2) Embedding:

$$Q_m = \frac{\sqrt{3}}{2} \left(\frac{4\pi}{e} \right),$$

and

SO(3) Embedding:

$$Q_m = \pm \sqrt{3} \left(\frac{4\pi}{e} \right),$$

where the sign is a function of what boundary condition we choose.

Chapter 6

Numerical Method - Collocation

I cannot take any credit for what follows in this section. I shall present a very simple-minded explanation of the idea behind collocation. For those interested in a more detailed explanation, I suggest they study the papers of U. Ascher, J. Christiansen and R.D. Russell (1977, 1978) who graciously allowed me to use their program.

Consider the single second order linear differential equation on the interval $a \leq x \leq b$,

$$L[y] = f(x) \tag{6.1}$$

where L is a second order linear differential operator¹,

$$L[y] \equiv y'' + a_1(x)y' + a_0(x)y, \tag{6.2}$$

and
$$y' \equiv \frac{dy}{dx} \tag{6.3}$$

with the boundary conditions,

$$y(a) = A, \quad y(b) = B. \tag{6.4}$$

¹Collocation can solve systems of non-linear equations of mixed orders, but we need not go into the gory detail in order to obtain an intuitive feel of how collocation works.

Assuming a well behaved² solution we have something that might look like this

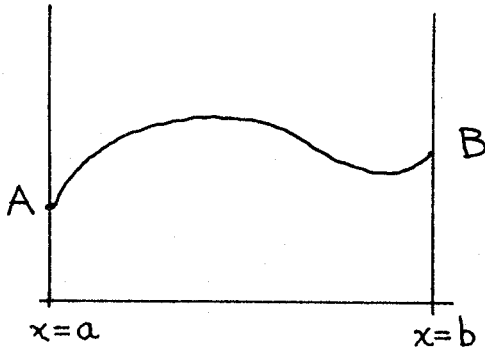


Fig. 6.1 A typical solution

Consider a polynomial

$$p(x) = \sum_{i=0}^{k+1} \alpha_i (x-a)^i, \quad (6.5)$$

which satisfies the differential equation at k interior points, $\{x_j\}_{j=1}^k$, $a < x_j < b$, and has the same boundary conditions as y . Explicitly,

$$p(a) = A, \quad p(b) = B, \quad (6.6)$$

$$L[p(x_j)] = f(x_j), \quad (6.7)$$

where $x_j \in (a,b)$, $j=1,2,\dots,k$.

²"Well-behaved" is to be interpreted as piecewise continuous and infinitely differentiable on the pieces. It seems that functions seen in physics tend to fall in this category.

This will yield a system of linear equations in the $(k+2)$ unknown constants $\alpha_0, \alpha_1, \alpha_2 \dots, \alpha_{k+1}$ which can be solved, in principle. The polynomial should, in some sense, "look like" the true solution $y(x)$.

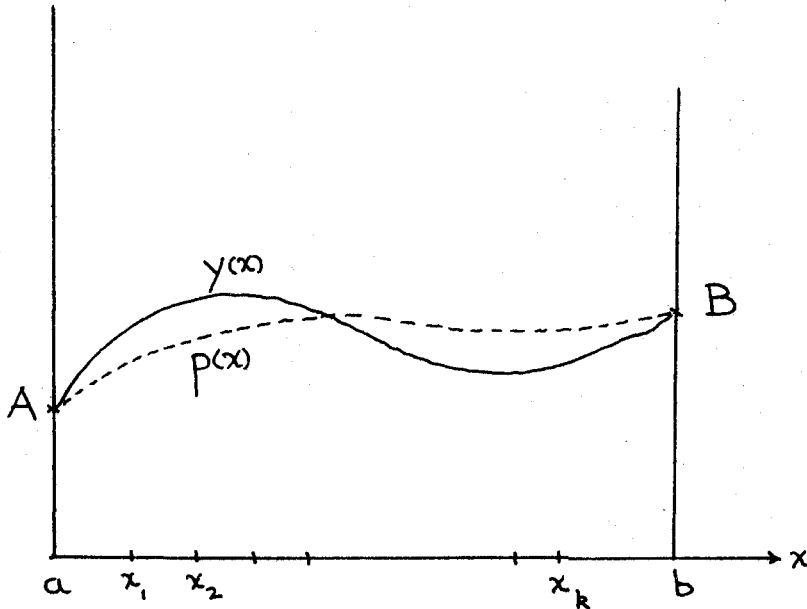


Fig. 6.2

A polynomial fit of the true solution.

If the solution $y(x)$ has "interesting" behavior, a large value of k would be required to make $p(x)$ a good approximation to the solution, thereby forcing one to solve very large and difficult systems of linear equations ((6.6), (6.7)). This method would become computationally expensive and prone to roundoff error. A better approach is to use more than one polynomial by breaking up our interval into subintervals.

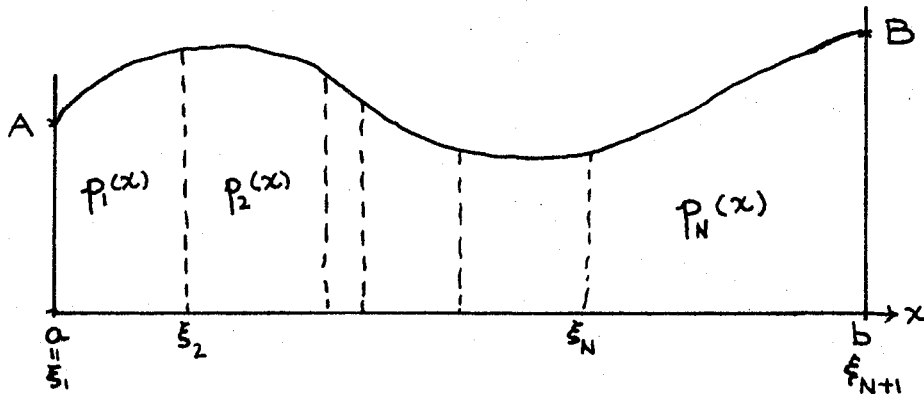


Fig. 6.3

Dividing of the interval $[a,b]$ into subintervals.

Consider N polynomials $p_1(x)$, $p_2(x)$, ..., $p_N(x)$ defined on the $(N+1)$ points dividing the subintervals $\xi_1, \xi_2, \dots, \xi_{N+1}$ called the mesh points. As can be seen from Fig. 6.3, the mesh points are chosen such that,

$$a = \xi_1 < \xi_2 < \dots < \xi_N < \xi_{N+1} = b. \quad (6.8)$$

Each polynomial is $(k+1)^{st}$ degree so we need to solve for $N(k+2)$ unknown polynomial coefficients. Consequently, we must impose an equal number of independent conditions.

$$I. \quad p_1(a) = A, \quad p_N(b) = B. \quad (6.9)$$

II. The function that we produce should be continuous in its zeroth and first derivatives.

$$P_{\mu}(\xi_{\mu+1}) = P_{\mu+1}(\xi_{\mu+1}), \quad (6.10)$$

$$P'_{\mu}(\xi_{\mu+1}) = P'_{\mu+1}(\xi_{\mu+1}). \quad \mu = 2, \dots, N \quad (6.11)$$

III. The P_{μ} 's should satisfy the differential equation at k points within their respective subintervals.

$$L[P_{\mu}(x_{\mu,j})] = f(x_{\mu,j}), \quad (6.12)$$

$$\mu = 1, 2, \dots, N \quad ; \quad j = 1, 2, \dots, k \quad ;$$

$$\xi_{\mu} < x_{\mu,1} < x_{\mu,2} < \dots < x_{\mu,k} < \xi_{\mu+1}.$$

These conditions yield $N(k+2)$ linear equations in as many unknowns. For comparable accuracy, this method yields a set of equations that are solved with less trouble than those of the first method presented.

For technical reasons the k points in each subinterval are chosen to be the points corresponding to the roots of the k^{th} Legendre polynomial. Explicitly,

$$P_k(\rho_j) = 0, \quad j = 1, 2, \dots, k \quad (6.13)$$

$\rho_j \in [-1, +1]$, these are called the Gauss points. Then one maps these points linearly from $[-1, 1]$ to the m^{th} subinterval, $[\xi_m, \xi_{m+1}]$. It has been shown by U. Ascher, J. Christiansen and R.D. Russell (1977) that this method gives a maximum error on the i^{th} subinterval of order h_i^6 , where h_i is the length of the i^{th} subinterval. Collocation has very nice convergence properties.

For the general case of N non-linear ordinary differential equations in N unknowns, the coefficients of the N polynomials are determined by the solution of a set of non-linear algebraic equations using a generalized Newton's method.

Now we will discuss some of the aspects of the numerical calculation as applied to the solution of the differential equation developed in chapters 4 and 5.

The differential equations are defined on the interval $0 \leq z < \infty$, but collocation requires a finite interval. To resolve this difficulty we map $[0, \infty)$ onto $[0, 1)$ by means of the transformation

$$z \longrightarrow x = \frac{z}{1+z} , \quad 0 \leq z < \infty , \quad 0 \leq x < 1 \quad (6.14)$$

At the same time the differential equations and integrals must be modified in the appropriate manner so that they are

expressed in terms³ of x . This transformation is not generally applicable to all problems. If the solutions on $[0, \infty)$ are not "extremely smooth", the problem becomes impossible on $[0, 1)$. For example, consider a function on $[0, \infty)$ that looks like the figure below,

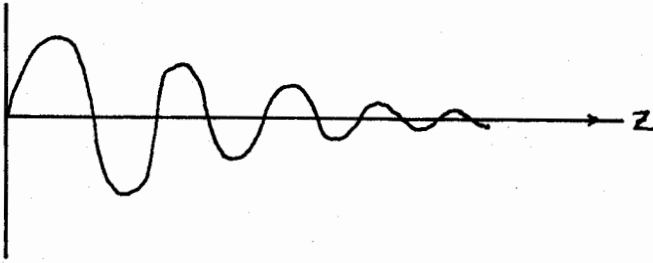


Fig. 6.4

Oscillatory behaviour on $[0, \infty)$.

after mapping onto $[0, 1)$ our function would have the form,

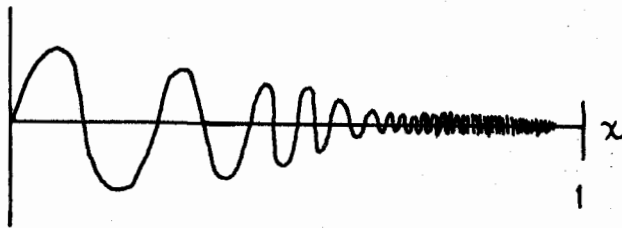


Fig. 6.5

Corresponding oscillatory behaviour on $[0, 1)$.

³If you do the conversion to the $[0, 1)$ interval, you will find that singularities are introduced at $x=1$. Fortunately, collocation is adaptable to this type of singularity on the boundaries so the equations could be solved.

which would be impossible to solve numerically. Fortunately, the functions that characterize monopoles have very smooth behavior in the asymptotic limit of $z \rightarrow \infty$ and this mapping procedure is valid. The computer solutions are plotted on this $[0,1)$ domain so one must be cautious as to what the function looks like as a function of the radius because of the distortion due to the mapping.

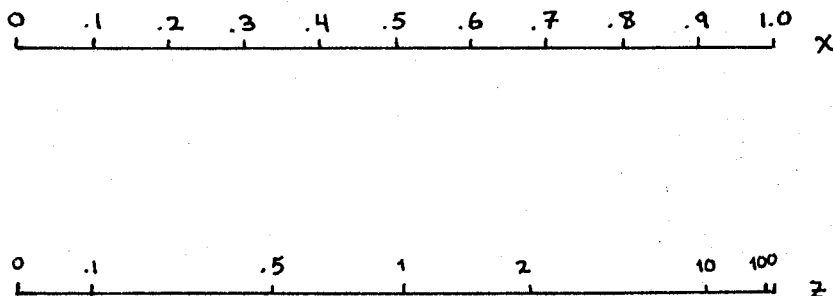


Fig. 6.6

Distortion effects of mapping $[0, \infty)$ onto $[0,1)$.

This figure shows the distorting effect which becomes very severe towards $x = 1$ ($z \rightarrow \infty$). Despite this distortion this mapping has the advantage of clearly displaying the essential behavior of the fields, i.e. extrema and limiting behavior.

For the $SU(2)$ monopole it was seen that two of the fields become asymptotically large as $z \rightarrow \infty$. Specifically,

$$F \xrightarrow{z \rightarrow \infty} z, \tag{6.15}$$

and
$$J \xrightarrow{z \rightarrow \infty} \eta z, \quad (6.16)$$

To keep the fields finite (in order to do numerical calculations) we define G and N such that,

$$F = (1+z) G, \quad (6.17)$$

$$J = (1+z) N, \quad (6.18)$$

with boundary conditions given by,

$$\begin{aligned} G(0) = 0, \quad N(0) = 0, \\ \lim_{z \rightarrow \infty} G(z) = 1, \quad \lim_{z \rightarrow \infty} N(z) = \eta. \end{aligned} \quad (6.19)$$

The functions plotted will be $G(x)$, $N(x)$ and $H(x)$.

As promised, we shall present a brief description of the numerical methods of Bais and Primack (1975) and Cutler and Wyld (1976). It is difficult to compare the merits of various numerical methods without a detailed investigation on the programs. Since we are not in possession of all the programs, it would be improper to say that collocation is superior. However, with collocation a much wider range of solutions is investigated than before (Cutler and Wyld have solutions for $\lambda/e^2 \leq 100$ whereas we have solutions for $\lambda/e^2 \leq 10^{11.5}$). Also, in the special cases where analytic solutions exist, there is

very good agreement with out numerical results.

Bais and Primack converted their system of differential equations to a set of integral equations by separating the linear and nonlinear terms in the equations,

$$L_i y_i = -f_i(x, y, y') \quad (\text{no summation on } i).$$

Where y_i are the various fields, the f_i are the nonlinear terms in the i^{th} equation, and the L_i are the linear differential operators. They then solve the linear homogeneous equations with homogeneous boundary conditions

$$L_i y_i = 0.$$

From these solutions they construct the Greens functions where

$$L_i G_i(x, x') = -\delta(x-x').$$

Then choosing solutions to the equations of the form,

$$y_i = a_i + z_i,$$

where the a_i are arbitrary but satisfy the inhomogeneous boundary conditions of the problem. The a_i can be chosen to be anything although in practice it should be an approximation of the exact solution y_i . The differential equation is now in terms of z_i which satisfies homogeneous boundary conditions,

$$L_i z_i = -f_i - P_i,$$

where $P_i = L_i a_i$ which is known. The integral equation becomes,

$$y_i(x) = a_i(x) + \int_a^b G_i(x, x') [f_i(x', y(x'), y'(x')) + P_i(x')] dx'$$

which is solved by iterating this integral equation.

Cutler and Wyld use an approach developed by Henyey et al. (1959) for astrophysical problems called relaxation.

Consider a system of M differential equations,

$$Q = f(x, \phi) \phi'' + g(x, \phi) \phi' + h(x, \phi) = 0,$$

with boundary conditions

$$\phi(a) = A, \quad \phi(b) = B,$$

where $Q(x)$, $\phi(x)$, A , B , $h(x, \phi)$ are column vectors and f and g are $M \times M$ matrices. Now by discretizing the domain we obtain a set of difference equations,

$$\begin{aligned} Q_j &\equiv Q(x_j, \phi_j) \\ &= f(x_j, \phi_j) \frac{d^2 \phi}{dx^2} \Big|_{x_j} + g(x_j, \phi_j) \frac{d\phi}{dx} \Big|_{x_j} + h(x_j, \phi_j) = 0, \end{aligned}$$

now approximating the derivatives,

$$\left. \frac{d^2\phi}{dx^2} \right|_{x_j} \approx \frac{1}{\Delta^2} (\phi_{j+1} - 2\phi_j + \phi_{j-1}),$$

$$\left. \frac{d\phi}{dx} \right|_{x_j} \approx \frac{1}{2\Delta} (\phi_{j+1} - \phi_{j-1}),$$

with

$$x_j = (j-1)\Delta + a, \quad j=1, N$$

$$\phi_j \equiv \phi(x_j), \quad j=1, N.$$

Substituting these approximations into the expression for $Q_j = 0$ we obtain $M \times N$ nonlinear equations for $\phi(x_j)$ which can be solved by a generalized Newton's method. This method does not converge when the differential equation becomes singular at the origin, where one is forced to constrain the first several points to vary like $\phi(x) \propto x^\alpha$.

Chapter 7

Results

The SU(2) dyon and SU(3) monopole solutions are calculated over a wide range of the parameter β and η (see chapters 4 and 5). When the fields along with their associated masses (and electric charges for the dyon) are plotted for various β , we find that as β becomes large the changes become small. In other words, the monopole/dyon fields approach an asymptotic configuration as $\beta \rightarrow \infty$ and their corresponding mass (and electric charge) approach an asymptotic value. This process, which we shall call saturation, is explained for the SU(2) dyon/monopole. We obtain the asymptotic field configurations, masses and electric charges for infinite β by requiring the energy (mass) finite. This assumption means that the scalar fields must be at their vacuum value everywhere (except perhaps the origin). The agreement between this method and the numerically obtained asymptotic values is very good. Saturation is observed for the SU(3) monopole (SU(2) embedding) but the asymptotic values could not be obtained for infinite β because the equations resisted solution. Unfortunately the SU(3) monopole (SO(3) embedding) could be solved only for values of β very close to zero. This was the most massive of all the monopoles investigated.

The equations and definitions of the fields associated with the SU(2) dyon are presented in detail in chapter 4 (with some modifications made in chapter 6). The SU(2) monopole

(a dyon with zero electric charge) may be considered as a special case of the family of SU(3) monopoles, so we will present its results as part of the discussion of SU(3).

We plot the fields $H(\chi)$, $N(\chi)$ and $G(\chi)$ on the interval $0 \leq \chi < 1$. Where χ is defined by

$$\chi = \frac{z}{z+1}, \quad (7.1)$$

where $z = e^{\nu r}$, (7.2)

so $\chi=0$ corresponds to the origin and $\chi=1$ corresponds to $r \rightarrow \infty$. It is helpful to remember that H governs the behaviour of the spatial components of the gauge fields. N governs the time component of the gauge fields and is closely associated with the electric charge of the dyon. Finally, G controls the scalar fields. These functions are smooth and bounded with boundary values:

$$\begin{aligned} H(r=0) &= 1, & H(r=\infty) &= 0, \\ N(r=0) &= 0, & N(r=\infty) &= \eta, \\ G(r=0) &= 0, & G(r=\infty) &= 1, \\ & & 0 \leq \eta < 1, \end{aligned} \quad (7.3)$$

where η is a free parameter which governs the behaviour of the time components of the gauge fields in the asymptotic limit.

Qualitatively, the higher the value of η , the greater the dyon charge (the $\eta=0$ case corresponds to the zero electric charge monopole).

FIG. 7.1, FIG. 7.2 and FIG. 7.3 show the fields for $\beta = 1.0$ and various values of η ranging from 0.1 to 0.99. It is of interest to note that the scalar fields ϕ^a , as governed by G, is insensitive to changes in η . The qualitative behaviour of the fields for various η is similar if β is held at a different fixed value.

