COUNTABLE INITIAL SEGMENTS OF THE DEGREES OF UNSOLVABILITY

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ABSTRACT

The problem of identifying the order types of the countable initial segments of the degrees of unsolvability was first tackled by Clifford Spector more than 20 years ago, and has since given rise to a series of papers. In this thesis a complete characterization of these order types is given by proving the following theorem: any countable upper semilattice with least element can be embedded as an initial segment of the degrees.

"À mes parents."

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Introduction and notations.

In this thesis we prove the following result: any countable upper semilattice with least element can be embedded as an initial segment of the degrees of unsolvability. This result settles the question of identifying the countable initial segments of the degrees. The first result in this direction was given by Spector [6] who showed the existence of a minimal degree, which is equivalent to the embedding of the two element chain in the degrees. This result was extended by Titgemeyer [8] to the embedding of all finite chains. Hugill [1] then obtained the result for countable chains and Lachlan [2] for countable distributive lattices. All of the work above was based on the fact one can obtain a nice representation of distributive lattices. Since no such representation were available for finite lattices in general, the method used had to be quite drastically changed. A first step in this direction is in Thomason [7]. Then Lerman [2] obtained a suitable 'representation in the limit' of finite lattices and was able to settle the embedding problem for those lattices. It is this work we extend, using the fact that any countable semilattice is the limit of a sequence of finite lattices.

An intuitive discussion of the necessary results from recursive function theory follows. Formal definitions can be found in Rogers [5].

Let N be the set of natural numbers. A and B will be subsets of N, and f and g will be functions from N to N.

A function f is <u>recursive</u> if there exists an effective procedure which calculates $f(n)$, for $n \in N$. By an effective procedure we mean a deterministic procedure that could be carried out, for example, by a finite state automata. If a procedure yields results on a subset of N only, then the corresponding function f is said to be <u>partial recursive</u>. A set A is <u>recursive</u> if its characteristic function C_A is recursive. A is <u>Turing reducible</u> to B (notation $A \leq_T B$) if there is a procedure that calculates C_A which is effective, except that at a certain finite number of times it might require information about membership in B: briefly, knowledge of C_B will yield knowledge of C_A. The central notion of effective procedure can be made precise in such a way that we obtain an enumeration of such procedures. A procedure is then noted $\{e\}$ if its rank in the enumeration is e. The fact that $A \leq_T B$ is then noted $A = \{e\}(C_B)$. In general $f \leq_T g$ means $< f > \leq_T < g >$, where $< f >$ is the graph of f. If both $A \leq_T B$ and $B \leq_T A$ then we say A is <u>Turing equivalent</u> to B (noted $A \simeq_T B$). Intuitively this means that to calculate membership in A or in B is of the same order of complexity. It can be shown that this relation is an equivalence relation on $\mathcal{P}(N)$. $\mathcal{P}(N)/\simeq_T$ is the set $\underset{\sim}{D}$ of <u>degrees of unsolvability</u>. Note that $\leq_T$ induces an order relation on $\underset{\sim}{D}$: if $\underset{\sim}{a} \in \underset{\sim}{D}$ and $\underset{\sim}{b} \in \underset{\sim}{D}$ then $\underset{\sim}{a} \leq_T \underset{\sim}{B}$ if there is $A \in \underset{\sim}{a}$ and $B \in \underset{\sim}{b}$ such that $A \leq_T B$. It is clear that $\underset{\sim}{D}$ has a smallest element $\underset{\sim}{0}$: it is the degree that contains all recursive sets. With the ordering given, $\underset{\sim}{D}$ is an upper semilattice (but not a lattice), i.e. given $\underset{\sim}{a}$ and $\underset{\sim}{b}$ we can find $\underset{\sim}{c}$ such that

$\underset{\sim}{c} \geq_T \underset{\sim}{a}$, $\underset{\sim}{c} \geq_T \underset{\sim}{b}$ and $(\underset{\sim}{d} \geq_T \underset{\sim}{a}$ and $\underset{\sim}{d} \geq_T \underset{\sim}{b} \rightarrow \underset{\sim}{d} \geq_T \underset{\sim}{c})$ we write $\underset{\sim}{c} = \underset{\sim}{a} \vee \underset{\sim}{b}$. An initial segment S of the degrees is a subset of the degrees such that $\underset{\sim}{0} \in S$ and if $a \in S$ and $\underset{\sim}{b} \leq_T a$ then $\underset{\sim}{b} \in S$.

The canonical index of a finite set $D = \{x_1, x_2, \ldots, x_n\}$ is the natural number $2^{x_1} + 2^{x_2} + \ldots + 2^{x_n}$. A sequence $< A_i, i \in N >$ of finite sets is said to be strongly recursively enumerable (strongly r.e.) if there is a recursive function h such that for $i \in N$, h(i) is the canonical index of A_i .

Say we have two sets of functions M and N and a pairing $P = \{(f,g),\ f \in M,\ g \in N\}$. Then this pairing is uniform if there is a procedure that will yield g given f so that $(f,g) \in P$: that is, there is $e \in \omega$ such that $g = \{e\}(f)$ for all $(f,g) \in P$.

The notion of recursive function and recursive sets can be extended is an obvious manner to functions, $f : N^n \rightarrow N$ and subsets of N^n . A recursive predicate is a recursive subset of N^n, for some $n \in N$.

§1. Weak representations and admissible extensions. This section is first concerned with an exposition of the fundamental concepts and representation theorem from Lerman [2]. Then we introduce the notion of admissible extension and some lemmas.

Let $(L,\leq)$ be a fixed lattice. For any set U let $(\mathcal{E}(U),\leq)$ denote the dual lattice of equivalence relations on U where $R_0 \leq R_1$ if and only if R_1 implies R_0. The join $R_0 \vee R_1$ in $(\mathcal{E}(U),\leq)$ is $R_0 \& R_1$, 0 is the universal relation and 1 is the identity relation. A <u>w-representation</u> of $(L,\leq)$ is a pair (F,U) such that U is a finite set, $U \neq \emptyset$, $F:(L,\leq) \to (\mathcal{E}(U),\leq)$ is a join embedding, $F(0) = 0$ and $F(1) = 1$. Let (F,U) and (G,V) be w-representations of $(L,\leq)$. We say (G,V) <u>extends</u> (F,U) written $(F,U) \subset (G,V)$ if $U \subset V$ and if for each $c \in L$, $F(c) = G(c)|U$. $(L,\leq)$ is said to be <u>sequentially representable</u> if there is a sequence $<(F_i,U_i) : i < \omega>$ of w-representations of $(L,\leq)$ such that for all $i < \omega$ (F_{i+1},U_{i+1}) extends (F_i,U_i) and

1.1 if $a,b,c \in L$ and $a \wedge b = c$, $u_0,u_1 \in U_i$ and $u_0 F_i(c)\, u_1$ then there is a sequence $\{v_0,v_1,v_2\}$ of elements of U_{i+1} such that $u_0\ F_{i+1}(a) v_0\ F_{i+1}(b) v_1\ F_{i+1}(a) v_2\ F_{i+1}(b)\ u_1$.

1.2 if $u_0,u_1,v_0,v_1 \in U_i$ and for all $b \in L$ $u_0\ F_i(b)\ u_1$ implies $v_0\ F_i(b)\ v_1$ then there are w_0, and embeddings h_0,h_1 of (F_i,U_i) in (F_{i+1},U_{i+1}) such that $h_0(u_0) = v_0$, $h_0(u_1) = h_1(u_0) = w_0$, $h_1(u_1) = v_1$. (The notion of embedding is the obvious analogue of the notion of extension).

THEOREM 1.3. $(L,\leq)$ has a sequential representation.

Proof: First let $D = \{1 = c_1, c_2, \ldots, c_n = 0\}$ be a distributive lattice. Find $\ell \in \omega$ and $C_1, \ldots, C_n \subset \ell$ such that $c_i \to C_i$ is a lattice embedding of $(L,\leq)$ into the Boolean algebra 2^ℓ such that $C_1 = \ell$, $c_n = \emptyset$, and $\ell - c_i$ has power > 1 for all i, $1 < i \leq n$. If we let $U = \ell$ and $F(c_j)$ be $\{(k,k) : k < \ell\} \cup ((\ell - C_j) \times (\ell - C_j))$ then (F,U) is a w-representation of $(L,\leq)$. But every finite lattice is a sub-semilattice of some distributive lattice with the same 0 and 1; thus we have (F_0, U_0).

