ON THREE TYPES OF SECTIONS IN TOPOLOGICAL SEQUENCE SPACES

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Abstract

This thesis contains a detailed account of sequence spaces which follows the works of several authors. We examine the significance of front, Cesàro, and unconditional sections and arrive at factorization theorems that characterize some topological properties of the space. Some applications to Fourier series are made using the results on Cesàro sections. It is shown that for a sequence x in an FK-space E, the series $\Sigma x_k \delta^k$ is subseries convergent if and only if x can be expressed as the coordinatewise product of a convergent null sequence and a sequence with unconditionally bounded sections in E. Finally a strong mean value property for a conservative matrix A is defined, and a necessary and sufficient condition for a sequence to have unconditionally bounded sections in the convergence field of A is obtained.

In memory of my grandmother

AUGUSTA DEE

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Chapter 1 - Preliminaries

§1. A Brief History

Topological summability theory probably has its roots in S. Mazur's 1929 article, [14]. He observed that if A is a triangle (a summation matrix with $a_{nk} = 0$ when $k > n$; $a_{nn} \neq 0$) and B is a stronger matrix ($c_B \supset c_A$; see Chapter 4, §4) then $\lim_B(x) = \lim_n (Bx)_n$ is a continuous linear functional on c_A. The next step along these lines was taken independently by Mazur-Orlicz [15] and by Zeller [29]. They extended these ideas to non-triangular matrices by attempting to give c_A a topology making each coordinate functional continuous and also such that applications to functional analysis could be made; for example to deduce the continuity of $\lim_B$ as above. From this work comes the result that every convergence field is an FK-space. (see §2).

One of the most exhaustive studies of convergence fields can be found in [25] where certain "distinguished subsets" of the convergence field are studied and investigated for their invariance, i.e. if $c_A = c_B$, are their corresponding subsets under consideration the same? One way of proving invariance for a subset of c_A is to show that it is characterized by a property of its FK-topology, as the most fundamental and singly most important aspect of any FK-space is its uniqueness of topology. This observation led to the formulation of many distinguished subsets in terms of topology in such generalized spaces as K-spaces. (see §2). For example, a conservative matrix A is said to be conull if $\lim_n \sum_k a_{nk} - \sum_k \lim_n a_{nk} = 0$. In 1965 A.K. Snyder proved A is conull if and only if $P_n \cdot e \to e$ weakly in c_A [24]. (see §2). It then follows naturally to call any FK-space conull if $P_n \cdot e \to e$ weakly.

One of the methods developed to study sequence spaces is illustrated in D.J.H. Garling's work [9] of which Chapter 2 is based. Prior to this most emphasis had been placed on duality theory. Instead Garling focussed his attention on front sections and multiplier spaces. Equally adaptable to the same method of analysis are the Cesàro sections and unconditional sections, introduced respectively by M. Buntinas [4] and J.J. Sember [23]. Chapter 3 follows Buntinas' work and Chapter 4, Sember's.

Although basically the same, the techniques employed in this thesis in studying a sequence space will vary slightly depending upon which kind of section is used as a tool. For example, the Cesàro section operators do not form a semi-group of linear operators, so they are not susceptible to the technique of local invariance developed in Chapter 2. However this does not really make their theory any less rich than the other two. We will see how each type of section operator determines an associated BK-space; it is from this space that many topological properties can be settled upon coordinatewise multiplication of the space in question by the associated BK-space.

§2. Notation and Terminology

In this section some notation and terminology is introduced which will be used throughout this thesis. A locally convex topological vector space V is a topological vector space with a neighborhood base at 0 consisting of convex subsets of V. For the standard definitions and results concerning topological vector spaces the reader is referred to A.P. and W. Robertson's book [19] and to G. Köthe's book [11]. We let ω denote the linear space of all real or complex sequences. A linear subspace of ω is called a sequence space. Let δ^k be the sequence with a 1 in the k-th position and 0's elsewhere, let e be the sequence with a 1 in each position, and let φ be the sequence space spanned by the set $\{\delta^k : k \in Z^+\}$ where Z^+ is the set of positive integers.

A K-space is a locally convex topological vector space such that the vector space is a sequence space for which the projection functional π_k, defined by $\pi_k(x) = x_k$, is continuous for each k in Z^+. An FK-space is a Fréchet K-space, i.e. a complete metrizable K-space, and a BK-space is a Banach K-space. A very good introduction to FK-spaces can be found in chapters 11.3 and 12.4 in A. Wilansky's book [26].

If S and A are subsets of ω we write $\Gamma\{A\}$ for the convex hull of A; $\bar{A}$ for the closure of A, when it is clear from the context which K-space A is a subset of; and

$$S\cdot A = \{x\cdot y : x \in S,\ y \in A\} = S(A)$$

where $x\cdot y$ is the coordinatewise product $(x_k y_k)_{k\in Z^+}$. Should A be a sequence space and $C\cdot A = A$, where C is the closed unit ball of bounded sequences (see §3), then A is called solid.

We define the following important sets of "section operators" on ω:

$$P = \{P_n : n \in Z^+\} \text{ where } P_n = \sum_{k=1}^{n} \delta^k ,$$

$$\sigma = \{\sigma^n : n \in Z^+\} \text{ where } \sigma^n = n^{-1} \sum_{k=1}^{n} P_k = \sum_{k=1}^{n} (1 - \frac{k-1}{n})\delta^k ,$$

$$H = \{h_F = \sum_{k \in F} \delta^k : F \text{ is a finite subset of } Z^+\} .$$

For a given x in ω, $P_n \cdot x$ (respectively $\sigma^n \cdot x$, $x_F = h_F \cdot x$) is called the n-th (front) section (respectively n-th Cesàro section, F-th unconditional section) of x. Thus the set of sections (respectively Cesàro sections, unconditional sections) of x is $P \cdot x$ (respectively $\sigma \cdot x$, $H \cdot x$).

Let E be a K-space. If x is in ω and $P \cdot x$ (respectively $\sigma \cdot x$, $H \cdot x$) is a bounded subset of E, x is said to have the property section boundedness (AB) (respectively Cesàro section boundedness (σB), unconditional section boundedness (UAB)). We will often use the following notation:

$$E_{AB} = \{x \in \omega : x \text{ has the property AB}\},$$

$$E_{\sigma B} = \{x \in \omega : x \text{ has the property } \sigma B\},$$

$$E_{UAB} = \{x \in \omega : x \text{ has the property UAB}\}.$$

Here it should be noted that E_{AB}, $E_{\sigma B}$, and E_{UAB} need not be contained in E. Also, the properties have not been listed according to strength but by chronological development. The reader is referred to Chapter 2,

§2 for the relative strengths of all the properties defined in this section. If for each f in E', where E' is the topological dual of E, the series $\sum_{k=1}^{\infty} x_k f(\delta^k)$ converges (equivalently $(P_n \cdot x)$ is Cauchy in $\sigma(E,E')$, the weakest topology on E which makes every $f \in E'$ continuous), then x is said to have the property functional section convergence (FAK). We write

$$E_{FAK} = \{x \in \omega : x \text{ has the property FAK}\}$$

and

$$E_{SAK} = \{x \in E : x \text{ has the property SAK}\},$$

where x is said to have the property weak section convergence (SAK) if $(P_n \cdot x)$ converges in $\sigma(E,E')$.

Let Φ denote the set of all finite subsets of the integers directed by set-theoretic inclusion. We define the properties unconditional functional section convergence (UFAK), unconditional weak section convergence (USAK), and unconditional section convergence (UAK).

$E_{UFAK} = \{x \in \omega : x \text{ has the property UFAK}\}$ where the property UFAK is Cauchiness of the net $(h_F \cdot x)_{F \in \Phi}$ in $\sigma(E,E')$,

$E_{USAK} = \{x \in E : x \text{ has the property USAK}\}$ where the property USAK is convergence of the net $(h_F \cdot x)_{F \in \Phi}$ in $\sigma(E,E')$,

and

$E_{UAK} = \{x \in E : x \text{ has the property UAK}\}$ where the property UAK is convergence of the net $(h_F \cdot x)_{F \in \Phi}$ in E.

The only other kinds of "section" convergence considered in this thesis are section convergence (AK) and Cesàro section convergence

(σK). The sequence x in E is said to have the property AK (respectively σK) if the sequence $(P_n \cdot x)$ (respectively $(\sigma^n \cdot x)$) converges in E. We adhere to the previous notation and write

$$E_{AK} = \{x \in E : x \text{ has the property AK}\}$$

and

$$E_{\sigma K} = \{x \in E : x \text{ has the property } \sigma K\}.$$

Finally, x in E is said to have the property section density (AD) if x belongs to the closure in E of φ. If T stands for an above mentioned property, a K-space E is said to be a T-space whenever each x in E has the property T. For the definition of symbols regarding specific sequence spaces used in the sequel and some of their properties the reader is referred to the next section.

We conclude this section by mentioning some of the functional-analytic theorems used in subsequent chapters.

I. Banach-Steinhaus Theorem. Let E be a barrelled topological vector space and F a locally convex Hausdorff topological vector space. If (T_n) is a sequence of continuous linear mappings of E into F which is pointwise convergent to T, then T is a continuous linear mapping [19, p.69].

II. Closed Graph Theorem. A linear mapping T from a Fréchet space E into a Fréchet space F is continuous if and only if whenever $x_n \to x_o$ in E and $T(x_n) \to y_o$ in F then $T(x_o) = y_o$ [11, p.168].

III. Mackey's Theorem. The bounded and weakly bounded subsets of a locally convex topological vector space are the same [11, p.254].

IV. Orlicz, Pettis, Grothendieck Theorem. In a locally convex Hausdorff space, if a series of elements has the property that every subseries is weakly convergent, then the series is unconditionally convergent (for the initial topology on the space) [18, p.153].

§3. Sequence Spaces: Subsets and Properties

(i) $bv = \{x \in \omega : \sum_{k=1}^{\infty} |x_k - x_{k+1}| < \infty\}$ - sequences of bounded variation. A BK, AB-space but not an AD-space with norm $\|x\|_{bv} = \sum_{k=1}^{\infty} |x_k - x_{k+1}| + \lim_k |x_k|$ [cf. 9].

$$B = \{x \in bv : \|x\|_{bv} \leq 1\}.$$

(ii) $bv_o = \{x \in bv : \lim_k x_k = 0\}$ - null sequences of bounded variation. A BK, AK-space with norm $\|\cdot\|_{bv}$.

$$B_o = \{x \in bv_o : \|x\|_{bv} \leq 1\}.$$

(iii) $\ell^1 = \{x \in \omega : \sum_{k=1}^{\infty} |x_k| < \infty\}$. A BK, UAK-space with norm $\|x\|_1 = \sum_{k=1}^{\infty} |x_k|$.

(iv) $\ell^\infty = \{x \in \omega : \sup_k |x_k| < \infty\}$ - bounded sequences. A BK, UAB-space but not an AD-space with norm $\|x\|_\infty = \sup_k |x_k|$.

$$C = \{x \in \ell^\infty : \|x\|_\infty \leq 1\}.$$

(v) $c = \{x \in \omega : \lim_k x_k$ exists$\}$ - convergent sequences. A BK, UAB-space but not an AD-space with norm $\|\cdot\|_\infty$.

$$C_\lambda = \{x \in c : \|x\|_\infty \leq 1\}.$$

(vi) $c_o = \{x \in c : \lim_k x_k = 0\}$ - convergent null sequences. A BK, UAK-space with norm $\|\cdot\|_\infty$.

$$C_o = \{x \in c_o : \|x\|_\infty \le 1\}.$$

$$C_\varphi^+ = \{x \in \varphi : 0 \le x_k \le 1,\ k = 1,2,\ldots\}.$$

$$C_\varphi = \{x \in \varphi : |x_k| \le 1,\ k = 1,2,\ldots\}.$$

(vii) $\sigma b = \{x \in \omega : \sup_n |\sum_{k=1}^{n} (\sigma^n \cdot x)_k| < \infty\}$. A BK-space [29, Satz 4.10] and a σB-space [4] with norm $\|x\|_{\sigma b} = \sup_n |\sum_{k=1}^{n} (\sigma^n \cdot x)_k|$.

(viii) $\sigma s = \{x \in \sigma b : \lim_{n\to\infty} \sum_{k=1}^{n} (\sigma^n \cdot x)_k$ exists$\}$. A BK-space [29] and a σK-space [13, Satz 2] with norm $\|\cdot\|_{\sigma b}$.

(ix) $q = \{x \in \ell^\infty : \sum_{k=1}^{\infty} k|\Delta^2 x_k| < \infty\}$ where $\Delta x_k = x_k - x_{k+1}$ and $\Delta^2 x_k = \Delta x_k - \Delta x_{k+1}$. The space of quasi-convex sequences is a BK, σB-space with norm $\|x\|_q = \sum_{k=1}^{\infty} k|\Delta^2 x_k| + \sup_k |x_k|$ [cf. 4]. If $x \in q$ then $n\Delta x_n \to 0$ [1,v.I,p.5],

$$\Delta x_n = \sum_{k=n}^{\infty} \Delta^2 x_k \quad [1,\text{v.II,p.202}],$$

and

$$\sum_{k=1}^{\infty} |\Delta x_k| \le \sum_{k=1}^{\infty} k|\Delta^2 x_k| \quad [\text{cf. } 4].$$

Hence $q \subset bv$ and $\|\cdot\|_{bv} \le \|\cdot\|_q$ on q.

$$Q = \{x \in q : \|x\|_q \le 1\}.$$

$m^2 = \{x \in \text{real } \omega : \Delta^2 x_k \ge 0,\ k = 1,2,\ldots\}$ - set of convex sequences.