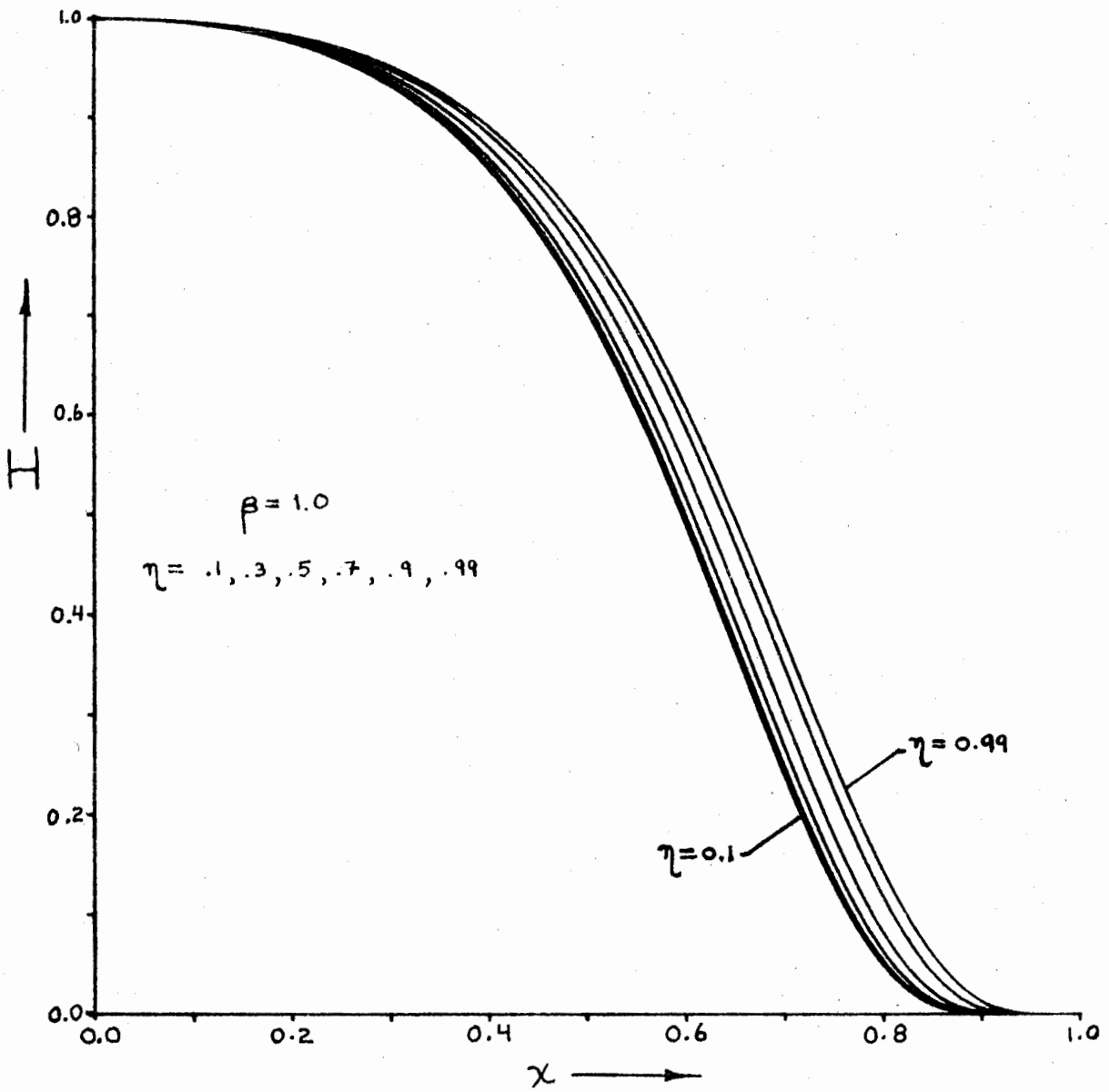


Fig.7.1 SU(2) dyon gauge fields (H) for various η

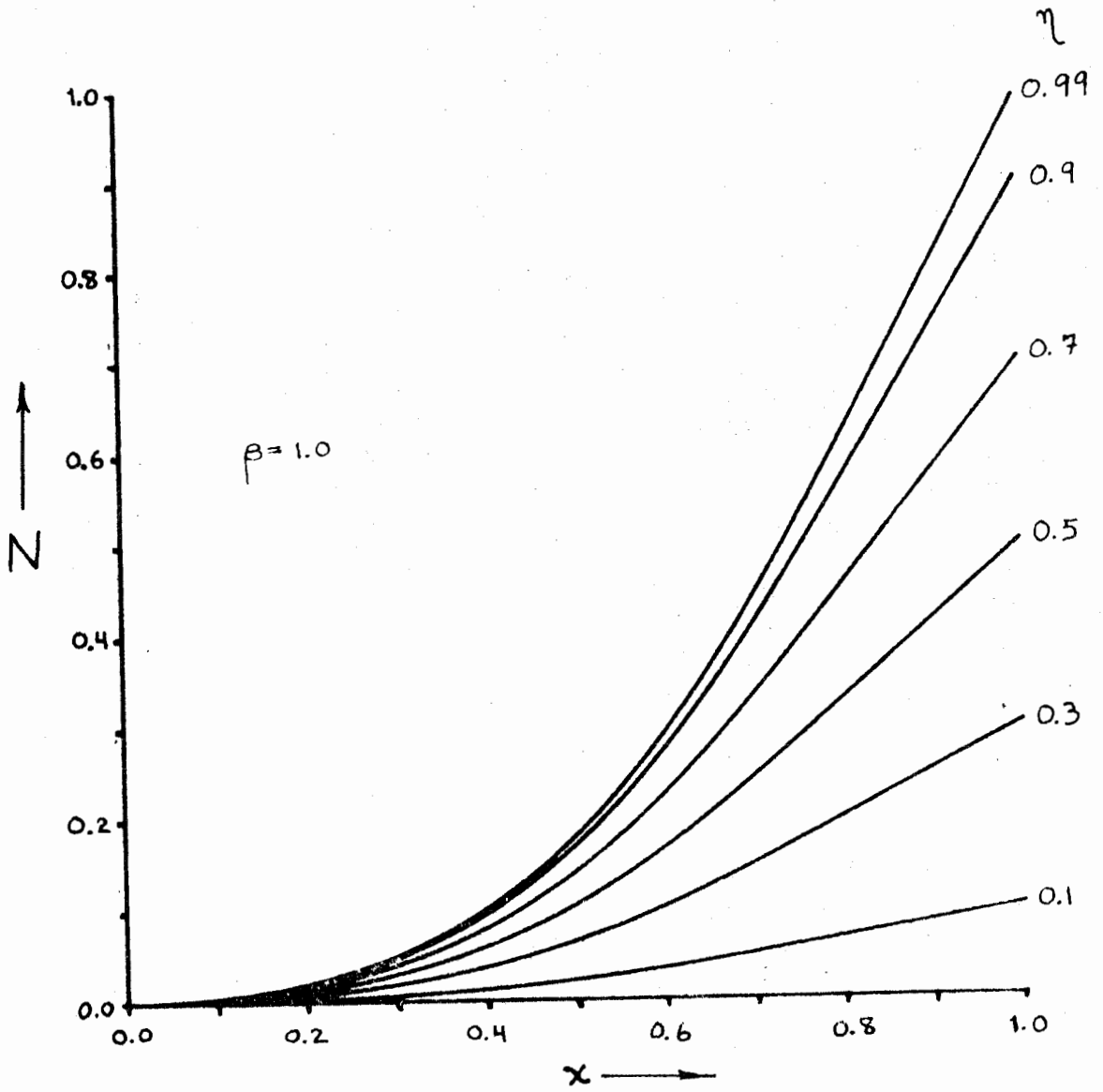


Fig.7.2 SU(2) dyon gauge fields (N) for various η

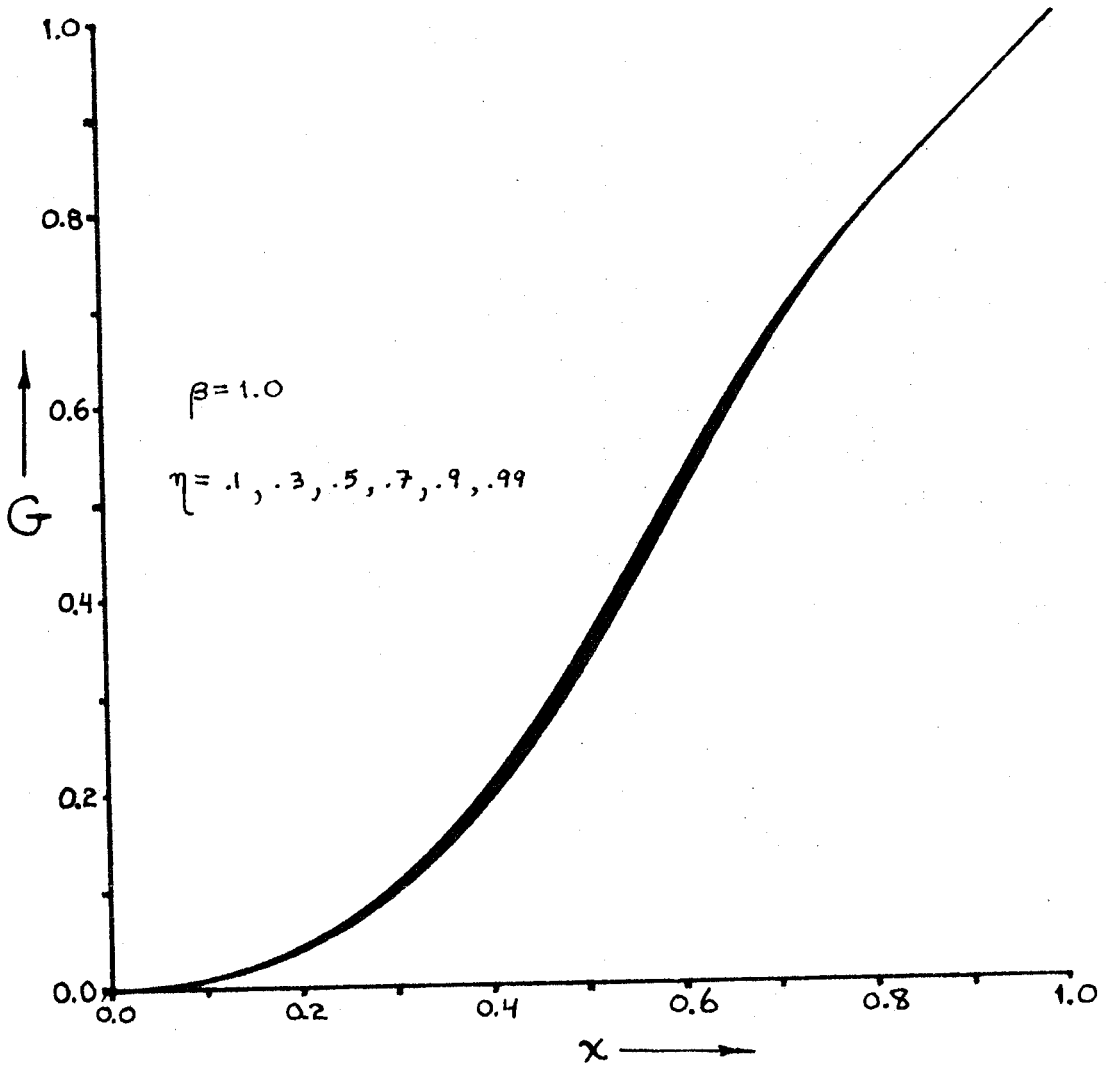


Fig.7.3 SU(2) dyon scalar fields (G) for various η

An extremely interesting phenomena is seen in figures 7.4, 7.5 and 7.6 where we plot the fields for $\eta = 0.9$ and several β 's ranging from zero up to 10^5 . As β becomes large, the fields approach a constant configuration. In terms of the scalar field G seems to approach the function

$$G_{\beta \rightarrow \infty} = \chi, \quad (7.4)$$

while H and N do not have an obvious analytic form in the $\beta \rightarrow \infty$ limit. Similar results are obtained for values of other than 0.9.

Using the definition of the electric field (4.40) along with Gauss' law we obtain the electric charge density ρ_e ,

$$r^2 \rho_e = -\frac{2}{e} \frac{JH^2}{r} = -\frac{2M_w}{e} \frac{NH^2}{\chi}.$$

Figure 7.7 shows the electric charge density to behave as $\frac{1}{r}$ near the origin and vanish as the radius becomes large. The electric charge is extended but finite. The curves shown are representative of all .

