PRICES OF RISKY ASSETS IN GENERAL EQUILIBRIUM

by

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ABSTRACT

The empirical evidence indicating that the intercept on the security market line is higher than the riskless rate of interest has motivated several extensions of the simple Capital Asset Pricing Model of Sharpe and Lintner, in an attempt to incorporate the effects of restrictions on borrowing and lending. Not all of these attempts have been satisfactory. In particular, Black (1972) does not consider the effect of a budget constraint on the individual's behaviour, despite the fact that he cannot borrow or lend; nor the implications of these constraints for the properties of general equilibrium. In this thesis, an equilibrium theory of investor's behaviour, which is general enough to include a wide variety of restrictions on borrowing and lending is presented. Some theoretical consequences of the restrictions and empirical implications of the theory are investigated.

A one period model is developed. Individuals who believe that future prices are Normally distributed - these beliefs are often called Gaussian - maximize the expected utility of end-of-the-period wealth subject to budget constraint. For each investor, an internal rate of discount can be defined, and his demand for risky assets is obtained as an explicit function of this rate. By assuming homogeneous beliefs, the aggregation of individuals' demands is performed and the expressions for the market clearing prices of risky assets obtained. The intercept on the security market line is a risk tolerance weighted sum of individuals' internal discount rates. Opportunities for borrowing and lending are incorporated into the theory in a straightforward way , by observing that individuals can, by borrowing and lending, lower

or raise their internal rates of discount until they are equal to the market rates.

It is shown that in the case where there is no borrowing or lending, but there exists a fixed supply of a riskless asset, the equilibrium is in general not pareto optimal.

Furthermore in equilibrium there exists a simple relationship between the prices of risky assets that is independent of the financing opportunities available to investors, a result obtained originally by Cheng [1977]. It is pointed out that in principle this relationship allows for the testing of the one period model in a one period context. While an explicit example of how this test should be carried out is not given, it is demonstrated how the model can be tested using the observed prices from any three periods. Finally we show that a simple natural generalization of the equilibrium relationship between prices holds in a world where investors have separable cubic utility functions, thereby establishing a link between the mean variance and linear risk tolerance approaches to asset pricing.

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Having made a beginning one can proceed in many different directions. The final direction that this thesis takes has been strongly influenced by my association with Professor Cheng. I would like to thank Pao Cheng for his very active encouragement and helpful advice.

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INTRODUCTION

An individual in deciding whether or not to purchase an asset will, among other things, be interested in the income he can expect to receive from the asset, the degree of uncertainty of the income, and at what time or times in the future he can expect to receive the income. The theory of decision-making under uncertainty, based on the expected utility hypothesis, allows for the determination of the amounts of various risky assets an individual would choose to purchase, given their prices. These demand relationships are of considerable interest in themselves and allow for the determination of the prices of risky assets in a market where the total supply of assets is fixed. Under certain simplifying assumptions the aggregation of individual investors' demands can be performed, leading to relationships between the prices of assets that must hold if the market is to be in equilibrium.

In the case where the assumptions lead to the simple capital asset pricing model of Sharpe [1964] and Lintner [1965], the security market line, perhaps the most famous of these relationships, is obtained. The security market line is rarely explicitly expressed as a relationship between prices. Instead it is written in a form that relates the expected return on an asset to the expected market risk of the asset. In equilibrium the expected return on an asset is linearly related to its market risk, where the intercept is the risk free rate of interest.

The relationship has been subjected to a great deal of empirical testing, a review of which has been given by Jensen [1972]. As a result of the tests, there seems to be general agreement that there exists a linear relationship between return and risk. At the conclusion of their empirical study of returns on the N.Y.S.E. [1935-68], Fama and Macbeth [1972] write:

> Thus we cannot reject the hypothesis that in making a portfolio decision, an investor should assume that the relationship between an assets portfolio risk and its expected return is linear.

However, there also appears to be general agreement that the empirical evidence shows that low risk securities are underpriced and high risk ones overpriced, from the point of view of the theory. The empirical security market line is flatter than the theoretical one, and cuts the returns axis at a higher return than the riskless rate.

The empirical evidence indicating that the intercept on the security market line is higher than the riskless rate of interest has motivated several extensions of the simple capital asset pricing model. In particular, Black [1972] considered two cases in which borrowing and lending at a riskless rate of interest are prohibited. In one case only risky assets are traded, while in the second case, investors can in addition trade a riskless asset whose total supply is fixed, but cannot sell it short. Brennan [1971] considered the case where the riskless borrowing and lending rates are not equal.

The effects of restrictions on borrowing and lending on market equilibrium are of considerable theoretical importance. For example, as shown in Chapter IV, in the case considered by Black, the prices of assets depend on the initial distribution of bonds among investors, and one distribution may be preferable to another.

In this thesis emphasis is placed on the prices of assets, and the relationship between prices in equilibrium, rather than on market relationships between investors' expectations. From the theoretical point of view prices are the basic unknowns that are to be determined, and from the empirical point of view can be observed.[(1)]

In Chapter III a one-period model for the pricing of risky assets is presented. Investors are assumed to have homogeneous Gaussian beliefs and are limited in their purchasing of assets by a budget constraint. The equilibrium prices are shown to depend upon the investors' budget constraints.

It turns out that it is possible to identify an internal rate of discount for each investor that depends on his budget constraint. In Chapter IV, borrowing and lending opportunities are thus easily incorporated into the theory, as investors, depending on their initial wealth and aversion towards risk, will equate their internal discount rates to the market borrowing and lending rates, provided their internal rates do not lie between the market rates.

In Chapter V it is shown that in equilibrium there exists a simple relationship between the prices of risky assets that is completely independent of the investors' budget constraints and aversion towards risk.

This relationship, in a somewhat different form, was originally obtained by Cheng [1977], who showed that, unlike the security market line, it is independent of the financial environment. Some empirical implications of this relationship are discussed.

It is also shown in Chapter V, that the simplest possible generalization of this relationship would hold in a market, where investors with the same beliefs have separable cubic utility functions.

Chapter I gives a brief review of the concepts employed in the one period model, while in Chapter II, some pertinent results of the Markowitz approach to the portfolio selection problem are presented.

CHAPTER I: Investment Decision Making Under Uncertainty: An Overview

The security market line is obtained from a two parameter, one period model such as that presented in Chapter III. In that model, the investors' decision concerns the allocation of a given amount of initial resources, among various risky assets that yield an uncertain amount of wealth at the end of one period. However the motive for transferring wealth from the present to the future is the desire to substitute future for present consumption. The time horizon relevant for the individual making such a decision is, in general, his lifetime.

It has been shown by Fama [1970] that although an individual faces a many-period decision problem, if his utility function over present and future consumption is strictly concave, his observed behaviour in the market will be indistinguishable from that of a risk-averse person with one-period horizon. The result depends on the assumption that the investor is far-sighted enough to have already planned his optimal strategies to cover all possible contingencies in the future. If it is assumed further that the individual's utility function over present and future consumption is independent of future states of the world, that is, depends only on consumption bundles available at future dates, and not on other circumstances, then his investment behavior is indistinguishable from that of an individual who maximizes the expected utility of end-of-the-period wealth.(2)

Basic assumptions:

Each investor is assumed to have a utility function in end-of-the-period wealth, W_1, with the usual properties, $U'(W_1) > 0$, $U''(W_1) < 0$. The condition $U'(W_1) > 0$ is the usual "more is preferred to less" assumption, while the condition $U''(W_1) < 0$ can be interpreted as representing an aversion toward risk (Arrow [1965], Pratt [1964]) on the part of the investor, given that he prefers more to less.

