ON THE OBERWOLFACH PROBLEM FOR CYCLES WITH MULTIPLE LENGTHS

by

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M.Sc., Simon Fraser University, 1993

THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

In the Department
of
Mathematics

APPROVAL

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Degree: Doctor of Philosophy

Title of thesis: On the Oberwolfach Problem for Cycles with Multiple Lengths

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Date Approved: February 6, 2004

Abstract

The complete graph of order n, denoted K_n, is the graph consisting of n vertices and $\binom{n}{2}$ edges, one for each unordered pair of vertices. For even n, the graph $K_n - I$ is obtained from K_n by removal of $n/2$ pairwise disjoint edges. A *2-factor* of a graph $G = (V, E)$ is a subset $F \subseteq E$ such that each vertex of G is incident with exactly 2 edges in F. Each connected component of the subgraph induced by F is a cycle of length at least three in G. The *shape* of F is the multiset $(a_1^{n_1}, \ldots, a_k^{n_k})$ of lengths of these cycles. (The symbol $a_i^{n_i}$ expresses that F has exactly n_i cycles of length a_i.)

The *Oberwolfach problem* $OP(a_1^{n_1}, \ldots, a_k^{n_k})$ is to determine whether the edges of K_n or $K_n - I$ can be partitioned into isomorphic 2-factors, each having a given shape $(a_1^{n_1}, \ldots, a_k^{n_k})$, where $n = \sum n_i a_i$. This problem arises in design problems such as the specification of tournaments and balanced circular arrangements.

The Oberwolfach problem has been solved in the uniform case (i.e., when $k = 1$). It is known that in this case the problem has a solution except when $a_1 = 3$ and $n_1 = 2$ or $n_1 = 4$. Much less is known when $k > 1$.

In this thesis, we focus on the cases $OP(a_1^{n_1}, a_2^{n_2})$ and $OP(3, a_2^{n_2}, a_3^{n_3})$. In particular, we prove that $OP(a, b)$ has a solution for all odd a and b, $a \neq b$. We also prove that $OP(a^n, b^n)$ has a solution for all odd a, b and n such that $n \neq 7, 11$ and $5 \leq a < b$.

We also prove that $OP(3, 4, n-7)$ and $OP(3, 6, n-9)$ have solutions for all n not congruent to one modulo 4. For $n = 4k$, we prove that the Oberwolfach problem has a solution in the more general case $OP(3, a, n-3-a)$.

A *graceful labeling* of a graph on n vertices with k edges is an injective mapping from the vertex set of the graph to the set $\{1, 2, 3, \ldots, k+1\}$ such that $\{|u - v| : (u, v) \in E\} = \{1, 2, 3, \ldots, k\}$.

Throughout the thesis, we prove and use various results on graceful labelings. In particular, we prove that, for an arbitrary label $1 \leq a \leq n$, a path on n vertices has a graceful labeling with one endpoint labeled a. We also conjecture on the necessary and sufficient conditions for the existence of a graceful labeling of a path on n vertices with the endpoints labeled a and b.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

1.1 Definitions and Notation

In this section, we introduce the definitions and notation used throughout this thesis.

For a graph G, $V(G)$ and $E(G)$ shall denote the sets of vertices and edges of G, respectively. H is a *subgraph* of G if H is a graph, $V(H)$ is a subset of $V(G)$ and $E(H)$ is a subset of $E(G)$. H will be called a *factor* of G if it is a subgraph of G with $V(H) = V(G)$. A graph whose vertices all have the same degree r is *r-regular*. An r-regular factor of a graph is its *r-factor*.

In our graphs, we will not allow loops, i.e., edges emanating from and terminating at the same vertex. We will not allow directed edges either.

A *multigraph* is a graph with at least one pair of distinct edges which have identical endpoints. Except where we specify otherwise explicitly, all graphs that we discuss are *not* multigraphs.

If H is a subgraph of G, then $G - H$ is the graph with $V(G - H) = V(G)$ and with $E(G - H) = E(G) \setminus E(H)$.

We say that two graphs G_1 and G_2 are isomorphic if there exists a bijection i from $V(G_1)$ onto $V(G_2)$ such that $(i(u), i(v))$ is an edge in G_2 if and only if (u, v) is an edge in G_1. If G_1 and G_2 are multigraphs, we will further require that the number of edges in G_1 connecting u with v be same as the number of edges in G_2 connecting $i(u)$ with $i(v)$.

For a given n, K_n shall denote the complete graph on n vertices. When n is even, we will denote by $K_n - I$ the graph K_n with a 1-factor removed (this graph is defined uniquely up to isomorphism). $\overline{K}_n$ is a graph on n vertices with no edges. λK_n is a multigraph in which each pair of vertices is connected by exactly λ distinct edges.

A *factorization* of a graph G is a list of factors of G which form a partition of $E(G)$. If $H_1, H_2, \ldots, H_m$ are graphs, then $OP(G; H_1, H_2, \ldots, H_m)$ denotes the problem of deciding whether G can be factored into m factors such that the ith factor is isomorphic to H_i for $1 \leq i \leq m$. The symbol "OP" abbreviates "Oberwolfach Problem". If all H_i are pairwise isomorphic, we write $OP(G; H)$ instead. Since, for a given G and H, there can be at most one m such that G decomposes into m factors each of which is isomorphic to H, we do not need to include a reference to m in the notation.

A 2-*factorization* of G is its decomposition into subgraphs each of which is a 2-factor of G. Obviously, a 2-factor is comprised of disjoint cycles. If H consists of n_i cycles of length a_i $(i = 1, \ldots, s)$, then we may write $OP(G; a_1^{n_1}, a_2^{n_2}, \ldots, a_s^{n_s})$ instead of $OP(G; H)$. Furthermore, we will usually omit the superscripts that are equal to 1. Finally, if G is equal to K_n (for odd n) or $K_n - I$ (for even n), then we will usually omit G in the notation. Thus, for example, the problem of deciding whether K_{19} can be factored into 2-factors, each of which consists of a 7-cycle and a 12-cycle is denoted $OP(7, 12)$.

In our constructions, we will make frequent use of permutations of vertices of the complete graph K_n or the complete graph less a 1-factor, $K_n - I$. In particular, we will frequently use the permutation α_n described below (when no confusion results, we may write α instead of α_n).

Let $t = \lfloor \frac{n-1}{2} \rfloor$. We will denote the vertices of K_n as $a_1, a_2, \ldots, a_t, b_1, b_2, \ldots, b_t, c_1$. For n even, we will denote the remaining vertex as c_2. Then

$$\alpha_n = (a_1, a_2, \ldots, a_t)(b_1, b_2, \ldots, b_t).$$

α_n induces a permutation of the edge-set of K_n. Note that we omit cycles of length one when describing a permutation. We will refer to that permutation as α_n also. Obviously, if H is a subgraph of K_n, then the image $\alpha_n(H)$ of H is a subgraph of K_n

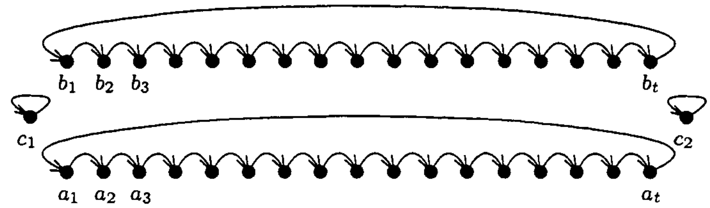

Figure 1.1: The action of α_n on $V(K_n)$, n even

isomorphic to H.

We will denote the two non-trivial vertex-orbits induced by α_n as follows:

$A = \{a_1, a_2, \ldots, a_t\}$,

$B = \{b_1, b_2, \ldots, b_t\}$,

The additional vertex orbits are trivial: $\{c_1\}$ and (when n is even) $\{c_2\}$. We will simply write c_1 and c_2 when no confusion may arise.

Further, we will denote the edge-orbits induced by α_n as follows:

1. AB_i is the orbit containing the edge (a_1, b_{1+i}) for $0 \leq i \leq t-1$,
2. A_i is the orbit containing the edge (a_1, a_{1+i}) for $1 \leq i \leq \lfloor \frac{t}{2} \rfloor$,
3. B_i is the orbit containing the edge (b_1, b_{1+i}) for $1 \leq i \leq \lfloor \frac{t}{2} \rfloor$,
4. AC_1 is the orbit containing the edge (a_1, c_1). In addition, when n is even, AC_2 is the orbit containing the edge (a_1, c_2).
5. BC_1 is the orbit containing the edge (b_1, c_1). In addition, when n is even, BC_2 is the orbit containing the edge (b_1, c_2).
6. C_0 is the orbit containing the edge (c_1, c_2) (this orbit exists only when n is even).

It is readily seen that all edge orbits have size t except that, when n is even, the orbit C_0 has size 1 and, when t is even, the orbits $A_{\frac{t}{2}}$ and $B_{\frac{t}{2}}$ have size $t/2$ each.

Further, we notice that in case $n \equiv 2 \pmod 4$ (i.e., when both n and t are even), the orbits $C_0, A_{\frac{t}{2}}, B_{\frac{t}{2}}$ induce a one-factor in K_n.

The *wreath product* $G \wr H$ is constructed as follows: Replace each vertex of G with a copy of H. In addition, connect two vertices in different copies of H if and only if the vertices in G that gave rise to the copies of H are connected in G. Formally, we have $V(G \wr H) = V(G) \times V(H)$ and $E(G \wr H) = \{((g_1, h_1), (g_2, h_2)) : g_1, g_2 \in V(G), h_1, h_2 \in V(H), (g_1, g_2) \in E(G)\} \cup \{((g, h_1), (g, h_2)) : g \in V(G), h_1, h_2 \in V(H), (h_1, h_2) \in E(H)\}$.

1.2 History and Motivation

The *Oberwolfach Problem* is due to Ringel [11]. As originally posed, the problem asks whether or not a complete graph on an odd number of vertices can be partitioned into identical subgraphs, each isomorphic to a given 2-factor.

Since then, the problem has been generalized and studied by a number of authors. In its most general form in the graph context, the (undirected) Oberwolfach Problem is the problem of deciding whether, given multigraphs $G, H_1, H_2, \ldots, H_m$, G can be partitioned into m factors isomorphic to $H_1, H_2, \ldots, H_m$, respectively.

In addition, a number of related problems have been studied by many authors. For example, Piotrowski [21] solves the bipartite uniform length cycle analogue of the Oberwolfach Problem. More recently, Liu [19, 20] has considered the complete multipartite case, again with uniform cycle lengths.

The so-called *Hamilton-Waterloo Problem* (introduced in [9], see also [2]) asks for a 2-factorization of the complete graph K_n, n odd, in which r of the 2-factors are isomorphic to a given 2-factor F_1, and s of the 2-factors are isomorphic to a given 2-factor F_2, where $r + s = \lfloor \frac{n}{2} \rfloor$.

