

SOME TERNARY FRAMES

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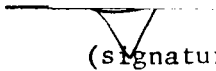
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## SOME TERNARY FRAMES

### ABSTRACT

This paper has three objectives: 1) to give a general account of the relationship between logics, systems, semantics and philosophy; 2) to investigate one specific structure (O-structures) in one kind of semantics (neighborhood or Scott-Montague semantics); and 3) to show completeness for certain logics proved sound with respect to O-structures and to look briefly at some revisions in the structures themselves.

Chapter I compares syntactic considerations in developing a logic (i.e. the relationship between primitive constants, axioms and theorems) with semantic considerations (the provision of model conditions for formulae and the notion of semantic validity) and explains why it is desirable for a logic's syntax and semantics to correspond in an exact fashion. To flesh out these general remarks, the propositional calculus (PC) is shown to be sound and complete with respect to PC semantics. Soundness and completeness for modal logics is then taken up. Modal logics differ from PC in containing constants that are non-truth-functional in nature. For this reason modal logics require semantics that incorporate a feature permitting the characterization of truth conditions for these formulae. Two kinds of modal semantics are explored: relational and neighborhood semantics.

Chapter II investigates one specific structure (an O-structure) in neighborhood semantics. An O-structure is

a pair  $\langle U, O \rangle$  where  $U \neq \emptyset$  and  $O$  is a function:  $U \rightarrow 2^{(u^2)}$ . The expression  $xO^u y$  reads "x is ..... than y for u" where "....." takes any transitive, irreflexive relation. An initial truth condition for  $\square$  is defined and it is shown that under this condition,  $O$ -structures can be made to yield the logics  $U.Con$ ,  $KD$  and others. Three philosophical interpretations for  $\square$  in  $U.Con$  and  $KD$  are elaborated. The first is borrowed directly from the literature; the others are developed from remarks by Quine.

Chapter III shows completeness for  $KD$  and  $T$ , looks at two other truth conditions for  $\square$  and suggests an  $O$ -structure refinement that redresses one intuitive deficiency in two of the interpretations for  $\square$  explored in Chapter II. An  $O^*$ -structure is defined as the pair  $\langle U, O^* \rangle$  where  $O^*: U^3 \rightarrow J$ . Finally,  $O^*$  structures are used to show that a notion of "degrees of belief" developed by F.P. Ramsey is a notion that preserves  $[K]$  and  $[D]$ .

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## CHAPTER ONE

### LOGICS AND SEMANTICS

#### 1) General Features

Logics are to be distinguished from formal systems that generate logics<sup>1</sup>. A logic  $L$  is a set of formulae called theorems. A formal system is an ordered triple  $S = \langle L, A, R \rangle$  where  $L$  is a language,  $A$  a set of axioms and  $R$  a set of rules called transformation rules (or rules of inference). To say that a formal system generates a logic is to say that  $L$  is the closure of  $A$  (expressed in  $L$ ) under  $R$ . The terminology in this work differs slightly from conventional presentations. It is customary to distinguish between primitive theorems (axioms) and derived theorems (theorems) of a logic and to use the generic term "thesis" to refer indiscriminately to members of either set. However, for purposes of this paper "thesis" and "theorem" will be used interchangeably. "Axiom" identifies primitive theorems or theses.

A language  $L$  in turn is a triple  $\langle A, K, F \rangle$  where  $A$  is the set of atomic formulae of the language,  $K$  the set of primitive logical constants and  $F$  the closure of  $A$  under a set of formations (or formation rules). The set of formations of a logic is given with reference to the set  $K$ .  $F$  is the set of well-formed formulae. "Formation" will be used to identify both the rule and the results of applying the rule (i.e., the formation result).

## Logic as Theory

Logic may be thought of as an analytic/explanatory theory or as a theory of meaning. Conceived as an analytic/explanatory theory (like chemical theory for example), a logical theory analyzes or explains the properties of formations proper to the theory, in terms of the theory's fundamental theoretic elements. The fundamental theoretic elements of a formal system are the set of atomic formulae (or atoms) and the set of primitive logical constants.

Formations (or results of formations) have two kinds of properties: the syntactic property of derivability and the semantic property of holding in a class of models. Both of these notions will be covered in detail later. Suffice it to note for now that a particular formation's having or lacking the property of derivability in a logic is explained by reference to the set of axioms and transformation rules of the system generating the logic. And the set of axioms, as shall also be seen shortly, is definitionally connected with the logical constants of the system. By the same token, a formation's having the semantic property of holding in a model depends upon the characterization of primitive constants and atomic formulae in that model. Because complex formations are constructed from primitive formations in accord with formation rules, their having the syntactic or semantic properties that they do have is to be explained in terms of the properties



of their primitive components.

Alternatively a logic may be thought of as elaborating a theory of meaning for its primitive constants, the elements of  $\underline{K}$ . Logical theories on this view state conditionally that if primitive constants are defined in certain ways, certain consequences will follow with respect to the way in which formations containing those constants may enter into our reasoning. Axioms, in this view, provide a syntactic specification of the meaning of primitive constants by instantiating formations which reflect the logical properties of those constants. The addition or deletion of axioms to a specification can identify different properties for those constants.

### The Notion of Atomicity

The notion of (explanatory) atomicity employed here is theory-relative. The theoretical elements of one theory may be non-atomic to another. The elements of the propositional calculus (or PC) for example, are, propositional letters and truth-functional constants. PC, consequently, is not capable of explaining how it is that formations containing quantifiers or non-truth-functional operators have the semantic or syntactic properties that they have. The predicate calculus, by contrast, is a logical theory adequate to the analysis of formations containing quantifiers while the set of modal logics is

capable of analyzing formations with other non-truth-functional operators. The primitives of predicate calculus are, minimally, predicate constants and individual variables and the set of primitive formations for modal logics contain non-truth-functional constants that are not contained in PC.

In addition, primitive constants designated as atomic to a theory are not even absolutely atomic to that theory because other constants interdefinable with the designated constants may take their place as primitives. The formations normally designated atomic in PC, for example " $\neg$ " and " $\vee$ ", can be replaced by " $\neg$ " and " $\wedge$ ", " $\neg$ " and " $\rightarrow$ ", the Sheffer stroke "/" or other combinations.

#### Constants, Systems and Logics

If developing a logic is conceived as developing the implications of a theory of meaning sketched for the logic's primitive constants, then axioms and transformation rules collectively, can be thought of in this context as providing joint definitions for those constants. That is, the axioms and rules represent some or other interpretation of those constants by instantiating some or other of their logical properties. Hence, although axioms and rules cannot be said in any straightforward sense to be wrong, they can be said to succeed or fail at representing certain desired properties.

If the set of axioms and transformation rules reflect exactly the logical properties of a set of constants relative

to a particular interpretation, the derived theorems of the logic represent exhaustively all the ways in which those constants (as interpreted) may be used in reasoning. In a very important sense, a logic may be said to mirror the logical properties of its constants. Every property of every interpreted constant is reflected in a theorem set adequate to the interpretation of those constants.

### Soundness, Consistency and Completeness<sup>2</sup>

A given axiomatization may miss its mark in two ways: it may be too restrictive or it may be too liberal. If too restrictive, then some axiom or transformation rule will be missing and the set of theses will not completely reflect the intended interpretation of the theory's constants. If too liberal, then some axiom or transformation rule will be present that permits derivations that do not reflect the intended interpretation.

An analogy with the phrasing of legal statutes is instructive. The point of legislation is to rule out certain forms of conduct (as not permissible) and to admit others. The task of the legislative draughtsperson is to capture the spirit or intent of the legislation in phrasing the statute so that all acceptable acts are legalized and all unacceptable acts are made illegal. The statute can go wrong by being too restrictive and failing to rule in all admissible acts or by being too liberal and admitting acts

that are not consistent with the intent of the legislation. Failure in the former direction can be characterized as intent-incompleteness, and in the latter as intent-inconsistency or intent-unsoundness. Overly restrictive or incomplete logics are defective in the sense that they are not strong enough to permit the derivation of all desirable formulae. Excessively liberal or unsound or inconsistent logics on the other hand are too strong in the sense that they permit derivations which they should not.

#### Detecting Theory Inadequacy - The Semantic Point

The only way in which the legislator can determine whether the letter of the law captures the spirit, is to let it loose upon the land to see if, in fact, the laws are too restrictive or too liberal. A defective statute may take years to be discovered. The logician provided with only an axiomatic version of a particular logic is in equally difficult straits when it comes to determining the adequacy of a system. If the only method is by test of derivation, it could take years to turn up an inconsistency. Even worse, failing to turn up anything for years would guarantee nothing because the number of possible derivations in a logic is infinite. Similarly, showing that a given axiomatization yields all desirable theorems requires a list of desirable theorems and a proof for each entry on the list. Since this list would be infinite, the logician encounters the same sort of problem

for completeness.

Providing a semantics for a logic provides an alternate way of stipulating the meaning of the logic's primitive constants. Instead of giving the meaning of a term operationally, primitive constants are defined in a semantics by presenting conditions in a model. In most cases, semantic conditions are truth conditions but this does not have to be the case. There are no restrictions on the sorts of models or the sort of model conditions that may be employed, just so long as the semantics is capable of characterizing the required properties of modelled formations.

In semantics concerned with truth conditions, modelled formations can be seen to possess or lack the property of validity. Very generally, a formation is valid in a semantics iff the formation holds in all models (is universally true) for the logic. Hence, the logician may use semantics to show: 1) that any given formation is a theorem by showing it to be true in all models (not false in any model); 2) that a system is sound by showing that its axioms are valid and that its transformation rules preserve validity; and 3) that a system is complete by showing that every valid formula is a thesis.

In terms of the metaphor with legal statutes, semantics can show that the phrasing of a statute protects the intent of the legislation (i.e. is sound) and takes the intent far enough (i.e. is complete). Having the set of theses correspond to the set of valid formulae guarantees that the

reflection of the constants in the logic matches their reflection in the semantics. Both the syntactic and semantic interpretations of those formations pick out the same logical properties.

## 2) Generating the Propositional Calculus<sup>~</sup> (PC) and PC Semantics

Turning from a very general description of logics and systems to a particular case, the propositional calculus can be generated by a plurality of systems. One system, PM (for Principia Mathematica)<sup>3</sup>, contains as a language, the following elements:

$\underline{AT}_{PM}$ : a set of atomic formulae designated by lower case letters from the middle portion of the alphabet (the propositional letters  $p, q, r$ , etc.). Semantically they tend to be treated as propositions, i.e. assigned exactly one of two distinct values, true or false.

$\underline{K}_{PM}$ : a set of formations functionally complete with respect to a truth-functional interpretation. The set of formations is complete in the sense that all truth-functions can be expressed in  $L_{PM}$ . The two constants normally designated as primitive are negation ( $\neg$ ), read "not" and disjunction ( $\vee$ ), read "or".

$\underline{F}_{PM}$ : predictably, the closure of  $\underline{AT}_{PM}$  under  $\underline{K}_{PM}$ .

$\underline{F}_{PM}$  may be defined recursively as the smallest set satisfying:

- 1)  $\underline{AT}_{PM} \subset \underline{F}_{PM}$
- 2)  $\forall \alpha, \alpha \in \underline{F}_{PM} \Rightarrow \neg \alpha \in \underline{F}_{PM}$

3)  $\forall \alpha, \forall \beta, \alpha \in \underline{F}_{PM}$  and  $\beta \in \underline{F}_{PM} \Rightarrow \alpha \vee \beta \in \underline{F}_{PM}$

The standard additional logical terms are defined in terms of " $\neg$ " and " $\vee$ " as follows:

Def.  $\wedge$ :  $\alpha \wedge \beta$  iff  $\neg(\neg\alpha \vee \neg\beta)$  " $\wedge$ " is read "and"

Def.  $\rightarrow$ :  $\alpha \rightarrow \beta$  iff  $\neg\alpha \vee \beta$  " $\rightarrow$ " is read "if...then..."

Def.  $\leftrightarrow$ :  $\alpha \leftrightarrow \beta$  iff  $\neg(\neg(\neg\alpha \vee \beta) \vee \neg(\neg\beta \vee \alpha))$  " $\leftrightarrow$ " is read "...if and only if..."

$\underline{R}_{PM}$  and  $\underline{A}_{PM}$

The PM transformation rules are:

1) Uniform Substitution [US]

For any complex formation  $\alpha$ , if  $\alpha$  is a PM thesis and  $\beta$  is the result of substitution some member of  $\underline{F}_{PM}$  for every occurrence of some propositional letter of  $\alpha$ , then  $\beta$  is also a thesis.

2) Modus Ponens [MP]

If  $\alpha$  is a PM thesis and  $\alpha \rightarrow \beta$  is a PM thesis, then  $\beta$  is a thesis of PM.

As noted earlier there are alternative axiom bases for PC, each adequate for the generation of all tautologies. The following base is a truncation of the axiom set of PM<sup>4</sup>:

1)  $(p \vee p) \rightarrow p$

2)  $p \rightarrow (p \vee q)$

$$3) (pvq) \rightarrow (qvp)$$

$$4) (p \rightarrow q) \rightarrow ((rvp) \rightarrow (rvq))$$

### Semantics for PC

A PC model is a non-empty set  $U$  and a valuation  $V$ . The elements of  $U$  (designated by lower case letters from the end of the alphabet -  $u, v, w$ , etc.) may be thought of intuitively as points which eventually function in a PC model to make certain propositions true and certain propositions false.  $V$  associates with each PC atomic formula a set of points in  $U$ . The set of points assigned by  $V$  to some formula  $\alpha$  in a model  $\mathcal{M}$  (in symbols  $V(\alpha)^{\mathcal{M}}$ ) is the truth set of  $\alpha$  in  $\mathcal{M}$  (in symbols  $\|\alpha\|^{\mathcal{M}}$ ).  $V$  may be conceived either as a function from  $\underline{At} \rightarrow \mathcal{P}(U)$  or from  $\underline{At} \rightarrow \underline{2}^U$ . In terms of the power set of  $U$ ,  $\mathcal{P}(U)$ ,  $V$  associates each atomic formula with some set of points in  $U$ . Considered exponentially,  $V$  associates each atomic formula with an element in the set of functions from  $U$  into  $\underline{2}$ , i.e. the set  $\{0,1\}$ . Each formula is assigned a value 0 or 1 at each point in  $U$ .

As there are an infinite number of ways of assigning formulae to points, there are an infinite number of PC models. There are no minimal restrictions on the sorts of constructible PC models except that  $U \neq \emptyset$ . In the simplest case  $U$  is a unit set  $\{x\}$  and a model, call it  $\mathcal{M}^*$ , gives to each atomic wff. the value 1 or 0 at  $x$  (and mention of



x can be suppressed). More complex models might be thought of as non-empty collections of models in this simplest sense. Every point in a PC model is itself a model in the simplest sense.

Truth conditions for all PC formations are defined recursively as follows:

$$1) \quad \frac{\mathcal{M}}{u} \alpha \Leftrightarrow u \in V(\alpha)$$

$$2) \quad \frac{\mathcal{M}}{u} \neg \alpha \Leftrightarrow \not\frac{\mathcal{M}}{u} \alpha$$

3)  $\frac{\mathcal{M}}{u} \alpha \vee \beta \Leftrightarrow \frac{\mathcal{M}}{u} \alpha \quad \text{or} \quad \frac{\mathcal{M}}{u} \beta$  For any  $\alpha$  and any model  $\mathcal{M}$ , the truth set of  $\alpha$  in  $\mathcal{M}$  ( $\|\alpha\|^{\mathcal{M}}$ ) is defined:  $\{u \in U: \frac{\mathcal{M}}{u} \alpha\}$ .

Where  $\cup$  and  $\cap$  designate set union and set intersection, truth conditions for PC formations may be defined as follows:

$$1) \quad \|\alpha \vee \beta\|^{\mathcal{M}} = \|\alpha\|^{\mathcal{M}} \cup \|\beta\|^{\mathcal{M}}$$

$$2) \quad \|\alpha \wedge \beta\|^{\mathcal{M}} = \|\alpha\|^{\mathcal{M}} \cap \|\beta\|^{\mathcal{M}}$$

$$3) \quad \|\alpha \rightarrow \beta\|^{\mathcal{M}} = \|\neg \alpha\|^{\mathcal{M}} \cup \|\beta\|^{\mathcal{M}}$$

$$4) \quad \|\alpha \leftrightarrow \beta\|^{\mathcal{M}} = (\|\neg \alpha\|^{\mathcal{M}} \cup \|\beta\|^{\mathcal{M}}) \cap (\|\neg \beta\|^{\mathcal{M}} \cup \|\alpha\|^{\mathcal{M}})$$

### Theoremhood, PC Derivability and Independence

A formula  $\alpha$  is a theorem of a logic  $L$  iff  $\alpha$  is derivable in  $L$ . As the concept of derivability plays a central role in what follows, it would be timely to shed some light on the notion at this point. Let  $\vdash_{PC} \alpha$  signify

that  $\alpha$  is derivable in PC.  $\alpha$  may be derivable from a set of hypotheses,  $\Sigma$ , or it may be derived from the null set,  $\phi$ . Derivability from a set  $\Sigma$  of hypotheses is defined as follows:  $\Sigma \vdash_{PC} \alpha$  iff there is a finite sequence of wffs.  $\beta_1 \dots \beta_N$  such that  $\alpha = \beta_N$  and for every  $\beta_i: 1 < i < N$ ,  $\beta_i$  is a substitution instance of a PC axiom or a member of  $\Sigma$  or follows from two earlier members of the sequence by [MP]. If  $\alpha$  is an axiom, then  $\alpha$  is derivable from itself or derivable simpliciter. This is represented by setting  $\Sigma = \phi$  i.e.  $\phi \vdash_{PC} \alpha$ .

Defined in this way the notion of PC derivability has in addition the following property:  $\Sigma \cup \{\alpha\} \vdash_{PC} \beta \Rightarrow \Sigma \vdash_{PC} \alpha \rightarrow \beta^5$ . This means that if  $\beta$  is derivable from a finite sequence of wffs.  $\Sigma'$  containing  $\alpha$  then  $\alpha \rightarrow \beta$  is derivable in turn from  $\Sigma$  (where  $\Sigma' = \Sigma \cup \{\alpha\}$ ).

The notion of derivability enters into a proof that the axiom set of a logic  $L$  is independent. If any formula  $\alpha$  is derivable from a set  $\Sigma$ ,  $\|\Sigma\| \subset \|\alpha\|$  ( $\|\Sigma\| \cap \|\beta_i\| \neq \emptyset$  ( $\beta_i \in \Sigma$ )). Hence, the proof that  $\alpha$  is not derivable from  $\Sigma$  in  $L$  is the proof that there exists some point  $u$  in a model for  $L$  where  $\frac{\mathcal{M}}{u} \Sigma$  and  $\frac{\mathcal{M}}{u} \not\vdash \alpha$ . An axiom set  $\{Ax_1, Ax_2, Ax_3\}$  of the logic  $L$  then is independent iff:

a)  $\{Ax_1, Ax_2\} \not\vdash_L Ax_3$  and

b)  $\{Ax_2, Ax_3\} \not\vdash_L Ax_1$  and

c)  $\{Ax_1, Ax_3\} \not\vdash_L Ax_2$ .