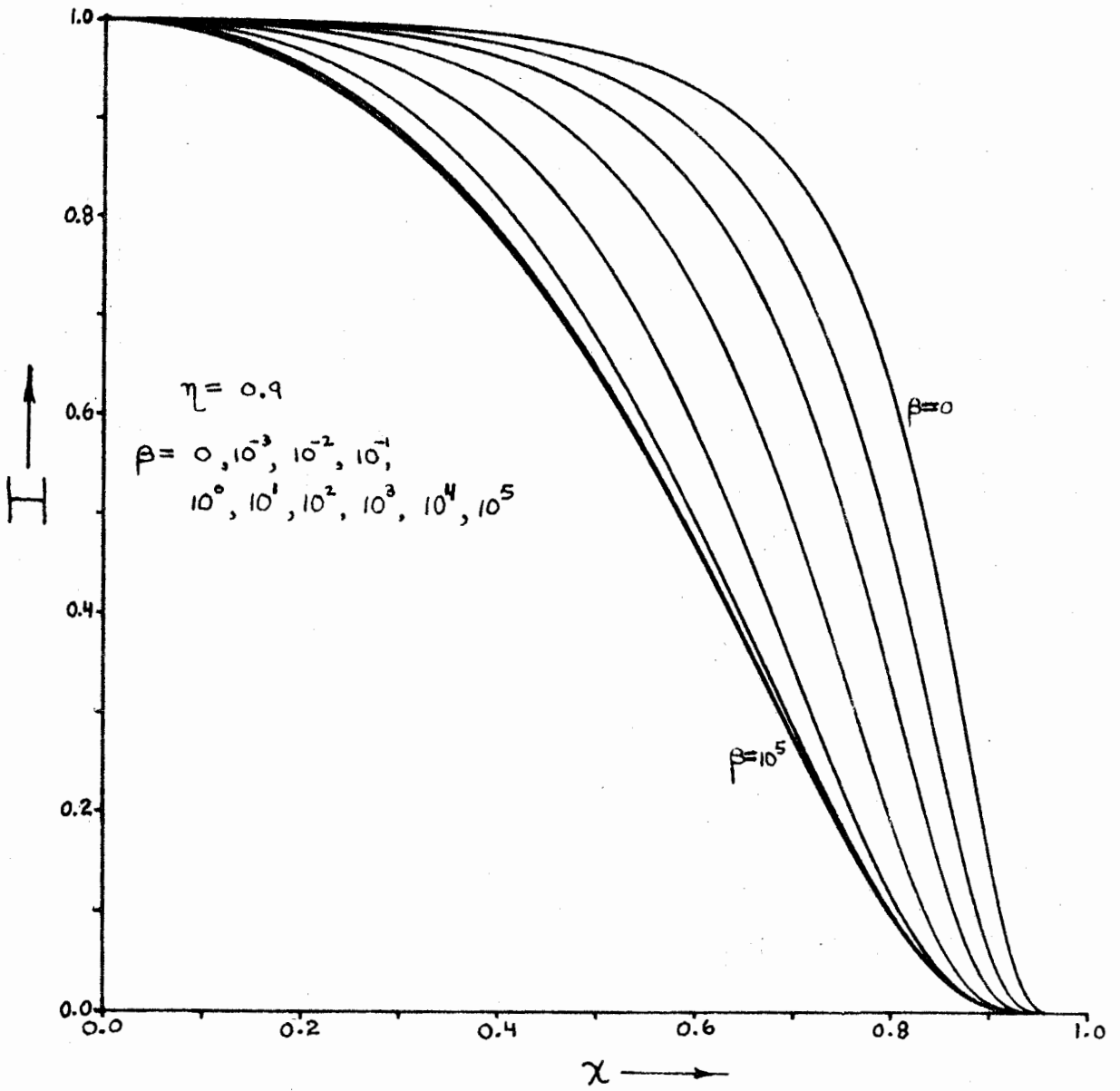


Fig.7.4 SU(2) dyon gauge fields (H) for various β

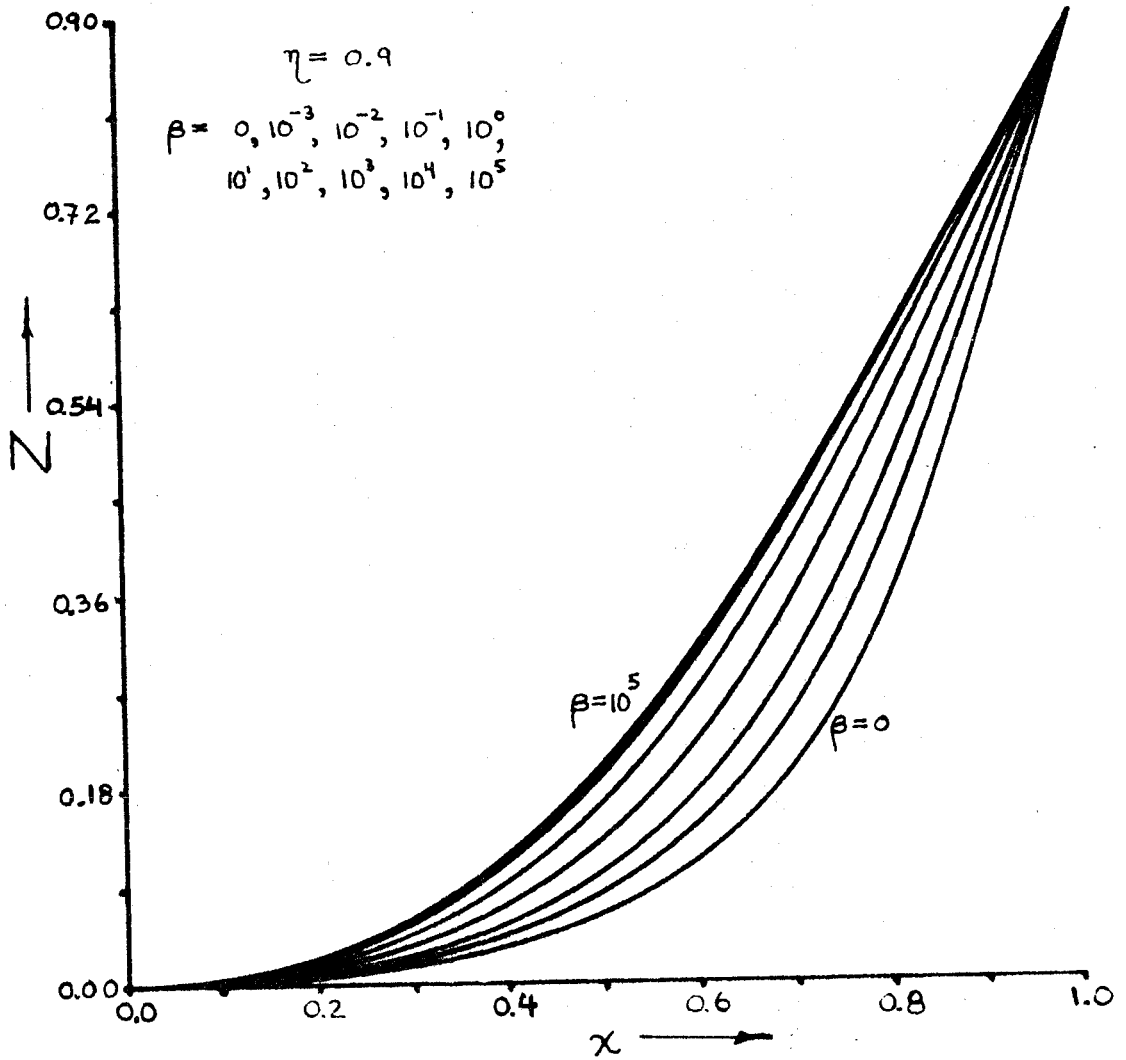


Fig.7.5 SU(2) dyon gauge fields (N) for various β

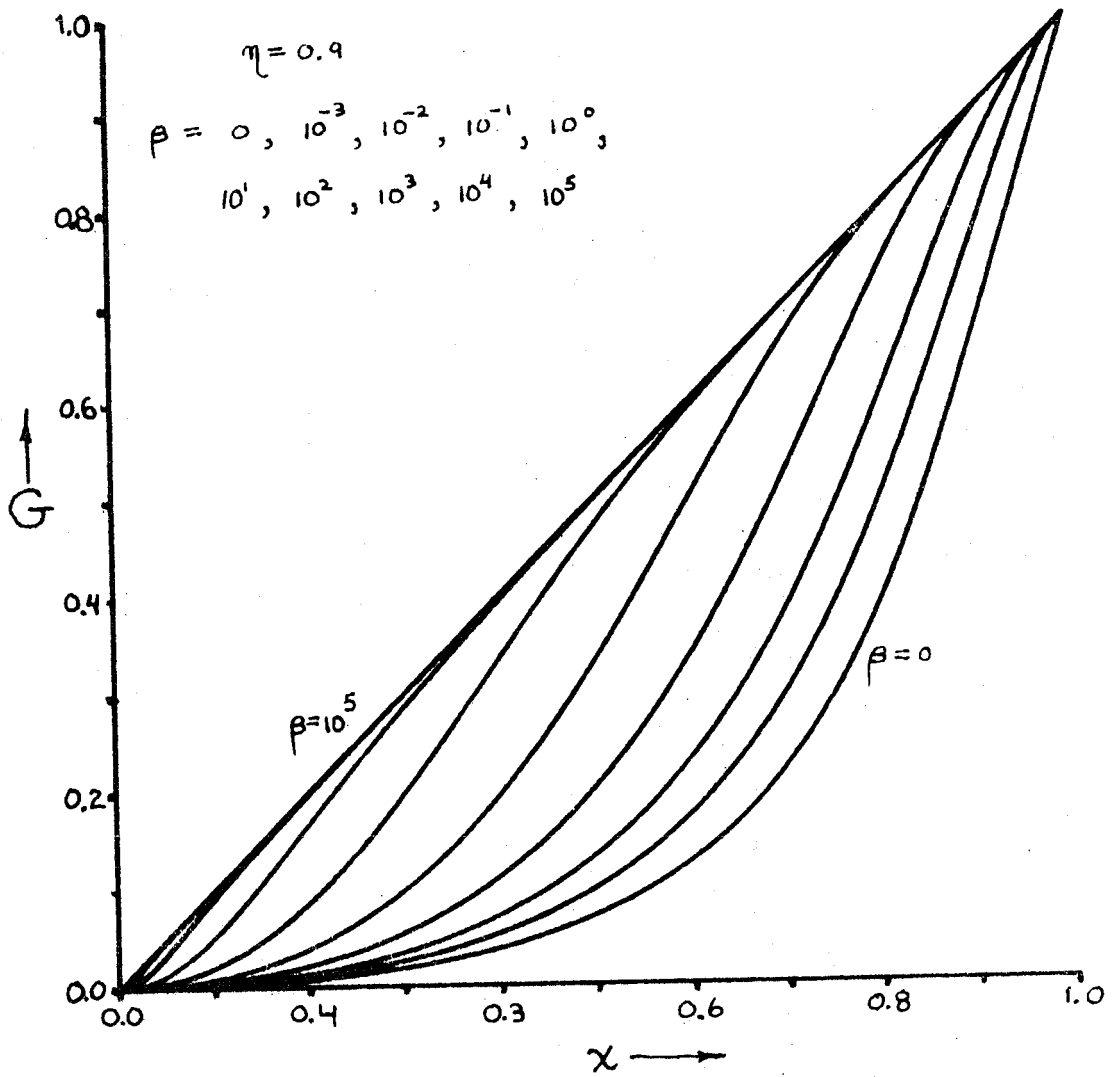


Fig.7.6 SU(2) dyon scalar fields (G) for various β

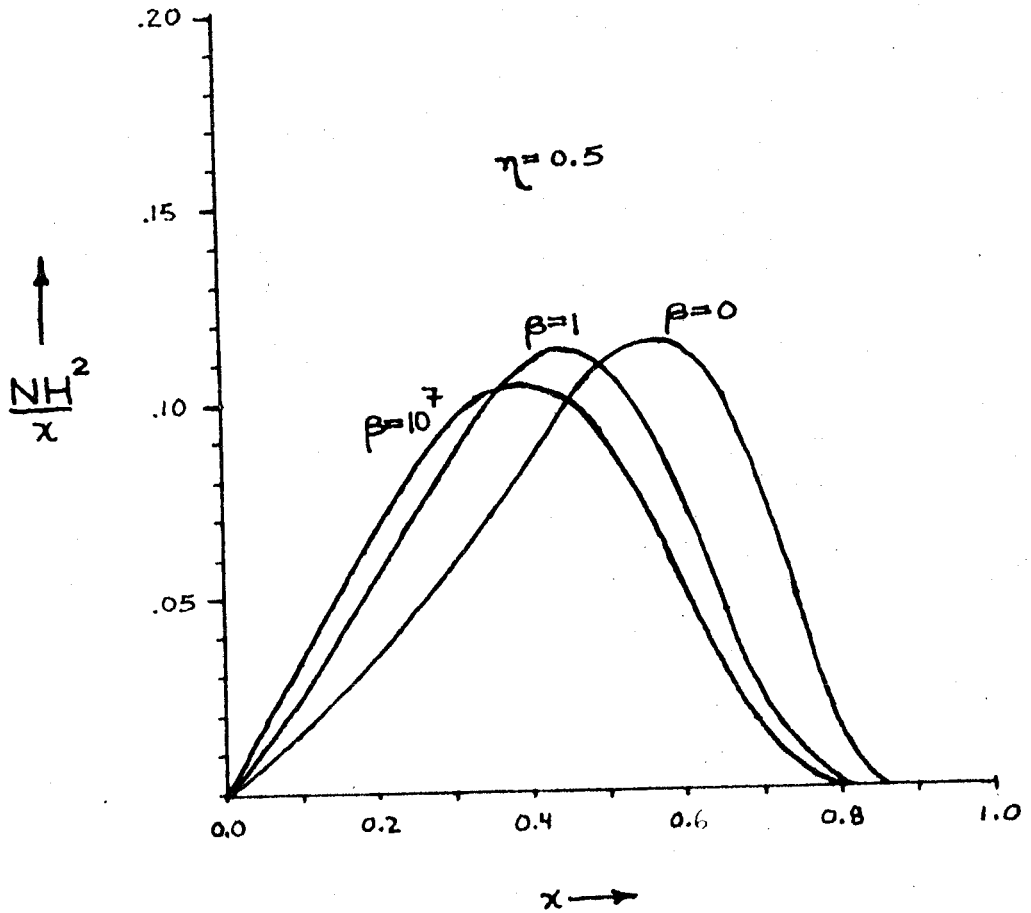


Fig. 7.7 Electric charge density of the SU(2) dyon.

The mass and charge of the dyon solutions are calculated from the values of the fields. Recall,

$$M = \frac{M_w}{\alpha} C(\beta, \eta),$$

where $C(\beta, \eta)$ is given by equation (4.25), and

$$Q_e = -\frac{8\pi}{e} \zeta(\beta, \eta)$$

where $\zeta(\beta, \eta)$ is given by equation (4.43). Figures 7.8 and 7.9 plot $\zeta(\beta, \eta)$ and $C(\beta, \eta)$ for various values of β and η .

The results are nearly in agreement with those obtained by Bais and Primack (1975).

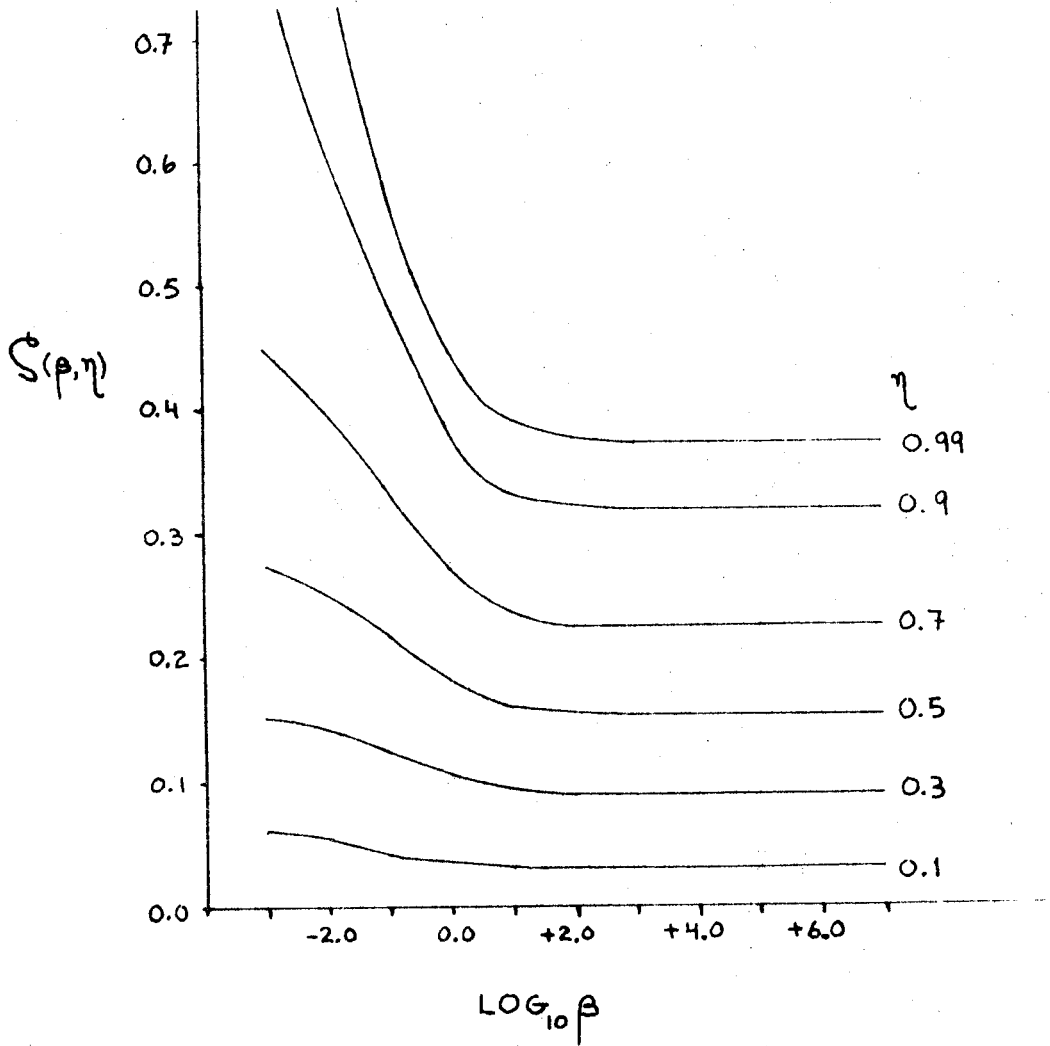


Fig.7.8 SU(2) dyon electric charge (S) for various β and η

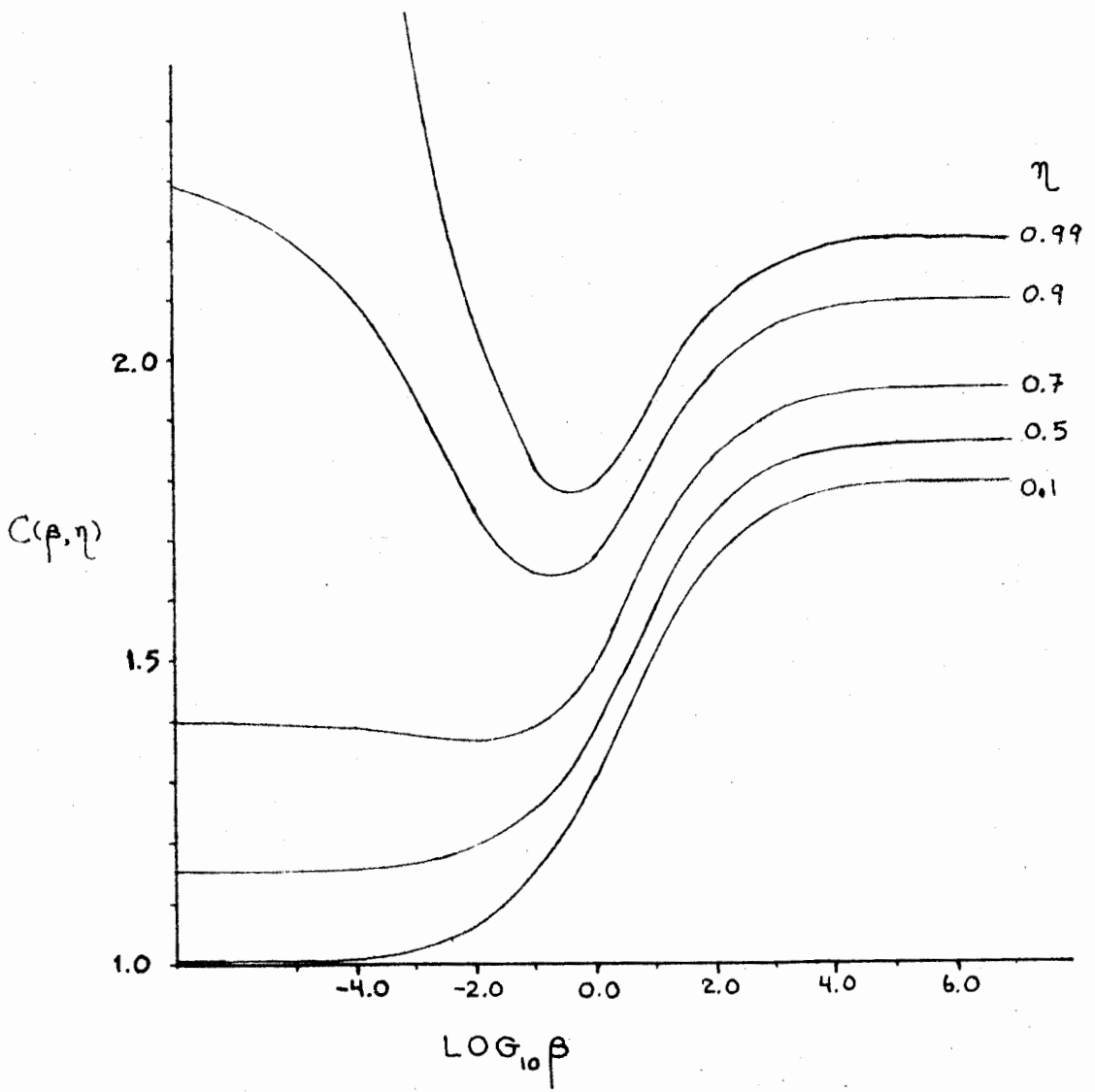


Fig.7.9 SU(2) dyon mass (C) for various β and η .

As β becomes large the mass and charge of the dyon "saturates" at finite values. Considering the behaviour of the fields for large β , this result should not be unexpected. Hereafter this phenomena shall be referred to as saturation. We say as β becomes large the dyon fields along with their associated masses and charges saturate. To explain this effect we must go back to our expression for the action, equation (4.18).

$$\begin{aligned}
 \mathcal{L}(\beta, \eta) = & \int_0^\infty dz \left[H'^2 + \frac{1}{2z^2} (H^2 - 1)^2 - \frac{H^2 J'^2}{z^2} - \frac{(zJ' - J)^2}{2z^2} \right. \\
 & \left. + \frac{F^2 H^2}{z^2} + \frac{(zF' - F)^2}{2z^2} + \frac{\beta}{4z^2} (F^2 - z^2)^2 \right] \quad (7.5)
 \end{aligned}$$

Consider the limit as $\beta \rightarrow \infty$. In order to minimize the action, or even keep it finite, the function $F(z)$ must approach the form,

$$F_{\beta \rightarrow \infty}(z) = z \quad (7.6)$$

This means that G takes the form,

$$G_{\beta \rightarrow \infty}(z) = \frac{F_{z \rightarrow \infty}(z)}{1+z} = \frac{z}{1+z} = \chi.$$

This process is displayed beautifully in FIG 7.6. Now consider

the limiting form of the action,

$$l_{\beta \rightarrow \infty}(\eta) = \int_0^{\infty} dz \left[H'^2 + \frac{(H^2-1)^2}{2z^2} - \frac{H^2 J^2}{z^2} - \frac{(zJ'-J)^2}{2z^2} + H^2 \right] \quad (7.7)$$

which yields the following differential equations,

$$z^2 H'' = H(H^2 - J^2 + z^2 - 1), \quad (7.8)$$

$$z^2 J'' = 2JH^2. \quad (7.9)$$

These equations were solved numerically and the corresponding mass and charge were calculated. The values obtained correspond extremely well with the limiting values for large β . Consider the following values obtained for the mass of the SU(2) dyon:

η	0.1	0.3	0.5	0.7	0.9	0.99
$\beta \rightarrow \infty$	1.791	1.815	1.866	1.954	2.103	2.207
$\beta = 10^5$	1.789	1.813	1.864	1.952	2.101	2.205

Table 7.1 SU(2) dyon mass in the large and infinite β limit for various η

Similarly, the values of the charge show good agreement:

	S					
η	0.1	.3	.5	.7	.9	.99
$\beta \rightarrow \infty$.0290	.0884	.1522	.2252	.3174	.3726
$\beta = 10^5$.0290	.0883	.1522	.2252	.3174	.3726

Table 7.2 SU(2) dyon charge in the large and infinite β limit for various η

Chapter 5 derives and explains the equations of the monopole solutions in an SU(3) gauge theory. There are two distinct embeddings of the subgroup SU(2) within the symmetry group of the Lagrangian density, SU(3). Corresponding to each of the embeddings there are different ansatze for the fields of the theory (for details see chapter 5). A crude summary of the various fields is given in the following table:

Embedding	Field	Controls	Boundary Conditions
SU(2)	F	scalar fields ϕ^1, ϕ^2, ϕ^3	$F(0) = 0, F(\infty) = 1$
	G	scalar field ϕ^0	$G(0) = 0, G(\infty) = 1$
	H	gauge fields $A_\mu^1, A_\mu^2, A_\mu^3$	$H(0) = 1, H(\infty) = 0$
SO(3)	\mathcal{F}	scalar fields ϕ_1	$\mathcal{F}(0) = 0, \mathcal{F}(\infty) = \sqrt{3}/4$
	\mathcal{D}	scalar fields ϕ_2	$\mathcal{D}(0) = 0, \mathcal{D}(\infty) = \sqrt{3}/4$
	\mathcal{H}_1	gauge fields	$\mathcal{H}_1(0) = 1, \mathcal{H}_1(\infty) = 1/\sqrt{8}$
	\mathcal{H}_2	gauge fields	$\mathcal{H}_2(0) = 0, \mathcal{H}_2(\infty) = -1/\sqrt{8}$

Table 7.3 SU(3) fields and their boundary conditions

The equations of motion are obtained by minimizing the mass integrals (5.71) and (5.73). The SU(2) embedding is investigated for the three cases obtained by Sinha (1976):

$$\left. \begin{array}{l} 1. \quad a=1, \quad b=0, \\ 2. \quad a=\frac{1}{2}, \quad b=\frac{\sqrt{3}}{2}, \end{array} \right\} U(1) \times U(1)$$

$$3. \quad a=\frac{\sqrt{3}}{2}, \quad b=-\frac{1}{2} \quad \left. \right\} U(2)$$

The mass for the SU(2) embedding is given by the expression,

$$\begin{aligned} C(\beta, a, b) = & \int_0^\infty dz \left[H'^2 + \frac{1}{2z^2} (1-H^2)^2 + a^2 F^2 H^2 \right. \\ & + \frac{z^2}{2} (a^2 F'^2 + b^2 G'^2) \\ & \left. + \frac{\beta z^2}{4} (a^2 F^2 + b^2 G^2 - 1)^2 \right] \end{aligned} \quad (7.10)$$

The first case (where $a = 1$ and $b = 0$) has the same mass and fields (to within a factor of z) as the SU(2) monopole¹. We

¹The SU(2) monopole is a special case of the SU(2) dyon and, strictly speaking, has little to do with the monopole corresponding to the SU(2) embedding in the gauge group SU(3).

have

$$\begin{aligned}
 C(\beta, l, 0) = & \int_0^\infty dz \left[H'^2 + \frac{1}{2z^2} (1-H^2)^2 \right. \\
 & + F^2 H^2 + \frac{z^2}{2} F'^2 \\
 & \left. + \frac{\beta z^2}{4} (F^2 - 1)^2 \right] \tag{7.11}
 \end{aligned}$$

From chapter 4, the SU(2) monopole is obtained by setting $J(z) = 0$ everywhere and the equations of motion now may be obtained by minimizing the mass integral (4.25),

$$\begin{aligned}
 C(\beta, \eta=0) = & \int_0^\infty dz \left[H'^2 + \frac{1}{2z^2} (H^2 - 1)^2 \right. \\
 & + \frac{1}{z^2} H^2 F^2 + \frac{1}{2z^2} (z F' - F)^2 \\
 & \left. + \frac{\beta}{4z^2} (F^2 - z^2)^2 \right] \tag{7.12}
 \end{aligned}$$

Now making the substitution,

$$F(z) = z F(z), \tag{7.13}$$

the two expressions for the mass become identical²,

²The expressions are identical because the boundary conditions for $F(z)$ ($F(0)=0$, $F \underset{z \rightarrow \infty}{\sim} z$), transforms to $F(0) = 0$ and $F(\infty) = 1$ which are the proper conditions for the SU(3) fields.

i.e.,

$$C(\beta, 1, 0) = C(\beta, \eta=0). \quad (7.14)$$

Thus in viewing the results of the $a=1, b=0$ case one may immediately apply these results to the $SU(2)$ monopole.

It is interesting to note that exact solutions have been discovered for the $SU(2)$ embedding for the special case of $\beta = 0$. The mass is given by,

$$C(0, a, b) = \int_0^\infty dz \left[H'^2 + \frac{1}{2z^2} (1-H^2)^2 + a^2 F^2 H^2 + \frac{z^2}{2} (a^2 F'^2 + b^2 G'^2) \right] \quad (7.15)$$

which yields the following equations of motion,

$$\begin{aligned} z^2 H'' &= H(H^2 + a^2 z^2 F^2 - 1), \\ (z^2 F')' &= 2FH^2 \\ (z^2 G')' &= 0. \end{aligned} \quad (7.16)$$

Using the boundary conditions from table 7.3 this set of equations has the analytic solution:

$$H(z) = \frac{az}{\sinh az},$$

$$F(z) = \coth az - \frac{1}{az} \quad (7.17)$$

$$G(z) = 1.$$

This solution was discovered by Prasad and Sommerfield (1975) for the SU(2) monopole ($a=1$, $b=0$) and later Czechowski (1976) extended the solution to the SU(2) embedding case ($a=\frac{\sqrt{3}}{2}$, $b=-\frac{1}{2}$). Substituting (7.23) into the mass integral (7.14) we obtain,

$$C(0, a, b) = |a|. \quad (7.18)$$

This solution provides a nice check for the numerically calculated results.

The fields of the SU(2) embedding in SU(3) are plotted for various β on figures 7.10 to 7.17. Notice in particular the saturation of the fields for large β .

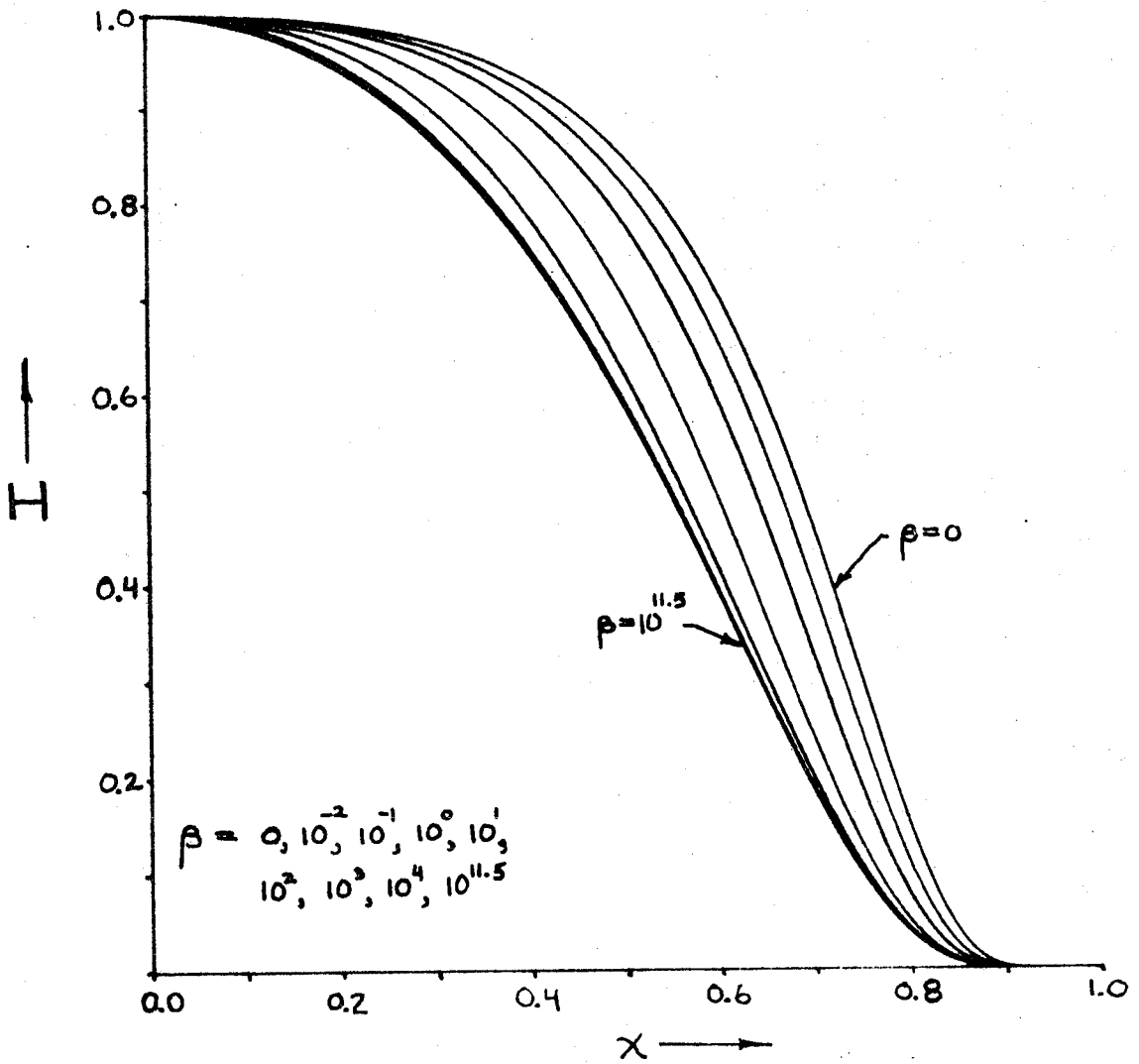


Fig.7.10 SU(3)/SU(2) monopole gauge fields (H) for various β
(SU(2) embedding: $a=1, b=0$)

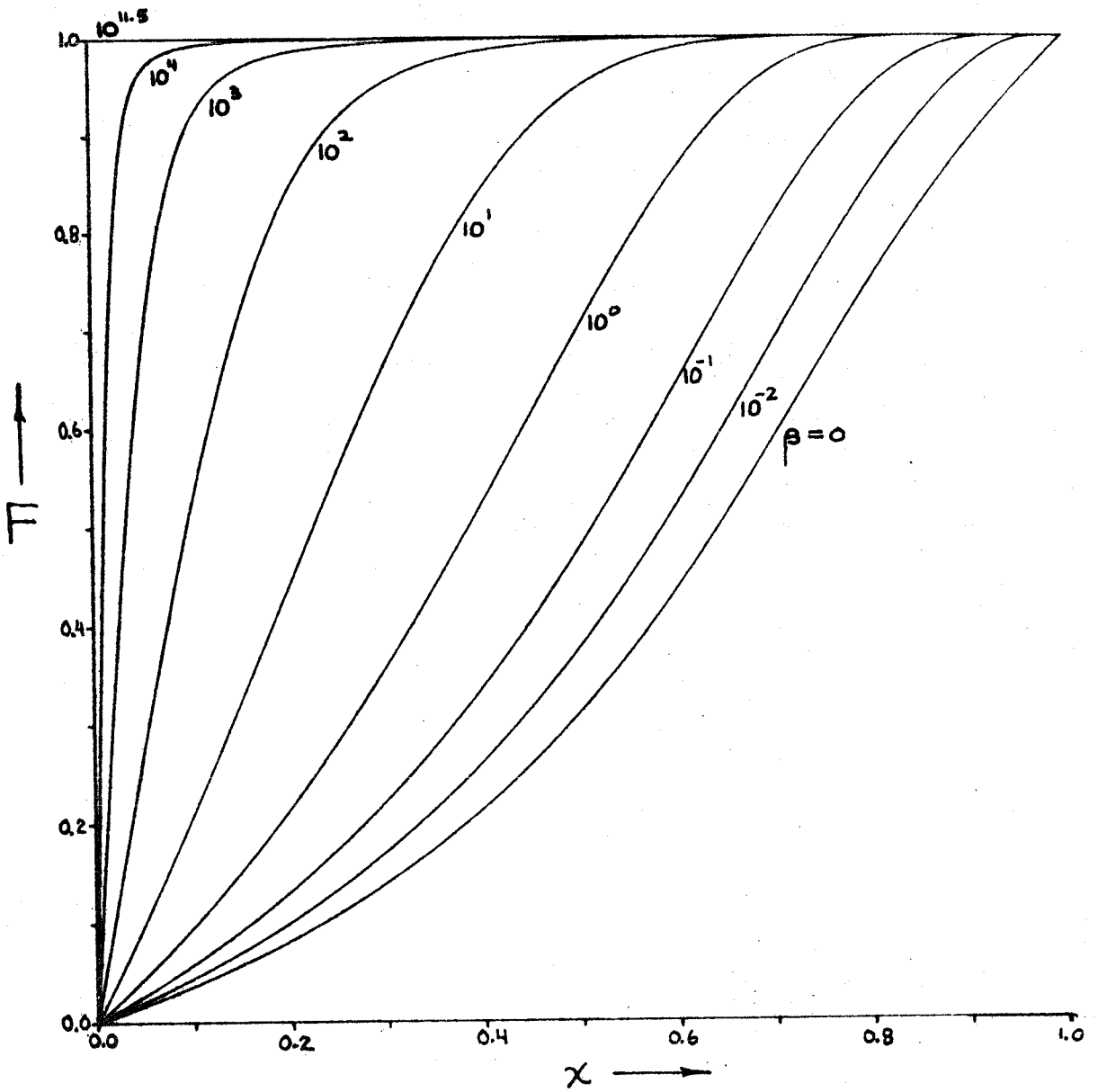


Fig.7.11 $SU(3)/SU(2)$ monopole scalar fields (F) for various β ($SU(2)$ embedding: $a=1, b=0$)

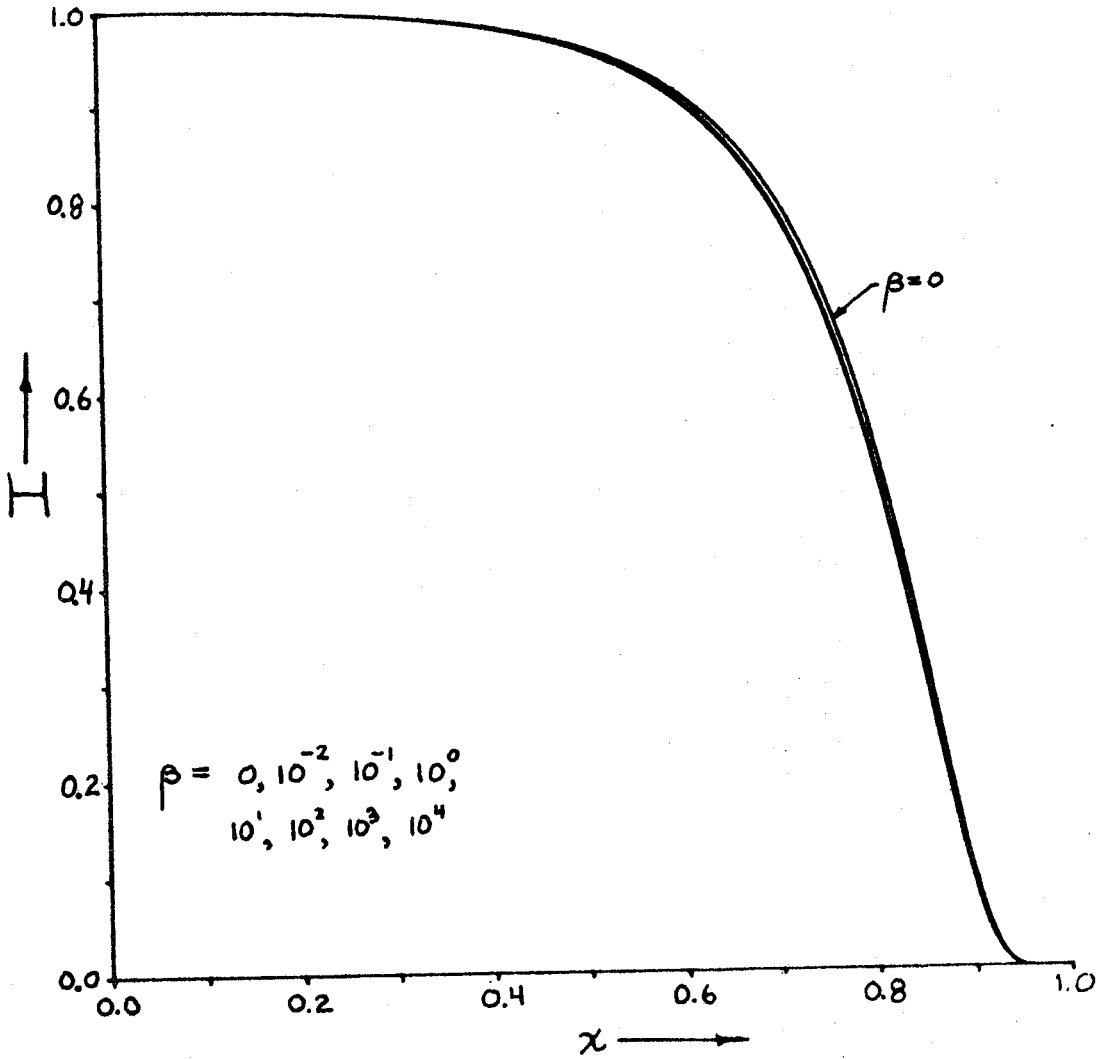


Fig.7.12 SU(3) monopole gauge fields (H) for various β (SU(2) embedding: $a=\frac{1}{2}, b=\frac{\sqrt{3}}{2}$)

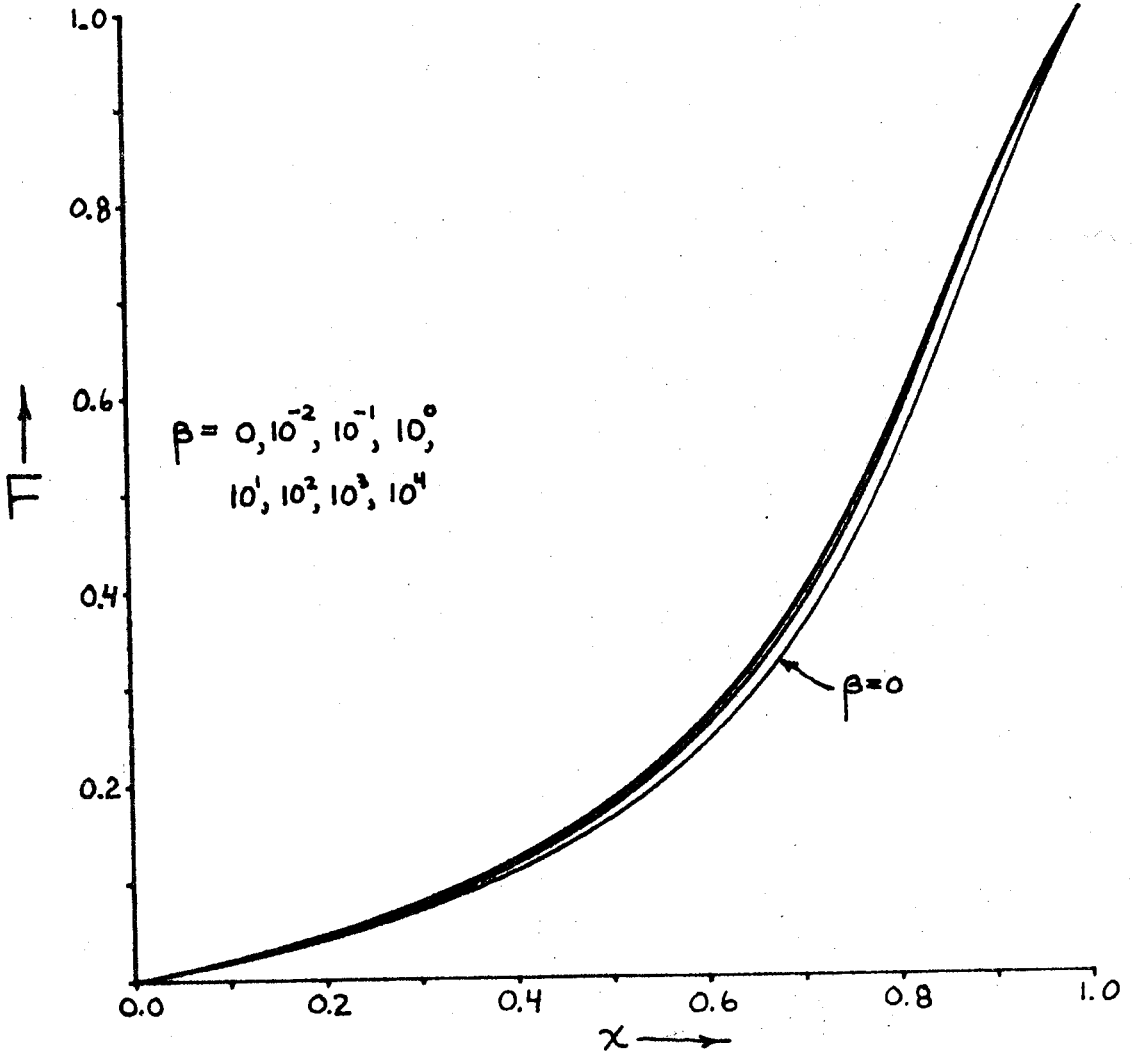


Fig.7.13 SU(3) monopole scalar fields (F) for various β (SU(2) embedding: $a=\frac{1}{2}$, $b=\frac{\sqrt{3}}{2}$).