The *Alspach Problem* asks for a decomposition of K_n or $K_n - I$ into cycles of given lengths $c_1, c_2, c_3, \ldots, c_s$, where $\sum c_i$ equals the total number of edges of K_n or $K_n - I$. The uniform cycle case has been settled (see [5, 14, 23, 24, 25]). In the non-uniform case, it is known (see [3]) that when all cycles have lengths three or five, then the decomposition exists as long as the trivial necessary conditions are satisfied.

In the thesis, we deal exclusively with the Oberwolfach Problem.

We will interchangeably use the terms "$OP(G; H_1, H_2, \ldots, H_m)$ has a solution" and "$OP(G; H_1, H_2, \ldots, H_m)$ exists". Further, we shall use the term "we construct $OP(G; H_1, H_2, \ldots, H_m)$" to mean that we construct a factorization solving $OP(G;$

$H_1, H_2, \ldots, H_m$).

In its general form, the Oberwolfach Problem appears to be very difficult. In this thesis, we will concentrate on the case when G is either a complete graph on n vertices (when n is odd) or a complete graph on n vertices less a one-factor (when n is even), and when all H_i are identical collections of cycles. Using our terminology, we will concentrate on the problem $OP(a_1^{n_1}, a_2^{n_2}, \ldots, a_s^{n_s})$.

Even this problem is difficult and only a partial progress has so far been made towards its solution. In this section, we will summarize the most important results found to date. We will then outline the methods and results presented in the later chapters.

The "uniform" case, i.e., the case $OP(a^k)$, has been completely settled.

Theorem 1.2.1 (**[6, 7, 15]**) *For $n \geq 3$, the $OP(a^k)$ has a solution whenever $a \cdot k = n$, except that $OP(3^2)$ and $OP(3^4)$ do not have solutions.*

The condition that $a \cdot k = n$ is the obvious necessary condition in the above theorem. The theorem says that with the exception of $OP(3^2)$ and $OP(3^4)$ (which are known to not have solutions [16]), the necessary condition is also sufficient.

The author extended this result to multigraphs:

Theorem 1.2.2 (**[12]**) *Let $\lambda \geq 1$ and let G be λK_n (or λK_n minus a 1-factor when n is even and λ is odd). Let further $a \geq 3$ and $m \geq 1$ be such that $a \cdot m = n$. Then $OP(G; a^m)$ has a solution if and only if none of the following are satisfied:*

1. $\lambda \equiv 2 \pmod 4, a = 3, m = 2,$
2. *λ is odd, $a = 3$, $m = 2$, or*
3. $\lambda = 1, a = 3, m = 4.$

Although Theorem 1.2.1 and methods used to prove it provide a powerful basis for decomposition of complete graphs into 2-factors, so far only limited progress has been made in generalizing the results for 2-factors with non-uniform length cycles.

Alspach [4] provides a good summary of known results. These results, together with more recent results [8, 13], provide the following theorem in the non-uniform cycle length case:

Theorem 1.2.3 (**[8, 13, 16, 17, 18]**) *The following Oberwolfach problems all have solutions:*

1. $OP(3, a)$ *for all odd* $a \geq 5$,
2. $OP(4, a)$ *for all even* $a \geq 4$,
3. $OP(a, a+2)$ *for all* $a \geq 3$,
4. $OP(a, a+1)$ *for all* $a \geq 3$, $a \neq 4$,
5. $OP(3, 8a-2)$ *for all* $a \geq 1$,
6. $OP(3, 4a, 4a)$ *for all* $a \geq 1$,
7. $OP(2a+1, 2a+1, 2a+2)$ *for all* $a \geq 1$,
8. $OP(a, a, 2\lceil a/2 \rceil + 2c)$ *for all* $a \geq 3$ *and each* $c = 0, 2, 3, \ldots$,
9. $OP(2a+1, 4^n)$ *for all* $a \geq 1$ *and all* $n \geq 1$,
10. $OP(2a+1, (4a)^n)$ *for all* $a \geq 1$ *and all* $n \geq 1$, *and*
11. $OP(a_1^n, a_2)$ *for all* $n \geq 1$, $a_1 \geq 3$, $a_2 \geq 4a_1 n - 1$.

All of the results in Theorem 1.2.3 deal with the cases when only two cycle lengths are present, i.e., with cases of the form $OP(a^n, b^m)$. Furthermore, with the exception of cases 8 and 11, either one of the cycle lengths is a constant, or the two cycle lengths are both based on the same parameter.

Cases 8 and 11 are the only two cases which allow for two cycle lengths based on two parameters. However, even in these two cases, the two cycle lengths must satisfy restrictive inequalities.

In this thesis, we focus on cases with two independent cycle lengths. In Chapter 2, we prove that $OP(a, b)$ has a solution for all distinct odd a and b.

In Chapter 3 we look at the problem of the form $OP(3, a, b)$. We first expand on the techniques introduced in Chapter 2 to prove that $OP(3, 4, s-7)$ and $OP(3, 6, s-9)$ have solutions for all s which are not of the form $s = 4k + 1$.

We then again look at the case with two independent cycle lengths, i.e., the case $OP(3, a, b)$. We succeed in solving this case for $3 + a + b \equiv 0 \pmod 4$.

In Chapter 4 we return to the case with two cycle lengths, this time in the context of $OP(a^n, b^n)$. We introduce additional techniques that enable us to tackle the problem of having more than 2 cycles in each 2-factor. We succeed in proving that, for sufficiently large odd a, b, and n, $OP(a^n, b^n)$ has a solution.

Finally, in Chapter 5 we introduce and discuss a conjecture on graceful labelings of paths.

Throughout the thesis, we frequently encounter situations where small cases have to be treated separately. Except where it is easy to check the small cases either manually or with a computer search using a simple exhaustive search algorithm, we cover small case proofs in the thesis.

Chapter 2

Two Cycles

2.1 Outline of the Construction

In this chapter, we will solve the Oberwolfach problem $OP(a, b)$ for odd a and b. In particular, we will prove that $OP(a, b)$ has a solution whenever a and b are odd integers satisfying $a, b \geq 3$ and $a \neq b$.

In this section, we outline the constructions that we use in this chapter. Let $n = a + b$. We consider K_n with its vertices labeled as in Chapter 1. Also, t and α will be as in Chapter 1.

The following obvious lemma provides the basis for constructing Oberwolfach Problem solutions through a construction of a base 2-factor.

Lemma 2.1.1 *Let n be even and let F be a 2-factor of K_n such that*

(i) F intersects only those edge-orbits of α that have size t, including each of the orbits $A_1, A_2, \ldots, A_{\lfloor \frac{t-1}{2} \rfloor}$, $B_1, B_2, \ldots, B_{\lfloor \frac{t-1}{2} \rfloor}$, AC_1, AC_2, BC_1 and BC_2, and

(ii) F does not intersect any of the edge-orbits of α in more than one edge.

Then $F, F\alpha, F\alpha^2, \ldots, F\alpha^{t-1}$ form a 2-factorization of a subgraph of K_n isomorphic to $K_n - I$. Further, all 2-factors in the 2-factorization are isomorphic to F. ∎

When F consists of 2 cycles, one with length a and the other with length b, then $F. F\alpha. F\alpha^2, \ldots, F\alpha^{t-1}$ solve $OP(a, b)$. Therefore, all that remains is to construct an appropriate F.

We construct F as a union of a number of paths and edges. We start with a path with t edges. The path has the form $a_i, b_j, a_{i+1}, b_{j-1}, a_{i+2}, b_{j-2}, \ldots$ (or $a_i, b_j, a_{i-1}, b_{j+1}, a_{i-2}, b_{j+2}, \ldots$). Figure 2.1 shows an example with $t = 20$, $i = 1$ and $j = 15$. In that figure, as in all figures throughout the thesis, the left edge of a vertex label is aligned with the corresponding vertex. For example, label '$a_1 = a_i$' corresponds to the first vertex in the bottom vertex row.

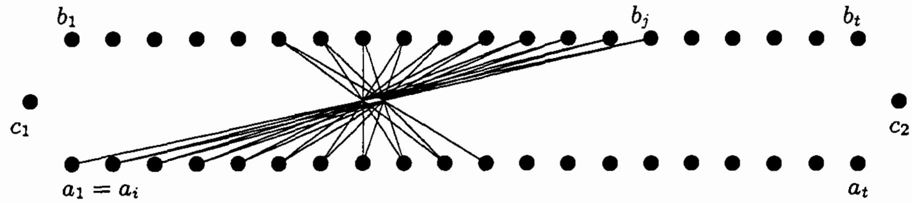

Figure 2.1: Construction of F — initial path

This path intersects all of the edge-orbits of α of the form AB_k, with each orbit intersected exactly once. We want to break this path into two subpaths P_1 and P_2, each of odd length. This can be done by a removal of an edge from the path when t is odd. When t is even, the removal of an edge results in two paths — one of an even length, the other of an odd length. We then append an extra edge to one of the endpoints of the even-length path to again create two odd-length paths. We choose the extra edge so that it is in the same orbit AB_k as the edge that was removed from the path. This ensures that the union of P_1 and P_2 intersects each of the edge-orbits of the form AB_k in at most one edge (see Figure 2.2).

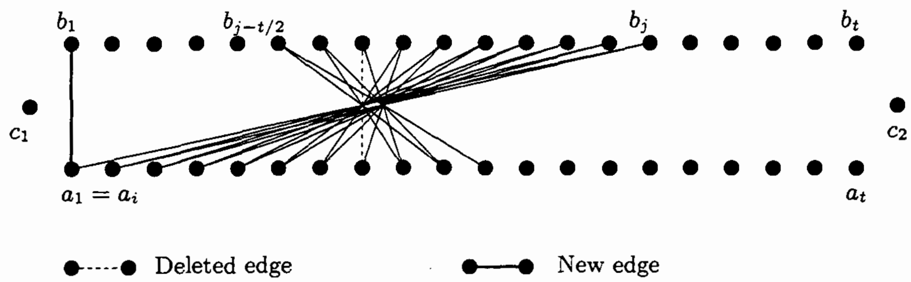

Figure 2.2: Construction of F — P_1 and P_2 when t is even

We now have two paths, each with one endpoint in A and the other in B. Also, these paths use up about one half of the vertices in A and B each. We still need to use the remaining vertices in A and B as well as the edge orbits of the form A_i and B_i.

We will use up these vertices and edge-orbits by creating 2 more paths - one in orbit A and the other in orbit B. These paths will have lengths approximately $t/2$ each. We will then attach these paths to the paths P_1 and P_2.