The investor cannot simply maximize the utility of final wealth since he does not know what his final wealth will be. He is assumed to have a subjective idea of the probability of occurrence of any given level of final wealth, and to maximize a weighted sum of the utilities for each possible value of final wealth; the weights in the sum being the subjective probabilities of that value of final wealth occurring. In other words, the individual maximizes the expected utility of final wealth.

Besides being a simple intuitive generalization of the concept of utility to the case of uncertainty, more importantly it can be shown that the expected utility hypothesis provides a preference ordering among risky alternatives (see for example Mossin [1973]) and is consistent with the investment behavior of the lifetime decision-maker described above.

Risk and the Relevant Time Period

An asset is defined to be risky if its end-of-the period price is not known with certainty. By this definition money and, in the manner described below, treasury bills or government bonds qualify as riskless assets. However money as a store of wealth is inferior to a government bond which pays a rate of return, and thus money plays no role in the theory with a riskless asset.

It is not the government bill itself which is riskless, but the bill in combination with the relevant time period. An individual with a time horizon of ten days, who buys a treasury bill ten days from maturity has acquired a riskless asset. The same individual purchasing a newly-issued sixty day treasury bill has acquired a risky asset. Theory does not provide us with a measure of the length of the period relevant for the individuals' immediate investment decision. Thus in the one-period model, the duration of the period is not defined.

Despite this, what is clearly intended is that the period is some interval over which individuals' tastes and beliefs can be considered as stable. It is assumed that this period is the same for all individuals.

Market Equilibrium

It is usual to assume that assets can be traded in arbitrarily small amounts. In accordance with the above description of investors' behaviour, facing any set of prices for assets in the market, each investor chooses to hold the amount of assets of every type that maximizes the expected utility of final wealth, where his final wealth is constrained by his debt obligations and his initial wealth. The actual prices at which all individuals trade are determined by the requirement that the actual amount of assets of each type that all individuals wish to hold must equal the total amount of outstanding assets of that type.

The term amount in the preceding paragraph refers to the physical amount of outstanding assets (e.g. the number of G.M. stocks outstanding) and not to the value of these assets. The investor has initially an endowment of a certain number of assets of each type; the value of this endowment is unknown before the market clearing prices are established. This point is rarely mentioned in the literature and it seems worthwile to repeat that in equilibrium the investors' initial wealth is not a given external parameter, but is determined by the theory.

Tastes and Beliefs

In following the prescription above for determining the prices of risky assets, we will not get very far unless we are prepared either to restrict further the form of the investors' utility function or alternatively to specify the nature of his beliefs about the possible outcomes of various investment decisions. In what follows we have chosen to leave unspecified the form of the investors' utility functions, other than that they are risk averse, and to assume that every investor has the same Gaussian beliefs (the precise meaning of this assumption is given in Chapter III).[3]

Alternatively we could have left the form of the investors' beliefs unspecified, and restricted their utility functions to be one of a broad class of utility functions known as linear in risk tolerance (Rubenstein [1974]). Either approach makes the problem of the determination of the market-clearing prices of risky assets tractable.

It is usual for economists to put as few restrictions on the utility function as possible, but this does not provide a case for the superiority of the first approach over the second. Both approaches are motivated by a desire for tractability of the mathematical problem and are complementary.

CHAPTER II: Markowitz Problem

In this chapter we review the Markowitz approach to the portfolio selection problem. Markowitz [1952] did not attempt to provide the complete mathematical solution of the problem which was given later by Merton [1970]. It turns out that many of the concepts and results arising in the solution of the Markowitz problem, occur also in the more general case where individuals are expected utility maximizers with Gaussian beliefs. Here we concentrate on those pertinent results.

Markowitz assumed that an individual has probability beliefs about the returns and covariances between returns of marketable securities. He further assumed that of all possible portfolios the investor will select the one, which for a given rate of return has the smallest variance. These portfolios are called efficient.

Let $\tilde{R}_j$ be 1+ rate of return on a security j. $\tilde{R}_j$ is a random variable. From all risky securities form a portfolio, k, and denote its return by

$$\tilde{R}_k = \sum_j \omega_{kj} \tilde{R}_j$$

where

$$\sum_j \omega_{kj} = 1$$

The ω_{kj} are the weights of the various securities in portfolio k. The problem is to minimize the variance of the portfolio k, for a given expected return by appropriately choosing the weights of the various securities in the portfolio.

Thus minimize

$$\mathrm{Var}(\tilde{R}_k) = \sum_i \sum_j \omega_{ki}\omega_{kj}\mathrm{Cov}(\tilde{R}_i, \tilde{R}_j) \tag{1}$$

where the expected return on portfolio k is

$$\varepsilon_k = \sum_j \omega_{kj} E(\tilde{R}_j) \tag{2}$$

and

$$\sum_j \omega_{kj} = 1 \tag{3}$$

The weights that minimize equation (1) subject to the constraints (2) and (3) define the efficient portfolios for investor k. We simply state the solution to the problem here.

Given the k^{th} investors' probability beliefs, there exist two well defined efficient portfolios which we call, using the notation of Black [1972], p and q. For investor k, every efficient portfolio can be represented as a linear combination of the portfolios p and q.

Let D_{ij} be the elements of the inverse of the matrix with elements $\mathrm{Cov}(\tilde{R}_i, \tilde{R}_j)$. Then the weights on the portfolios p and q are defined to be:

$$\omega_{pi} \equiv \frac{\sum_j D_{ij} E(\tilde{R}_j)}{\sum_i \sum_j D_{ij} E(R_j)} \tag{4}$$

and

$$\omega_{qi} \equiv \frac{\sum_j D_{ij}}{\sum_i \sum_j D_{ij}} \tag{5}$$

Let ε_p and ε_q be the expected returns on the portfolios p and q; and let σ_p^2 and σ_q^2 be their variances. Then from the definitions (4) and (5):

$$\sigma_q^2 = \frac{1}{c}, \quad \varepsilon_q = \frac{a}{c} \tag{6}$$

and

$$\sigma_p^2 = \frac{b}{a^2}, \quad \varepsilon_p = \frac{b}{a} \tag{7}$$

where in the notation of Merton [1972] $a \equiv \sum_{ij} D_{ij}E(R_j)$; $b \equiv \sum_i \sum_j D_{ij}E(\tilde{R}_i)E(\tilde{R}_j)$; $c \equiv \sum_{ij} D_{ij}$.

While it is true that any efficient portfolio can be represented as a linear combination of any two other efficient portfolios, the portfolios p and q have special properties. Thus portfolio q has a constant covariance with any security i

$$\text{Cov}(i, q) = \sigma^2 q \tag{8}$$

and the expected return on any security i is proportional to its covariance with portfolio p.

$$E(\tilde{R}_i) = \frac{\varepsilon_q}{\sigma_q^2} \text{Cov}(i, p) = a \text{ Cov}(i, p) \tag{7}$$

It also follows that for any efficient portfolio k

$$\frac{\sigma_k^2}{\sigma_q^2} - \frac{(\varepsilon_k - \varepsilon_q)^2}{\varepsilon_q(\varepsilon_p - \varepsilon_q)} = 1 \tag{8}$$

In ε_k, σ_k space (8) is an hyperbola. Equation (8) is sketched in figure I. We draw attention to the following points that follow from (8):

(1) q is the absolute minimum variance portfolio.