## Soundness, Consistency and Completeness for PC

A classical logic  $L$  is consistent if no formula of the form  $\alpha \wedge \neg \alpha$  is a theorem of  $L$  and sound if every  $L$ -theorem is valid. The notion of soundness is usually stronger than that of consistency, since if a logic is sound (all theorems are valid) it cannot contain an inconsistent formula (a formula that is false at all points in all models) as a theorem. However, every consistent formula need not be valid so consistency does not guarantee soundness. This essay will be concerned with the stronger notion of soundness. The proof that a logic is sound, as noted earlier, is the proof that each axiom is valid and each transformation rule is validity preserving. Every derived theorem must be valid under these conditions.

Recall that a logic is complete iff every formula valid in the semantics for the logic is a thesis of the logic. If a logic  $L$  is incomplete, then, some formula  $\alpha$  will be valid (true at all points in all models) but not a theorem of  $L$ .  $\alpha$  is valid iff  $\neg \alpha$  is false at all points in all  $L$ -models.  $\alpha$  is not a theorem iff  $\neg \alpha$  is not  $L$ -inconsistent, i.e.  $\neg \alpha$  is  $L$ -consistent. If it can be shown that all consistent formulae of a logic are true at some point in a model, the logic is shown to be weakly complete.

## Proving Soundness and Completeness for PC

### 1) Soundness

APC

$$1) \quad \frac{}{PC} (p \vee p) \rightarrow p$$

Proof:

1) if  $(p \vee p) \rightarrow p$  is not valid  $\exists u$  in some PC model:

$$\frac{M}{u} p \vee p, \quad \not\frac{M}{u} p$$

2) if  $\frac{M}{u} p \vee p$ , then  $\frac{M}{u} p$  contrary to assumption,

3) therefore  $\frac{}{PC} (p \vee p) \rightarrow p$

And similarly for axioms 2, 3 and 4.

Modus Ponens:  $\frac{}{PC} \alpha \rightarrow \beta$  and  $\frac{}{PC} \alpha$ , then  $\frac{}{PC} \beta$

1) if  $\alpha \rightarrow \beta$  is PC valid,  $\|\alpha\| \subseteq \|\beta\|$  in all PC models,

2) if  $\alpha$  is PC valid, then  $\|\alpha\| = U$  in all PC models,

3) therefore  $\|\beta\| \supseteq U$  in all PC models, i.e.  $\|\beta\| = U^{PC}$

4) and  $\beta$  is therefore valid in PC.

*In. 15.20*  
Universal Substitution:

Assume  $\alpha$  is valid. Form  $\beta$  from  $\alpha$  by substituting the wff.  $\gamma$  for every occurrence of some atomic formula,

say  $p$  in  $\alpha$ . If  $\beta$  is not valid,  $\exists u$  in some PC model:  $\not\models_u^{\mathcal{M}} \beta$ . We can create another PC model  $\mathcal{M}'$  on  $U$  such that every propositional letter  $q_i$  in  $\alpha$  other than  $p$ :  $\models q_i \models^{\mathcal{M}'} = \models q_i \models^{\mathcal{M}}$  and  $\models p \models^{\mathcal{M}'} = \models \neg \models^{\mathcal{M}}$ . In this case  $\not\models_u^{\mathcal{M}'} \alpha$ , contrary to the hypothesis that  $\alpha$  is PC valid.

## 2) Completeness

Completeness proofs at present tend to follow a model provided by Leon Henkin<sup>6</sup> utilizing the notion of a maximal consistent set. A set of formulae is consistent iff for any formula  $\alpha$  in the set,  $\neg \alpha$  is not also in the set and maximal iff every wff.  $\alpha$  not in the set is inconsistent with the set. A formula  $\alpha$  is inconsistent with a set  $\Sigma$  iff  $\Sigma \vee \{\alpha\} \not\models_{\perp}$ . Hence, there are a number of maximally consistent sets (maxi-sets) constructible for every logic with negation since for every atomic formulae  $\alpha$  there exists a wff.  $\neg \alpha$ , inconsistent with  $\alpha$  but consistent in the logic and thus an element of another maxi-set. A formula  $\alpha$  is consistent in a logic  $L$  iff  $\not\models_L \neg \alpha$ .

Maxi-sets so constructed have a number of crucial properties. Every  $L$ -consistent set of wffs. will have a maximally consistent  $L$ -extension.  $L$ -inconsistent formulae will be in no maxi-set because they are  $L$ -inconsistent and hence consistent in no  $L$ -set. All  $L$ -theses will be in all  $L$ -maxi-sets since the only formulae inconsistent with  $L$ -theses are  $L$ -inconsistent formulae and these, as noted,

are absent from all L-maxi-sets. Every non-thesis will be missing from at least one L-maxi-set (i.e. all those sets containing the negation of that non-thesis). If some wff.  $\alpha$  is derivable from a set of wffs.  $\Sigma$ , every maximally consistent extension of  $\Sigma$  will contain  $\alpha$ . Otherwise, it must contain  $\neg\alpha$  (or fail to be maximal) and fail to be consistent.

Maxi-sets work in completeness proofs as points in the special PC model  $\langle U_{PC}, V_{PC} \rangle$  called the canonical model. What is special about this model is that its universe is the set of PC maxi-sets and its valuation assigns to each atomic formula  $\alpha$  of PC the set of maxi-sets containing  $\alpha$ . With this sort of model we can prove PC to be complete by showing that any PC consistent formula will be true at at least one point in  $U_{PC}$ . If so, then there exists no PC consistent formula false at all points in all models and therefore no valid non-theorem of PC.

Informally: if PC is not complete  $\exists\alpha : \alpha$  is a non-theorem of PC, valid in PC semantics.

1) If valid in PC semantics, then  $\alpha$  is true at all points in all PC models,

2) If  $\alpha$  is true at all points in all PC models,  
 $\neg\alpha$  is false at all points in all PC models,

3) If  $\alpha$  is a non-theorem of PC,  $\neg\alpha$  is consistent in PC,

4) Take the canonical model  $\langle U_{PC}, V_{PC} \rangle$  where  $U_{PC}$  = the set of all PC maxi-sets and  $V_{PC}$  the function:  $\underline{At}_{PC} \rightarrow 2^{U_{PC}}$ ,

5) By 1) because  $\langle U_{PC}, V_{PC} \rangle$  is a PC model,  $\forall u \in U_{PC}$ ,  $2 \in u$ ,

6) If  $\neg a$  is PC consistent,  $\neg a$  has a maximally consistent extension, i.e.  $\exists x \in U_{PC}: \neg a \in x$ . By 5)  $a \in x$ , therefore  $a \wedge \neg a \in x$ ,

7) But  $x$  is consistent by construction. Therefore PC is complete.

Formally:

To prove PC completeness is to prove that the characterization of truth at a point in the canonical model extends to every PC wff. That is, we must prove that  $V_{PC}$  (a function from  $\underline{At}_{PC} \rightarrow 2^{U_{PC}}$  such that  $\forall a \in \underline{At}_{PC}, \forall u \in U_{PC}$   $\frac{\mathcal{M}}{u}PCa \Leftrightarrow a \in u$ ) extends to all PC formulae. This proof is called the proof for the fundamental theorem and it goes by mathematical induction on the structure of PC wffs. beginning with  $\underline{At}_{PC}: \forall a \in F_{PC}, \forall u \in U_{PC}, \frac{\mathcal{M}}{u}PC \Leftrightarrow a \in u$ .

1) [At]: by definition of  $V_{PC}$ , the theorem holds for all atomic wffs.,

2) [ $\neg$ ]: let  $\alpha = \neg \beta$ , where  $\beta \in \underline{At}_{PC}$   $\frac{\mathcal{M}}{u}PC \neg \beta \Leftrightarrow \neg \beta \in u$

a) Assume  $\neg \beta \notin u$ . Therefore  $\neg \beta$  is inconsistent with  $u$  and  $\beta \in u$  because  $u$  is maximal. By the induction hypothesis  $\frac{\mathcal{M}}{u}PC \beta$ , therefore  $\not\frac{\mathcal{M}}{u}PC \neg \beta$  contrary to the assumption,  $\frac{\mathcal{M}}{u}PC \neg \beta$ .  
Therefore  $\neg \beta \in u$ .  $\neg \beta \in u = \frac{\mathcal{M}}{u}PC \neg \beta$

b)  $\not\vdash_u^M PC \neg \beta$  therefore  $\vdash_u^M PC \beta$  therefore by the induction hypothesis  $\beta \in u$  therefore  $\neg \beta \notin u$  ( $u$  is consistent).

3) [v]: let  $\alpha = (\beta \vee \gamma)$ , where  $\beta, \gamma \in \underline{At}_{PC}$   
 $\vdash_u^M PC \beta \vee \gamma \Rightarrow \beta \vee \gamma \in u$

a) Assume  $\beta \vee \gamma \notin u$ , then  $\beta \notin u$  and  $\gamma \notin u$ . If  $\beta \notin u$ , then  $\not\vdash_u^M PC \beta$  by the induction hypothesis. If  $\gamma \notin u$ , then  $\not\vdash_u^M PC \gamma$  by the induction hypothesis and  $\not\vdash_u^M PC \beta \vee \gamma$ .  
 $\beta \vee \gamma \in u \Rightarrow \vdash_u^M PC \beta \vee \gamma$

b) Suppose  $\beta \vee \gamma \in u$  therefore  $\beta \in u$  or  $\gamma \in u$  ( $u$  is maximal) therefore by the induction hypothesis  $\vdash_u^M PC \beta$  or  $\vdash_u^M PC \gamma$  therefore  $\vdash_u^M PC \beta \vee \gamma$ .

Since all PC wffs. are reducible to wffs. of the form  $\alpha, \neg \alpha, \alpha \vee \beta$  the fundamental theorem holds for all wffs. of PC and PC is complete. In this way, then, providing a semantics for a system can determine whether the logic generated by the system is sound and complete.

### 3) Modal Logics

A modal logic is a logic whose underlying language contains the constant " $\Box$ ". Each of the modal logics considered in this paper include PC. These logics, then, are not in any sense alternatives to PC but extensions of PC. A logic  $L'$  is an extension of another logic  $L$  iff  $\{\alpha: \vdash_L \alpha\} \subseteq \{\alpha: \vdash_{L'} \alpha\}$ .  $L'$  is said to include  $L$  if  $L'$  is an extension



of L.

PC was seen to be a logical theory capable of reflecting certain properties of " $\neg$ " and " $\vee$ " and the complex terms " $\wedge$ ", " $\rightarrow$ " and " $\leftrightarrow$ ". As this interpretation is truth-functional in nature, PC is described as the theory of truth-functions<sup>7</sup>. Modal constants are not truth-functional so a system capable of generating a logic adequate to modal terms will have to undergo some alterations.

#### The Generic Modal Formation ' $\square$ '

More than one species of modality has been identified. Familiar in this class are the alethic modalities (necessity, possibility and contingency), epistemic/doxastic modalities (knowledge and belief), deontic modalities (obligation, permission) and the temporal modalities (always, sometimes). This list is by no means exhaustive. To simplify proceedings, the operator ' $\square$ ' will be treated as a general modal formation, interpretable in any mode depending upon the system under study. The idea of using ' $\square$ ' generically to represent a range of modal notions is a comparatively recent development in the history of modal logic. Seeing modality more generally has been of important heuristic advantage both to the study of specific modal concepts and to the study of modal logics as a discipline in itself.

The business of providing natural language interpretations for modal concepts characterized in different logics is

complicated. It appears doubtful, even at this late date, that any of the standard modal logics sit completely comfortably with interpretations proposed for them. Logicians should be continually cautioned against accepting too readily the suggestion that a particular logic reflects some everyday notion in its entirety. Which modal logics go with which interpretations is an important and difficult question for the philosophically minded logician.

#### Some Simple Modal Extensions of PC

A simple modal logic can be constructed from PC by adding the modal rule of inference [RE] (the rule of extensionality):  $\vdash_L(\alpha \leftrightarrow \beta) \Rightarrow \vdash_L(\Box\alpha \leftrightarrow \Box\beta)$ , this is Segerberg's system E<sup>8</sup> and although,  $\underline{A}_E = \underline{A}_{PC}$ ,  $\{\alpha: \vdash_E\alpha\} \supset \{\alpha: \vdash_{PC}\alpha\}$  Hence E contains PC. To take a trivial example,  $\vdash_E\Box(\alpha \rightarrow \beta) \leftrightarrow \Box(\neg\alpha \vee \beta)$  which is not a PC thesis since  $\Box(\alpha \rightarrow \beta)$  is not a PC wff.

Systems extending, in turn, from E are established by adding additional modal transformation rules and axioms. The other standard modal transformation rules are the Rule of Regularity [RR],  $\vdash_L(\alpha \rightarrow \beta) \Rightarrow \vdash_L(\Box\alpha \rightarrow \Box\beta)$  and the Rule of Necessitation [RN],  $\vdash_L\alpha \Rightarrow \vdash_L\Box\alpha$ . Two characteristic modal axioms are [D]  $\vdash_L\Box\alpha \rightarrow \neg\Box\neg\alpha$  and [K]  $\vdash_L\Box(\alpha \wedge \beta) \rightarrow (\Box\alpha \wedge \Box\beta)$ . These rules and axioms combine to generate additional logics as follows:

Base	+	<u>R</u>	+	<u>A</u>	=	System
E		[RR]				C

C	[RN]	[K]	K
K		[D]	KD

Seegerberg has identified several other systems in addition to E, C and  $K^9$  and an infinite class of logics weaker than K but stronger than C has been discovered<sup>10</sup>. It is not clear which bits of English, if any, these latter logics can be taken to represent.

#### 4) Relational Semantics for Modal Logics

The introduction of non-truth-functional formations creates a problem for a semantics that defines truth conditions only in terms of combinations of truth-values of atomic components. The task facing the modal semanticist is the construction of a semantic condition that will specify truth conditions for non-truth-functional formations and guarantee the validity of desired modal principles. Development in the semantics for modal logics received a major impetus from the works of Saul Kripke<sup>11</sup> and Jaakko Hintikka<sup>12</sup>. It is the work of these logicians which has focused attention upon the sequence of systems mentioned above since the Kripkean and Hintikkan approaches force us to accept the basic principles of these systems, [K], [RR] and [RN].

Kripke's semantics utilize a notion of "possible worlds" which has historical precursors in Leibniz<sup>13</sup> and

Wittgenstein<sup>14</sup>. For Leibniz, our world or (monad), was the best of the many possible monads that God could have brought into existence. Leibniz uses the idea of "possible worlds" in this context to show that man has free will. Adam, according to Leibniz, did not have to sin because the monad in which Adam does not sin is a possible, though not actual monad. That he did not sin is purely a contingent feature of this monad. Hence, Adam was not committed to sinning for any reasons of logic.

Although Leibniz uses the notion of possible worlds in this way to support his doctrine of free will, it is rather curious to note that he did not make more of the connection between the ideas of presence in all, at least one, or absence from all possible monads and the ideas of "necessity", "contingency" and "self-contradiction". Necessary propositions were propositions with contradictory denials. Contingent propositions could be denied without contradiction. But Leibniz never exploited the ties between these ideas and the notion of truth or falsity in a class of possible worlds<sup>15</sup>.

Wittgenstein, by contrast, although not using the notion of "possible worlds" per se to distinguish truths of logic (tautologies) from truths of science (accidentally true propositions), did employ a similar notion in the Tractatus specifically to make this distinction. For Wittgenstein, a proposition determines a place in logical

space (3.42). Tautologies leave all space to actuality. Contradictions fill all logical space (4.463). Wittgenstein's notion of logical space or, more specifically, a set of places in logical space, correlates with the notion of "possible worlds". The whole of logical space could be regarded set theoretically as the union of the set of possible world contexts (i.e. points in a model prior to valuation). A place in logical space is a context which would verify a proposition if that context happened to be an element of the set of facts comprising the world (1.0).

The notion of a possible world in Kripke/Hintikkaan semantics is indexical in character. That is, "possibility" is a two-place predicate capable of characterization in terms of the usual stock of relational properties, i.e. reflexivity, transitivity, etc. In specifying properties for this relation, as we shall see, different logics, and hence, different logical properties for modal constants, are specified.

### Kripke

Kripke takes the set  $\underline{K}$  of possible worlds and selects an element  $G$  as the real world. Any set of possible worlds in  $\underline{K}$  is only possible relative to some index world (it may or may not be  $G$ ) if they stand in the relation  $R$  with that world. Given any two worlds  $H_1 R H_2$  (read  $H_2$  is possible relative to  $H_1$ ), every proposition true in  $H_2$  is possible

in  $H_1$  but not vice versa. Not every proposition possible in  $H_1$  is true in  $H_2$  but true in some  $H'$  such that  $H_1RH'$ . If  $\alpha$  is necessary in  $H_1$ ,  $\alpha$  must be true in all worlds  $H'$  such that  $H_1RH'$ .

The relation  $R$  can have many properties but it cannot fail to have reflexivity in Kripke's view. Each world must be possible relative to itself since if some formula is true in a world it is at least possible in that world. Hence, each world must be possible relative to itself and  $R$  in all Kripke structures is reflexive. And this means, as we shall see, that the principle  $[T]: \Box p \rightarrow p$  is inescapable in Kripke semantics. In fact the weakest logic that can be modelled in a Kripke structure is the system  $T$ .

More formally, a normal Kripke structure is a triple  $\langle G, \underline{K}, \underline{R} \rangle$  where  $\underline{K} \neq \emptyset$  and  $G \in \underline{K}$  and  $\underline{R} \subseteq K^2$ . A model on a normal Kripke structure is the quadruple  $\langle G, \underline{K}, \underline{R}, \varphi \rangle$  where  $\varphi: \{PxH\} \rightarrow \{T, F\}$ .  $P$  is the set of atomic sub-formulae of  $\alpha$  and  $H$  the set of elements of  $\underline{K}$ . That is, for every atomic sub-formula of  $\alpha$ ,  $\varphi(P, H) = T$  or  $\varphi(P, H) = F$ .  $\langle G, \underline{K}, \underline{R}, \varphi \rangle$  may be "extended" to all wffs. of PC by replacing  $P$  with a variable that ranges over all formulae of the system and defining  $T$  or  $F$  conditions as follows:

$$[\varphi \underline{At}]: \forall \beta \in \underline{At}, \forall H \in \underline{K}, \varphi(\beta, H) = T \text{ or } \varphi(\beta, H) = F$$

$$[\varphi \neg]: \varphi(\neg\beta, H) = T \text{ iff } \varphi(\beta, H) = F, \text{ otherwise}$$

$$\varphi(\neg\beta, H) = F$$

$$[\varphi \vee]: \varphi(\beta \vee \gamma, H) = T \text{ iff } \varphi(\beta, H) = T \text{ or } \varphi(\gamma, H) = T$$

The rules for " $\forall$ ", " $\rightarrow$ ", " $\leftrightarrow$ " are derived in the usual manner.  $\Box$  is defined:  $[\Box\alpha]: \varphi(\Box\alpha, H) = T$  iff  $\forall H' \in K: HRH', \varphi(\alpha, H') = T$ .