For a simple example, consider the construction in Figure 2.2 (here, t is even). We will want a path on $t/2$ vertices $b_{j+1}, b_{j+2}, b_{j+3}, \ldots, b_t, b_1, b_2, \ldots, b_{j-t/2}$ such that its edges intersect each of the edge-orbits $B_1, B_2, \ldots, B_{t/2-1}$ exactly once and such that b_1 is one of its endpoints. The existence of such a path can be seen to be equivalent to the existence of a *graceful labeling* of a path on $t/2$ vertices, with one endpoint given label $t + 1 - j$. We will prove that such graceful labelings exist.

While this example is particularly simple, it describes the approach we will use to construct the paths in A and B. However, in order to connect these paths with P_1 and P_2 and to ensure that all lengths work as desired, we will need additional graceful labeling results. We prove these results in Section 2.2. With the help of these results, we will be able to construct the desired paths.

The new paths will connect to the existing paths P_1 and P_2 to create two paths of lengths $a - 2$ and $b - 2$. Again, these paths will each have one endpoint in A and the other in B. We will connect one of the paths to vertex c_1 and the other to c_2. This creates F (and uses up the edge orbits AC_1, BC_1, AC_2 and BC_2).

While this description does not delve in the detail of constructing paths in A and B and connecting them to P_1 and P_2, it gives the general idea of the construction.

In order to cover all cases $OP(a, b)$ for odd a and b, we have to be able to control the length of the cycles we create. The construction allows us to control this length in two ways.

First, we are more or less free to choose the edge that is removed when P_1 and P_2 are created, i.e., we can control the relative length of P_1 and P_2.

Second, the two paths that are constructed in the orbits A and B may be connected to either the same path P_i or, alternatively, one path may be connected to P_1 and the other to P_2. The former approach will cover the cases of the form $OP(m, n - m)$ where $m < n/4$. The latter approach will provide solution to the cases $OP(m, n - m)$

with $n/4 < m < n/2$.

The actual construction in Section 2.3 will also vary depending on the congruence class of n modulo 8.

2.2 Graceful Labelings

Let G be a graph with n vertices and with k edges. A *graceful labeling* of G is an injection from the vertices of G to $\{1, 2, \ldots, k+1\}$ such that $\{|u - v| : (u, v) \in E(G)\} = \{1, 2, \ldots, k\}$.

Graceful labelings were first introduced by Rosa [22] who called them b-valuations. Golomb [10] first introduced the term graceful labeling. It is easily seen that every path has a graceful labeling. Less is known about the structure of graceful labelings on paths.

We need the following lemma to construct the paths that we will use in the base 2-factor F in our construction of $OP(a, b)$. To see how Lemma 2.2.1 might be used, note that a graceful labeling of a path on s vertices, $s < (n-1)/2$, can be used to construct a path with vertices $a_i, a_{i+1}, \ldots, a_{i+s-1}$ (or $b_i, b_{i+1}, \ldots b_{i+s-1}$) such that the edges of the path intersect each of the orbits $A_1, A_2, \ldots, A_{s-1}$ (or $B_1, B_2, \ldots, B_{s-1}$) exactly once.

Lemma 2.2.1 *For every n and m such that $1 \leq m \leq n$, there is a graceful labeling of the path on n vertices in which one of the endpoints is assigned label m.*

This lemma is easy to believe. In fact, we conjecture in Chapter 5 that a stronger result is true. However, in spite of its apparent simplicity, we were unable to locate a proof of Lemma 2.2.1. Abrham [1] proves the result for all m when n is odd and for all $m \leq (n-2)/4$ (which, due to symmetry, implies the result for $m \geq (3n+6)/4$) when n is even. In fact, Abrham proves a stronger result for these pairs of n and m. However, his methods do not provide an obvious extension to cover the remaining cases in Lemma 2.2.1. Simoson and Simoson [26] conjecture that Lemma 2.2.1 is true.

In order to prove Lemma 2.2.1, we identify the vertices of a path with their labels. We then need to prove that there is a path on the vertex set $\{1, 2, 3, \ldots, n\}$ such that one of the endpoints is m and such that $\{|u-v| : (u,v) \in E(G)\} = \{1,2,3,\ldots,n-1\}$.

We will build a gracefully labeled path one edge at a time, starting with the longest edge (i.e., the edge with the maximum absolute value of the difference between the labels of its endpoints) and working our way down to the shortest edge. We will make sure that, at each step, the graph constructed up to that point will satisfy certain invariant conditions that will enable us to continue the construction.

Given n, and given s, t and k satisfying $0 \leq s < k < t \leq n+1$, consider the following sets of graphs.

1. $S_1^{s,t}(k)$ is the set of all graphs G such that
   (a) G is a path with the vertex set equal to $\{1,2,\ldots,s\} \cup \{k\} \cup \{t, t+1, \ldots, n\}$,
   (b) one endpoint of G is m and the other endpoint is k (G may consist of a single vertex, in which case $m = k$), and
   (c) $\{|i-j| : (i,j) \in E(G)\} = \{t-s-1, t-s, t-s+1, \ldots, n-1\}$.

2. $S_2^{s,t}(u;k)$ is the set of all graphs G such that
   (a) G is a union of two vertex-disjoint paths P_1 and P_2 (P_2 may consist of a single vertex, P_1 contains at least one edge) with the vertex set of G equal to $\{1,2,\ldots,s\} \cup \{k\} \cup \{t, t+1, \ldots, n\}$,
   (b) one endpoint of P_1 is u,
   (c) one endpoint of P_2 is m, and the other is k,
   (d) $\{|i-j| : (i,j) \in E(G)\} = \{t-s, t-s+1, t-s+2, \ldots, n-1\}$, and
   (e) if a is the other endpoint of P_1, then either $a \leq s$ and $a + (t-s-1) > s$ or $a \geq t$ and $a - (t-s-1) < t$.

3. $S_3^{s,t}(u,v;w;k)$ is the set of all graphs G such that
   (a) G is a union of three vertex-disjoint paths P_1, P_2 and P_3 (P_3 may consist of a single vertex, P_1 and P_2 contain at least one edge each) with the vertex set of G equal to $\{1,2,\ldots,s\} \cup \{k\} \cup \{t, t+1, \ldots, n\}$,

(b) one endpoint of P_1 is w,

(c) the endpoints of P_2 are u and v,

(d) the endpoints of P_3 are m and k,

(e) $\{|i-j| : (i,j) \in E(G)\} = \{t-s+1, t-s+2, t-s+3, \ldots, n-1\}$, and

(f) if a is the other endpoint of P_1, then either $a < s$ and $a + (t-s) > s$ or $a > t$ and $a - (t-s) < t$.

We will further partition the set $S_2^{s,t}(u;k)$ into two subsets depending on a. The set $S_2^{s,t}(u;k;-)$ consists of all those graphs in $S_2^{s,t}(u;k)$ for which $a \leq s$. Similarly, $S_2^{s,t}(u;k;+)$ contains those graphs where $a \geq t$.

We use the same rule to partition each $S_3^{s,t}(u,v;w;k)$ into $S_3^{s,t}(u,v;w;k;-)$ and $S_3^{s,t}(u,v;w;k;+)$.

We denote by $S^{s,t}$ the union of all $S_1^{s,t}(k)$, $S_2^{s,t}(u;k)$ and $S_3^{s,t}(u,v;w;k)$.

We say that G is *interesting* if, for some s and t, G is either in $S_1^{s,t}(k)$, or in $S_2^{s,t}(u;k)$ for $u \in \{s,t\}$, or in $S_3^{s,t}(u,v;w;k)$ for $(u,v;w) \in \{(s-1,s;t),(s-1,t;s),(s,t;s-1),(s,t;t+1),(s,t+1;t),(t,t+1;s)\}$.

We say that G is *very interesting* if it is interesting and it does not belong to $S_3^{s,t}(s,t;t+1;k;-)$ with $a+t-s = s+2$ and with $s+2 < k < t-2$, nor to $S_3^{s,t}(s,t;s-1;k;+)$ with $a-(t-s) = t-2$ and with $t-2 > k > s+2$. Finally, we say that a very interesting G is *m-very interesting* if m is as in the above definitions.

We now prove the following lemma.

Lemma 2.2.2 *For a given n, let $G \in S^{s,t}$ be m-very interesting with $t-s \geq 6$. Then G extends to a gracefully labeled path on n vertices with one endpoint labeled m.*

Proof. From the definition of $S^{s,t}$, we note that "everything important" happens in the vertices ranging from $s-1$ to $t+1$ and that, given s and t, we may ignore n.

It is quite tedious but still possible to manually check that the lemma holds true for $6 \leq t-s \leq 8$ (it is easy to check this on a computer).

Suppose the lemma is false and choose a counterexample with the smallest $t-s$. Tables 2.1–2.3 show how to extend any m-very interesting graph to another m-very

Table 2.1: Extension of $S_1^{s,t}(k)$

Case	k	Append Edges	Result
$S_1^{s,t}(k)$	$s+1$	$(s+1,t-1)$	$S_1^{s+1,t}(t-1)$
	$t-1$	$(s+1,t-1)$	$S_1^{s,t-1}(s)$
	other	$(s+1,t-1)$	$S_2^{s+1,t-1}(s+1;k)$

interesting graph (due to symmetry, we omitted the cases of the form $S_2^{s,t}(u;k;+)$ and $S_3^{s,t}(u,v;w;k;+)$ from the tables).

Note that some of the cases in the table lead to a graph of the form $S_3^{s',t'}(s',t';t'+1;k)$ or of the form $S_3^{s',t'}(s',t';s'-1;k)$. Such graphs could potentially not be m-very interesting. However, a review of our construction in these cases confirms that none of the constructed graphs fall into the non-very interesting category. The cases that result in graphs of a different form are obviously very interesting too.

Also note that we need $t-s \geq 9$ in order for all constructions in the tables to work. In particular, $t-s \geq 9$ is required for the case $S_3^{s,t}(s,t;t+1;k;-)$ with $a+(t-s) = s+2$ and with $k = t-2$.

