DOMINATION IN GRAPHS

by

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Abstract

A *dominating set* in a graph G is a set of vertices D such that each vertex is either in D or has a neighbour in D. A partition of $V(G)$ such that each class is a dominating set in G is called a *domatic partition* of G.

In this thesis, we first briefly survey a variety of known results in the field, presenting fundamentals as well as more recent concepts in domination. In particular, we turn our attention to ordinary domination, factor domination (where D dominates every given spanning subgraph of G), and distance domination (where a vertex not in D is within a given distance from D).

Our main contributions are in the area of factor domination. We first prove several probabilistic bounds on the size of a factor domatic partition as well as the size of a smallest factor dominating set. Then, using well-known Beck's method, we obtain a polynomial time randomized algorithm that constructs such a partition. We also give similar bounds and algorithms for other related problems.

Acknowledgements

First, I would like to thank my supervisor Ladislav Stacho for his kindness, patience, and support.

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Chapter 1

Introduction

1.1 Motivation

Domination is an area in graph theory with an extensive research activity. In 1998, a book [HHS98] on domination has been published which lists 1222 papers in this area. In general, a dominating set in a graph is a set of vertices D such that each vertex is either in D or is adjacent to a vertex in D. We will give all fundamental definitions in the next chapter. The historical roots of domination is said to be the following chess problem. Consider an 8×8 chessboard on which a queen can move any number of squares vertically, horizontally, or diagonally. Figure 1.1 shows the squares that a queen can attack or dominate. One is interested to find the minimum number of queens needed on the chessboard such that all squares are either occupied or can be attacked by a queen. In Figure 1.2, five queens are shown who dominate all the squares.

To model the queens problem on a graph, let G represent the chessboard such that each vertex corresponds to a square, and there is an edge connecting two vertices if and only if the corresponding squares are separated by any number of squares horizontally, vertically, or diagonally. Such a set of queens in fact represents a dominating set.

For another motivation of this concept, consider a bipartite graph where one part represents people, the other part represents jobs, and the edges represent the skills of

		X					X
		X				X	
		X			X		
X		X		X			
	X	X	X				
X	X	Q	X	X	X	X	X
	X	X	X				
X		X		X			

Figure 1.1: Squares attacked by a Queen.

							Q
					Q		
				Q			
			Q				
	Q						

Figure 1.2: 5 dominating queens.

each person. Each person may take more than one job. One is interested to find the minimum number of people such that are jobs are occupied. As shown in Figure 1.3, {Jane, John} form a minimum size dominating set.

The concept of dominating set occurs in variety of problems. The puzzles above are only interesting examples. A number of these problems are motivated by communication network problems, for example. The communication network includes a set of nodes, where one node can communicate with another if it is directly connected to that node. In order to send a message directly from a set of nodes to all others, one needs to choose this set such that all other nodes are connected to at least one node in the set. Now, such a set is a dominating set in a graph which represents the network. For other applications of domination, the facility location problem, land surveying, and routings can be mentioned.

An essential part of the motivations in this field is based on the varieties of domination. There are more than 75 variations of domination cited in [HHS98]. These variations are mainly formed by imposing additional conditions on D, $V(G) - D$, or

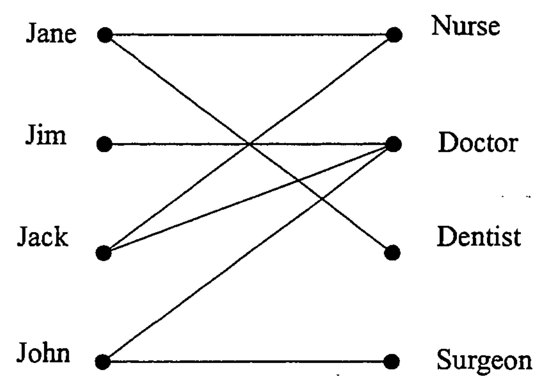

Figure 1.3: Dominated jobs.

$V(G)$. We will review some of these variations in the next chapter, but our main focus is two type of domination, namely, factor domination and distance domination.

1.2 Thesis Outline

In the first part of this thesis we survey fundamentals and known results in the area of domination, mainly focused on bounding the minimum size of a dominating set in a graph. Beside ordinary domination, we also survey varieties of domination which are mainly formed by assuming some conditions on the dominating set or the vertices that are not in the dominating set.

We also survey the domatic partition of a graph which is a partition of vertices of a graph such that each class is a dominating set, focusing on bounding the maximum number of classes in such a partition. Moreover, we survey two varieties of the domatic partition; factor domatic partition and distance domatic partition.

Our main results in this area is presented in the second part of the thesis. Fedor et al. in [FGPS05] presented some probabilistic bounds on the size of a domatic partition of a graph, using conditional probabilistic methods and Lovász Local Lemma. They also obtained a deterministic and a randomized algorithm which in fact construct such partitions. We use similar methods to the factor domatic partition.

In Section 3.1, we first use the method of conditional probabilities and obtain a lower bound of $\left\lfloor \left(1 - o(\frac{\ln\ln kn}{\ln kn})\right) \frac{\delta_{min}+1}{\ln kn} \right\rfloor$ on the size of a factor domatic partition, where n denotes number of the vertices of G, k denotes number of factors, and δ_{min} is the minimum degree over all factors. Then we present a deterministic algorithm which in fact constructs a factor domatic partition of this size.

Also, using the Lovász Local Lemma we obtain another lower bound of $\left\lfloor \left(\frac{1}{3} - o(1)\right) \frac{\delta_{min}+1}{\ln(k\Delta(G))} \right\rfloor$ on the size of a factor domatic partition and we use the well-known Beck's method to obtain a randomized algorithm to construct such partition with high probability.

In Section 3.2, we extend our results to the distance version.

Using probabilistic methods Alon and Spencer [AS00] obtained an upper bound of $n\,\frac{1+\ln(\delta(G)+1)}{\delta(G)+1}$ on the size of a dominating set. In Section 3.3, we use a similar method and obtain a probabilistic upper bound of $\left\lceil n\,\frac{1+\ln(k(\delta_{min}+1))}{(\delta_{min}+1)} \right\rceil$ on the size of a factor dominating set. We also extend this to the general case of distance domination.

A different aspect of domination problems was introduced and studied in [AFL+03]. In Section 3.4, we consider this variant under the assumption that some vertices of the graph are pre-colored.

1.3 Basic Definitions

We cover some basic definitions and notations here. We will define others when necessary. For definitions which are not covered here refer to any graph theory book.

A graph $G = (V(G), E(G))$ consists of a vertex set $V(G)$ and edge set $E(G)$. Let $n = |V(G)|$ denote the order of G. In a graph G, the degree of a vertex v is the number of vertices adjacent to v, denoted by $d_G(v)$. The minimum and maximum degree of a graph are denoted by $\delta(G)$ and $\Delta(G)$ respectively. A vertex v is an isolated vertex if and only if $d_G(v) = 0$. A graph is connected if for every pair of vertices u and v there is a $u - v$ path in the graph. If G is connected, then the distance between two vertices u and v is the minimum length of a $u - v$ path in G, denoted by $d_G(u, v)$.

Let $N_G(v)$ denote the set of neighbours of a vertex $v \in V(G)$, and let $N_G[v] = N_G(v) \cup \{v\}$ be the closed neighbourhood of v in G. Let $d_G[v] = |N_G[v]| = d_G(v) + 1$.

Chapter 2

Known Results

2.1 Domination

A *dominating set* D is a set of vertices such that each vertex of G is either in D or has at least one neighbour in D. The minimum cardinality of such a set is called the *domination number* of G, $\gamma(G)$. In Figure 2.1, filled vertices form a minimum size dominating set in the Petersen line graph. Therefore, $\gamma(L(P)) = 3$.

The problem of determining the size of a minimum dominating set is NP-complete [GJ79]. In fact, the problem remains NP-complete even when restricted to certain classes of graphs such as bipartite graphs and chordal graphs. However, there are interesting classes of graphs such as trees, interval graphs, and cographs for which $\gamma(G)$ can be computed in polynomial time.

In the following, we will concentrate on bounds on the domination number $\gamma(G)$ in terms of order, maximum, and minimum degree of G, all of which have been studied widely. It can be seen directly from the definition that $1 \leq \gamma(G) \leq n$. The following examples show that this bound is sharp. Let G be a graph with $\Delta(G) = n - 1$, then the vertex of maximum degree dominates all other vertices in G and therefore $\gamma(G) = 1$. For the lower bound, let G be an edgeless graph, then the dominating set must contain all the vertices, and $\gamma(G) = n$.

By the following trivial argument, better bounds on $\gamma(G)$ in terms of order and

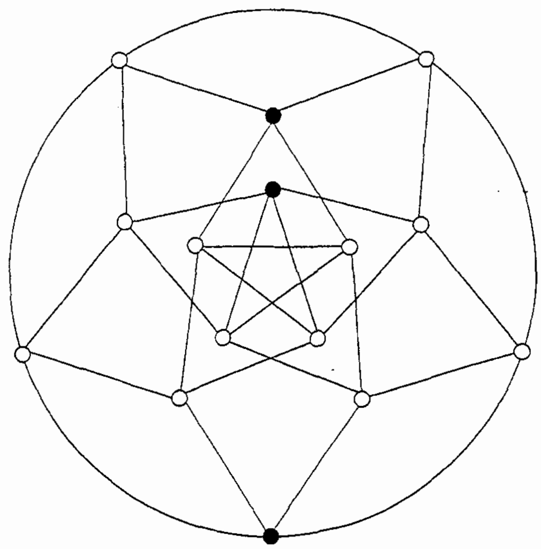

Figure 2.1: A minimum dominating set in $L(P)$.

degrees of vertices of G can be obtained. Let v be a vertex of maximum degree in G. Since v dominates itself and all vertices in its neighbourhood, $\Delta(G)+1$ vertices are dominated by v and the trivial upper bound follows. For the lower bound, since each vertex can dominate at most $\Delta(G)$ other vertices and itself, the lower bound follows.

Theorem 2.1 *[WAS79] [Ber62] For any graph G,*

$$\left\lceil \frac{n}{1+\Delta(G)} \right\rceil \leq \gamma(G) \leq n - \Delta(G).$$

■

There is a chain of upper bounds on $\gamma(G)$ in terms of order of G obtained by restricting $\delta(G)$. Ore gave the first result in this direction for graphs without isolated vertices.

Theorem 2.2 *[Ore62] For any graph G with $\delta(G) \geq 1$,*

$$\gamma(G) \leq \frac{n}{2}.$$

■

McCuaig and Shepherd made an improvement on the upper bound for the connected graphs with $\delta(G) \geq 2$, unless G is one of seven exceptional graphs shown in Figure 2.2.

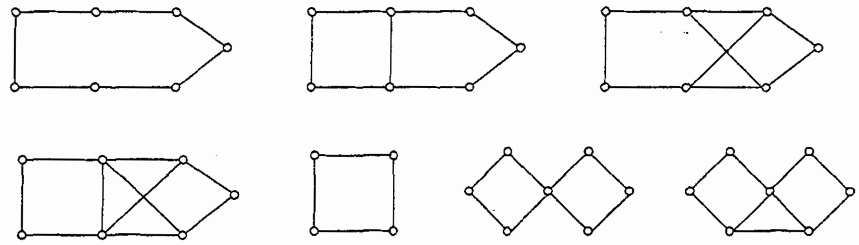

Figure 2.2: The family graph A

Theorem 2.3 *[MS89] If G is a connected graph with $\delta(G) \geq 2$ and $G \notin A$, then*

$$\gamma(G) \leq \frac{2n}{5}.$$

■

Reed made another improvement on the upper bound on $\gamma(G)$ for the connected graphs with $\delta(G) \geq 3$.

Theorem 2.4 *[Ree96] If G is a connected graph with $\delta(G) \geq 3$, then*

$$\gamma(G) \leq \frac{3n}{8}.$$

■

Based on these results, the following conjecture has been proposed in [HHS98].

Conjecture 2.5 *[HHS98] For any graph G with $\delta(G) \geq k$,*

$$\gamma(G) \leq \frac{kn}{3k-1}.$$

■

Caro and Roditty proved the conjecture for $\delta(G) \geq 7$. In fact, they obtained a better bound than in Conjecture 2.5.

Theorem 2.6 *[CR85] [CR90] For any graph with $\delta(G) \geq 7$,*

$$\gamma(G) \leq n\left[1 - \delta(G)\left(\frac{1}{\delta(G)+1}\right)^{1+\frac{1}{\delta(G)}}\right].$$

■

The Conjecture 2.5 remains open for $4 \leq k \leq 6$. However, the conjecture is proved for 4-regular graphs.

Theorem 2.7 *[LS04] If G is a 4-regular graph of order n, then*

$$\gamma(G) \leq \frac{4n}{11}.$$

■

With probabilistic methods, Alon and Spencer gave the following general bound.

Theorem 2.8 *[AS00] For any graph G with $\delta(G) \geq 1$,*

$$\gamma(G) \leq n\frac{1+\ln(\delta(G)+1)}{\delta(G)+1}.$$

■

2.2 Varieties of Domination

2.2.1 Factor and Global Domination

In this section we will discuss domination extended in a natural way, where a set of vertices is a dominating set in several edge-disjoint spanning subgraphs rather than just in the original graph. In order to be able to explain these concepts, we need some definitions. A *factor* of G is a spanning subgraph of G. A *k-factoring* of G is a set of

k factors $f = \{G_1, G_2, \ldots, G_k\}$, whose union is G. Figure 2.3 and Figure 2.4 show a 2-factoring of C_3P_5. A *factor dominating set* with respect to f is a set of vertices D which is a dominating set in each factor G_i, for $1 \leq i \leq k$. The minimum cardinality of a factor dominating set with respect to f is called the *factor domination number* $\gamma(G, f)$. In Figure 2.5, filled vertices form a minimum size factor dominating set for the given 2-factoring. Hence, $\gamma(C_3P_5, f) = 7$ when f is as shown.