(2) p is the portfolio located where the tangent to the efficient frontier (as the upper half of the hyperbola is called) drawn from the origin touches the frontier.

(3) No line can be drawn tangent to the efficient frontier from a point ε on the ε_k axis when $\varepsilon > \varepsilon_q$.

(4) A line drawn tangent to the efficient frontier at point t, for example, intersects the ε_k axis at a height ε_t^* where

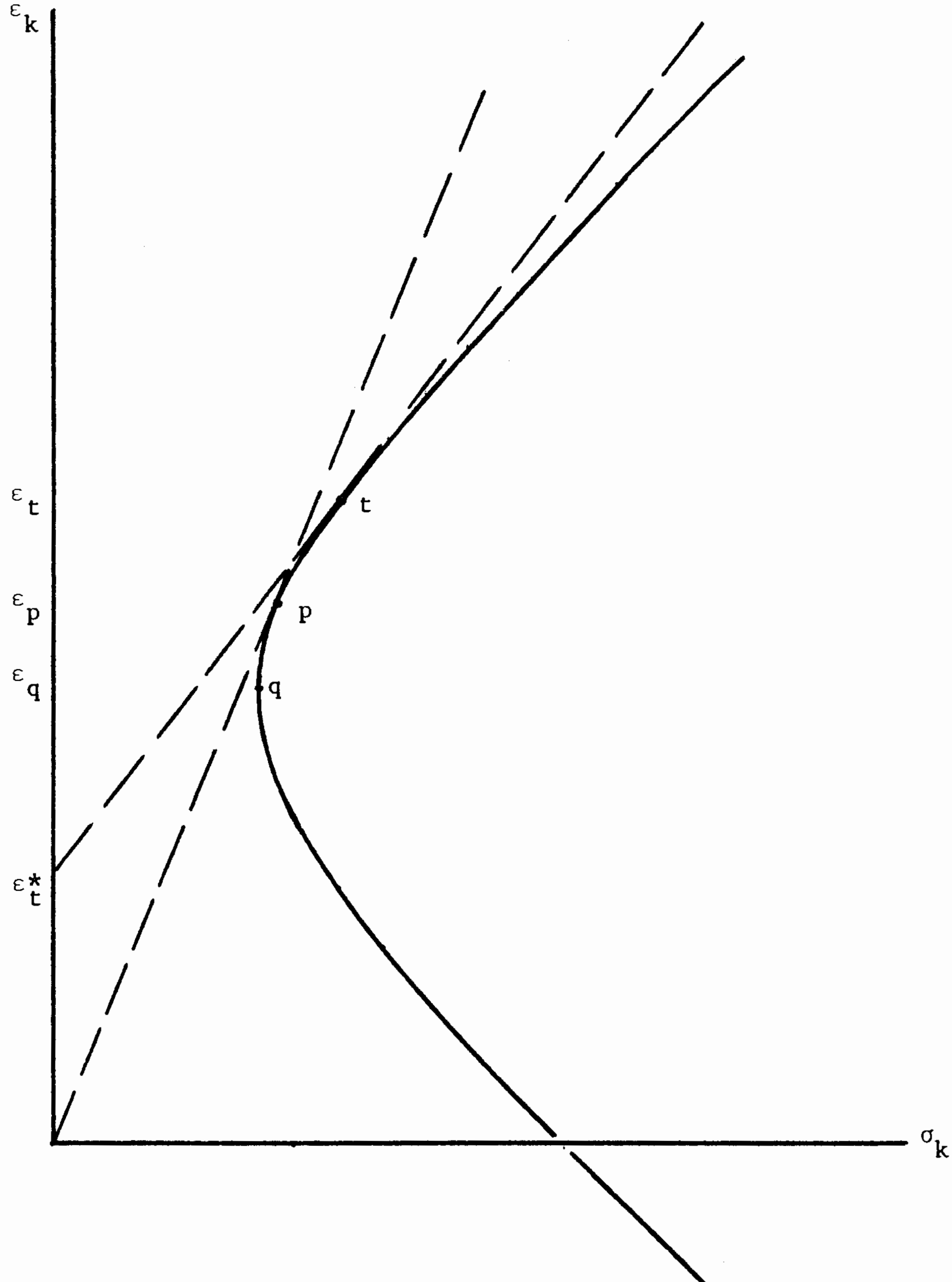
$$\varepsilon_t^* = \frac{\varepsilon_q(\varepsilon_p - \varepsilon_t)}{(\varepsilon_q - \varepsilon_t)} .$$

and for any security or portfolio i Cov(i, t) = 0 when E(i) = ε_t^*.

(5) For all points t on the efficient frontier

$$\varepsilon_t^* < \varepsilon_q .$$

FIGURE I



The Markowitz Frontier defined by

$$\frac{\sigma_k^2}{\sigma_q^2} - \frac{[\varepsilon_k - \varepsilon_q]^2}{\varepsilon_q[\varepsilon_p - \varepsilon_q]} = 1$$

CHAPTER III: Relative Prices of Risky Securities in General Equilibrium

We would consider a world in which individuals have an initial endowment of securities, there being S different types of securities. Initially trading of securities is permitted among the individuals who are aware of a deadline after which all trading must cease for a period of time.

The securities are risky because at the end of the period, the world may be in any one of a number of states (denoted by θ_1, θ_2, ...) and the prices of the securities at that time will depend upon which state the world is in. If the world is in the state θ at the end of the period we denote the prices at that time by $p_1(\theta)$, $p_2(\theta) \ldots p_s(\theta)$.

The problem is to determine the prices of the securities $\{p_1 \ldots p_s\}$ and the amount of each security held by any individual k $\{N_{k1} \ldots N_{ks}\}$ at the trading deadline.

In order to obtain a solution to this problem we make the following assumptions:

(1) Each individual, k, has a utility function $U_k(W_{1k})$ in end of the period wealth where

$$U_k' > 0; \; U_k'' < 0$$

The end of the period wealth can be written

$$W_k(\theta) = \sum_{i=1}^{s} p_i(\theta) N_{ki} + C_k$$

where if there are borrowing and lending opportunities, C_k depends upon a decision variable, i.e. if k decides to borrow B_k dollars at rate b, $C_k = -bB_k$. If there are no such opportunities $C_k = 0$.

(2) Each individual assigns a probability $\pi_k(\theta)$ to the occurance of each future state of the world θ.

(3) Each individual faces a constraint on his purchase of securities which we write as

$$\sum_j p_j N_{kj} = Q_k.$$

Thus, for example if k decides to borrow B_k,

$$Q_k = \sum_j p_j N^o_{kj} + B_k$$

where N^o_{kj} is the endowment of security j to individual k.(4)

(4) Each individual maximizes the expected utility of his end of the period wealth subject to the constraint (3).