The extension patterns provided in the tables reduce $t-s$ by anywhere between 0 and 3. Also, if the first extension does not reduce $t-s$, then the second extension does. The extended graph $G' \in S^{s',t'}$ is then m-very interesting with $t'-s' \geq 6$. Therefore, by the minimality of $t-s$, G' extends further to a graceful labeling with one endpoint labeled m, which gives a contradiction. ∎

Table 2.2: Extension of $S_2^{s,t}(u;k)$

Case	$a+(t-s-1)$	k	Append Edges	Result
$S_2^{s,t}(s;k)$	$s+1$	$s+1$	$(a,s+1),(s,t-2)$	$S_1^{s+1,t}(t-2)$
		other	$(a,s+1)$	$S_2^{s+1,t}(s+1;k)$
	other	$t-1$	$(s,t-1),(a,a+t-s-2)$	$S_1^{s,t-1}(a+t-s-2)$
		other	$(s,t-1)$	$S_2^{s,t-1}(t-1;k)$
$S_2^{s,t}(t;k)$	$s+1$	$s+1$	$(a,s+1)$	$S_1^{s+1,t+1}(t)$
		other	$(a,s+1)$	$S_2^{s+1,t}(t;k)$
	$s+2$	$s+1$	$(a,s+2),(s+1,t-1)$	$S_2^{s+2,t}(s+2;t-1)$
		$s+2$	$(a,s+2),(s+1,t-1),(s+3,t)$	$S_2^{s+2,t-1}(t-1;s+3)$
		$t-1$	$(a,s+2),(s+1,t-1),(s+1,t-2)$	$S_2^{s+2,t-1}(s+2;t-2)$
		other	$(a,s+2),(s+1,t-1)$	$S_3^{s+2,t-1}(s+1,t-1;s+2;k)$
	other	$s+1$	$(s+1,t),(a,a+t-s-2)$	$S_1^{s+1,t}(a+t-s-2)$
		other	$(s+1,t)$	$S_2^{s+1,t}(s+1;k)$

Table 2.3: Extension of $S_3^{s,t}(u,v;w;k)$

Case	$a+(t-s)$	k	Append Edges	Result
$S_3^{s,t}(s-1,s;t;k)$	$s+1$	$t-1$	$(a,s+1),(s,t-1),(s-1,t-3)$	$S_2^{s+1,t-1}(s+1;t-3)$
		other	$(s-1,t-1),(a,s)$	$S_2^{s,t-1}(t-1;k)$
	$s+2$	$t-1$	$(s-1,t-1),(a,s+1),(s,t-2)$	$S_2^{s+1,t-1}(s+1;t-2)$
		other	$(s-1,t-1)$	$S_3^{s,t-1}(s,t-1;t;k)$
	$s+3$	$s+1$	$(s,t),(s-1,t-2),(a,s+1)$	$S_1^{s+1,t}(t-2)$
		$s+2$	$(s,t),(s-1,t-2),(a,s+1),(s+2,t-1),(s+3,t-1)$	$S_2^{s+2,t-2}(t-2;s+3)$
		$t-2$	$(s,t),(a,s+2),(s+1,t-1)$	$S_3^{s+2,t-1}(s+1,t-1;s+2;t-2)$
		$t-1$	$(s,t),(s-1,t-2),(a,s+1),(s+2,t-1)$	$S_2^{s+1,t-2}(s+1;s+2)$
		other	$(s,t),(s-1,t-2),(a,s+1),(s+2,t-1)$	$S_3^{s+2,t-2}(s+1,t-2;s+2;k)$
	$s+4$	$t-1$	$(s-1,t-1),(s+1,t),(a,s+2),(s,t-3)$	$S_2^{s+2,t-1}(s+2,t-3)$
		other	same as $s+2$	same as $s+2$
	other	$t-1$	$(s-1,t-1),(s+1,t),(s,t-2)$	$S_2^{s+1,t-1}(s+1;t-2)$
		other	same as $s+2$	same as $s+2$
$S_3^{s,t}(s-1,t;s;k)$	$s+1$	$s+1$	$(a,s+1),(s,t-1)$	$S_2^{s+1,t}(t;t-1)$
		other	$(a,s+1)$	$S_3^{s+1,t}(s,s+1;t;k)$
	$s+2$	$s+1$	$(s,t),(a,s+1),(s-1,t-3)$	$S_1^{s+1,t}(t-3)$

Case	$a+(t-s)$	k	Append Edges	Result
		other	$(s,t),(a,s+1)$	$S_2^{s+1,t}(s+1;k)$
	other	$t-1$	$(s-1,t-1),(s+1,t)$	$S_2^{s,t-1}(s;s+1)$
		other	$(s-1,t-1)$	$S_3^{s,t-1}(t-1,t;s;k)$
$S_3^{s,t}(s,t;s-1;k)$	$s+1$	$t-1$	$(s-1,t-1),(a,s),(t,s+2)$	$S_1^{s,t-1}(s+2)$
		other	$(s-1,t-1),(a,s)$	$S_2^{s,t-1}(t-1;k)$
	other	$t-1$	$(s-1,t-1),(a,a+t-s-1)$	$S_2^{s,t-1}(s;a+t-s-1)$
		other	$(s-1,t-1)$	$S_3^{s,t-1}(s,t;t-1;k)$
$S_3^{s,t}(s,t;t+1;k)$	$s+1$	$s+1$	$(a,s+1),(t+1,s+2)$	$S_2^{s+1,t}(t;s+2)$
		other	$(a,s+1)$	$S_3^{s+1,t}(s,t;s+1;k)$
	$s+2$	$s+1$	$(a,s+2),(s+1,t),(s,t-2)$	$S_2^{s+2,t}(s+2;t-2)$
		$s+2$	$(a,s+2),(s+1,t),(s+1,t-1),(s+4,t+1)$	$S_2^{s+2,t-1}(t-1;s+4)$
		$t-1$	$(a,s+2),(s+1,t),(s+1,t-1),(s,t-3)$	$S_2^{s+2,t-1}(s+2;t-3)$
		$t-2$	$(a,s+2),(s,t-1),(s+1,t-1),(s+1,t-2),(s+4,t)$	$S_2^{s+2,t-2}(s+2;s+4)$
		other	none (not very interesting)	no construction
	other	$s+1$	$(s+1,t+1),(a,a+t-s-1)$	$S_2^{s+1,t}(t;a+t-s-1)$
		other	$(s+1,t+1)$	$S_3^{s+1,t}(s,t;s+1;k)$
$S_3^{s,t}(s,t+1;t;k)$	$s+1$	$s+1$	$(a,s+1)$	$S_2^{s+1,t+1}(t+1;t)$
		other	$(a,s+1)$	$S_3^{s+1,t}(s+1,t;s;k)$
	$s+2$	$s+1$	$(s,t),(a,s+1),(s+3,t+1)$	$S_1^{s+1,t}(s+3)$

Case	$a+(t-s)$	k	Append Edges	Result
		other	$(s,t),(a,s+1)$	$S_2^{s+1,t}(s+1;k)$
	$s+3$	$s+1$	$(s+1,t+1),(a,s+2),(s,t-2)$	$S_2^{s+2,t}(s+2;t-2)$
		other	$(s+1,t+1)$	$S_3^{s+1,t}(s,s+1;t;k)$
	other	$s+1$	$(s+1,t+1),(s,t-1)$	$S_2^{s+1,t}(t;t-1)$
		other	same as $s+3$	same as $s+3$
$S_3^{s,t}(t,t+1;s;k)$	$s+1$	$s+1$	$(a,s+1),(s,t-1)$	$S_2^{s+1,t}(t;t-1)$
		other	$(a,s+1)$	$S_3^{s+1,t}(s,s+1;t;k)$
	$s+2$	$s+1$	$(s,t),(a,s+1),(s+3,t+1)$	$S_1^{s+1,t}(s+3)$
		other	$(s+1,t+1),(a,s+1)$	$S_2^{s+1,t}(t;k)$
	$s+3$	$s+1$	$(s+1,t+1),(a,s+2)$	$S_2^{s+2,t+1}(s+2;t)$
		other	$(s+1,t+1)$	$S_3^{s+1,t}(s+1,t;s;k)$
	$s+4$	$s+1$	$(s+1,t+1),(s,t-1),(a,s+2),(s+3,t)$	$S_2^{s+2,t-1}(s+2;s+3)$
		other	same as $s+3$	same as $s+3$
	other	$s+1$	$(s+1,t+1),(s,t-1),(s+2,t)$	$S_2^{s+1,t-1}(t-1;s+2)$
		other	same as $s+3$	same as $s+3$

Proof of Lemma 2.2.1. The lemma is easily checked for $n < 5$. For $n \geq 5$, note that the graph consisting of the single vertex m is m-very interesting with $t - s = (n+1) - 0 = n + 1 \geq 6$. The result now follows by Lemma 2.2.2.■

The following describes a special gracefully labeled union of paths in K_n which will be used in a later construction.

Lemma 2.2.3 *Let $n \geq 4$ and let $V = \{1, 2, \ldots, n\}$. Let further $\{u, v, w\} = \{1, 2, 3\}$. Then, unless $n = 4$ and $w = 1$, or $n = 5$ and $w = 2$, there exist paths P_u, P_v and P_w (with some of the paths possibly consisting of a single vertex) such that*

(i) $V(P_u) \cup V(P_v) \cup V(P_w) = V$,

(ii) u, v and w are endpoints of P_u, P_v and P_w, respectively,

(iii) P_u, P_v and P_w are disjoint except that P_w and one of P_u and P_v share a common endpoint distinct from u, v and w,

(iv) $\{|a - b| : (a, b) \in E(P_u) \cup E(P_v) \cup E(P_w)\} = \{1, 2, \ldots, n-2\}$ as multisets.

Proof. We will prove the statement by induction on n. Cases $n = 4, 5$ and 6 are easy to check, so we will assume that $n \geq 7$.

Case $w = 1$: By symmetry, we may assume that $u = 2$ and $v = 3$.

(a) We will start building P_u, P_v and P_w as follows: set $w_1 = n-1, u_1 = n-2, v_1 = n$, and let P_u, P_v and P_w contain the edges uu_1, vv_1 and ww_1, respectively. The lengths of these edges are $n-4$, $n-3$ and $n-2$, respectively. These can be extended to paths P_u, P_v and P_w if and only if the lemma holds for $n-3$ (with $w = 2$). To see this, note that the $n-3$ vertices to which the lemma is applied are the vertices $4, 5, 6, \ldots, n$, that w_1 plays the role of w, and that one will think of the vertices as being labeled in the opposite direction, i.e., the vertex $w_1 = n-1$ is viewed as having label 2. See Figure 2.3 for a visual description of the construction.

(b) Alternatively, we may start by building P_u, P_v and P_w as follows: $w_1 = n - 2, u_1 = n$ and $v_1 = n - 1$, and, as before, let the edges uu_1, vv_1 and ww_1 be in P_u, P_v and P_w, respectively. Paths P_u, P_v and P_w can now be constructed if and only if the lemma holds for $n-3$ (this time with $w = 3$).

Note that we only need this alternative construction when $n = 8$, as in that case there are no required paths for $n - 3 = 5$ with $w = 2$.

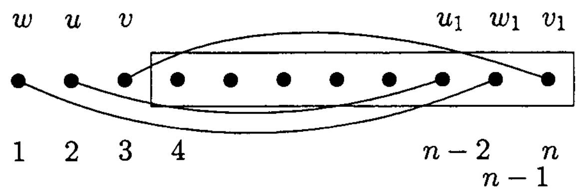

Figure 2.3: Lemma 2.2.3

Therefore, we have proved that if the lemma is true for $n-3$, then it is also true for n with $w = 1$.

Cases $w = 2$ and $w = 3$ use similar arguments as Case $w = 1$ and we omit their proofs.

Results from the above three cases, together with the fact that the lemma holds true when $4 \le n \le 6$, imply the lemma in general. ■

The following lemma also has a graceful labeling flavour. It can be used to construct special paths in either the orbit A or the orbit B.