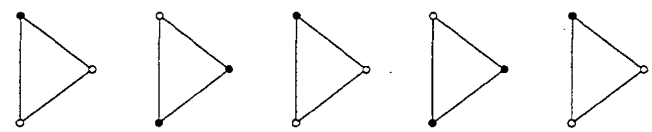

Figure 2.3: G_1.

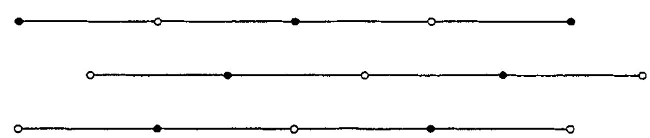

Figure 2.4: G_2.

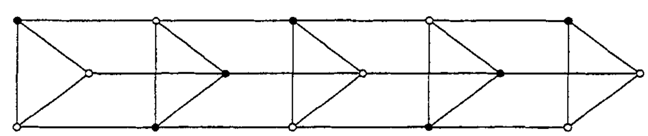

Figure 2.5: A minimum factor dominating set for a given 2-factoring of C_3P_5.

Let $\delta_{min} = \min\{d_{G_i}(v) : v \in V(G), 1 \leq i \leq k\}$ be the smallest minimum degree among all factors $G_1, \ldots, G_k$ and let $\Delta_{max} = \max\{d_{G_i}(v) : v \in V(G), 1 \leq i \leq k\}$ be the largest maximum degree over all factors $G_1, \ldots, G_k$.

Factor domination has several interesting applications in many network communication problem such as the following : How can one send a message from a subset of nodes of a network and have it received after one hop by all other nodes using only links of some private subnetwork for security or redundancy reasons. To model this problem, let the communication network be represented by a graph G where vertices of G correspond to nodes of the network and edges correspond to links joining nodes which can communicate directly, and finally k edge-disjoint factors of G represent k

private subnetworks. Therefore, the factor domination number represents the minimum number of nodes needed to send a message from, such that all other nodes receive the message in each subnetwork independently in one hop.

The concept of factor domination was first introduced by Brigham and Dutton [BD90] and Sampathkumar [Sam89] independently. Carrington and Brigham [CB92] have shown that the factor domination decision problem is NP-complete even when the factors are greatly simplified (for example, when each factor is a path). Brigham and Dutton presented several bounds on factor domination number in terms of degrees of G and other graphical invariants. Here we will only concentrate on bounds using the degrees of G. It can be observed directly from the definitions that,

$$\max_{1\leq i\leq k}\gamma(G_i)\leq\gamma(G,f)\leq\sum_{i=1}^{k}\gamma(G_i). \tag{2.1}$$

As we have seen in the previous section, there are numerous bounds on ordinary domination number in terms of order and minimum degree of G. In contrast to this, for a general graph G only the following trivial bound on factor domination number in terms of order and minimum degree over all factors was given by Brigham and Dutton [BD90]. For any k-factoring of f of G we have;

$$\gamma(G,f)\leq n-\delta_{min}. \tag{2.2}$$

This follows easily. Indeed, let D be a set of $n-\delta_{min}$ vertices in a graph G and let v be an arbitrary vertex which is not in D. Obviously vertex v has at least δ_{min} neighbours in each factor but since there are only $\delta_{min}-1$ vertices different from v which are not in D, v has at least one neighbour in D in any factor of G. Therefore, D is a factor dominating set in G with respect to f.

They also proved a lower bound of $\frac{nk}{\Delta(G)+k}$ which again follows easily. Indeed, let D be a factor dominating set. Since each vertex not in D has at least k edges to D (one edge in each factor to guarantee its domination), there are $k(n-\gamma(G,f))$ such edges and so $k(n-\gamma(G,f))\leq\sum_{v\in D}d_G(v)\leq\Delta(G)\gamma(G,f)$, and the bound follows. Hence, they obtained the following fundamental bounds for general graphs.

Theorem 2.9 *[BD90] For any k-factoring f of a graph G of order n,*

$$\frac{nk}{\Delta(G)+k} \leq \gamma(G,f) \leq n - \delta_{min}.$$

■

It is obvious that in the graphs with $\Delta(G) < k$, each vertex is isolated in at least one factor, and hence it has to be in D; so $\gamma(G, f) = n$. Brigham and Dutton have restricted their attention to the class of graphs with maximum degree at least k and they obtained some new bounds on $\gamma(G, f)$.

Theorem 2.10 *[BD90] For any k-factoring f of a graph G, if* $\Delta(G) \geq k$, *then* $k \leq \gamma(G, f) \leq \gamma(G) + k + 2$; *otherwise* $\gamma(G, f) = n$. ■

Next, we will continue with this approach to factor domination problem by restricting k, the number of factors. Brigham and Dutton [BD90], and also Sampathkumar [Sam89] independently introduced the concept of *global domination* which is the case of factor domination of K_n with a 2-factoring. In other words, a *global dominating -set* is a dominating set in both G and $\bar{G}$. The minimum cardinality of a global dominating set is called *global domination number* $\gamma_g(G)$. There are many bounds on global domination number $\gamma_g(G)$ in terms of many graphical invariants, beside those obtained from $\gamma(K_n, f)$ by setting $k = 2$.

Dankelmann and Laskar presented the following corollary in [DL03].

Corollary 2.11 *[DL03] Let* $f = \{G_1, G_2\}$ *be a 2-factoring of the complete graph* K_n, *if* $\delta_{min} \geq 2$, *then*

$$\gamma(K_n, f) \leq n\frac{\ln(\delta_{min}+1) + \ln 2 + 1}{\delta_{min}+1}.$$

■

They also conjectured the following upper bound when $\delta_{min} \geq 2$.

Conjecture 2.12 *[DL03] Let* $f = \{G_1, G_2\}$ *be a 2-factoring of the complete graph* K_n. *If* $\delta_{min} \geq 2$, *then*

$$\gamma(G, f) \leq \frac{3n}{5}.$$

■

In the following we include some other bounds on $\gamma_g(G)$ in terms of order, minimum and maximum degree, and ordinary domination number of G. After considering $\gamma_g(G)$ for some classes of graphs, Brigham and Dutton, noticed that if G is disconnected, then any dominating set must contain at least one vertex from each component and such a set clearly dominates $\bar{G}$. Hence,

Theorem 2.13 *[BD90] If either G or $\bar{G}$ is disconnected, then*

$$\gamma_g(G) = \max\{\gamma(G), \gamma(\bar{G})\}.$$

■

We conclude this section by a strengthening the result of Theorem 2.9 which states that $\gamma_g(G) \leq n - \min\{\delta(G), \delta(\bar{G})\}$.

Theorem 2.14 *[BD90] For any graph G either $\gamma_g(G) = \max\{\gamma(G), \gamma(\bar{G})\}$ or $\gamma_g(G) \leq \min\{\Delta(G), \Delta(\bar{G})\} + 1$.* ■

Theorem 2.15 *[BD90] For any graph G if $\delta(G) = \delta(\bar{G}) \leq 2$, then $\gamma_g(G) \leq \delta(G) + 2$, otherwise $\gamma_g(G) \leq \max\{\delta(G), \delta(\bar{G})\} + 1$.* ■

2.2.2 Distance Domination

The concept of domination can be extended into distance version which is more applicable in practical problems. For example consider the communication network problem. Here, a transmitting group is a subset of those cities that are able to transmit messages to every city in the network, via communication links, by at most l hops.

An *l-dominating set* is a set D_l such that every vertex in $V - D_l$ is within distance l from at least one vertex in D_l. The minimum cardinality of a distance dominating set is called *l-domination number* $\gamma_l(G)$. In Figure 2.2.2, filled vertices form a 3-dominating set of minimum size in P_5P_7. Hence, $\gamma_3(P_5P_7) = 3$.

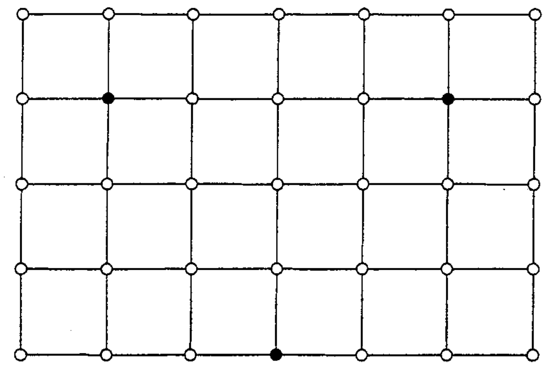

Figure 2.6: A minimum 3-dominating set in P_5P_7.

The problem of finding l-dominating sets of relatively small sizes is important in a variety of contexts such as problems of placement of the minimum number of objects (hospitals, police,etc) within desired distance of a given population.

The concept of distance domination was first introduced by Meir and Moon [MM75]. The problem of finding $\gamma_l(G)$ appears to be very difficult and only few results are known. In fact, this problem is NP-complete as shown in [CN84]. The restriction of the problem to special classes of graphs such as bipartite or chordal graphs of diameter $2r + 1$, has been studied and the problem remains NP-complete [CN84]. This motivated research on achieving good bounds for distance domination.

It is obvious that for any graph G, $\gamma_l(G) \leq \gamma(G)$, and therefore all upper bounds on ordinary domination number can be applied to the l-domination number. Thus, if G is a connected graph with $\delta(G) \geq 2$, and is not one of the seven exceptional graphs mentioned in previous section, then $\gamma_l(G) \leq \frac{2n}{5}$. The case $l = 2$ was considered in [SSE02], where it has been shown that for the seven exceptional graphs, $\gamma_2(G) \leq \frac{2n}{5}$ as well. Hence,

Theorem 2.16 *[SSE02] If G is a connected graph and $\delta(G) \geq 2$, then*

$$\gamma_2(G) \leq \frac{2n}{5}.$$

■

A stronger upper bound on $\gamma_2(G)$ is obtained in [SSE02].

Theorem 2.17 *[SSE02] If each component of a graph G contains at least 3 vertices, then*

$$\gamma_2(G) \leq \lfloor \frac{n}{3} \rfloor.$$

■

From the results on ordinary domination it can be concluded that, for any graph with no isolated vertex, $\gamma_l(G) \leq \frac{n}{2}$. This bound can be improved for 2-domination number to a bound which might be even better than Theorem 2.17.

Theorem 2.18 *[SSE02] If a graph G has no isolated vertex, then*

$$\gamma_2(G) \leq \frac{n - \Delta(G) + 1}{2}.$$

■

In the rest of this section we discuss an algorithmic approach to this problem. The most obvious algorithm to find a small l-dominating set of G is a greedy algorithm: At each iteration, choose a vertex v of current maximum degree in the current graph and put it into set D_l. Then remove v and all vertices which are l-dominated by v. Repeat this iteration until the degree of each vertex in the remaining graph is zero. Toward the end of the algorithm there are many vertices which are not dominated with degree zero and they must be moved into D_l. Therefore, this algorithm might have a poor performance.

Obviously for any connected graph of order $n < l+1$, any vertex of G l-dominates every vertex of G and so $\gamma_l(G) = 1$. Thus, in the following theorem we may assume $n \geq l+1$.

Theorem 2.19 *[HOS91] If G is a connected graph of order $n \geq l+1$, $l \geq 1$, then*

$$\gamma_l(G) \leq \frac{n}{l+1}.$$

■

The algorithmic proof of this theorem suggests the following algorithm which is in fact presented in [HOS96]. This algorithm finds an l-dominating set of size at most $\lfloor \frac{n}{l+1} \rfloor$ in a connected graph of order $n \geq l+1$.

Algorithm 2.20 *[HOS96] Let $l \geq 1$ be an integer and G a connected graph of order $n \geq l+1$.*

1. Find a spanning tree T of G. Set $D_l \leftarrow \emptyset$.

2. If radius of T is at most l, then let v be a central vertex of T, output $D_l \cup \{v\}$ and stop. Otherwise continue.

3. Let $d = diam(T)$ and find a path $u_0, u_1, \ldots, u_d$ of length d in T. Set $D_l \leftarrow D_l \cup \{u_l\}$ and let T be the component containing u_{l+1} in $T - u_l u_{l+1}$ and return to step 2. ■

Another argument for $\lfloor \frac{n}{l+1} \rfloor$ as an upper bound is presented in [KP98] which follows. Let T be a rooted spanning tree of G and let $D_1, \ldots, D_{l+1}$ be a partition of $V(G)$ such that for $0 \leq k \leq l$, each vertex in D_{k+1} is x_k edges away from the root in T and $x_k \mod (l+1) = k$. This partition can be constructed by traversing T breath-first from the root and assigning every new layer of vertices circularly to the sets $D_1, \ldots, D_{l+1}$. It can be shown that each of these sets is an l-dominating set. Since the algorithm partitions $V(G)$ and $n \geq l+1$, there is at least one partition which has no more than $\lfloor \frac{n}{l+1} \rfloor$ vertices.

Penso and Barbosa [PB04] considered the problem of finding an l-dominating set in G having no more than $\lfloor \frac{n}{l+1} \rfloor$ vertices by means of synchronous distributed computation on G in $O(l \log^* n)$ time. The algorithm in the first stage partitions G into trees of a rooted spanning forest, where each tree has at least $l+1$ vertices and $O(l)$ height. The second stage approaches the tree as described in the argument above and partitions its vertices.

2.2.3 Connected Domination

Connected dominating set is a dominating set which induces a connected subgraph. Since each dominating set has at least one vertex in each component of G, only connected graphs have a connected dominating set. Therefore, in this section we may assume all graphs are connected. The minimum cardinality of a connected dominating set is called *connected domination number* $\gamma_c(G)$. In Figure 2.7, filled vertices form a connected dominating set of minimum size and therefore, $\gamma_c(G) = 3$.

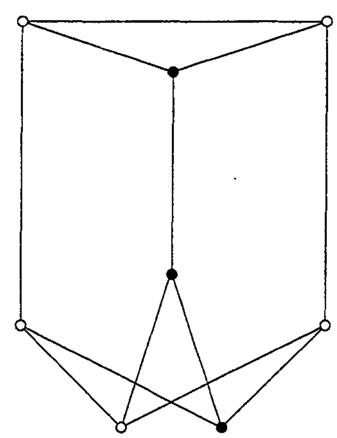

Figure 2.7: A minimum connected dominating set.