Suppose (F_i, U_i) has been found, that $a \wedge b = c$ in L and $u_0 F_i(c) u_1$. To satisfy 1.1 for just one pair $\langle u_0, u_1 \rangle$ we can take $U_{i+1} = U_i \cup \{v_0, v_1, v_2\}$ where v_0, v_1, v_2 are distinct and are not elements of U_i. For $d \in L$ let $F_{i+1}(d)$ be the least equivalence relation on U_{i+1} extending $F_i(d)$, including $\{(u_0,v_0), (v_1,v_2)\}$ if $d \leq a$, and including $\{(v_2,u_1), (v_0,v_1)\}$ if $d \leq b$. (Note that by "least equivalence relation" we mean least as a set of pairs). Then (F_{i+1}, U_{i+1}) extends (F_i, U_i) and satisfies 1.1 for $\langle u_0, u_1 \rangle$. For future reference we call such an extension an <u>extension of type 1</u>. By iteration we can satisfy all pairs $\langle u_0, u_1 \rangle$ and all triples $\langle a,b,c \rangle$.

Now suppose u_0, u_1, v_0, v_1 are as in the hypothesis of 1.2. Let (F_i^0, U_i^0), (F_i^1, U_i^1), be copies of (F_i, U_i) under bijective maps $h_j : U_i \to U_i^j$ for $j < 2$. Let U_i, U_i^0, U_i^1 be pairwise disjoint except that v_0 is to be identified with $h_0(u_0)$, $h_0(u_1)$ with $h_1(u_0)$, $h_1(u_1)$ with v_1. Let $U_{i+1} = U_i \cup U_i^0 \cup U_i^1$ and for each $c \in L$ let $F_{i+1}(c)$ be the least equivalence relation extending each of $F_i(c)$, $F_i^0(c)$, $F_i^1(c)$.

Note that no pair of elements of U_i is in $F_{i+1}(c)$ that is not already in $F_i(c)$. Indeed such a new pair $\langle x,y \rangle$ would arise from a situation where $x\ F_i(c)\ v_0$ and $y\ F_i(c)\ v_1$, or $x\ F_i(c)\ v_1$ and $y\ F_i(c)\ v_0$. From the hypothesis of 1.2 and the $h_i (i < 2)$, being isomorphisms it cannot happen that $v_0\ F_{i+1}(c)\ v_1$ without $v_0\ F_i(c)\ v_1$. Then (F_{i+1}, U_{i+1}) extends (F_i, U_i) and satisfies 1.2 for u_0, u_1, v_0, v_1. For future reference we call any such extension an extension of type II. Again by iteration of this procedure we can completely satisfy 1.2.

Let (G,W) and (F,V) both be w-representations of L with $(G,W) \supset (F,V)$, we call (G,W) an admissible extension of (F,V) if

$$(x)_{x\in W}\ (Ey)_{y\in V}\ (z)_{z\in V}\ (a)_{a\in L}\ (xG(a)z \rightarrow xG(a)y) .$$

To put this definition another way: $(G,W) \supset (F,V)$ is an admissible extension of (F,V) means that with each $x \in W$ we can associate some $y \in V$ such that for all $a \in L$, if x is $G(a)$-equivalent to any member of V it is $G(a)$-equivalent to y. The purpose of this notion will become clear in the next section.

We write $L^* \subset L$ to mean L^* is a sublattice of L with the same 0 and 1. If (G,W) is a w-representation of L and if $L^* \subset L$ then $(G \| L^*, W)$ is the obvious w-representation of L^*.

We now list five lemmas; proofs of lemmas 1.4 and 1.6 are given in §5.

LEMMA 1.4. Extensions of types I and II are admissible.

LEMMA 1.5. If (G,W) is an admissible extension of (F,V) and (F,V) is an admissible extension of (H,U), then (G,W) is an admissible extension of (H,U).

Proof: Say $x \in W$, $z \in U$, $a \in L$ and $xG(a)z$. By hypothesis we know there is $y_0 \in V$ s.t. $(v)_{\in V} (b)_{\in L} (xG(b)v \to xG(b)y_0)$. Since $U \subset V$ we have $xG(a)y_0$. Then $y_0F(a)z$ and by hypothesis we can find $y_1 \in U$ such that $(v)_{v\in U}(b)_{b\in L}(y_0 F(b)v \to y_0 F(b)y_1)$, and thus $y_0 F(a)y_1$. But then $xG(a)y_1$. Since y_1 is independent of a and z, we are done.

LEMMA 1.6. Let (G,W) and (H,U) be extensions of (F,V) such that (G,W) is an admissible extension of (F,V) and $W \cap U = V$. Define $V' = W \cup U$ and for each $a \in L$ let $F'(a)$ be the least equivalence relation on V' which includes $G(a)$ and $H(a)$. Then (F',V') is a w-representation of L. (F',V') is an admissible extension of (H,U) and is an extension of (G,W). Further, if (H,U) is also an admissible extension of (F,V), then (F',V') is an admissible extension of (F,V).

LEMMA 1.7. If (G,W) and (F,V) are w-representations of L and (G,W) is an admissible extension of (F,V) and $L^* \subset L$ then $(G\|L^*,W)$ is an admissible extension of $(F\|L^*,V)$.

Proof: Immediate.

LEMMA 1.8. Let (G,W) and (F,V) be w-representations of L with $W \cap V = \emptyset$. Then they have a common admissible extension.

Proof: Let $U = W \cup V$. If $a \in L$ and $a \neq 0$ let $H(a)$ be the least equivalence relation on $W \cup V$ extending both $F(a)$ and $G(a)$. Let $H(0)$ be the universal relation on U. Since

$$(x)_{\in W} (y)_{\in V} (xH(a \vee b)\ y \leftrightarrow a = b = 0 \leftrightarrow xH(a)y \ \&\ xH(b)y),$$

(H,U) is a w-representation of L and by construction it is an extension of both (G,W) and (F,V) . Since $(x)_{\in W\setminus V}(y)_{\in V}(xH(a)y \rightarrow a = 0)$ it follows that (H,U) is an admissible extension of (F,V). Similarly for (G,W).

§2. Representation of a countable lattice by an array of w-representations of finite sublattices. Say we are given a countable upper semilattice with zero $(L_\omega,\leq)$; we may suppose it has a greatest element, otherwise we add one. Then $(L_\omega,\leq) = \lim_{i\to\infty}(L_i,\leq)$ where $\langle(L_i,\leq): i < \omega\rangle$ is an ascending sequence of finite lattices each with the same 0 and 1, such that "join" in $(L_{i+1},\leq)$ extends "join" in $(L_i,\leq)$.

In this section we show how to build an array

$$\langle\langle(F_{i,0},U_{i,0}) : \ell(0) \leq i < \omega\rangle, \ldots, \langle(F_{i,k},U_{i,k}) : \ell(k) \leq i < \omega\rangle, \ldots\rangle,$$

where $\ell(0) = 0$ and $\ell(k) \leq \ell(k+1)$ for all k , satisfying the following conditions:

(C1) For each j the sequence $\langle(F_{k,j}, U_{k,j}) : \ell(j) \leq k < \omega\rangle$ is almost a sequential representation of $(L_j,\leq)$ meaning that there is a strictly increasing sequence $\ell(j) = n(0) < n(1) < \ldots$ such that $\langle(F_{n(k),j}, U_{n(k),j}) : k < \omega\rangle$ is a sequential representation of $(L_j,\leq)$ and, for all k ,

$$n(k) \leq \ell < n(k+1) \rightarrow (F_{\ell,j}, U_{\ell,j}) = (F_{n(k),j}, U_{n(k),j}) .$$

(C2) For each pair $\langle i, j+1\rangle$ such that $i \geq \ell(j+1)$ and $i,j < \omega$, for all u,v, in $U_{i,j+1}$ and for each $c \in L_j$,

$$U_{i,j} \supset U_{i,j+1} \;\&\; uF_{i,j+1}(c)\,v \leftrightarrow uF_{i,j}(c)\,v .$$

(C3) Each column is recursive, i.e. for each j there is a bijection g from $\cup\{U_{i,j} : \ell(j) \leq i < \omega\}$ onto ω such that $\langle g(U_{i+\ell(j),j}) : i < \omega \rangle$ is a strongly r.e. sequence of finite sets, and $g^{-1}(x)\ F_{y+\ell(j),j}(c)g^{-1}(z)$ is a recursive predicate of x,y and z for each $c \in L_j$.