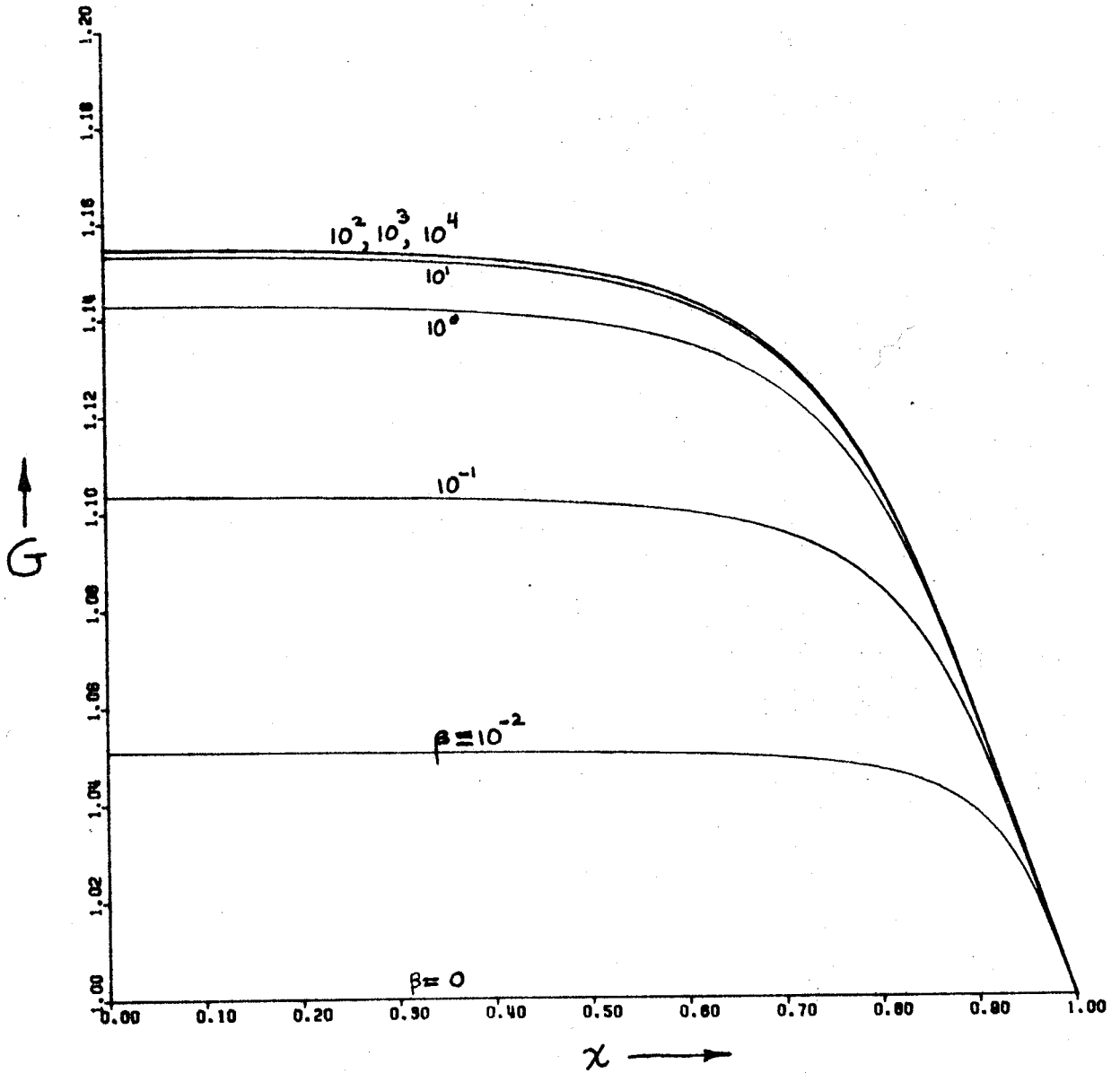


Fig.7.14 SU(3) monopole scalar fields (G) for various β (SU(2) embedding: $a = \frac{1}{2}$, $b = \frac{\sqrt{3}}{2}$)

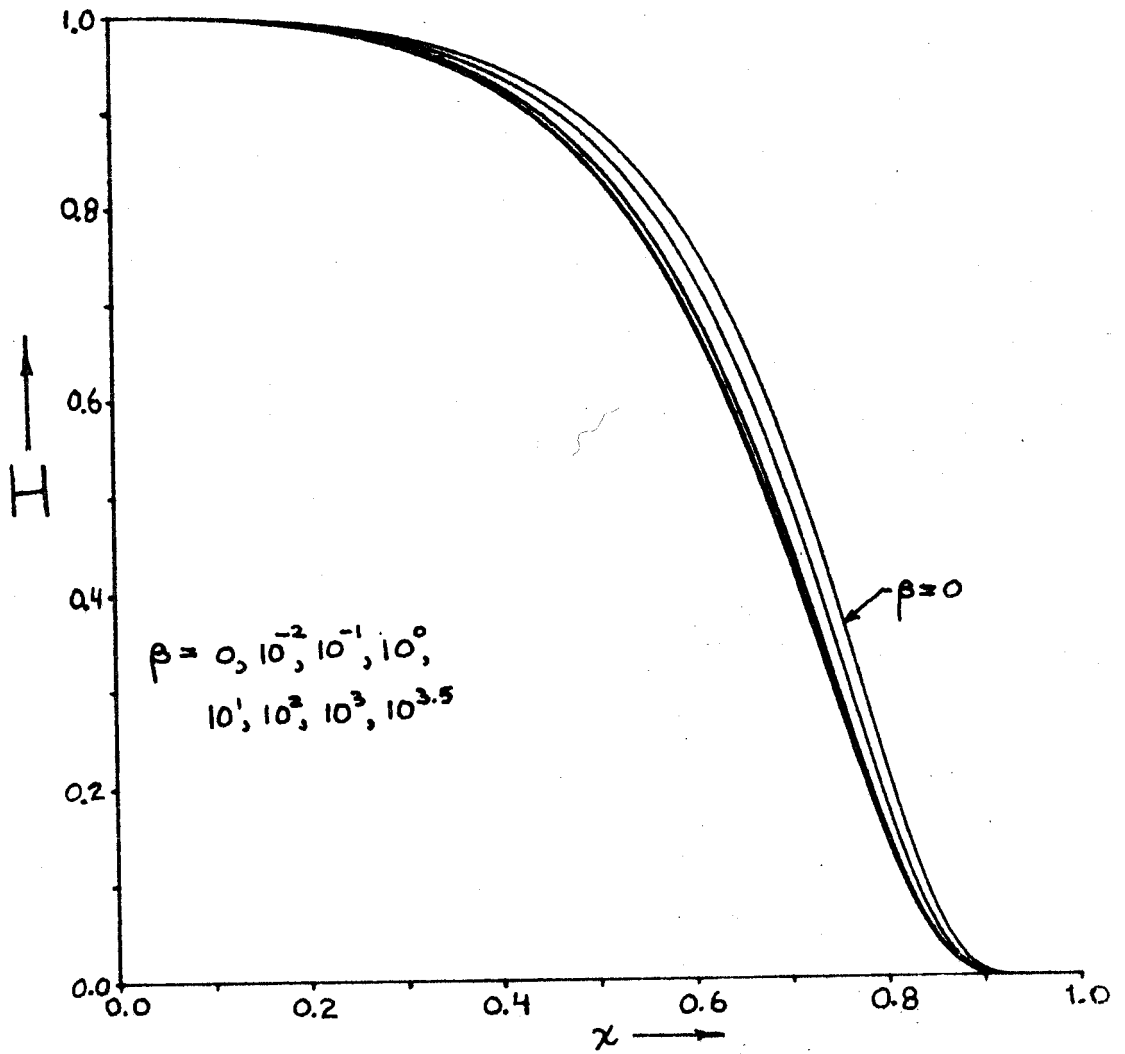


Fig.7.15 SU(3) monopole gauge fields (H) for various β (SU(2) embedding: $a=\frac{\sqrt{3}}{2}, b=-\frac{1}{2}$)

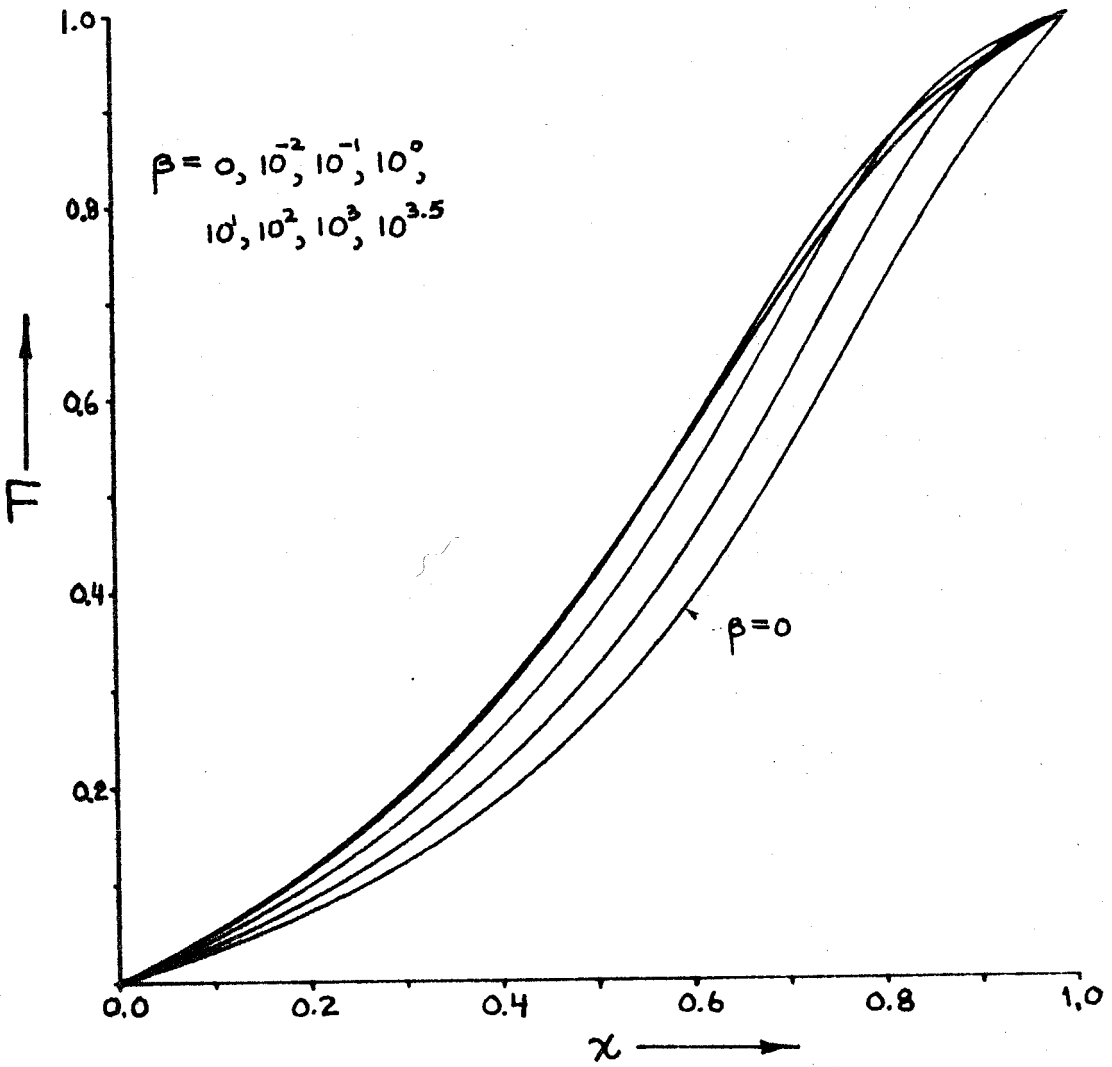


Fig.7.16 SU(3) monopole scalar fields (F) for various β (SU(2) embedding: $a = \frac{\sqrt{3}}{2}$, $b = -\frac{1}{2}$)

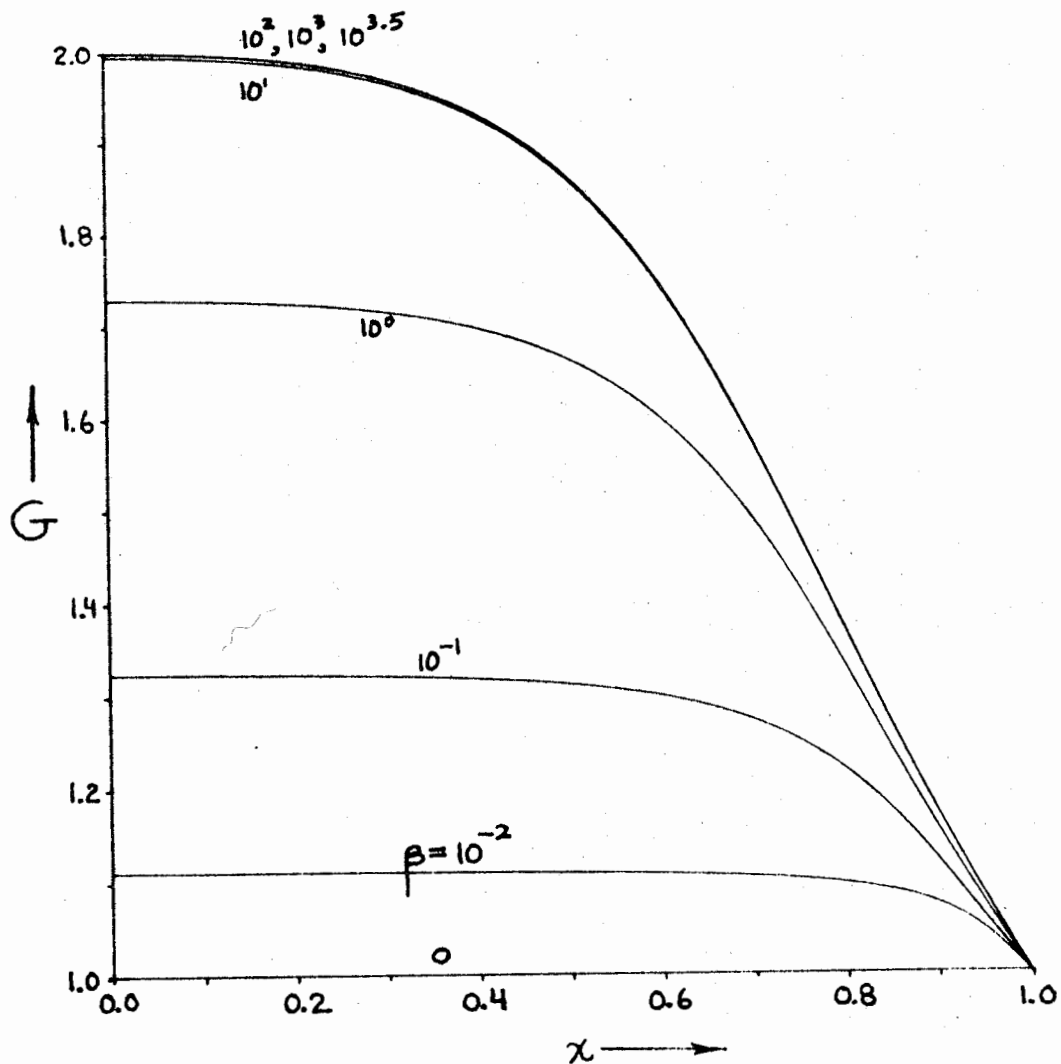


Fig.7.17 SU(3) monopole scalar fields (G) for various β (SU(2) embedding: $a = \frac{\sqrt{5}}{2}$, $b = -\frac{1}{2}$)

The various masses of the monopole solution are displayed on Fig. 7.18. Because of the logarithmic nature of the plot, the masses for $\beta = 0$ are not plotted. It turns out that the numerically calculated mass confirms the analytically predicted value (equation (7.18)) to one part in 10^6 in each case. This is a nice endorsement of collocation as an integration scheme. One may see that the mass saturates in each case.

The mass of the SU(2) monopole ($a=1, b=0$) as plotted in Fig. 7.18 corresponds extremely well with the results given by Cutler and Wyld (1974) for the range $0.1 \leq \beta \leq 100$.