A direct application of connected domination is again in the computer networks where a connected dominating set serves as a communication backbone in network. For example in a cellphone network.

The concept of connected domination was introduced by Sampathkumar and Walikar [SW79] and they showed that if H is a connected spanning subgraph of G, then $\gamma_c(G) \leq \gamma_c(H)$, since every connected dominating set of H is also a connected dominating set of G.

It has been shown that the problem of deciding wether a connected dominating set of size M exist such that $\gamma_c(G) \geq M$ is NP-complete in [GJ79]. This problem is equivalent to the problem of finding a spanning tree that maximizes the number of leaves, since a set is a connected dominating set if and only if its complement is a set of leaves of a spanning tree. Considering this, Hedetniemi and Laskar [HL84] showed that for any graph of order $n \geq 3$, $\gamma_c(G) \leq n - 2$, since any tree T has at least 2 leaves.

Kleiman and West [KW91] studied the number of leaves in spanning trees of a connected graph and they showed that if G is a connected graph with $\delta(G) \geq k$, then it has a spanning tree with at least $n - 3\lfloor\frac{n}{k+1}\rfloor + 2$ leaves. As discussed above, the complement of this set of leaves is a connected dominating set. Hence,

Theorem 2.21 *[HHS98] For any connected graph G with $\delta(G) \geq k$,*

$$\gamma_c(G) \leq 3\lfloor\frac{n}{n+k}\rfloor - 2.$$

■

In the previous section we showed that $\gamma(G) \leq n - \Delta(G)$. Hedetniemi and Laskar showed that the same upper bound remains valid for connected domination number.

Theorem 2.22 *[HL84] For any connected graph G,*

$$\gamma_c(G) \leq n - \Delta(G).$$

■

There is a nice relation between $\gamma(G)$ and $\gamma_c(G)$ that was observed by Duchet and Meyniel in [DM82]. As we discussed before the lower bound, $\gamma(G) \leq \gamma_c(G)$, is trivial and the upper bound, $\gamma_c(G) \leq 3\gamma(G) - 2$, follows since any dominating set in a connected graph can be turned into a connected dominating set by joining at most $\gamma(G) - 1$ induced components through at most $\gamma(G)$ paths, each of length at most 3. Each such path adds at most 2 new vertices into a connected dominating set. Hence,

Theorem 2.23 *[DM82] If G is a connected graph, then*

$$\gamma(G) \leq \gamma_c(G) \leq 3\gamma(G) - 2.$$

■

Caro et al. sharpened this result in the following theorem.

Theorem 2.24 *[CWY00] If G is a connected graph and D is a dominating set of G that induces a subgraph with t components, then $\gamma_c(G) \leq |D| + 2t - 2$. In particular,*

$$\gamma(G) \leq \gamma_c(G) \leq 3\gamma(G) - 2.$$

■

As we have seen in Theorem 2.8 Alon [AS00] showed that for any graph G with no isolated vertex, $\gamma(G) \leq n\frac{1+\ln(\delta(G)+1)}{\delta(G)+1}$. Caro et al. by more complicated argument showed that the bound holds for connected dominating sets.

Theorem 2.25 *[CWY00] For any connected graph G of order n,*

$$\gamma_c(G) \leq (1+o(1))n\frac{\ln\delta(G)}{\delta(G)}.$$

■

An r-connected graph is a graph G, such that by deleting any $r-1$ vertices it remains connected. An *r-connected dominating set* is a dominating set that induces an r-connected subgraph, denoted by $\gamma_{r-c}(G)$. Clearly, only r-connected graphs can have r-connected dominating sets. For any r-connected graph $\gamma_{r-c}(G) \geq 1$, since $V(G)$ is an r-connected dominating set. Directly from Theorem 2.25 can be seen that $\gamma_{r-c}(G) \leq (1+o(1))n\frac{\ln\delta(G)}{\delta(G)}$. Consequently, Caro and Yuster [CY03] proposed the following conjecture for r-connected graphs.

Conjecture 2.26 *[CY03] Let r be a fixed positive integer. If G is an r-connected graph of order n, then*

$$\gamma_{r-c}(G) \leq (1+o(1))n\frac{\ln\delta(G)}{\delta(G)}.$$

■

Theorem 2.25 proves the conjecture for $r=1$. Caro and Yuster proved their conjecture for $r=2$. The problem remains open for $r \geq 3$.

In [DMP⁺05] a distributed algorithm for removing all cycles of length at most $1+2\log n$ for any graph G such that the remaining graph stays connected is presented. Based on this algorithm the following result is obtained.

Theorem 2.27 *[DMP⁺05] There is a randomized distributed $O(\log^2 n)$-time algorithm for computing a $O(\log\Delta(G))$-approximation to the minimal connected dominating set.* ■

2.2.4 Dominating Cycles

A *dominating cycle* is a cycle in which every vertex is in the neighbourhood of a vertex on the cycle. The minimum cardinality of a dominating cycle is denoted by $\gamma_{cy}(G)$. In Figure 2.8, filled vertices form a dominating cycle of minimum size and hence, $\gamma_{cy}(G) = 3$.

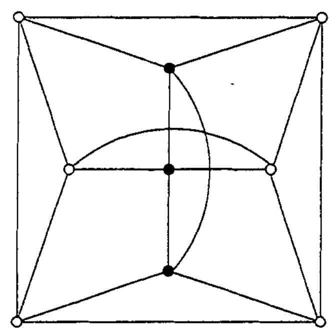

Figure 2.8: A minimum dominating cycle.

Lesniak and Williamson [LFW77] introduced the concept of dominating cycles. The problem is shown to be NP-complete even when restricted to planar graphs [PS81]. However, there are polynomial time algorithms for few classes of graphs such as circular-arc graphs.

It is obvious that not all graphs have dominating cycles. Therefore, it makes sense to study sufficient conditions for the existence of a dominating cycle. Let $\sigma_k(G)$ denote the minimum value of the degree sum of any k independent vertices of G. Bondy and Fan gave a sufficient condition for the existence of a dominating cycle.

Theorem 2.28 *[BF87] Let G be a k-connected graph, $k \geq 2$, If $\sigma_k(G) \geq n-2k$, then G has a dominating cycle.* ■

A resulting corollary of this theorem gives a condition on minimum degree of G.

Corollary 2.29 *[BF87] Let G be a k-connected graph, $k \geq 2$, if $\delta(G) \geq \frac{n-2k}{k+1}$, then G has a dominating cycle.* ■

Bondy gives the following sufficient condition for each longest cycle of a 2-connected graph to be a dominating cycle.

Theorem 2.30 *[Bon81] Let G be a 2-connected graph of order $n \geq 3$ with $\sigma_3(G) \geq n+2$, then each longest cycle of G is a dominating cycle.* ■

Let $\kappa(G)$ denote the connectivity of the graph G. The following condition for a 3-connected graph is given in [STW01].

Theorem 2.31 *[STW01] Let G be a 3-connected graph of order $n \geq 3$ with $\sigma_4(G) \geq n + 2\kappa(G)$, then G contains a longest cycle which is a dominating cycle.* ■

A stronger conclusion is shown in [LLT05].

Theorem 2.32 *[LLT05] Let G be a 3-connected graph of order $n \geq 3$ with $\sigma_4(G) \geq n + 2\kappa(G)$, then each longest cycle of G is a dominating cycle.* ■

Theorem 2.33 *[LLT05] Let G be a 3-connected graph of order $n \geq 13$, if $\sigma_4(G) \geq \frac{4n}{3} + \frac{5}{3}$, then each longest cycle of G is a dominating cycle.* ■

2.2.5 Other Varieties of Domination

Total domination

Total dominating set is a set of vertices such that each vertex $v \in V$ is in open neighbourhood of a vertex in the set. Note that in total domination vertex v does not dominate itself and so it is required that there be no isolated vertex. The minimum cardinality of a total dominating set is called the *total domination number* $\gamma_t(G)$. The decision problem to determine the total domination number of a graph is known to be NP-complete. The upper bound on the total domination number in terms of order of the graph has been studied continuously and we present here a chain of these results. Cockayne et al. gave the first bound in this chain.

Theorem 2.34 *[CDH80] If G is a connected graph of order $n \geq 3$, then*

$$\gamma_t(G) \leq \frac{2n}{3}.$$

■

Henning et al. improved the bound by excluding some classes of graphs.

Theorem 2.35 *[Hen00] If G is a connected graph of order n with $\delta(G) \geq 2$ and $G \notin \{C_3, C_5, C_6, C_{10}\}$, then*

$$\gamma_t(G) \leq \frac{4n}{7}.$$

■

Favaron et al made another improvement for $\delta(G) \geq 3$.

Theorem 2.36 *[FHMP00] If G is a connected graph of order n with $\delta(G) \geq 3$, then*

$$\gamma_t(G) \leq \frac{7n}{13}.$$

■

Finally, the best known upper bound on $\gamma_t(G)$ in terms of order is represented in the following theorem.

Theorem 2.37 *[AEMF+04] If G is a graph of order $n \geq 3$, then*

$$\gamma_t(G) \leq \frac{n}{2}.$$

■

Paired domination

Paired dominating set is a dominating set whose induced subgraph has a perfect matching. From the definition it requires that there be no isolated vertices. Every paired dominating set is a total dominating set. The *paired domination number* is the minimum cardinality of a paired dominating set and denoted by $\gamma_{pr}(G)$. The paired dominating set problem is also shown to be NP-complete [HS98]. Haynes and Slater presented the following sharp bounds on $\gamma_{pr}(G)$.

Theorem 2.38 *[HS98] If a graph G has no isolated vertices, then*

$$2 \leq \gamma_{pr}(G) \leq n.$$

■

By conditioning n and $\delta(G)$, a better upper bound was obtained.

Theorem 2.39 *[HS98] If G is connected and $n \geq 6$ and $\delta(G) \geq 2$, then*

$$\gamma_{pr}(G) \leq \frac{2n}{3}.$$

■

k-domination

A *k-dominating set* is a set of vertices D such that each vertex in $V(G) - D$ is dominated by at least k vertices in D for a fixed positive integer k. The minimum cardinality of a k-dominating set is called *k-domination number* $\gamma_k(G)$. Fink and Jacobson presented the following upper bound in terms of order, maximum degree and k.

Theorem 2.40 *[FJ85] For any graph G,*

$$\gamma_k(G) \geq \frac{kn}{\Delta(G) + k}.$$

■

Cockayne et al. obtained the following upper bound in terms of order and k.

Theorem 2.41 *[CGS85] For any graph G if $\delta \geq k$, then*

$$\gamma_k(G) \leq \frac{kn}{k+1}.$$

■

k-tuple domination

k-tuple domination is a variation of k-domination which was introduced by Harary and Haynes. A *k-tuple dominating set* is a set of vertices D such that each vertex in V is dominated by at least k vertices in D. Therefore $k \leq \delta(G) + 1$. The *k-tuple domination number* is the minimum cardinality of a k-tuple dominating set and denoted by $\gamma_{\times k}(G)$. It is obvious that a graph G has a k-tuple dominating set if and only if $\delta \geq k - 1$.

The main application of k-tuple domination in network is for fault tolerance or mobility. Where, a node can use a service only if it is replicated on it or its neighbourhood. So, each node need to have k copies of the service available in its closed neighbourhood.

2.3 Domatic Number and Variations

An interesting variant of domination problem is to ask how many dominating sets one can pack into a given graph G. Such type of packing questions are common for many problems and we will survey some results in this direction, mostly related to domination parameters discussed in previous sections. The main question is how to partition the vertex set of a graph into maximum number of disjoint dominating sets. The word "domatic" is created from two words "dominating" and "chromatic" since the definition of it is related to both domination and colouring concepts.

A *domatic partition* is a partition of $V(G)$ such that each class of the partition is a dominating set in G. The maximum number of classes in a domatic partition is called the *domatic number* of G, denoted by $D(G)$. In any graph G, $\{V(G)\}$ is a domatic partition and therefore, $D(G) \geq 1$. This bound is sharp since for any graph G, with an isolated vertex, $D(G) = 1$. It also follows from the definitions of domination number and domatic number of a graph that $\gamma(G) \cdot D(G) \leq n$, therefore, $D(G) \leq \frac{n}{\gamma(G)}$.

The concept of domatic partition arises in various areas. In particular, in the problem of communication networks that we discussed in previous sections, domatic number of a graph represents the maximum number of disjoint transmitting groups. Another application of domatic number is related to the task of distributing resources in a computer network in the most economic way. Suppose, for example, resources are to be distributed in a computer network in such a way that expensive services are quickly accessible in the neighbourhood of each vertex. If every vertex can serve a single resource only, then the maximum number of resources that can be supported equals the domatic number in the graph representing the network.

The concept of domatic number was introduced by Cockayne and Hedetniemi in [CH77] and they presented the following trivial bound on $D(G)$.

Theorem 2.42 *[CH77] For any graph G,*

$$D(G) \leq \delta(G) + 1.$$

■

This follows since if $D(G) > \delta(G) + 1$, there is a class in such a partition which does not dominate at least one vertex, in particular a vertex of minimum degree, which is a contradiction. Motivated by this bound, Zelinka [Zel83] was interested in finding a lower bound on $D(G)$ in terms of $\delta(G)$. However he showed that such a bound does not exist.

Theorem 2.43 *[Zel83] For each non-zero cardinal number p there exist a graph G in which each vertex has degree at least p and whose domatic number is 2. If p is finite, then there exist both a finite graph and an infinite graph with this property.* ■

Nevertheless, he achieved a bound on $D(G)$ in terms of order and minimum degree of G. For this, consider a graph G and its complement $\bar{G}$. If a vertex has degree r in G it has degree $n-r-1$ in $\bar{G}$. Therefore, the maximum degree in $\bar{G}$ is $n-\delta(G)-1$. Now, let D be a subset of $V(G)$ with $n-\delta(G)$ vertices. Since the maximum degree in $\bar{G}$ is $n-\delta(G)-1$, each vertex v which is not in D is adjacent to at most $n-\delta(G)-1$ vertices in D and v is not adjacent to at least one vertex in D in $\bar{G}$. This implies that v is adjacent to at least one vertex in D in G which guarantees its domination. Thus, any set of $n - \delta(G)$ vertices is a dominating set in G. Now considering any partition of $V(G)$ into classes of $n - \delta(G)$ vertices the following bound on $D(G)$ is obtained.