$\alpha$  is true for Kripke if  $\varphi(\alpha, G) = T$ , false if  $\varphi(\alpha, G) = F$ . A formula  $\alpha$  is satisfiable if  $\varphi(\alpha, H) = T$  for some  $H$  in at least one model and valid if  $\varphi(\alpha, H) = T$  for all  $H$  in all models.

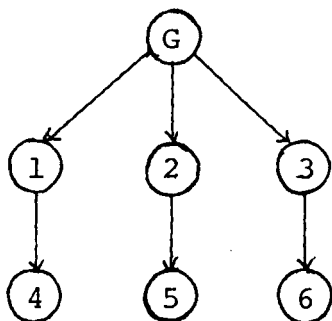
### How R Works to Provide Semantics for Different Modal Systems

Because the relation  $R$  defines the truth conditions for  $\Box$ , imposing conditions on  $R$  in certain cases affects the set of valid principles. In this way, the character of  $R$  alters which modal formulae will be theorems. The effect of imposing certain standard conditions, as the following diagrams show, is to add or delete elements to the set designated possible relative to a given index world. This means that certain formulae will hold in certain worlds under different  $R$  conditions. To preserve the connection with Kripke, we will identify the index world in each diagram as the world " $G$ ".

In a Kripke structure, of course, there may be an infinite number of sets of possible worlds relative to a particular index world.  $GRI$  is represented diagrammatically as:



$K = \{G, 1, 2, 3, 4, 5, 6\}$



a) If  $R$  has no properties at all (and this structure would not be a Kripke structure because  $R$  is always reflexive), the set of alternatives to  $G = \{1, 2, 3\}$  and the total set of pairs in the  $R$  relation is  $\{\langle G, 1 \rangle, \langle G, 2 \rangle, \langle G, 3 \rangle, \langle 1, 4 \rangle, \langle 2, 5 \rangle, \langle 3, 6 \rangle\} = R$ .

Notice the formula  $[K]$ ,  $\Box p \wedge \Box q \Rightarrow \Box(p \wedge q)$ , must hold in all relational models regardless of the imposed properties:

Proof: 1) Assume  $[K]$  does not hold,  $\varphi(\Box p, 6) = T$ ,  
 $\varphi(\Box q, 6) = T$ ,  $\varphi(\Box(p \wedge q), 6) = F$

2) If  $\alpha(\Box p, 6) = T$  and  $\varphi(\Box q, 6) = T$ ,  $\forall H': GRH'$ ,  
 $\varphi(p, H') = T$  and  $\varphi(q, H') = T$

3) If  $\varphi(\Box(p \wedge q), 6) = F$ ,  $\exists H' \in K: GRH'$  and  $\varphi(p, H') = F$   
 or  $\varphi(q, H') = F$

Which is inconsistent with 2), hence  $[K]$  must hold in all relational models.



b) If  $R$  is reflexive, secondum Kripke,  $\langle G, G \rangle \in \underline{R}$ ,  
 $\langle 1, 1 \rangle \in \underline{R}$ , ..... ,  $\langle 6, 6 \rangle \in \underline{R}$  and  $[T]: \Box p \rightarrow p$ , holds:

Proof:

1) If  $\varphi(\Box p, 6) = T$ ,  $\varphi(p, 6) = T$  because  $\langle G, G \rangle \in \underline{R}$

Hence the weakest logic modellable in Kripke's structures  
 is the logic  $T$ .

c) If  $R$  is transitive only, the set of alternatives  
 to  $G$  also includes  $\{4, 5, 6\}$ , i.e.  $\underline{R} = \{\langle G, 1 \rangle, \langle G, 2 \rangle,$   
 $\langle G, 3 \rangle, \langle G, 4 \rangle, \langle G, 5 \rangle, \langle G, 6 \rangle, \langle 1, 4 \rangle, \langle 2, 5 \rangle, \langle 3, 6 \rangle\}$ .

$[K]$  must hold.  $[T]$  fails. The distinguishing formula of  $S4$ :  
 $\Box p \rightarrow \Box \Box p$ , holds:

Proof:

1) If  $\varphi(\Box p, G) = T$ ,  $\forall H': GRH' \varphi(p, H') = T$

2) If  $\varphi(\Box \Box p, G) = F$ ,  $\exists H': GRH'$  and  $\varphi(\Box p, H') = F$

3) If  $\varphi(\Box p, H') = F$ ,  $\exists H'': H'RH''$  and  $\varphi(p, H'') = F$

4) But if  $R$  is transitive and  $GRH'$  and  $H'RH''$ ,  $GRH'' \forall H'':$

$H'RH''$

5) Therefore,  $\exists H'': \varphi(p, H'') = F$  and  $\varphi(p, H'') = T$

which is not possible.

d) If  $R$  is symmetrical the Brouwer formula holds:

$p \rightarrow \Box \Diamond p$  and  $\{\langle 1, 6 \rangle, \langle 2, 6 \rangle, \langle 3, 6 \rangle, \langle 4, 1 \rangle, \langle 5, 2 \rangle, \langle 6, 3 \rangle\} \subset \underline{R}$

1) If  $\varphi(\Box \Diamond p, G) = F$ ,  $\exists H': GRH'$  and  $\varphi(\Diamond p, H') = F$

1a) If  $\varphi(\Diamond p, H') = F$ ,  $\forall H'': H'RH''$ ,  $\varphi(p, H'') = F$

2) If  $GRH'$  and  $R$  is symmetrical, then  $H'RG$ , i.e.

$G \in H''$

3) Therefore  $\phi(p, G) = F$  and  $\phi(p, G) = T$  (by hypothesis) so  $\vdash p \rightarrow \Box \Diamond p$ .

e) And finally if  $R$  is an equivalence relation (reflexive, transitive and symmetrical) the distinguishing formula of S5 is valid:  $\Diamond p \rightarrow \Box \Diamond p$

1) If  $\phi(\Box \Diamond p, G) = F$ , then  $\exists H': GRH'$  and  $\phi(\Diamond p, H') = F$

2) If  $\phi(\Diamond p, H') = F, \forall H'': H'RH'', \phi(p, H'') = F$

3) If  $\phi(\Diamond p, G) = T, \exists H''': GRH'''$  and  $\phi(p, H''') = T$

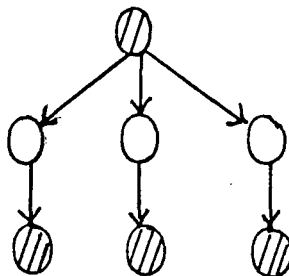
4) But  $\forall H''': GRH''', H'RH'''$  because  $R$  is equivalent.

5) Therefore  $\phi(p, H''') = F$  (by 2), which is impossible.

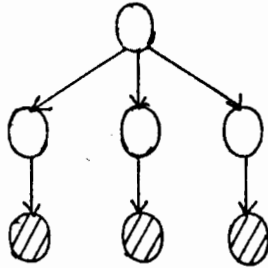
Hence, [5] holds under equivalence.

Altering the  $R$  relation alters which worlds are possible relative to a given world and hence alters which modal formulae must be true in which worlds. Letting the shaded circles represent non-alternativeness to  $G$ , the spectacle in a simple visual representation looks as follows:

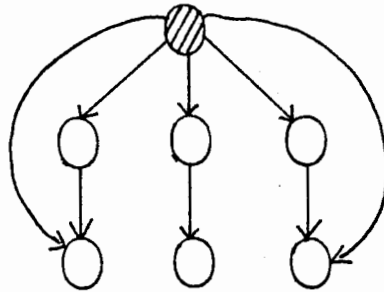
1) No properties



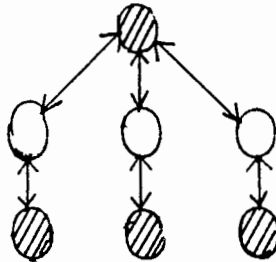
## 2) Reflexivity



## 3) Transitivity



## 4) Symmetry



5) With equivalence each circle is connected to all other circles by  $\updownarrow$ . Each circle is alternative to itself and no circle is shaded.

Hintikka and  $\Omega$

Hintikka adopted a similar approach with the notion of a "special" model system ( $\Omega$ ) composed of configurations of model sets ( $u, \mu$ , etc.) and the alternativeness relation  $H$  on these configurations. Model sets in a system are to be

construed intuitively as partial descriptions of states of affairs. They are partial in the sense that each model set in the system does not evaluate all atomic wffs. of the logic. Only those atomic wffs. relevant to the particular formation under semantic analysis are evaluated. Hintikka differs from Kripke in dispensing with the notion of truth and a fortiori truth in the fixed world  $G$  in favour of the notion of satisfiability. A formula is satisfiable in a model system  $\Omega$  if it is imbeddable in a model set  $u$  where  $u \vdash \Omega$  and  $\Omega$  is construed under the following conditions:

$$[C\neg]: \forall \alpha \in \underline{At}, \alpha \in \mu \in \Omega \quad \text{iff} \quad \neg \alpha \notin \mu$$

$$[C\vee]: (\alpha \vee \beta) \in \mu \in \Omega \quad \text{iff} \quad \alpha \in \mu \in \Omega \text{ or } \beta \in \mu \in \Omega$$

$$[C\Box]: \Box \alpha \in \mu \in \Omega \quad \text{iff} \quad \forall u: \mu H u \in \Omega, \alpha \in u$$

The notion of the validity of  $\alpha$  becomes the non-imbeddability of the negation of  $\alpha$  in any  $\mu \in \Omega$ . The familiar restrictions of reflexivity, symmetry, transitivity, etc. are placed on  $H$  alternativeness to create appropriate structures for  $T$ ,  $S4$ ,  $S5$ , etc. Hintikka also notes that the effect of altering conditions on  $H$ -alternativeness can also be achieved by specifying direct conditions on model sets. For example, given  $C\Box$  above, adding the condition  $C^*\Box, \Box \alpha \in \mu \in \Omega \quad \text{iff} \quad \alpha \in \mu$ , specifies a condition parallel to reflexivity in Kripke structures. Conditions on model sets parallel to symmetry and transitivity are as follows:

Sym.  $(\Box\alpha \in \mu \in \Omega) \quad (uH\mu \in \Omega) \Rightarrow (\Box\alpha \in \mu \in \Omega)$

Trans.  $(\Box\alpha \in \mu \in \Omega) \quad (uH\mu \in \Omega) \Rightarrow (\Box\alpha \in \mu \in \Omega)$

A Hintikkaan model structure, then, is a pair,  $\langle \Omega, H \rangle$  where  $\Omega$  is a non-empty set and  $H$  a relation defined over  $\Omega$ . A model on  $\langle \Omega, H \rangle$  is the triple  $\langle \Omega, H, V \rangle$  where  $V$  is a partial function:  $\{\underline{At}^* x\Omega\} \rightarrow \{S, U\}$ , which determines for each relevant (\*) atomic formula the value satisfiable,  $S$ , or unsatisfiable  $U$ . That is,  $V(\alpha, \mu) = U$  or  $S$ . That Hintikkaan models do not evaluate all atomic wffs. for each  $\mu \in \Omega$  does not represent an inadequacy in any respect. Descriptions of  $\mu \in \Omega$  are partial but each can be made relevant to any complex wff.  $V(\underline{At})$  is extended to all members of  $\underline{F}$  through satisfiability conditions for:

$[V \neg]: \neg\alpha \in \mu \in \Omega$  iff  $\alpha \notin \mu \in \Omega$

$[V \vee]: \alpha \vee \beta \in \mu \in \Omega$  iff  $\alpha \in \mu \in \Omega$  or  $\beta \in \mu \in \Omega$

### A Set Theoretic Relational Structure

Both Kripke and Hintikkaan structures can be seen as versions of a general relational structure  $\langle U, R \rangle$  where  $U \neq \emptyset$  and  $R \subset U^2$ , where  $U^2$  is the cartesian product of  $U$  with itself. For any given point  $u \in U$ ,  $R$  identifies the set of ordered pairs  $\langle u, v \rangle \in R$  or  $\{v: uRv\}$ . A model is a triple  $\langle U, R, V \rangle$ , where  $V: \underline{At} \rightarrow 2^U$ .  $V$  evaluates  $\underline{At}, \neg$ , and  $\vee$  as did the structures for PC semantics, and  $\Box$  as follows:

$$[V \Box]: \frac{\mathcal{M}}{u} \Box \alpha \quad \text{iff } \forall u, v \in U, uRv \Rightarrow \frac{\mathcal{M}}{v} \alpha$$

This structure differs from Kripke in eliminating the notion of a fixed world  $G$  and relieving  $R$  of constant reflexivity, and from Hintikka in evaluating all members of  $\underline{At}$  at each point. But in the important semantic respects it is equivalent to both.

#### 5) Neighborhood or Scott/Montague Semantics for Modal Logics

Neighborhood models are constructed for modal logics by adding to PC semantics a feature that makes possible, in the same way that the relation  $R$  (or  $H$ ) in relational semantics made possible, the specification of truth-conditions for non-truth-functional constants. As  $R$  (or  $H$ ) associate with each object in the structure some collection of objects, the function  $N$  in neighborhood semantics associates with each object some collection of collections of objects. As collections of objects provide the meat of the truth conditions for  $\Box$  in Kripke and Hintikka, so the collections of collections of objects provide the truth conditions for  $\Box$  in Scott/Montague semantics.

#### The Neighborhood Function

The neighborhood function  $N: U \rightarrow 2^{(2^U)}$  associates each point with a collection of point-sets of  $U$ . The collection

of point-sets for  $u$  is the collection of neighborhoods for  $u$ . A neighborhood, then, is a single subset of  $U$  or one element of the power set of  $U$ ,  $\mathcal{P}(U)$ . The function  $N$ , although single valued, is not necessarily unique. For any two points in  $U$  say,  $u$  and  $v$ , it is possible that the collection of neighborhoods for  $u$  is the collection for  $v$ . A neighborhood in the collection for  $u$  may also be in the collection for  $v$ . That is, it is possible that  $\underline{a} \in N_u \cap N_v$  for some  $\underline{a} \subseteq U$ .

### Neighborhood Frames and Models

The ordered pair  $\langle U, N \rangle$  is called a neighborhood frame  $\mathfrak{F}$ . Value assignments on a neighborhood frame are specified in the usual manner by the function  $\underline{a}_t \rightarrow \underline{2}^U$ . The triple  $\langle U, N, V \rangle$  is a model on the frame  $\mathfrak{F}$ . The truth of a formulae  $\alpha$  at a point in a model  $\mathfrak{M}$  is given by adopting the standard PC conditions for truth-functional constants and for  $\alpha$ :

$$[\forall \alpha]: \frac{\mathfrak{M}}{u} \alpha \text{ iff } \exists \underline{a} \in N(u): \underline{a} = \{V: V \subseteq U \text{ and } \frac{\mathfrak{M}}{V} \alpha\}$$

$$\text{or } \frac{\mathfrak{M}}{u} \alpha \text{ iff } \exists \underline{a} \in N(u): \underline{a} = \|\alpha\|_{\mathfrak{M}}$$

A formula  $\alpha$  receives the ' $\alpha$ ' operator at any point in the model just in case the truth set of  $\alpha$  is an element of the collection of neighborhoods defined for that point. To be one element in a collection of neighborhoods is, of course, to be a neighborhood.

## A Simple Neighborhood Model

Consider the simplified neighborhood model  $\mathcal{M}'$  below, where  $U' = \{u, v\}$  and  $V'(\alpha) = \{u, v\}$ ,  $V'(\beta) = \{u\}$  the values  $N'(u)$ ,  $N'(v)$  could be indicated by horizontal straight arrows from  $U$  to  $\underline{2}(2^U)$

$U'$	$2(2^{U'})$
$u \cdot \alpha, \beta$	1 {u}
	2 {v}
	3 {u,v}
	4 $\{\emptyset\}$
$v \cdot \alpha$	5 {u}, {v}
	6 {u}, {v,u}
	7 {u}, $\{\emptyset\}$
	8 {v}, {u,v}
	9 {v}, $\{\emptyset\}$
	10 {u,v}, $\{\emptyset\}$
	11 {u}, {v}, {u,v}
	12 {u}, {v}, $\{\emptyset\}$
	13 {u}, {u,v}, $\{\emptyset\}$
	14 {v}, {u,v}, $\{\emptyset\}$
	15 {u}, {v}, {u,v}, $\{\emptyset\}$
	16 $\emptyset$

$\alpha$  is true in  $\mathcal{M}'$  at all points in  $U'$  and the formula  $\beta \rightarrow \alpha$  is true at all points (since  $\alpha$  is true at all points,  $\neg\beta \vee \alpha$  is true at all points).  $\|\beta\|$ , i.e.  $\{u\}$ ,  $\subset \|\alpha\|$ , i.e.  $\{u, v\}$ . The formula  $\alpha\alpha$  is true at  $u$  or  $v$  if and only if the function  $N'$  selects for  $u$  or  $v$  some element of  $2(2^U)$  containing  $\|\alpha\| = \{u, v\}$ . In  $\mathcal{M}'$  these elements include the sets numbered 3, 6, 8, 10, 11, 13, 14, 15. For the formula  $\alpha\beta$ , the sets 1, 5, 6, 7, 11, 12, 13, 15. Note that  $\alpha\beta$  may be true at  $v$  without  $\beta$  being true at  $v$ . If  $\alpha\alpha$  and  $\alpha\beta$  are true at  $u$  or  $v$ , then  $N'(u)$  or  $N'(v)$  must contain 6, 11, 13, 15.



If  $\alpha$  and  $\neg\beta$  at  $u$  or  $y$  then  $N'(u), N'(y)$  contain 3, 8, 10, 14. And if  $\alpha, \beta$ , at  $u$  but  $\alpha$  and  $\neg\beta$  at  $v$ ,  $N'(u)$  contains 6, 11, 13, 15 and  $N'(v)$  contains 3, 8, 10, 14. If  $\alpha$  or  $\beta$  at  $u$  or  $v$ , then  $N'(u)$  or  $N'(v)$  is the union of the sets satisfying  $\alpha, \beta$ , into  $\{1, 3, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15\}$ . And so on for  $\alpha \rightarrow \beta$  and  $\neg\beta \rightarrow \alpha$ .