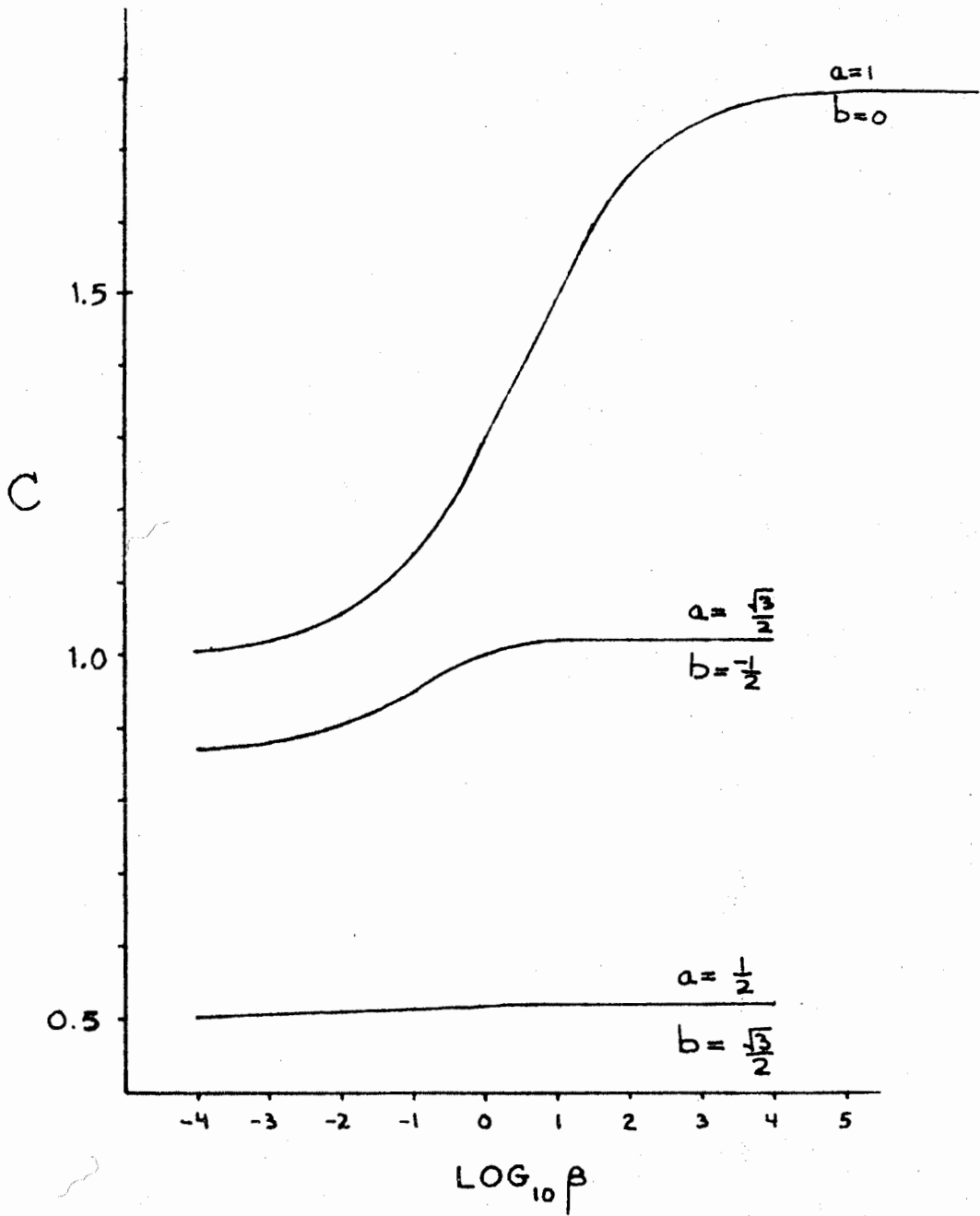


Fig.7.18 SU(3) monopole mass (C) for various β
(SU(2) embeddings)

The explanation of the saturation follows the same argument as presented for the SU(2) dyon. Recall the integral expression for the mass (7.10),

$$C(\beta, a, b) = \int_0^\infty dz \left[H'^2 + \frac{1}{2z^2} (1-H^2)^2 + a^2 F^2 H^2 + \frac{z^2}{2} (a^2 F'^2 + b^2 G'^2) + \frac{\beta z^2}{4} (a^2 F^2 + b^2 G^2 - 1)^2 \right]$$

In the limit as $\beta \rightarrow \infty$ the fields must conspire to eliminate the last term in the integral (in fact it was found numerically that this term becomes small as β becomes large). This implies that F and G are no longer independent and they must be such that,

$$\beta \rightarrow \infty: \quad a^2 F_\infty^2(z) + b^2 G_\infty^2(z) = 1. \quad (7.19)$$

At this point we express G_∞ in terms of F_∞ , substitute back into C and then solve the corresponding Euler-Lagrange equations. Unfortunately we could not obtain a solution for the equations. But, I feel this approach is correct and the technical difficulties associated with obtaining a solution will be overcome in the future. However, the case of the SU(2) monopole ($a=1, b=0$) is explained. The mass integral (7.11),

$$C(\beta, 1, 0) = \int_0^{\infty} dz \left[H'^2 + \frac{1}{2z^2} (1-H^2)^2 + F^2 H^2 + \frac{z^2}{2} F'^2 + \frac{\beta z^2}{4} (F^2 - 1)^2 \right]$$

has the corresponding Euler-Lagrange equations,

$$z^2 H'' = H (H^2 + \alpha^2 z^2 F^2 - 1),$$

$$(z^2 F')' = F (2H^2 + \beta z^2 (F^2 - 1)).$$

In order to keep the mass finite as $\beta \rightarrow \infty$, the last term in the integral must be suppressed. The solution

$$F(z) = 1$$

is not acceptable because it violates the boundary condition $F(0) = 0$. So we propose a solution of the form,

$$F \approx 1 - e^{-\alpha z}, \quad \alpha z \gg 1. \quad (7.20)$$

Substituting this form into the differential equation for F we get,

$$(z^2 F')' \approx \alpha z (2 - \alpha z) e^{-\alpha z} \approx 2H^2 (1 - e^{-\alpha z}) + \beta z^2 (1 - e^{-\alpha z}) [-2e^{-\alpha z} + e^{-2\alpha z}].$$

Assuming $H = 0(\beta^0)$ we find that for $\alpha z \gg 1$ we must have,

$$\alpha^2 = 2\beta \quad \Rightarrow \quad \alpha = \sqrt{2\beta} \quad (7.21)$$

$$F \approx 1 - e^{-\sqrt{2\beta} z}.$$

This function becomes very sharp as $\beta \rightarrow \infty$.

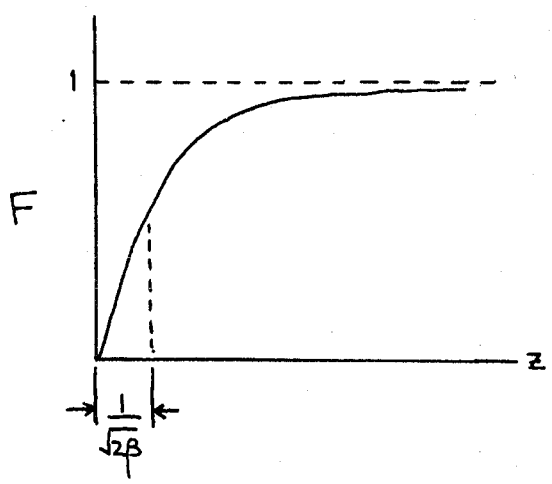


Fig.7.19 Large β form of SU(3)/SU(2) monopole scalar fields (F)
(SU(2) embedding: $a=1, b=0$)

From Fig. 7.11 this seems to be an appropriate description for the behaviour of F for large β . Substituting the form (7.21) in the mass integral and taking the limit as $\beta \rightarrow \infty$ we find all terms involving F vanish. The limiting mass integral is given by,

$$C(\beta \rightarrow \infty, 1, 0) = \int_0^{\infty} \left[H'^2 + \frac{1}{2z^2} (H^2 - 1)^2 + H^2 \right] dz \quad (7.22)$$

To explain saturation physically consider the form of the scalar potential as $\beta = \lambda/e^2$ becomes large. Recall that the symmetry breaking potential takes the form,

$$\begin{aligned} V(\phi) &= \frac{\lambda}{4} (\phi^2 - v^2)^2 \\ &= \frac{\beta M_w^2}{4} \left(\left(\frac{\phi}{v} \right)^2 - 1 \right)^2, \end{aligned}$$

where

$$M_w = ev,$$

and e , the coupling constant are considered fixed constants.

The scalar potential looks something like this,

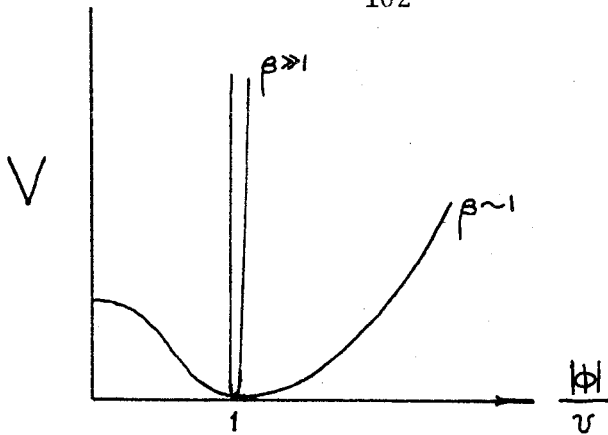


Fig.7.20 Large β form of the scalar potential V

As $\beta \rightarrow \infty$ the scalar field becomes restricted to its vacuum value by a huge potential barrier. If the boundary condition on the field at the origin is not the vacuum value then the field will decay very quickly (instantaneously as $\beta \rightarrow \infty$) to the vacuum value. It seems we obtain a finite energy version of the point monopole. So, as β becomes large the scalar field becomes more restricted and the change in the field should become smaller. This is exactly what is observed. There are several questions one may ask about saturation. Is this phenomena restricted to monopole solutions alone? What are the implications to a full quantum theory? Is there any physical significance? These questions are not answered here.

The SU(3) (SO(3) embedding) is solved only for values of near zero. The corresponding fields are shown in figures 7.21, 7.22, 7.23 and 7.24. The second gauge field H_2 has an extremum at a point other than the boundaries. This is unique among all the fields we have plotted. This phenomena is not explained as yet. The mass is calculated from equation

(5.73) and we find,

$$C(0) = 3.342398.$$

This monopole is certainly the most massive of any investigated. We feel that solutions to the $SO(3)$ embedding can be solved for larger β with some modifications of the numerical method. I am sure that we will again encounter saturation.

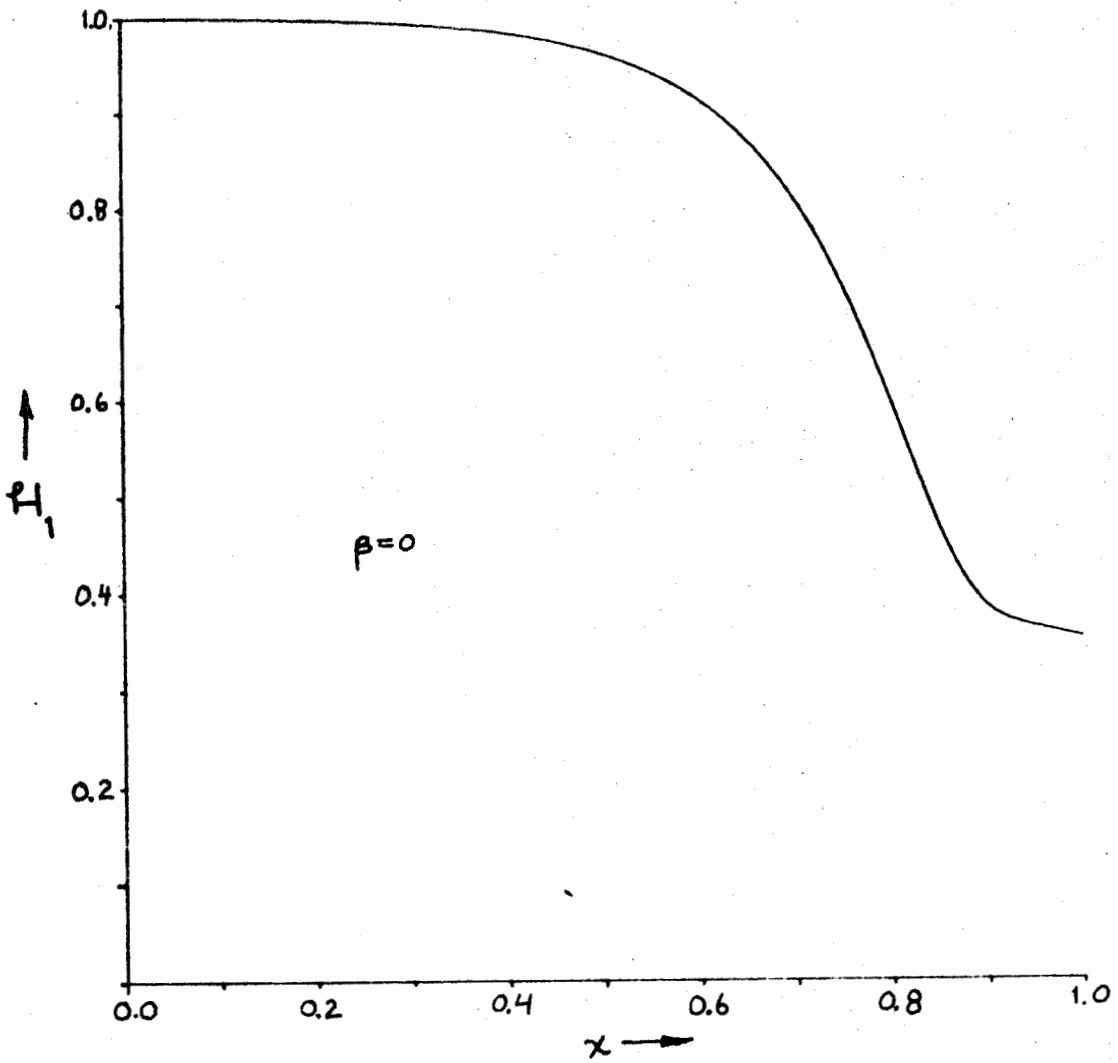


Fig.7.21 SU(3) monopole gauge fields (H_1) for $\beta=0$
(SO(3) embedding)

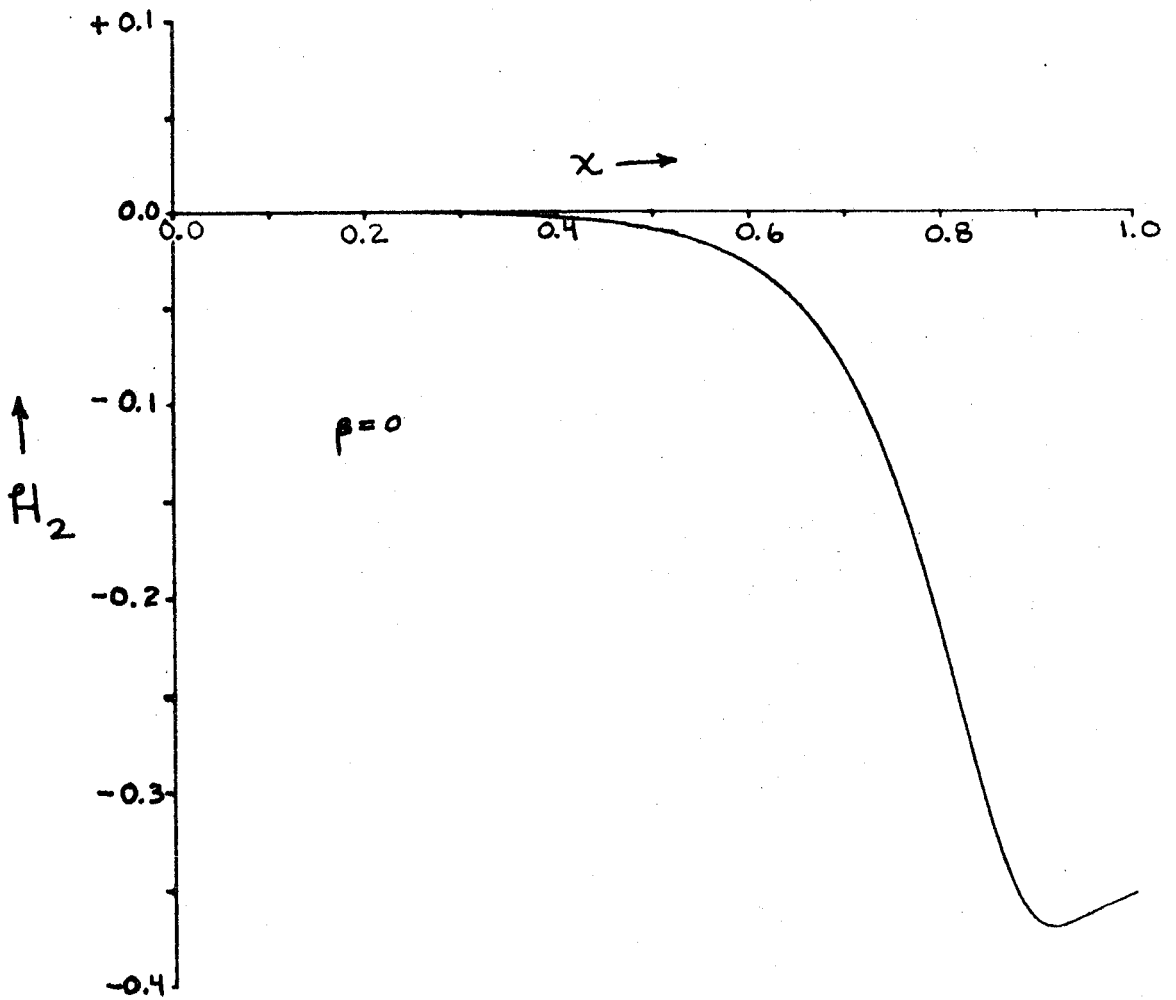


Fig.7. 22 SU(3) monopole gauge fields (H_2) for $\beta=0$
(SO(3) embedding)