Theorem 2.44 *[Zel83] For any graph G of order n,*

$$D(G) \geq \left\lfloor \frac{n}{n - \delta(G)} \right\rfloor.$$

■

Since all previous results are only bounds on $D(G)$, it would be very interesting and important to have an algorithm that actually constructs such a partition.

The problem of determining the domatic number is known to be NP-complete [HH98]. One of the best (exponential) algorithms for determining the domatic number is provided in [FGPS05]. It is based on an (exponential) bound on the number of all minimal dominating sets: in any n-vertex graph there are at most 1.7697^n minimal dominating sets. Using this bound Fedor et al developed an $O(1.7697^n)$ algorithm that finds all such sets. Finally, based on this algorithm, they derived an $O(2.8805^n)$ algorithm which finds the domatic number of any n-vertex graph. Hence we have the following theorem.

Theorem 2.45 *[FGPS05] The domatic number of a graph G of order n can be computed in time $O(2.8805^n)$.* ■

Although this is an exact algorithm it is an exponential time algorithm which is not very practical. Therefore, there has been some effort to achieve an approximation of $D(G)$.

Such an approximation is presented in [FHKS03]. A randomized algorithm is provided which finds a domatic partition of size $(1 - o(1))\frac{\delta(G)+1}{\ln n}$ using randomized assignment. Afterwards the assignment is derandomized using the method of conditional probabilities. Hence, using this method and based on Theorem 2.42 a $\frac{1-o(1)}{\ln n}$-approximation is obtained.

Theorem 2.46 *[FHKS03] Any graph admits a polynomial time constructible domatic partition of size*

$$\left(1 - o\left(\frac{\ln \ln n}{\ln n}\right)\right)\frac{\delta(G)+1}{\ln n}.$$

■

Fedor et al obtained another algorithm that finds a domatic partition of size at least $\frac{\delta(G)}{c \ln \Delta(G)}$, for some constant c, by application of the Lovász Local Lemma for the randomized assignment algorithm.

Theorem 2.47 *[FHKS03] Any graph admits a domatic partition of size*

$$\left(\frac{1}{3} - o(1)\right)\frac{\delta(G)}{\ln \Delta(G)}$$

where $o(1) \to 0$ *as* $\Delta(G) \to \infty$. ■

Next we will consider two variations of the domatic partition, in particular, l-domatic partition and factor domatic partition. We also add a new variant here which is a combination of distance and factor domination, factor l-domatic partition. Note that our results in next the section concern these extensions. In particular, we extend the previous two results to factor domatic partitions.

An l-*domatic* partition is a partition of $V(G)$ where each class is an l-dominating set in G, $l \geq 1$. The maximum number of classes in such a partition is called l-*domatic* number, $D_l(G)$. This concept was first introduced by Borowiecki and Kuzak [BK80]. Zelinka obtained several results which compare the values of $D_l(G)$ for different values of l. In the following we include some of these results.

From the definition of an l-dominating set it is clear that an l-dominating set in G is also a k-dominating set in G, when $l < k$. Hence,

Theorem 2.48 *[Zel83] Let l and k be positive integers and $l < k$. For any graph G,*

$$D_l(G) \leq D_k(G).$$

■

A similar behavior is observed for the spanning subgraph of G since the distance of any two vertices in G is less than or equal to their distance in any spanning subgraph of G.

Theorem 2.49 *[Zel83] Let G be a graph and H be a spanning subgraph of G, then*

$$D_l(G) \geq D_l(H).$$

■

A more complicated argument gives the following lower bound.

Theorem 2.50 *[Zel83] Let G be a connected graph of order n, then*

$$D_l(G) \geq \min\{n, l+1\}.$$

■

The next variation that we discuss here is factor domatic partition. A *factor domatic partition* is a partition of $V(G)$ where each class is a factor dominating set for a given k-factoring f of G. The maximum number of classes in a factor domatic partition is called *factor domatic number* $D_f(G)$, where $f = \{G_1, G_2, \ldots, G_k\}$ is a k-factoring of G.

Determining $D_f(G)$ for general graphs seems hard, therefore various restrictions have been considered. The restriction into special class of graphs, namely complete graphs and complete bipartite graphs have been studied by Haynes and Henning in [HH00]. They presented some upper bounds on the sum and product of the domatic numbers of factors of complete graphs and complete bipartite graphs.

Theorem 2.51 *[HH00] Let $f = \{G_1, G_2, \ldots, G_k\}$ be a k-factoring of K_n, $n \geq 3$. Then*

$$k \leq \sum_{i=1}^{k} D(G_i) \leq n + k - 1.$$

■

For any 2-factoring of a complete graph K_n, Cockayne and Hedetniemi gave the following bound on the sum of domatic numbers, which is also a direct corollary of Theorem 2.51 by setting $k = 2$.

Theorem 2.52 *[CH77] For any graph G and its complement $\bar{G}$,*

$$D(G) + D(\bar{G}) \leq n + 1.$$

■

Also the following upper bound on the product of the domatic numbers is given in [DHH99].

Theorem 2.53 *[DHH99] For any graph G and its complement $\bar{G}$,*

$$D(G) \cdot D(\bar{G}) \leq \frac{n^2}{4}.$$

■

Consequently, Haynes and Henning [HH00] considered the simple case of 2-factoring for the complete bipartite graph $K_{s,s}$ and achieved the following bounds on the sum and product of domatic numbers of factors.

Theorem 2.54 *[HH00] Let $f = \{G_1, G_2\}$ be a 2-factoring of $K_{s,s}$, $s \geq 2$. Then*

$$D(G_1) + D(G_2) \leq s + 2.$$

■

Theorem 2.55 *[HH00] Let $f = \{G_1, G_2\}$ be a 2-factoring of $K_{s,s}$, $s \geq 2$. Then*

$$D(G_1) \cdot D(G_2) \leq \begin{cases} 2s & \text{for } s \leq 9, \\ \lfloor \frac{s}{2} \rfloor^2 & \text{for } s \geq 10. \end{cases}$$

and the bounds are sharp. ■

Haynes and Henning also studied the problem for the special case of 3-factoring of a complete graph K_n. The following bound on the sum of the domatic numbers of 3 factors of K_n is obtained by setting $k = 3$ in Theorem 2.51. Here, we include an example which shows that the following bound is sharp. Let $G_1 \cong 2K_{\frac{n}{2}}$ and $G_2 \cong \frac{n}{2}K_2$. Thus, G_3 is obtained from a complete bipartite graph $K_{\frac{n}{2},\frac{n}{2}}$ by removing the edges of 1-factor. Therefore

$$D(G_1) + D(G_2) + D(G_3) = \frac{n}{2} + 2 + \frac{n}{2} = n + 2.$$

Theorem 2.56 *[HH00] Let $f = \{G_1, G_2, G_3\}$ be a 3-factoring of K_n, $n \geq 3$. Then*

$$3 \leq D(G_1) + D(G_2) + D(G_3) \leq n + 2$$

and the bound is sharp. ■

They also presented a bound on the product of domatic numbers for a given 3-factoring of K_n with large n.

Theorem 2.57 *[HH00] Let $f = \{G_1, G_2, G_3\}$ be a 3-factoring of K_n. Let $n \geq 27$ be an odd integer with $n \notin \{29, 35, 37, 53\}$ or let $n \geq 42$ be an even integer with $n \notin \{44, 50, 52, 56\}$. Then*

$$D(G_1) \cdot D(G_2) \cdot D(G_3) \leq \lfloor \frac{n}{3} \rfloor^3.$$

■

The next variation is a combination of factor domatic partition and l-domatic partition, called factor l-domatic partition. A factor l-domatic partition is a partition of $V(G)$ where each class is both a factor dominating set and an l-dominating set. The maximum number of classes in such a partition is called factor l-domatic number. This variant has been studied in [AFL+03]. However, the authors assumed that the all factors are connected and not necessarily edge disjoint. Also, rather than finding the maximum number of factor l-dominating sets for a fixed l, they fixed the number of such sets and found the minimum l for that number of sets.

Formally, let $\alpha(t,k)$ denote the minimum distance l for the given integers t and k such that every k-factoring of every graph on at least t vertices admits a factor l-domatic partition with t classes. In the following, we include some bounds on $\alpha(t,k)$.

By considering spanning tree of each factor and using Hall's Theorem the following bound is obtained.

Theorem 2.58 *[AFL+03] For every $k \geq 2$ and $t \geq k$,*

$$\alpha(t,k) \leq \left\lceil \frac{3(kt-1)}{2} \right\rceil.$$

∎

Using a probabilistic argument, in particular, Lovász Local Lemma, they obtained a better bound.

Theorem 2.59 *[AFL+03] For every $k \geq 1$ and $t \geq 1$,*

$$\alpha(t,k) \leq O(t\log(kt)).$$

∎

A lower bound is also presented when $t \geq 2$ and $k \geq 4$.

Theorem 2.60 *[AFL+03] If $t \geq 2$ and $k \geq 4$, then*

$$\alpha(t,k) \geq \Omega(t\log k).$$

∎

These bounds are not very tight and tightening them remains an open problem.

Chapter 3

Our Results on Factor and Distance Domination

In this chapter we include our main results which are extensions of bounds given in Theorem 2.46 and Theorem 2.47 and their constructive algorithms. In particular, we bound the factor domatic number in Section 3.1 and factor l-domatic number in Section 3.2. In Section 3.3 we extend Alon's bound on $\gamma(G)$ given in Theorem 2.8 to a bound on the size of the factor dominating sets in a factor domatic partition. Finally, we consider the problem of minimizing $\alpha(t, k)$ as it will be discussed in Section 3.4 and consider the problem when some vertices have been precoloured. We find a condition on the precolouring such that the same upper bound as given in Theorem 2.58 holds for $\alpha(t, k)$.

3.1 Factor Domatic Partition

We defined factor dominating sets in Section 2.1 and factor domatic partition in Section 2.3. We mentioned some bounds on the sum and product of the domatic number of factors of a graph in a given k-factoring of G. In Theorem 2.46, Feige et al. [FHKS03] showed that any graph admits a polynomial time constructible domatic partition of size $\left\lfloor \left(1 - o(\frac{\ln \ln n}{\ln n})\right) \frac{\delta(G)+1}{\ln n} \right\rfloor$. In the following, we show a similar bound holds for factor domatic number, using the same probabilistic method. We use the same method to estimate a lower bound on factor domatic number for any given k-factoring of a graph G. Then we describe a deterministic algorithm to construct a factor domatic partition of this size.

Let $f = \{G_1, \ldots, G_k\}$ be a given k-factoring of G. Let $[h] = \{1, 2, \ldots, h\}$ be a set of colours. A random colouring of G is an assignment of colours to vertices such that the probability of assigning colour $c \in [h]$ to vertex $v \in V(G)$ is $\frac{1}{h}$. The factor domatic partition of G for a given k-factoring of G can now be reformulated as a colouring such that each colour class forms a dominating set in each factor. For a triple (v, c, i) where $v \in V(G)$, $c \in [h]$, and the index of factor G_i, $1 \leq i \leq k$, let $I_{v,c,i}$ be the following random indicator variable

$$I_{v,c,i} = \begin{cases} 1 & \text{if there is no vertex of colour } c \text{ in } N_{G_i}[v], \\ 0 & \text{otherwise.} \end{cases}$$

If $I_{v,c,i} = 1$ then we say v does not see colour c in $N_{G_i}[v]$. This means that the vertices with colour c do not form a dominating set in the factor G_i. For an event A, let $P[A]$ denote the probability of A, and $E[A]$ denote the expectation of A.

It can easily be seen that $D_f(G) \leq \delta_{min}+1$, because otherwise a vertex of minimum degree will not be dominated by at least one set in at least one factor. Therefore, the following theorem gives a $\frac{1-o(\frac{\ln\ln kn}{\ln kn})}{\ln kn}$-approximation of $D_f(G)$.

Theorem 3.1 *For any k-factoring f of a graph G,*

$$D_f(G) \geq \left\lfloor \left(1 - o(\frac{\ln\ln kn}{\ln kn})\right) \frac{\delta_{min}+1}{\ln kn} \right\rfloor.$$

Proof:

Let $f = \{G_1, \ldots, G_k\}$ be a k-factoring of G and let $h = \left\lfloor \frac{\delta_{min}+1}{\ln(kn \ln kn)} \right\rfloor$. Randomly colour each vertex with one of h colours. Now for any triple (v, c, i), $E[I_{v,c,i}] = P[I_{v,c,i} = 1]$ and $P[I_{v,c,i} = 1] = (1 - \frac{1}{h})^{d_{G_i}[v]}$. This follows since each colour c appears with probability $\frac{1}{h}$, so the colour c does not appear with probability $1 - \frac{1}{h}$, and there are $d_{G_i}[v]$ vertices in the closed neighbourhood of v. So the probability that none of these vertices has colour c is $(1-\frac{1}{h})^{d_{G_i}[v]}$. Using the fact that $\delta_{min} < d_{G_i}[v]$ we have

$$P[I_{v,c,i}=1] = (1-\frac{1}{h})^{d_{G_i}[v]} \leq (1-\frac{1}{h})^{\delta_{min}+1} \leq e^{\frac{-(\delta_{min}+1)}{h}}$$
$$\leq e^{-\ln(kn\ln kn)} \leq e^{\ln\frac{1}{kn\ln kn}} \leq \frac{1}{kn\ln kn}.$$

So, the expected total number of bad events $A_{v,c,i}$ is at most

$$\sum_{v\in V(G)}\sum_{c\in[h]}\sum_{i=1}^{k} P[I_{v,c,i}=1] = nhk\frac{1}{kn\ln kn} = \frac{h}{\ln kn}. \tag{3.1}$$

Hence, the expected number of colours that do not form factor dominating sets is at most $\frac{h}{\ln kn}$. Thus the expected number of colours which do form a factor dominating set is at least

$$h - \frac{h}{\ln kn} = (\delta_{min}+1)\left(\frac{1}{\ln(kn\ln kn)} - \frac{1}{\ln(kn\ln kn)\ln kn}\right)$$
$$= (\delta_{min}+1)\left(\frac{\ln kn - 1}{(\ln kn + \ln\ln kn)(\ln kn)}\right)$$
$$= \frac{\delta_{min}+1}{\ln kn}\left(1 - \frac{1+\ln\ln kn}{\ln(kn\ln kn)}\right).$$

Now, simply add vertices with colours that do not form a factor dominating set into a colour class that forms a factor dominating set. In this way we obtain a factor domatic partition of G for the given k-factoring f of G for which

$$D_f(G) \geq \left\lfloor \left(1 - o(\frac{\ln\ln kn}{\ln kn})\right)\frac{\delta_{min}+1}{\ln kn}\right\rfloor.$$

■

In Theorem 3.1 we only found a lower bound on $D_f(G)$, and it is important to find an algorithm which constructs a factor domatic partition of this size. In the following we present such an algorithm. Let $A_{v,c,i}$ be the event that there is no vertex of colour c in $N_{G_i}[v]$. Indeed, $I_{v,c,i}=1$ if and only if $A_{v,c,i}$ holds.