(5) The total number of securities of any type N^T_i (i = 1...s) is fixed $N^T_i = \sum_k N_{ki} = \sum_k N^o_{ki}$

Given these assumptions a succinct statement of the solution of the problem is: facing any set of prices p_j, the k^{th} individual chooses the number of securities N_{ki} that maximizes

$$\sum_\theta \pi_k(\theta) U_k\{\sum_{i=1}^{s} p_i(\theta) N_{ki} + C_k\} + \lambda_k(Q_k - \sum_j p_j N_{kj}) \qquad (1)$$

Since the N_{ki} that satisfy (1), must in equilibrium satisfy

$$\sum_k N_{ki} = \sum_k N^o_{ki} = N^T_i \ (i = 1, \ldots, s) \qquad (2)$$

the s equations (2) may be used to determine the s prices, provided these equations are independent. That these equations may not be independent follows from the fact that the N_{ki} appearing in (2) are the optimal choices of individuals and therefore are constrainted to obey the equation

$$\sum_k \sum_i p_i N_{ki} = \sum_k Q_k \qquad (3)$$

for any prices whatsoever. Thus, in the case where the individuals' initial wealth is held in the form of risky assets, and there are no borrowing and lending opportunities available, equation (3) becomes

$$\sum_k \sum_i p_i N_{ki} = \sum_i p_i N_i^T$$

for any set of prices p_i whatsoever; and so only s-1 of the market clearing conditions (2) are independent.

If there are borrowing and lending opportunities available then there is an extra decision that every individual must make in solving his optimization problem. In this case equation (3) becomes

$$\sum_k \sum_i p_i N_{ki} = \sum_i p_i N_i^T + \sum_k B_k - \sum_k L_k$$

where B_k and L_k are the amounts borrowed and lent by individual k, which will vary depending on the prices p_i. Now the s equations (2) are independent and imply (via equation (3)) that in equilibrium

$$\sum_k B_k = \sum_k L_k$$

Gaussian Beliefs

The probabilities assigned to future states of the world can be viewed as probabilities assigned to future prices. We will be interested in the special case where each individual assigns the same joint normal distribution to future prices. This does not preclude the possibility that this distribution is contingent on present prices (see below). In this case equation (1) can be written as

$$\{\max_{N_{ki}}\} f^k(E(W_k), V(W_k)) + \lambda_k(Q_k - \sum_j p_j N_{kj}) \tag{4}$$

where

$$E(W_k) = \sum_{i=1}^{s} N_{ki} E(\tilde{p}_i) + C_k$$

$$V(W_k) = \sum_i \sum_j N_{ki} N_{kj} \mathrm{Cov}(\tilde{p}_i, \tilde{p}_j)$$

$E(\tilde{p}_i)$ is the expected future price of security i, and $\mathrm{Cov}(\tilde{p}_i, \tilde{p}_j)$ is the covariance among future prices i and j, assigned by individuals. The numbers $\mathrm{Cov}(\tilde{p}_i, \tilde{p}_j)$, form a symmetric matrix, which we call C and denote its elements by C_{ij}. The function f^k have the properties

$$f_1^k \equiv \frac{\partial f^k}{\partial E(W_k)} > 0; \quad f_2^k \equiv \frac{\partial f^k}{\partial V(W_k)} < 0$$

From (4) we obtain the N_{ki} that maximize the individual's expected utility:

$$f_1^k E(\tilde{p}_i) + 2f_2^k \sum_j N_{kj} C_{ij} - \lambda_k p_i = 0 \tag{5}$$

or

$$N_{ki} = \frac{1}{a_k} \sum_j C^{-1}_{ij} E(\tilde{p}_j) - \frac{1}{a_k} \frac{\lambda k}{f^k_1} \sum_j C^{-1}_{ij} p_j \qquad (6)$$

where C^{-1}_{ij} are the elements of the inverse of the variance covariance matrix C, and

$$\frac{1}{a_k} \equiv \frac{-f^k_1}{2f^k_2} > 0$$

Since the N_{ki} that satisfy (5) must satisfy (2) in equilibrium, we obtain from (5) expressions for the market clearing prices:

$$p_i = \frac{E(\tilde{p}_i) - \gamma \sum_j C_{ij} N^T_j}{r_z} \qquad (7)$$

where $\gamma^{-1} \equiv \sum_k \frac{1}{a_k}$

and $r_z \equiv \gamma \sum_k (-\lambda k/2f^k_2)$

Individual's Internal Rate of Discount

In order to obtain some insight into the meaning of equation (7), let us rewrite equation (5), which describes the k^{th} individual's optimizing behaviour, in a form which is reminiscent of equation(7):

$$p_i = \frac{E(\tilde{p}_i) - a_k \sum_j C_{ij} N_{kj}}{\lambda_k / f^k_1} \qquad (8)$$

Note that in equation (8) f^k_1 is a function of the decision variables N_{kj}. Let the individual k be holding an arbitrary bundle of assets denoted by N_{kj}. Then we can interpret equation (8) as telling us what the prices of these securities would have to be in order that the individual would be willing to hold them. Consider an individual, content with his holdings of risky securities, who inherits a security that with certainty is worth one dollar at the end of the period. Then if one dollar is an insignificant fraction of his total wealth, equation (8) tells us what the price of this security would have to be in order that the individual would be indifferent between holding it, and selling it. According to equation (8) this price is f_1^k/λ_k. In other words, for the k^{th} individual in equilibrium λ_k/f_1^k is the rate at which he discounts an extra dollar of certain future income. In equilibrium at the margin all certain future dollars will be discounted at the same rate. Thus the numerator of equation (8) is the certainty equivalent for individual k, of an uncertain future income of value $\tilde{p}_i$. and the denominator in his internal rate of discount of one dollar to be received with certainty at the end of the period.

If the k^{th} individual is free to borrow and lend at a riskless rate of interest r_f, then he will borrow or lend and readjust his holdings of risky assets until in equilibrium his internal rate of discount is equal to the market rate, i.e. until $\lambda_k/f_1^k = r_f$.[5]

From equation (7), the market value of an asset, is equal to the expected value of the asset, corrected for risk and discounted at a rate which is the same for all assets. The market discount rate r_z is an average (a risk tolerance weighted average) of individuals' internal discount rates.

$$r_z \equiv \gamma \sum_k (-\lambda_k/2f_2^k) = \frac{1}{\sum_k \frac{1}{a_k}} \sum_k \frac{1}{a_k} (\lambda_k/f_1^k)$$

When individuals can equate their internal rates to the market rate $r_z = r_f$.

However if individuals cannot equate their rates of discount to a single market rate, as for example when borrowing and lending rates are unequal, then these discount rates will depend on their aversion towards risk and their initial wealth. In this case equation (7) is an implicit expression for the prices of risky assets, since the right hand side depends on investors' initial wealth and thus on prices.

Prices of Risky Assets when Beliefs are Contingent on Present Prices

In the previous section it was assumed that individuals form their expectations about future prices independently of present prices. But quite often in the financial literature it is assumed that rates of return are assessed independently of present prices. For example in empirical work the assumption is often made that the expected rates of return, $E(\tilde{R}_i)$, and the covariances of these rates, $Cov(\tilde{R}_i, \tilde{R}_j)$ are stationary over time. Let us divide equation (8) by p_i to obtain

$$\lambda_k/f_1^k = E(\tilde{R}_i) - a_k \sum_j Cov(\tilde{R}_i, \tilde{R}_j) p_j N_{kj} \tag{9}$$

where $\tilde{R}_i \equiv \tilde{p}_i / p_i$

In Chapter II we introduced the matrix D_{ij} which is the inverse of the matrix $Cov(\tilde{R}_i, \tilde{R}_j)$. In terms of this matrix we have from (9)

$$\begin{aligned} N_{kj} &= \frac{1}{a_k}\frac{1}{p_j}\{\sum_i D_{ji} E(\tilde{R}_i) - \frac{\lambda_k}{f_1^k}\sum_i D_{ji}\} \\ &= \frac{1}{a_k}\frac{1}{p_j}\{\frac{\varepsilon_q}{\sigma_q^2}\omega_{pj} - \frac{\lambda_k}{f_1^k}\frac{1}{\sigma_q^2}\omega_{qj}\} \end{aligned} \tag{10}$$

where ω_{pj} and ω_{qj} are the weights on the portfolios p and q defined in Chapter II. Imposing the market clearing conditions on equations (10) we obtain the equilibrium prices

$$p_i = \gamma^{-1}\frac{1}{N_i^T}\{\frac{\varepsilon_q}{\sigma_q^2}\omega_{pi} - r_z\frac{1}{\sigma_q^2}\omega_{qi}\} \qquad (11)$$

We see from equation (9) that facing any set of prices, the individual will choose to hold the number of securities N_{kj} that equates

$$E(\tilde{R}_i) - a_k\sum_j Cov(\tilde{R}_i, \tilde{R}_j)p_j N_{kj}$$

to λ_k/f_1^k for all securities i; and that if there exists a riskless security, then if it is possible the k^{th} individual will equate λ_k/f_1^k to r_f.