$m_o^2 = \{x \in m^2 : \lim_k x_k = 0\}$ - (positive) convex null sequences.

$n^2 = \{x \in \text{real } \omega : \Delta^2 x_k \le 0,\ k = 1,2,\ldots\}$ - set of concave sequences.

Every bounded convex sequence is an element of q [1,v.1, p.5], and every real valued sequence in q is the difference of two positive convex bounded sequences [16]. It follows that q (even in the complex case) is the linear span of the positive bounded convex sequences.

(x) $q_o = q \cap c_o$ - quasi-convex null sequences. A BK, σK-space with norm $\|\cdot\|_q$ [cf. 4]. q_o is the linear span of m_o^2 ..

$$Q_o = \{x \in q_o : \|x\|_q \le 1\}.$$

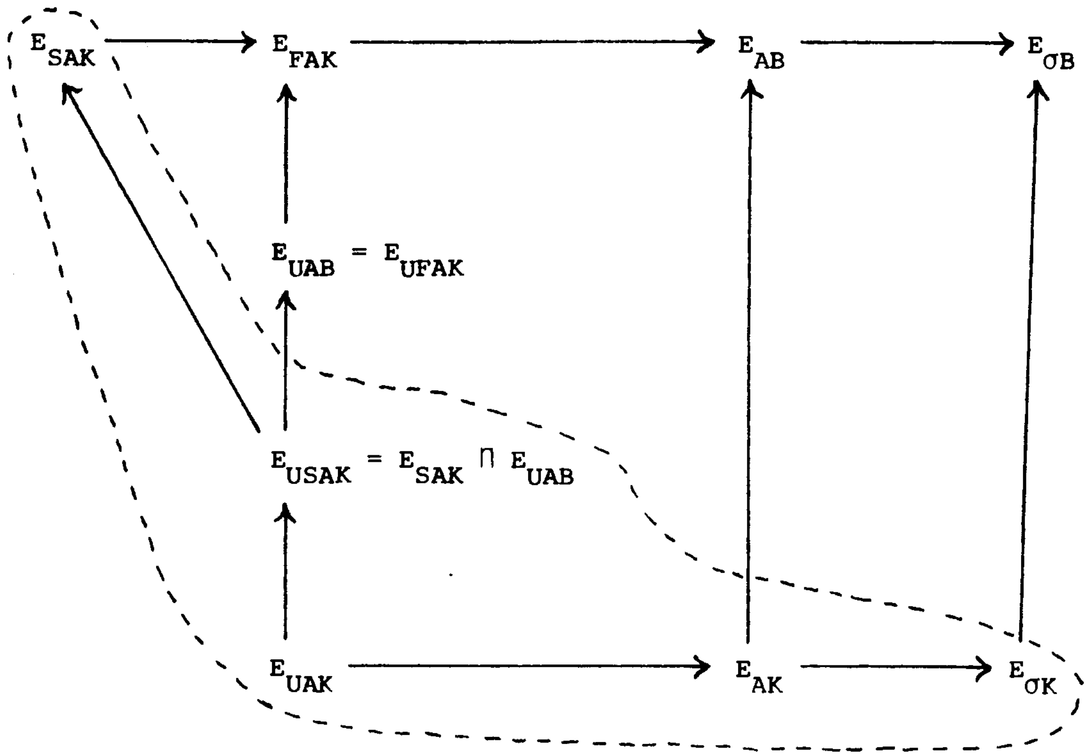
Chapter 2 - Front Sections and Invariance

§1. Introduction

The material in this chapter is essentially contained in D.J.H. Garling's paper [9] with the exception of Theorem 1 which is a modification of Theorem 6 in [5]. Also in §2 there is a diagram illustrating the relationships among the sets E_T , for various properties T, where E is a K-space. In §3 the significance in a K-space of the sets B and C are examined. The last section continues with the work of §3 after imposing several restrictions on the topology of the K-space. The result $E_{AK} = bv_o \cdot E_{AB}$ is obtained for an FK-space E and serves as a model for the corresponding section operators σ and H.

§2. Generalities

The following diagram illustrates the general set-theoretic relationships of the subsets E_T for an arbitrary K-space E. The arrow represents inclusion with the set resting at the head of the arrow the larger one. The dotted lines enclose those subsets E_T which are always contained in E.



The diagram follows immediately from the definitions of the subsets except for $E_{UAB} = E_{UFAK}$ and $E_{USAK} = E_{SAK} \cap E_{UAB}$ which are proved in §2, Chapter 4, and for $E_{AB} \subset E_{\sigma B}$ and $E_{AK} \subset E_{\sigma K}$ which are shown now. Let $x \in E_{AB}$ and p be a continuous semi-norm on E. Since $\sigma^n = n^{-1} \sum_{k=1}^{n} P_k$,

$$p(\sigma^n \cdot x) = n^{-1} p(\sum_{k=1}^{n} P_k \cdot x) \leq \max_{1 \leq k \leq n} p(P_k \cdot x).$$

Taking the supremum over $n \in Z^+$ shows x is in $E_{\sigma B}$. If x is in E_{AK} and p is a continuous semi-norm on E, then

$$p(\sigma^n \cdot x - x) = p(n^{-1}(\sum_{k=1}^{n} P_k \cdot x - nx)) \le n^{-1} \sum_{k=1}^{n} p(P_k \cdot x - x).$$

Given $\varepsilon > 0$, choose $M \in Z^+$ so that $n > M$ implies $p(P_n \cdot x - x) < \varepsilon/2$ and then $K \in Z^+$ so that $M[\sup_j p(P_j \cdot x - x)](M+K)^{-1} < \varepsilon/2$. Then for $n > M + K$

$$p(\sigma^n \cdot x - x) < M[\sup_{1 \le j \le M} p(P_j \cdot x - x)](M+K)^{-1} + n^{-1}(n-M) \cdot \varepsilon/2 < \varepsilon.$$

Before getting into the main theme of this chapter it is advantageous to prove some results on making new K-space from old ones. Later we will apply Theorem 1 to our three sets of section operators to deduce the factorization of the corresponding "section convergence space" in an FK-space. For example, the corresponding P-factorization is $E_{AK} = bv_o \cdot E_{AB}$.

<u>Theorem 1</u>. Let (E, τ) be a K-space whose topology is generated by the set $\mathcal{P}$ of semi-norms p. Let $\mathcal{F} \subset \varphi$ such that for each $n \in Z^+$, $0 < \sup_{t \in \mathcal{F}} |t_n| < \infty$, and define $E_{\mathcal{F}B} = \{x \in \omega : p_{\mathcal{F}}(x) = \sup_{t \in \mathcal{F}} p(t \cdot x) < \infty$, for each p in $\mathcal{P}\}$. Then $E_{\mathcal{F}B}$ with topology generated by the set $P_{\mathcal{F}} = \{p_{\mathcal{F}} : p \in \mathcal{P}\}$ of semi-norms is a sequentially complete K-space.

<u>Proof</u>. $E_{\mathcal{F}B}$ is a K-space since, for the semi-norm $p(x) = |x_k|$, we have $p_{\mathcal{F}}(x) = \sup_{t \in \mathcal{F}} |t_k| p(x)$ with $0 < \sup_{t \in \mathcal{F}} |t_k| < \infty$. Let $(x^{(n)})$ be a Cauchy sequence in $E_{\mathcal{F}B}$ with coordinatewise limit x. For each continuous semi-norm p on E and each $t \in \mathcal{F}$, there exists $m \in Z^+$ such that

$$p(t\cdot x - t\cdot x^{(m)}) \le 1,$$

as $x^{(m)}$ converges coordinatewise to x and $t \in \varphi$. Then

$$p(t\cdot x) \le p(t\cdot x - t\cdot x^{(m)}) + p(t\cdot x^{(m)}) \le 1 + p_{\mathcal{F}}(x^{(m)}).$$

It follows

$$p_{\mathcal{F}}(x) \le 1 + \sup_m p_{\mathcal{F}}(x^{(m)}) < \infty, \quad \text{and so} \quad x \in E_{\mathcal{F}B}.$$

For each $\varepsilon > 0$, there exists $N \in Z^+$ such that $p_{\mathcal{F}}(x^{(n)} - x^{(m)}) < \varepsilon/2$ whenever $n,m > N$. Also for each $t \in \mathcal{F}$, there exists $m > N$ such that $p(t\cdot x - t\cdot x^{(m)}) < \varepsilon/2$. Hence, for all $n > N$,

$$p(t\cdot x - t\cdot x^{(n)}) \le p(t\cdot x - t\cdot x^{(m)}) + p(t\cdot x^{(m)} - t\cdot x^{(n)}) < \varepsilon .$$

Thus, for all $n > N$, $p_{\mathcal{F}}(x-x^{(n)}) < \varepsilon$. Hence $(x^{(n)})$ converges to x in $E_{\mathcal{F}B}$.

Corollary. If E is a metrizable K-space, then $E_{\mathcal{F}B}$ is an FK-space.

Proof. A K-space is metrizable if and only if its topology is generated by a countable number of semi-norms. An FK-space is a sequentially complete metrizable K-space.

Let X and Y be subsets of ω. X is called Y-invariant if $Y\cdot X = X$. A K-space E is called locally Y-invariant if Y is an equicontinuous set of linear operators from E into itself.

Notice if e belongs to Y, locally Y-invariant implies Y-invariant. If $Y' \subset Y$ and E is locally Y-invariant, then E is locally Y'-invariant. Usually the set Y will be a semi-group of linear operators on E which is a subset of a BK-space. For each of the three sets of section operators we will develop the following association of BK-spaces:

$$P \leftrightarrow bv, \qquad \sigma \leftrightarrow q, \qquad H \leftrightarrow c.$$

Under the right circumstances each of the above listed spaces will be very informative with regard to the corresponding section operators action on a K-space. We should also remark that locally P-invariant and locally H-invariant spaces are AB- and UAB-spaces respectively. As previously mentioned we will be chiefly concerned with a semi-group of linear operators on a K-space E; when considering local invariance this will always be the case. The terminology is explained by the next theorem.

Theorem 2. Suppose that E is a topological vector space and that D is a semi-group of linear operators from E into itself. D is equicontinuous if and only if there is a base $\mathcal{U}$ of neighborhoods with the property that $d(U) \subset U$ for each d in D and U in $\mathcal{U}$. If further E is locally convex, D is equicontinuous if and only if the topology of E is defined by a collection $\mathcal{P}$ of semi-norms with the property that $p(x) \geq p(d(x))$ for each p in $\mathcal{P}$, d in D, and x in E.

Proof. The condition is certainly sufficient. Suppose that D is equicontinuous. Let $\mathcal{V}$ be a base of neighborhoods of 0. For each V in $\mathcal{V}$ let $U = \bigcap_{d \in D} d^{-1}(V) \cap V$. Since D is equicontinuous, U is a neighborhood of 0. If $x \in U$ then $x \in V$ and $d(x) \in V$ for each d in D. Since D is a semi-group, $d(x) \in U$, so that $d(U) \subset U$. The collection $\mathcal{U}$ of all such neighborhoods is clearly a base of neighborhoods so that the condition is necessary. If E is locally convex, $\mathcal{U}$ can be taken to be a collection of closed absolutely convex neighborhoods of 0 simply by starting with $\mathcal{V}$ as such. The characterization in terms of semi-norms is obtained by setting $\mathcal{P} = \{p_U : p_U$ is the gauge of U in $\mathcal{U}\}$.

Corollary 1. If (E,τ) is a K-space the following are equivalent:

(i) E is locally P-invariant;

(ii) E has a base $\mathcal{U}$ of (closed) absolutely convex neighborhoods of 0 with the property that $P(U) \subset U$, for each U in $\mathcal{U}$;

(iii) τ is defined by a collection $\mathcal{P}$ of semi-norms with the property that $p(x) \geq p(P_n \cdot x)$ for each p in $\mathcal{P}$, n in Z^+, and x in E.

Corollaries 2, 3, and 4. Locally H-invariant, locally B-invariant, and locally solid K-spaces can be characterized in an analogous way.

The Cesàro section operators do not form a semi-group of linear operators, so an analogous statement about locally σ-invariant does not follow.

Proposition 1. Suppose that E is a locally P-invariant (respectively locally H-invariant) space. An element x of E has the property AK (respectively UAK) if and only if it has the property AD.

Proof. We prove only the case where E is locally P-invariant. The other case is similar. Clearly if x has the property AK it has the property AD. Suppose x has the property AD and U is a basic absolutely convex neighborhood of 0 for which $P(U) \subset U$. There exists $y \in \varphi$ such that $x - y \in \frac{1}{2}U$. There exists an integer n_o such that $P_n \cdot y = y$ for $n \geq n_o$. Then if $n \geq n_o$

$$x - P_n \cdot x = x - y + P_n(y-x) \in \tfrac{1}{2}U + \tfrac{1}{2}U = U,$$

so that x has the property AK.