The effect of placing restrictions on  $N$  is illustrated in this simplified model by determining the set of possible values of  $N(u)$  and  $N(v)$ . Assume  $\alpha\beta$  holds at  $u$ . Now, if  $\beta \rightarrow \alpha$  at all points in this model and if we restrict  $N'$  by closing it under supersets, then  $N'(v)$  must also contain as elements all supersets of  $\|\beta\| = \{v\}$ . In  $M'$  the only superset of  $\{v\}$  is  $\{u, v\}$ . So, if closed under supersets, the possible values for  $N'(u)$  are limited to 6, 11, 13, 15.  $N'$  cannot select for any point in  $U$  where  $\alpha\beta$  holds an element in  $2(2^U)$  containing  $\|\beta\|$  but not  $\|\alpha\|$ . As conditions on  $N$  are frame conditions, no matter what model is being considered, where  $N(u)$  is closed under supersets,  $[RR]$  must also hold. The collection of neighborhoods cannot contain any set as an element without also containing its supersets. If some formula implies another at every point in every model in a class of frames closed under supersets, if the former is in the scope of  $\square$  at a point, so must the latter.

The simple model  $M'$  is actually a quasi-model since the valuation has not been made in a class of frames but over a range of possible classes of frames, each possibility

indicated by a different set of restrictions on  $N'$ . In addition to closure under supersets, other standard conditions on frame functions required to yield the characteristic modal formulae noted earlier on Page 20 are as follows:

Characteristic Formulae	Frame Condition
[RE] (Rule of Extensionality)	$N(u)$ closed under set incl.
$(\vdash \alpha \leftrightarrow \beta) \Rightarrow (\vdash \Box \alpha \leftrightarrow \Box \beta)$	$\forall \underline{a}, \underline{b} \subseteq U, \forall u \in U, \underline{b} \subseteq N(u) \Rightarrow (\underline{a} \subseteq \underline{b} \Rightarrow \underline{a} \in N(u))$
[RN] (Rule of Necessitation)	$\forall u \in U, U \in N(u)$
$\vdash \alpha \Rightarrow \vdash \Box \alpha$	(the weaker condition $N(u) \neq \emptyset$ validates [RN] if $N(u)$ is closed under supersets)
[K] (in honor of Kripke)	$N(u)$ is closed under finite intersections
$(\Box p \wedge \Box q) \rightarrow \Box(p \wedge q)$	$\forall \underline{a}, \underline{b} \subseteq U, \forall u \in U, \underline{a}, \underline{b} \in N(u) \Rightarrow \underline{a} \cap \underline{b} \in N(u)$
[D] (Deontic Rule)	$\forall \underline{a} \subseteq U, \forall u \in U, \underline{a} \in N(u) \Rightarrow U - \underline{a} \notin N(u)$
$\Box p \rightarrow \neg \Box \neg p$	

### Validity in Neighborhood Structures

A formula holds in a particular model if it is true at all points in that model. A formula is valid on a frame if the formula holds in every model on that frame. Further, a

formula may be valid over a class of frames if it is valid on every frame in that class. Models in which particular formulae fail are known as counter-models for those formulae. A given formula may hold in a model but fail to hold in a frame or hold in a model and a frame but fail to hold over a class of frames of which that frame is a member. Where it holds in a frame it holds for all models on that frame.

#### Soundness and Completeness in Neighborhood Frames

As with PC, a proof that the axioms of a logic are valid and that the transformation rules of the logic preserve validity in the class of frames for the logic, proves that the logic is sound. Similarly for completeness. With respect to a class of frames  $C$  and a logic  $L$ , if every formation valid in  $C$  is a thesis of  $L$ , then  $L$  is complete with respect to  $C$ . The number of formations valid in a class of frames exhausts the number of possible theses for any logic with respect to those frames.

As soundness proofs for modal logics do not differ significantly from the proof for PC soundness, the proof is omitted. Completeness proofs, however, are more complicated. First, because modal logics contain constants that PC does not, the initial step towards completeness requires a proof that the idea of truth at a point in the modal canonical model also holds for modal constants. In addition, a proof

is required to show that the canonical frame constructed for the system is in the class of frames for the system. This proof consists in the demonstration that the canonical neighborhood function has the same properties as all neighborhood functions in frames of that class.

The Fundamental Theorem for all "Classical"<sup>16</sup> Modal Logics

The proof that the concept of truth at a point in the canonical model holds for all permitted formations of a modal logic is the proof that the fundamental theorem holds for all formations of that modal logic. It follows the PC proof with the addition of a fourth step, the proof for formulae of the form  $\Box\beta$ .

A canonical neighborhood model for a modal logic  $L$  is a triple  $\langle U_L, N_L, V_L \rangle$  where  $U_L$  denotes the set of all  $L$ -maximal sets of formulae,  $N_L$  the canonical neighborhood function from  $U_L \rightarrow 2(2^{U_L})$ , and  $V_L$  the canonical valuation from  $\text{At}_L \rightarrow 2^{(U_L)}$ . Let  $|\alpha|_L$  denote the set of maxi-sets in  $U_L$  containing  $\alpha$ .  $N$  defines for each  $u \in U_L$  the value:

$$N_L(u) = \{ \underline{a} : \underline{a} \subseteq U_L \text{ and for some wff. } \alpha, \Box\alpha \in u \text{ and } \underline{a} = |\alpha|_L \}$$

The function  $V_L$  assigns each  $\text{At}_L$  to a set of maxi-sets such that:

$$V_L(\alpha) = |\alpha| = \{ v : v \subseteq U_L \text{ and } \frac{\mathfrak{M}}{V}\alpha \} = \|\alpha\|^L$$

The fundamental theorem is the proof that for all wffs.:

$$\frac{\mathfrak{M}}{u}\alpha \quad \text{iff } \alpha \in u \text{ or } (\|\alpha\|^{\mathfrak{M}L} = |\alpha|_L)$$

The proof is the same for PC with the additional step to show that the theorem holds for formulae of the form,  $\Box\beta$ .

Proof:  $[V_L]$  set  $\alpha = \Box\beta$  where  $\beta$  is atomic

=

1) Assume  $\Box\beta \in u$ , then by definition of  $N_L(u): |\beta| \in N_L(u)$ .

By hypothesis  $|\beta| = \|\beta\|$ , hence  $\|\beta\| \in N_L(u)$  so  $\frac{\mathcal{M}}{u}\Box\beta$

=

2) Assume  $\frac{\mathcal{M}}{u}\Box\beta$ , then  $\exists \underline{a} \in N_L(u): \underline{a} = |\beta|$ . By the induction hypothesis  $|\beta| = \|\beta\|$ . By definition  $N_L(u)$ , if  $\underline{a} \in N_L(u)$ ,  $\exists \gamma: \Box\gamma \in u$  and  $\underline{a} = |\gamma|$ . Thus  $|\beta| = |\gamma|$ . Hence,

$\vdash_L \beta \leftrightarrow \gamma$ .

Now we know that  $\Box\gamma \in u$  and we know that  $\vdash_L \beta \leftrightarrow \gamma$ . If we had licence for the inference from  $\vdash_L \Box\beta \leftrightarrow \gamma$  to  $\vdash_L \Box\beta \leftrightarrow \Box\gamma$  the proof would be complete for it would follow that  $\Box\beta \in u$ . Now the one rule that is preserved in all semantics based on neighborhood frames is the rule of extensionality. Neighborhood semantics is founded on extensional set theory which holds that sets with the same members are identical. Equivalent formulae have identical truth sets. What holds with respect to the elements of the truth sets of one, by extensionality holds for the other. Hence, we have the licence for the inference and the fundamental theorem for "classical" modal logics (logics with [RE]) is proven.

### Suitability in Canonical Models

In addition to proving the fundamental theorem for a

logic with respect to a semantics, it is also necessary to show that the canonical model is, indeed, a model in the class of frames for that logic. As noted on Page 36, classes of frames in neighborhood semantics are determined or defined by conditions obtaining for the function that structures frames in that class. What properties obtain for the function in a class of frames is determined in turn by the character of the relation that determines the truth conditions for  $\square$  in models on those frames. This means, as we shall see in Chapter Three, that the relation in the canonical model must have the same properties as the relation in all models in that class of frames. Which is to say that the canonical model must be suitable.

### Relational Frames and Models

There is a connection between the sets of alternatives in relational semantics and the collection of neighborhoods in Scott/Montague semantics. The connection, quite simply, is this: the neighborhood collection for any point is a superset of the alternative set for that point. The following illustration makes this clear. Let the simplified universe  $U = \{u, v, w, x, y\}$

Assume  $\square\alpha, \square\beta, \square\gamma$  at  $u, v, w$

$u.$      $\alpha, \beta, \gamma$

$v.$      $\alpha, \beta, \gamma$

$w.$      $\alpha, \beta, \gamma$

x.  $\alpha$

y.  $\beta$

In Kripke relational semantics, if  $\Box\beta, \Box\gamma$  at  $u$ ,  $v$ ,  $w$  the set of alternatives to  $u = \{u, v, w\}$  but not  $x$  and  $y$  because  $\frac{\mathcal{M}}{x} \Vdash \beta$ ,  $\frac{\mathcal{M}}{y} \Vdash \gamma$ . In neighborhood semantics, if  $\Box\alpha$ ,  $\Box\beta, \Box\gamma$  at  $u$ , then  $\|\alpha\|$ ,  $\|\beta\|$ ,  $\|\gamma\| \in N(u)$ .

$$\|\alpha\| = \{u, v, w, x\}$$

$$\|\beta\| = \{u, v, w, \neg, y\}$$

$$\|\gamma\| = \{u, v, w, \neg, \neg\}$$

$$\text{Now, } \bigcap \|\alpha\|, \|\beta\|, \|\gamma\| = \{u, v, w\} = R_u$$

Hence, a Scott/Montague semantics where the neighborhood collection is a filter (i.e. non-empty and closed under intersection and subsets), will validate formulae exactly like relational structures (without reflexivity). We can also define  $N(u) = \{\underline{a} : uRx \subseteq \underline{a}\}$ . The points in  $\bigcap N(u)$  are just the points which have every  $\alpha$  true such that  $\Box\alpha$  is true at  $u$ . Restricting  $N(u)$  in this way is just one way of interrelating neighborhood and relational semantics. Chapter Two will explore some other ways.

### Summary

Chapter One has explored in a general way the nature of logics, formal systems and semantics. It has been noted that there is a critical connection between the logical properties of the primitive constants of a logic and the

eventual complexions of the logic itself. A logic reflects the properties given its primitive constants by an interpretation. Whether a given axiomatic presentation succeeds in capturing a particular interpretation is determined by examining the theorem set produced. If the constant happens to be a term that operates in a natural language, the logic need only be compared with the ways in which the term actually operates in that language. We have also seen that semantics provides the logician with tools to evaluate the adequacy of a presentation from a formal point of view. Proving soundness and completeness for a system guarantees that that system's presentation meets the minimal formal requirements.

It remains to Chapter Two to draw out more explicitly the relationship between semantics and philosophy. Philosophy, broadly conceived, is the business of conceptual analysis and the business of conceptual analysis is the business of stating as nearly as we can the truth conditions for sentences containing terms of philosophical interest. Providing a formal semantic interpretation for these concepts can generate formal systems by rendering certain generating principles (axioms and transformation rules) inescapable. Hence, semantics produces for philosophy one means of determining whether a particular piece of analysis is correct. We check the logic against the everyday use of the concept. Logics, in a syntactic sense, are generated by systems, and



in a semantic sense by classes of structures. Semantics in turn are generated by representing a notion in a model.

The advantage of Scott/Montague semantics over relational semantics is that philosophical intuitions about the meanings of words can be imposed directly onto a semantic model. In relational semantics the R or H relation does the formal job for a range of systems but the semantic structures themselves stand in need of philosophical elucidation and interpretation. How R's being reflexive or transitive relates to the T and S4 informal interpretations of  $\Box$  is unclear. But as we shall see in Chapter Two, neighborhood semantics have direct and intuitive connections.

## CHAPTER TWO

TERNARY FRAMES ON STRICT ORDERING RELATIONS: THEIR  
APPLICATIONS TO THE CONCEPT OF BELIEF AND COUNTERFACTUAL  
CONDITIONALS

Chapter Two explores one specific Scott/Montague structure based on a ternary, strict ordering relation. This semantics has been proposed as a semantics for deontic logic<sup>1</sup>. Here, its application to the concept of belief and counterfactual conditionals will be investigated. Chapter Three suggests some possible revisions to repair intuitive deficiencies for these interpretations and investigates the question of completeness proofs for some logics explored in this chapter.

## 1) A Strict Ordering Relation -0-

A strict ordering structure is an ordered pair  $\langle U, O \rangle$  where  $U$  is a non-empty set and  $O$  a function:  $U \rightarrow 2^{(U^2)}$ .  $O^u$  is transitive and irreflexive. The expression  $xO^u y$  is read "x is ..... than y for u" where the "....." takes any transitive, irreflexive relation (better than, greater than, more than, etc.). The set  $2^{(U^2)} = P(U^2)$ . A model on an  $O$  structure is a triple  $U, O, V$ , where  $V$  is a function:  $\underline{At} \rightarrow 2^U$ . Truth conditions for truth functional and  $\square$  formulae are as follows:

$$\models_{\mathcal{M}} \neg : [\neg] : \models_{\mathcal{M}} \neg a \quad \text{iff} \quad \not\models_{\mathcal{M}} a$$

$$\models_{\mathcal{M}} \vee : [\vee] : \models_{\mathcal{M}} a \vee b \quad \text{iff} \quad \models_{\mathcal{M}} a \quad \text{or} \quad \models_{\mathcal{M}} b$$

$$\models_{\mathcal{M}} \square : [\square] : \models_{\mathcal{M}} \square a \quad \text{iff} \quad \forall x : \not\models_{\mathcal{M}} a_x, \exists y : \models_{\mathcal{M}} a_y \quad \text{and} \quad y O^u x$$

The function  $\odot(u)$ , in turn, is defined for each point  $u \in U$  as  $\{\underline{a} \subseteq U : \forall x \notin \underline{a}, \exists y \in \underline{a} \text{ and } y O^u x\}$ . An alternative frame condition for  $\square$  is defined in terms of  $\odot(u)$  as follows:  $u \in V(\square a)$  iff  $\|a\| \in \odot(u)$ .

### O-Structures as Generalized Relational Structures

An O-structure can be seen as a generalized Kripke structure in the following way:  $\langle U, P \rangle$  is an O-type structure iff  $U \neq \emptyset$  and  $P: U \rightarrow 2(U^N)$ . A Kripke structure is an O-type structure for  $N = 1$ . Each  $u \in U$  is connected with a set of singletons in  $U$  (Kripke alternatives to  $u$ ). That is,  $P$  assigns to each  $u \in U$  a unary relation  $P^u$ . A model on a Kripke O-type structure evaluates  $\square$  as follows:  $\models_{\mathcal{M}} \square a$  iff  $\forall u, x \in U, x P^u = \not\models_{\mathcal{M}} a_x$ . An O-type structure is an O-structure proper when  $N = 2$ , i.e. the pair  $\langle U, P \rangle$ , where  $P: U \rightarrow 2(U^2)$ .

### Rules of Inference in O-Structures

Since the truth conditions in an O-model for "-" and " $\wedge$ " are the standard conditions for these operators in the PC semantics examined earlier, proofs that modus ponens

and uniform substitution preserve validity are omitted. A proof for the rule of extensionality is also omitted since all logics on Scott/Montague semantics are obedient to [RE]. Proofs for the rule of necessity and the rule of regularity are as follows:

[RN]:  $\vdash \alpha \Rightarrow \vdash \Box \alpha$

1) If  $\vdash \alpha$  and  $\not\vdash \Box \alpha$ , then for all  $u \in U$ ,  $\frac{M}{u} \alpha$  and for some  $u \in U$   $\not\frac{M}{u} \Box \alpha$

2)  $\not\frac{M}{u} \Box \alpha$  iff  $\exists x: \not\frac{M}{x} \alpha$ ,  $\forall y: \frac{M}{y} \alpha$  and  $\sim y O^u x$

3) But,  $\vdash \alpha$  so  $\|\alpha\| = U$  and  $\|\neg \alpha\| = \emptyset$

4) Therefore  $\sim \exists x: \not\frac{M}{x} \alpha$ , a fortiori  $\sim (\exists x: \not\frac{M}{x} \alpha, \forall y: \frac{M}{y} \alpha$  and  $\sim y O^u x$ ) contrary to 2).

[RR]:  $\vdash (\alpha \rightarrow \beta) \Rightarrow \vdash (\Box \alpha \rightarrow \Box \beta)$

1) If  $\vdash (\alpha \rightarrow \beta)$ ,  $\not\vdash (\Box \alpha \rightarrow \Box \beta)$ , for all  $u \in U$ ,  $\frac{M}{u} \alpha \rightarrow \beta$ , and for some  $u \in U$ ,  $\frac{M}{u} \Box \alpha$ ,  $\not\frac{M}{u} \Box \beta$ .

2) If  $\vdash (\alpha \rightarrow \beta)$ ,  $\|\alpha\| \subseteq \|\beta\|$  in all models

3) If  $\frac{M}{u} \Box \alpha$  then  $\|\alpha\| \in \odot$

4)  $\odot(u)$  is closed under supersets:  $\forall \underline{a} \in \odot(u)$ ,  $\forall \underline{b}: \underline{b} \supseteq \underline{a}$ ,  $\underline{b} \in \odot(u)$  so  $\|\beta\| \in \odot(u)$  if  $\|\alpha\| \in \odot(u)$

5) By def.  $\odot$ ,  $\|\beta\| \in \odot u$  iff  $\frac{M}{u} \Box \beta$

6) Therefore  $\frac{M}{u} \Box \beta$  whenever  $\frac{M}{u} \Box \alpha$ , contrary to 1.

O-structures, then verify the standard K rules of inference, [RE], [RR] and [RN].

### Theorems in O-Structures

The formula [Con] ( $\neg \perp$ ) holds in O-structures with the truth conditions for  $\square$  provided. The formulae  $\perp$  (the false) is equivalent to any formula of the form,  $\alpha \wedge \neg \alpha$ , i.e.

$$\|\alpha\| = \|\alpha \wedge \neg \alpha\| = \emptyset.$$

#### [Con] $\neg \perp$

- 1) If  $\neg \perp$  is false, for some  $u \in U$ ,  $\frac{M}{u} \perp$
- 2) If  $\frac{M}{u} \perp$ ,  $\|\perp\| \in \odot(u)$
- 3)  $\forall x: \frac{M}{x} \perp, \exists y \frac{M}{y} \perp$  and  $y O^u x$
- 4)  $\|\perp\| = \emptyset$  therefore  $\sim \exists y \frac{M}{y} \perp$ , a fortiori  $\sim (\forall x: \frac{M}{x} \perp, \exists y: \frac{M}{y} \perp$  and  $y O^u x)$  inconsistent with 3)

However, the formula [D] fails:  $\square p \rightarrow \neg \square \neg p$

#### [D]

- 1) If  $\frac{M}{u} \square p, \forall x \notin \|p\|, \exists y \in \|p\|$  and  $y O^u x$
- 2) If  $\frac{M}{u} \neg(\neg \square \neg p)$  then  $\frac{M}{u} \square \neg p$  and  $\forall y \in \|p\|, \exists x \notin \|p\|$  and  $x O^u y$

1) and 2) are inconsistent if finiteness conditions are placed on points in  $\|\alpha\|$  and  $\|\neg \alpha\|$ . If these sets

are infinite it is quite possible that for every point  $x$  in  $\|\neg\alpha\|$  there exists a point  $y$  in  $\|\alpha\|$  such that  $yO^u x$  and for every point  $y$  in  $\|\alpha\|$  there exists some  $x$  in  $\|\neg\alpha\|$  such that  $xO^u y$ . If finiteness conditions are imposed and  $\frac{\mathcal{M}}{u} \models \alpha$  then  $\exists \underline{a}: \underline{a} \subseteq \|\alpha\|$  and  $\forall x \in \underline{a}, \forall x \notin \underline{a}, xO^u y$ .  $\frac{\mathcal{M}}{u} \models \neg\alpha$  iff  $\exists z \in \|\neg\alpha\|, \forall w \in \|\alpha\|, zO^u w$ . But  $z \in \underline{a}'$ , therefore  $\forall x \in \underline{a}, \sim zO^u x$  and  $\neg \square \neg \alpha$ . Hence [D] holds if finiteness conditions are imposed.