We can rewrite equation (11) in its more familiar form. Dividing equation (7) by p_1 and rearranging we get

$$E(R_i) = r_z + \gamma\sum_j Cov(\tilde{R}_i, \tilde{R}_j)p_j N_j^T = r_z + \gamma V_T Cov(\tilde{R}_i, \tilde{R}_m) \qquad (12)$$

where $\tilde{R}_m$ is the return on the market portfolio with weights $\frac{p_i N_i^T}{V_T}$ where $V_T \equiv \sum_i p_i N_i^T$. From (12) we obtain

$$\gamma V_T = (E(\tilde{R}_m) - r_z)/\sigma_m^2$$

and thus

$$E(\tilde{R}_i) = r_z + E(\tilde{R}_m - r_z)Cov(\tilde{R}_i, \tilde{R}_m)/\sigma_m^2 \qquad (13)$$

Equation (13) is the familiar security market line. If people are free to borrow and lend at a riskless rate of interest then $r_z = r_f$. More generally r_z is the expected return on a portfolio that is uncorrelated with the market portfolio.

Summary

We summarize the results of this chapter here and use the opportunity to introduce a somewhat more compact notation. Let p_0 be the vector of beginning of the period prices and $\bar{p}_1$ the expected end of the period prices, and let n_k be the fraction of each firm that the k^{th} individual chooses to hold. Then when end of the period prices are assessed independently of current prices

$$n_k = \frac{1}{a_k}\{C^{-1}\bar{p}_1 - {}^{\lambda}k/f_1^k C^{-1}p_0\} \quad (14)$$

$$p_0 = (\bar{p}_1 - \gamma b)/r_z \quad (15)$$

where $b \equiv C\iota$ and ι is the column vector of ones.

Let ω_p and ω_q be the vectors of weights of the portfolios p and q and let P_0 be a diagonal matrix of the beginning of period prices. Then when rates of return are assessed independently of current prices

$$n_k = \frac{1}{a_k} P_0^{-1}[a\omega_p - {}^{\lambda}k/f_1^k\ C\omega_q] \quad (16)$$

$$p_0 = \gamma^{-1}\{a\omega_p - r_z c\omega_q\} \quad (17)$$

where $a \equiv {}^{\varepsilon}q/\sigma_q^2$ and $C \equiv 1/\sigma_q^2$ which is the notation of Merton [19].

We make two observations on these equations. Equations (14) and (16) for the optimal number of securities held by an individual facing a given set of prices are identical. This must be the case, for given any set of current prices, and expectations about future prices, expected returns are determined.

However equations (15) and (17) are not identical. In this case present prices are not given but are determined by the equations. For example, according to equation (15) the ratio of any two prices is independent of r_z, while this is certainly not true of the prices determined by (17).

Illustration

As an illustration consider the case where every investor has a quadratic utility function. Then

$$f^k(E_k, V_k) = E_k - \gamma_k(E_k^2 + V_k^2)$$

and

$$\frac{1}{a_k} = \frac{1}{2\gamma_k} - E_k$$

thus in general a_k depends upon the investors' optimal holding of securities. When the market is in equilibrium

$$\gamma^{-1} = \sum_k \frac{1}{a_k} = \sum_k \frac{1}{2\gamma_k} - \sum_i \bar{p}_{1i} \tag{18}$$

Substituting for γ^{-1} into equation (15) we obtain

$$p_0 = (\bar{p}_1 - \frac{1}{\sum_k \frac{1}{2\gamma_k} - \iota'\bar{p}_1} b)/r_z \tag{15a}$$

However in the case where expected returns are assessed independently of current prices, γ^{-1} depends on p_0. From equation (18)

$$\gamma^{-1} = \sum_k \frac{1}{2\gamma_k} - \mu' p_0 \tag{19}$$

where μ is the vector of expected returns. Substituting (19) into

(17) we obtain

$$p_0 = \{\sum_k \frac{1}{2\gamma_k} - \mu' p_0\}\{a\omega_p - r_z c\omega_q\}$$

Solving for $\mu' p_0$

$$\mu' p_0 = \frac{b - r_z a}{1+b - r_z a} \sum_k \frac{1}{2\gamma_k}$$

where $\quad b \equiv \mu' D^{-1} \mu$

Thus

$$p_0 = \frac{1}{1+b - r_z a} \sum_k \frac{1}{2\gamma_k} \{a\omega_p - r_z c\omega_q\} \qquad (17a)$$

Thus the same utility function gives quite different valuation formula for the risky assets. For example, while a rise in r_z causes all prices to fall according to (15a), according to (17a) some will rise.

In the example we treated r_z as if it was independent of prices. This will in general only be the case if individuals are free to borrow and lend at an exogeneously given rate of interest. In the next chapter we consider certain restrictions on borrowing and lending.

CHAPTER IV: Market Equilibrium With Various Riskless Assets

In the last chapter we noted that a_k, which is a measure of the individual's aversion towards risk, was in general a function of the final holding of securities. When expectations are assessed independently of current prices, this means that the a_k are themselves a function of the initial prices of risky assets. In order to keep things manageable we now assume that the a_k are independent of final wealth and its variance. This is equivalent to assuming an exponential utility function for each investor.

Individuals' Budget Constraint

Consider an individual who is prohibited from borrowing or lending at riskless rates. The optimal holding of assets for this individual is from equation II(16)

$$n_k = \frac{1}{a_k} P_0^{-1}[a\omega_p - {}^{\lambda}k/f_1^k c\omega_q] \qquad (1)$$

If n_{0k} is the vector of his endowment of risky assets, then his budget constraint may be written as

$$W_{0k} = n'_{0k}p_0 = n'_k p_0 \qquad (2)$$

Let us use a subscript, 0, to denote the individuals' internal rate of discount when he cannot borrow or lend. Then (1) and (2) imply that this rate is

$$({}^{\lambda}k/f_1^k)_0 = \frac{(a - a_k W_{0k})}{c} \qquad (3)$$

Borrowing and Lending

If we now allow individuals to borrow at the riskless rate r_b and lend at the riskless rate r_ℓ, they will if possible borrow or lend until their internal rates of discount are equal to the market rate. Thus if

$$\left(\lambda^k / f_1^k\right)_0 > r_b \tag{4}$$

the individual will borrow lowering his internal rate of discount until

$$\lambda^k / f_1^k = r_b \tag{5}$$

Denote an individual who borrows by the subscript, b. Substituting (5) into (1) we obtain the total value of risky assets held by b.

$$n_b' p_0 = \frac{1}{a_b}(a - r_b c) \tag{6}$$

According to equation (4), for borrowers

$$\frac{a - a_b W_{0b}}{c} > r_b,$$

or

$$\frac{a - cr_b}{a_b} > W_{0b} \tag{7}$$

Thus, from (6) and (7)

$$n_b' p_0 > W_{0b},$$

which is just another way of stating what is meant by the term borrower. The total value of risky assets held by borrowers is

$$\sum_b n_b' p_0 = \gamma_B^{-1}[a - r_b c] \tag{8}$$

where

$$\gamma_B^{-1} = \sum_b \frac{1}{a_b}$$

the summation being over all members of the borrowing group.