§3. Invariance of K-spaces

We set aside the operators σ and H temporarily, after first noting that in a K-space the members of the sets of operators P, σ, and H are continuous; being the finite sum of products of scalar multiples of projections and δ^k sequences.

Proposition 2. Suppose that E is a K-space, and that x is in E. Consider the four following statements:

(a) $C \cdot x \subset E_{AK}$;

(b) $C \cdot x$ is a bounded subset of E;

(c) $B_o \cdot x \subset E_{AK}$;

(d) $B \cdot x$ is a bounded subset of E.

We have the following implications:

$$(a) \Rightarrow (b) \Rightarrow (c) \Leftrightarrow (d).$$

Proof. Suppose that $C \cdot x \subset E_{AK}$. Let $G = \mathrm{span}(C \cdot x)$, with the subspace topology. G is an AK-space, so if f is a continuous linear functional on G, f can be written in the form $f(y) = \sum_{i=1}^{\infty} y_i f(\delta^i)$. In particular if $c \in C$, $f(c \cdot x) = \sum_{i=1}^{\infty} c_i x_i \, f(\delta^i)$; as this is true for each $c \in C$, $\sum_{i=1}^{\infty} |x_i f(\delta^i)|$ is convergent, and $|f(c \cdot x)| \leq \sum_{i=1}^{\infty} |x_i f(\delta^i)|$. Thus $C \cdot x$ is weakly bounded in G hence weakly bounded in E. By Mackey's theorem $C \cdot x$ is bounded in E. Thus (a) $\Rightarrow$ (b). Since $C \supset B$ we have (b) $\Rightarrow$ (d). Suppose that $B \cdot x$ is a bounded subset of E. In particular $B_o \cdot x$ is a bounded subset of E. If $b \in bv_o$, let

$T(b) = b\cdot x$. T is a linear map from bv_o into E, which is continuous since $T(B_o)$ is bounded [20, p.23]. Then $P_n(b\cdot x) = T(P_n\cdot b) \to T(b) = b\cdot x$ since $P_n\cdot b \to b$ in bv_o. Hence (d) ⇒ (c). We now show (c) ⇒ (d). Let $R = \text{span}(B_o\cdot x)$ with the subspace topology. As before, any continuous linear functional on R can be written in the form

$$f(y) = \sum_{i=1}^{\infty} y_i f(\delta^i). \tag{2.1}$$

For $\lambda \in B_o$,

$$\left|\sum_{i=1}^{k} \lambda_i x_i f(\delta^i)\right| = \left|\sum_{i=1}^{k-1} \{(\lambda_i-\lambda_{i+1}) \sum_{j=1}^{i} x_j f(\delta^j)\} + \lambda_k \sum_{i=1}^{k} x_i f(\delta^i)\right|$$

$$\le \left(\sum_{i=1}^{k-1} |\lambda_i-\lambda_{i+1}| + |\lambda_k|\right) \sup_n \left|\sum_{i=1}^{n} x_i f(\delta^i)\right|.$$

Therefore

$$\left|\sum_{i=1}^{\infty} \lambda_i x_i f(\delta^i)\right| \le \|\lambda\|_{bv} \sup_n \left|\sum_{i=1}^{n} x_i f(\delta^i)\right|. \tag{2.2}$$

Now suppose $\sup_n \left|\sum_{i=1}^{n} x_i f(\delta^i)\right| = \infty$. Then there exists an increasing sequence (n_i) of positive integers such that

$$\left|\sum_{j=n_i+1}^{n_{i+1}} x_j f(\delta^j)\right| \ge 4^i + 2^i \sum_{j=1}^{n_i} |x_j f(\delta^j)|.$$

Let $z_j = x_j$ for $j \le n_1$ and $z_j = 2^{-i}x_j$ for $n_i < j \le n_{i+1}$, $i = 1,2,\ldots$. Then $z \in B_o\cdot x$, and

$$\left|\sum_{j=1}^{n_{i+1}} z_j f(\delta^j)\right| \ge \left|\sum_{j=n_i+1}^{n_{i+1}} z_j f(\delta^j)\right| - \sum_{j=1}^{n_i} |z_j f(\delta^j)|$$

$$\ge 2^{-i}\left|\sum_{j=n_i+1}^{n_{i+1}} x_j f(\delta^j)\right| - \sum_{j=1}^{n_i} |x_j f(\delta^j)| \ge 2^i.$$

This contradicts (2.1) so we conclude from (2.2) that $B_o \cdot x$ is weakly bounded in R, and it follows that $B_o \cdot x$, and so $B \cdot x$ is bounded in E.

The following examples show there are no further implications:

(b) $\not\Rightarrow$ (a). Let $E = \ell^\infty$ with the supremum norm topology. $C \cdot e$ is a bounded subset of ℓ^∞, but $(1,0,1,0,\ldots)\cdot e = (1,0,1,0,\ldots)$ does not have the property AK.

(d) $\not\Rightarrow$ (b). Let $E = bv$ with norm $\|\cdot\|_{bv}$. Then $B \cdot e$ is a bounded subset of bv, but $(1,0,1,0,\ldots)\cdot e$ is not even a member of bv.

We immediately obtain the following corollaries which stress the power of Proposition 2 in the presence of an AK-space.

<u>Corollary 1</u>. If E is a solid (respectively B-invariant) AK-space, $C \cdot x$ (respectively $B \cdot x$) is bounded for each x in E.

<u>Corollary 2</u>. If E is a B_o-invariant K-space, E is an AK-space if and only if $B \cdot x$ is bounded for each x in E.

<u>Proposition 3</u>. Suppose E is a B-invariant locally P-invariant K-space. E is locally B-invariant if and only if $B \cdot x$ is bounded for each x in E.

<u>Proof</u>. The condition is certainly necessary. Suppose that it is satisfied, and let $\mathcal{P}$ be a defining collection of semi-norms satisfying the conditions of Corollary 1, Theorem 2. Suppose that $p \in \mathcal{P}$, $x \in E$, and $\theta \in B$. We can write $\theta = \lambda \cdot e + \psi$, where $\psi \in B_o$ and $|\lambda| + \|\psi\| \le 1$. Thus we have $p(\theta \cdot x) \le |\lambda| p(x) + p(\psi \cdot x)$. Now $\psi \cdot x$ has the property AK (Propostion 2) so that

$$p(\psi \cdot x) = \sup_n p(P_n \cdot (\psi \cdot x))$$

$$= \sup_n p\left(\sum_{i=1}^{n-1} (\psi_i - \psi_{i+1}) P_i \cdot x + \psi_n P_n \cdot x\right)$$

$$\leq \sup_n \left(\left(\sum_{i=1}^{n-1} |\psi_i - \psi_{i+1}| + |\psi_n|\right) \sup_{i \leq n} p(P_i \cdot x)\right)$$

$$= \|\psi\| \sup_n p(P_n \cdot x) \leq \|\psi\| p(x) .$$

Hence $p(\theta \cdot x) \leq p(x)$ and E is locally B-invariant (Corollary 3, Theorem 2).

Corollary. If E is a B-invariant AK-space, E is locally P-invariant if and only if E is locally B-invariant.

Proof. Apply Corollary 1, Proposition 2.

Proposition 3 and its corollary do not extend to solid spaces however, For example, let τ be a topology on ℓ^∞ defined by the semi-norms $\{p_x\}_{x \in \ell^1}$, where

$$p_x(y) = \sup_n \left| \sum_{i=1}^{n} x_i y_i \right| .$$

(ℓ^∞, τ) is a locally P-invariant solid AK-space, and $C \cdot x$ is bounded for each x in ℓ^∞ (Corollary 1, Theorem 2). (ℓ^∞, τ) is not locally solid (Corollary 4, Theorem 2).

As far as replacing B by C_λ and P by H in Proposition 3, it seems likely to go through considering the association $P \leftrightarrow bv$ and $H \leftrightarrow c$, although I have been unable to prove it.

§4. Further Topological Conditions

In §3 we saw, in general, that B-invariant and AK are two properties that are especially strong together in a K-space. Here we examine the situation when E satisfies one or more of the following conditions:

(a) E is sequentially complete;

(b) E is barrelled;

(c) E is a Fréchet space.

Spaces of the above type are extremely nice as they will often reduce otherwise unrelated properties to a single one. For example when E is a sequentially complete K-space, an AB-space is B-invariant (Corollary 2, Proposition 6); so a sequentially complete AK-space is a priori B-invariant. Conditions (a) and (b) will be useful when E is an AB-space. (c) allows use of the closed graph theorem, producing correspondingly more powerful results.

Proposition 4. (i) If E is a barrelled K-space, E is locally P-invariant (respectively locally H-invariant) if and only if E is an AB(respectively UAB)-space.

(ii) If E is a barrelled B-invariant K-space, E is locally B-invariant if and only if $B\cdot x$ is bounded for each x in E.

Proof. The first result follows since any pointwise bounded set of continuous linear maps from a barrelled space into a locally convex space is equicontinuous. For (ii) just apply Proposition 3 and Proposition 4, (i).

Corollary 1. Suppose that E is a barrelled K-space with $E \subset E_{AB}$. Then the following are equivalent:

(i) $x \in E_{AK}$;

(ii) $x \in E_{SAK}$;

(iii) x has the property AD.

Proof. Proposition 4, (i) together with Proposition 1 show that (i) is equivalent to (iii). (i) implies (ii) in any K-space, so we are left to prove (ii) implies (i). x in E_{SAK} implies x belongs to the weak closure of $\{P_n \cdot x\}$ and hence to the weak closure of the convex hull of $\{P_n \cdot x\}$, $\Gamma\{P_n \cdot x\}$. As $\Gamma\{P_n \cdot x\} \subset \varphi$, and since the closure in the original topology and the weak closure of convex sets agree in a locally convex space, x has the property AD, hence belongs to E_{AK}.

Corollary 2. A barrelled B-invariant AK-space is locally B-invariant.

Proof. See Corollary 1, Proposition 2.

Proposition 5. A barrelled solid AK-space E is locally solid.

Proof. Let $c \in C$. Define for $m \in Z^+$,

$$f_m(x) = \sum_{i=1}^{m} c_i x_i \delta^i \qquad (x \in E).$$

Since E is an AK-space $f_m(x) \to c \cdot x$. By the Banach-Steinhaus theorem $f(x) = c \cdot x$ is a continuous linear mapping from E into E. The result now follows from Corollary 1, Proposition 2 and the definitions.

The next result guarantees we can "complete" the set P to B_o in a sequentially complete space in the sense that $P \cdot x$ being bounded insures $B_o \cdot x$ is also bounded.

Proposition 6. Suppose that E is a sequentially complete K-space. Let x in E_{AB}. Then $B_o \cdot x$ is bounded and $B_o \cdot x \subset E_{AK}$.

Proof. Suppose x has bounded sections. Let $z = \lambda \cdot x$ with $\lambda \in B_o$, and let p be a continuous semi-norm on E. Since $P_n \cdot z = \sum_{i=1}^{n-1} (\lambda_i - \lambda_{i+1}) P_i \cdot x + \lambda_n P_n \cdot x$, we have

$$p(P_n\cdot z - P_m\cdot z) = p\left(\sum_{i=m}^{n-1}(\lambda_i-\lambda_{i+1})P_i\cdot x + \lambda_n P_n\cdot x - \lambda_m P_m\cdot x\right)$$

$$\le \sum_{i=m}^{n-1}|\lambda_i-\lambda_{i+1}|p(P_i\cdot x) + |\lambda_n|p(P_n\cdot x) + |\lambda_m|p(P_m\cdot x)$$

$$\le \left(|\lambda_m| + \sum_{i=m}^{n-1}|\lambda_i-\lambda_{i+1}| + |\lambda_n|\right)\sup_{m\le i\le n} p(P_i\cdot x)$$

for $m < n$. It follows that $(P_i\cdot z)$ is a Cauchy sequence in E; as E is a K-space $(P_i\cdot z)$ must converge to $z \in E$, thus $B_o\cdot x \subset E_{AK}$. The result follows from the proof of (c) $\Rightarrow$ (d) of Proposition 2.

Corollary 1. If E is a sequentially complete K-space and x is in E, $P\cdot x$ is bounded if and only if $B\cdot x$ is bounded.

Corollary 2. If E is a sequentially complete AB-space, E is B-invariant and $B\cdot x$ is bounded for each x in E.

Corollary 3. If E is a sequentially complete locally P-invariant space, E is locally B-invariant.

Proof. Apply Proposition 3.

Notice here, with the additional hypothesis of sequential completeness, we did not have to assume E was B-invariant and $B\cdot x$ was bounded for each x in E as in Proposition 3.

Corollary 4. If E is a sequentially complete B_o-invariant K-space, E is an AK-space if and only if E is an AB-space.

Proof. $E \subset E_{AB}$ implies $E = B_o\cdot E \subset E_{AK}$.