Assume the columns of the array before the k^{th} have already been constructed so that C1, C2 and C3 are satisfied and so that for $n = k - 1$ the n^{th} column has the following embedding property with respect to L_n . If $i \geq \ell(n)$, $(F,V) \subset (F_{i,n}, U_{i,n})$, and (G,W) is an admissible extension of (F,V) then there exists $j > i$ and an embedding of (G,W) in $(F_{j,n}, U_{j,n})$ which is the identity on (F,V). The commutative Diagram 1 illustrates the embedding property. In the diagram $\subseteq$ signifies an injection and $\cong$ signifies an embedding.

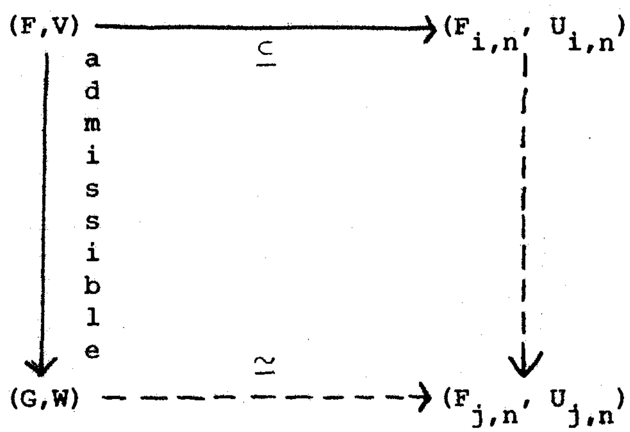

Diagram 1

Now we shall show how to effectively construct the k^{th} column such that C1, C2 and C3 remain satisfied, and the k^{th} column has the embedding property with respect to L_k.

We shall only treat the case $k > 0$; the modifications required when $k = 0$ will be readily apparent.

We first show how to obtain $(F_{\ell(k),k}, U_{\ell(k),k})$. The commutative Diagram 2 reflects the argument. Let (G,W) be a w-representation

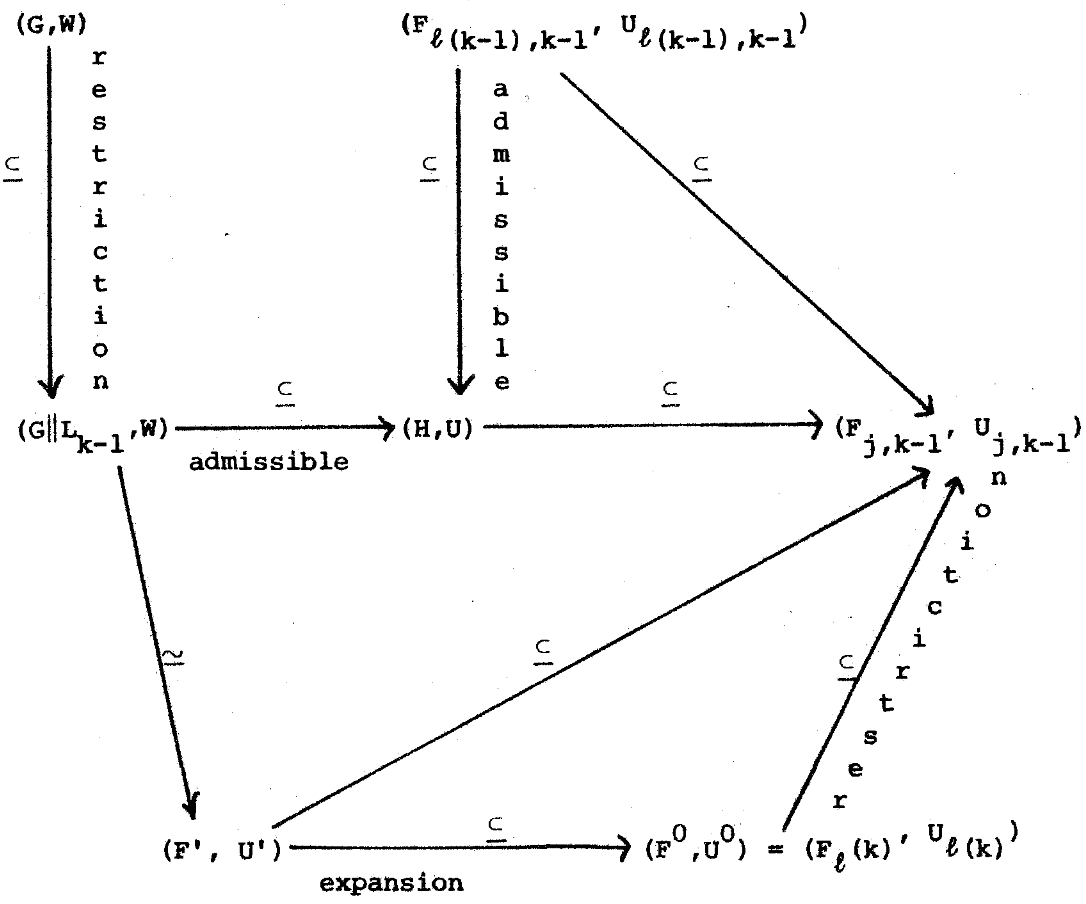

Diagram 2

of L_k such that $W \cap U_{\ell(k-1),k-1} = \emptyset$. By Lemma 1.8, we can find (H,U) an admissible extension of both $(F_{\ell(k-1),k-1}, U_{\ell(k-1),k-1})$ and $(G\|L_{k-1}, W)$. By the induction hypothesis there is an embedding of (H,U) in $(F_{j,k-1}, U_{j,k-1})$, for some $j > \ell(k-1)$, which is the identity on $U_{\ell(k-1),k-1}$. Let (F', U') be the copy of $(G\|L_{k-1}, W)$ within $(F_{j,k-1}, U_{j,k-1})$. Expand it to (F^0, U^0) a w-representation of L_k . Then take (F^0, U^0) as $(F_{\ell(k),k}, U_{\ell(k),k})$.

We now proceed with the general step in the construction of the k^{th} column; by convention we write F_n, U_n for $F_{n,k}$, $U_{n,k}$ respectively. So suppose we have $(F,V) \subset (F_n, U_n)$ and (G,W) an admissible extension of (F,V), where without loss of generality we can assume $W \cap U_n = V$. To follow the argument the reader should look at the commutative Diagram 3. Let (H,U) be obtained from (F_n, U_n) following the procedures of theorem 1.3, such that (H,U) is a suitable successor for (F_n, U_n) in a sequential

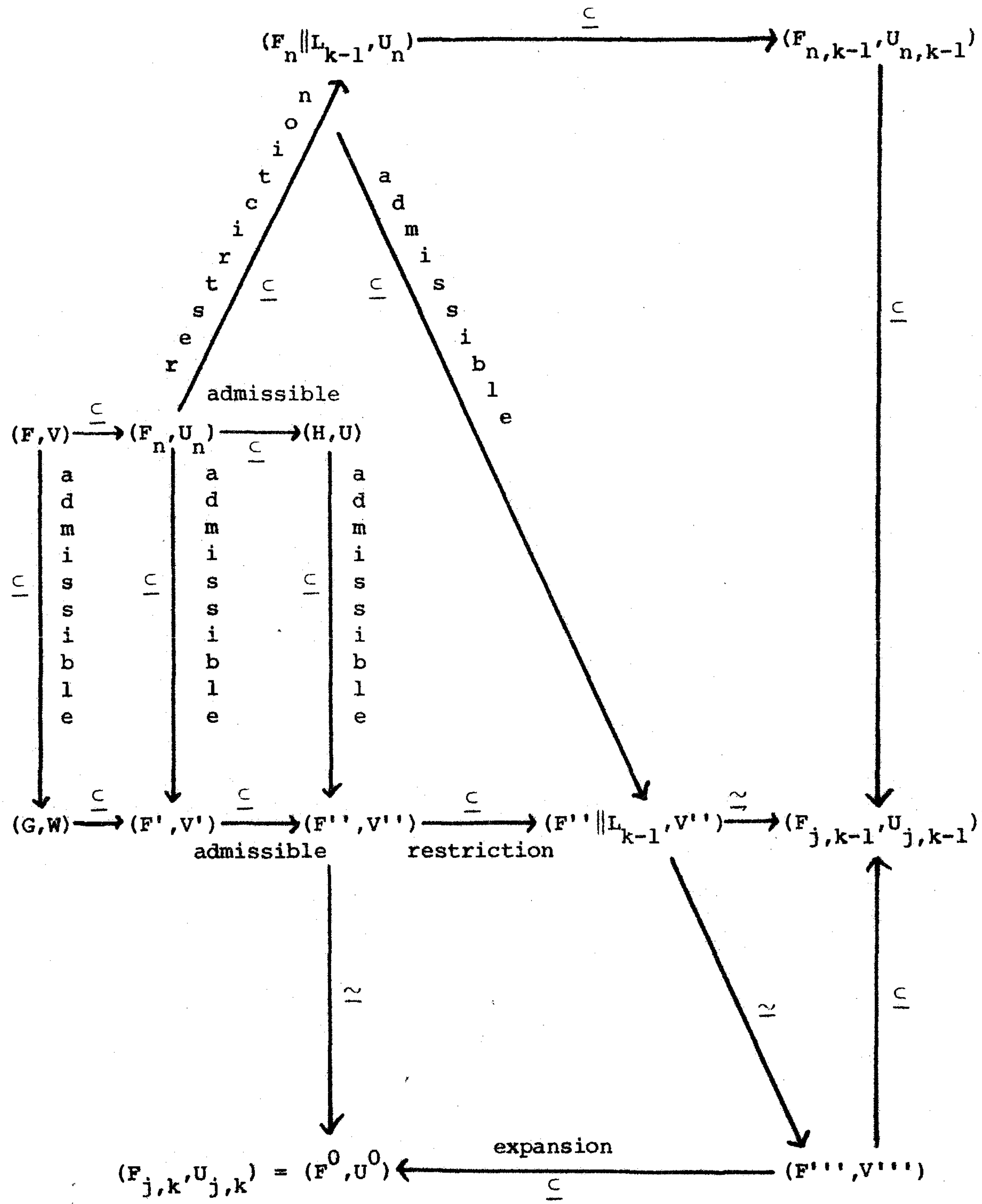
(F_n‖L_{k-1}, U_n)
(F_{n,k-1}, U_{n,k-1})
restriction
admissible
(F,V)
(F_n,U_n)
(H,U)
(G,W)
(F',V')
(F'',V'')
(F''‖L_{k-1}, V'')
(F_{j,k-1}, U_{j,k-1})
admissible
restriction
expansion
(F_{j,k}, U_{j,k}) = (F^0, U^0)
(F''', V''')