Lemma 2.2.4 *Let $V = V(K_s) = \{1, 2, \ldots, s\}$, where $s \ge 4$, and m be an integer satisfying $0 \le m \le s-3$. Then there is $s' \in V$ and a path P in K_s such that*

(i) $V(P) = V \setminus \{s'\}$,

(ii) $\{|a-b| : (a,b) \in E(P)\} = \{1, 2, \ldots, s-1\} \setminus \{s'+m\}$, and

(iii) one of the endpoints of P is 1.

Proof. The proof will be by induction on s. It can be checked that the lemma holds true when $s \le 7$. So we will assume $s \ge 8$. If $m \le s-6$, then we start constructing P by using edges $(1, s)$ and $(s, 2)$. This path can be completed so as to satisfy the lemma if and only if the lemma holds with $s-2$ in place of s and with $m+1$ in place of m (here, we apply the induction hypothesis to the vertex set $\{2, 3, 4, \ldots, s-1\}$ which we view as being relabeled by subtracting 1 from each label). Since $m \le s-6$, we have $0 \le m+1 \le (s-2)-3$, and thus the induction hypothesis applies.

Therefore, we may assume that $m > s-6$. Since $m \le s-3$, we have $m \in \{s-5, s-4, s-3\}$.

Case 1. $m = s - 3$. We set $s' = 2$. Hence $s' + m = s - 1$. The path P will start with the edge $(1, s-1)$. Proving that this may be extended to P as required is now equivalent to showing that one can find a path P' with $V(P') = \{3, 4, \ldots, s\}$, with one endvertex equal to $s - 1$, and with $\{|a - b| : (a, b) \in E(P')\} = \{1, 2, \ldots, s - 3\}$. This is equivalent to the existence of a graceful labeling of a path on $s - 2$ vertices such that one of the endpoints has label 2; this exists by Lemma 2.2.1.

Case 2. $m = s - 4$. The procedure is similar to Case 1. We again set $s' = 2$. Hence, this time $s' + m = s - 2$. The path P will start with the edge $(1, s)$. Proving that this may be extended to P as required is now equivalent to showing that there is a path P' with $V(P') = \{3, 4, \ldots, s\}$, with one endvertex equal to s, and with $\{|a - b| : (a, b) \in E(P')\} = \{1, 2, \ldots, s - 3\}$. This is then equivalent to the existence of a graceful labeling of a path on $s - 2$ vertices such that one of the endpoints is given label 1; this again exists by Lemma 2.2.1.

Case 3. $m = s - 5$. We now set $s' = 3$ (thus $s' + m = s - 2$). The path P will start with the edge $(1, s)$, and will also use the edges $(2, s - 1)$, and $(2, s - 2)$. We observe that proving that these three edges extend to a P as required is equivalent to Lemma 2.2.3 with $n = s - 3 \geq 5$ and $w = 1$. For the purposes of Lemma 2.2.3, one uses vertices $4, 5, 6, \ldots, s$ and views them as being labeled in the opposite direction (vertex s is viewed as being relabeled with 1, vertex $s - 1$ is relabeled 2, and so on).

Figure 2.4 illustrates this case.

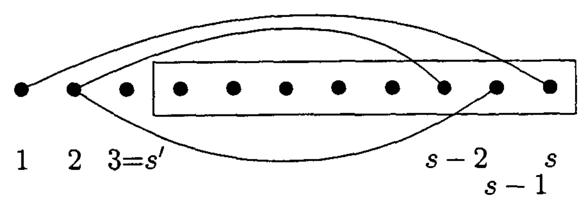

Figure 2.4: Lemma 2.2.4

■

2.3 Construction of $OP(a, b)$

Lemma 2.3.1 *Let $n \geq 18$, $n \equiv 2 \pmod 8$ and let m be an odd integer such that $n/4 < m < n/2$. Then there exists an $OP(m, n-m)$*

Proof. In this case we have $n = 2t+2$ and $4|t$ for some $t \geq 8$. For $s = 1, 3, 5, \ldots, \frac{t}{2}-1$, we define five paths in K_{2t+2}. Paths P_1 and P_2 are described in the following tables. Note that in the tables, as well as in similar tables throughout the thesis, we list the vertices of each path in the first column and the intersected edge orbits in the second column. For example, the edge $(a_{t/2+1}, b_{(t/2+1)-s})$ of P_1 is in the orbit AB_{t-s}.

P_1:

$a_{t/2+1}$	AB_{t-s}
$b_{(t/2+1)-s}$	AB_{t-s+1}
$a_{(t/2+1)-1}$	AB_{t-s+2}
$b_{(t/2+1)-(s-1)}$	AB_{t-s+3}
$\vdots$	$\vdots$
$a_{(t/2+1)-\frac{s-1}{2}}$	AB_{t-1}
$b_{(t/2+1)-\frac{s+1}{2}}$	

P_2:

$a_{(t/2+1)-\frac{s+1}{2}}$	AB_1
$b_{(t/2+1)-\frac{s+1}{2}+1}$	AB_2
$a_{(t/2+1)-\frac{s+1}{2}-1}$	AB_3
$\vdots$	$\vdots$
b_{t-s}	AB_{t-s-1}
a_1	AB_0
b_1	

P_3: $V(P_3) = \{b_{t-s+1}, b_{t-s+2}, \ldots, b_t, b_1, b_2, \ldots, b_{t/2-s}\}$, the edge-orbits intersected by the edges of P_3 are $B_1, B_2, B_3, \ldots, B_{t/2-1}$. One of the endpoints of P_3 is b_1. The existence of such a path is equivalent to the existence of a graceful labeling of a path with $t/2$ vertices with one of the endvertices labeled with $t/2 - s$. Such a path exists by Lemma 2.2.1.

P_4 and P_5: The path P_4 contains exactly one edge joining $a_{(t/2+1)-\frac{s+1}{2}}$ to a_r for some $r \in \{t/2+2, t/2+3, \ldots, t\}$. The path P_5 has $a_{t/2+1}$ as one of its endpoints and is such that $V(P_5) = \{a_{t/2+1}, a_{t/2+2}, \ldots, a_t\} \setminus \{a_r\}$, and such that the edges of P_4 and P_5 intersect each of the orbits $A_1, A_2, \ldots, A_{t/2-1}$ exactly once. The existence of these P_4 and P_5 is guaranteed by Lemma 2.2.4 (here we are using $t \geq 8$).

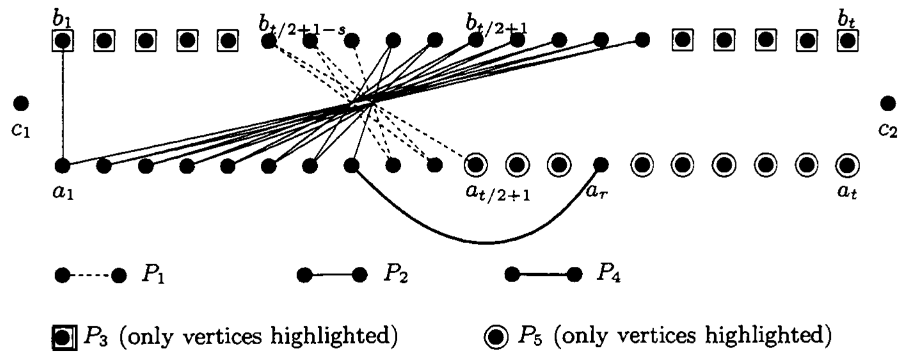

Figure 2.5: Example: $n = 42$, $t = 20$, $s = 5$, $n/4 < m < n/2$

The union P of P_1 and P_5 is a path of length $s + t/2 - 2$ with one endpoint in A and the other endpoint in B. Similarly, the union P' of P_2, P_3 and P_4 is a path of length $(3t)/2 - s$. Again, one of its endpoints is in A, while the other is in B.

Figure 2.5 shows how this construction works.

We now join vertex c_1 to the endpoints of P, and join c_2 to the endpoints of P'. In this way we obtain a 2-factor F, with cycle lengths $t/2 + s$ and $(3t)/2 - s + 2$, respectively. It is easily checked that the edge set of F intersects each edge-orbit under α in at most one edge, and that the only orbits it misses are the orbits $A_{t/2}, B_{t/2}$ and C_0. Therefore, $F, F\alpha, F\alpha^2, \ldots, F\alpha^{t-1}$ form a 2-factorization of $K_n - I$.

Since $t/2 + s$ varies over all odd integers m such that $n/4 < m < n/2$ as s varies over the set $\{1, 3, 5, \ldots, t/2 - 1\}$, we are done. ■

Lemma 2.3.2 *Let $n \equiv 2 \pmod 8$ and let $m \geq 3$ be an odd integer such that $m < n/4$. Then there is a solution to $OP(m, n - m)$.*

Proof. As before, $n = 2t + 2$, where $4|t$, $t \geq 8$. For $s = 1, 3, 5, \ldots, \frac{t}{2} - 3$, we set P_1, P_2 and P_3 as in Lemma 2.3.1.

P_4 contains only the edge $(a_{(t/2+1)-\frac{s+1}{2}}, a_{t-\frac{s+1}{2}})$. This edge belongs to $A_{t/2-1}$.

P_5 is a path such that $V(P_5) = \{a_{t/2+2}, a_{t/2+3}, \ldots, a_t\}$, and such that one of its endpoints is $a_{t-\frac{s+1}{2}}$. The edge-orbits intersected by the edges of P_5 are $A_1, A_2, \ldots, A_{t/2-2}$. The existence of such a P_5 is equivalent to the existence of a graceful labeling of the path of length $t/2-2$, one of whose endpoints is labeled $(s+3)/2$, which is guaranteed by Lemma 2.2.1.

This construction is depicted in Figure 2.6. One may note that the construction is similar to the construction in Lemma 2.3.1. The main difference is that paths P_4 and P_5 are modified so that the path P_5 is attached to the union of the paths P_2, P_3 and P_4 instead of the path P_1.

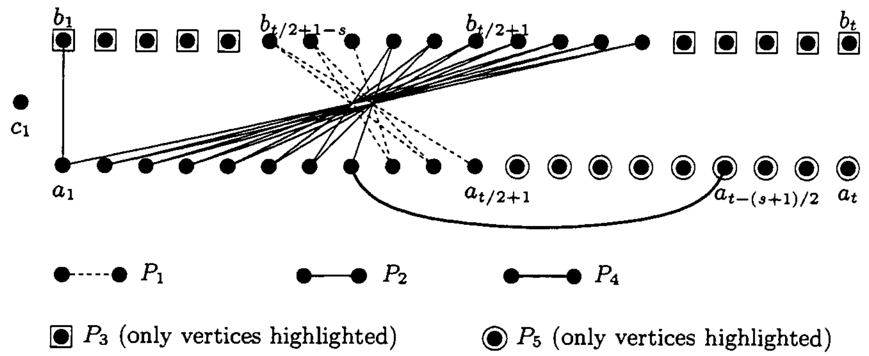

Figure 2.6: Example: $n = 42$, $t = 20$, $s = 5$, $m < n/4$

The union P of P_2, P_3, P_4 and P_5 is a path of length $2t-s-2$ with one of the endpoints in A, and the other endpoint in B. Similarly, the path P_1 has one of its endpoints in A, and the other in B, and is of length s.