Similarly an individual, ℓ, will be a lender if

$$\frac{a - a_\ell W_{0\ell}}{c} < r_\ell$$

or

$$\frac{a - cr_\ell}{a_\ell} < W_{0\ell} \tag{9}$$

The total value of risky assets held by the lending group is

$$\sum_\ell n_\ell' p_0 = \gamma_L^{-1}[a - r_\ell c] \tag{10}$$

where

$$\gamma_L^{-1} \equiv \sum_\ell \frac{1}{a_\ell}$$

<u>Individuals Who Specialize in Risky Assets</u>

If $r_b > r_\ell$ then there may well be individuals who neither wish to borrow or lend. For these individuals, denoted by s, it follows from (7) and (9) that

$$\frac{a - cr_b}{a_s} \leq W_{0s} \leq \frac{a - cr_\ell}{a_s} \tag{11}$$

Since they neither borrow or lend

$$\lambda_s / f_1^s = \frac{a - a_s W_{0s}}{c} \tag{12}$$

and their portfolios of risky assets are given by

$$n_s = \frac{1}{a_s} P_0^{-1}[a\omega_p - (a - a_s W_{0s})\omega_q] \tag{13}$$

The total value of these securities is

$$\sum_s n_s' p_0 = \gamma_s^{-1}[a - (a - \gamma_s^{-1} \sum_s W_{0s})] \tag{14}$$

Equilibrium

When the riskless borrowing and lending rates are not equal investors, depending on their initial wealth and attitude towards risk fall into one of three groups, borrowers, lenders, or specializers in riskless assets. Equilibrium in this case has been considered by Brennan [1971] and in more detail by Cheng [1977], who also considered the case of endogenous borrowing rates. Black [1972] considered the case of equilibrium with no riskless assets. However, Cheng [1977] has shown that in this case the equilibrium prices of risky assets cannot be determined. The essential reason for this as mentioned in the discussion in Chapter II, is that the s market clearing conditions are not independent, because of the overall budget constraint. Black also discussed equilibrium in the case when there is a riskless asset in fixed supply, which cannot be sold short. We consider this case in more detail here. Black assumed that expectations are assessed independently of current prices and we make that assumption here.

It is worth mentioning one important point about the pricing equations we have obtained. We have obtained explicit expressions for the prices of securities where $r_z = r_f$ and investors' utility functions are either exponential or quadratic. On the basis of these equations [e.g. II(17a)] we cannot conclude that one distribution of securities among individuals is preferable to another. In the technical jargon the equilibrium is pareto optimal. However this is not the case if there is an outstanding supply of riskless assets and no borrowing as we shall show.

Let b_{0k} be the fraction of bonds outstanding, that are initially held by the k^{th} individual, and let b_k be his optimal holding. We cannot determine the s prices of the risky assets and the price of the bonds since the s+1 market clearing conditions are not independent. Thus let r be the exogenously given return on the bonds, and let B be the total value of the bonds, also given exogenously. The k^{th} individual's budget constraint is

$$n'_{0k}p_0 + b_{0k}B = n'_k p_0 + b_k B \tag{15}$$

where $b_k \geq 0$.

Let us call individuals who choose to hold a positive number of bonds lenders. For these individuals the constraint $b_k \geq 0$ is not binding and thus for lenders

$$\lambda_\ell / f_1^\ell = r$$

For those individual who do not choose to hold bonds, i.e., $b_k = 0$, we have from the budget constraint (15)

$$\frac{\lambda_s}{f_1^s} = (a - a_s W_{0s})/c \tag{16}$$

where $W_{0s} = n'_{0s}p_0 + b_{0k}B$ (17)

Thus a lender's portfolio of risky securities is given by (1) with $\lambda_\ell/f_1^\ell = r$ and those who do not choose to hold bonds have a portfolio given by (13). Thus summing these equations over all investors and apply the market clearing conditions $\sum_\ell n_\ell + \sum_s n_s = 2$, we obtain:

$$p_0 = \gamma^{-1}[a\omega_p - \gamma(a\gamma_s^{-1} + rc\gamma_L^{-1} \sum_s W_{0s})\omega_q] \tag{18}$$

where $\sum_s W_{0s} = \sum_s n'_{0s}p_0 + \sum_s b_{0s}B = \sum_s n'_{0s}p_0 + B_{0s}$

Collecting terms in p_0 on the left hand side

$$\{I - \omega_q \sum n'_{0s}\}p_0 = \gamma^{-1}[a\omega_p - c\bar{r}\omega_q] \tag{19}$$

where $$\bar{r} \equiv \gamma\{\gamma_s^{-1}\frac{a}{c} + r\gamma_L^{-1} - \frac{B_{0s}}{C}\} \tag{20}$$

Multiplying (19) by n'_{0s} and summing over s we obtain

$$\sum_s n'_{0s}p_0 = \frac{\gamma^{-1}[a \sum_s n'_{0s}\omega_p - c\bar{r} \sum_s n'_{0s}\omega_q]}{1 - \sum_s n'_{0s}\omega_q} \tag{21}$$

Substituting (21) back into (19) we obtain the equilibrium prices

$$p_0 = \gamma^{-1}[a\omega_p - cr_z\omega_q] \tag{22}$$

where

$$r_z = \frac{[\bar{r} - \frac{a}{c} \sum_s n'_{0s}\omega_p]}{1 - \sum_s n'_{0s}\omega_q} \tag{23}$$

It is clear that r_z and hence the prices depend upon the group membership and the initial distribution of bonds between the two groups, since by equation (20) $\bar{r}$ depends upon the value of the bonds initially held by individuals in group s.

Distribution Effects

Keeping the set membership unchanged, increment the initial bond holdings of the non-lending group by a small amount δB_{0s}. By equation (23), there will be a change in the r_z given by

$$\delta r_z = \frac{\delta \bar{r}}{1 - \sum_s n'_{0s}\omega_q} = -\frac{\gamma}{c}\,\delta B_{0s}\,\frac{1}{1 - \sum_s n'_{0s}\omega_q},$$

using equation (20). The change in r_z induces a change in the equilibrium prices, through equation (22), which are increased by an amount

$$\delta p_0 = \frac{1}{1 - \sum_s n'_{0s}\omega_q}\,\delta B_{0s}\omega_q \qquad (24)$$

The result of increasing the non lenders holding of bonds is to increase the total wealth of that group by an amount

$$\delta B_{0s} + \sum_s n'_{0s}\delta p_0 = \delta B_{0s}\{1 + \frac{n'_{0s}\omega_q}{1 - \sum_s n'_{0s}\omega_q}\}$$

$$= \delta B_{0s}\{\frac{1}{1 - \sum_s n'_{0s}\omega_q}\} > \delta B_{0s}$$

We can consider the increase in bond holdings of the non lending group to have been transferred to them from the lending group. The lending group is then poorer by an amount δB_{0s}, but because equilibrium prices have risen by an amount given by (24), the initial value of their risky endowment has increased by an amount

$$\sum_{\ell} n'_{0\ell}\,\delta p_0 = \frac{1}{1 - \sum_{s} n'_{0s}\,\omega_q}\,\delta B_{0s} \sum_{\ell} n'_{0\ell}\,\omega_q = \delta B_{0s}$$

Thus the total market value of the lending group's endowment has not changed. The result of the transfer of bonds, has been to increase the market value of the endowment of one group, and leave the market value of the endowment of the other group unchanged. The initial distribution was not pareto-optimal.