Diagram 3

representation of L_k . By lemmas 1.4 and 1.5, (H,U) is an admissible extension of (F_n, U_n) where without loss of generality we can assume $W \cap U = V$. By lemmas 1.6 we form (F',V') a common extension of (F_n,U_n) and (G,W) with $V' = U_n \cup W$; from lemma 1.6, (F',V') is an admissible extension of (F_n,U_n). Applying lemma 1.6 again to (H,U) and (F',V') seen as admissible extensions of (F_n,U_n) we obtain (F",V"), a common extension of (H,U) and (F',V'). Further (F",V") will be an admissible extension of (F_n,U_n) by the last sentence of lemma 1.6. Now by lemma 1.7, $(F''\|L_{k-1},V'')$ is an admissible extension of $(F_n\|L_{k-1},U_n)$. Also by the induction hypothesis (C2) we have $(F_n\|L_{k-1},U_n) \subset (F_{n,k-1},U_{n,k-1})$. Since by assumption the (k - 1)-th row has the embedding property we can find $j > n$ and an embedding of $(F''\|L_{k-1},V'')$ in $(F_{j,k-1},U_{j,k-1})$ which is the identity on U_n . Let (F"',V"') be the corresponding copy of $(F''\|L_{k-1},V'')$ in $(F_{j,k-1},U_{j,k-1})$. Expand (F"',V"') to (F^0,V^0) a w-representation of L_k, such that there is an isomorphism of (F",V") onto (F^0,V^0) which is the identity on U_n . Extend the k-th column by letting $(F_{j,k},U_{j,k})$ be (F^0,V^0) and letting $(F_{i,k},U_{i,k})$ be (F_n,U_n) for each i, $n < i < j$. Notice that (F^0,V^0) is a suitable successor of (F_n,U_n) in a sequential representation of L_k because there is an embedding of (H,U) in (F^0,V^0) which is the identity on U_n . Further there is clearly an embedding of (G,W) in (F^0,V^0) which is the identity on V .

We have shown how to ensure that the k^{th} column has the embedding property with respect to (F,V), $(F_{n,k},U_{n,k})$, and (G,W) and we have extended the k^{th} column by a finite number of rows in the process.

If $(F^*,V^*) \subset (F_{n,k},U_{n,k})$ and (G^*,W^*) is an admissible extension of (F^*,V^*), then $(F^*,V^*) \subset (F_{j,k},U_{j,k})$, whence by further extending the k^{th} column we can satisfy the embedding property with respect to (F^*,V^*), $(F_{n,k},U_{n,k})$, and (G^*,W^*). Since infinitely many stages are available it is easy to ensure that the k^{th} column when completed satisfies the embedding property completely.

§3. The trees corresponding to sequential representations. This section is concerned with the concept of tree and related definitions. Again all of this is contained, implicitly or explicitly, in Lerman [1]. Let $<(F_i,U_i) : i < \omega>$ be a sequential representation of a finite lattice $(L,\leq)$, where $U_i \subset \omega$ for $i < \omega$. Further let $<(F_i,U_i) : i < \omega>$ be recursive in the sense that, for each $c \in L$, $xF_i(c)y$ is a recursive predicate of x, i and y, and the cardinality of U_i is a recursive function of i. Relative to this representation, a <u>tree</u> is a pair $<\phi, \eta>$ where η is a strictly increasing recursive function, $\eta(0) = 0$ and ϕ is a 2-ary recursive partial function such that, for all i,j,k,

$$\eta(i) \leq j < \eta(i+1) \rightarrow [\phi(k,j) \text{ is defined} \leftrightarrow k \in U_i]$$

<u>and</u>, for all k_0, k_1, i,

$$k_0,k_1 \in U_i \rightarrow (Ej)(\eta(i) \leq j < \eta(i+1) \wedge \phi(k_0,j) \neq \phi(k_1,j)).$$

$\psi \in {}^{\omega}\omega$ <u>is on</u> T if there exists $\theta \in {}^{\omega}\omega$ called the <u>signature of</u> ψ such that, for all i,j,

$$\eta(i) \leq j < \eta(i+1) \rightarrow \psi(j) = \phi(\theta(i), j) .$$

$\mathcal{J}(T)$ denotes the set of all such ψ. We say $\sigma \in {}^{<\omega}\omega$ <u>is on</u> T if there exists $\theta \in {}^{<\omega}\omega$ called the <u>signature of</u> σ such that $\ell h(\sigma) = \eta(\ell h(\theta))$ and

$$\eta(i) \leq j < \eta(i+1) \rightarrow \sigma(j) = \phi(\theta(i), j) .$$

$\mathcal{J}^{<\omega}(T)$ denotes the set of all such σ.

Let $\triangleleft$ denote the lexicographical order of signatures, i.e. $\theta \triangleleft \theta'$ if and only if

$$(\theta = \theta') \vee (Ei)(\theta(i) < \theta'(i) \wedge (i' < i)(\theta(i') = \theta'(i))).$$

Let $\psi_0, \psi_1 \in \mathcal{J}(T)$ and θ_0, θ_1 be the respective signatures of ψ_0, ψ_1. Let $c \in L$. We say ψ_0 <u>and</u> ψ_1 <u>are equivalent mod</u> c, written $\psi_0 \equiv_T \psi_1 \pmod{c}$, if $\theta_0(i)\ F_i(c)\ \theta_1(i)$ for all $i < \omega$. If $T^* = \langle \phi^*, \eta^* \rangle$ and $T = \langle \phi, \eta \rangle$ are trees relative to the same representation then T^* is an <u>acceptable subtree</u> of T if $rng(\eta^*) \subset rng(\eta)$, $\mathcal{J}(T^*) \subset \mathcal{J}(T)$, and for all $c, \psi_0, \psi_1 \in F(T^*) \rightarrow (\psi_0 \equiv_{T^*} \psi_1 \pmod{c} \leftrightarrow \psi_0 \equiv_T \psi_1 \pmod{c})$. From the form of the definition the notion of being an acceptable subtree is transitive.