If we join c_1 to the endpoints of P, and join c_2 to the endpoints of P_1, we will obtain a 2-factor F whose cycles have lengths $2t-s$ and $s+2$, respectively. As in Lemma 2.3.1, $F, F\alpha, F\alpha^2, \ldots, F\alpha^{t-1}$ form a 2-factorization of $K_n - I$.

As s varies over the set $\{1, 3, 5, \ldots, t/2-3\}$, $s+2$ varies over all odd integers m, $3 \leq m \leq n/4$, which ends the proof. ■

Lemma 2.3.3 *Let $n \equiv 6 \pmod 8$, and let $m \geq 3$ be an odd integer such that $m < n/4$. Then there is a solution to $OP(m, n-m)$.*

Proof. We may write $n = 2t+2$, where $t \geq 10$ is congruent to 2 modulo 4. For $s = 1, 3, \ldots, \frac{t}{2}-2$, we set P_1, P_2, P_3, P_4 and P_5 as in Lemma 2.3.2. The same reasoning as there shows that this lemma holds true. ■

Lemma 2.3.4 *Let $n \equiv 6 \pmod 8$, and let m be an odd integer such that $n/4 < m < n/2$. Then there is a solution to $OP(m, n-m)$.*

Proof. We write $n = 2t+2$, where t is congruent to 2 modulo 4. For $s = 1, 3, \ldots, \frac{t}{2}-2$, we set P_1, P_2 and P_3 as in Lemma 2.3.1.

Let P_4 be a path such that $V(P_4) = \{a_{t/2+1}, a_{t/2+2}, \ldots, a_t\}$, where one of the endpoints of P_4 is $a_{t/2+1}$. Further, let P_4 be such that its edges intersect the orbits $A_1, A_2, \ldots, A_{t/2-1}$. The existence of such a path is equivalent to the existence of a graceful labeling of a path on $t/2$ vertices, with one of the endpoints being given label 1, and as such is guaranteed by Lemma 2.2.1.

The union P of P_2 and P_3 is a path of length $(3t)/2 - s - 1$, with one of its endpoints in A and the other in B. Similarly, P', the union of P_1 and P_4, is a path of length $t/2 + s - 1$, with one endpoint in A, and the other in B.

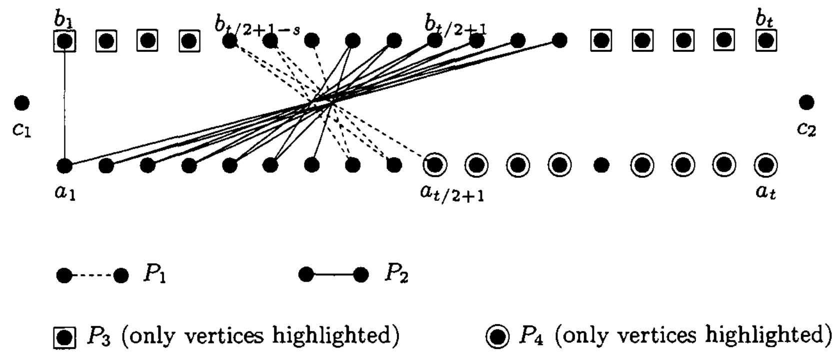

Figure 2.7: Example: $n = 38$, $t = 18$, $s = 5$, $n/4 < m < n/2$

If we connect c_1 to the endpoints of P, and c_2 to the endpoints of P', we obtain a 2-factor F, with the cycles of lengths $(3t)/2 - s + 1$ and $t/2 + s + 1$. As before, $F, F\alpha, F\alpha^2, \ldots, F\alpha^{t-1}$ form a 2-factorization of $K_n - I$.

As s ranges over $\{1, 3, 5, \ldots, t/2 - 2\}$, $m = t/2 + s + 1$ ranges over all odd integers for which $n/4 < m < n/2$, which proves this lemma. ∎

Lemma 2.3.5 *Let $n \equiv 0 \pmod 8$, and let $m \geq 3$ be an odd integer such that $m < n/4$. Then there is a solution to $OP(m, n-m)$.*

Proof. We may write $n = 2t + 2$, where t is congruent to 3 modulo 4. For $s = 1, 3, 5, \dots, \frac{t-5}{2}$, we define

P_1 :			P_2 :	
$a_{\frac{t+1}{2}}$	AB_0		$a_{\frac{t+1}{2}-\frac{s+1}{2}}$	AB_{s+1}
$b_{\frac{t+1}{2}}$	AB_1		$b_{\frac{t+1}{2}+\frac{s+1}{2}}$	AB_{s+2}
$a_{\frac{t+1}{2}-1}$	AB_2		$a_{\frac{t+1}{2}-\frac{s+3}{2}}$	AB_{s+3}
$b_{\frac{t+1}{2}+1}$	AB_3		$b_{\frac{t+1}{2}+\frac{s+3}{2}}$	AB_{s+4}
$a_{\frac{t+1}{2}-2}$	AB_4		$\vdots$	$\vdots$
$b_{\frac{t+1}{2}+2}$	AB_5		a_1	AB_{t-1}
$\vdots$	$\vdots$		b_t	
$a_{\frac{t+1}{2}-\frac{s-1}{2}}$	AB_{s-1}			
$b_{\frac{t+1}{2}+\frac{s-1}{2}}$				

P_3: $V(P_3) = \{b_t, b_1, b_2, b_3, \dots, b_{\frac{t-1}{2}}\}$. The edge-orbits intersected by the edges of P_3 are the orbits $B_1, B_2, B_3, \dots, B_{\frac{t-1}{2}}$, and one of the endvertices is b_t. The existence of such a path is equivalent to a graceful labeling of a path on $\frac{t+1}{2}$ vertices, with one endpoint given label 1, and thus follows from Lemma 2.2.1.

P_4 : The only edge in P_4 is the edge $(a_{\frac{t+1}{2}-\frac{s+1}{2}}, a_{t-\frac{s+1}{2}})$. This edge belongs to $A_{\frac{t-1}{2}}$. Note that $t - (s+1)/2 > (t+1)/2$ and, therefore, $a_{t-(s+1)/2}$ is not in $V(P_1) \cup V(P_2)$.

P_5: $V(P_5) = \{a_{\frac{t+1}{2}+1}, a_{\frac{t+1}{2}+2}, \dots, a_t\}$. The edge-orbits intersected by the edges of P_5 are the orbits $A_1, A_2, \dots, A_{\frac{t-3}{2}}$, and one of the endpoints is $a_{t-\frac{s+1}{2}}$. The existence of such a path is equivalent to the existence of a graceful labeling of a path on $\frac{t-1}{2}$ vertices, with one endpoint being given label $\frac{s+3}{2}$, and as such is guaranteed by Lemma 2.2.1. Figure 2.8 illustrates this construction.

The union P of P_2, P_3, P_4 and P_5 is a path of length $2t - s - 2$, with one endpoint in A, and the other in B. P_1 is a path of length s, with one endpoint in A, and the other in B. If we now connect vertex c_1 to the endpoints of P, and the vertex c_2 to the endpoints of P_1, we will obtain a 2-factor F, whose cycle lengths are $2t - s$ and $s + 2$, respectively. One can check that the edge-set of F intersect each edge-orbit

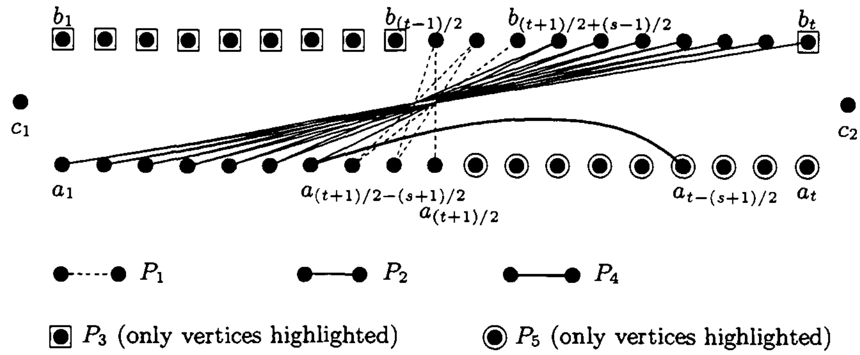

Figure 2.8: Example: $n = 40$, $t = 19$, $s = 5$, $m < n/4$

in at most one edge, and that the orbits AB_s and C_0 are missed entirely. Therefore, $F, F\alpha, F\alpha^2, \ldots, F\alpha^{t-1}$ form a 2-factorization of $K_n - I$.

Since $s + 2$ varies over all odd integers m, $3 \leq m < n/4$, as s varies over $\{1, 3, 5, \ldots, \frac{t-5}{2}\}$, we are done. ■

Lemma 2.3.6 *Let $n \equiv 0 \pmod 8$, and let m be an odd integer such that $n/4 < m < n/2$. Then there is a solution to $OP(m, n-m)$.*

Proof. Again, $n = 2t+2$, where t is congruent to 3 modulo 4. For $s = 1, 3, 5, \ldots, \frac{t-1}{2}$, we define P_1 and P_2 as in Lemma 2.3.5.

P_3: $V(P_3) = \{b_1, b_2, \ldots, b_{\frac{t-1}{2}}\}$. The edge-orbits intersected by the edges of P_3 are the orbits $B_1, B_2, B_3, \ldots, B_{\frac{t-3}{2}}$, and one of the endpoints of P_3 is the vertex $b_{\frac{s+1}{2}}$. This path is equivalent to a graceful labeling of a path on $\frac{t-1}{2}$ vertices, with one of the endpoints being given label $\frac{s+1}{2}$, and its existence is once again guaranteed by Lemma 2.2.1.

P_4 contains only one edge, namely the edge $(b_{\frac{s+1}{2}}, b_{\frac{t+1}{2}+\frac{s-1}{2}})$. This edge belongs to $B_{\frac{t-1}{2}}$.

P_5: $V(P_5) = \{a_{\frac{t+1}{2}}, a_{\frac{t+1}{2}+1}, a_{\frac{t+1}{2}+2}, \ldots, a_t\}$. The edge-orbits intersected by the edges of P_5 are the orbits $A_1, A_2, A_3, \ldots, A_{\frac{t-1}{2}}$, and one of the endvertices is $a_{\frac{t+1}{2}}$. A path like this is equivalent to a graceful labeling of a path on $\frac{t+1}{2}$ vertices, with one of the endpoints assigned label 1. The existence of such a labeling is guaranteed by Lemma 2.2.1.