[K] fails where  $\underline{a}, \underline{b} \in \mathcal{O}(u)$  and  $\underline{a} \cap \underline{b} = \emptyset$ . If  $\underline{a} \cap \underline{b} = \emptyset, \neg \exists y \in \underline{a} \cap \underline{b}$  and  $\neg yO^u x$  for all  $x \in (\underline{a} \cap \underline{b})^1$ . [K], is also restored if finiteness conditions on points in  $u$  because the possibility of disjunct sets is ruled out. Placing an appropriate restriction on  $O^u$  also preserves [D] and [K], avoiding the model condition restriction on points in all models<sup>2</sup>. Before considering other formulae it will assist the enterprise in the long run to simplify the proof technique for modal formulae of a certain type.

### Proof Technique

Because  $\textcircled{O}(u)$  is closed under supersets, some formula  $\alpha$ 's occurring in the scope of  $\square$  in the antecedent guarantees that  $\forall \beta: \|\beta\| \supseteq \|\alpha\|, \square\beta$ . When the  $\square$  appears both in the antecedent and consequent positions in some formulae  $\gamma$ ,  $\gamma$  is a theorem if the formulae within the scope of the occurrences of  $\square$  in  $\gamma$  are in the appropriate set theoretic relation, i.e.  $\forall \alpha, \forall \beta \|\alpha\| \subseteq \|\beta\| \Rightarrow (\square\alpha \rightarrow \square\beta)$ . Moreover,

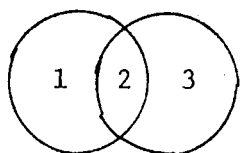
this is also a necessary condition if  $\Box\alpha \rightarrow \Box\beta$  since if  $\|\beta\| \supseteq \|\alpha\|$ ,  $\Box\beta$  may not obtain. Hence  $\forall\alpha, \beta \Box\alpha \rightarrow \Box\beta$  iff  $\|\alpha\| \subseteq \|\beta\|$ .

These properties yield a proof technique in the following way:

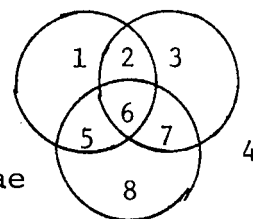
1) If we assume the antecedent of a conditional to be true where the antecedent contains some formula  $\alpha$  in the scope of a modal operator, certain results will follow for  $\forall x \in \|\neg\alpha\|$  and  $\exists y \in \|\alpha\|$ .

2) If some formulae  $\beta$  in the consequent is in the appropriate relation to  $\alpha$ , i.e.  $\|\neg\alpha\| \supseteq \|\neg\beta\|$  and  $\|\alpha\| \subseteq \|\beta\|$ ,  $\Box\beta$  will hold where  $\Box\alpha$  holds.

3) Use a Venn diagram numbered accordingly for two or three formulae as follows to plot  $\|\alpha\|$  and  $\|\beta\|$ .



2 formulae



3 formulae

4) If  $\|\alpha\|$ ,  $\|\beta\|$ ,  $\|\neg\alpha\|$ ,  $\|\neg\beta\|$  satisfy the appropriate schema,  $\vdash \Box\alpha \rightarrow \Box\beta$ . In each case the truth set of the negated formula in the scope of  $\Box$  (i.e.  $\|\neg\alpha\|$ ) will be listed first. The truth set of the formula itself will be listed second.

$\square p \rightarrow \square (q \rightarrow p)$

Antecedent: {3,4} {1,2}

Consequent: {3} {1,2,4}

Schema:



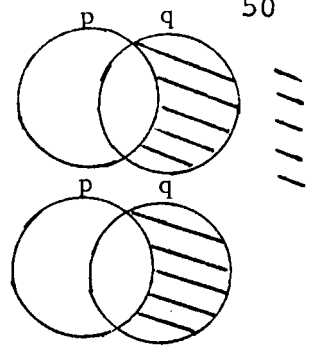
Required Schema:



Converse:

D

C



$\square \neg p \rightarrow \square (p \rightarrow q)$

Antecedent: {1,2} {3,4}

Consequent: {1} {2,3,4}

Schema:



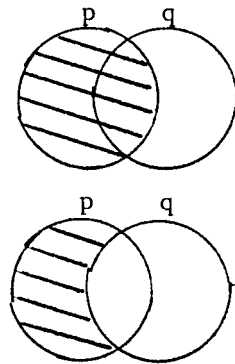
Required Schema:



Converse:

D

C

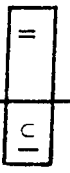


$\square (p \rightarrow q) \rightarrow \square (\neg q \rightarrow \neg p)$  (Transposition)

Antecedent: {1} {2,3,4}

Consequent: {1} {2,3,4}

Schema:



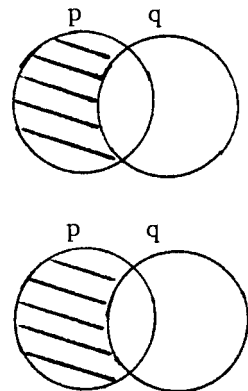
Required Schema:



Converse:

=

=





$$\Box(p \rightarrow (qrr)) \rightarrow \Box(p \rightarrow q)$$

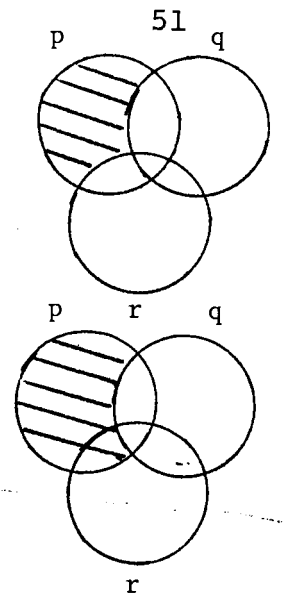
Antecedent: {1} {2,3,4,5,6,7,8}

Consequent: {1,5} {2,3,4,6,7,8}

Schema:  $\supset$   $\subset$

Required Schema:  $\begin{array}{|c|} \hline \subset \\ \hline \end{array}$   $\begin{array}{|c|} \hline \supset \\ \hline \end{array}$

Converse:  $\begin{array}{|c|} \hline \subset \\ \hline \end{array}$   $\begin{array}{|c|} \hline \supset \\ \hline \end{array}$



$$\Box p \rightarrow \Box(pvq)$$

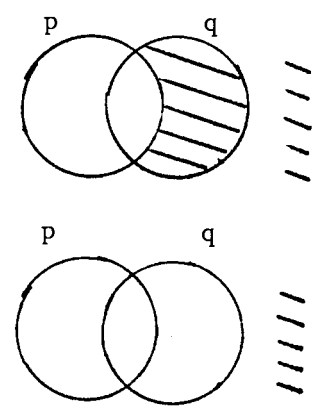
Antecedent: {3,4} {1,2}

Consequent: {4} {1,2,3}

Schema:  $\begin{array}{|c|} \hline \subset \\ \hline \end{array}$   $\begin{array}{|c|} \hline \supset \\ \hline \end{array}$

Required Schema:  $\begin{array}{|c|} \hline \subset \\ \hline \end{array}$   $\begin{array}{|c|} \hline \supset \\ \hline \end{array}$

Converse:  $\supset$   $\subset$



$$(\Box(p \rightarrow q) \wedge \Box(q \rightarrow r)) \rightarrow \Box(p \rightarrow r) \text{ (Transposition)}$$

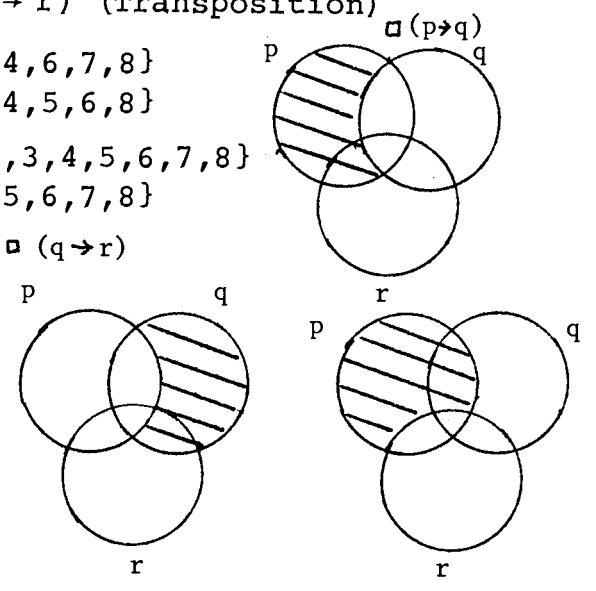
Antecedent: {1,5} {2,3,4,6,7,8} {3,7} {1,2,4,5,6,8}

Consequent:  $U = \{1,3,5,7\}$  {1,2,3,4,5,6,7,8} {1,2} {3,4,5,6,7,8}

Schema: -  $\subset$   $\supset$   $\supset$   $\subset$

Required Schema:  $\subset$   $\supset$

Converse: -  $\supset$



$$\Box((p \vee q) \rightarrow r) \rightarrow \Box(p \rightarrow r)$$

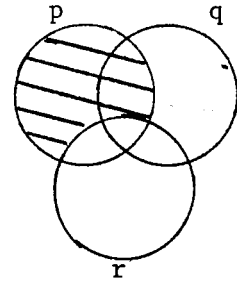
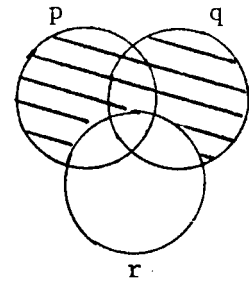
$$\text{Antecedent: } \{1,2,3\} \quad \{4,5,6,7,8\}$$

$$\text{Consequent: } \{1,2\} \quad \{3,4,5,6,7,8\}$$

$$\text{Schema: } \quad \subset \quad \supset$$

$$\text{Required Schema: } \quad \underline{\subset} \quad \underline{\supset}$$

$$\text{Converse: } \quad \supset \quad \subset$$



### Summary - Core Theorems

#### Rules/Theorems

[RE], [RR], [RN]

[Con], [K Converse]

#### Non-Theorems

$$\Box p \rightarrow \Box(q \rightarrow p)$$

$$\Box \neg p \rightarrow \Box(p \rightarrow q)$$

$$\Box(p \rightarrow q) \rightarrow \Box(\neg q \rightarrow \neg p) \quad (\Box \text{ transposition})$$

$$\Box(p \rightarrow q) \rightarrow \Box((p \rightarrow q) \vee r)$$

$$\Box p \rightarrow \Box(p \vee q)$$

$$\Box((p \vee q) \rightarrow r) \rightarrow \Box(p \rightarrow r)$$

$$\Box(q \rightarrow p) \rightarrow \Box p$$

$$\Box(p \rightarrow q) \rightarrow \Box \neg p$$

$$\Box((p \rightarrow q) \vee r) \rightarrow \Box(p \rightarrow q)$$

$$\Box(p \vee q) \rightarrow \Box p$$

$$\Box(p \rightarrow r) \rightarrow \Box((p \vee q) \rightarrow r)$$

$$(\Box(p \rightarrow q) \wedge \Box(q \rightarrow r)) \rightarrow \Box(p \rightarrow r)$$

$$\Box(p \rightarrow r) \rightarrow (\Box(p \rightarrow q) \wedge \Box(q \rightarrow r))$$

Because this logic contains the usual rules of inference and the distinctive theorem [Con] it has been called U.Con.

Note that U.Con becomes the logic KD when appropriate finiteness conditions are added.

## 2) '□' as Obligation in O-Structures

The key to interpreting '□' is obligation in O-structures lies in the principle of utility which, stated coarsely, deems an act good if its advantages outweigh its disadvantages<sup>3</sup>. On this view, the moral agent, in determining whether a given act is obligatory, is conceived as being confronted by a number of different possible states of affairs, any of which he may bring about by some or other action. To bring about a state of affairs by a specific action is in the same stroke to refrain from bringing about certain other possible states of affairs. A particular act is obligatory according to this thesis if for every way in which the agent may refrain from performing the action there exists some way of performing it which will make the world a better place. Better worlds are those with greater or more numerous advantages or slighter or less numerous disadvantages. The judgements made then, on this view, are judgements about the comparative advantages and disadvantages to be realized as consequences of acting or failing to act in a particular way.

This view is reflected in models on O-structures in the following way:  $\|\neg\alpha\|$  represents all non-contradictory ways in which the agent may refrain from performing  $\alpha$  while,  $\|\alpha\|$ , represents the set of ways in which  $\alpha$  may be performed. The structure is a relativistic one, each point having associated with it a peculiar ordering of

points in the universe.  $yB^u x$  is read "y is better for u than x".  $B^u$  as required, is transitive and irreflexive. What makes some point better than another is either the comparatively greater advantages at the one or the comparatively fewer disadvantages associated with the other.

Two points in this account require clarification. First, the notion of "ways of performing an action". Carnap's idea of a state description elucidates the notion considerably. The ways of performing some act  $\psi$  correspond to all those possible states whose description contain some description which entails that some minimal description of  $\psi$  obtains. Each element in the set of possibilities represents one way of performing  $\psi$ . A person's stooping on one knee at most six inches from the Queen for instance, entails his stooping before the Queen at least as far as every infinite part of every other inch between him and the Queen. Hence, corresponding to any way of performing  $\psi$  there will be an infinite set of possible state descriptions containing a  $\psi$  description, each of which may pick out a different way of performing  $\psi$ .

Secondly, it is safe to assume on this account that if an individual correctly determines that every possible way of avoiding an action creates less good than some performance of the action, that individual ought to perform that action. Whether we may make the further assumption that moral individuals are as moral as they can be if their

private determinations, given their native imagination, intellectual capacity and the evidential grounds available to them, are as correct as they can be under the circumstances, is not clear from the text. The text professes no interest in extending deontic characterizations of acts to agents.

### Obligation and the Core Theorems

As noted above, without finiteness conditions on O-structures, [Con] is retained as a thesis but [K] and [D] do not hold. [Con] represents the uncontroversial principle that contradictory acts are never obligatory. If they were, of course, no agent could discharge his moral duty. No person could be thoroughly good. [D],  $\Box p \rightarrow \neg \Box \neg p$ , on the other hand, means that it is false that some act and its contradictory are both obligatory ( $\neg(\Box p \wedge \Box \neg p)$ ) and this simply does not square with the deontic facts of life. Moral dilemmas are constructed of just such stuff. In defence of [D] it might be argued that it is a sine qua non of a perfect moral code (or at least a perfectly comfortable moral code) that if one out to do p, then it follows that one out not perform not-p. Indeed, just this view represents Sir David Ross' conception of morality<sup>4</sup>. It is the task of the philosopher, on Ross' view, to sort out prima facie contradictory obligations. All such obligations are only prima facie on Ross' view. It is a measure of the change in attitude towards the nature of moral codes and morality

that [D] is no longer regarded as an unassailable moral principle. When obligations are inextricably bound up with the religious sanctions of omniscient, omni-benevolent gods, the motivation to preserve [D] as a principle is considerably stronger. After all, a god who wanted his subjects in heaven shouldn't even allow the possibility of irresoluble moral dilemmas. Construed as guides to social harmony that have evolved over centuries of human experience, moral codes that do not preclude dilemmas are quite understandable. Constructed by comparatively weak-minded, muddling individuals with limited experience, it is not surprising that not all possibilities are clearly envisioned in formulating a particular code. Moral conflicts are a fact of life. It is a measure of the strength of [K] as a principle of deontic logic that it collapses the distinction between these enormously different formulae. [Con] with an aggregation principle as strong as [K] gives [D] as a principle and [D] as we have seen, is too strong according to current views of the nature of morality and moral codes.

The only other formulae that might seem strange on first deontic reading is  $\Box p \rightarrow \Box(p \rightarrow q)$ . However, if " $\rightarrow$ " is replaced with " $\vee$ " and " $\neg$ ", the formulae correctly suggests that if  $\neg p$  is obligatory then either  $\neg p$  or  $q$  is obligatory. The core theorems of O-structures, then, are compatible with a deontic interpretation of  $\Box$ , even without finiteness conditions.

### 3) □ as Justified Belief

The inspiration for the characterization of belief to follow derives from Quine's well-known fabric metaphor<sup>5</sup>. Quine sees the scientist and the layman at par in their struggle to come to terms with sense information, differing only with respect to the relative conceptual sophistication and self-conscious attitude characteristic of the scientist but not the layman. The "scientific" principles of predictive power, conformity to observation, simplicity, consistency and familiarity of principle guiding the scientist in theory construction are the principles lurking behind the layman's dealing with his own "surface irritation". Quine sees the whole of human epistemology as a subtly inter-connected collection of theories and beliefs, changing from moment to moment as new bits of "irritation" create pressures for refinement and alteration within the fabric. New theories and revisions are woven into the fabric as they demonstrate conformity to the "scientific" principles noted above. Dominating the fabric is a criterion of consistency which, although indulgent of temporary violations, insists upon the ultimate amelioration of intertheory and theory/data conflict.

### A Species of Belief

To say that an individual believes something in the sense suggested by Quine's metaphor is to say that a certain

sort of connection between the object of belief and that individual's epistemology has been made. An individual epistemology, in the sense employed here, is a body of beliefs reflecting some internal organization. The organizing principles are expressed in terms that can be understood only against the background of other beliefs comprising the individual's epistemology. Personal beliefs and theories are constructed in the shadow of private epistemologies and reflect the experience, tacit "theory", and logical and mathematical vicissitudes of individual believers. New candidates for belief are evaluated and admitted if they engage and reinforce the existing pattern of beliefs satisfactorily. What one accepts, another rejects and both may be justified in doing so given the unique complexion of their epistemologies. Propositions worthy of some individual's belief, on this view, are doxastically comfortable to him. They fit in with and reinforce in their own way, that individual's epistemology.