Theorem 3.2 *Any graph G with a given k-factoring f of G admits a polynomial time constructible domatic partition of size*

$$\left\lfloor \left(1 - o(\frac{\ln \ln kn}{\ln kn})\right) \frac{\delta_{min} + 1}{\ln kn} \right\rfloor.$$

■

In order to prove Theorem 3.2, we provide a constructive algorithm in the following.

Algorithm 3.3 *The Factor Domatic Partition Algorithm*

Input: *A graph G of order n with a k-factoring $f = \{G_1, \ldots, G_k\}$.*

Output: *A factor domatic partition of expected size given in Theorem 3.2 for the k-factoring f of G.*

1. Order vertices arbitrarily as $v_1, \ldots, v_n$ and process them in this order.

2. Colour v_1 randomly with one of the colours in $[h]$, where h is as given in Theorem 3.1.

3. Suppose $v_1, \ldots, v_j$ have been coloured with colours $c_1, \ldots, c_j$. Colour v_{j+1} as follows. Denote the conditional probability of the event $A_{v,c,i}$ given that vertices $v_1, \ldots, v_j$ have already been coloured with colours $c_1, \ldots, c_j$ by $P[A_{v,c,i}|c_1, \ldots, c_j]$. This probability is zero if there is a vertex v_z, $z \le j$ in $N_{G_i}[v]$ that has been coloured with colour c. Otherwise it is $(1 - \frac{1}{h})^m$, where m is the number of uncoloured neighbours of v in G_i, i.e. $m = |N_{G_i}[v] \cap \{v_{j+1}, \ldots, v_n\}|$. Formally,

$$P[A_{v,c,i}|c_1, \ldots, c_j] = \begin{cases} 0 & \text{if } \exists z \le j,\ v_z \in N_{G_i}[v] \text{ and } c_z = c, \\ (1 - \frac{1}{h})^m & \text{otherwise.} \end{cases}$$

Let $g(c_1, \ldots, c_j)$ denote the expected total number of bad events, i.e. for which $I_{v,c,i} = 1$, after colouring all vertices, given that $v_1, \ldots, v_j$ have been coloured with colours $c_1, \ldots, c_j$ respectively. So

$$g(c_1, \ldots, c_j) = \sum_{v \in V(G)} \sum_{c \in [h]} \sum_{i=1}^{k} P[A_{v,c,i} | c_1, \ldots, c_j].$$

Now colour vertex v_{j+1} such that $g(c_1, \ldots, c_{j+1}) \leq g(c_1, \ldots, c_j)$. ■

We need to show that such a colour for v_{j+1} exists.

Lemma 3.4 *Suppose $v_1, \ldots, v_j$ have been coloured with colours $c_1, \ldots, c_j$ respectively. We can always colour v_{j+1} with a colour $c_{j+1} \in [h]$ such that $g(c_1, \ldots, c_j, c_{j+1}) \leq g(c_1, \ldots, c_j)$.*

Proof:

The statement will obviously follow if we show that $g(c_1, \ldots, c_j)$ is a convex combination of $\{g(c_1, \ldots, c_j, z) : z \in [h]\}$ because then there is a colour c_{j+1} such that $g(c_1, \ldots, c_j, c_{j+1}) \leq g(c_1, \ldots, c_j)$. We have

$$\begin{aligned}
\sum_{z \in [h]} g(c_1, \ldots, c_j, z) &= \sum_{z \in [h]} \sum_{v \in V(G)} \sum_{c \in [h]} \sum_{i=1}^{k} P[A_{v,c,i} | c_1, \ldots, c_j, z] \\
&= \sum_{v \in V(G)} \sum_{c \in [h]} \sum_{i=1}^{k} \sum_{z \in [h]} P[A_{v,c,i} | c_1, \ldots, c_j, z] \\
&= \sum_{v \in V(G)} \sum_{c \in [h]} \sum_{i=1}^{k} h P[A_{v,c,i} | c_1, \ldots, c_j] \\
&= h g(c_1, \ldots, c_j)
\end{aligned}$$

Therefore, $g(c_1, \ldots, c_j) = \frac{1}{h} \sum_{z \in [h]} g(c_1, \ldots, c_j, z)$ and the lemma is proved.

■

In the following we use a different method of constructing the colouring that we used in Theorem 3.1. This method is a standard way to provide an algorithm for the Lovász Local Lemma, originally introduced by Beck [Bec91] and leads to a randomized algorithm. The Lovász Local Lemma only shows that an event holds with positive probability. It might be important to provide a polynomial time (at least randomized) algorithm for the corresponding problem. This method was successfully used by Fiege et al. in [FHKS03] to derandomize the probabilistic lower bound given in Theorem 2.47 on the domatic number, $D(G)$, of a graph G. We use this method to develop a randomized polynomial time algorithm that constructs a factor domatic partition of a given graph with a k-factoring. Following the general method, we first bound the expected size of such a partition using the Lovász Local Lemma. This is done in Lemma 3.6. Then we apply the method of Beck to construct such a partition with high probability in Theorem 3.7.

For the reader's convenience, we introduce the Lovász Local Lemma. If we have n mutually independent events where each occurs with probability $0 \le p \le 1$, then the probability that none of these events occurs is $(1-p)^n$. In the case when the n events have rare dependencies, the Lovász Local Lemma [EL75] shows that under some conditions no event holds with positive probability. The following lemma is the symmetric case of the Lovász Local Lemma.

Lemma 3.5 *[EL75] Let $A_1, \ldots, A_n$ be n events in a probability space. A graph $G = (V, E)$ on the set of vertices $\{1, \ldots, n\}$ corresponding to events $A_1, \ldots, A_n$ is called a dependency graph for the events $A_1, \ldots, A_n$ if for each i, $1 \le i \le n$, the event A_i is mutually independent of all events $\{A_j : (i,j) \notin E\}$. Suppose that $G = (V, E)$ is a dependency graph for the above events and suppose that the maximum degree in G is d. Suppose further that $P[A_i] \le p$ for all $1 \le i \le n$. If $ep(d+1) \le 1$, then*

$$P[\bigwedge_{i=1}^{n} \bar{A}_i] > 0,$$

where e is the basis of the natural logarithm. ■

Now, we are ready to prove our lower bound on $D_f(G)$.

Lemma 3.6 *For any k-factoring f of a graph G,*

$$D_f(G) \ge \left\lfloor \left(\frac{1}{3} - o(1)\right) \frac{\delta_{min} + 1}{\ln(k\Delta(G))} \right\rfloor.$$

Proof: Let $f = \{G_1, \ldots, G_k\}$ be a k-factoring of G and let $h = \left\lfloor \frac{\delta_{min}+1}{3\ln(3^{\frac{1}{3}}k^{\frac{1}{3}}\Delta(G))} \right\rfloor$. Randomly colour vertices with colours in $[h]$. For each triple of a vertex, colour, and index of a factor (v, c, i), let $A_{v,c,i}$ be the same event as defined in the proof of Theorem 3.1. The probability of the bad event $A_{v,c,i}$ is

$$P[A_{v,c,i}] = (1-\frac{1}{h})^{d_{G_i}[v]} \leq (1-\frac{1}{h})^{\delta_{min}+1} \leq e^{\frac{-(\delta_{min}+1)}{h}}$$
$$\leq e^{-3\ln(3^{\frac{1}{3}}k^{\frac{1}{3}}\Delta(G))} \leq \frac{1}{3k\Delta(G)^3}.$$

Each bad event $A_{v,c,i}$ corresponds to the vertex set $N_{G_i}[v]$. Therefore, the dependency translates into the intersection property for the corresponding vertex sets, and two events are dependent if and only if their corresponding vertex sets have a non-empty intersection. Thus, for each event $A_{v,c,i}$ there are at most

$$kh\Big[d_{G_i}[v] + d_{G_i}(v)(\Delta(G)-1)\Big]$$

dependent events. This is sum of the maximum number of vertices at distance two or less from v over all factors and colours. Since $d_{G_i}(v) \leq \Delta_{max}$ and

$$h \leq \Delta(G) - 1 \leq \Delta(G) - \frac{k\Delta(G)+1}{k(\Delta^2(G)+1)}$$

we have

$$kh\Big[d_{G_i}[v] + d_{G_i}(v)(\Delta(G)-1)\Big] \leq k\Delta(G) - \frac{k\Delta(G)+1}{k(\Delta^2(G)+1)}(\Delta^2(G)+1)$$
$$< k\Delta^3(G) - 1.$$

Hence, $d = k\Delta^3(G) - 1$ and $p = \frac{1}{3k\Delta^3(G)}$. Since we have

$$ep(d+1) = e\frac{1}{3k\Delta^3(G)}(k\Delta^3(G)) = \frac{e}{3} < 1,$$

by Lovász Local Lemma, the probability that no bad event holds is greater than zero, which guarantees a factor domatic partition of size at least $\left\lfloor \left(\frac{1}{3} - o(1)\right)\frac{\delta_{min}+1}{\ln(k\Delta(G))} \right\rfloor$.

■

Theorem 3.7 *For any graph G with a given k-factoring $f = \{G_1, \ldots, G_k\}$, there exists a polynomial time randomized algorithm that constructs a factor domatic partition of size*

$$\left\lfloor \left(\frac{1}{21} - o(1)\right) \frac{\delta_{min}}{\ln(k\Delta(G))} \right\rfloor$$

with probability at least $\frac{1}{4}$. ■

In the following we present an algorithm to construct a factor domatic partition of the size given in Theorem 3.7 with some loss in the constant that works with high probability. First we define three classes of vertices for a given partial colouring of some vertices. In the following definitions, $h = \left\lfloor \frac{\delta_{min}}{c \ln k\Delta(G)} \right\rfloor$ as given in Lemma 3.6, where c is a large enough constant ($c \geq 30$ will do). An undominated factor for vertex v is a factor in which v does not see all h colours in its closed neighbourhood. A *good* vertex is a vertex that has no undominated factor. A *dangerous* vertex is a vertex which has at least one undominated factor and in each such factor it has at least $\frac{1}{3}$ of its neighbours coloured. A *neutral* is a vertex that has at least one undominated factor and in each such factor it has less than $\frac{1}{3}$ of its neighbours coloured.

Phase One

1. Consider an arbitrary ordering of vertices, say $v_1, \ldots, v_n$, and process them in this order.

2. Randomly colour v_1 with one of h colours.

3. If the current vertex or one of its neighbours is dangerous, freeze it (do not colour it). Otherwise, randomly colour it with one of h colours.

After the above randomized assignment we will have two different partitions over vertices. The first partition is {Coloured, Frozen} and the second partition is {Good, Dangerous, Neutral}. A vertex which is both coloured and good has already been dominated in all factors; i.e. does not have an undominated factor, and we do not need to reconsider it. Remaining vertices will be considered in the second phase. In particular, we continue the algorithm with vertices that are either dangerous, neutral or frozen and call them *saved* vertices. Some of the saved vertices have already been coloured and we do not recolour them in future, since this may influence good vertices. From the third step of this phase it is obvious that a dangerous vertex cannot be coloured. Therefore, the only coloured vertices among saved vertices are neutral vertices. Moreover, at least $\frac{2}{3}$ of neighbours of each dangerous or neutral vertex are frozen(uncoloured) in each of its undominated factors. This is obvious for neutral vertices directly from the definition. For dangerous vertices it follows: assume that $\frac{1}{3}$ of neighbours of v have already been coloured in its undominated factor. Therefore, v becomes dangerous and from the third step of phase one the rest of v's uncoloured neighbours will be frozen.

Now we consider the subgraph induced by the saved vertices and run the next phase on each of its connected component. We claim that the order of the largest component of the subgraph induced by saved vertices is $O(\Delta(G)^6 \ln kn)$ with probability at least $\frac{1}{2}$ and we will prove this later in Theorem 3.13. Therefore, we may assume that the order of each connected component is at most $n_2 = O(\Delta^6(G) \ln kn)$.

Phase Two

1. Consider each connected component of the subgraph induced by saved vertices.
2. Repeat all three steps of phase one on each component separately.

After running this phase we will again have two partitions over vertices. {Coloured, Frozen} and {Good, Dangerous, Neutral} of phase two. Similar to phase one, we do not reconsider a vertex which is both coloured and good. Remaining vertices will be considered in phase three. In particular, we continue the algorithm with the new set of saved vertices. As before, some of the saved vertices are coloured and we do not recolour them in the next phase. Moreover, at least $\frac{1}{3}$ of neighbours of each dangerous or neutral vertex are frozen(uncoloured) in each of its undominated factors. Indeed, from the argument in the previous phase we know that before running phase two at least $\frac{2}{3}$ of neighbours of each dangerous or neutral vertex are frozen(uncoloured) in each of its undominated factors. In phase two at most additional $\frac{1}{3}$ of neighbours of each dangerous or neutral vertex may be coloured in each such factor.