Let $T = \langle \phi, \eta \rangle$ be a tree, $\sigma \in \mathcal{J}^{<\omega}(T)$, and $\ell h(\sigma) = \eta(m)$. Then T^σ is to be the unique tree $(\phi^\sigma, \eta^\sigma)$ such that $\eta^\sigma(i) = \eta(m+i)$ for all $i > 0$, $\phi^\sigma(k,j) = \sigma(j)$ if $k \in U_0$ and $j < \eta(m)$ and $\phi^\sigma(k,j) = \phi(k,j)$ whenever $j \geq \eta(m)$ and $\phi^\sigma(k,j)$ is defined. We may observe that T^σ is an acceptable subtree of T. Let $\psi_0, \psi_1 \in \mathcal{J}(T^\sigma)$, θ_0, θ_1

be the respective signatures of ψ_0, ψ_1 in T, and $\theta_0^\sigma, \theta_1^\sigma$ be the respective signatures of ψ_0, ψ_1 in T^σ. Then $\theta_0 | m = \theta_1 | m$, $\theta_1^\sigma(i) = \theta_0(i + m)$ for $i < \omega$, and $\theta_1^\sigma(i) = \theta_1(i + m)$ for $i < \omega$. Now, for each $c \in L, \psi_0 \equiv_T \psi_1 \pmod c \leftrightarrow (i)\ (\theta_0(i)\ F_i(c)\ \theta_1(i)) \leftrightarrow (i)_{i \geq m}\ (\theta_0(i)\ F_i(c)\ \theta_1(i)) \leftrightarrow (i)\ (\theta_0^\sigma(i)\ F_{i+m}(c)\ \theta_1^\sigma(i)) \leftrightarrow (i)\ (\theta_0^\sigma(i)\ F_i(c)\ \theta_1^\sigma(i)) \leftrightarrow \psi_0 \equiv_{T^\sigma} \psi_1 \pmod c$, where the next to last equivalence follows from (F_{i+m}, U_{i+m}) being an extension of (F_i, U_i).

Let ψ be on the tree $T = < \phi, \eta >$. For $c \in L$ define $\psi_{T,c}$ to be the member of $\mathcal{J}(T)$ equivalent to $\psi \pmod c$ whose signature $\theta_{T,c}$ is least possible with respect to $\lhd$.

The following lemmas are due to Lerman. The proofs of lemmas 3.2 and 3.3 are deferred until section 5. The trees and subtrees are of course all relative to the same sequential representation of L, say $<<F_i, U_i> : i < \omega >$.

LEMMA 3.1. If T is an acceptable subtree of T_0 then given $c \in L$ there exist i and j such that $\psi_{T_0,c} = \{i\}(\psi_{T,c})$ and $\psi_{T,c} = \{j\}(\psi_{T_{0,c}})$ for all $\psi \in \mathcal{J}(T)$.

Note. We use interchangeably the notations $\{e\}^\psi$ and $\{e\}(\psi)$, where $e \in \omega$ and $\psi \in {}^\omega\omega$; $\{e\}^\psi(n)$ and $\{e\}(\psi,n)$ will then also be alternate notations.

Proof. ψ and $\psi_{T,c}$ are equivalent mod c in T. Hence ψ and $\psi_{T,c}$

are equivalent mod c in T_0 by the definition of acceptable subtree. Thus $\psi_{T_0,c} = (\psi_{T,c})_{T_0,c}$ whence $\psi_{T_0,c}$ is recursive in $\psi_{T,c}$. Since the procedure is uniform there thus exists i such that $\psi_{T_0,c} = \{i\}(\psi_{T,c})$.

Conversely, given $\psi_{T_0,c}$ we can effectively compute ψ^* on T such that ψ^* is equivalent to $\psi_{T_0,c}$ mod c on T_0, and such that ψ^* is least possible with respect to $\lhd$ on T. Two points are important here: Firstly, $\psi \equiv_{T_0} \psi_{T_0,c}$ (mod c) which means that some possibility for ψ^* exists. Secondly,

$$\psi^* \equiv_{T_0} \psi_{T_0,c} \text{ (mod c)} \leftrightarrow (i)\ (\theta^*(i)\ F_i(c)\ \theta_{T_0,c}\ (i))$$

where θ^* and $\theta_{T_0,c}$ are the respective signatures of ψ^* and $\psi_{T_0,c}$ on T_0. This means that there is a least such ψ^* which can be effectively computed from $\psi_{T_0,c}$. Now $\psi^* \equiv_{T_0} \psi$ (mod c) since equivalence mod c is transitive. Since T is an acceptable subtree of T_0, $\psi^* \equiv_T \psi$(mod c) which means $\psi_{T,c} = (\psi^*)_{T,c}$. Thus $\psi_{T,c}$ is recursive in ψ^* and hence in $\psi_{T_0,c}$. Again the procedure is uniform and thus there exists j such that $\psi_{T,c} = \{j\}(\psi_{T_0,c})$.

LEMMA 3.2. If T is a tree, $b \nleq c$ in L, and $e \in \omega$, then there is an acceptable subtree T_0 of T such that, for any $\psi \in \mathcal{J}(T_0)$, $\psi_{T,c} \neq \{e\}(\psi_{T,b})$.

LEMMA 3.3. If T_0 is a tree and $e \in \omega$, there exists an acceptable subtree T of T_0 and $a \in L$ such that either, for all

$\psi \in \mathcal{J}(T)$, $\{e\}^{\psi}$ is total and $\{e\}^{\psi}$ has the same degree as $\psi_{T,a}$, or $\{e\}^{\psi}$ is total for no $\psi \in \mathcal{J}(T)$.

§4. Proof of the embedding theorem. Let a j-tree be a tree in the sense of (F_0^j, U_0^j), (F_i^j, U_1^j), ..., where this is the sequential representation of L_j obtained from the j^{th}-column of the array by deleting repetitions.

LEMMA 4.1. Given a k-tree $T_0 = < \phi_0, \eta_0 >$ there exists a $(k + 1)$-tree $T = < \phi, \eta >$ such that $\mathcal{J}(T) \subset \mathcal{J}(T_0)$ and for each $c \in L_k$ there exist $i, j \in \omega$ such that if $\psi \in \mathcal{J}(T)$ then $\psi_{T_0,c} = \{i\}(\psi_{T,c})$ and $\psi_{T,c} = \{j\}(\psi_{T_0,c})$.

Proof. Say $<(F_i, U_i) : 0 \leq i < \omega >$ and $<(G_i, W_i) : 0 \leq i < \omega >$ are respectively the sequential representations of L_k and L_{k+1} obtained from the k^{th} and $(k + 1)$-th column of the array (the representations have been renamed here). Let $m(-1) = 0$. For $i \geq 0$ denote by $m(i)$ the first index greater than $m(i - 1)$ such that $W_i \subset U_{m(i)}$. Then for all $u, v \in W_i$ and $c \in L_k$, $uF_{m(i)}(c)\, v \leftrightarrow uG_i(c)v$. When building the i-th node of T we let $\eta(i + 1) = \eta_0(m(i)+1)$. If $s(t)$ is the least element of U_i, where $\eta(i) \leq t < \eta(i+1)$, then we define $\phi(\ell,t) = \phi_0(s(t),t)$ when $\eta(i) \leq t < \eta_0(m(i))$ and $\ell \in W_i$, and $\phi(\ell,t) = \phi_0(\ell,t)$ when $\eta_0(m(i)) \leq t < \eta(i + 1) = \eta_0(m(i) + 1)$.

Note that if $\psi_0, \psi_1 \in \mathcal{J}(T)$, $c \in L_K$ and if θ_i and θ_1^0 $(i < 2)$ are the respective signatures of $\psi_i (i < 2)$ on T and T_0 we have, for each $i \in \omega, \theta_0(i)\ G_i(c)\ \theta_1(i) \leftrightarrow \theta_0^0\ (m(i))\ F_{m(i)}(c)\ \theta_1^0\ (m(i))$. Using this we can complete the proof by imitating the proof of lemma 3.1.

We are now ready to exhibit an embedding of L_ω as an initial segment of the degrees. In order to do this we shall construct a sequence of trees $T_0, T_0', T_1, T_1', \ldots$ such that, for each i, T_1' is an acceptable subtree of T_i, T_i and T_i' are i-trees and T_{i+1} comes from T_1' by the canonical method for passing from an i-tree to an (i + 1)-tree which is provided by Lemma 4.1. The sequence will further be such if $\psi \in \cap\ \{\mathcal{J}(T_i) : i < \omega\}$ then the map $c \rightarrow \deg\ (\psi_{T_j, c})$ is an embedding of the desired kind, where for any $c \in L_\omega = \cup\{L_i : i < \omega\}$ we suppose j to be the least number such that $c \in L_j$. For any $\psi \in \mathcal{J}(T_j)$ let ψ_c be $\psi_{T_j, c}$. Note that if $a = b \vee c$ in L_ω then $\deg(\psi_a) = \deg(\psi_b) \vee \deg(\psi_c)$ for any $\psi \in \mathcal{J}(T_i)$ where $a, b, c \in L_j$. Indeed let $\psi_a', \psi_b', \psi_c'$ denote $\psi_{T_j, a}, \psi_{T_j, b}, \psi_{T_j, c}$, and let $\theta, \theta_a, \theta_b, \theta_c$ be the respective signatures of $\psi, \psi_a', \psi_b', \psi_c'$ on T_j. Now $\theta_a(i) = \mu k[kG_i(a)\ \theta(i)]$ and similarly for b and c, where $<(G_i, W_i) : 0 \leq i < \omega>$ is the sequential representation of L_j. Hence $\theta_a(i) = \mu k[k\ G_i(b)\ \theta(i)\ \&\ k\ G_i(c)\ \theta(i)] = \mu k[k\ G_i(b)\ \theta_b(i)\ \&\ k\ G_i(c)\ \theta_c(i)]$. Thus ψ_a' is recursive in $\{\psi_b', \psi_c'\}$. Conversely $\theta_b(i) = \mu k[k\ G_i(b)\ \theta(i)] = \mu k[k\ G_i(b)\ \theta_a(i)]$ since $\theta_a(i)\ G_i(a)\ \theta(i)$ implies $\theta_a(i)\ G_i(b)\ \theta(i)$. Hence ψ_b' is recursive in ψ_a' and similarly ψ_c' is recursive in ψ_a'.