Corollary 5. If E is a barrelled sequentially complete K-space, E is an AB-space if and only if E is locally B-invariant.

For the remainder of this chapter we treat exclusively FK-spaces.

Lemma 1. If E is an FK-space, $E_{AK} \subset B_o\cdot E$.

Proof. Suppose $x \in E_{AK}$. Let (p_i) be an increasing sequence of defining semi-norms for the topology of E. There exists an increasing sequence (n_j) of positive integers such that

$$p_j(x-P_n \cdot x) < 4^{-j} \quad \text{for} \quad n \geq n_j .$$

Let $z_i = x_i$ for $1 \leq i \leq n_1$

and let $z_i = 2^j x_i$ for $n_j < i \leq n_{j+1}$.

If $n_1 < r < s$, where $n_j < r \leq n_{j+1}$ and $n_k < s \leq n_{k+1}$,

$$p_1(P_s \cdot z - P_r \cdot z) \leq p_1(P_{n_{j+1}} \cdot z - P_r \cdot z) + p_1\Big(\sum_{m=j+1}^{k-1} (P_{n_{m+1}} \cdot z - P_{n_m} \cdot z) + P_s \cdot z - P_{n_k} \cdot z\Big)$$

$$\leq p_1(P_{n_{j+1}} \cdot z - P_r \cdot z) + \sum_{m=j+1}^{k-1} p_1(P_{n_{m+1}} \cdot z - P_{n_m} \cdot z) + p_1(P_s \cdot z - P_{n_k} \cdot z)$$

$$\leq 2^j p_j(P_{n_{j+1}} \cdot x - P_r \cdot x) + \sum_{m=j+1}^{k-1} 2^m p_m(P_{n_{m+1}} \cdot x - P_{n_m} \cdot x) + 2^k p_k(P_s \cdot x - P_{n_k} \cdot x)$$

$$\leq 2 \sum_{m=j}^{k} 2^{-m} \leq 2^{2-j} .$$

Hence $(P_n \cdot z)$ is a Cauchy sequence in E which converges to z, since E is an FK-space. Thus $z \in E$; since $x \in B_o \cdot z$, $x \in B_o \cdot E$.

Theorem 3. Suppose that E is an FK-space. The following are equivalent:

(i) E is B-invariant;

(ii) E is locally B-invariant;

(iii) E is an AB-space;

(iv) $B_o \cdot E$ is a closed linear subspace of E;

(v) $B_o \cdot E = E_{AK}$.

Proof. The equivalency of (ii) and (iii) follows from Corollary 2, Proposition 6. Suppose E is B-invariant and $x \in E$. Define

$$T_x : bv \to E \quad \text{by} \quad T_x(y) = x \cdot y.$$

T_x is clearly linear, and, since E is a K-space has a closed graph. By the closed graph theorem T_x is continuous, and so, $T_x(B) = B \cdot x$ is bounded in E. As E is barrelled it is locally B-invariant (Proposition 4, (ii)). The converse is trivial, hence we can now conclude the equivalence of (i), (ii), and (iii). By Lemma 1 $E_{AK} \subset B_o \cdot E$. If E is locally B-invariant $B_o \cdot E \subset E_{AK}$ by Proposition 2, so that (ii) implies (v). If E is locally B-invariant, $B_o \cdot E = E_{AK}$ = closure (φ) by (ii) implies (v) and Proposition 1; so $B_o \cdot E$ is a closed linear subspace of E. Finally (i) is implied by either (iv) or (v).

Corollary 1. Suppose E is an FK-space. The following statements are equivalent:

(i) E is a B-invariant AD-space;

(ii) E is an AK-space;

(iii) E is B_o-invariant.

Corollary 2. Suppose E is a solid FK-space. E is locally solid, and $E_{AK} = C_o \cdot E$.

Proof. Let $T_x : \ell^\infty \to E$ be defined by $T_x(y) = x \cdot y$, where $x \in E$. As in the proof of Theorem 3, T_x is a continuous linear map and so $T_x(C) = C \cdot x$ is a bounded subset of E. The map $S_x : E \to E$ defined by $S_x(y) = x \cdot y$ is linear for each x in C. It is also continuous by the closed graph theorem. This shows E is locally solid, and $C_o \cdot E = B_o \cdot E = E_{AK}$ since E is solid.

It follows from Theorem 3 that if E is a B-invariant FK-space, E_{AK} is a closed linear subspace of E. That the converse is false is illustrated by the following.

<u>Example</u>: c_o is an FK-space with the property AK if endowed with the norm $\|\cdot\|_\infty$. Let $z = (2^i)$ and let $E = c_o \oplus \operatorname{span}(z)$; if $x = y + \lambda z \in E$, let $p(x) = \|y\|_\infty$ and $q(x) = |\lambda|$. Then E is an FK-space under the semi-norms $\{p,q\}$. E is not B-invariant $((z^{-i}) \in B$ and $(2^{-i})\cdot z = e)$, while $E_{AK} = c_o$ which is closed in E.

<u>Lemma 2</u>. Let E be a sequentially complete K-space. Then $(E_{AB}, \mathcal{P}_P)_{AK} \equiv (E_{PB}, \mathcal{P}_P)_{AK} = E_{AK}$. (For the notation see Theorem 1).

<u>Proof</u>. Let $x \in E_{AK}$ and let $p \in \mathcal{P}$. There exists $K \in Z^+$ such that if $n,k \geq K$, $p(P_k\cdot x - P_n\cdot x) < \varepsilon$; where $\varepsilon > 0$ is a prescribed constant given in advance. Then

$$p_P(P_k\cdot x - x) = \sup_n p(P_n\cdot(P_k\cdot x - x)) = \sup_{n>k} p(P_k\cdot x - P_n\cdot x) < \varepsilon$$

if $k \geq K$. Thus $E_{AK} \subset (E_{AB})_{AK}$. Let $x \in (E_{AB})_{AK}$ let $p \in \mathcal{P}$, and let $\varepsilon > 0$ be given. If $k,j \geq K$ where $K \in Z^+$ is chosen so that $p_P(P_k\cdot x - P_j\cdot x) < \varepsilon$ whenever $k,j \geq K$, then

$$p(P_k\cdot x - P_j\cdot x) \leq \sup_m p(P_m\, (P_k\cdot x - P_j\cdot x)) = p_P(P_x\cdot x - P_j\cdot x) < \varepsilon .$$

Thus $(P_k\cdot x)$ is a Cauchy sequence in E; since E is a sequentially complete K-space $(P_k\cdot x)$ converges to x in E.

We now prove the factorization theorem relating the sequences with convergent sections to the sequences with bounded sections via the BK-space associated with P, bv. Although this theorem is attributed to Garling because of [9], this is, so far as I know, the first time an explicit proof appears in print.

Theorem 4. If E is an FK-space, then $E_{AK} = bv_o \cdot E_{AB}$.

Proof. By Theorem 1 $(E_{AB}, \mathcal{P}_P)$ is an FK-space. Let $x \in E_{AB}$ and let $p \in \mathcal{P}$. We have

$$\sup_n p_P(P_n \cdot x) = \sup_n \sup_m p(P_m \cdot (P_n \cdot x)) = \sup_n \sup_{m \leq n} p(P_m \cdot (P_n \cdot x))$$

$$= \sup_n p(P_n \cdot x) < \infty,$$

showing $E_{AB} \subset (E_{AB}, \mathcal{P}_P)_{AB}$. Applying Theorem 3 to $(E_{AB}, \mathcal{P}_P)$ and using Lemma 2, we obtain $B_o \cdot E_{AB} = (E_{AB})_{AK} = E_{AK}$.

Chapter 3 - Cesàro Sections

§1. Introduction

Here we treat Cesàro section boundedness of a sequence and obtain analogous factorizations for FK-spaces to those derived in Chapter 2. In each instance a suitable subset of the quasi-convex sequences q is used in place of the corresponding subset of bv. It is shown in an FK-space that those sequences with Cesàro section convergence are exactly the product of sequences in q_o with sequences having Cesàro section boundedness. In §4 the main results (§3) are applied to obtain theorems on multiplier spaces; §5 contains more applications of §3, this time extending some results of Littlewood and Paley, and Salem concerning certain factorizations under convolution.

§2. Basic Facts

When dealing with Cesàro sections we will have need to consider the properties of the quasi-convex sequences q (see Chapter 1, §3), as they are the associated BK-space of the set σ. The first two lemmas are stated without proof since their verification involves only simple computations.

Lemma 1. Let (m_k) be any strictly increasing sequence of positive real numbers and let (N_k) be a strictly increasing sequence of integers for which $N_1 = 1$ and for all $k \in \{2,3,\dots\}$,

$$N_{k+1}(m_k - m_{k-1}) \geq N_k(m_{k+1} - m_{k-1}) - N_{k-1}(m_{k+1} - m_k).$$

Let y be the sequence which, for each k, $y_{N_k} = m_k$ and is linear between N_k and N_{k+1}. Then y is concave and the sequence λ given by $\lambda_n = y_n^{-1}$ is a convex null sequence.

Lemma 2. For all $\lambda \in \omega$,

$$P_n \cdot \lambda = \sum_{k=1}^{n-1} k\Delta^2\lambda_k \sigma^k + (n-1)\Delta\lambda_n \sigma^{n-1} + \lambda_n P_n \tag{3.1}$$

$$= \sum_{k=1}^{n-1} k\Delta^2\lambda_k \sigma^k + n\lambda_n \sigma^n - (n-1)\lambda_{n+1}\sigma^{n-1} . \tag{3.2}$$

The first proposition indicates the compatibility of q with the Cesàro sections σ. Corollary 3 then illustrates the role of q in a specialized situation as the associated BK-space for the Cesàro sections σ.

Proposition 1. For any semi-norm p on a sequence space E containing $\sigma \cdot x$,

$$p_\sigma(x) \equiv \sup_n p(\sigma^n \cdot x) = \sup_n p_\sigma(\sigma^n \cdot x) = \sup_{\lambda \in Q} p_\sigma(\lambda \cdot x) .$$

Proof. $p_\sigma(x) = \sup_n p(\sigma^n \cdot x) = \sup_n \lim_{m\to\infty} p(\sigma^m \cdot \sigma^n \cdot x)$

$$\le \sup_n \sup_m p(\sigma^m \cdot \sigma^n \cdot x) = \sup_n p_\sigma(\sigma^n \cdot x) .$$

For the reverse inequality, let $\lambda = \sigma^n \cdot \sigma^m$. Then by Lemma 2 (3.2)

$$\sup_n p_\sigma(\sigma^n \cdot x) = \sup_{m\le n} p(\sigma^n \cdot \sigma^m \cdot x) = \sup_{m\le n} p\Big(\sum_{k=1}^{m-1} k\Delta^2\lambda_k (\sigma^k \cdot x) + m\lambda_m (\sigma^m \cdot x)\Big)$$

$$\le \sup_{m\le n}\Big(\frac{2}{mn} \sum_{k=1}^{m-1} k + \Big(1 - \frac{m-1}{n}\Big)\Big) \sup_k p(\sigma^k \cdot x)$$

$$= \sup_{m\le n}\Big(1 - \frac{2}{m} + \frac{2}{mn}\Big) \sup_k p(\sigma^k \cdot x) = \sup_k p(\sigma^k \cdot x) = p_\sigma(x) .$$

Thus $p_\sigma(x) = \sup_n p_\sigma(\sigma^n \cdot x)$.

For $\lambda \in Q$,

$$\begin{aligned} p(\lambda\cdot\sigma^n\cdot x) &= \lim_{m\to\infty} p(P_m\cdot\lambda\cdot\sigma^n\cdot x) \\ &= \lim_{m\to\infty} p\Big(\sum_{k=1}^{m-1} k\Delta^2\lambda_k(\sigma^k\cdot\sigma^n\cdot x) + (m-1)\Delta\lambda_m(\sigma^{m-1}\cdot\sigma^n\cdot x) + \lambda_m(\sigma^n\cdot x)\Big) \\ &\le \lim_{m\to\infty} \Big(\sum_{k=1}^{m-1} |k\Delta^2\lambda_k| + (m-1)|\Delta\lambda_m| + |\lambda_m|\Big)p_\sigma(x) \\ &\le \|\lambda\|_q\, p_\sigma(x), \end{aligned}$$

where Lemma 2 (3.1) was used and the fact that if $x \in q$ then $\Delta x_n = \sum_{k=n}^{\infty} \Delta^2 x_k$. Taking the supremum over n yields $\sup_{\lambda\in Q} p_\sigma(\lambda\cdot x) \le p_\sigma(x)$, while the converse inequality follows as $e \in Q$.

Corollary 1. For any K-space E, $E_{\sigma B} = (E_{\sigma B})_{\sigma B}$.

Although the corresponding result was not stated for P, it does hold, and its proof is rather simple.

Corollary 2. For any K-space E, $E_{\sigma B}$ is a σB-space.

Corollary 3. For any K-space E, $E_{\sigma B} = q\cdot E_{\sigma B}$.

The next result is the stepping stone to the corresponding result achieved in Lemma 2, Chapter 2.

Proposition 2. For any K-space E, $E_{\sigma K} \subset (E_{\sigma B})_{\sigma K}$.