We continue the algorithm by considering each connected component of the subgraph induced by the new saved vertices in the next phase. Again by Theorem 3.13, we may assume that with probability at least $\frac{1}{2}$, the order of the largest component of the subgraph induced by new set of saved vertices is at most

$$\begin{aligned} n_3 = O(\Delta^6(G) \ln kn_2) &= O(\Delta^6(G) \ln k(\Delta^6(G) \ln kn)) \\ &= O(\Delta^6(G)(\ln \Delta(G) + \ln \ln kn)). \end{aligned}$$

Phase Three

Case I $\Delta(G) > \ln \ln kn$. Then, $n_3 = O(\Delta^7(G))$. As we have discussed above after running phase two at least $\frac{1}{3}$ of neighbours of each dangerous or neutral vertex are frozen(uncoloured) in its undominated factor. This implies that the minimum degree of each component in phase three is at least $\frac{\delta_{min}}{3}$. To complete the colouring we can use Algorithm 3.3 on each component and therefore, from Theorem 3.2 we get a factor domatic partition of size $\frac{\delta_{min}}{3 \ln kn_3}$. Since $kn_3 = O(k\Delta(G))$, we will have a factor domatic partition of size $\frac{\delta_{min}}{21 \ln k\Delta(G)}$.

Case II $\Delta(G) \leq \ln \ln kn$. Then, $n_3 = O((\ln \ln kn)^7)$. After running two previous phases the components in third phase are fairly small and also $\Delta(G) \leq \ln \ln kn$. Therefore, Lemma 3.6 guarantees the existence of a factor domatic partition of size $\frac{\delta_{min}}{\ln k\Delta(G)}$. We use an exhaustive search on all vertices to find a factor domatic partition and it will take polynomial time to find such a domatic partition. Since each of n_3 vertices can be in at most $O(\frac{\delta_{min}}{\ln k\Delta(G)})$ sets, hence

$$n_3^{O(\frac{\delta_{min}}{\ln k\Delta(G)})} \leq (\ln \ln kn)^{O(\ln \ln kn)} \leq poly(n).$$

In order to prove the correctness of the above algorithm, we need to show that with probability at least $\frac{1}{2}$, the order of the largest connected component of the subgraph induced by saved vertices is $O(\Delta^6(G) \ln kn)$. In other words, we show that with probability at most $\frac{1}{2}$, a connected component of the subgraph induced by saved vertices with order at least $O(\Delta^6(G) \ln kn)$ exists. Such a component is called a large component. Now to estimate the order of a large component, it is enough to estimate the order of a spanning tree of it. For this, we estimate the number of saved vertices with "large" pairwise mutual distances. In our arguments, 7 will be large enough distance.

We need few definitions before we start. A set of vertices is 7-separated if the mutual distance of its elements is at least 7 in G. Note that if the distance of a pair of vertices is at least l in G, then it is at least l in each factor of G as well. Hence, a 7-separated set in G is a 7-separated set in each factor of G. A 7-separated set of t vertices is called a bad t-set if it becomes connected after joining all pairs of vertices of distance exactly 7 by an edge. There is a relation between bad t-sets and large components of the subgraph induced by saved vertices after Phase One. In particular, if such a component has order $t\Delta^6(G)$ or more, then it contains a bad t-set.

Lemma 3.8 *Any connected subgraph of G of order at least $t\Delta^6(G)$ contains at least one bad t-set.*

Proof: Let H be a connected subgraph of G with at least $t\Delta^6(G)$ vertices. Choose a vertex v and remove all vertices in H of distance at most 6 from v. Now choose the next vertex which has distance 7 from v and repeat this procedure in the remainder of H. Repeat this until all vertices of H are either chosen or removed.

∎

In order to find the maximum number of bad t-sets in G, we first count distinct spanning rooted trees and then we find the number of distinct realizations of each of these trees in G as a bad t-set. The following bound on the number of distinct rooted trees is known, but we include details for convenience.

Lemma 3.9 *There are at most 4^{t-1} distinct rooted trees on t vertices.*

Proof: Order t vertices of a tree T in lexicographic breath-first order, assign a two digits code to each vertex except the root as follows. The first digit is 1 if the vertex has the same parent as the previous vertex in the order and it is 0 otherwise. The second digit is 1 if it has a child and 0 if not. It is obvious that a tree T can be presented as a unique list of codes described above in breath first order. Now we will show that any list of codes in this order, represents at most one tree.

Starting form the root, each code whose first digit is 0 is the first vertex in a new level since it does not have the same parent as the previous vertex. So, the first code whose first digit is 0 starts the first level (vertices of distance 1 form the root), and the codes whose first digit is 1 after that code and before the next level and all have the same parent. If the second digit of a code is 1 label it *open*, and if it is 0 label it as a leaf. Now, start the next level with the next code whose first digit is 0, then if first digit of next code is 1 it has the same parent of the previous vertex, if it is 0 place this vertex as a child of an open vertex from the previous level. Repeat these steps till the end of the list. Using this algorithm we can construct a unique shape of tree and the claim is proved. There are 4 different 2 digits code, and there exists $(t-1)$ non-root vertices, therefore number of distinct trees on t vertices is at most 4^{t-1}.

■

Now we are ready to find the number of bad t-sets in G.

Lemma 3.10 *Any graph G contains at most $n(4\Delta^7(G))^t$ bad t-sets.*

Proof: For each of the 4^{t-1} distinct rooted trees on t vertices, we count the number of their realizations. There are n possibilities for choosing the root in G. Since the maximum number of vertices of distance 7 from a fixed vertex in G is $\Delta^7(G)$, there are at most $\Delta^7(G)$ possibilities for choosing each new vertex, following the pattern given by the tree. Therefore, the total number of bad t-sets in G is at most $n(4\Delta^7(G))^t$.

■

In order to estimate the probability that a connected component of the subgraph induced by saved vertices is large, we find the probability that its spanning tree is large. We show a stronger result, in particular, we estimate the probability that all vertices of a 7-separated set are saved. For this, we first show the following.

Lemma 3.11 *For any set S of t vertices of pairwise distance at least 3, the probability that all vertices in S are dangerous is at most*

$$\left(kh(1-\frac{1}{h})^{\frac{\delta_{min}}{3}}\right)^t.$$

Proof: Let $S = \{v_1, \ldots, v_t\}$ be a set of vertices of pairwise distance at least 3 in G. It is obvious that a vertex which is in the neighbourhood of S is a neighbour of exactly one vertex in S. Otherwise, S would contain a pair of vertices of distance 2, which is a contradiction. Consider the colouring procedure during the Phase One. Let $\vec{H} = (H_1, \ldots, H_t)$ be a fixed sequence of factors in $f = \{G_1, \ldots, G_k\}$. We let $\vec{H}$ be a prescribed factor pattern for which $v_1, \ldots, v_t$ are dangerous after phase one. Let S_i be a random variable that denotes the set of nonfrozen neighbours of $v_i \in S$ in H_i for $1 \le i \le t$. Note that S_i is the set of exactly those neighbours of v_i in H_i which will receive a colour after phase one. As we pointed out above, $S_i \cap S_j = \emptyset$ for all $i \neq j$. It is easy to see that if v_i is dangerous, then $|S_i| \le \frac{\delta_{min}}{3}$. Let $\vec{c} = (c_1, \ldots, c_t)$ be a fixed colour sequence. We let $\vec{c}$ be a prescribed colour pattern of missing colours at neighbours of vertices in S in $\vec{H}$. In particular, let X_i be the event that S_i is missing colour $c_i \in \vec{c}$ after phase one. Since colours are assigned randomly and independently for each vertex in phase one, S_i avoids the colour c_i with probability $(1-\frac{1}{h})^{|S_i|}$, and since $|S_i| \le \frac{\delta_{min}}{3}$, we have

$$P[X_i] \le (1-\frac{1}{h})^{\frac{\delta_{min}}{3}}.$$

The probability that each S_i is missing the colour c_i is the probability that the events X_i hold simultaneously for all $1 \le i \le t$. Hence,

$$P\left[\prod_{i=1}^{t} X_i\right] \le (1-\frac{1}{h})^{\frac{t\delta_{min}}{3}}.$$

We have estimated the probability that each vertex in S will become dangerous after phase one with the assumption that for each $1 \le i \le t$, the factor H_i is an undominated factor for v_i (missing colour c_i) and v_i has at least $\frac{\delta_{min}}{3}$ coloured neighbours in H_i after phase one. Since there are k^t possible factor patterns $\vec{H}$ and h^t possible colour patterns $\vec{c}$, the probability that every vertex in S becomes dangerous after phase one for at least one such choice of factors and colours is at most

$$k^t h^t (1 - \frac{1}{h})^{\frac{t\delta_{min}}{3}}.$$

■

Using this result, we next find the probability that all vertices of a 7-separated set are saved.

Lemma 3.12 *For any 7-separated set S of t vertices the probability that all vertices in S are saved is at most*

$$\left(2.5\Delta(G)kh(1 - \frac{1}{h})^{\frac{\delta_{min}}{3}}\right)^t.$$

Proof: Let $S = \{v_1, \ldots, v_t\}$. If v is a saved vertex after phase one, then at least one of the following holds

1. v is dangerous,
2. v has at least one dangerous neighbour, or
3. v is neutral.

Let $X(v)$ be the indicator random variable for the event that the vertex v is dangerous and let U_v be the set of v's undominated factors. Note that if $X(v) = 1$, then $U_v \neq \emptyset$. We will represent each of the above three cases as a linear combination of $X()$. By definition, if a vertex v is dangerous, then $X(v) = 1$. If v has at least one dangerous neighbour, say u, then $X(u) = 1$ and therefore

$$\sum_{u \in N_G(v)} X(u) \geq 1.$$

Finally, if v is neutral, then it has at least $\frac{2d_{G_i}(v)}{3}$ frozen neighbours in each of its undominated factors $G_i \in U_v$. Moreover, each such frozen neighbour is either dangerous or it has at least one dangerous neighbour. In the former case, v has a dangerous neighbour and we again have case 2. In the later case, we have

$$\frac{1}{|U_v|}\sum_{G_i\in U_v}\frac{\sum_{u\in N_{G_i}(v)}\sum_{w\in N_G(u)}X(w)}{\frac{2d_{G_i}(v)}{3}}\geq 1.$$

Let

$$Y(v)=X(v)+\sum_{u\in N_G(v)}X(u)+\frac{1}{|U_v|}\sum_{G_i\in U_v}\frac{\sum_{u\in N_{G_i}(v)}\sum_{w\in N_G(u)}X(w)}{\frac{2d_{G_i}(v)}{3}}.$$

It is easy to see that if v is a saved vertex, then $Y(v)\geq 1$. Therefore, the probability that all t vertices in S are saved can be upper bonded by the probability that $Y(v_i)\geq 1$ for all $1\leq i\leq t$, formally $P\Big[\text{all vertices in } S \text{ are saved}\Big]\leq P\Big[\prod_{i=1}^{t}Y(v_i)\geq 1\Big]$. Now, by Markov's inequality we have

$$P\Big[\prod_{i=1}^{t}Y(v_i)\geq 1\Big]\leq E\Big[\prod_{i=1}^{t}Y(v_i)\Big].$$

From the definition of $Y()$, it can be seen that it is linear in $X()$. Therefore,

$$E\Big[\prod_{i=1}^{t}Y(v_i)\Big]=\sum_{j=1}^{p}c_jE\Big[\prod_{i=1}^{t}X(w_{j_i})\Big],$$

where w_{j_i} is either v_i or it is in $N_G(v_i)$ or it is in $N_{G,2}(v_i)$. Note that since S is 7-separated, all w_{j_i} for fixed j are distinct. Since $X(v)$ is an indicator random variable,

$$E\Big[\prod_{i=1}^{t}X(w_{j_i})\Big]=P\Big[X(w_{j_1})=X(w_{j_2})=\cdots=X(w_{j_t})=1\Big].$$

Since S is a 7-separated set, pairwise distances of the vertices in $\{w_{j_1},\ldots,w_{j_t}\}$ are at least 3 for each $1\leq j\leq p$. Therefore, using Lemma 3.11, we have

$$P\Big[X(w_{j_1})=\cdots=X(w_{j_t})=1\Big]\leq\Big(kh(1-\frac{1}{h})^{\frac{\delta_{min}}{3}}\Big)^t.$$

Putting all these together we have

$$E\Big[\prod_{i=1}^{t} Y(v_i)\Big] \leq (c_1 + \cdots + c_p)\Big(kh(1-\frac{1}{h})^{\frac{\delta_{min}}{3}}\Big)^t.$$

We can upper bound $c_1 + \cdots + c_p$ as follows: Since the sum comes from the product of $Y(v_i)$s and for each v_i we have at most

$$1 + \Delta(G) + \frac{1}{|U_{v_i}|} \cdot |U_{v_i}| \cdot \frac{d_{G_i}(v_i)(\Delta(G) - 1)}{\frac{2d_{G_i}(v_i)}{3}}$$

terms in its definition, the sum of the coefficients can be upper bounded by

$$\Big(1 + \Delta(G) + \frac{3}{2}(\Delta(G) - 1)\Big)^t.$$

Hence,

$$\begin{aligned} E\Big[\prod_{i=1}^{t} Y(v_i)\Big] &\leq \Big(1 + \Delta(G) + \frac{3}{2}(\Delta(G) - 1)\Big)^t \Big(kh(1-\frac{1}{h})^{\frac{\delta_{min}}{3}}\Big)^t \\ &\leq \Big(2.5\Delta(G)kh(1-\frac{1}{h})^{\frac{\delta_{min}}{3}}\Big)^t. \end{aligned}$$

■

Finally, our main claim about the size of a largest connected component of saved vertices is proved in the following theorem.