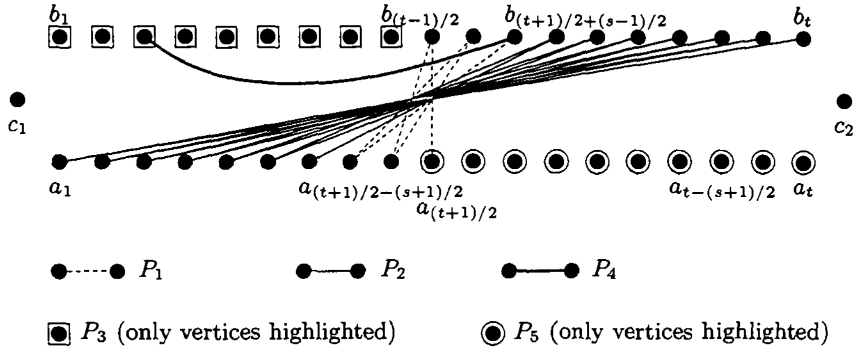

Figure 2.9: Example: $n = 40$, $t = 19$, $s = 5$, $n/4 < m < n/2$

Figure 2.9 illustrates the construction.

The union P of paths P_1, P_3, P_4 and P_5 has one of its endpoints in A, the other in B, and its length is $t + s - 1$, while the length of P_2 is $t - s - 1$ (and also, one of its endpoints is in A, while the other is in B). If we join vertex c_1 to the endpoints of P, and join vertex c_2 to the endpoints of P_2, we obtain a 2-factor F, whose two cycles have lengths $t + s + 1$ and $t - s + 1$. As in the earlier lemmas, this 2-factor gives rise to a 2-factorization of $K_n - I$. Since $t - s + 1$ ranges over all odd m such that $n/4 < m < n/2$ when s ranges over $\{1, 3, 5, \ldots, \frac{t-1}{2}\}$, we are done. ■

Lemma 2.3.7 *Let* $n \equiv 4 \pmod 8$, *and let* m *be an odd integer such that* $3 \leq m \leq n/4$. *Then* $OP(m, n-m)$ *has a solution.*

Proof. The proof of this lemma is essentially same as that of Lemma 2.3.5, with the only difference that now $s = 1, 3, 5, \ldots, \frac{t-3}{2}$.■

Lemma 2.3.8 *Let* $n \equiv 4 \pmod 8$, *and let* m *be an odd integer such that* $n/4 < m < n/2$. *Then* $OP(m, n-m)$ *has a solution.*

Proof. This is essentially same as Lemma 2.3.6, except that $s = 1, 3, 5, \ldots, \frac{t-3}{2}$. ■

Theorem 2.3.9 combines Lemmas 2.3.1–2.3.8 and covers the remaining small cases.

Theorem 2.3.9 *Let* $a, b \geq 3$, $a \neq b$ *be odd integers. Then* $OP(a, b)$ *has a solution.*

Proof. Lemmas 2.3.1, 2.3.2, 2.3.3, 2.3.4, 2.3.5, 2.3.6, 2.3.7 and 2.3.8 cover all pairs (a, b) except for the pair $(3, 7)$. In this case, we will construct a 2-factor F such that $F, F\alpha, F\alpha^2, F\alpha^3$ form the desired 2-factorization:

$$F = \{(a_1, b_2, b_1), (a_2, a_3, c_2, b_3, a_4, c_1, b_4)\}.$$

It is easily seen that this 2-factor is as desired. ■

Chapter 3

Three Cycles

The skeleton of the constructions used in this chapter is similar to that described in Section 2.1. However, the factor F now comprises 3 cycles and, in Sections 3.1 and 3.2, we allow odd n of the form $n = 4k + 3$. In order to accommodate the third cycle, we prove additional results with graceful labeling flavour. Also, we can no longer rely on the paths P_1 and P_2 exactly as constructed in Section 2.1. We have to tweak their construction a bit at times in order to create an appropriate F. The general idea is the same though. Start with paths that use up most of the edges in the orbits AB_i and combine them with additional paths and/or cycles to create F.

In Sections 3.1 and 3.2 we look at the cases $OP(3, 4, n-7)$ and $OP(3, 6, n-9)$, respectively. In Section 3.3 we consider the case $OP(3, a, n-3-a)$.

3.1 Construction of $OP(3, 4, n-7)$

Lemma 3.1.1 *Let $n \geq 16$ be an integer divisible by 4. Then $OP(3, 4, n-7)$ has a solution.*

Proof. As usual, we write $n = 2t + 2$. First, we will construct the $(n-7)$-cycle.

P_1 :	
a_3	$AB_{\frac{t-7}{2}}$
$b_{\frac{t-1}{2}}$	$AB_{\frac{t-9}{2}}$
a_4	$AB_{\frac{t-11}{2}}$
$b_{\frac{t-3}{2}}$	$AB_{\frac{t-13}{2}}$
$\vdots$	$\vdots$
$\vdots$	AB_1
$\vdots$	AB_0
$\vdots$	AB_{t-1}
$\vdots$	AB_{t-2}
$\vdots$	$\vdots$
b_4	$AB_{\frac{t+9}{2}}$
$a_{\frac{t-1}{2}}$	$AB_{\frac{t+7}{2}}$
b_3	

P_2: P_2 will contain the edges $(a_3, a_{\frac{t+5}{2}})$ and $(a_{\frac{t+5}{2}}, a_1)$. These two edges contribute to $A_{\frac{t-1}{2}}$ and $A_{\frac{t-3}{2}}$.

P_3: P_3 will be a path such that $V(P_3) = \{a_{\frac{t+9}{2}}, a_{\frac{t+11}{2}}, \ldots, a_t, a_1, a_2\}$, such that a_1 is one of its endpoints, and such that the orbits intersected are $A_1, A_2, A_3, \ldots, A_{\frac{t-5}{2}}$ (here we are using $n \geq 16$). The existence of this path is equivalent to the existence of a graceful labeling of a path on $\frac{t-3}{2}$ vertices, with one of the endpoints given label 2, and hence P_3 exists.

P_4: P_4 will contain the edge $(b_3, b_{\frac{t+5}{2}})$, which belongs to $B_{\frac{t-1}{2}}$.

P_5: $V(P_5) = \{b_{\frac{t+1}{2}}, b_{\frac{t+3}{2}}, b_{\frac{t+5}{2}}, \ldots, b_{t-1}\}$, $b_{\frac{t+5}{2}}$ is one of the endpoints of P_5, and the orbits used are $B_1, B_2, B_3, \ldots, B_{\frac{t-3}{2}}$. Again, the existence of this path is guaranteed by its equivalence with the existence of the appropriate graceful labeling.

We set P to be the union of paths P_1, P_2, P_3, P_4 and P_5. It can be checked that P is a path of length $n-9$, and that its endpoints are in the sets A and B, respectively. The construction of P is illustrated in Figure 3.1.

If C_1 is the cycle constructed by connecting c_1 to the endpoints of P, then C_1 is an $(n-7)$-cycle, and one may observe that it doesn't intersect any of the edge-orbits in more than one edge. Further, one may observe that the vertices and edge-orbits missed by C_1 are

$$a_{\frac{t+1}{2}}, a_{\frac{t+3}{2}}, a_{\frac{t+7}{2}}, b_1, b_2, b_t, c_2, \text{ and}$$

$$AB_{\frac{t-5}{2}}, AB_{\frac{t-3}{2}}, AB_{\frac{t-1}{2}}, AB_{\frac{t+1}{2}}, AB_{\frac{t+3}{2}}, AB_{\frac{t+5}{2}}, AC_2, BC_2, C_0,$$

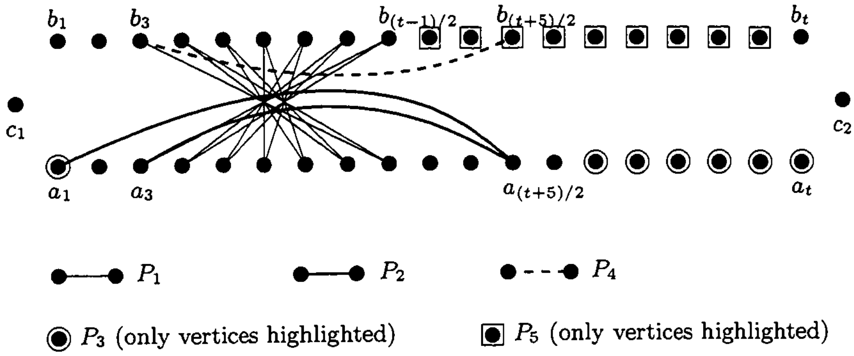

Figure 3.1: Construction of P for $OP(3,4,33)$

respectively.

If we set C_2 to be the 3-cycle $a_{\frac{t+7}{2}}, b_1, c_2$ (using the orbits $AB_{\frac{t-5}{2}}, AC_2$ and BC_2), and C_3 the 4-cycle $a_{\frac{t+1}{2}}, b_t, a_{\frac{t+3}{2}}, b_2$ (using the orbits $AB_{\frac{t-3}{2}}$,$AB_{\frac{t-1}{2}}$,$AB_{\frac{t+1}{2}}$ and $AB_{\frac{t+3}{2}}$), then C_1, C_2 and C_3 form a 2-factor F which gives rise to an $OP(3,4,n-7)$. ∎

Lemma 3.1.2 *Let $n \geq 66$, $n \equiv 2 \pmod 4$. Then $OP(3,4,n-7)$ has a solution.*

Proof. As usual, $n = 2t+2$.

P_1 :	
$a_{\frac{t+4}{2}}$	$AB_{\frac{t-6}{2}}$
b_{t-1}	$AB_{\frac{t-8}{2}}$
$a_{\frac{t+6}{2}}$	$AB_{\frac{t-10}{2}}$
b_{t-2}	$AB_{\frac{t-12}{2}}$
$\vdots$	$\vdots$
$\vdots$	AB_1
$\vdots$	AB_0
$\vdots$	AB_{t-1}
$\vdots$	AB_{t-2}
$\vdots$	$\vdots$
a_{t-1}	$AB_{\frac{t+6}{2}}$
$b_{\frac{t+4}{2}}$	

P_2 :	
$b_{\frac{t+4}{2}}$	$B_{\frac{t-2}{2}}$
b_3	$B_{\frac{t-4}{2}}$
$b_{\frac{t+2}{2}}$	$B_{\frac{t-6}{2}}$
b_4	$B_{\frac{t-10}{2}}$
$b_{\frac{t-2}{2}}$	$B_{\frac{t-12}{2}}$
b_5	$B_{\frac{t-14}{2}}$
$b_{\frac{t-4}{2}}$	$B_{\frac{t-16}{2}}$
b_6	$B_{\frac{t-18}{2}}$
$b_{\frac{t-6}{2}}$	$B_{\frac{t-8}{2}}$
b_1	$B_{\frac{t-20}{2}}$
$b_{\frac{t-18}{2}}$	

P_4 :	
$a_{\frac{t+4}{2}}$	$A_{\frac{t-4}{2}}$
a_t	$A_{\frac{t-2}{2}}$
$a_{\frac{t-2}{2}}$	$A_{\frac{t-8}{2}}$
a_3	$A_{\frac{t-10}{2}}$
$a_{\frac{t-4}{2}}$	$A_{\frac{t-6}{2}}$
a_1	$A_{\frac{t-12}{2}}$
$a_{\frac{t-10}{2}}$	

P_2 is a path only if $\frac{t-18}{2} \geq 7$, or, equivalently, if $n \geq 66$.