Proof. Using Lemma 2 (3.2) and the following facts:

(a) if $n \le m$ then $\Delta^2(\sigma^n\cdot\sigma^m - \sigma^n)_k = \frac{2}{mn}$ for $1 \le k \le n-1$;

(b) if $n > m$ then

$$\Delta^2(\sigma^n\cdot\sigma^m - \sigma^n)_k = \frac{2}{mn} \quad \text{for } 1 \le k \le m-1,$$

$$\Delta^2(\sigma^n\cdot\sigma^m - \sigma^n)_k = \frac{n-m+1}{mn} \quad \text{for } k = m,$$

and

$$\Delta^2(\sigma^n \cdot \sigma^m - \sigma^n)_k = 0 \quad \text{for} \quad m+1 \le k \le n-1;$$

it is not difficult to show that

$$\sigma^n \cdot (\sigma^m \cdot x - x) = \sum_{k=1}^{n-1} k\Delta^2(\sigma^n \cdot \sigma^m - \sigma^n)_k(\sigma^k \cdot x - x) + ((\sigma^m)_n - 1)(\sigma^n \cdot x - x). \tag{3.3}$$

Suppose p is a continuous semi-norm on E, and $p(\sigma^n \cdot x - x) \to 0$. For $\varepsilon > 0$, let N be chosen such that, for all $k \ge N$,

$$p(\sigma^k \cdot x - x) < \varepsilon/4 ,$$

and let $M > N$ be chosen such that

$$\frac{N}{M} \sup_{k \le N} p(\sigma^k \cdot x - x) < \varepsilon/2 .$$

Suppose $m > M$. We consider three cases: if $n \le N$ then using (3.3)

$$\begin{aligned} p(\sigma^n \cdot (\sigma^m \cdot x - x)) &\le \frac{2}{mn} \sum_{k=1}^{n-1} k\, p(\sigma^k \cdot x - x) + \frac{n-1}{m}\, p(\sigma^n \cdot x - x) \\ &\le \frac{2(n-1)}{m} \sup_{k \le n} p(\sigma^k \cdot x - x) < \varepsilon ; \end{aligned}$$

if $N < n \le m$ then

$$\begin{aligned} p(\sigma^n \cdot (\sigma^m \cdot x - x)) &\le \frac{2}{mn} \sum_{k=1}^{N} k\, p(\sigma^k \cdot x - x) + \frac{2}{mn} \sum_{k=N+1}^{n-1} k\, p(\sigma^k \cdot x - x) \\ &+ \frac{n-1}{m}\, p(\sigma^n \cdot x - x) \le \frac{N}{m} \sup_{k \le N} p(\sigma^k \cdot x - x) + \frac{2(n-1)}{m} \sup_{k > N} p(\sigma^k \cdot x - x) < \varepsilon; \end{aligned}$$

if $m < n$ then

$$\begin{aligned} p(\sigma^n \cdot (\sigma^m \cdot x - x)) &\le \frac{2}{mn} \sum_{k=1}^{N} k\, p(\sigma^k \cdot x - x) + \frac{2}{mn} \sum_{k=N+1}^{m-1} k\, p(\sigma^k \cdot x - x) \\ &+ (1 - \frac{m-1}{n})p(\sigma^m \cdot x - x) + p(\sigma^n \cdot x - x) \\ &\le \frac{N}{m} \sup_{k \le N} p(\sigma^k \cdot x - x) + 2 \sup_{k > N} p(\sigma^k \cdot x - x) < \varepsilon. \end{aligned}$$

Thus, whenever $m > M$, $p_\sigma(\sigma^m \cdot x - x) < \varepsilon$.

It is not always true that $(E_{\sigma B})_{\sigma K} = E_{\sigma K}$ or, for that matter, that $(E_{AB})_{AK} = E_{AK}$ as the following construction indicates. Let E be any K-space with the property σK (respectively AK) properly containing φ and let $x \in E - \varphi$. By Zorn's Lemma there exists a maximal subspace $F \supset \varphi$ such that $x \notin F$. Clearly, if F is given the subspace topology,

$$(F_{\sigma B})_{\sigma K} = (E_{\sigma B})_{\sigma K} \supset E \supsetneq F = F_{\sigma K}$$

(respectively $(F_{AB})_{AK} = (E_{AB})_{AK} \supset E \supsetneq F = F_{AK}$).

<u>Proposition 3</u>. If E is a sequentially complete K-space then $E_{\sigma K} = (E_{\sigma B})_{\sigma K}$.

<u>Proof</u>. Considering Proposition 2, it suffices to show that if $(\sigma^n \cdot x)$ is a Cauchy sequence of $E_{\sigma B}$ then it is a Cauchy sequence of E. But this is clear since for every continuous semi-norm p,

$$p(\sigma^n \cdot x - \sigma^m \cdot x) = \lim_k p(\sigma^k \cdot (\sigma^n \cdot x - \sigma^m \cdot x)) \le p_\sigma(\sigma^n \cdot x - \sigma^m \cdot x).$$

<u>Proposition 4</u>. Let E be a K-space. A sequence x is in $(E_{\sigma B})_{\sigma K}$ if and only if x has the property AD with respect to the topology of $E_{\sigma B}$.

<u>Proof</u>. Let p be a continuous semi-norm on E. If x has the property AD with respect to $E_{\sigma B}$ then there exists a y in φ such that

$$p_\sigma(x-y) < \varepsilon/3 .$$

Since $\varphi \subset (E_{\sigma B})_{\sigma K}$ there exists an $N \in Z^+$ such that

$$p_\sigma(y - \sigma^n \cdot y) < \varepsilon/3 \quad \text{for every } n > N.$$

We have by Proposition 1,

$$p_\sigma(x-\sigma^n\cdot x) \leq p_\sigma(x-y) + p_\sigma(y-\sigma^n\cdot y) + p_\sigma(\sigma^n\cdot(y-x)) < \varepsilon$$

for every $n > N$. The converse is immediate.

Corollary. If E is an FK-space then $E_{\sigma K}$ is an FK-space under the topology of $E_{\sigma B}$.

Proof. It follows from the corollary to Theorem 1, Chapter 2 that $E_{\sigma B}$ is an FK-space. By Proposition 3 $E_{\sigma K} = (E_{\sigma B})_{\sigma K}$ which is closed in $E_{\sigma B}$ by Proposition 4.

§3. Main Results

Theorem 1 expresses the property σB in terms of the associated space q. Again similar results can be found in Chapter 2 for the property AB and bv.

Theorem 1. Let E be an FK-space. The following statements are equivalent:

(i) $x \in E_{\sigma B}$;

(ii) for each continuous linear functional f on E, the sequence y defined by $y_k = f(x\cdot\delta^k)$ is an element of σb;

(iii) $q_o\cdot x \subset E_{\sigma K}$;

(iv) $q_o\cdot x \subset E$;

(v) $Q_o\cdot x$ is a bounded subset of E;

(vi) $Q\cdot x$ is a bounded subset of $E_{\sigma B}$.

Proof. (iv) ⇒ (iii): Let $T : q_o \to E$ be given by $T(\lambda) = \lambda\cdot x$. Since E is a K-space T has a closed graph and hence is continuous by the closed graph theorem. Since q_o is a σK-space,

$$\sigma^n\cdot\lambda\cdot x = T(\sigma^n\cdot\lambda) \to T(\lambda) = \lambda\cdot x \quad \text{for every} \quad \lambda \in q_o .$$

(iii) $\Rightarrow$ (iv) is clear.

(iv) $\Rightarrow$ (v): $T(Q_o) = Q_o\cdot x$ is bounded since T is continuous.

(v) $\Rightarrow$ (i) follows as $\sigma \subset 2Q_o$.

(i) $\Leftrightarrow$ (vi) follows from Proposition 1.

(i) $\Leftrightarrow$ (ii): By Mackey's theorem, (i) is equivalent to the condition $\sigma\cdot x$ is a weakly bounded subset of E. That is, for each continuous linear functional f on E,

$$\sup_n |f(\sigma^n\cdot x)| = \sup_n \Big|\sum_{k=1}^{n} (1 - \frac{k-1}{n})f(x\cdot\delta^k)\Big| = \|y\|_{\sigma b} < \infty .$$

(i) $\Rightarrow$ (iii): If $x \in E_{\sigma B}$, then by Proposition 1, Corollary 3 $q_o\cdot x \subset E_{\sigma B}$. Thus, by (iv) $\Rightarrow$ (iii) ($E_{\sigma B}$ is an FK-space by Corollary to Theorem 1, Chapter 2) and Propostion 3, $q_o\cdot x \subset (E_{\sigma B})_{\sigma K} = E_{\sigma K}$.

Corollary. If $x \in E$ then the following statements may be added to the list in Theorem 1:

(vii) $q\cdot x \subset E$;

(viii) $Q\cdot x$ is a bounded subset of E.

Proof. (iv) $\Leftrightarrow$ (vii) and (viii) $\Rightarrow$ (v) are immediate. That $Q\cdot x \subset \Gamma\{2(\{x\} \cup Q_o\cdot x)\}$ yields (v) $\Rightarrow$ (viii).

Next we examine the space E as a whole.

Theorem 2. Let E be an FK-space. The following statements are equivalent:

(i) E is a σB-space;

(ii) $E = q\cdot E$;

(iii) $q_o\cdot E \subset E_{\sigma K}$;

(iv) $q_o \cdot E = E_{\sigma K}$;

(v) $m_o^2 \cdot E = E_{\sigma K}$.

Proof. The equivalence of (i), (ii), and (iii) follows from the corollary to Theorem 1. Suppose condition (iii) holds. To conclude that condition (v) holds it is sufficient to show that for any FK-space E, $E_{\sigma K} \subset m_o^2 \cdot E_{\sigma K}$, as $E_{\sigma K} \subset E$. By the corollary to Proposition 4 and Propostion 1, $E_{\sigma K}$ is an FK-space with a set of defining semi-norms $P = \{p_m : m \in Z^+\}$ such that

$$p_m(x) = \sup_n p_m(\sigma^n \cdot x)$$

and

$$p_m(x) \leq p_{m+1}(x)$$

where $m \in Z^+$ and $x \in E_{\sigma K}$.

Let $x \in E_{\sigma K}$ and set $(m_k) = (2^k)$. For each positive integer k choose N_k such that

(a) all N_k are integers

(b) $1 = N_1 < N_2 < \ldots$;

(c) $N_{k+1} \geq 3N_k - 2N_{k-1} \quad (k = 2,3,4,\ldots)$;

(d) $p_j(\sigma^r \cdot x - \sigma^s \cdot x) < 4^{-j}$ whenever $r,s \geq N_j$.

Under these conditions construct sequences y and λ as in Lemma 1. Then $x = \lambda \cdot (y \cdot x)$ where $\lambda \in m_o^2$, and it remains to show $y \cdot x \in E_{\sigma K}$. Consider the semi-norm p_m. If $N_{k+1} \geq s \geq r \geq N_k \geq N_i > N_m$ then by Lemma 2 (3.1),

$$p_m(\sigma^r \cdot y \cdot x - \sigma^s \cdot y \cdot x) \leq \sum_{t=1}^{s-1} t|\Delta^2 y_t| p_m(\sigma^t \cdot (\sigma^r \cdot x - \sigma^s \cdot x))$$
$$+ (s-1)|\Delta y_s| p_m(\sigma^{s-1} \cdot (\sigma^r \cdot x - \sigma^s \cdot x)) + |y_s| p_m(\sigma^r \cdot x - \sigma^s \cdot x)$$

$$\leq \left(\sum_{t=1}^{s-1} t(-\Delta^2 y_t) + (s-1)(-\Delta y_s) + y_s \right) p_m(\sigma^r \cdot x - \sigma^s \cdot x)$$

$$= (-y_1 + 2y_s) p_m(\sigma^r \cdot x - \sigma^s \cdot x) < (-2+2^{k+2})(4^{-k}) < 2^{-k+2}.$$

Thus, for $s \geq r \geq N_i > N_m$,

$$p_m(\sigma^r \cdot y \cdot x - \sigma^s \cdot y \cdot x) < \sum_{t=i}^{\infty} 2^{-t+2} = 2^{-i+3} .$$

Hence $y \cdot x \in E_{\sigma K}$ since $E_{\sigma K}$ is an FK-space. (v) implies (iv) since the linear span of m_o^2 is q_o ; while (iv) implies (iii) is immediate.

<u>Corollary 1</u>. If E is an FK-space then $E_{\sigma K} = q_o \cdot E_{\sigma B}$.

<u>Proof</u>. $E_{\sigma B}$ is an FK-space with $E_{\sigma B} = (E_{\sigma B})_{\sigma B}$ and $E_{\sigma K} = (E_{\sigma B})_{\sigma K}$. Thus $E_{\sigma K} = (E_{\sigma B})_{\sigma K} = q_o \cdot E_{\sigma B}$.

Corollary 1 is the expected counterpart to Theorem 4, Chapter 2.

<u>Corollary 2</u>. If E is an FK-space the following statements are equivalent:

(i) E is a σK-space;

(ii) $E = q_o \cdot E$;

(iii) $E = m_o^2 \cdot E$;

(iv) E is an AD-space and a σB-space;

(v) E is an AD-space and $q_o \cdot E \subset E$.