Thus using Lemmas 3.1 and 4.1 we obtain $\deg(\psi_a) = \deg(\psi_b) \vee \deg(\psi_c)$. In particular if $b \geq c$ we have $\deg(\psi_b) \geq \deg(\psi_c)$.

Consider $\omega \cup \{< e,b,c > : e \in \omega,\ b,c \in L,\ b \not\geq c\}$ and suppose $<\mathcal{C}_i : i < \omega >$ is an enumeration of this set such that if $\mathcal{C}_i = < e,b,c >$ then b,c are both in L_i . Now if $\mathcal{C}_i = e$ we choose T_i' according to Lemma 3.3 such that, for all $\psi \in \mathcal{J}(T_1')$, $\{e\}^{\psi}$ is either not total or has the same degree as ψ_a for some $a \in L_i$. If $\mathcal{C}_i = < e,b,c >$ it is a consequence of Lemmas 3.1 and 4.1 that, for all $\psi \in \mathcal{J}(T_i)$, ψ_b is uniformly recursive in $\psi_{T_i,b}$ and $\psi_{T_i,c}$ is uniformly recursive in ψ_c . Hence there exists e' such that, for all $\psi \in \mathcal{J}(T_i)$, $\psi_c = \{e\}^{\psi_b}$ implies $\psi_{T_i,c} = \{e'\}(\psi_{T_i,b})$. Thus we apply Lemma 3.2 to T_i, using e' instead of e to obtain T_1' . This step ensures that, for all $\psi \in \mathcal{J}(T_i')$, $\psi_c \neq \{e\}^{\psi_b}$. Now the proof will be over as soon as we see that $\cap \{\mathcal{J}(T_i) : i < \omega >$ is not empty. But $\mathcal{J}(T_0)$ can be regarded as a product of countably many finite sets of integers, and if we give each of these finite set the discrete topology then $\mathcal{J}(T_0)$ is compact. But by construction each $\mathcal{J}(T_i)$ is a closed subset of $\mathcal{J}(T_0)$: thus $\cap \{\mathcal{J}(T_i) : i < \omega\} \neq \emptyset$.

§5. We now attend to the lemmas left without proof in the previous sections.

Proof of Lemma 1.4. The first case is when (G,W) is an extension of type I of (F,V). Say $W\setminus V = \{v_0, v_1, v_2\}$ and (G,W) is obtained from $u_0, u_1 \in V$ and $a,b,c \in L$, by the construction specified in the proof of Theorem 1.3 for the satisfaction of 1.1. If $v_0G(e)z$ for $z \in V$, $e \in L$, then it must be that $e \leq a$: thus $v_0G(e)u_0$ and we can take u_0 for y in the definition of admissible extension. Similarly for v_2. Now if $v_1G(e)z$ it must be that $e \leq a$ and $e \leq b$ and thus $v_1G(e)u_0$: we then take u_0 for y.

The second case is when (G,W) is an extension of type II of (F,V) ; we use the notation of the final part of the proof of theorem 1.3, except that (G,W), (F,V) and V^j $(j < 2)$ replace (F_{i+1}, U_{i+1}), (F_i, U_i) and U_i^j $(j < 2)$ respectively. Remember that $W = V \cup V^0 \cup V^1$ and that $u_0, u_1, v_0, v_1 \in V$ satisfy

$$u_0F(c)u_1 \rightarrow v_0F(c)v_1, \quad c \in L .$$

Note that the following implications hold: if $a \in L$,

$$[(x \in V^0 \ \& \ xG(a)z \ \& \ z \in V) \rightarrow zG(a)v_0]$$

$$[(x \in V^1 \ \& \ xG(a)z \ \& \ z \in V) \rightarrow zG(a)v_1] .$$

Thus in the first case pick v_0 for y in the definition of admissible extension; in the second case pick v_1.

Proof of Lemma 1.6. We first show (F',V') is a w-representation of L. Say $a,b \in L$, $x,y \in V'$, $a \leq b$ and $xF'(b)y$. Since $G(a) \leq G(b)$ and $H(a) \leq H(b)$, we also obtain $F'(a) \leq F'(b)$. We now have to show if $a,b \in L$, $x,y \in V'$ then $x\,(F'(a)\ \&\ F'(b))y$ implies $x\,F'(a \vee b)y$. The only non-trivial case is when $x \in W\setminus V$ and $y \in U\setminus V$. We then know there exist $z_0, z_1 \in V$ such that ($x\ G(a) z_0$ and $z_0 H(a) y$) and ($x G(b) z_1$ and $z_1 H(b) y$). Since (G,W) is an admissible extension of (F,V) we can find $z \in V$ such that $xG(a)z$ and $xG(b)z$: thus $x\ F'(a \vee b)y$. By a similar argument we can show (F',V') is an admissible extension of (H,U); that it is an extension of (G,W) is clear. The last part of the lemma is also immediate from lemma 1.5.

Proof of Lemma 3.2. Let e be fixed. It will be sufficient to show that there exists $\sigma \in \mathcal{J}^{<\omega}(T)$ such that for any $\psi \in F(T^{\sigma})$, $\psi_{T,c} \neq \{e\}(\psi_{T,b})$.

Remember that $<< F_i, U_i > : i < \omega >$ is the sequential representation of L we consider. Since F_0 is a join embedding of $(L,\leq)$ in $(\mathcal{S}(U_0),\leq)$ there exist $m,n \in U_0$ such that $m\ F_0(b) n$ but not $m\ F_0(c) n$. Let $\tau, \tau' \in \mathcal{J}^{<\omega}(T)$ be chosen such that their signatures are $< m >, < n >$ respectively. Then τ and τ' are equivalent mod b but not mod c in T and we have $\tau_{T,b} = \tau'_{T,b}$ and $\tau_{T,c} \neq \tau'_{T,c}$. Choose $j < \eta(1)$ such that $\tau_{T,c}(j) \neq \tau'_{T,c}(j)$. If there is no $\psi \supset \tau$ on T such that $\{e\}(\psi_{T,b}, j)$ is defined, let $\sigma = \tau$. Otherwise there exists $\rho \in \mathcal{J}^{<\omega}(T)$ such that $\rho \supset \tau$ and $\{e\}(\rho_{T,b}, j)$ is defined. If $\{e\}(\rho_{T,b}, j) \neq \tau_{T,c}(j)$ let $\sigma = \rho$. If $\{e\}(\rho_{T,b}, j) = \tau_{T,c}(j)$, let $\sigma(k) = \tau'(k)$ for $k < \eta(1)$ and $\sigma(k) = \rho(k)$ for $k \geq \eta(1)$. In either case we have ensured that $\{e\}(\sigma_{T,b}, j)$ is defined and $\neq \sigma_{T,c}(j)$. Thus the same will be true when σ is replaced by any $\psi \in \mathcal{J}(T^{\sigma})$.

The rest of this section is devoted to the proof of Lemma 3.3.