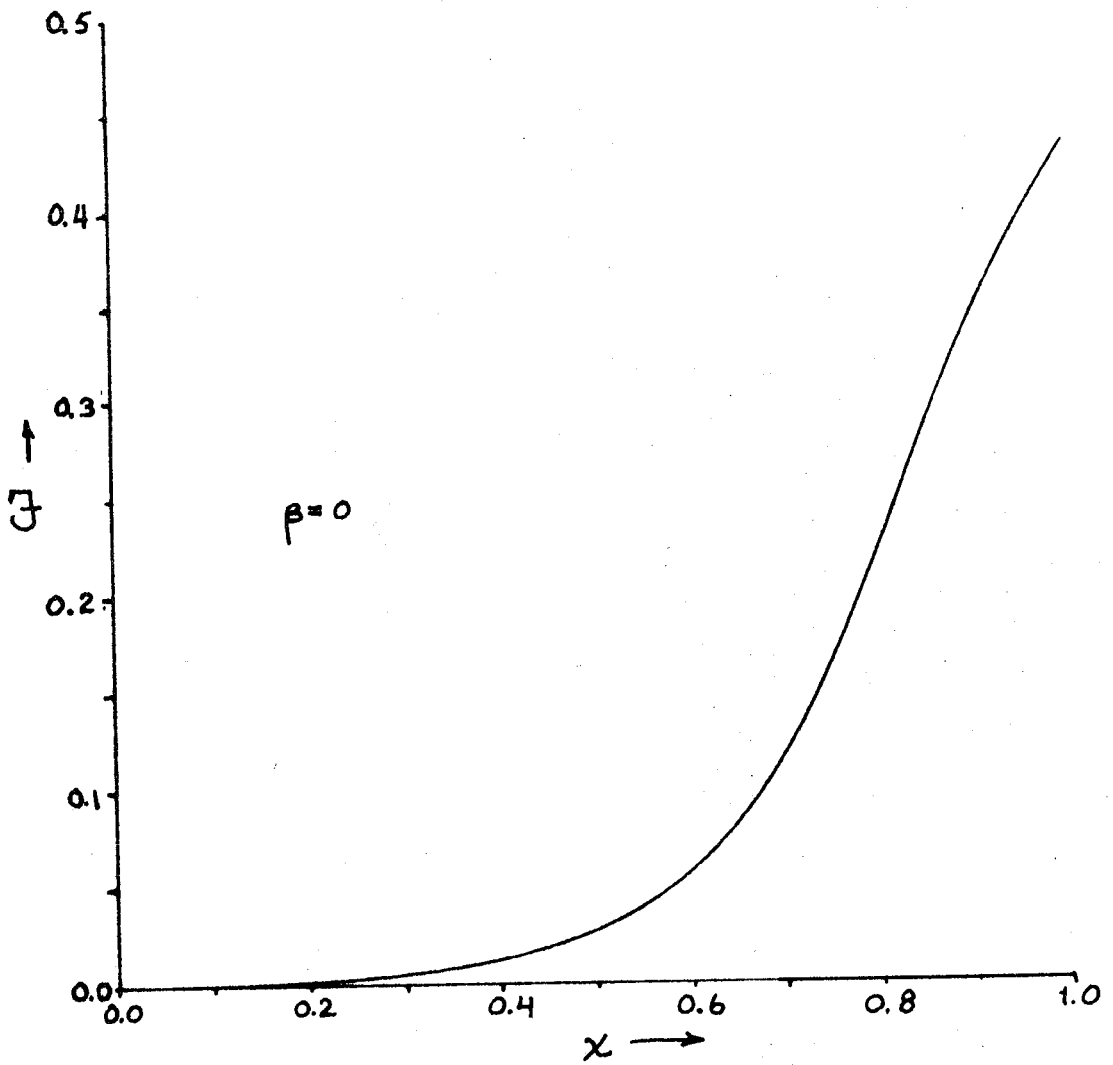


Fig.7.23 SU(3) monopole scalar fields (\mathcal{F}) for $\beta=0$
(SO(3) embedding)

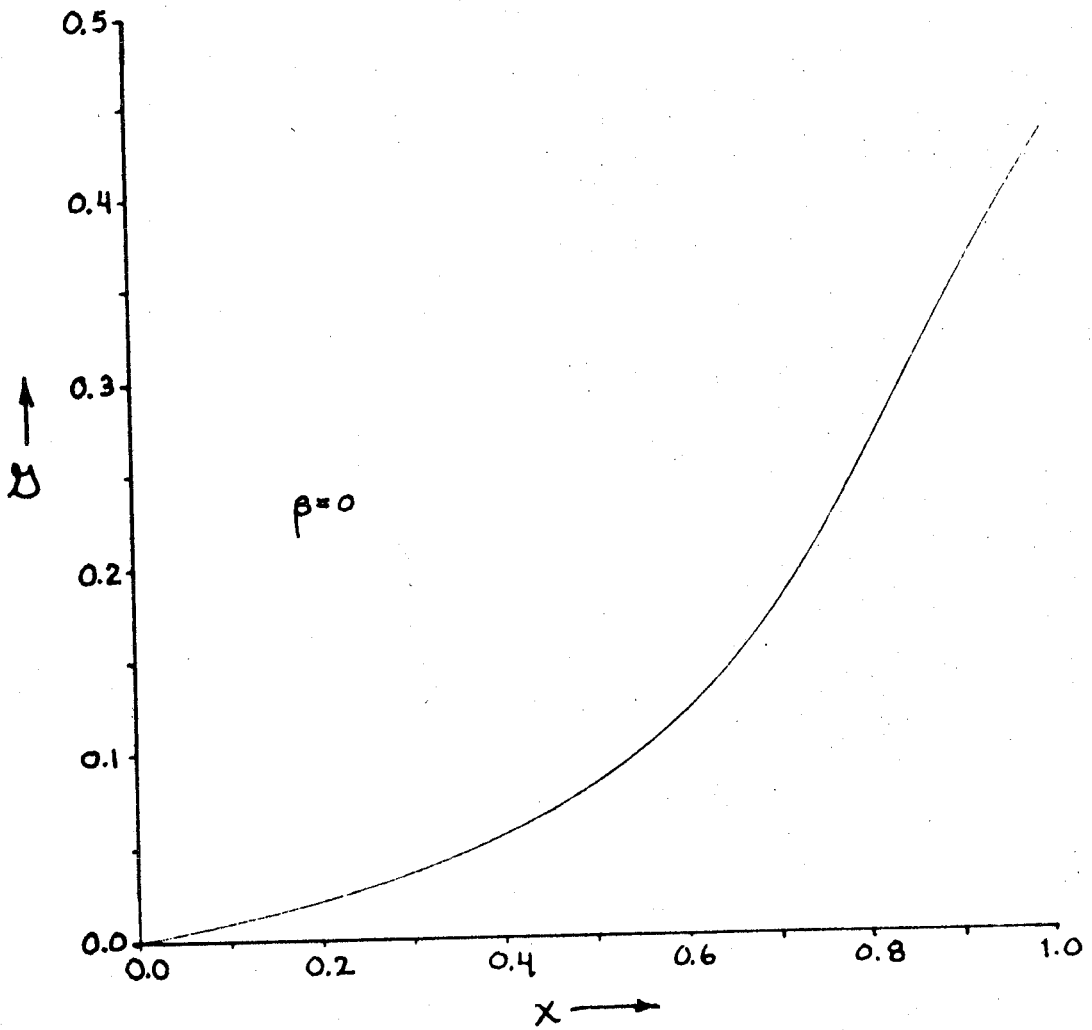


Fig.7.24 SU(3) monopole scalar fields (\mathcal{Y}) for $\beta=0$
(SO(3) embedding)

Chapter 8

Conclusions

In this thesis we have determined all the spherically-symmetric SU(2)-dyon and SU(3) monopole solutions using a numerical method known as collocation. The various solutions and their properties (masses and electric charges) are determined for a wide range of $\beta = \frac{\lambda}{e^2}$ (where $\frac{\lambda}{4}$ is the coefficient of the quartic term in the scalar potential and e is the gauge coupling constant). Previously, only SU(2) monopole and dyon solutions have been investigated numerically. Bais and Primack (1975) considered SU(2) dyon solutions for $0 \leq \beta \leq 10$, while Cutler and Wyld (1976) considered SU(2) monopole solutions for $0 \leq \beta \leq 100$. In these ranges of β our results are in substantial agreement with these computations. However, we have extended the range of β for the SU(2) monopole and dyon to $0 \leq \beta \leq 10^{11.5}$. To our knowledge this thesis represents the only numerical investigation of SU(3) monopole solutions so far.

All physical results depend on the value of the mass of the intermediate vector boson, M_W . This particle has not been observed. 't Hooft (1974) points out that in models by Geogi and Glashow $M_W \approx 53\text{GeV}$. Assuming that this is a reasonable estimate we now have a mass and length scale for our solutions. The mass is expressed as,

$$M = \frac{M_W}{\alpha} C \quad (8.1)$$

where $\alpha = 1/137$. From our results we find that,

$$0.5 \leq C \leq 4,$$

so we find that the monopoles could have a mass of the order of 10^4 GeV which could soon be within experimental reach. The length scale for our solutions are expressed in terms of z (later mapped to x) where,

$$z \equiv e v r \equiv M_w r, \quad (8.2)$$

so $z=1$ corresponds to a radius of

$$r_0 = \frac{1}{M_w} \rightarrow \frac{\hbar c}{M_w} \approx 4 \times 10^{-18} \text{ m.}$$

Thus, our scale of length for the monopole is much smaller than the nuclear radius.

The SU(2) dyon is a point magnetic pole with magnetic charge $\frac{4\pi}{e}$. The electric charge density behaves as $\frac{1}{r}$ near the origin and decays to zero as the radius becomes large. The total electric charge is finite and we find a continuum of values for each value of β .

There are various types of SU(3) magnetic monopoles which are distinguished by: 1. which of two distinct embeddings of the SU(2) subgroup of SU(3) they correspond to; 2. the symmetry of the vacuum. The two embeddings are referred to

as SU(3) and SO(3) while there are two possible vacua, U(1)xU(1) and SU(2)xU(1). We consider the SU(2) embedding for both the U(1)xU(1) and SU(2)xU(1) vacua for $0 \leq \beta \leq 10^5$. The SO(3) embedding is solved for the SU(2)xU(1) vacuum only for $\beta=0$. The distinctions between the various monopoles are discussed in detail by Corrigan et al. (1976) and Sinha (1976). We use their ansätze as a basis for our calculations.

We find that as β becomes large, the solutions along with their corresponding masses (and electric charge) change very little. We call this effect saturation and it is observed for both SU(2) monopole/dyon and SU(3) monopole solutions. A physical explanation of saturation is given. From the explanation we obtain a set of equations that are independent of β which, in principle, can be solved to obtain the infinite β monopole/dyon solutions. These equations are solved for the SU(2) monopole and dyon and give an extremely good agreement to the corresponding large β solutions.

Saturation allows us to give lower and upper bounds to monopole and dyon masses and electric charges. For example, the SU(2) monopole mass is found to lie in the range,

$$1.00 \leq \frac{M}{(M_w/\alpha)} < 1.79,$$

The SU(2) embeddings of SU(3) have the mass spectrums,

$$0.50 \leq \frac{M}{(M_w/\alpha)} < 0.53 \quad (\text{U(1)xU(1) vacuum}),$$

and $0.87 < \frac{M}{(M_w/\alpha)} < 1.03$ (SU(2)xU(1) vacuum).

The monopole corresponding to the SO(3) embedding has a mass for $\beta = 0$ of

$$\frac{M}{(M_w/\alpha)} = 3.34$$

We have shown that the mass of a monopole is a good indication of the symmetry group to which it corresponds. The experimental observation of magnetic monopoles would certainly confirm at least the qualitative correctness of these spontaneously broken theories.

Appendix A

Symmetry Breaking and the Higgs Mechanism¹

Consider the simple example of a Lagrangian density of the form,

$$\mathcal{L} = \frac{1}{2} (D_\mu \phi)^* (D^\mu \phi) - \frac{\mu^2}{2} \phi^* \phi + \frac{\lambda}{4} (\phi^* \phi)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad (\text{A.1})$$

where,

$$D_\mu \phi = \partial_\mu \phi - ie A_\mu \phi, \quad (\text{A.2})$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (\text{A.3})$$

Under local gauge transformations:

$$\phi(x) \rightarrow \phi'(x) = \text{EXP}[-i\theta(x)] \phi(x),$$

$$\phi^*(x) \rightarrow \phi'^*(x) = \text{EXP}[+i\theta(x)] \phi^*(x), \quad (\text{A.4})$$

$$A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) - \frac{1}{e} \partial_\mu \theta(x)$$

while \mathcal{L} is invariant under this U(1) symmetry.

If we had written $+\frac{\mu^2}{2} \phi^* \phi$ in \mathcal{L} instead, the Higgs potential would have had a potential term $V(\phi)$ of the form,

¹This whole appendix borrows heavily from Abers and Lee (1973).

$$V(\phi) = + \frac{\mu^2}{2} \phi^* \phi + \frac{\lambda}{4} (\phi^* \phi)^2, \quad (\text{A.5})$$

which would be represented by,

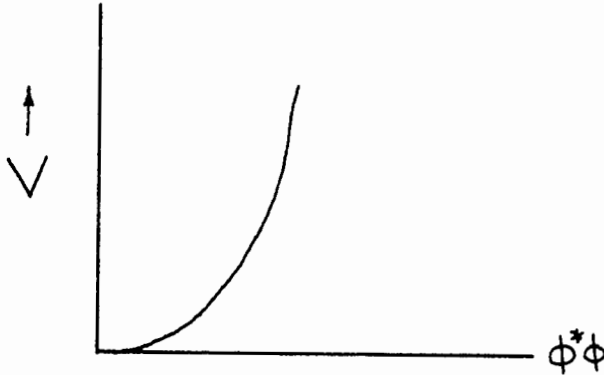


Fig. A.1

A non-symmetry-breaking scalar potential.

and the vacuum expectation value of $|\phi|$ would be zero. This is the Lagrangian density for charged scalar electrodynamics. However the potential in (A.1) has the form,

$$V(\phi) = - \frac{\mu^2}{2} \phi^* \phi + \frac{\lambda}{4} (\phi^* \phi)^2, \quad (\text{A.6})$$

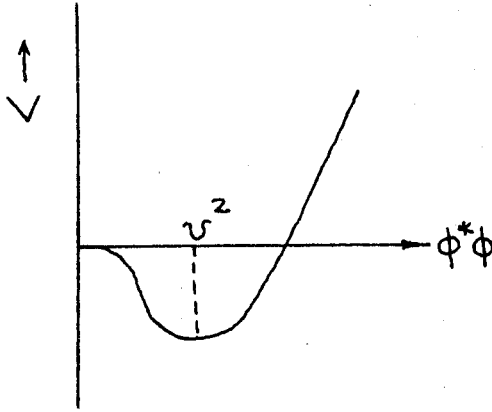


Fig. A.2

A symmetry-breaking scalar potential.

where
$$v^2 = \frac{\mu^2}{\lambda} . \tag{A.7}$$

The vacuum expectation value of $|\phi|$ would be v . We could choose the vacuum in some region to be,

$$\phi_{vac} = v , \quad v \in \mathbb{R} , \tag{A.8}$$

Consider small perturbations about this vacuum value,

$$\phi = (v + \epsilon) \text{EXP} [i \xi / v] \tag{A.9}$$

$$\approx v + \epsilon + i \xi + \text{higher order terms} .$$

The exponential in (A.9) corresponds to a gauge transformation of the vacuum value of the scalar field ϕ_{vac} .² The vacuum is not invariant under the U(1) symmetry of the Lagrangian density and we say that we have a spontaneously broken U(1) symmetry. \mathcal{L} can now be written,

$$\begin{aligned} \mathcal{L} = & -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \partial_{\mu} \epsilon \partial^{\mu} \epsilon - \frac{1}{2} \mu^2 \epsilon + \frac{1}{2} \partial_{\mu} \xi \partial^{\mu} \xi \\ & + \frac{e^2 v^2}{2} A_{\mu} A^{\mu} - e v A_{\mu} \partial^{\mu} \xi + \text{higher order terms.} \end{aligned} \quad (\text{A.10})$$

The ϵ field has mass³ $\sqrt{2}\mu$, but the fields A_{μ} and ξ have been mixed up, making interpretation difficult. To clear this up, consider the following gauge transformation:

$$\phi(x) \rightarrow \phi'(x) = e^{-i\xi/\hbar} \phi = (v + \epsilon), \quad (\text{A.11})$$

$$A_{\mu} \rightarrow A'_{\mu} = A_{\mu} - \frac{1}{e v} \partial_{\mu} \xi. \quad (\text{A.12})$$

\mathcal{L} is invariant under these transformations so we may write,

²We will often use the word vacuum in place of the vacuum value of the scalar field.

³The Lagrangian density producing a Klein-Gordon equation for a real scalar particle of mass m is

$$\mathcal{L} = \frac{1}{2} (\partial_{\mu} \phi \partial^{\mu} \phi - m^2 \phi^2)$$

which gives $(\square + m^2)\phi = 0$.

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}' F^{\mu\nu}' + \frac{1}{2} \partial^\mu \epsilon \partial_\mu \epsilon + \frac{1}{2} e^2 v^2 A_\mu' A'^\mu + \frac{1}{2} e^2 A_\mu'^2 \epsilon (2v + \epsilon) - \lambda e^2 v^2 \left(1 + \frac{\epsilon}{2v}\right)^2. \quad (\text{A.13})$$

In this gauge there are no coupling terms of the form $A'^\mu \partial_\mu \epsilon$ so the masses may be read off the quadratic terms. There is a scalar ϵ -meson, with mass $\sqrt{2}\mu$, a massive vector meson A_μ' , with mass $M_w = ev$, and no particle corresponding to ξ which has been gauged away. From equation (A.12) we see ξ has disappeared into the longitudinal component of the vector field in the new gauge.

In this example, we started with a Lagrangian density invariant under a U(1) group of symmetry transformations with a corresponding single gauge field. In the vacuum the symmetry was completely broken and the vector field acquired mass ev .

Now we shall look at the general features of a spontaneously-broken gauge model. Consider a Lagrangian density invariant under local gauge transformations of some group G of dimension N. There are n scalar fields which transform under an n-dimensional representation, along with N gauge fields⁴. Suppose the symmetry breaking leaves the vacuum invariant under an M dimensional subgroup H of G. Thus,

⁴I shall give only the features of the Higgs mechanism. For those interested in a general proof see Abers and Lee (1973).

there are M generators satisfying $L^a \phi_{\text{vac}} = 0$. Under the Higgs mechanism we will then have M massless vector fields, $(N-M)$ massive vector fields and $(n-(N-M))$ scalar mesons (these objects do not include the monopole).

In the previous example we have $N=1$, $M=0$, $n=2$ giving the proper results.

In chapters 4 and 5 we use this general result to determine the number of massless vector fields to be expected in the various vacua.

Appendix B

Boundary Conditions on the SU(2) Fields

This section is devoted to an explanation of the boundary conditions (4.22). A useful approach is to demand that the mass be finite. Recall equation (4.25),

$$C(\beta, \eta) = \int_0^\infty dz \left[H'^2 + \frac{(H^2-1)^2}{2z^2} + \frac{H^2}{z^2} (J^2+F^2) + \frac{(zJ'-J)^2 + (zF'-F)^2}{z^2} + \frac{\beta}{4z^2} (F^2 - z^2)^2 \right]$$

since each term in the integrand should give a finite contribution as z goes to zero, the second and third term in the integrand implies:

$$H(0) = \pm 1, \tag{B.1}$$

(we shall use +1) and

$$J(0) = 0, \tag{B.2}$$

$$F(0) = 0. \tag{B.3}$$

At large radii the solutions must satisfy the point monopole conditions (3.20) and (3.21). The scalar field must be such that the potential is a minimum so (4.9) implies,

$$\phi^a \phi^a \xrightarrow{r \rightarrow \infty} v^2,$$

substituting in (4.11) gives,

$$F \xrightarrow{z \rightarrow \infty} \pm e^{\nu r} = \pm z, \quad (\text{B.4})$$

(we shall use +z).

The third term in C then implies that,

$$H \xrightarrow{z \rightarrow \infty} 0 \quad (\text{B.5})$$

The differential equation for J (4.20) in the limit of $z \rightarrow \infty$ has the form,

$$J'' \rightarrow 0,$$

so $J \underset{z \rightarrow \infty}{\sim} \eta z,$

where η is a constant. Now we see that (4.19) has the form,

$$H'' \xrightarrow{z \rightarrow \infty} (1 - \eta^2) H,$$

in order for H to decay to zero we must have $\eta^2 < 1$. Thus, the final boundary condition is given by,

$$J \underset{z \rightarrow \infty}{\sim} \eta z, \quad (\text{B.6})$$

$$|\eta| < 1,$$

(we shall choose $\eta > 0$).