<u>Proof</u>. Evidently (i), (ii), and (iii) are equivalent while (i) implies (iv) and (iv) implies (v). If (v) holds then $q \cdot E = E$ and so E is an AD-space and a σB-space. Proposition 3 and Proposition 4 together with the topology of $E_{\sigma B}$ restricted to E being weaker than the FK-topology of E gives $E = E_{\sigma K}$.

§4. Some Applications to Multipliers

Another interpretation of the invariance results above lead to consideration of spaces naturally called multiplier spaces. For every $E, F \subset \omega$, where F is a sequence space, we denote by $(E \to F)$ the set $\{x \in \omega : x \cdot E \subset F\}$. That F is a sequence space insures $(E \to F)$ is a sequence space. The following multiplier results are simple consequences of the theorems in the previous section.

Propostion 5. Suppose $E, F, (E \to F)$ are FK-spaces. If F is a σB-space then $(E \to F)$ is a σB-space. If, further, $F = F_{\sigma B}$ then $(E \to F) = (E \to F)_{\sigma B}$.

Proof. If F is a σB-space, $q \cdot F = F$. Thus $q \cdot (E \to F) \subset (E \to q \cdot F) = (E \to F)$. By Theorem 2 $(E \to F)$ is a σB-space. If, further, $F = F_{\sigma B}$ then, by Theorem 2, $q_o \cdot (E \to F)_{\sigma B} \cdot E = (E \to F)_{\sigma K} \cdot E \subset F$. Applying Theorem 1 to $q_o \cdot x \in F$ where $x \in (E \to F)_{\sigma B} \cdot E$ yields $(E \to F)_{\sigma B} \cdot E \subset F_{\sigma B} = F$. Thus $(E \to F)_{\sigma B} \subset (E \to F)$.

Proposition 6. Suppose $E \cdot F = G$, where E, F, G are FK-spaces. If either E or F is a σK-space (respectively σB-space) then G is a σK-space (respectively σB-space).

Proof. If E is a σK-space then $q_o \cdot G = q_o \cdot (E \cdot F) = (q_o \cdot E) \cdot F = E \cdot F = G$. The proofs are similar for the other cases.

Proposition 7. Suppose E and F are FK-spaces. If both E and F are σB-spaces then $(E \to F) \subset (E_{\sigma K} \to F_{\sigma K})$.

Proof. If $x \cdot E \subset F$ then $x \cdot E_{\sigma K} = x \cdot q_o \cdot E \subset q_o \cdot F = F_{\sigma K}$.

Corollary. Suppose E and F are FK-spaces. If E is a σK-space and F is a σB-space then $(E \to F) = (E \to F_{\sigma K}) = (E \to F_{\sigma B})$.

Proposition 8. Suppose E is an FK-space. If $F \supset q_o$ then $(F \to E) \subset E_{\sigma B}$.

Proof. If $F \supset q_o$ then $x \cdot F \subset E$ implies $x \cdot q_o \subset E$. Thus, by Theorem 1, $x \in E_{\sigma B}$.

Although these are all the results we give on multipliers, there are more available for "free". Simply adjust the statements of those results obtained above to be compatible with another type of section.

§5. Some Applications to Fourier Series

Let C^∞ denote the space of 2π-periodic complex valued functions on $\mathcal{R}$ which have derivatives of all orders. For each $k \in Z^+$, we define the semi-norm

$$p_k(f) = \sup_{0 \le j \le k} \sup_{0 \le x \le 2\pi} |D^j f(x)|$$

where $D^j f$ denotes the j-th derivative of f. Under this collection of semi-norms, C^∞ is a Fréchet space. A distribution is a continuous linear functional on C^∞. The linear space of all distributions will be denoted by $\mathcal{D}$.

If F is a distribution and satisfies an inequality of the form $|F(u)| \le K\|u\|_\infty$ where $u \in C^\infty$, K is a constant, and $\|\cdot\|_\infty$ is the uniform norm, then F is termed a Radon measure and can be expressed as an integral with respect to some uniquely determined regular Borel measure m, viz. $F(u) = \int_0^{2\pi} u(x)\,dm(x)$ [10, p.177]. The space of Radon measures will be denoted by $\mathcal{M}$. It should be noted that every $f \in \mathcal{L}^1(0,2\pi)$ can be identified with an element of $\mathcal{M}$ by $F(u) = \frac{1}{2\pi}\int_0^{2\pi} f(x)\cdot u(x)\,dx$ since $\left|\frac{1}{2\pi}\int_0^{2\pi} f(x)\cdot u(x)\,dx\right| \le \|f\|_1 \cdot \|u\|_\infty$. By $\hat{\mathcal{D}}$ we shall denote the set of sequences of Fourier coefficients $\hat{F}$ of distributions F. Here, $\hat{F}(n)$, the n-th Fourier coefficient of F, is given by $F(e^{-inx})$. It can be shown $\hat{\mathcal{D}}$ is the set of tempered sequences $\{x = (x_k)_{k \in Z^+} : \text{for some } n, x_k = O(|k|^n)\}$ [7, p.65].

Let E be a locally convex Hausdorff space of distributions defined by a collection P of semi-norms such that $\hat{E} = \{\hat{F} : F \in E\}$ is a "K-space" defined by the semi-norms $p(\hat{F}) = p(F)$, $p \in P$. A couple of remarks are in order before proceeding further. First, K-space refers to the analogue of our earlier definition with sequences defined on Z rather than on Z^+. With this understanding the concepts of AB, AK, σK, etc. can be modified so as to apply to $\hat{E}$. For example, here

$$P_n = \sum_{|k| \le n} \delta^k \quad \text{and} \quad \sigma^n = \sum_{|k| \le n} \left(1 - \frac{|k|}{n+1}\right)\delta^k .$$

Second, that the semi-norms on $\hat{E}$ are well-defined. It is shown in [7, p.65] that the distributions F and G are the same whenever $\hat{F}(n) = \hat{G}(n)$ $(n \in Z)$. Let $E_{\sigma K} = \{F \in E : \hat{F} \in \hat{E}_{\sigma K}\}$ and $E_{\sigma B} = \{F \in \mathcal{D} : \hat{F} \in \hat{E}_{\sigma B}\}$. We say E is a σK-space if $E = E_{\sigma K}$ and E is a σB-space if $E \subset E_{\sigma B}$. We denote by $\mathcal{Q}$ (respectively $\mathcal{Q}_o$, M_o^2) the set of Fourier series $\sum_{n \in Z} \lambda_n e^{inx}$ where $\lambda_n = \lambda_{-n} (n \in Z)$ and where $(\lambda_n)_{n \in Z^+}$ is an element of q (respectively q_o, m_o^2).

For the convolution $F*G$ of two distributions we have $\widehat{F*G} = \hat{F} \cdot \hat{G}$ [7, p.73]. Thus the main results, when restricted to tempered sequences, can easily be translated to spaces of distributions by replacing the operation of coordinatewise multiplication with convolution. For example if $\hat{E}$ is an FK-space then:

(3.4) $\mathcal{E} = \mathcal{E}_{\sigma K} \Leftrightarrow \mathcal{E} = \mathcal{Q}_o * \mathcal{E} \Leftrightarrow \mathcal{E} = M_o^2 * \mathcal{E}$;

and

(3.5) $\mathcal{E} \subset \mathcal{E}_{\sigma B} \Leftrightarrow \mathcal{E} = \mathcal{Q} * \mathcal{E} \Leftrightarrow \mathcal{E}_{\sigma K} = \mathcal{Q}_o * \mathcal{E}$.

In 1937 Littlewood and Paley [12] proved the factorization $L^p \subset L^1 * L^p$ $(1 < p < \infty)$. Since $\mathcal{Q}_o \subset L^1$ [30, v.I, p.183], and $L^p_{\sigma K} = L^p$ $(1 < p < \infty)$ this factorization statement is contained in (3.4).

In 1945 Salem [21] proved that if $f \in C$, the space of continuous 2π-periodic functions, (respectively L^1) then there exists a sequence y such that $y_n = y_{-n}$ $(n \in Z)$ and $(y_n)_{n \in Z^+}$ is positive, concave, and increasing to infinity such that $\hat{f} \cdot y \in \hat{C}$ (respectively $\hat{L}^1$). Since $C = C_{\sigma K}$ and $L^1 = L^1_{\sigma K}$ these results are contained in (3.4) using the fact that if $(\lambda_n) \in m_o^2$, then (λ_n^{-1}) is positive, concave, and increasing to infinity.

Chapter 4 - Unconditional Sections

§1. Introduction

This chapter draws its substance from John J. Sember's paper [23] with the exception of those results listed in [17], done jointly by the previously mentioned author and myself. General facts concerning unconditionally bounded sections are established in §2. In §3 we see that the space of convergent sequences c is the BK-space corresponding to bv and q of the preceeding two chapters. As with the other kinds of section convergence, we obtain the result that for any FK-space E, $E_{UAK} = c_o \cdot E_{UAB}$. The final section contains some miscellaneous examples and a criterion as to when a sequence has unconditionally bounded sections in a convergence field.

§2. Unconditional Section Boundedness

We begin by plugging one of the two gaps remaining in the diagram in Chapter 2, §2.

Theorem 1. If E is a K-space, then

$$E_{UAB} = \{x \in \omega : \Sigma|x_k f(\delta^k)| < \infty \text{ for every } f \in E'\} = E_{UFAK} .$$

Proof. By Mackey's theorem, $H\cdot x$ is bounded if and only if it is weakly bounded. Thus $x \in E_{UAB}$ if and only if, for each $f \in E'$, $\{\sum_{k\in F} x_k f(\delta^k) : F \in \Phi\}$ is a bounded set of complex numbers. This last condition is equivalent to the absolute convergence of $\Sigma x_k f(\delta^k)$. Certainly the absolute convergence of $\Sigma x_k f(\delta^k)$, for $f \in E'$, implies $H\cdot x$ is weakly unconditionally Cauchy. Conversely, if $H \cdot x$ is weakly unconditionally Cauchy and U is any weak neighborhood of 0, then there is a set $F_o \in \Phi$ such that $x_F = h_F \cdot x \in U$ whenever $F \cap F_o = \emptyset$ and $F \in \Phi$. Let $A = \{x_G : G \subset F_o\}$; then A is finite and, for any F in Φ,

$$x_F = x_{F \cap F_o} + x_{F \setminus F_o} \in A + U.$$

Thus the set of all x_F is weakly precompact which is equivalent to bounded in a locally convex Hausdorff space.

The conditional case for a K-space E is in general related by $E_{FAK} \subset E_{AB}$, and the inclusion is proper as is illustrated by example 5, §4.

Corollary. Let x be any sequence with unconditionally bounded sections in a K-space E. Then x has the property FAK.

Completing the previously mentioned diagram, we next observe that if a sequence with unconditionally bounded sections has section convergence in the weak topology, then this convergence is unconditional.

Theorem 2. If E is a K-space, then $E_{UAB} \cap E_{SAK} = E_{USAK}$.

Proof. If $H \cdot x$ is bounded and $\Sigma x_k \delta^k$ converges weakly to x, then for $\varepsilon > 0$ and $f \in E'$ we can choose $N \in Z^+$ sufficiently large so that $\sum_{k=N+1}^{\infty} |x_k f(\delta^k)| < \varepsilon/2$ and $|f(x) - \sum_{k=1}^{N} x_k f(\delta^k)| < \varepsilon/2$. Then if F is any finite subset of the positive integers containing $\{1,2,\ldots,N\}$, we have

$$|f(x) - f(\sum_{k \in F} x_k \delta^k)| \leq |f(x) - \sum_{k=1}^{N} x_k f(\delta^k)| + \sum_{k=N+1}^{\infty} |x_k f(\delta^k)| < \varepsilon$$

and it follows that $H \cdot x \to x$ in the weak topology. The reverse inclusion is immediate using Theorem 1.

Lemma 1. The convex hull of $H \cdot x$, where x is any sequence, is $C_\Phi^+ \cdot x$.

Proof. If $y \in \Gamma\{H \cdot x\}$ then $y = \sum_{j=1}^{k} \mu_j h^{(j)} \cdot x$ where $\mu_j \geq 0$ for each $j = 1,2,\ldots,k$ and $\sum_{j=1}^{k} \mu_j = 1$. Letting $F = F_1 \cup F_2 \ldots \cup F_k$, where $F_j = \{n : h_n^{(j)} = 1\}$, and $\lambda_n = \sum_{j=1}^{k} \{\mu_j : n \in F_j\}$, we can write $y = \sum_{n \in F} \lambda_n x_n \delta^n$, where $0 \leq \lambda_n \leq 1$ for each n. Thus $y \in C_\Phi^+ \cdot x$.