Theorem 3.13 *Every connected component of the subgraph induced by saved vertices has at least $\Omega(\Delta^6(G) \ln kn)$ vertices, with probability at most $\frac{1}{2}$.*

Proof: Let H be a connected component of a subgraph induced by saved vertices witch has at least $t\Delta^6(G)$ vertices. We specify t later. As shown in Lemma 3.8, such a connected component contains at least one bad t-set. By Lemma 3.10, there are at most $n\Big(4\Delta^7(G)\Big)^t$ bad t-sets in G. From Lemma 3.12, each such bad t-set contains only saved vertices with the probability at most

$$\begin{aligned}\left(2.5\Delta(G)kh(1-\frac{1}{h})^{\frac{\delta_{min}}{3}}\right)^t &\leq \left(2.5\Delta(G)khe^{-\frac{\delta_{min}}{3h}}\right)^t \\ &\leq \left(2.5\Delta(G)k\frac{\delta_{min}}{c(\ln k\Delta(G))}e^{\frac{-c(\ln k\Delta(G))}{3}}\right)^t \\ &\leq \left(2.5\Delta(G)k\Delta(G)\frac{1}{k^{10}\Delta(G)^{10}}\right)^t \\ &\leq \left(\frac{k\Delta(G)^{-8}}{4}\right)^t.\end{aligned}$$

Hence, the product of these two upper bounds on the number of saved vertices in G and the probability that each such set contains only saved vertices, is an upper bound on the probability that a connected component of order at least $\Omega(t\Delta^6(G))$ contains only saved vertices. Therefore, by letting $t = \frac{\ln(2kn)}{\ln\Delta(G)}$ in this bound, we have

$$\begin{aligned}n(4\Delta^7(G))^t(\frac{k\Delta(G)^{-8}}{4})^t &\leq \frac{nk}{\Delta(G)^t} \\ &= \frac{nk}{\Delta(G)^{\frac{\ln(2kn)}{\ln\Delta(G)}}} \\ &= \frac{1}{2}.\end{aligned}$$

Note that we have proved a slightly stronger result since here $t < \ln kn$.

■

3.2 Factor l-domatic Partition

In the survey, we presented definitions and some properties of l-domatic partition, namely the partition of $V(G)$ such that each class is an l-dominating set. We also defined a variation in the survey which is a combination of factor domination and distance domination, called factor l-domatic partition. In the following we generalize the given lower bounds on the factor domatic number given in Theorem 3.1 and

Lemma 3.6, to bounds on the factor l-domatic number $D_{f,l}(G)$. Before we start, we need some definitions and notation.

The *closed l-neighbourhood* of a vertex v in G, $N_{G,l}[v]$, is the set of all vertices at distance at most l from v in G. Let $d_{G,l}(v) = |N_{G,l}(v)|$ denote the number of vertices in the l-neighbourhood of v in G. In the following bounds we use the same method as in the proof of Theorem 3.1 and Lemma 3.6. In particular, we again use a random colouring. Hence, similarly, for a triple (v, c, i) where $v \in V(G)$, $c \in [h]$ and $1 \leq i \leq k$ is the index of factor G_i, let $I_{v,c,i}$ be the following indicator random variable,

$$I_{v,c,i} = \begin{cases} 1 & \text{if there is no vertex of colour } c \text{ in } N_{G_i,l}[v], \\ 0 & \text{otherwise.} \end{cases}$$

If $I_{v,c,i} = 1$, then v does not see colour c in $N_{G_i,l}[v]$. This means that the colour c is not forming an l-dominating set in the factor G_i.

It is obvious that the same lower bounds as in Theorem 3.1 and Lemma 3.6 hold for $D_{f,l}(G)$ as well. One would expect that considering the distance $l > 1$ will lead to better lower bounds. However, since we need to lower bound $d_{G_i,l}[v]$, if a component of the factor G_i is a large clique, then $N_{G_i,l}[v] = N_{G_i}[v]$, and consequently we cannot hope for a better lower bound on $D_{f,l}(G)$ than already given in Theorem 3.1 and Lemma 3.6. This example shows that possibly these lower bounds will depend on the girth $g(G)$ of the graph G. If $\lfloor \frac{g(G)-1}{2} \rfloor \geq l$, then there will be no cycle in $N_{G_i,l}[v]$ and consequently no two vertices in $N_{G_i,l}[v]$ will have a common neighbour. Therefore, we can lower bound $d_{G_i,l}[v]$ by

$$d_{G_i,l}[v] \geq 1 + \delta_{min} + \delta_{min}(\delta_{min} - 1) + \cdots + \delta_{min}(\delta_{min} - 1)^{l-1}.$$

If $\lfloor \frac{g(G)-1}{2} \rfloor < l$, then the lower bound on $d_{G_i,l}[v]$ needs to be in terms of $g(G)$, since we can guarantee no cycles only in $N_{G_i,\lfloor \frac{g(G)-1}{2} \rfloor}[v]$. In this case the lower bound will be

$$d_{G_i,l}[v] \geq 1 + \delta_{min} + \delta_{min}(\delta_{min} - 1) + \cdots + \delta_{min}(\delta_{min} - 1)^{\lfloor \frac{g(G)-1}{2} \rfloor - 1}.$$

For convenience, we define

$$\tilde{g} = \min\left\{l, \left\lfloor \frac{g(G) - 1}{2} \right\rfloor\right\}.$$

Then, we have

$$d_{G_i,l}[v] \geq 1 + \delta_{min} + \delta_{min}(\delta_{min} - 1) + \cdots + \delta_{min}(\delta_{min} - 1)^{\tilde{g}-1}.$$

Lemma 3.14 *For a given k-factoring f of a graph G,*

$$D_{f,l}(G) \geq \left\lfloor \left(1 - o(\frac{\ln \ln kn}{\ln kn})\right) \frac{(\delta_{min} - 1)^{\tilde{g}}}{\ln kn} \right\rfloor.$$

Proof: Let $f = \{G_1, \ldots, G_k\}$ be a k-factoring of G and let $h = \left\lfloor \frac{(\delta_{min}-1)^{\tilde{g}}}{\ln(kn \ln kn)} \right\rfloor$. Colour each vertex randomly with one of h available colours. For any triple (v, c, i), $E[I_{v,c,i}] = P[I_{v,c,i} = 1]$ and $P[I_{v,c,i} = 1] = (1 - \frac{1}{h})^{d_{G_i,l}[v]}$, since the colour c does not appear with probability $(1 - \frac{1}{h})$ and there are $d_{G_i,l}[v]$ vertices in the closed l-neighbourhood of v in G_i. Now to lower bound $d_{G_i,l}[v]$, we find the minimum number of vertices in a closed l-neighbourhood of v. As shown above,

$$d_{G_i,l}[v] \geq 1 + \delta_{min} + \delta_{min}(\delta_{min} - 1) + \cdots + \delta_{min}(\delta_{min} - 1)^{\tilde{g}-1},$$

and since

$$1 + \delta_{min} \sum_{j=0}^{\tilde{g}-1} (\delta_{min} - 1)^j = 1 + \delta_{min} \frac{(\delta_{min} - 1)^{\tilde{g}} - 1}{\delta_{min} - 2} \geq (\delta_{min} - 1)^{\tilde{g}},$$

using this in the probability above, we get

$$\begin{aligned} P[I_{v,c,i} = 1] &\leq (1 - \frac{1}{h})^{(\delta_{min}-1)^{\tilde{g}}} \leq e^{-\frac{(\delta_{min}-1)^{\tilde{g}}}{h}} \\ &\leq e^{-\ln(kn \ln kn)} \leq \frac{1}{kn \ln kn}. \end{aligned}$$

Thus, the expected total number of vertices which are not l-dominated in some factor G_i is at most

$$\sum_{v \in V(G)} \sum_{c \in [h]} \sum_{i=1}^{k} P[I_{v,c,i} = 1] = nhk \frac{1}{kn \ln kn} = \frac{h}{\ln kn}.$$

So the expected number of colours which do not form a factor l-dominating set is at most $\frac{h}{\ln kn}$. Thus, the expected number of colours which form a factor l-dominating set for the given k-factoring is at least

$$\begin{aligned} h - \frac{h}{\ln kn} &= (\delta_{min} - 1)^{\bar{g}} \left(\frac{1}{\ln(kn \ln kn)} - \frac{1}{\ln(kn \ln kn) \ln kn} \right) \\ &= (\delta_{min} - 1)^{\bar{g}} \left(\frac{\ln kn - 1}{\ln kn + \ln \ln kn \ln kn} \right) \\ &= \frac{(\delta_{min} - 1)^{\bar{g}}}{\ln kn} \left(1 - \frac{1 + \ln \ln kn}{\ln(kn \ln kn)} \right). \end{aligned}$$

Now, to form the partition, simply add vertices with colours that do not form a factor l-dominating set, into some class which does so. Hence we obtain a factor l-domatic partition of G such that

$$D_{f,l}(G) \geq \left\lfloor \left(1 - o(\frac{\ln \ln kn}{\ln kn}) \right) \frac{(\delta_{min} - 1)^{\bar{g}}}{\ln kn} \right\rfloor.$$

■

Before we get to the algorithmic aspect of the bound found above, we need the following definition. Let $B_{v,c,i}$ be the event that the vertex v does not see the colour c in $N_{G_i,l}[v]$. Note that $I_{v,c,i} = 1$, if and only if $B_{v,c,i}$ holds.

Theorem 3.15 *Any graph G with a given k-factoring f admits a polynomial time constructible factor l-domatic partition of size*

$$\left\lfloor \left(1 - o(\frac{\ln \ln kn}{\ln kn}) \right) \frac{(\delta_{min} - 1)^{\bar{g}}}{\ln kn} \right\rfloor.$$

■

Algorithm 3.16 *The Factor l-domatic Partition Algorithm*

Input: *A graph G of order n with a k-factoring $f = \{G_1, \ldots, G_k\}$.*

Output: *A factor l-domatic partition of expected size given in Theorem 3.15 for the k-factoring f of G.*

1. Order the vertices arbitrarily $v_1, \dots, v_n$ and process them in this order.

2. Colour v_1 randomly with one of the colours in $[h]$, where h is as given in Lemma 3.14.

3. Suppose $v_1, \dots, v_j$ have been coloured with colours $c_1, \dots, c_j$. Colour v_{j+1} as follows. Denote the conditional probability of the event $B_{v,c,i}$, given that vertices $v_1, \dots, v_j$ have already been coloured with colours $c_1, \dots, c_j$ by $P[B_{v,c,i}|c_1, \dots, c_j]$. This probability is zero if there exists a vertex v_z, $z \leq j$ in $N_{G_i,l}[v]$ that has been coloured with colour c. Otherwise it is $(1 - \frac{1}{h})^m$, where m is the number of uncoloured vertices in $N_{G_i,l}[v]$, i.e. $m = |N_{G_i,l}[v] \cap \{v_{j+1}, \dots, v_n\}|$. Formally,

$$P[B_{v,c,i}|c_1, \dots, c_j] = \begin{cases} 0 & \text{if } \exists z \leq j,\ v_z \in N_{G_i,l}[v] \text{ and} c_z = c, \\ (1 - \frac{1}{h})^m, & \text{otherwise.} \end{cases}$$

Let $g(c_1, \dots, c_j)$ denote the expected total number of these bad events for which $I_{v,c,i} = 1$ after colouring all vertices, given that $v_1, \dots, v_j$ have already been coloured with colours $c_1, \dots, c_j$, respectively, i.e.

$$g(c_1, \dots, c_j) = \sum_{v \in V(G)} \sum_{c \in [h]} \sum_{i=1}^{k} P[B_{v,c,i}|c_1, \dots, c_j].$$

Now colour the vertex v_{j+1} with colour c_{j+1} such that

$$g(c_1, \dots, c_{j+1}) \leq g(c_1, \dots, c_j).$$

■

The proof of correctness goes in lines similar to the proof of Lemma 3.4. Again, as shown before, $g(c_1, \dots, c_j)$ is a convex combination of $g(c_1, \dots, c_{j+1})$ for $c_{j+1} \in [h]$. Therefore, we can always colour v_{j+1} with a colour c_{j+1} such that $g(c_1, \dots, c_j, c_{j+1}) \leq g(c_1, \dots, c_j)$. The rest of the proof is the same.

Our last result in this section will be another lower bound on $D_{f,l}(G)$; this time we use Lovász Local Lemma. Our technique is a generalization of the one used in Lemma 3.6, for the factor domatic number.