#### Believable Propositions

The connection between O-structures and the sense of belief suggested by Quine's metaphor can be brought out in the following way. The doxastic agent is depicted as choosing between various candidates with respect to the degree of doxastic comfort conferred, given the information available to him at the time. To determine comparative



degree of comfort, the agent considers the class of worlds in which he imagines  $\alpha$  to be false with the class of worlds in which he imagines  $\alpha$  to be true. As there are an infinite number of propositions true at each world, it is assumed that the individual attends only to some relevant subset of formulae when making his comparative judgement-between worlds. If, for every case in which  $\alpha$  is imagined to fail, there exists a case in which  $\alpha$  is hypothesized to hold and it is the case that the case where  $\alpha$  holds is more comfortable on the grounds available to the agent at the time, the agent is justified in believing  $\alpha$ , otherwise not. To be more doxastically comfortable is to be more acceptable or more likely (to be true) to the agent on the epistemic grounds available to him at the time.

#### Ways in Which Propositions may be False

If some proposition  $\alpha$  is false,  $\neg\alpha$  is true. The contradictory of any proposition  $\alpha$ <sup>6</sup> is given by the disjunction of the set of contraries of  $\alpha$ . If  $\neg\alpha$  is true at a point, then some  $\beta$  contrary to  $\alpha$  is true at that point. Hence, an agent's sorting through cases where  $\neg\alpha$  is imagined to hold is equivalent to his sorting through cases where some  $\alpha$ -contrary holds. Exhausting every contrary in the range exhausts the range. Determining that  $\alpha$ , on balance, is more likely true or more doxastically comfortable than  $\neg\alpha$ , is the determination that no worlds containing an

$\alpha$ -contrary are more likely or doxastically comfortable than some world containing  $\alpha$  on the epistemic grounds available at the time.

### Justified Beliefs and Scientific Hypotheses

The status of justified belief on the proposed analysis is like the status of hypotheses in scientific discourse. Philosophers of science distinguish, if somewhat vaguely, between hypotheses of a highly "theoretical" nature and hypotheses of the nature of "experimental laws"<sup>7</sup>. Experimental laws state relationships between observable characteristics of some subject matter. Hypotheses of this nature are constructed as non-demonstrative (i.e. inductive or non-deductive) inferences from sets of observation statements. Observation statements tend either to support or discredit certain hypotheses accordingly as they increase or decrease the probability of the hypotheses in question. A hypothesis is more probable as its truth is made more likely. The measure of probability in the case of experimental laws is usually statistical in nature, i.e. the ratio of the number of confirming instances to the number of disconfirming instances.

Hypotheses of a highly "theoretical" nature, on the other hand, do not state relationships between observable traits of some subject matter and do not draw their worldly support in any narrow statistical sense. These sorts of

hypotheses (like statements about the molecular constitution of matter<sup>8</sup>) contain fundamental concepts that are utilized and retained so long as they prove useful in helping science come to terms with sense experience. These fundamental notions provide character to sense experience and for this reason are liable to modification at the hand of recalcitrant sense experience, i.e. experience that is stubbornly inconsistent with the theory. Theoretical statements thus interact in a very intimate way with the "facts", deriving their probability not in the narrow statistical sense of "experimental" statements, but in a wider sense which counts them as increasingly likely as they prove increasingly (or at least steadfastly) useful to science. Hence, although theoretical and experimental hypotheses may both be thought of as probable in a wide sense of being supported by, (or made more likely by) the pattern of worldly events, each passes the test of probability in a different way.

And so it is with justified beliefs. They may be of the nature of "experimental laws", susceptible to statistics in the same way as experimental laws. My belief that my dog is suffering some internal disorder is a case in point. Or they may be deeper in nature, functioning as organizing principles in an epistemology. Some religious beliefs, I presume, are cases of this sort.

The rigour with which the agent pursues his hypothesis is a function of the character of the agent, his inclinations

at the time and the circumstances. As more and more experience is brought to bear on a hypothesis over a period of time, confidence grows and the putative belief becomes more deeply entrenched in the agent's personal fabric. To repeat what Quine's metaphor suggests, justified beliefs are intimately connected with the network of beliefs of their proprietors.

### An O-Structure for Belief

An O-structure for belief remains an ordered pair  $\langle U, C \rangle$  where  $U$  is a non-empty set and  $C$  a function from  $U$  into  $2^{(U^2)}$  which determines for each point  $u \in U$  a transitive and irreflexive relation  $C^u$ . The expression  $x C^u y$  is read "x is more comfortable than y on the evidence available at v" where comfort is to be taken in a wide sense to mean "more likely in a doxastic context".

This logic of belief will be indexed to individuals since the truth conditions for the operator  $\square$  are formulated in terms of subjective assignments. However, the logic will hold for all individuals so reference to particular individuals will be suppressed throughout.

### Core Theorems and Justified Belief

In determining what theorems and inference rules are suitable it is important to remember that we are dealing with a special sense of belief. Some formulae acceptable in this limited sense will not be true of belief in its

widest sense. In developing any logic for belief a tension arises in trying to preserve certain opaque features of belief and to have at least some formulae holding as theorems. Interpreted doxastically, the weakest modal logic E (containing only the modal rule of inference [RE]), commits an individual to believing every proposition proved equivalent to each proposition he does believe. An operator is opaque if it creates contexts which do not always permit the intersubstitution of co-designative terms, salva veritate. Hence, if  $\Box$  is an opaque operator, there exists some proposition  $\alpha$  identical to some other proposition  $\beta$  except that  $\alpha$  contains a term  $\alpha^*$  where  $\beta$  contains a term  $\beta^*$  and  $\alpha^*$  and  $\beta^*$  are co-designative and  $\Box\alpha \not\equiv \Box\beta$  or  $\Box\beta \not\equiv \Box\alpha$ . Hence, [RE] must fail in this case because it fails for  $\alpha$  and  $\beta$ . Hence, where  $\Box$  is opaque, the weakest rule of inference and the weakest classical logic is ruled out. There does not seem to be much hope, therefore, for a modal logic for an opaque concept of belief.

However, there are two approaches that we can try to force some middle ground and provide  $\Box$  with opacity and allow at least some inference rules. The first yields an inference rule weaker than [RE] and requires additional assumptions about the beliefs that people hold. The second requires reading  $\Box$  in a way that frees the term from "existential import".

First, the question of additional beliefs. In an interpretation of belief that pays the fullest due to

opacity, one is aware of his beliefs in the sense that protesting honestly that he does not believe  $\beta$  when he admits to believing  $\alpha$  (where  $\alpha \rightarrow \beta$ ) means that he does not in fact believe  $\beta$ . Where  $\square$  is transparent, of course, he believes  $\beta$  despite his heartfelt protestations to the contrary. Now, we can create a weaker RE-like rule that provides a middle ground by including an additional assumption: namely, that our agent also believes (in the wide sense of belief, i.e. not necessarily justified) that  $\alpha \leftrightarrow \beta$ . Using  $\boxtimes$  to indicate the wide sense of belief, the new [RE]-like principle is stated:  $\vdash (\alpha \leftrightarrow \beta) \Rightarrow \vdash (\boxtimes(\alpha \leftrightarrow \beta) \rightarrow (\square\alpha \leftrightarrow \square\beta))$ .

This suggests more mildly that if  $\alpha$  is proved equivalent to  $\beta$  and  $\underline{a}$  believes that  $\alpha$  is equivalent to  $\beta$ , then  $\underline{a}$  justifiably believes  $\alpha$  iff he justifiably believes  $\beta$ . Imposing the second conjunct in the antecedent provides an important link between  $\alpha$  and  $\beta$  and  $\underline{a}$ 's doxology. It is this sort of link that is alleged to be missing in arguments supporting the opaque nature of the concept of belief. Arthur may believe that Venus is the morning star but disbelieve that the evening star is the morning star precisely because he is not acquainted with the vital information that "Venus" and "the evening star" are co-designative. But provide that information and give a reasonable degree of rationality to Arthur and his rejecting the identity assertion becomes another matter.

[RN] and [RR], of course, are also too strong for similar reasons. [RN] commits every individual to believing every PC theorem and [RR] commits individuals to believing all logical consequences of propositions they do believe. Certainly a belief that a theorem of PC is true is justifiable for any PC theorem but this does not mean that everyone believes all PC theorems, even in the narrow sense of belief at issue. What does follow is that if any individual believes a PC theorem in the wide sense, then his belief will be justifiable. Using  $\boxtimes$  once again to indicate the wide sense of belief, the [RN]-like principle is  $\vdash \alpha \Rightarrow \vdash (\boxtimes^* \alpha \rightarrow \square \alpha)$ .<sup>9</sup> [RR] is too strong for exactly the same reasons that [RE] is too strong. A weaker [RR]-like principle that is acceptable is  $\vdash (\alpha \rightarrow \beta) \Rightarrow \vdash (\boxtimes (\alpha \rightarrow \beta) \rightarrow (\square \alpha \rightarrow \square \beta))$ . All three inference rules, then, are too strong for justified belief because they commit individuals to beliefs they might not have but all three can be weakened in one way by making additional assumptions about what else agents believe in the wide sense of belief.

The second approach yields [RE], [RR] and [RN] and involves reading formulae of the form  $\square \alpha$  in a conditional rather than the customary existential mode. Normally,  $\square \alpha$ , where  $\square \alpha$  represents the sense of "justified belief" elaborated here (and reference to individuals is suppressed), reads "a believes (justifiably) that  $\alpha$ ". This reading clearly entails the existence of some  $\alpha$  believed by a.

However, if we interpret  $\square$  conditionally, commitment to existing  $\alpha$ 's believed by a evaporate, i.e. read as "if a believed  $\alpha$ , a would be justified in believing  $\alpha$ " or, using just the modal term "would" without the "if...then" clause, "a's belief in  $\alpha$  would be justified". [RE], [RR] and [RN] then become:

1) [RE] if  $\alpha$  is proved equivalent to  $\beta$ . Then if a's belief in  $\alpha$  would be justified, a's belief in  $\beta$  would be justified and vice versa.

2) [RN] if  $\alpha$  is a thesis, then a's belief in  $\alpha$  would be justified.

3) [RR] if  $\alpha \rightarrow \beta$  then if a's belief in  $\alpha$  would be justified, a's belief in  $\beta$  would be justified.

Changing our way of talking with  $\square$ , though grammatically awkward, frees  $\square$  from committing agents to beliefs they might not have. Using the modal "would",  $\square\alpha$  allows us to say about certain propositions, precisely because of the character of their truth conditions, that they are bound to be believed justifiably if they are believed at all. Hence, with these modifications, [RE], [RN] and [RR] though obviously false of belief in the widest sense, are at least defensible with justified belief.



## [Con] and the Failure of [K] and [D]

In some respects [Con] is desirable as a principle of justified belief since, no matter what contradiction is at issue, belief in that contradiction cannot be justified. Any consistent formula holds a better chance of being true than any formula of the form  $\alpha \wedge \neg \alpha$ . The presence of [Con] does not mean that people, in the wide sense of belief cannot believe contradictory propositions. It suggests the weaker condition that if they do, their belief cannot be justified in the sense specified. However, even this weaker condition idealizes the nature of believers. It gives people too much credit with respect to their capacity to pick out logical fissures in propositions they believe.

The failure of [K] and [D] surprisingly, is compatible with justified belief. If a logic of belief is to recognize the distinction between believing a contradictory proposition  $\Box(\alpha \wedge \neg \alpha)$  and having inconsistent beliefs,  $\Box\alpha \wedge \Box\neg\alpha$ , [K] or its converse must fail. There is no question that people can have justifiable beliefs that are inconsistent. Most people have experienced the misfortune of having exactly this fact pointed out to them at some time in the heat of argument. However, from having inconsistent beliefs it does not follow that one believes a contradictory proposition. Showing that an individual who believes  $\alpha$  and believes  $\neg\alpha$  is in fact believing two propositions which are inconsistent, is usually sufficient to show that his position must be

abandoned. Quite clearly, if his believing  $\alpha$  and believing  $\neg\alpha$  meant that he believed  $\alpha \wedge \neg\alpha$ , showing that he does believe would not cause him to abandon his position.

After all, his opponent would be merely pointing out something that he already believes. But in no case is it true that he believes the contradiction just because his beliefs are not consistent. Put another way, if belief in two propositions is individually justifiable, it does not follow that belief in the propositions conjoined is justifiable. [Con] insists that inconsistent conjunctions cannot be believed with justification.

The failure of [D] gives individuals the liberty of holding inconsistent beliefs. The question to be addressed is, how can belief in  $\alpha$  and belief in  $\neg\alpha$  at the same time ever be justified? The answer lies in the subjective nature of probability assignments. People can, and do, make mistakes and fail to see logical connections between beliefs. For any two propositions  $\alpha$  and  $\beta$  where  $\beta \rightarrow \neg\alpha$ , that  $\beta \rightarrow \neg\alpha$  may not be obvious on first or even second or third reading. The grounds for defending the dismissal of [K] centered on admitting formulae of the form  $\Box\alpha \wedge \Box\neg\alpha$ . The loss of [D] goes hand in hand with the loss of [K] and the presence of [Con]. In any consistent logic, if [Con] obtains then [K] obtains iff [D] obtains. [D] eliminates formulae of the form  $\Box\alpha \wedge \Box\neg\alpha$  which, with [K], yield  $\Box(\alpha \wedge \neg\alpha)$  which is inconsistent with [Con].

The other core formulae bear up well under interpretation of  $\Box$  as justified belief if we replace conditionals in the scope of modal operators with " $\neg$ ", and " $\vee$ ". The only oddity is the absence of transitivity. In the wide sense of belief we would not want transitivity but in the narrow sense the presence of transitivity does seem to be in keeping with the rather ideal nature of believers dictated by the extensional character of the semantics. The semantic structure idealizes utility and probability judgements of individuals as the syntax idealizes the agent's system of beliefs and moral judgements. Transitivity nevertheless does fail but its loss does not seem serious for a logic of justified belief.

#### Other Formulae

The case for the failure of  $\Box$  to distribute through disjunction interprets well for belief. Interpreting  $\Box$  alethically, a disjunction's being necessarily true does not entail either one disjunct or the other's being necessarily true. Disjoining any contingent proposition and its negation always produces a necessarily true complex proposition. Interpreted deontically two acts, neither of which is independently obligatory can "disjoin" to become obligatory. For example, it is not obligatory for any individual not to borrow a book from a library nor is it obligatory for any individual to return a book to a library or pay a fine. However, it is obligatory that if one borrows a book, one

returns the book or pays the fine<sup>10</sup>. Using def. " $\rightarrow$ " on the last statement, it is obligatory that one does not borrow a book or one returns a book or pays a fine. Hence, deontic operators align with alethic operators with respect to distribution through disjunction.

The case of belief aligns with these operators since for some girl Cindy unknown to our agent, our agent is bound (by [RN]) to believe the complex proposition "Cindy has red hair or Cindy does not have red hair" without believing either that Cindy has red hair or that Cindy does not have red hair.  $\Box(pvq) \rightarrow \Box pv \Box q$  is not valid in O-structures:

Proof:

- 1) If  $\frac{M}{U} \Box(pvq)$ , then  $\forall x \neq \|p\| \cup \|q\|$ ,  $\exists y \in \|p\| \cup \|q\|$  and  $yC^u x$
- 2) Take some point  $z \in \|\neg p\| \cup \|q\|$  and  $z' \in \|\neg q\| \cup \|p\|$
- 3) It is quite possible that  $\forall y \in \|p\|$ ,  $zC^u y$  and  $\forall w \in \|q\|$ ,  $z'C^u w$  providing that finiteness conditions are not imposed,
- 4) Therefore  $\neg \Box p \wedge \neg \Box q$  is not inconsistent with the assumption.

The converse,  $\Box pv \Box q \rightarrow \Box(pvq)$  does hold by [RR].

With the converse of [K] the alethic, deontic and doxastic operators are alike in that a necessary condition of a conjunction's being necessary, obligatory or believable is that at least each conjunct be necessary, obligatory or believable. If only one is, the other might be impossible, forbidden or unbelievable and hence preclude the conjunctions being necessary, obligatory or believable.

In keeping with the non-factive character of belief<sup>11</sup>, the characteristic formula of the systems T and S4 are undesirable. The [T] formula:  $\Box p \rightarrow p$  renders every believable proposition true. [4],  $\Box p \rightarrow \Box \Box p$ , makes all beliefs transparent to one's intellect and rules out the possibility of self-deception. In the simplest case, self-deception may be formalized as  $\Box p \wedge \Box \neg p \wedge \neg \Box \Box p$ . The third rider is required to give the self-deceiver's protestation that he does not believe p a point since  $\Box \Box p \rightarrow \neg \Box \Box p$ . If  $\neg \Box \Box p$  does not obtain, he's not even deceiving himself. By S4 then  $\Box \Box p \wedge \Box \Box \neg p \wedge \neg \Box \Box p$  and by simplification  $\Box \Box p \wedge \neg \Box \Box p$ . But by double negation (which holds in O-structures) and def. " $\rightarrow$ ",  $\neg(p \wedge \neg p)$ . Substituting  $\Box \Box p/p$ ,  $\neg(\Box \Box p \wedge \neg \Box \Box p)$ . Hence, self-deception as described would be impossible with [4].

The weaker principle [T\*]  $\Box(\Box p \rightarrow p)$  is provocative. People do believe that the propositions they believe are true but it is going too far to suggest that each individual believes that all of his beliefs are true. Every individual has some dark doxological corner containing propositions

of minimal credibility; propositions which he believes to a minimal degree while admitting that he could well be mistaken. Hence [T\*] is too strong.

Finally, the paradoxes (like [Con]) give too much credit to individuals for logical acumen. If the individual understands the notion of " $\rightarrow$ " and he believes that what he believes is true, he believes that what he believes follows from any proposition whatsoever. However, most people are not familiar with this logical characteristic of " $\rightarrow$ ".

### Summary

O-structures without finiteness conditions provide an idealistic interpretation of belief if  $\Box\alpha$  is read "a's belief that  $\alpha$  would be justified". [RE], [RR] and [RN], though objectionable if  $\Box$  is read with existential import are tolerable under the proposed reading. The failure of [K] and [D] was seen as a boon to a logic that distinguished between one's having inconsistent beliefs and believing contradictions. The presence of [Con], though in some respects desirable, is even too strong for justified belief.

It is interesting to note that O-structures, with modifications, can also be used to provide an epistemic interpretation for  $\Box$ . Imposing finiteness conditions, as shown, restores [D], which is a requirement of epistemic logic. In addition, as shall be seen shortly, [T] can be validated by replacing the notion of a subjective judgement

with an objective relation.