The argument above assumed that the set membership was left unchanged. If however because of wealth effects the group membership changes, the mathematical analysis becomes very complicated. But there is no need to go through a mathematical analysis to show that the optimal distribution of bonds is the one where initially they are all held by members of the non lending group.

Notice that it is only the value of the bonds initially held by the non-lenders that has any effect on prices. The total value of the outstanding bonds is not directly relevant for the determination of prices, (it has indirect relevance in establishing the group membership). From the formal point of view, the bonds of value B_{0s}, entering equation (20) could just as well be thought of as having been issued by the members of the non-lending group. The opportunity to issue bonds is most appreciated by individuals with high internal rates of discount, and it is those individuals with the very highest rates who end up being members of the non-lending group. The optimal distribution of bonds is the one which leads to an equilibrium in which the members of the non-lending group initially held all of the bonds.

The fact that we can if we wish look upon B_{0s} in equation (20) as having been issued by the members of the non-lending group, means in effect that we can look upon them as having been given the privilege of borrowing. If we allow individuals to issue a restricted number of bonds, then the resulting equilibrium will be pareto-optimal, since it will only be those with the very highest rates of discount who will issue them.

However in general with a fixed number of outstanding riskless bonds, the competitive equilibrium prices depend upon their distribution among investors, and there appears to be no reason that B_{0s} should be equal to the total outstanding number of bonds (see equation (28) below).

The total value of the lender's endowment of risky securities is from equation (22)

$$\sum_{\ell} n'_{0\ell} p_0 = \gamma^{-1}[a \sum_{\ell} n'_{0\ell}\omega_p - cr_z \sum_{\ell} n'_{0\ell}\omega_q] \tag{25}$$

Since:

$$\sum_{s} n'_{0s}\omega_q + \sum_{\ell} n'_{0\ell}\omega_q = \iota'\omega_q = 1$$

substituting for r_z, as given by (23), in (25)

$$\sum_{\ell} n'_{0\ell} p_0 = \gamma^{-1}[a \sum_{\ell} n'_{0\ell}\omega_p + a \sum_{s} n'_{0s}\omega_p - c\bar{r}]$$

$$= \gamma^{-1}[a - c\bar{r}] \tag{26}$$

On the other hand the total value of risky assets held by the lenders in equilibrium is given by the equation (10),

$$\sum_{\ell} n'_{\ell} p_0 = \gamma_L^{-1}[a - rc] \tag{27}$$

The difference between (26) and (27) is the value of the bonds initially held by members of the group s.

$$\therefore \quad B_{0s} = \gamma^{-1}[a - c\bar{r}] - \gamma_L^{-1}[a - rc] \tag{28}$$

bur from equation (11)

$$\sum_{s} n'_{0s} p_0 + B_{0s} \leq \gamma_s^{-1}[a - rc] \tag{29}$$

Substituting (28) into (29)

$$\sum_{s} n'_{0s} p_0 \leq \gamma^{-1} c[\bar{r} - r] \tag{30}$$

However

$$\sum_{s} n'_{0s} p_0 = \iota' p_0 - \sum_{\ell}{}' n_{\ell} p_0$$

$$= \gamma^{-1} c[\bar{r} - r_z] \geq 0 \tag{31}$$

Equations (30) and (31) together imply

$$\bar{r} \geq r_z \geq r$$

CHAPTER V: Structure of Market Prices in Equilibrium

In this chapter we will look at the relationships between prices implied by the valuation equations (15) and (17) of Chapter III. These relationships are of interest from both the theoretical and empirical point of view. Consider first equation (15) of Chapter II, which holds when future prices are assessed independently of present prices

$$p_0 = \frac{(\bar{p}_1 - \gamma b)}{r_z} \tag{1}$$

According to equation (1), in equilibrium the vectors, p_0, $\bar{p}_1$ and b are linearly dependent. Consider the matrix, M, whose rows are the vectors p_0, $\bar{p}_1$ and b

$$M = \begin{bmatrix} p_{01}, & p_{02}, & \ldots, & p_{0s} \\ \bar{p}_{11}, & \bar{p}_{12}, & \ldots, & \bar{p}_{1s} \\ b_1 \,, & b_2 \,, & \ldots, & b_s \end{bmatrix} \tag{2}$$

Since, in equilibrium, the rows of M are linearly dependent, M is at most of rank 2, (in general it is of rank 2). Therefore the determinant of any 3 x 3 submatrix of M is equal to zero. That is, for any i, j and k

$$\det \begin{bmatrix} p_{0i} & p_{0j} & p_{0k} \\ \bar{p}_{1i} & \bar{p}_{1j} & \bar{p}_{1k} \\ b_i & b_j & b_k \end{bmatrix} = 0 \tag{3}$$

If we define

$$m_{ij} \equiv \bar{p}_{1i}b_j - \bar{p}_{1j}b_i \tag{4}$$

then equation (3) can be written

$$m_{ij}p_{0k} + m_{jk}p_{0i} + m_{ki}p_{oj} = 0 \tag{5}$$

for all i, j, k.

Equation (5) is independent of γ and r_z. It is therefore independent of the investors' attitudes towards risk and their budget constraints. In a market of risk averse individuals with homogeneous Gaussian beliefs about future prices, equation (5) always holds in equilibrium, independently of the financial environment. For this reason equation (5) has been called the invariance law of prices by Cheng [1977].

In equation (5), one could substitute for p_{0j} in terms of two other prices, p_{0m} and $p_{0\ell}$, for example, thereby obtaining an equilibrium relationship between 4 prices. This process could be continued, until a linear relationship is obtained between all prices. Equation (5) is the smallest possible such relationship between prices. This motivates the following question: what theory of human behaviour denies equation (5) and instead predicts that the smallest possible linear relationship between prices is of the form

$$M_{ijk}p_{\ell} + M_{jk\ell}p_i + M_{k\ell i}p_j + M_{\ell ij}p_k = 0? \tag{6}$$

Rubenstein [1973] has shown that in a market with homogeneous but unspecified beliefs, where individuals have utility functions which are cubic and linear in risk tolerance

$$E(\tilde{R}_j) = r_f + \lambda_2 \mathrm{Cov}(\tilde{R}_j, \tilde{R}_M) + \lambda_3 \mathrm{Cos}(\tilde{R}_j, \tilde{R}_M, \tilde{R}_M) \qquad (7)$$

where Cos is the coskewness operator. Multiplying (7) by p_{0j} and rearranging we obtain

$$p_0 = \frac{1}{r_f}\{p_1 - \gamma_1 b + \gamma_2 \delta\} \qquad (8)$$

where δ is a vector with elements

$$\delta_i = \sum_j \sum_k \mathrm{Cos}(\tilde{p}_i, \tilde{p}_j, \tilde{p}_k)$$

and, γ_1 and γ_2 are market parameters independent of the equilibrium prices, when beliefs are assessed independently of present prices. According to equation (8), in equilibrium, the vectors p_0, $\bar{p}_1$, b and δ are linearly dependent, thus for any i, j, k and ℓ

$$\det. \begin{bmatrix} p_{0i} & p_{0j} & p_{ok} & p_{0\ell} \\ \bar{p}_{1i} & \bar{p}_{1j} & \bar{p}_{1k} & \bar{p}_{1\ell} \\ b_i & b_j & b_k & b_\ell \\ \delta_i & \delta_j & \delta_k & \delta_\ell \end{bmatrix} = 0 \qquad (9)$$