Suppose that there exists σ on T_0 and $k \in \omega$ such that $\{e\}^{\psi}(k)$ is not defined for any ψ on T_0, $\psi \supset \sigma$. Then we may take T to be T_0^{σ}. Thus suppose that for every σ on T_0 and $k \in \omega$ there exists $\psi \supset \sigma$ such that $\{e\}^{\psi}(k)$ is defined. We may construct an acceptable subtree T of T_0, such that for all ψ on T $\{e\}^{\psi}$ is total, as follows. Suppose $\eta|(i+1)$ has been defined and that $\phi(k,j)$ has been defined for all k,j satisfying

$$(Ei' < i)\ [\eta(i') \le j < \eta(i'+1)\ \&\ k \in U_{i'}]\ .$$

Let $\sigma_0, \sigma_1, \ldots, \sigma_{m-1}$ be an enumeration of the strings on T of length $\eta(i)$. For $\alpha, \beta \in {}^{<\omega}\omega$ let $\alpha \frown \beta$ denote the string obtained by the concatenation of β to the right of α. We can choose $\tau_0, \tau_1, \ldots \tau_{m-1}$ in turn such that $\sigma_j \frown \tau_0 \frown \ldots \frown \tau_j$ is on T_0 and

$$\{e\}(\sigma_j \frown \tau_0 \frown \ldots \frown \tau_j, i)$$

is defined for each $j < m$. Let $\tau = \tau_0 \frown \ldots \frown \tau_{m-1}$. Let $\eta(i+1) = \eta_0(i^* + 1)$ where i^* is the least number such that $\eta(i) + \ell h(\tau) = \eta_0(i^*)$. If $\eta(i) \le j < \eta_0(i^*)$ and $k \in U_i$, define $\phi(k,j) = \tau(j - \eta(i))$. If $\eta_0(i^*) \le j < \eta_0(i^* + 1)$ and $k \in U_i$, define $\phi(k,j) = \phi_0(k,j)$. This completes the induction step in the definition of T. It is easy to check that η and ϕ and recursive and that T is an acceptable subtree of T_0. For any $\psi \in \mathcal{J}(T)$ there exists $j < m$ such that $\sigma_j \subset \psi$, whence $\sigma_j \frown \tau \subset \psi$, whence $\{e\}^{\psi}(i)$ is defined from the way in which we constructed the i^{th} node.

Thus without loss of generality we may suppose that, for all $\psi \in \mathcal{T}(T_0)$, $\{e\}^{\psi}$ is total. We make this assumption below.

Let $S(T_0)$ consist of all $b \in L$ which satisfy

$$(\psi_0)(\psi_1)(\psi_0,\psi_1 \in \mathcal{T}(T_0) \ \& \ \psi_0 \equiv_{T_0} \psi_1 \ (\text{mod } b) . \rightarrow . \{e\}^{\psi_0} = \{e\}^{\psi_1}) .$$

Without loss of generality suppose that $S(T_0) = S(T_0^{\sigma})$ for all $\sigma \in \mathcal{T}^{<\omega}(T_0)$. Otherwise replace T_0 by T_0^{σ} for some $\sigma \in \mathcal{T}^{<\omega}(T_0)$ such that $S(T_0^{\sigma})$ is maximal with respect to inclusion.

LEMMA 5.1. If $a,b, \in S(T_0)$ and $a \wedge b = c$ then $c \in S(T_0)$.

Proof. This is where we use clause 1.1 of the definition of sequential representation. For proof by contradiction suppose $a \wedge b = c$, a and b are in $S(T_0)$ and $c \notin S(T_0)$. Choose any $\sigma \in \mathcal{T}^{<\omega}(T_0)$ such that $\ell h(\sigma) > 0$. Then $c \notin S(T_0^{\sigma})$ since $S(T_0) = S(T_0^{\sigma})$. Therefore there exist $\psi_0, \psi_1 \in \mathcal{T}(T_0^{\sigma})$ such that $\{e\}^{\psi_0} \neq \{e\}^{\psi_1}$ and $\psi_0 \equiv_{T_0^{\sigma}} \psi_1$ (mod c). Since T_0^{σ} is an acceptable subtree of T we have $\psi_0 \equiv_{T_0} \psi_1$ (mod c). Since there must exist $k \in \omega$ such that $\{e\}^{\psi_0}(k)$ and $\{e\}^{\psi_1}(k)$ are defined and distinct, we can suppose that ψ_0 and ψ_1 agree except at a finite number of nodes. By interpolation we can suppose that ψ_0 and ψ_1 differ on exactly one node, say the $(i + 1)$th node $[\eta_0(i + 1), \eta_0(i + 2))$. (Note that ψ_0 and ψ_1 cannot differ on the

0^{th} node of T_0 because $\ell h(\sigma) > 0$ and ψ_0 and ψ_1 are both on T_0^σ). Let θ_0 and θ_1 be the respective signature of ψ_0 and ψ_1 on T_0. Then $\theta_0(i+1)F_{i+1}(c)\ \theta_1(i+1)$ since $\psi_0 \equiv_{T_0} \psi_1 \pmod{c}$. Let $[\eta_0(i+1), \eta_0(i+2))$ be the h-th node of T_0^σ; then by the definition of T_0^σ we see that $\theta_0(i+1)$, $\theta_1(i+1)$ are elements of U_h. Since $\ell h(\sigma) > 0$, $h \leq i$ and hence $U_h \subset U_i$. It follows that $\theta_0(i+1)$, $\theta_1(i+1)$ are elements of U_i and since (F_{i+1}, U_{i+1}) extends (F_i, U_i) we have $\theta_0(i+1)\ F_i(c)\ \theta_1(i+1)$. From 1.1 there exist $k_0, k_1, k_2 \in U_{i+1}$ such that $\theta_0(i+1)\ F_{i+1}(a)\ k_0\ F_{i+1}(b)\ k_1\ F_{i+1}(a)\ k_2\ F_{i+1}(b)\ \theta_1(i+1)$. For each $j < 3$ let ϕ_j be the member of $\mathcal{J}(T_0)$ whose signature θ_j' agrees with θ_0 and θ_1 except at $i+1$ and satisfies $\theta_j'(i+1) = k_j$. Then $\psi_0 \equiv_{T_0} \phi_0$ and $\phi_1 \equiv_{T_0} \phi_2 \pmod{a}$ and $\phi_2 \equiv_{T_0} \phi_1$ and $\phi_2 \equiv_{T_0} \psi_1 \pmod{b}$. Since $a, b \in S(T_0)$ we deduce that $\{e\}^{\psi_0} = \{e\}^{\phi_0}$, $\{e\}^{\phi_0} = \{e\}^{\phi_2}$, $\{e\}^{\phi_0} = \{e\}^{\phi_1}$, $\{e\}^{\phi_2} = \{e\}^{\psi_1}$. It follows that $\{e\}^{\psi_0} = \{e\}^{\psi_1}$, contradiction.

From the lemma just proved it follows that $S(T_0)$ has a least member, i.e. least in the sense of $(L, \leq)$. Below c denotes the least member of $S(T_0)$. Since $c \in S(T_0)$, if $\psi_0 \equiv_{T_0} \psi_1 \pmod{c}$ for ψ_0, $\psi_1 \in \mathcal{J}(T_0)$, then $\{e\}^{\psi_0} = \{e\}^{\psi_1}$. Therefore $\psi \in \mathcal{J}(T_0) \rightarrow \deg(\{e\}^{\psi}) \leq \deg(\psi_{T_0,c})$ because $\psi \equiv_{T_0} \psi_{T_0,c} \pmod{c}$. We now prove

LEMMA 5.2. There exists an acceptable subtree T of T_0 such that $\psi \in \mathcal{J}(T) \rightarrow \deg(\psi_{T_0,c}) \leq \deg(\{e\}^{\psi})$.

Proof. This is where we use clause 1.2 of the definition of sequential representation. For induction suppose $\eta|(i+1)$ has been defined and that $\phi(k,j)$ has been defined for each pair $< k,j >$ satisfying $(Eh < i)(\eta(h) \le j < \eta(h+1)\ \&\ k \in U_h)$. Suppose that the definition of acceptable subtree has been satisfied so far, i.e.

(i) $\mathrm{Rng}(\eta|(i+1)) \subset \mathrm{Rng}\ \eta_0$,

(ii) if σ_0, σ_1 are strings of T of length $\eta(i)$ then

σ_0, σ_1 are strings on T_0 and, for each $b \in L$,

$\sigma_0 \equiv_{T_0} \sigma_1 \pmod b \leftrightarrow \sigma_0 \equiv_T \sigma_1 \pmod b$.