#### 4) Counterfactuals

What sets the subjunctive conditional apart from the conditional, according to Quine<sup>12</sup> is not just the falsity of its antecedent but the fact that the conditional can be "seriously entertained and affirmed or denied in full cognizance of the falsity of the antecedent. Subjunctive conditionals depend on dramatic projection: we feign belief in the antecedent to see how convincing we then find the consequent". The problem with counterfactuals is that construed as standard truth functions (conditionals in the indicative mood) all counterfactuals are true for the very good reason that, by hypothesis, their antecedents are always false. This is problematic because we definitely want some, in fact a great many, counterfactuals to be false. The proposition "if Socrates were alive today he would be Greek" is much more convincing than "if Socrates were alive today he would have wings on his back". The first commands common assent, except perhaps for the keen historian who has discovered that Socrates actually never was a Greek at all but a clandestine Phoenician, and the second common dissent, except for our historian, an orthopedician on the side, who also discovered that the Phoenician Socrates had prepubescent wing buds on the tips of his scapulae.

The key to interpreting counterfactuals on O-structures

lies in the notion of increasing similarity and a comparative judgement between worlds at which both the antecedent and consequent hold (the relevant set of affirming counterfactual conditions in a sense of relevance to be spelled out shortly) and the set of points where the antecedent holds and the consequent fails. The structures allow us to construe counterfactuals either subjectively or objectively by referring to or omitting reference to similarity judgements by individuals. To preserve the link with belief, we shall construe them first subjectively.

#### A Modified O-Structure for Counterfactuals

One proposition,  $\alpha$  counterfactually implies another  $\beta$ , at some point  $u$  for some individual  $\underline{a}$  iff for every hypothesized circumstance  $x$  in which  $\underline{a}$  conceives the conditional to fail ( $x \in \|\alpha \wedge \neg \beta\|$ ), there exists another hypothesized circumstance  $y$  in which  $\underline{a}$  conceives the conditional to hold relevantly ( $y \in \|\alpha \wedge \beta\|$ ), such that the circumstance at which the conditional holds relevantly is more similar to  $u$  in  $\underline{a}$ 's judgement on the grounds available to  $\underline{a}$  at  $u$ . Where " $\alpha \rangle \beta$ " is read " $\alpha$  counterfactually implies  $\beta$ ":  $\frac{\mathcal{M}}{u} \alpha \rangle \beta$  iff  $\forall x: \frac{\mathcal{M}}{x} \alpha \wedge \neg \beta$ ,  $\exists y: \frac{\mathcal{M}}{y} \alpha \wedge \beta$  and  $y S^u x$ .

The function  $\mathcal{S}(u)$  remains essentially the same  $\mathcal{S}(u) = \{\underline{a}^* \subseteq U: \forall x \neq a, \exists y \in a^* \text{ and } y S^u x\}$  where  $\underline{a}^*$  designates the relevant set of confirming cases.



The difference in this approach, of course, lies in the notion of a relevant case of confirming contexts for counterfactuals. Other analyses have construed counterfactuals as follows:  $\frac{M}{u} \Box(\alpha \rightarrow \beta)$  iff  $\|\alpha \rightarrow \beta\| \in \mathcal{S}(u)$  where  $\|\alpha \rightarrow \beta\| \in \mathcal{S}(u)$  iff  $\forall x \in \|\alpha \wedge \neg \beta\|, \exists y \in \|\alpha \rightarrow \beta\|, yS^u x$ .

The idea of the relevant case is to keep our counterfactual analyst from sorting through the latter two cases that plainly do not meet Quine's condition of "feigning belief in the antecedent".

Whether a proposition is counterfactually true or not on the subjective analysis is a function of each analyst's estimates as to which circumstances are most similar to the analyst's conception of his world. In each comparative case the constant condition is represented by the truth of the antecedent. Competing consequents are projected into possible worlds under this condition and the analyst employs, exactly as in the case of belief, his own epistemological network in making his comparative judgement. A proposition counterfactually true to one individual may be counterfactually false to another with a different conception of the way things are at a given time. Socrates and his wing buds are a case in point.

#### Core Theorems and Modified O-Structures

Does the relevance factor affect theoremhood in modified O-structures?

[RN],  $\vdash (\alpha \rightarrow \beta) \Rightarrow \vdash (\alpha) \beta$ , remains by vacuity. [RR],  $\vdash ((\alpha \rightarrow \beta) \rightarrow (\gamma \rightarrow \delta)) \Rightarrow \vdash ((\alpha) \beta) \rightarrow (\gamma \delta)$ , holds.  $\odot$  (u) remains closed under supersets since if  $\vdash (\alpha \rightarrow \beta) \rightarrow (\gamma \rightarrow \delta)$  and  $\|\alpha \rightarrow \beta\| \subseteq \|\gamma \rightarrow \delta\|$ , every point outside of  $\|\gamma \wedge \delta\|$  will be outside  $\|\alpha \wedge \beta\|$ . Any  $y \in \|\alpha \rightarrow \beta\|$  will be in  $\|\gamma \wedge \delta\|$  including  $\|\gamma \wedge \delta\|$ . So [RR] continues to hold. [Con] also holds since  $\frac{\exists}{u} \perp$  iff  $\exists y \in \|\perp\|$ , which is not the case. [K] and [D] fail counterfactually for the same reason they fail in unmodified O-structures.

The remaining core theorems diagram as follows (where  $::$  indicates the non-relevant cases):

$(pvq \ r) \rightarrow (p \ r)$

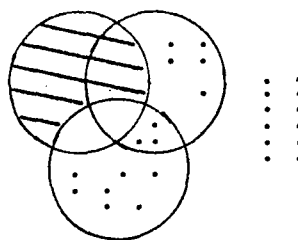
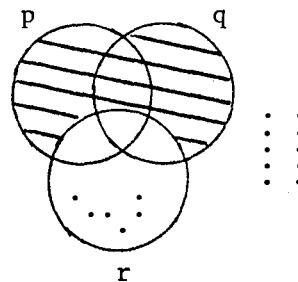
Antecedent:            {1,2,3}    {5,6,7}

Consequent:            {1,2}        {5,6}

Schema:                     $\subset$              $\subset$

Required Schema:         $\subset$              $\supset$

Converse:                     $\supset$              $\supset$

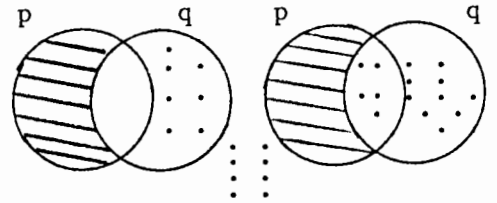


Transposition:  $(p \triangleright q) \rightarrow (\neg q \rightarrow \neg p)$

Antecedent:     {1}        {2}

Consequent:    {1}        {4}

Schema:                =        -



So transposition fails.

Transitivity:  $(p \triangleright q) \wedge (q \triangleright r) \rightarrow (p \triangleright r)$

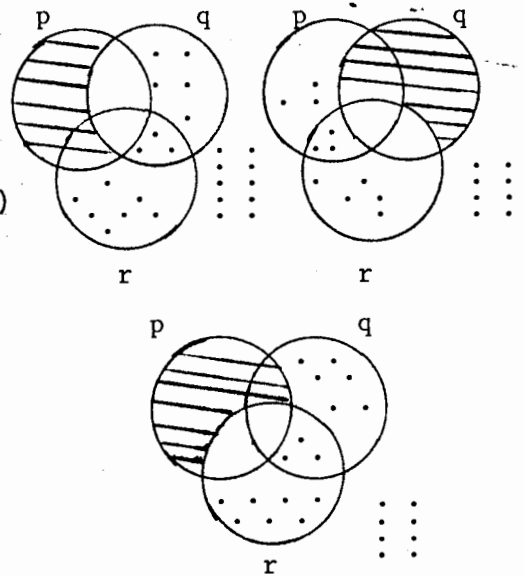
Antecedent:    {1,5}        {2,6}

                  {2,3}        {6,7}

          U = {1,2,3,5}    {2,6,7}

Consequent:    {1,2}        {5,6}

Schema:                c        -



So transitivity fails.

To sum up the, we can see that modified O-structures for  $\square$ , yield, [RN], [RR], [Con] but [D], [K], transposition, transitivity and  $((pvq) \triangleright r) \rightarrow (p \triangleright r)$  are ruled out. The loss of the last three is a boon to counterfactual logic. The loss of [D], however, is a serious flaw. We do not want it to be the case that one formula p does and does not counterfactually imply some other formula q. This is rectified by placing finiteness conditions on the modified O-structure to yield [D] and [K] once again.

## Similarity as an Objective Relation

A desirable counterfactual theorem is the theorem [T]. [T] suggests that if a counterfactual holds between two formulae at a point, the conditional in the indicative mood holds between these formulae as well. Put another way, it is a necessary condition of any counterfactuals being true at a point that the corresponding conditional must be true at that point. And this makes perfect sense. If a conditional is false at a point, no one would argue that the counterfactual in question has any plausibility given the way things are at all. Conceding the truth of the counterfactual amounts to conceding the truth of the conditional should circumstances have arisen to permit the truth of the antecedent of that conditional.

However, where the  $S^u$  relation is subjective [T] does not hold since it does not follow that the actual world is more similar to every individual analyst's conception of the way things are than any other world he may conjure up. That is, the condition on  $\mathcal{S}(u)$  adequate to validate [T],  $\forall \underline{a} \subseteq U, \forall u \in U, \underline{a} \in \mathcal{S}(u) \rightarrow u \in \underline{a}$  does not hold.

If we objectivize our truth condition for  $\square$ , however, by dropping reference to the estimates of individual analysts, the subjectivity problem disappears and [T] is validated. There is no world more similar to a given world than that world itself. Hence, each index world is an element of its own neighborhood collection. Proving [T] with this

condition is straightforward:

Proof: If  $\frac{m}{u} \Box p$ , then  $\|p\| \in \mathcal{S}(u)$  therefore  $u \in \|p\|$  by the [T] condition.

This condition also validates [D] as follows:

[D]  $\Box p \rightarrow \neg \Box \neg p$

- 1) If  $\frac{m}{u} \Box p$ , then  $\|p\| \in \mathcal{S}(u)$  and  $u \in \|p\|$
- 2) If  $\neg(\neg \Box \neg p)$ ,  $\frac{m}{u} \Box \neg p$  and  $u \in \|\neg p\|$ ,  $\in$  i.e.  $u \notin \|p\|$

It is interesting to note that failure to use the relevance condition with an underlying objective relation means that no counterfactual can be false at any point. The antecedent of every counterfactual is false by definition. If irrelevant cases are not ruled out, a counterfactual would fail where there exists some case where the antecedent holds and the consequent fails more objectively similar than every case where the antecedent fails and the consequent holds. Now, as seen above, there are no worlds more similar to the given world than the given world itself. The antecedent of every counterfactual must fail in the given world or else it is not a counterfactual. Hence, no counterfactual is false since for every case in which it fails there exists at least one case where it holds (the given world), which cannot be less similar than any point where it fails.

## Summary

Chapter Two has shown by example how an interpretation of the meaning of two concepts can lead directly to the development of a formal semantics and to the creation of systems for those concepts. In the case of belief, U.Con was seen to be a system that offered some advantages over traditional interpretations in freeing the concept from [K]. However, even U.Con, as witness the presence of [Con], still tends to create a somewhat artificial and idealized sense of belief.

The case of O-structures and counterfactuals, with the introduction of a relevant set of confirming cases, offers an advantage in relieving the notion of transitivity and transposition. It was also shown that replacing the subjective character of similarity judgements with an objective relation creates a semantics and a system for epistemic notions. It remains to Chapter Three to investigate the question of completeness for the systems yielded in Chapter Two and to suggest some ways of modifying O-structures better to suit the notions of belief and counterfactual implication

## CHAPTER THREE

COMPLETENESS AND  $\mathcal{O}$ -STRUCTURE ADJUSTMENTS

- 1) Completeness for KD, T and Alternate Truth Conditions for  $\square$

KD

a) Construct the canonical model  $\langle U_{KD}, O_{KD}, V_{KD} \rangle$ , where  $U_{KD}$  designates the set of KD maximal sets and  $O_{KD}$  is the function:  $U_{KD} \rightarrow 2^{(U_{KD})^2}$  which determines for each  $u \in U_{KD}$  a relation  $O_{KD}^u$ .  $\forall x, \forall y, x O_{KD}^u y$  iff  $\square(u) \cap y \subset \square(u) \cap x$  where  $\square(x) = \{\alpha : \square \alpha \in x\}$ . That is, some maxi-set  $x$  is the left-most member of the  $O_{KD}^u$  relation with another set  $y$  iff the set of formulae in the scope of  $\square$  at  $u$  in  $y$  forms a proper subset of the set of  $\square$  formulae at  $u$  in  $x$ . The function  $\odot_{KD}(u)$  may be defined as  $\{\underline{a} : \underline{a} \subseteq U_{KD} \text{ and } \forall y \neq \underline{a}, \exists x \in \underline{a} \text{ and } x O_{KD}^u y\}$ .

KD is complete if  $\langle U_{KD}, O_{KD}, V_{KD} \rangle$  is a model in the class of frames for KD and the fundamental theorem holds for the truth conditions for  $\square$ :  $\frac{\mathcal{M}}{U} \square \alpha$  iff  $\forall x \neq \|\alpha\|, \exists y$  and  $y O_x^u$ .  $\langle U_{KD}, O_{KD}, V_{KD} \rangle$  is suitable if  $O_{KD}^u$  is irreflexive, transitive and, in addition, has some property (\*) to correspond to the placing of finiteness conditions on the structures. As seen on page 48, if a finiteness condition is employed, [K] and [D] hold.

b) The fundamental theorem:  $\frac{\mathcal{M}}{u} \Box \alpha \Leftrightarrow \Box \alpha \in u$  where  
 $V_{KD}(\alpha) = |\alpha|_{KD}, \forall \alpha \in \underline{At}_{KD} =$

Suppose  $\Box \alpha \notin u$  and show  $\frac{\mathcal{M}}{u} \Box \alpha$ , i.e.  $\exists x \in \|\neg \alpha\|$ :  
 $\forall y \in \|\alpha\|, \exists y O_{KD}^u x$ . If  $\{\Box(u) \cup \{\neg \alpha\}\}$  is consistent,

i) if  $\Box(u) \cup \{\neg \alpha\}$  is not consistent, then  $\vdash_{KD} \beta_1 \wedge \dots \wedge \beta_N \rightarrow \alpha$

ii) by [RR] and [K],  $\vdash_{KD} \Box \beta_1 \dots \wedge \Box \beta_N \rightarrow \Box \alpha$

iii) therefore  $\Box \alpha \in u$

iv) but  $\Box \alpha \notin u$  by assumption and [D] is in  $u$  because  
it is a KD theorem, therefore  $\{\Box(u) \cup \{\neg \alpha\}\}$  is consistent  
and  $\frac{\mathcal{M}}{u} \Box \alpha$

By contraposition,  $\frac{\mathcal{M}}{u} \Box \alpha \Rightarrow \Box \alpha \in u$

=

Suppose  $\Box \alpha \in u$  and show  $\frac{\mathcal{M}}{u} \Box \alpha$ , i.e.  $\forall x \notin \|\alpha\|$ ,

$\exists y \in \|\alpha\|: y O_{KD}^u x$

i) consider any arbitrary  $x \notin \|\alpha\|$

ii) show that  $\{x \cap \Box(u)\} \cup \{\alpha\}$  is consistent; if  
consistent then  $\frac{\mathcal{M}}{u} \Box \alpha$

iii) if not consistent, then  $\exists u: \{\beta_1 \dots \beta_N\} \subseteq \{x \cap \Box(u)\}$   
 $\cup \{\alpha\}$  is inconsistent

iv)  $\vdash_{KD} \beta_1 \dots \wedge \beta_N \rightarrow \neg \alpha$

v) by [RR] and [K]  $\vdash_{KD} \Box \beta_1 \dots \wedge \Box \beta_N \rightarrow \Box \neg \alpha$

vi) therefore  $\Box \neg \alpha \in u$

vii) but  $\Box \alpha \in u$  by assumption and [D] is in  $u$  so  
 $\{x \cap \Box(u)\} \cup \{\alpha\}$  is consistent and  $\frac{\mathcal{M}}{u} \Box \alpha$



c)  $O_{KD}^u$  is irreflexive.

If  $O_{KD}^u$  is not irreflexive, then there could exist some maxi-set in  $U_{KD}$  such that  $xO_{KD}^u x$ . If  $xO_{KD}^u x$  then the set of formulae  $\Box$  at  $u$  in  $x$  must form a proper subset of the set of formulae  $\Box$  at  $u$  in  $y$  (by def.  $O_{KD}^u$ ). In extensional set theory, no set can be identical to a proper subset of itself (because identical sets have identical members). So  $O_{KD}^u$  is irreflexive.

d)  $O_{KD}^u$  is transitive.

Assume  $xO_{KD}^u y$  and  $yO_{KD}^u z$  and  $\neg xO_{KD}^u z$ . If  $\neg xO_{KD}^u z$ , then  $(\Box(u) \cap z \not\subseteq \Box(u) \cap x)$ . But if  $yO_{KD}^u z$ , then  $\Box(u) \cap z \subseteq \Box(u) \cap y$ . If  $xO_{KD}^u y$ , then  $\Box(u) \cap y \subseteq \Box(u) \cap x$ . Therefore, because  $\subseteq$  is transitive  $\Box(u) \cap z \subseteq \Box(u) \cap x$ , contrary to assumption and  $O_{KD}^u$  is transitive.

Because there are several ways to place finiteness conditions on O-structures, the proof that  $O_{KD}^u$  has the property (\*) is omitted. For a version that places this condition on the relation  $O^u$ , see "A Utilitarian Semantics for Deontic Logic"<sup>1</sup>.

T

Construct the canonical model for the system T. T is complete iff the model  $\langle U_T, O_T, V_T \rangle$  is a model on a frame in the class of frames for T.  $\langle U_T, O_T, V_T \rangle$  is such a model if  $O_T^u$  is irreflexive, transitive and in addition, possesses a property that gives the structures the

characteristic of "indexical inclusion";  $\forall \underline{a}, \underline{a} \in Nu \Rightarrow u \in \underline{a}$ .  
 Where  $O^u$  has the property of "vanity", i.e.  $\forall x, \forall u \in U, \neg xO^u u$ ,  
 the structures will have indexical inclusion. The fundamental  
 theorem holds for T because the truth conditions for  $\Box$  remain  
 unaltered.  $\langle U_T, O_T, V_T \rangle$  is suitable:

a)  $O_T^u$  is irreflexive and transitive by the steps  
 c) and d) for KD above.

b) Assume that  $\exists x: xO_T^u u$ . Then  $\exists \beta \in x: \Box \beta \in u$  and  
 $\beta \neq u$ . If  $\beta \neq u$  then  $\neg \beta \in u$  because  $u$  is maximal. But  
 because [T] holds,  $\{\alpha: \Box \alpha \in u\} \subseteq u$ . Therefore,  $\beta \in u$  and  
 $\neg \beta \in u$ , contrary to  $u$ 's being consistent. Therefore,  $O_T^u$   
 is vain and the canonical function has "indexical inclusion".