Conversely if $y = \sum_{n \in F} \lambda_n x_n \delta^n$ with $0 \leq \lambda_n \leq 1$ and F is finite, then the λ_n can be arranged so that $0 \leq \lambda_{n(1)} \leq \lambda_{n(2)} \leq \ldots \leq \lambda_{n(k)} \leq 1$ and we can write $y = \lambda_{n(1)} \sum_{i=1}^{k} x_{n(i)} \delta^{n(i)} + \sum_{j=2}^{k} (\lambda_{n(j)} - \lambda_{n(j-1)}) \sum_{i=j}^{k} x_{n(i)} \delta^{n(i)} + (1-\lambda_{n(k)}) \cdot 0$ which is in the convex hull of $H \cdot x$.

Proposition 1. If E is a K-space and x is any sequence, the following are equivalent:

(i) x has unconditionally bounded sections in E;

(ii) $C_\Phi^+ \cdot x$ is bounded in E;

(iii) $C_\Phi \cdot x$ is bounded in E.

Proof. In view of Lemma 1 it suffices to show (ii) $\Rightarrow$ (iii). As $C_\Phi \cdot x \subset C_\Phi^+ \cdot x - C_\Phi^+ \cdot x + iC_\Phi^+ \cdot x - iC_\Phi^+ \cdot x$ we are done; since in a locally convex space scalar multiples and finite sums of bounded sets are bounded.

The next result is the analogue in the unconditional setting of Corollary 1, Proposition 4, Chapter 2.

Proposition 2. Suppose E is a locally H-invariant K-space. Then the following are equivalent:

(i) $x \in E_{UAK}$;

(ii) $x \in E_{SAK}$;

(iii) x has the property AD.

Proof. (i) $\Leftrightarrow$ (iii) follows from Proposition 1, Chapter 2. If $x \in E_{SAK}$ then x belongs to the weak closure of $\{P_n \cdot x\}$; hence to the weak closure of $\Gamma\{H \cdot x\} = C_\Phi^+ \cdot x$. Since the closure in the original topology and the weak closure of convex sets agree in a locally convex space, x has the property AD and so belongs to E_{UAK}. (i) $\Rightarrow$ (ii) is clear.

Corollary. Let E be an FK-space such that $E \subset E_{UAB}$. Then $E_{AK} = E_{UAK}$.

That the converse of the above corollary is false is illustrated by the example following Corollary 2, Theorem 3, Chapter 2.

§3. Sequential Completeness

We have just seen that under ordinary circumstances $H \cdot x$ bounded implies $C_\Phi^+ \cdot x$ is bounded. If E is sequentially complete much more can be said, as the following result paralleling Proposition 6, Chapter 2 guarantees.

Theorem 3. Let E be a sequentially complete K-space and let x belong to E_{UAB}. Then $C_o \cdot x$ is bounded in E and $C_o \cdot x \subset E_{UAK}$.

Proof. Suppose $H \cdot x$ is bounded in E. Let $\eta \in C_o$ and p a continuous semi-norm on E. By Proposition 1 $C_\Phi \cdot x$ is bounded, so there exists $M > 0$ such that $p(t \cdot x) < M$ for each $t \in C_\Phi$.

We observe that, if $m > n$,

$$P_m(\eta\cdot x) - P_n(\eta\cdot x) = (0,0,\dots,0,\eta_{n+1}x_{n+1},\eta_{n+2}x_{n+2},\dots,\eta_m x_m,0,0,\dots)$$

$$= (\max_{n+1\le k\le m} |\eta_k|)\eta'\cdot x,$$

where $\eta' \in C_\Phi$. Therefore

$$p(P_m(\eta\cdot x) - P_n(\eta\cdot x)) = (\max_{n+1\le k\le m} |\eta_k|)p(\eta'\cdot x) < (\max_{n+1\le k\le m} |\eta_k|)M$$

and, since $\eta \in C_o$, it follows that $(P_n(\eta\cdot x))$ is a Cauchy sequence in E. As E is a sequentially complete K-space $(P_n(\eta\cdot x))$ must converge to $\eta\cdot x$. Thus every element of $C_o\cdot x$ is contained in the closure of the bounded set $C_\Phi\cdot x$. Since E is locally convex it follows that $C_o\cdot x$ is bounded. Now let $\varepsilon > 0$ be given. Since $P_n(\eta\cdot x) \to \eta\cdot x$ we can find $N \in Z^+$ such that, for $n \ge N$,

$$p(P_n(\eta\cdot x) - \eta\cdot x) < \varepsilon/2$$

and $$|\eta_n| < \varepsilon/2M .$$

Then if F is any finite subset of positive integers containing $\{1,2,\dots,N\}$, we have

$$p(\sum_{k\in F} (\eta\cdot x)_k\delta^k - \eta\cdot x) \le p(\sum_{k\in F\setminus\{1,2,\dots,N\}} (\eta\cdot x)_k\delta^k) + p(P_N(\eta\cdot x)-\eta\cdot x)$$

$$\le (\max_{k\in F\setminus\{1,2,\dots,N\}} |\eta_k|)p(\tau\cdot x) + \varepsilon/2 \quad (\text{where } \tau \in C_\Phi)$$

$$< (\varepsilon/2M)M + \varepsilon/2 = \varepsilon ,$$

and therefore $\eta\cdot x \in E_{UAK}$.

<u>Corollary 1</u>. Let E be a sequentially complete K-space. If $\Sigma|f(\delta^k)| < \infty$ for each $f \in E'$, then $c_o \subset E$.

Proof. The condition $\Sigma|f(\delta^k)| < \infty$ is clearly the same, from Theorem 1, as $H \cdot e$ is bounded in E.

Corollary 2. An FK-space E contains c_o if and only if $\Sigma|f(\delta^k)| < \infty$ for each $f \in E'$.

Proof. The sufficiency of the condition is given by Corollary 1. The necessity follows from the fact that the relative topology of E on c_o is weaker than the usual supremum norm topology on c_o [26, Cor.1, p.203] and $H \cdot e$ being bounded in c_o with the usual topology.

Corollary 3. Suppose E is an FK-space. E is a conservative conull space if and only if $e \in E_{USAK}$.

Proof. By definition, E is conull if $e \in E_{SAK}$. If $c \subset E$ then Corollary 2 implies $e \in E_{UAB}$. The necessity now follows from Theorem 2. Conversely, $e \in E_{USAK}$ yields E is a conservative conull space, by appeal to Corollary 1 and the definition of conull.

Theorem 3 asserts that $C_o \cdot E \subset E_{UAK}$ in a sequentially complete K-space in which every element has unconditionally bounded sections. If E is an FK-space, we have

Proposition 3. Let E be an FK-space in which every element has unconditionally bounded sections. Then $C_o \cdot E = E_{UAK}$.

Proof. The inclusion $C_o \cdot E \subset E_{UAK}$ follows from Theorem 3. Lemma 1, Chapter 2 says $E_{AK} \subset B_o \cdot E$. Since we always have $E_{UAK} \subset E_{AK}$ and $B_o \cdot E \subset C_o \cdot E$, the result follows.

Proposition 4. Let E be a sequentially complete K-space and let $x \in E \cap E_{UAB}$. Then $C_\lambda \cdot x \subset E \cap E_{UAB}$.

Proof. If $\tau \in C_\lambda$, we can write $\tau = L \cdot e + \eta$, where $|L| \leq 1$ and $\eta \in C_o$. Thus $\tau \cdot x = Lx + \eta \cdot x$. But $\eta \cdot x \in E_{UAK}$ by Theorem 3 and

$x \in E_{UAB}$ by hypothesis, and the result follows.

Since it is always true that $E \subset C_\lambda \cdot E$, we then have

Corollary. If E is a sequentially complete K-space in which every element has unconditionally bounded sections, then $E = cE$.

The next theorem shows the converse of the above corollary holds in an FK-space. Thus we will have established that c is the associated BK-space for the section operators H in the same sense that bv corresponded to P in Chapter 2 and that q corresponded to σ in Chapter 3.

Theorem 4. Let E be an FK-space. The following are equivalent:

(i) E is locally C_λ-invariant;

(ii) $E = c \cdot E$;

(iii) $\Sigma |x_k f(\delta^k)| < \infty$ for every $x \in E$, $f \in E'$;

(iv) $E \subset E_{UAB}$;

(v) $H \cdot x$ is bounded for every $x \in E$.

Proof. The equivalence of (iii), (iv), and (v) follows from Theorem 1, and the corollary to Proposition 3 shows that (iv) $\Rightarrow$ (ii). If $E = c \cdot E$, then for each x in E the linear mapping $T_x : c \to E$ defined by $T_x(\tau) = \tau \cdot x$ in continuous by the closed graph theorem. Consequently it maps bounded sets onto bounded sets, so that $T_x(H) = H \cdot x$ is bounded in E for every x in E. Thus (ii) $\Rightarrow$ (iv). As (i) $\Rightarrow$ (ii) is clear, it remains to show that (ii) $\Rightarrow$ (i). For each τ in C_λ the linear mapping $S_\tau : E \to E$ defined by $S_\tau(x) = \tau \cdot x$ is continuous by the closed graph theorem. The proof of (ii) $\Rightarrow$ (iv) shows that $C_\lambda \cdot x = T_x(C_\lambda)$ is a bounded subset of E for each x in E. Since any pointwise bounded collection of continuous linear mappings from a barrelled space into a locally convex space is equicontinuous, E is locally C_λ-invariant.

<u>Corollary 1</u>. Every sequence in a solid FK-space has the property UFAK.

<u>Proof</u>. Solid implies c-invariant.

<u>Corollary 2</u>. If E is a solid FK-space, then $E_{UAK} = C_o \cdot E$.

<u>Proof</u>. This follows from Proposition 3 and Theorem 4.

<u>Corollary 3</u>. In a c-invariant FK-space, the series $\Sigma x_k \delta^k$ converges weakly if and only if it converges unconditionally.

<u>Proof</u>. Theorem 4 implies E is locally H-invariant. Proposition 2 now gives the result.

<u>Lemma 2</u>. Let (E,τ) be a K-space. If x has unconditional section convergence in (E,τ), then x has unconditional section convergence in $(E_{HB}, \mathcal{P}_H) \equiv (E_{UAB}, \mathcal{P}_H)$. (Notation is made clear by Theorem 1, Chapter 2).

<u>Proof</u>. Let $x \in E_{UAK}$, $p \in \mathcal{P}$, and $\varepsilon > 0$. There exists $F_o \in \Phi$ such that if $F \in \Phi$ and $F \supset F_o$, then $p(h_F \cdot x - x) < \varepsilon/4$. If $G \in \Phi$ and $G \cap F_o = \emptyset$, then

$$\begin{aligned} p(h_G \cdot x) &= p((x_G + x_{F_o}) - x_{F_o}) \\ &= p(x_{F'} - x_{F_o}) \quad \text{where } F' \supset F_o \\ &\le p(x_{F'} - x) + p(x - x_{F_o}) < \varepsilon/2 . \end{aligned}$$

For $h \in H$,

$$\begin{aligned} p_H(h \cdot x - x) &= \sup_{h' \in H} p(h' \cdot (h \cdot x - x)) = \sup_{h' \in H} p(h \cdot h' \cdot x - h' \cdot x) \\ &= \sup_{h' \in H} p\Big(\sum_{k \in F_o} (h \cdot h' \cdot x)_k \delta^k + \sum_{k \notin F_o} (h \cdot h' \cdot x)_k \delta^k - \sum_{k \in F_o} (h' \cdot x)_k \delta^k - \sum_{k \notin F_o} (h' \cdot x)_k \delta^k \Big) . \end{aligned}$$

So whenever $F \supset F_o$,

$$p_H(h_F \cdot x - x) = \sup_{h' \in H} p(\sum_{k \notin F_o} (h_F \cdot h' \cdot x)_k \delta^k - \sum_{k \notin F_o} (h' \cdot x)_k \delta^k)$$

$$\leq \sup_{h' \in H} (p(\sum_{k \notin F_o} (h_F \cdot h' \cdot x)_k \delta^k) + p(\sum_{k \notin F_o} (h' \cdot x)_k \delta^k))$$

$$\leq 2 \sup_{\substack{h' \in H \\ h'_k = 0, k \in F_o}} p(h' \cdot x) < 2 \cdot \varepsilon/2 = \varepsilon .$$

Thus $E_{UAK} \subset (E_{UAB})_{UAK}$.

We are now ready to prove the unconditional result analagous to Theorem 4, Chapter 2 and Corollary 1, Theorem 2, Chapter 3.

<u>Theorem 5</u>. Let E be an FK-space. Then

$$\{x \in E : \Sigma x_k \delta^k \text{ is subseries convergent}\} = c_o \cdot E_{UAB} .$$

<u>Proof</u>. If $x \in E_{UAB}$, then

$$\sup_{h' \in H} p_H(h' \cdot x) = \sup_{h' \in H} \sup_{h \in H} p(h \cdot h' \cdot x) = \sup_{h' \in H} p(h' \cdot x) < \infty,$$

and so E_{UAB} is an FK-space (corollary of Theorem 1, Chapter 2) with the property UAB. Applying Proposition 3 to E_{UAB} , Lemma 2 to $(E_{UAB})_{UAK}$, and Theorem 3 to E_{UAB} , we obtain

$$c_o \cdot E_{UAB} = (E_{UAB})_{UAK} \supset E_{UAK} \supset c_o \cdot E_{UAB} .$$

In a sequentially complete K-space $\Sigma x_k \delta^k$ is unconditionally convergent if and only if it is subseries convergent [cf.7, p.78]; whence the result follows.