Lemma 3.17 *For a given k-factoring f of a graph G,*

$$D_{f,l}(G) \geq \left\lfloor \left(\frac{1}{2l+1} - o(1)\right) \frac{(\delta_{min}-1)^{\bar{g}}}{\ln(k\Delta(G))} \right\rfloor.$$

Proof: Let $f = \{G_1, G_2, \ldots, G_k\}$ be a k-factoring of G and let

$$h = \left\lfloor \frac{(\delta_{min}-1)^{\bar{g}}}{(2l+1)\ln((2l+1)^{\frac{1}{2l+1}} k^{\frac{1}{2l+1}} \Delta(G))} \right\rfloor.$$

Randomly colour vertices with colours in $[h]$. For each triple of a vertex, colour, and index of a factor (v, c, i), let $B_{v,c,i}$ be the event as defined in Theorem 3.15. Using the bound on $d_{G_i}[v]$ from Lemma 3.14, the probability of the bad event $B_{v,c,i}$ is at most

$$\begin{aligned} P[B_{v,c,i}] &= (1-\frac{1}{h})^{d_{G_i,l}[v]} \leq (1-\frac{1}{h})^{(\delta_{min}-1)^{\bar{g}}} \\ &\leq e^{-\frac{(\delta_{min}-1)^{\bar{g}}}{h}} \leq e^{-(2l+1)\ln((2l+1)^{\frac{1}{2l+1}} k^{\frac{1}{2l+1}} \Delta(G))} \\ &\leq \frac{1}{k(2l+1)\Delta(G)^{2l+1}}. \end{aligned}$$

Each event $B_{v,c,i}$ corresponds to the vertex set $N_{G_i,l}[v]$. Therefore, the dependency translates into the intersection property for the corresponding vertex sets, and two events are dependent if and only if their corresponding vertex sets have a non-empty intersection. Thus, for each event $B_{v,c,i}$ there are at most

$$khd_{G,2l}[v]$$

dependent events, since there are at most $d_{G,2l}[v]$ vertices which are "close enough" to v and for each such vertex there can be a dependent event for any of k factors and any of h colours. Therefore, we have

$$\begin{aligned} khd_{G,2l}[v] &\leq kh[1+\Delta(G)+\Delta(G)(\Delta(G)-1)+\cdots+\Delta(G)(\Delta(G)-1)^{2l-1}] \\ &\leq kh[1+\Delta(G)\sum_{j=0}^{2l-1}(\Delta(G)-1)^j] \\ &\leq kh[1+\Delta(G)\frac{(\Delta(G)-1)^{2l}-1}{\Delta(G)-2}] \\ &\leq k\Delta(G)^{2l+1}-1 \end{aligned}$$

Hence, $d = k\Delta(G)^{2l+1} - 1$ and $p = \frac{1}{k(2l+1)\Delta(G)^{2l+1}}$ in Lovász Local Lemma. Since we have

$$\begin{aligned} ep(d+1) &= e\,\frac{1}{k(2l+1)\Delta(G)^{2l+1}} \times k\Delta(G)^{2l+1} \\ &= \frac{e}{2l+1} \leq 1, \end{aligned}$$

by Lovász Local Lemma, the probability that no bad event holds is greater than zero, which guarantees a factor l-domatic partition of size at least

$$\left\lfloor \left(\frac{1}{2l+1} - o(1)\right) \frac{(\delta_{min}-1)^{\bar{g}}}{\ln(k\Delta(G))} \right\rfloor.$$

∎

3.3 Factor Domination and Factor l-domination Number

We have previously seen the upper bound of $n - \Delta(G)$ on $\gamma(G)$ provided in Theorem 2.1, and similarly, in Theorem 2.9, the upper bound of $n - \delta_{min}$ on $\gamma(G, f)$.

We have also seen a probabilistic upper bound of $n\,\frac{1+\ln(\delta(G)+1)}{\delta(G)+1}$ on $\gamma(G)$ in Theorem 2.8. In the following we will also use probabilistic method to upper bound $\gamma(G, f)$ and $\gamma_l(G, f)$. We will prove a slightly stronger result, in which we will assume that the

k-factoring f contains not necessarily edge-disjoint factors of G. (Recall that factors are required to be edge-disjoint in the original definition of k-factoring.)

Theorem 3.18 *For a given k-factoring f of G with $\delta_{min} \geq 1$,*

$$\gamma(G, f) \leq \left\lceil n \frac{1 + \ln(k(\delta_{min} + 1))}{(\delta_{min} + 1)} \right\rceil.$$

Proof: Let $f = \{G_1, \ldots, G_k\}$ be a k-factoring of G. Let X be a set of vertices picked randomly from G with probability p, $0 \leq p \leq 1$. It is easy to see that the expected size of X is np. Let Y_i be the set of vertices that are not in X nor in $N_{G_i}(X)$, for all $1 \leq i \leq k$. To estimate the expected size of Y_i, we note that the probability for a vertex to be in Y_i is

$$(1-p)^{d_{G_i}[v]} \leq (1-p)^{\delta_{min}+1}.$$

Therefore, the expected size of Y_i is at most $n(1-p)^{\delta_{min}+1}$. Since in each factor G_i a vertex is either in X, or has a neighbour in X, or it is in Y_i, obviously $X \cup Y_i$ is a dominating set in the factor G_i for $1 \leq i \leq k$. Therefore, by the linearity of the expectation, $Z = X \cup \bigcup_{i=1}^{k} Y_i$ forms a factor dominating set of expected size

$$E(|Z|) = E(|X|) + \sum_{i=1}^{k} E(|Y_i|) \leq np + nk(1-p)^{\delta_{min}+1}. \tag{3.2}$$

Since $1 - p \leq e^{-p}$,

$$E(|Z|) \leq np + nke^{-p(\delta_{min}+1)}.$$

By setting $p = \frac{\ln(k(\delta_{min}+1))}{\delta_{min}+1}$, we have

$$\begin{aligned} E(|Z|) &\leq n \frac{\ln(k(\delta_{min} + 1))}{\delta_{min} + 1} + \frac{n}{\delta_{min} + 1} \\ &\leq n \frac{1 + \ln(k(\delta_{min} + 1))}{\delta_{min} + 1}. \end{aligned}$$

Therefore, there exists a choice for X which gives the desired bound on $\gamma(G, f)$.

■

Note that the lower bound of $\frac{nk}{\Delta(G)+k}$ on $\gamma(G,f)$ provided in Theorem 2.9 assumes that the factors are edge-disjoint. Now, we have $\Delta(G) \geq k\delta_{min}$, and hence the lower bound is indeed smaller than our upper bound. Without this assumption the lower bound becomes $\frac{n}{\Delta(G)+1}$, since in this case a vertex not in D does not necessarily need k edges to D and hence,

$$n - \gamma(G,f) \leq \sum_{v \in D} d_G(v) \leq \Delta(G)\gamma(G,f) \Longrightarrow \gamma(G,f) \leq \frac{n}{\Delta(G)+1}.$$

We first give the following simple upper bound on $\gamma_l(G,f)$.

Observation 3.19 *For any k-factoring f of G,*

$$\gamma_l(G,f) \leq n - (\delta_{min} - 1)^{\bar{g}}.$$

Proof: Let $f = \{G_1, \ldots, G_k\}$ be a k-factoring of G. Let D be a set of $n - (\delta_{min} - 1)^{\bar{g}}$ vertices. Each vertex v not in D has at least $(\delta_{min} - 1)^{\bar{g}}$ vertices in its l-neighbourhood in each factor. However, there are only $(\delta_{min} - 1)^{\bar{g}} - 1$ vertices different from v which are not in D. Therefore, v has at least one l-neighbour in D in each factor, and hence, D is a factor l-dominating set of G.

■

Now we improve this bound using a probabilistic argument.

Theorem 3.20 *For a given k-factoring f of G with $\delta_{min} \geq 1$,*

$$\gamma_l(G,f) \leq \left\lceil n \frac{1 + \ln(k(\delta_{min} - 1)^{\bar{g}})}{(\delta_{min} - 1)^{\bar{g}}} \right\rceil.$$

Proof: Let $f = \{G_1, \ldots, G_k\}$ be a k-factoring of G. Let X be a set of vertices picked randomly with probability p, $0 \leq p \leq 1$. It is easy to see that the expected size of X is np. Let Y_i be the set of vertices that are not in X nor in $N_{G_i,l}(X)$, for all $1 \leq i \leq k$.

To estimate the expected size of Y_i, we note that the probability for a vertex to be in Y_i is

$$(1-p)^{d_{G_i,l}[v]} \leq (1-p)^{(\delta_{min}-1)^{\tilde{g}}},$$

Where $\tilde{g}$ is as defined in Section 3.2.

Therefore, the expected size of Y_i is at most $n(1-p)^{(\delta_{min}-1)^{\tilde{g}}}$. Since in each factor a vertex is either in X, or $N_{G_i,l}(X)$, or it is in Y_i, obviously $X \cup Y_i$ is an l-dominating set in G_i for $1 \leq i \leq k$. Therefore, by the linearity of expectation, $Z = X \cup \bigcup_{i=1}^{k} Y_i$ forms a factor l-dominating set of expected size

$$E(|Z|) = E(|X|) + \sum_{i=1}^{k} E(|Y_i|) \leq np + nk(1-p)^{(\delta_{min}-1)^{\tilde{g}}}. \tag{3.3}$$

Since $1-p \leq e^{-p}$ and this is a close bound when $0 \leq p \leq 1$,

$$E(|Z|) \leq np + nke^{-p(\delta_{min}-1)^{\tilde{g}}}.$$

By setting $p = \frac{\ln(k(\delta_{min}-1)^{\tilde{g}})}{(\delta_{min}-1)^{\tilde{g}}}$, we have

$$\begin{aligned} E(|Z|) &\leq n\frac{\ln(k(\delta_{min}-1)^{\tilde{g}})}{(\delta_{min}-1)^{\tilde{g}}} + \frac{n}{(\delta_{min}-1)^{\tilde{g}}} \\ &\leq n\frac{1+\ln(k(\delta_{min}-1)^{\tilde{g}})}{(\delta_{min}-1)^{\tilde{g}}}. \end{aligned}$$

Therefore, there exists a choice for X which gives the desired bound on $\gamma_l(G, f)$.

■

3.4 Factor Domination with Precolored Vertices

In this section, we will turn our attention to a variant of the factor l-domination introduced in Section 2.3. In particular, recall the definition of $\alpha(t, k)$: given integers k and t, let $\alpha(t, k)$ denote the minimum distance l such that every k-factoring of every graph on at least t vertices admits a factor l-domatic partition with t classes. Recall also that in this problem factors are connected but not necessarily edge-disjoint. We will keep this assumption in this section.

In the following, we will study bounds on $\alpha(t,k)$ under the assumption that some vertices of G are already pre-coloured (assigned to specific classes). Our goal is to find conditions on the pre-colouring which will allow $\alpha(t,k)$ be a function of only t and k (but not n).

Recall the bound $\alpha(t,k) \leq \lceil \frac{3(kt-1)}{2} \rceil$, for every $k \geq 2$ and $t \geq k$, given in Theorem 2.58. The following simple example shows that if the distance of any two pre-coloured vertices with the same colour is at least l, and $l < t$, then $\alpha(t,k)$ depends on n.

Example 3.21 *Consider a k-factoring of a Hamiltonian graph G such that one of the factors is its Hamilton path. Starting from one end of the path, say v, colour vertices repeatedly $1, 2, \ldots, l, 1, 2, \ldots, l$, leaving $t-l$ vertices uncoloured. Note that the last segment may not use all l colours; see Figure 3.1.*

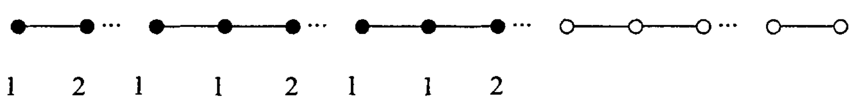

Figure 3.1: A factor of a Hamiltonian graph.

The remaining $t-l$ colours need to dominate the pre-coloured vertices and there are exactly $t-l$ uncoloured vertices where these colours can appear. In particular, the colour assigned to the last vertex will only $(n-1)$-dominate v. Thus, we have

$$\alpha(t,k) \geq n-1.$$

■

Note that the condition $l < t$ in this example is essential since otherwise there may be some uncoloured vertices in between the pre-coloured vertices. Interestingly, we will show that if the distance of any two pre-coloured vertices with the same colour is at least l and $l \leq t$, then $\alpha(t,k) \leq \lceil \frac{3(kt-1)}{2} \rceil$. This is exactly the same bound as in Theorem 2.58.

We need the following lemma from [AFL$^+$03] for the main result.

Lemma 3.22 *[AFL$^+$03] For any tree T on vertex set V, $|V| \geq k \geq 1$, there exists a partition of V into $V_1, \ldots, V_p$ such that for every $1 \leq i \leq p$, V_i contains a subset B_i such that $|B_i| = k$, and for every $u \in V_i$ and $v \in B_i$,*

$$d_T(u, v) \leq \left\lceil \frac{3(k-1)}{2} \right\rceil.$$

■

Directly from the lemma one can observe that the subgraph of T induced by B_i, i.e. $T[B_i]$ has diameter of at most $k-1$.

Theorem 3.23 *Let G be a graph with pre-coloured vertices such that the distance of any two pre-coloured vertices of the same colour is at least t. Then $\alpha(t, k) \leq \lceil \frac{3(kt-1)}{2} \rceil$, for every $k \geq 2$ and $t \geq k$.*

Proof: Let $f = \{G_1, \ldots, G_k\}$ be a k-factoring of G. If $|V(G)| < kt$, then any colouring will be a factor $\lceil \frac{3(kt-1)}{2} \rceil$-domatic colouring, since the distance of any two vertices in any factor is at most $|V(G)| - 1 < kt - 1 < \lceil \frac{3(kt-1)}{2} \rceil$. If $|V(G)| \geq kt$, then we apply Lemma 3.22 on a spanning subtree of each factor G_i. Therefore, for each factor G_i there is a partition of $V(G)$ into $V_{i1}, \ldots, V_{i,p_i}$ such that each V_{ij}, $1 \leq j \leq p_i$, contains a subset B_{ij}, where $|B_{ij}| = kt$, the diameter of B_{ij} is at most kt and for every $u \in V_{ij}$ and $v \in B_{ij}$,

$$d_{G_i}(u, v) \leq \lceil \frac{3(kt-1)}{2} \rceil.$$

Partition each B_{ij} into t blocks of size k such that each block B_{ijl}, $1 \leq l \leq t$, contains either uncoloured vertices or vertices coloured l. This is possible since the distance of any two pre-coloured vertices with the same colour is at least t and so there are at most k vertices of colour l in each B_{ij}.

Construct a bipartite graph H with parts B and V such that

$$B = \{B_{ijl} | 1 \leq i \leq k, 1 \leq j \leq P_i, 1 \leq l \leq t\}$$

and $V = V(G)$. Join a vertex $v \in V$ to $B_{ijl} \in B$ if $v \in B_{ijl}$. Since each block B_{ijl} contains k vertices so for each $B_{ijl} \in B$, $d_H(B_{ijl}) = k$. Moreover, each vertex is in at most k different blocks, so for each $v \in V$, $d_H(v) \leq k$. Hence, using Hall's Theorem for this bipartite graph, there is a matching that saturates B.

We construct the required colouring as follows : if a vertex that is matched to a block B_{ijl} is uncoloured, colour it with l; if it is pre-coloured, then from the construction of the bipartite graph, it is pre-coloured with l. Using this method, each B_{ijl} will contain vertices coloured only l and therefore, each B_{ij} will receive all t colours. So, each B_{ij} forms a factor $\lceil \frac{3(kt-1)}{2} \rceil$-dominating set.

∎

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