Equation (9) implies equation (6) where

$$M_{jk\ell} = \det. \begin{bmatrix} \bar{p}_{1j} & \bar{p}_{1k} & \bar{p}_{1\ell} \\ b_j & b_k & b_\ell \\ \delta_j & \delta_k & \delta_\ell \end{bmatrix}$$

Empirical Implications

The model presented in this thesis is a one period model. It would be preferable in testing such a model, to test it in a one period, rather than within a multiperiod context. In principle equation (5) allows for such a test. Equation (5) says that for any three prices observed in the market, p_i, p_j, p_k, it is possible to find three numbers m_{ij}, m_{jk} and m_{ki}, with a form given by equation (4) such that

$$m_{ij}p_k + m_{jk}p_i + m_{ki}p_j = 0$$

for all i, j and k. This is not an empty statement, because, for example, if individual have separable cubic utility functions then according to equation (6) it will not be possible to find such numbers. The problem of carrying out this test is one of efficiency rather than principle, and hopefully this can be overcome.

The capital asset pricing model can be tested over time by assuming stationary expected returns and covariances. In this case equation (17) of Chapter II is the appropriate pricing equation,

$$p_0 = \gamma^{-1}(a\omega_p - cr_z\omega_q) \tag{10}$$

According to equation (10) in equilibrium, the vectors p_0, ω_p and ω_q are linearly dependent, and thus for any i, j and k

$$\text{det.} \begin{bmatrix} p_{0i} & p_{0j} & p_{0k} \\ \omega_{pi} & \omega_{pj} & \omega_{pk} \\ \omega_{qi} & \omega_{qj} & \omega_{qk} \end{bmatrix} = 0 \tag{11}$$

By defining

$$\ell_{jk} \equiv \omega_{pj}\omega_{qk} - \omega_{pk}\omega_{qj} \tag{12}$$

Equation (9) can be written

$$\ell_{ij}p_{0k} + \ell_{kj}p_{0r} + \ell_{ki}p_{0j} = 0 \tag{13}$$

Cheng [1977] has suggested testing (13) by regressing one price on two or more prices over time and looking for a zero intercept. A different type of test, which concentrates on using as few time periods as possible can be devised.

Let p_x, p_y and p_z denote the prices of securities on three different dates. Then since by assumption the ℓ_{ij} are stationary equation (13) implies

$$\begin{bmatrix} p_{xi} & p_{xj} & p_{xk} \\ p_{yi} & p_{yj} & p_{yk} \\ p_{zi} & p_{zj} & p_{zk} \end{bmatrix} \begin{bmatrix} \ell_{jk} \\ \ell_{ki} \\ \ell_{ij} \end{bmatrix} = 0 \tag{14}$$

Since there is no reason to expect that the ℓ_{ij} are zero, equation (14) implies that for any three prices on any three dates

$$\text{det.} \begin{bmatrix} p_{xi} & p_{xj} & p_{xk} \\ p_{yi} & p_{yj} & p_{yk} \\ p_{zi} & p_{zj} & p_{zk} \end{bmatrix} = 0 \tag{15}$$

Anyone can convince himself that (15) does not hold by looking in the newspaper. But this is not the issue. The issue is, by how much are we prepared to let the left hand side of (15) differ from zero before we reject the simple capital asset pricing model, or the concept of stationary beliefs as adequate descriptions of investors' behaviour.

NOTES

1. The security market line involves investors' expectations. However, if beliefs are stable over time there exists relationships between market clearing prices that are independent of investors' beliefs (see chapter V).

2. There are in addition more subtle requirements, as has been pointed out by Ziemba [74, 77]. For example, it is necessary that the return distributions of the various assets be linearly independent, Ziemba [77].

3. For this approach to be valid, it is necessary that the utility function be defined over the complete range of the return distribution, and that it can be integrated with the normal distribution. A discussion of these points has been given by Karl Borch in The Economics of Uncertainty (Princeton University Press, 1968).

4. This form of the budget constraint may cause some confusion. The B_k should be thought of as some level of borrowing, not necessarily the optimal level. The constraint on the investors' purchases establishes for him an internal rate of discount. The optimal level of borrowing is that which equates the investors' internal rate of discount to the market borrowing rate. It should be noted that one could include in Q_k the transactions costs incurred by an investor participating in the market.

5. As will be seen in Chapter III, equation (7), an individual with a relatively small amount of initial wealth will always be a borrower (since we must have $a/c \equiv \varepsilon_q > r_b$ for an equilibrium to

exist, according to the comments on figure 1, chapter II). Lenders on the other hand are relatively rich, and they will never wish to loan out more than their initial wealth when $r_\ell < \varepsilon_q$.

REFERENCES

Arrow, K.J. 1965. "The Theory of Risk Aversion." In Essays in the Theory of Risk-Bearing. pp. 90-120. Chicago: Markham Publishing Co.

Black, F. 1972. "Capital Market Equilibrium with Restricted Borrowing." Journal of Business. July.

Brennan, M.J. 1971. "Capital Market Equilibrium with Divergent Borrowing and Lending Rates." Journal of Financial and Quantitative Analysis. December.

Cheng, P. 1977. "Capital Market Equilibrium with Endogenous Borrowing Rates." Simon Fraser University, Department of Economics and Commerce Discussion Paper.

Fama, E.F. 1970. "Multiperiod Consumption-Investment Decisions." The American Economic Review. March.

__________ and MacBeth, J. 1972. "Risk, Return and Equilibrium: Empirical Tests." Journal of Political Economy. May.

Jensen, M.C. 1972. "Capital Markets: Theory and Evidence." Bell Journal of Economics and Management Science. Autumn.

Lintner, J. 1965. "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets." Review of Economics and Statistics. February.

Markowitz, H.M. 1952. "Portfolio Selection." Journal of Finance. March.

Merton, R.C. 1970. "An Analytic Derivation of the Efficient Portfolio Frontier." Working Paper No. 493-70, Sloan School of Management, Massachusetts Institute of Technology. Reprinted in Journal of Finance and Quantitative Analysis, September, 1972.

Mossin, J. 1973. Theory of Financial Markets. Prentice-Hall Publishing Co. Chapter 2.

Pratt, J. 1964. "Risk Aversion in the Small and in the Large." Econometrica. January - April.

Rubenstein, M.E. 1973. "The Fundamental Theorem of Parameter-Preference Security Valuation." Journal of Financial and Quantitative Analysis. January.

__________. 1974. "An Aggregation Theorem for Security Markets." Journal of Financial Economics. September.

Sharpe, W.F. 1964. "Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk." Journal of Finance. September.

Ziemba, W.T. 1974. "The Behaviour of a Firm Subject to Stochastic Regulatory Review: Comment," Bell Journal of Economics, Autumn.

__________ 1977. "Multiperiod Consumption-Investment Decisions: Further Comments," The American Economic Review, September.