The logic T, then, is complete with respect to the  
 class of frames on strict ordering relations where  $O^u$  is vain.

It is interesting to note that altering truth conditions  
 for  $\Box$  by replacing the existential quantifier with a  
 universal quantifier also yields a semantics that preserves  
 [K], [D], and [T] (where  $O^u$  is vain) without imposing a  
 finiteness condition. Truth conditions for  $\Box$  become:

$$\frac{\mathcal{M}}{u} \Box \alpha \quad \text{iff } \forall x \neq \|\alpha\|, \forall y \in \|\alpha\|, yO^u x.$$

Strengthening the truth conditions for  $\Box$  in this way  
 has the effect of totally ordering the collections of  
 neighborhoods for each point by set inclusion.

Proof: using arbitrary sets  $\underline{a}$ ,  $\underline{b}$ , where  $\underline{a} \neq \underline{b}$ ,  
 assume  $\underline{a}, \underline{b} \in \odot(u)$  and  $\underline{a} \not\subseteq \underline{b}$  and  $\underline{b} \not\subseteq \underline{a}$

$$a) \quad \underline{a} \in \odot u \Rightarrow \exists y \in \underline{a} - \underline{b}: \forall w \in \underline{b}, wO^u y$$

$$b) \quad \underline{b} \in \odot u \Rightarrow \exists z \in \underline{b} - \underline{a}: \forall w \in \underline{a}, wO^u z$$

$$c) \quad \text{but } y \in \underline{a} \text{ and } z \in \underline{b}$$

$$d) \quad \text{therefore } zO^u y \quad yO^u z \text{ which is impossible because}$$

$O^u$  is asymmetric. Therefore, neighborhoods under the  $\forall x$ ,  
 $\forall y$  condition are totally ordered by set inclusion.

Substituting these conditions affects the core theorems  
 and inference rules.  $\odot(u)$  is no longer closed under supersets.  
 This means that [RR] fails to hold although [RE] and [RN]  
 (vacuously) remain.

Theorems under the new conditions vary considerably.  
 [K],  $\square T$  and [T] (where  $O^u$  is vain) remain. [Con] and [D]  
 are omitted.  $\square \perp$ , the [RR]-like principle  $\vdash \square \alpha \quad \vdash \square \beta \Rightarrow \vdash \alpha \rightarrow \beta$   
 or  $\vdash \beta \rightarrow \alpha$  and the principle  $\vdash \alpha$  or  $\vdash \neg \alpha$  or  $\vdash \neg(\square \alpha \wedge \square \neg \alpha)$  are  
 added.

$\square T$ ,  $\square \perp$ , Con]:  $\square T$  holds vacuously.  $\square \perp$  also holds  
 vacuously:

$$a) \quad \frac{\mathcal{M}}{u} \square \perp \text{ iff } \|\perp\| \in \odot(u), \quad b) \quad \|\perp\| \notin \odot(u) \text{ iff}$$

$$\exists y \in \|\perp\| \quad \text{and} \quad \exists x \notin \|\perp\| \quad \text{and} \quad \neg yO^u x, \quad c) \quad \neg \exists y \in \|\perp\| \quad \text{because}$$

$$\|\perp\| = \emptyset, \quad d) \quad \text{therefore } \|\perp\| \in \odot(u) \text{ and } \frac{\mathcal{M}}{u} \square \perp \quad \forall u \in U.$$

[Con] must fail if  $\vdash \square \perp$ .

[K]

[K] holds because:

a)  $\forall \underline{a}, \underline{b} \in \odot(u), \underline{a} \cap \underline{b} = \underline{a}$  or  $\underline{b}$

b) either  $\underline{a} \cap \underline{b} \neq \emptyset$  and  $\underline{a} \cap \underline{b} \in \odot(u)$  or

c)  $\underline{a} \cap \underline{b} \neq \emptyset$ . But  $\emptyset \in \odot(u)$ . Therefore,  $\underline{a} \cap \underline{b} \in \odot(u)$

and [K] holds.

[D]

In the normal case a set and its complement cannot be in the subset relation. However, when  $\underline{a} = U$  and  $\underline{a}' = \emptyset$ ,  $\underline{a}, \underline{a}' \in \odot(u)$  and  $\underline{a} \supseteq \underline{a}'$ . So [D] fails. Note that  $\Box T$  and [D] (if [D] held) would yield  $\neg \Box T = \neg \Box \perp$  when  $\Box \perp$  holds vacuously.

The [RR]-like principle holds because if  $\Box \alpha$  and  $\Box \beta$  at all points in all models,  $\|\alpha\| \subset \|\beta\|$  or  $\|\beta\| \subset \|\alpha\|$  at all points in all models under the  $\forall x, \forall y$  condition. The principle  $\vdash \alpha$  or  $\vdash \neg \alpha$  or  $\neg(\Box \alpha \wedge \Box \neg \alpha)$  holds because if some arbitrary set  $\underline{a} \neq U$  or  $\emptyset$ , then  $\underline{a}$  and  $\underline{a}'$  are not both in  $\odot(u)$ .

Interpreting  $\Box$  epistemically under  $\forall x, \forall y$  provides a concept of knowledge with some degree of opacity since individuals are no longer committed to knowing that certain formulae are true just because they are entailed by other formulae known to be true by that individual.

However,  $\forall x, \forall y$  is epistemically awkward on the traditional concept of knowledge. If the truth of  $\alpha$  is a necessary condition of anyone's knowing  $\alpha$ , no one could know any  $\alpha$  of the form  $\beta \wedge \neg \beta$ . The presence of  $\Box \perp$  would be even more embarrassing if [RR] held. Because contradictions entail all propositions, everyone would know everything.

The presence of both  $\Box \perp$  and  $\Box T$  suggests that a more plausible interpretation for  $\Box$  might be some notion of non-contingency. Tautologies and contradictions are non-contingent. [K] and the [RR]-like principle are both requirements under this interpretation, as is the absence of [D] and [Con]. The absence of [RR] accords with this interpretation as well, since [RR] doesn't make sense when  $\alpha, \beta$  are contingent and  $\alpha \rightarrow \beta$ . The problem with the interpretation of  $\Box$  as non-contingency is that it is possible for some  $\underline{a}$ :  $\underline{a} \in \textcircled{u}$  ( $\underline{a} \neq U$   $\underline{a} \neq \emptyset$ ). Non-contingent formulae are either tautologies or contradictions. However, some non-logical notion of non-contingency might be appropriate.

The only truth condition quantifier combination that has not been experimented with is  $\exists x, \exists y$ . Defining  $\Box$  in this way means that [RE] holds but [RN] and [RR] both fail. [RN] fails because  $\neg \exists x \neq \|\alpha\|$  when  $\vdash \alpha$  and [RR] because superset closure fails. [Con] continues to hold because  $\perp$  is still impossible at a point. Transposition also holds but the other core theorems fail. This logic might be

considered a better logic for justified belief because [RR] and [RN] do not prevail. Because the usual inference rules do not obtain, call this logic E.Con.

## 2) Degrees of Believing and Counterfactual Implication

On the structures presented, the sense of belief and counterfactual implication presented commits some proposition  $\alpha$  to belief or counterfactual implication just so long as there is some point in  $\|\alpha\|$  or  $\|\alpha\|$  \* more comfortable or more similar than every point where  $\alpha$  does not obtain. This certainly seems farfetched in cases where the points in  $\|\alpha\|$  on balance are just slightly better off in comparison with the points in  $\|\neg\alpha\|$ . We can well imagine an agent's belief being "suspended" in certain cases. The structures would be of greater philosophical interest if they could provide a measure of strength of belief and counterfactual implication and if they could be made to provide some middle ground that is doxastically/counterfactually neutral. For the sake of perspecuity this approach will be developed for the concept of belief only. What goes for belief, however, goes for " $\rangle$ " as well.

### A Degree Function

What is required is that the structures be altered to at least reflect degrees of belief. Let  $O^*$  be a function:

$U^3 \rightarrow J$ , which assigns to each point triple in  $U$  a value in the closed unit interval (0-1). For every pair of points related to every index point in  $U$ ,  $O^*$  provides a value which indicates the degree to which one point is more comfortable (relative to some given point) than another.

$O^*$  not only yields values of comparative comfort for each point pair relative to each point in  $U$  but can also yield a value of absolute comparative comfort for a proposition at a point. As we have seen, a proposition may be false in an infinite number of ways. All of these ways, however, are not equally comfortable for an agent with a given bank of evidence. Some will be more comfortable than others. The degree to which some proposition will be believed the, is at most, the degree to which that proposition is more comfortable than its nearest competitor. Degree of absolute comparative comfort is defined:  $\frac{91}{u} B_{\alpha}^N \Leftrightarrow \forall y \neq \|\alpha\|, \exists x \in \|\alpha\| : C^{\circ}(x, y) \gg N$  where " $B_{\alpha}^N$ " reads " $\alpha$  is believed to  $N^{\circ}$ " and " $C^{\circ}(x, y)$ " reads "degree to which  $x$  is more comfortable than  $y$ ".

Each point will yield a degree of absolute comparative comfort for each formula believed at that point (i.e. for each formula in the scope of  $\mathbb{M}$  at that point). This factor is not the only factor in determining how much confidence an individual is willing to place in a belief. It may be true about some  $\alpha$  that  $\alpha$  is much more comfortable to  $\underline{a}$  on the basis of the evidence but  $\underline{a}$  may still refrain from

placing confidence in  $\alpha$  because a is not sure that all relevant evidence is in or that the evidence is suspect or that his calculations are accurate. A more complete account of a degree function would build in these factors in describing how degree of comfort determinations are made. Let us assume some such account for  $O^*$  so that value of absolute comparative probability really does reflect degree of agent confidence in a proposition.

#### Believable, Credible and Unbelievable Propositions

$O^*$  would yield doxastically neutral territory if we selected a mid-range in  $(0-1)$ , the upper value to indicate belief threshold and the lower indicating disbelief threshold. The area between would indicate the neutral zone peculiar to those propositions that we neither believe nor disbelieve at a given time. Propositions in this zone may be christened credible (because not disbelieved). Credible propositions yield absolute comparative J-values above the threshold of disbelief. If a disbelieves  $\alpha$ , he does not find  $\alpha$  credible. If  $\alpha$  is not believed by a,  $\alpha$  may be credible or  $\alpha$  may not be credible. The level of individual thresholds is relative to individuals. Some individuals make doxastic commitments on evidence where others refuse. Some people are more easily duped than others. The relationship between disbelief, credibility and belief suggested here, parallels the relationship between impossible, possible, and necessarily



true propositions. Where  $T^B$  = threshold of belief,  $T^D$  = threshold of disbelief,  $\underline{a}$  believes  $\alpha$  iff  $B_\alpha^\circ > T^B$ , disbelieves  $\alpha$  iff  $B_\alpha^\circ < T^D$ , is doxastically neutral towards  $\alpha$  iff  $T^D \leq B_\alpha^\circ \leq T^B$ , finds  $\alpha$  credible iff  $B_\alpha^\circ \gg T^D$  and finds  $\alpha$  dubitable iff  $B_\alpha^\circ \ll T^B$ .

### Theoremhood in O\* Structures

Theoremhood in O\* structures will be determined by methods of computing degrees of belief for complex propositions. F.P. Ramsey<sup>2</sup> has suggested computations where formulae are evaluated in the range (0-1) with a doxastically neutral ground at mid-point:

$$B^\circ(\neg\alpha) = 1 - B^\circ(\alpha)$$

$$B^\circ(\alpha \wedge \beta) = B^\circ(\alpha) \times B^\circ(\beta, \text{ given full belief in } \alpha)$$

Belief in  $\beta$ , given  $\alpha$ , is to be understood as the amount of confidence an agent would place in  $\beta$  on the agent's assumption that  $\alpha$  is true. According to Ramsey's scheme, the agent has full belief (degree 1) in all propositions assumed by him to be true. Ramsey makes an assumption about the rationality of believers in assuming that if  $\alpha$  is assumed to be true by some agent, the agent's degree of belief in  $\neg\alpha$  will therefore be 0. This does not have to hold for all believers (as witness the preceding objections to [Con] as too strong a principle for doxastic logic).

Ramsey's suggestions indicate that the notion of belief operating in his account differs from the notion of justified belief as elaborated in Chapter Two. The suggested calculations not only verify [Con] but also [D].

[Con]:  $\neg \Box \perp$

- a)  $\underline{a}$  believes  $\alpha$  iff  $B^\circ_\alpha > T^B$
- b)  $\forall \underline{a}, T^B > .5$  (on Ramsey's scale .5 is neither belief nor disbelief)
- c)  $\forall \alpha: \alpha = (\beta \wedge \neg \beta), B^\circ_\alpha = B^\circ_\beta \times B^\circ(\neg \beta, \text{given full belief in } \beta)$
- d)  $B^\circ(\neg \beta, \text{given } B^\circ_\beta = 1) = 0$
- e) therefore  $\forall \underline{a}, \neg \Box \perp$

[D]:  $\Box \alpha \rightarrow \neg \Box \neg \alpha$

- a) if [D] fails then  $\Box \alpha \wedge \Box \neg \alpha$  is possible
- b)  $\Box \alpha \rightarrow B^\circ_\alpha > .5$
- c)  $B^\circ(\neg \alpha) = 1 - N(\text{where } N > .5) = M(\text{where } M < .5)$
- d)  $B^\circ(\neg \alpha) < .5$  and  $B^\circ(\neg \alpha) < T^B$
- e)  $\forall \underline{a}, \forall \alpha, \neg(\Box \alpha \wedge \Box \neg \alpha)$

Interestingly, [K] fails where the individual's belief in two individual propositions is independent and minimal:

- a) assume  $B^\circ(\alpha) = .5^+ \text{ min.}, B^\circ(\beta) = .5^+ \text{ min.}$
- b) because the product of two numbers  $u, m: 0 < u, m < 1$  is always less than either number,  $B^\circ(\alpha \wedge \beta) = B^\circ(.5^+ \times .5^+ = .25^+)$

[K] in fact, fails for all independent beliefs whose sums are, for example, less than 1.4 because the product of any two numbers summing to 1.4 will always be less than .5 (i.e.,  $1 \times .4 = .4$ ,  $.7 \times .7 = .49$ ,  $.8 \times .6 = .48$ , etc.). As the threshold rises, individuals must have close to full belief in propositions before they can accept the conjunction of those propositions. [K] will hold in certain cases where thresholds are low while failing with higher cases.

To make  $O^*$  yield [D], [Con] and [K] some revisions will have to be made. However,  $B^\circ(\neg\alpha)$  is to be characterized, it cannot be the case that  $B^\circ(\neg\alpha) = 1 - B^\circ(\alpha)$ . If  $\alpha$  is barely more comfortable, on balance, than  $\neg\alpha$ ,  $B^\circ(\neg\alpha)$  will approximate 1 as closely as  $B^\circ(\alpha)$  approximates 0. Because  $O^u$  is asymmetrical, if some point  $y$ , on balance, is more comfortable to any degree than another  $x$ ,  $\neg x O^u y$  to any degree. This consideration requires restricting the value of  $\neg\alpha$  as follows:  $(B^\circ(\alpha) > 0) \Rightarrow (B^\circ(\neg\alpha) = 0)$ . This restriction gives [D] providing  $T^B > 0$ . Adopting Ramsey's suggestion for " $\wedge$ ", [K] is yielded in all cases provided, as above, that  $T^B > 0$  (because the product of two numbers greater than 0 must itself be greater than 0). [Con] holds because of the restrictions imposed on " $\neg$ ". Defining a threshold of belief greater than 0 will have no effect on [D] or [Con] but will rule out [K] in certain areas.

Although  $O^*$  can be made to yield [K], [D] and [Con], models on the structure do not yield corresponding degrees of disbelief. This can be rectified by altering the range of  $O^*$  so that  $O^*: U^3 \rightarrow (-1, 1)$ . This will provide corresponding degrees of disbelief and the definition of  $T^D$ .

### Summary

We have seen that completeness can be shown in O-structures under certain conditions with the logics KD and T and that altering the truth conditions for  $\Box$  yields other logics (not yet proved complete). It has also been shown that Ramsey's notion of belief is at least a D notion and, in certain cases, a KD notion. Altering O-structures by redefining  $O^*$  as:  $U^3 \rightarrow J$  or  $U^3 \rightarrow (-1, 1)$  can preserve [D], [K] and [Con] and in addition permit a definition of the notions of belief threshold and disbelief threshold.

## References

## Chapter One

1. Following Segerberg in An Essay in Classical Modal Logic, p.2. This paper will be concerned exclusively with propositional logics.
2. The notions of soundness and consistency are distinguished later.
3. Whitehead and Russell, 1910-1913.
4. The PM associative axiom was shown to be dependent by Bernays.
5. This result is known as the Deduction Theorem. For a proof of same, see A Primer of Modal Logic, Jennings and Schotch.
6. "The completeness of the first-order functional calculus" in the Journal of Symbolic Logic, vol. 15 (1950), pp. 81-91.
7. An Introduction to Modal Logic by Hughes and Creswell, p.7.
8. An Essay in Classical Modal Logic, p.7.
9. Ref. 8.
10. By Jennings, Schotch and Johnston in "First Order Relational Frame Theory for Quasi-aggregative Modal Logics", unpublished.
11. See Kripke, S.A. in bibliography.
12. In Knowledge and Belief and the "Modes of Modality".
13. Leibniz Selections ed. by Philip P. Wiener.
14. Tractatus Logico Philosophicus.
15. Quine, in From a Logical Point of View (p.20) notes that Leibniz at least saw this connection.
16. Segerberg defines a "classical" modal logic as a logic containing [RE] as an inference rule in An Essay in Classical Modal Logic, p.7.

## Chapter Two

1. Jennings, R.E., "A Utilitarian Semantics for Deontic Logic" in the Journal of Philosophical Logic, 1974, pp. 445-456.
2. See ref. 1, p. 452.
3. Ref. 1, p. 448.
4. Van Fraassen in "Values and the Heart's Command", J.P.L., LXX no. 1, p.8 suggests this remark.
5. Quine, W.V.O., Word and Object, Chapter 1 and "Two Dogmas of Empiricism" in From a Logical Point of View.
6. Except where p has infinitely many contraries. In this case, the set of p-contraries, strictly speaking, cannot be given.
7. Nagel, Ernest in The Structure of Science, pp. 80-82.
8. Ref. 7, p. 81.
9. Where \* is a function satisfying ( $\vdash \alpha \leftrightarrow \vdash * \alpha$ ).
10. This example is from G.H. von Wright.
11. A factive context is a context that satisfies [T].
12. Word and Object, p. 222.

## Chapter Three

1. In J.P.L., vol. 3, pp. 454-455.
2. "Degrees of Belief" in Probability, Confirmation and Simplicity ed. by M. Foster and M. Martin, pp. 115-117.

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