<u>Corollary</u>. If E is an FK-space then E_{UAK} is solid.

Finally we observe that in a weakly sequentially complete K-space the property of having unconditionally bounded sections is especially strong.

Theorem 6. Let E be a weakly sequentially complete K-space and let x be any sequence. Then x has unconditionally bounded sections in E if and only if $\Sigma x_k \delta^k$ converges to x unconditionally in E (i.e., $E_{UAB} = E_{UAK}$).

Proof. If $H \cdot x$ is bounded then $H \cdot x$ is weakly Cauchy, from Theorem 1. It follows $P \cdot x$ is weakly Cauchy, hence weakly convergent to x. By Theorem 2 $\Sigma x_k \delta^k = x$ weakly unconditionally and, since E is weakly sequentially complete, $\Sigma x_k \delta^k$ is weakly subseries convergent [cf.7, p.78]. By the Orlicz, Pettis, Grothendieck theorem $\Sigma x_k \delta^k$ is subseries convergent, hence unconditionally convergent, to x in the original topology.

§4. Examples and Applications in Convergence Fields

Given an infinite matrix $A = (a_{nk})$ with complex entries it is possible to make

$$c_A = \{x \in \omega : Ax = (\sum_{k=1}^{\infty} a_{nk}x_k)_{n \in Z^+} \in c\}$$

into an FK-space by letting the following semi-norms generate its topology:

(i) $|\Pi_n|(x) = |x_n|$ $\qquad n = 1,2,\ldots$;

(ii) $q_n(x) = \sup_{r \in Z^+} |\sum_{k=1}^{r} a_{nk}x_k|$ $\qquad n = 1,2,\ldots$;

(iii) $p(x) = \sup_{n \in Z^+} |\sum_{k=1}^{\infty} a_{nk}x_k|$.

c_A is called the convergence field of A and has been widely studied.

Of special interest are those matrices with the "mean value property". A conservative matrix A is said to have the mean value property (MVP) in case

$$B = \{x \in c_A : \sup_{n,r \in Z^+} \left|\sum_{k=1}^{r} a_{nk}x_k\right| < \infty\}$$

fills up c_A. Bennett [2] and Sember [22] showed independently that $B = \{x \in c_A : x$ has bounded sections in $c_A\}$. Thus A has the MVP if and only if $c_A \subset (c_A)_{AB}$.

The above considerations lead to a "strong mean value property". The conservative matrix A is said to have the strong mean value property (SMVP) if

$$U = \{x \in c_A : \sup_{\substack{F \in \Phi_+ \\ n \in Z^+}} \left|\sum_{k \in F} a_{nk}x_k\right| < \infty\}$$

fills up c_A.

<u>Proposition 5</u>. $(c_A)_{UAB} = \{x \in \omega : \sup_{\substack{F \in \Phi_+ \\ n \in Z^+}} \left|\sum_{k \in F} a_{nk}x_k\right| < \infty\}$.

<u>Proof</u>. Let $x \in \omega$. Then

$$\sup_{\substack{F \in \Phi_+ \\ n \in Z^+}} \left|\sum_{k \in F} a_{nk}x_k\right| = \sup_{h \in H} \sup_{n \in Z^+} \left|\sum_{k=1}^{\infty} a_{nk}(h\cdot x)_k\right| \tag{4.1}$$

and the right hand side of (4.1) is finite if and only if $A\{H\cdot x\}$ is bounded in c. Thus it is sufficient to show that $A\{H\cdot x\}$ being bounded in c is equivalent to $H\cdot x$ being bounded in c_A. As any matrix map between FK-spaces is continuous as well as linear [26, Cor.5, p.264], it follows that the boundedness of $H\cdot x$ insures $A\{H\cdot x\}$ is bounded. Suppose $A\{H\cdot x\}$ is a bounded subset of c. We must show the semi-norms of types (i), (ii), and (iii) are bounded with $H \cdot x$ in their arguments to conclude the proof. That $A\{H \cdot x\}$ is bounded in c is precisely the same as $\sup_{h \in H} p(h \cdot x) < \infty$, and $|x_n|$ serves as a bound for the $|\pi_n|$ of type (i). Since

$$(4.2) \qquad \sup_{h \in H} q_n(h \cdot x) = \sup_{h \in H} \sup_{r \in Z^+} \left| \sum_{k=1}^{r} a_{nk} (h \cdot x)_k \right| = \sup_{h \in H} \left| \sum_{k=1}^{\infty} a_{nk} (h \cdot x)_k \right|,$$

and (4.1) is assumed to be finite; (4.2) is certainly finite.

Corollary. The conservative matrix A has the SMVP if and only if every element of c_A has the property UAB.

Examples:

1. Let A be the matrix with all zero entries. Then $c_A = \omega$ and A has the SMVP.
2. Let $A = I$, the identity matrix. Then $c_A = c$ and A has the SMVP.
3. It is well known the (C,1)-matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \end{bmatrix}$$

has the MVP [cf. 25, p.346]. However the following construction due to Allen R. Freedman shows a "large" portion of c_A = Cesàro summable sequences has the property UAB, while A does not have the SMVP.

Let $t \in c$ and $a \in \ell^\infty \cap c_A$. Clearly $t \cdot a \in \ell^\infty$. If $t_n \to L$ then $(t_n - L) \in c_o$ and $(t_n - L) \cdot a \in c_o$. Thus $t \cdot a = (t_n - L) \cdot a + La$ is in $c_o + c_A \subset c_A$. This shows $c \cdot (\ell^\infty \cap c_A) = \ell^\infty \cap c_A$. Since $\ell^\infty \cap c_A$ is an FK-space, Theorem 4 shows $\ell^\infty \cap c_A$ is a UAB-space. As the topology of c_A restricted to $\ell^\infty \cap c_A$ is weaker than $\ell^\infty \cap c_A$'s FK-topology, $\ell^\infty \cap c_A \subset (c_A)_{UAB}$. To construct a Cesàro summable sequence b without unconditionally bounded sections, let a be a nondecreasing positive sequence converging to infinity with $\frac{a_n}{n}$ converging to 0. Let $b = (b_n)$ where $b_n = (-1)^{n+1}a_n$ for $n = 1,2,\ldots$. Noting that $|n^{-1}\sum_{k=1}^{n} b_k| \le \frac{|b_n|}{n} = \frac{a_n}{n} \to 0$ it is apparent that $b \in c_A$, i.e. is Cesàro summable. Define $d = (d_n)$ by $d_n = (-1)^{n+1}a_n^{-\frac{1}{2}}$ for $n = 1,2,\ldots$. Then $d_n \to 0$ but $d \cdot b = (\sqrt{a_n})$ cannot be in c_A since any monotone Cesàro summable sequence is convergent.

4. Let

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{3} & 0 & \cdots \\ \cdots & & & & \end{bmatrix} .$$

c_A has Schauder basis $\{u,\delta^1,\delta^2,\ldots\}$ where $u = (n)$. $c_A = \overline{c_o} \oplus u$ and $(c_A)_{SAK} = \overline{c_o}$ [25, ex.3, p.344]. Clearly A has the SMVP, therefore c_A has the property UAB; hence c_A is locally H-invariant. We deduce, by Proposition 2, that $(c_A)_{UAK} = (c_A)_{SAK}$.

5. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & \cdots \\ 1 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & \frac{1}{8} & \frac{1}{8} & 0 & \cdots \\ \cdots & & & & & & & & & \end{bmatrix} .$$

Since A is conservative $F = B \cap I$ where $F = c_A \cap (c_A)_{FAK}$, $B = c_A \cap (c_A)_{AB}$, and $I = \{x \in c_A : \Sigma x_k a_k$ exists; $a_k = \lim_n a_{nk}\}$ [26, Lemma 4.3, p.331]. It is easily checked that for $x = (1, -1, 2, -2, 4, -4, 8, -8, \ldots)$, $x \in B \smallsetminus I$. Thus $(c_A)_{FAK} \subsetneq (c_A)_{AB}$ [28, p.27].

6. A sequence in $(c_A)_{AK}$ but not in $(c_A)_{UAB}$. Note that $x \in (c_A)_{AK}$ if and only if $A(P_n \cdot x) \to Ax$ in c [3, Thm.1, p.107]. Thus for

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & \frac{1}{4} & 0 & 0 & 0 & \cdots \\ 1 & \frac{1}{4} & \frac{1}{9} & 0 & 0 & \cdots \\ 1 & \frac{1}{4} & \frac{1}{9} & \frac{1}{16} & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \end{bmatrix}$$

and $x = (1, -2, 3, -4, \ldots)$ we have $x \in (c_A)_{AK}$. Since $A\{H \cdot x\}$ is unbounded in c, $x \notin (c_A)_{UAB}$.

7. There are many BK-spaces having the property of unconditional section boundedness, i.e. c-invariant, that are not solid [9]. For example, $|\sigma_1| = \{x \in \omega :$ there exists L such that $n^{-1} \sum_{k=1}^{n} |x_k - L| \to 0\}$ is such a space with $\|x\|_{|\sigma_1|} = \sup_n n^{-1} \sum_{k=1}^{n} |x_k|$. $|\sigma_1|$ is clearly a UAB-space and $(1,0,1,0,\ldots) = (1,0,1,0,\ldots) \cdot e$ is not in $|\sigma_1|$ so $|\sigma_1|$ is not solid.

Bibliography

1. Bary, N.: A treatise on trigonometric series, vols. 1 and 2. New York: Permagon Press 1964.

2. Bennett, G.: Distinguished subsets and summability invariants. Studia Math. T. XL., 225-234 (1971).

3. Bennett, G., Sember, J.J., and Wilansky, A.: Sections of sequences in matrix domains. Transactions New York Academy of Sciences 34(2), 107-112 (1972).

4. Buntinas, M.: Convergent and bounded Cesàro sections in FK-spaces. Math. Z. 121, 191-200 (1971).

5. Buntinas, M.: On Toeplitz sections in sequence spaces. Math. Proc. Camb. Phil. Soc. 78, 451-460 (1975).

6. Day, M.M.: Normed linear spaces. New York Heidelberg Berlin: Springer-Verlag 1973.

7. Edwards, R.E.: Fourier series - a modern introduction, vol. 2. New York: Holt, Rinehart and Winston 1967.

8. Freedman, A.R., Sember, J.J., and Raphael, M.: Some Cesàro-type summability spaces, Notices A.M.S., June 1977.

9. Garling, D.J.H.: On topological sequence spaces. Proc. Camb. Phil. Soc. 63, 997-1019 (1967).

10. Hewitt, E. and Stromberg, K.: Real and abstract analysis. New York Heidelberg Berlin: Springer-Verlag 1975.

11. Köthe, G.: Topological vector spaces I. New York Inc.: Springer-Verlag 1969.

12. Littlewood, J.E. and Paley, R.E.A.C.: Theorems on Fourier series and power series (III). Proc. London Math. Soc. (2), 43, 105-126 (1937).

13. Lorentz, G.G. and Zeller, K.: Abschnittslimitierbarkeit und der Satz von Hardy-Bohr. Arch. der Math. 15, 208-213 (1964).

14. Mazur, S.: Eine Anwendung der Theorie der Operationen bei der Untersuchung der Toeplitzschen Limitierungsverfahren. Studia Math. 2, 40-50 (1930).

15. Mazur, S. and Orlicz, W.: On linear methods of summability. Studia Math 14, 129-160 (1954).

16. Orlicz, W.: Über K-fach monotone Folgen. Studia Math. 6, 149-159 (1936).

17. Raphael, M. and Sember, J.J.: Some further results on unconditional section boundedness. Notices A.M.S., April 1977, p.293.

18. Robertson, A.P.: On unconditional convergence in topological vector spaces. Pro. Roy. Soc. Edin., Sect. A68 (II), 145-157 (1969).

19. Robertson, A.P. and W.: Topological vector spaces. Cambridge University Press 1973.

20. Rudin, W.: Functional Analysis. McGraw-Hill 1973.

21. Salem, R.: Sur les transformations des séries de Fourier. Fundamenta Math. 33, 108-114 (1945).

22. Sember, J.J.: The associative part of a convergence domain is invariant. Canadian Mathematical Bulletin 13 (1), 147-148 (1970).

23. Sember, J.J.: Rocky Mountain J. of Math. (to appear).

24. Snyder, A.K.: Conull and coregular FK-spaces. Math. Z. 90, 376-381 (1965).

25. Wilansky, A.: Distinguished subsets and summability invariants. Jour. d'Analyse Math. 12, 327-350 (1964).

26. Wilansky, A.: Functional Analysis. New York 1964.

27. Wilansky, A.: From triangular matrices to separated inductive limits. Studia Math. T. XXXI., 469-479 (1968).

28. Wilansky, A.: Inclusion theorems and distinguished subspaces of FK-spaces. notes by F.P. Cass, unpublished.

29. Zeller, K.: Allgemeine Eigenschaften von Limitierungsverfahren. Math. Z. 53, 463-487 (1951).

30. Zygmund, A.: Trigonometric Series, vol. I London: Cambridge University Press 1968.