Suppose further that an effective method of computing $\psi_{T,c}|\eta(i)$ from $\{e\}^{\psi}$ for each ψ on T has already been given. Let $\{<\sigma_s, k_{0,s}, k_{1,s}>: s < t\}$ be an enumeration of all triples satisfying: $\sigma_s \in \mathcal{T}^{<\omega}(T)$, $\ell h(\sigma_s) = \eta(i)$, $k_{0,s}$ and $k_{1,s}$ are members of U_i and not $k_{0,s}\ F_i(c) k_{1,s}$. We construct the i^{th} node in t stages numbered $0,1,\ldots,t-1$. Let $\eta(i) = \eta_0(i^*)$. Let

$$\tau_{0,k}(x) = \begin{cases} \phi_0(k, x+\eta(i)) & \text{if } x < \eta_0(i^*+1) - \eta(i), \\ \text{undefined otherwise} \end{cases}$$

for $k \in U_i$. Immediately prior to stage s we shall have defined strings $\tau_{s,k}$, $k \in U_i$, all of the same length such that $\sigma_u \frown \tau_{s,k}$ is on T_0 for each $u < t$ and $k \in U_i$. Further for each $a \in L$ we shall have

$$\sigma_u \frown \tau_{s,k} \equiv_{T_0} \sigma_u \frown \tau_{s,m} \pmod a \leftrightarrow k\ F_i(a) m \text{ for } u < t \text{ and } k,m \in U_i .$$

(This means that we have not yet violated the condition corresponding to (ii) above for node $i + 1$ of T). We shall choose $\tau_{s+1,k} \supset \tau_{s,k}$ for $k \in U_i$. After all the stages s, $s < t$, we shall let $\eta(i + 1)$ be the common length of all the strings $\sigma_u \frown \tau_{t,k}$ and we shall define $\phi(k,j) = \tau_{t,k}(j - \eta(i))$ for all $k \in U_i$ and j in $\eta(i) \leq j < \eta(i + 1)$. Thus the strings on T of length $\eta(i + 1)$ will be just all those of the form $\sigma_u \frown \tau_{t,k}$. Our aim in defining $\tau_{s+1,k}$ for $k \in U_i$ is to code in $\{e\}^{\psi}$ for ψ on T an effective way of distinguishing between ψ's which extend $\sigma_s \frown \tau_{s+1,\, k_{0,s}}$ and ψ's which extend $\sigma_s \frown \tau_{s+1,\, k_{1,s}}$.

Stage s. To simplify the notation we write σ, k_0 and k_1 for σ_s, $k_{0,s}$ and $k_{1,s}$. Let $\sigma \frown \tau_{s,k_0}$, $\sigma \frown \tau_{s,k_1}$ be denoted τ and τ' respectively. Assume $\ell h(\tau) > 0$; this requires a trivial modification of the construction when $i = 0$, i.e. when we are constructing the 0^{th} node of T. Let b be the greatest element of L such that $k_0 F_i(b) k_1$. Notice that $c \not\leq b$, whence $b \notin S(T_0)$. We can now find effectively τ_0, τ_1 and k' such that $\tau \frown \tau_0$, $\tau \frown \tau_1 \in \mathcal{J}^{<\omega}(T_0{}^{\tau})$, $\ell h(\tau_0) = \ell h(\tau_1)$, $\{e\}(\tau \frown \tau_0, k')$ and $\{e\}(\tau \frown \tau_1, k')$ are both defined and different, and $\tau \frown \tau_0 \equiv_{T_0} \tau \frown \tau_1$ (mod b). Further we may suppose $\tau \frown \tau_0$ and $\tau \frown \tau_1$ differ only in the $(p + 1)$-th node of T_0 by interpolation. Let $\tau \frown \tau_0$, $\tau \frown \tau_1$ agree with $\lambda x\, \phi_0(q_0,x)$, $\lambda x\, \phi_0(q_1,x)$ respectively on the $(p + 1)$-th node of T_0; then $q_0 F_{p+1}(b) q_1$. Thus, for all $a \in L$, $k_0 F_{p+1}(a) k_1 \rightarrow q_0 F_{p+1}(a) q_1$. Now $i \leq p$ whence $k_0, k_1 \in U_p$, and $\ell h(\tau) > 0$ whence $q_0, q_1 \in U_p$. Since $(F_{p+1}, U_{p+1}) \supset (F_p, U_p)$, for all $a \in L$ we have $k_0 F_p(a)k_1 \to q_0 F_p(a)q_1$. By 1.2 there exist embeddings $h_0, h_1 : (F_p, U_p) \to (F_{p+1}, U_{p+1})$ such that $q_0 = h_0(k_0)$, $h_0(k_1) = h_1(k_0)$, $h_1(k_1) = q_1$. For $j < 2$ and $r < 2$ let $\tau \frown \tau_r^j$ be the same as $\tau \frown \tau_r$ except that $\tau \frown \tau_r^j$ is to agree with $\lambda x\, \phi_0(h_j(k_r),x)$ on the $(p + 1)$-th node of T_0. We may suppose that $\{e\}(\tau \frown \tau_r^j, k')$ is defined for each $j < 2$ and $r < 2$; otherwise we may extend each of τ_0 and τ_1 in exactly the same way until this supposition is justified. From the equations above we have $\tau \frown \tau_0 = \tau \frown \tau_0^0$, $\tau \frown \tau_1^0 = \tau \frown \tau_0^1$, $\tau \frown \tau_1^1 = \tau \frown \tau_1$. Also, as we have noted above, $\{e\}(\tau \frown \tau_0, k')$ and $\{e\}(\tau \frown \tau_1, k')$ are defined and different.

It follows at once that for some $j < 2$,

$$\{e\}(\tau \frown \tau_0^j, k') \text{ and } \{e\}(\tau \frown \tau_1^j, k') \text{ are defined and different.}$$

Fix such a j. We may suppose that $\{e\}(\tau' \frown \tau_r^j, k')$ is defined for $r < 2$. We now define $\tau_{s+1,k}$ for $k \in U_i$.

Case 1. $\{e\}(\tau \frown \tau_1^j, k') \neq \{e\}(\tau' \frown \tau_1^j, k')$. Then let $\tau_{s+1,k} = \tau_{s,k} \frown \tau_1^j$ for all $k \in U_i$.

Case 2. Otherwise. Then let $\tau_{s+1,k}$ be the string on T_0 which is exactly the same as $\tau_{s,k} \frown \tau_0$ (and $\tau_{s,k} \frown \tau_1$) except that on the $(p + 1)$-th node of T_0 it is to agree with $\lambda x\, \phi_0(h_j(k),x)$. In this way $\tau_{s+1,k}$ is fixed for all $k \in U_i$. Note that in every case $\{e\}(\sigma \frown \tau_{s+1,k_0}, k')$ and $\{e\}(\sigma \frown \tau_{s+1,k_1}, k')$ are defined and unequal. Further we shall have $\sigma_u \frown \tau_{s+1,k} \equiv_{T_0} \sigma_u \frown \tau_{s+1,m} \pmod{a} \leftrightarrow$

$k\ F_i(a)m$ for all $a \in L$, $u < t$ and $k, m \in U_i$. This completes stage s and the construction of T . Recall that after the last stage, namely stage $t - 1$, we define $\eta(i + 1)$ to be the common length of all the strings $\sigma_u \frown \tau_{t,k}$ and let $\phi(k,j) = \tau_{t,k}\ j - \eta(i)$ for all $k \in U_i$ and j in $\eta(i) \leq j < \eta(i + 1)$.

How do we compute $\psi_{T,c}$ from $\{e\}^{\psi}$ for $\psi \in \mathcal{J}(T)$? For induction we may suppose that we have already computed $\psi_{T,c}|\eta(i)$. Let k be the unique member in U_i such that ψ agrees with $\lambda x\ \phi(k,x)$ in $[\eta(i), \eta(i + 1))$; it suffices to compute k mod $F_i(c)$. Recall from the definition of c that if $\psi \in \mathcal{J}(T)$, $\tau \in \mathcal{J}^{<\omega}(T)$, $\{e\}^{\tau}(k')$ is defined, and ψ extends τ mod c in T , then $\{e\}^{\psi}(k') = \{e\}^{\tau}(k')$. For any k_0, k_1 in U_i which are inequivalent mod $F_i(c)$ we can find s such that $< \sigma_s, k_{0,s}, k_{1,s} > = < \psi_{T,c}|\eta(i), k_0, k_1 >$. Suppose that $k \in \{k_0, k_1\}$; then ψ extends either $\sigma_s \frown \tau_{s+1,k_0}$ or $\sigma_s \frown \tau_{s+1,k_1}$ mod c on T . Thus by looking through the construction of T to stage s of the formation of node i , and by comparing $\{e\}^{\psi}(k')$ with $\{e\}(\sigma_s \frown \tau_{s+1,k_0}, k')$ and $\{e\}(\sigma_s \frown \tau_{s+1,k_1}, k')$ we can tell whether $k = k_0$ or $k = k_1$. By looking at all pairs k_0, k_1 inequivalent mod F_i (c) we can thus compute k mod $F_i(c)$ from $\{e\}^{\psi}$.

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