Appendix C

Boundary Conditions on SU(3) Fields

First consider the SU(2) embedding with mass integral (5.71),

$$C(\beta, a, b) = \int_0^\infty dz \left[H'^2 + \frac{(1-H^2)^2}{2z^2} + a^2 F^2 H^2 + \frac{z^2}{2} (a^2 F'^2 + b^2 G'^2) + \frac{\beta z^2}{4} (a^2 F^2 + b^2 G^2 - 1)^2 \right]. \quad (C.1)$$

The second term in the integrand implies,

$$H(0) = \pm 1, \quad (C.2)$$

where we shall choose the positive value. The last term in (C.1) implies,

$$a^2 F^2(\infty) + b^2 G^2(\infty) - 1 = 0,$$

together with (5.47)

$$a^2 + b^2 = 1,$$

gives $F(\infty) = \pm 1,$ (C.3)

and $G(\infty) = \pm 1,$ (C.4)

where we shall use +1 in both cases. Now the third term in

(C.1) requires,

$$H(\infty) = 0. \tag{C.5}$$

Assuming our solutions are regular at the origin we may expand them in a power series:

$$\begin{aligned} H &= 1 + h_1 z + h_2 z^2 + \dots, \\ F &= f_0 + f_1 z + f_2 z^2 + \dots, \\ G &= g_0 + g_1 z + g_2 z^2 + \dots, \end{aligned} \tag{C.6}$$

Substituting in the differential equations (5.75), (5.76) and (5.77) we find to second order in z ,

$$\begin{aligned} H &\approx 1 + h_2 z^2, \\ F &\approx f_1 z, \\ G &\approx g_0 + g_2 z^2, \end{aligned} \tag{C.7}$$

So we obtain our final boundary conditions

$$F(0) = 0 \tag{C.8}$$

$$G'(0) = 0. \tag{C.9}$$

The $SO(3)$ embedding requires a more extensive investigation.

From (5.50) and (5.69) we see that the scalar field has the form

$$\Phi = v \begin{pmatrix} \mathcal{F}(n^2 - \frac{1}{3}) & \mathcal{F}n^2 + i\mathcal{D}n^3 & \mathcal{F}n^3 - i\mathcal{D}n^2 \\ \mathcal{F}n^2 - i\mathcal{D}n^3 & \mathcal{F}(n^2 - \frac{1}{3}) & \mathcal{F}n^3 + i\mathcal{D}n^2 \\ \mathcal{F}n^3 + i\mathcal{D}n^2 & \mathcal{F}n^3 - i\mathcal{D}n^2 & \mathcal{F}(n^2 - \frac{1}{3}) \end{pmatrix} \quad (\text{C.10})$$

with eigenvalues,

$$v \frac{2}{3} \mathcal{F}, \quad -v \left(\frac{\mathcal{F}}{3} \pm \mathcal{D} \right)$$

From the discussion in chapter 5 we demand that these approach, in the asymptotic limit, the same eigenvalues as $+\frac{v}{2} \lambda_3$,

i.e. $\frac{v}{2\sqrt{3}}, \frac{v}{2\sqrt{3}}, -\frac{v}{\sqrt{3}}$. Which implies,

$$\mathcal{F}(\infty) = \frac{\sqrt{3}}{4}, \quad \mathcal{D}(\infty) = \frac{\sqrt{3}}{4}, \quad (\text{C.11})$$

or,
$$\mathcal{F}(\infty) = \frac{\sqrt{3}}{2}, \quad \mathcal{D}(\infty) = 0, \quad (\text{C.12})$$

We will have to distinguish between these cases later.

The mass integral that we wish to minimize (and be finite) is (5.73),

$$\begin{aligned}
 C(\beta) = \int_0^\infty dz \left[4 \left(\mathcal{H}_1'^2 + \mathcal{H}_2'^2 + \frac{(\mathcal{H}_1^2 + \mathcal{H}_2^2 - 1)^2 + 12 \mathcal{H}_1^2 \mathcal{H}_2^2}{2z^2} \right) \right. \\
 + 2z^2 \left(\frac{\mathcal{F}'^2}{3} + \mathcal{Y}'^2 \right) \\
 + 4 \left((\mathcal{F}\mathcal{H}_1 + \mathcal{Y}\mathcal{H}_2)^2 + (\mathcal{F}\mathcal{H}_2 + \mathcal{Y}\mathcal{H}_1)^2 \right) \\
 \left. + \frac{\beta z^2}{4} \left(\frac{4}{3} \mathcal{F}^2 + 4\mathcal{Y}^2 - 1 \right)^2 \right]. \tag{C.13}
 \end{aligned}$$

To minimize C, the term

$$\mathcal{H} \equiv (\mathcal{H}_1^2 + \mathcal{H}_2^2 - 1)^2 + 12 \mathcal{H}_1^2 \mathcal{H}_2^2 \tag{C.14}$$

must approach its minimum value as $z \rightarrow \infty$. The extrema of \mathcal{H} are given by

$\mathcal{H}_1(\infty)$	$\mathcal{H}_2(\infty)$	\mathcal{H}
0	0	1
0 ±1	±1 0	0
± $\frac{1}{\sqrt{8}}$	± $\frac{1}{\sqrt{8}}$	$\frac{15}{16}$
± $\frac{1}{\sqrt{8}}$	∓ $\frac{1}{\sqrt{8}}$	

Table C.1

Possible asymptotic values for the SU(3) gauge fields (SO(3) embedding) and their contribution to the monopole mass.

We reject the $H = 1$ case because it does not minimize C. Also we reject the $H = 0$ case because we required additionally that,

$$Z(\omega) H_1(\omega) + Y(\omega) H_2(\omega) = 0 \quad (C.15)$$

and
$$Z(\omega) H_2(\omega) + Y(\omega) H_1(\omega) = 0 \quad (C.16)$$

These conditions cannot be satisfied for (C.11) or (C.12) under the $H = 0$ conditions. For $H = \frac{15}{16}$ the only acceptable set of conditions is (C.11),

$$Z(\omega) = Y(\omega) = \frac{\sqrt{3}}{4},$$

and
$$H_1(\omega) = -H_2(\omega) = \pm \frac{1}{\sqrt{8}}, \quad (C.17)$$

where we will choose the plus sign.

As $z \rightarrow 0$ we must have,

$$H_1(0) H_2(0) = 0, \quad (C.18)$$

and
$$H_1^2(0) + H_2^2(0) = 1, \quad (C.19)$$

which has the solutions:

$H_1(0)$	$H_2(0)$
0	± 1
± 1	0

Table C.2

Possible boundary conditions for the SU(3) gauge fields (SO(3) embedding) at the origin.

Not all of these are compatible with (C.17). Consider the following "hand-wavy" argument.

It is reasonable that to minimize C, the functions $H_1(z)$ and $H_2(z)$ should approach their respective asymptotic values as close to the origin, as possible. This principle will favour some boundary conditions over others. Consider the following two situations:

(i) $H_1(0) = 1$, $H_1(\infty) = +\frac{1}{\sqrt{8}}$,

(ii) $H_1(0) = 1$, $H_1(\infty) = -\frac{1}{\sqrt{8}}$

which are depicted on the figure below,

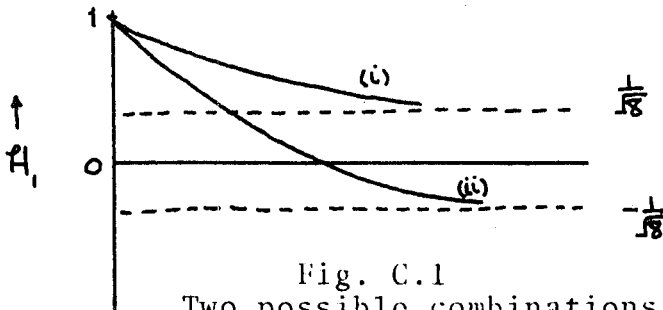


Fig. C.1
Two possible combinations of boundary conditions for the SU(3) gauge fields (SO(3) embedding).

It may be argued that (i) will be of lower energy than (ii) because:

1. A solution like (i) will be closer to its asymptotic limit, on average, than (ii).
2. The derivative contribution to C will tend to be larger for (ii) than (i).

On the basis of these arguments we reject boundary conditions of type (ii). We choose

$$\begin{aligned} \mathcal{H}_1(0) = 1, \quad \mathcal{H}_1(\infty) = \frac{1}{\sqrt{8}}, \\ \mathcal{H}_2(0) = 0, \quad \mathcal{H}_2(\infty) = -\frac{1}{\sqrt{8}}. \end{aligned} \tag{C.20}$$

Expanding the functions near the origin in four power series we find from the differential equations that:

$$\begin{aligned} \mathcal{H}_1 &\approx 1 + \dots, \\ \mathcal{H}_2 &\approx h_2 z + \dots, \\ \mathcal{F} &\approx f_1 z + \dots, \\ \mathcal{D} &\approx g_1 z + \dots, \end{aligned}$$

and so we require

$$\mathcal{F}(0) = \mathcal{D}(0) = 0 \quad (C.21)$$

There are many other boundary conditions that would yield the same energy due to the fact that C is invariant under a number of transformations, eg.

$$\mathcal{F} \rightarrow \mathcal{F},$$

$$\mathcal{D} \rightarrow \mathcal{D},$$

$$\mathcal{H}_1 \rightarrow -\mathcal{H}_1,$$

$$\mathcal{H}_2 \rightarrow -\mathcal{H}_2,$$

etc. ...

Appendix D

The Gell-Mann Matrices (see Gell-Mann (1964))

$$\begin{aligned} \lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda_2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} & \lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} & \lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ \lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} & \lambda_8 &= \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{2}{\sqrt{3}} \end{pmatrix} \end{aligned}$$

$$\text{Tr } \lambda_i \lambda_j = 2 \delta_{ij}$$

$$[\lambda_i, \lambda_j] = 2i f_{ijk} \lambda_k \quad (\text{Lie Product})$$

$$\{\lambda_i, \lambda_j\} = \frac{4}{3} \delta_{ij} 1 + 2d_{ijk} \lambda_k \quad (\text{Anticommutator})$$

ijk	f_{ijk}	ijk	d_{ijk}
123	1	118	$1/\sqrt{3}$
147	$1/2$	146	$1/2$
156	$-1/2$	157	$1/2$
246	$1/2$	228	$1/\sqrt{3}$
257	$1/2$	247	$-1/2$
345	$1/2$	256	$1/2$
367	$-1/2$	338	$1/\sqrt{3}$
458	$\sqrt{3}/2$	344	$1/2$
678	$\sqrt{3}/2$	355	$1/2$
		366	$-1/2$
		377	$-1/2$
		448	$-1/(2\sqrt{3})$
		558	$-1/(2\sqrt{3})$
		668	$-1/(2\sqrt{3})$
		778	$-1/(2\sqrt{3})$
		888	$-1/\sqrt{3}$

Table D.1

SU(3) structure constants f_{ijk} and anticommutator coefficients d_{ijk} .

Appendix E

Homotopy Theory

This entire section is a summary of a very nice review by Marciano and Pagels (1978). We shall present a very terse and incomplete introduction to this subject.

Suppose we have two manifolds X and Y with a set of continuous maps from X into Y ,

$$f: X \mapsto Y \tag{E.1}$$

or $f(x) = y, \quad x \in X, \quad y \in Y.$

Two mappings are called homotopic if they can be continuously distorted into each other. In other words, there exists a set of maps $F(x,t)$ (parameterized by $t, 0 \leq t \leq 1$) called a homotopy which is continuous in both x and t , such that

$$F(x,0) = f_0(x), \quad F(x,1) = f_1(x). \tag{E.2}$$

We say f_0 is homotopic to f_1 which we shall denote by,

$$f_0 \sim f_1.$$

It is trivial to show \sim is an equivalence relation, i.e. if $f_0 \sim f_1$ and $f_1 \sim f_2$ then $f_0 \sim f_2$. Homotopically equivalent maps form a class denoted by $\{f\}$.

Suppose X is the closed interval $I = [0,1]$ with the

endpoints identified. This space is topologically equivalent to a circle S^1 with a reference point x_0 identified with 0 and 1. Consider continuous maps where $f(0) = f(1) = y_0$ (a fixed point in Y), then the equivalence classes of maps $\{f\}, \{g\}, \dots$ from $S^1 \rightarrow Y$ form a group. The identity $\{e\}$ is the class of maps homotopic to the constant map C ,

$$C(x) = y_0, \quad \forall x \in X. \quad (E.3)$$

The inverse of $\{f\}$ is $\{f^{-1}\}$, where

$$f^{-1}(x) = f(1-x) \quad (E.4)$$

Group multiplication is defined by,

$$\{f\} * \{g\} = \{f \cdot g\} \quad (E.5)$$

$$f \cdot g(x) = \begin{cases} f(2x) & 0 \leq x \leq \frac{1}{2} \\ g(2x-1) & \frac{1}{2} \leq x \leq 1 \end{cases} \quad (E.6)$$

We call this the first homotopy group of Y denoted by $\pi_1(Y)$.

As an example, consider the space

$$Y = \mathbb{R}^2 - (0,0)$$

(the real plane with a hole in it). A group multiplication looks like,

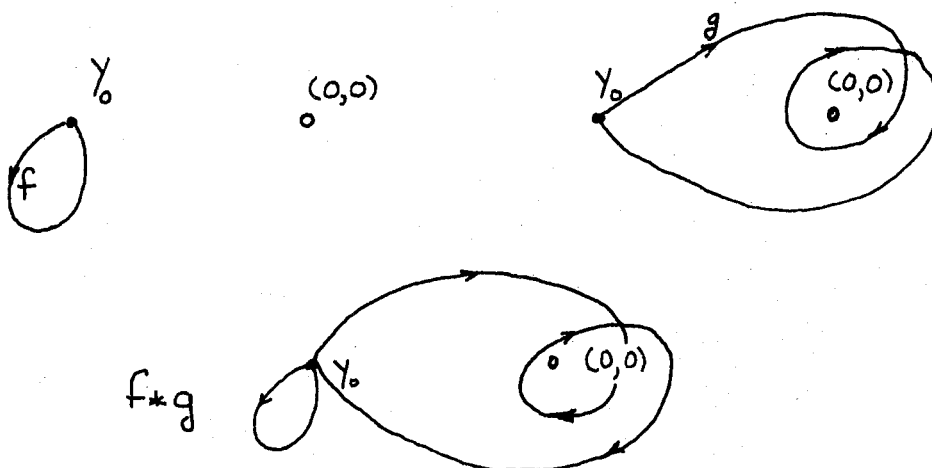


Fig. E.1

Homotopy group multiplication.

Loops that do not enclose (0,0) can be shrunk continuously to the point y_0 . These maps belong to the class of the identity $\{e\} \equiv \{0\}$, while loops that encircle (0,0) n times clockwise belong to a separate class denoted by $\{n\}$ (counterclockwise loops belong to $\{-1\}, \{-2\}, \dots$). So we have,

$$\pi_1(\mathbb{R}^2 - (0,0)) = \mathbb{Z} \quad (\text{the set of integers}) \quad (\text{E.7})$$

with addition representing the group product. The integer n (from $\{n\}$) is called the winding number.

We may generalize the mapping to $X=S^n$ (the n -dimensional sphere) so the classes of mappings with one fixed point

$f(x_0) = y_0$ form the n^{th} homotopy group denoted by $\pi_n(Y)$.

Mathematicians have studied this subject for some time.

Two well-known results important to the theory of monopoles are:

$$\pi_n(S^n) = \mathbb{Z}, \quad (\text{E.8})$$

also, if G is a simply connected compact Lie group and H is a subgroup of G , then

$$\pi_2(G/H) = \pi_1(H) \quad (\text{E.9})$$

where G/H is the space of cosets. Coleman (1975) has shown that equation (E.9) is useful in deciding whether or not a particular non-abelian gauge theory may contain topologically stable soliton solutions (magnetic monopoles). That is, if $\pi_2(G/H)$ is non-trivial then the theory has the possibility (but no guarantee) that a topologically stable soliton exists. However, if $\pi_2(G/h)$ is trivial then no soliton will exist.

Consider the following examples used in this thesis:

$$G = \text{SU}(2), \quad H = \text{U}(1), \quad (\text{E.10})$$

$$\pi_2(\text{SU}(2)/\text{U}(1)) = \pi_1(\text{U}(1)) = \mathbb{Z} \quad (\text{E.11})$$

which means there is the possibility of an infinite variety of

monopoles (only one has been found).

$$G = SU(3), \quad H = SU(2) \times U(1), \quad (E.12)$$

$$\text{then } \pi_2(G/H) = \pi_1(H) = \pi_1(SU(2)) + \pi_1(U(1)) = \mathbb{Z} \quad (E.13)$$

$$\text{and } G = SU(3), \quad H = U(1) \times U(1), \quad (E.14)$$

$$\text{then } \pi_2(G/h) = \mathbb{Z} + \mathbb{Z}, \quad (E.15)$$

which implies $SU(3)$ contains two different monopoles corresponding to the different vacua.

Appendix F

Notation Conventions

Throughout the text we shall be using natural units defined by

$$\hbar = c = 1 \tag{F.1}$$

The dimensions of the fundamental quantities become, $[E] = [M] = [L]^{-1} = [T]^{-1} = (\text{unit energy})$ and $[Q] = 1$, (electric and magnetic charge are dimensionless). Also, we shall use the natural system of electromagnetic units where Maxwell's equations take the form,

$$\partial^\mu F_{\mu\nu} = j_\nu \tag{F.2}$$

An important result of these conventions is that the fine structure constant takes the form,

$$\alpha = \frac{e^2}{4\pi} \approx \frac{1}{137} \tag{F.3}$$

Space-time points of Minkowski space will be represented by the contravariant coordinates,

$$x^\mu = (x^0, x^1, x^2, x^3) \equiv (x^0, \underline{x}) \tag{F.4}$$

where

$$x^0 = t, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z. \tag{F.5}$$

The metric tensor will be taken to be,

$$g_{00} = -g_{11} = -g_{22} = -g_{33} = 1 \quad (\text{F.6})$$

$$g_{\mu\nu} = 0 \quad (\mu \neq \nu).$$

As a general rule Greek indices μ, ν, λ, \dots run from zero to three, while the Latin indices i, j, k, \dots run from one to three. Any repeated index will be summed over its range unless stated otherwise.

Vectors in Euclidean three-space shall be denoted by a "squiggle", \underline{x} . The directional cosines are written,

$$n^i \equiv \frac{x^i}{|\underline{x}|} = \frac{x^i}{\sqrt{x^j x^j}}. \quad (\text{F.7})$$

We may have vectors in another vector space a "color-space". The components of this space shall be denoted by the indices a, b, c, \dots etc. A vector in this space is denoted by an arrow on top,

$$\vec{\phi}, \vec{A}_\mu$$

Sometimes a matrix notation is simpler. We convert to matrix notation by summing the components of a color-vector with the matrices of a representation of a group G whose dimension is equal to the dimension of the color-space.

$$\Phi \equiv \vec{\phi} \cdot \vec{L} \equiv \phi^a L^a \quad (\text{F.8})$$

$$A_\mu \equiv \vec{A}_\mu \cdot \vec{L} \equiv A_\mu^a L^a \quad (\text{F.9})$$

where $a=1, \dots, n$ $n = \dim G$,
and L^a are finite dimensional matrices.

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