P_3: $V(P_3) = \{b_7, b_8, b_9, \ldots, b_{\frac{t-8}{2}}\}$, one of the endpoints of P_3 is $b_{\frac{t-18}{2}}$, and the edge-orbits used are $B_1, B_2, B_3, \ldots, B_{\frac{t-22}{2}}$. This path is equivalent to a graceful labeling of a path on $\frac{t-20}{2}$ vertices such that one of the endpoints is given label 6. The existence of such a path is guaranteed by Lemma 2.2.1.

P_5: $V(P_5) = \{a_4, a_5, a_6, \ldots, a_{\frac{t-6}{2}}\}$, one of the endpoints of P_5 is $a_{\frac{t-10}{2}}$, and the edge-orbits used are $A_1, A_2, A_3, \ldots, A_{\frac{t-14}{2}}$. This is equivalent to a graceful labeling of a path on $\frac{t-12}{2}$ vertices, with one of the endvertices being given label 3. The existence of such a P_5 is guaranteed by Lemma 2.2.1.

The union P of paths P_1 through P_5 is a path of length $n - 9$, with one of the endpoints in A and the other in B. See Figure 3.2 for a visual description of the construction.

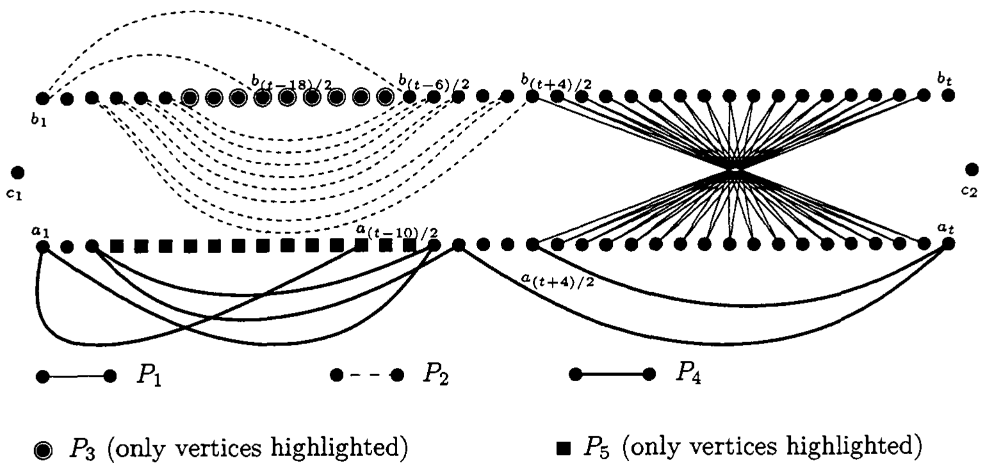

Figure 3.2: Construction of P for $OP(3, 4, 71)$

If we let C_1 be the cycle obtained by connecting c_1 to the endpoints of P, then C_1 has length $n - 7$, intersects each edge-orbit in at most one edge, and misses the following vertices and edge-orbits.

$$a_2, a_{\frac{t}{2}}, a_{\frac{t+2}{2}}, b_2, b_{\frac{t}{2}}, b_t, c_2,$$

$$AB_{\frac{t-4}{2}}, AB_{\frac{t-2}{2}}, AB_{\frac{t}{2}}, AB_{\frac{t+2}{2}}, AB_{\frac{t+4}{2}}, AC_2, BC_2, A_{\frac{t}{2}}, B_{\frac{t}{2}}, C_0.$$

If we let C_2 be the 3-cycle $a_2, b_{\frac{t}{2}}, c_2$ (intersecting the orbits $AB_{\frac{t-4}{2}}$, AC_2 and BC_2), and C_3 be the 4-cycle $a_{\frac{t}{2}}, b_t, a_{\frac{t+2}{2}}, b_2$ (intersecting the orbits $AB_{\frac{t-2}{2}}$, $AB_{\frac{t}{2}}$, $AB_{\frac{t+2}{2}}$ and $AB_{\frac{t+4}{2}}$), then C_1, C_2 and C_3 form a 2-factor which gives rise to $OP(3,4,n-7)$. ■

When n was even, we used either c_1 or c_2 for the 3-cycle. When $n \equiv 3 \pmod 4$, we only have c_1. We would like to be able to use c_1 for the $n-7$ cycle. Therefore, the 3-cycle will be either in A or in B. We will use Lemma 3.1.3 to provide us with such a 3-cycle.

Lemma 3.1.3 *Let $n \geq 7$ and let $V = \{1,2,3,\ldots,n\}$. Then there is a path P and a 3-cycle C such that*

(i) $V(P) \cup V(C) = V$,

(ii) $V(P) \cap V(C) = \emptyset$,

(iii) $\{|u-v| : (u,v) \in E(P) \cup E(C)\} = V \setminus \{n\}$, and

(iv) one of the endpoints of P is 5.

Proof. The proof will be done by induction on n. It can be checked that the lemma holds true for $7 \leq n \leq 15$. So we may assume that $n \geq 16$. Now, let P_1 be the path $5, n-3, 4, n-2, 3, n-1, 2, n, 1, n-8$. The statement will follow if we can prove that there exist a path P_2 and a 3-cycle C' such that

(i) $V(P_2) \cup V(C') = \{6,7,8,\ldots,n-4\}$,

(ii) $V(P) \cap V(C) = \emptyset$,

(iii) $\{|u-v| : (u,v) \in E(P_2) \cup E(C')\} = \{1,2,3,\ldots,n-10\}$, and

(iv) one of the endpoints of P_2 is $n-8$.

It is readily seen that the existence of such P_2 and C' is equivalent to the statement of the lemma for $n-9$ (see Figure 3.3). Since $n \geq 16$, $n-9 \geq 7$, and thus P_2 and C' exist by the induction hypothesis.

We may now set $P = P_1 \cup P_2$ and $C = C'$ to complete the proof. ■

Lemma 3.1.4 *Let $n \geq 63$, $n \equiv 3 \pmod 4$. Then $OP(3,4,n-7)$ has a solution.*

Proof. We write $n = 2t+1$, where t is an odd integer. Set

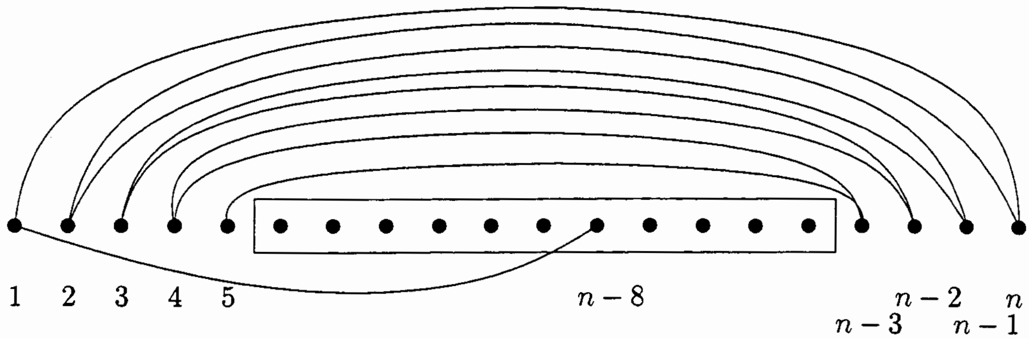

Figure 3.3: Inductive construction in Lemma 3.1.3

P_1 :	
a_2	$AB_{\frac{t-5}{2}}$
$b_{\frac{t-1}{2}}$	$AB_{\frac{t-7}{2}}$
a_3	$AB_{\frac{t-9}{2}}$
$b_{\frac{t-3}{2}}$	$AB_{\frac{t-11}{2}}$
a_4	$AB_{\frac{t-13}{2}}$
$b_{\frac{t-5}{2}}$	$AB_{\frac{t-15}{2}}$
$\vdots$	$\vdots$
$\vdots$	AB_1
$\vdots$	AB_0
$\vdots$	AB_{t-1}
$\vdots$	AB_{t-2}
$\vdots$	$\vdots$
b_3	$AB_{\frac{t+7}{2}}$
$a_{\frac{t-1}{2}}$	

P_2 :	
$b_{\frac{t+5}{2}}$	$B_{\frac{t-3}{2}}$
b_1	$B_{\frac{t-1}{2}}$
$b_{\frac{t+3}{2}}$	$B_{\frac{t-5}{2}}$
b_{t-1}	$B_{\frac{t-9}{2}}$
$b_{\frac{t+7}{2}}$	$B_{\frac{t-11}{2}}$
b_{t-2}	$B_{\frac{t-13}{2}}$
$b_{\frac{t+9}{2}}$	$B_{\frac{t-15}{2}}$
b_{t-3}	$B_{\frac{t-7}{2}}$
$b_{\frac{t+1}{2}}$	$B_{\frac{t-17}{2}}$
b_{t-8}	

P_4 :	
a_2	$A_{\frac{t-1}{2}}$
$a_{\frac{t+5}{2}}$	$A_{\frac{t-3}{2}}$
a_1	$A_{\frac{t-5}{2}}$
$a_{\frac{t+7}{2}}$	$A_{\frac{t-7}{2}}$
a_t	

P_3 and C_1: P_3 is a path and C_1 is a 3-cycle such that
(i) $V(P_3) \cup V(C_1) = \{b_{\frac{t+11}{2}}, b_{\frac{t+13}{2}}, b_{\frac{t+15}{2}}, \ldots, b_{t-4}\}$,
(ii) $V(P_3) \cap V(C_1) = \emptyset$,
(iii) $\{|i-j| : (u_i, u_j) \in E(P_3) \cup E(C_1)\} = \{1, 2, 3, \ldots, \frac{t-19}{2}\}$, and
(iv) one of the endpoints of P_3 is b_{t-8}.

For $n \geq 63$, the right-hand side of (i) is a set with at least 7 elements, and thus the existence of P_3 and C_1 follows from Lemma 3.1.3.

P_5 consists of a single edge, namely $(a_t, b_{\frac{t+5}{2}})$. This edge belongs to $AB_{\frac{t+5}{2}}$.

P_6 consists of the edge $(a_{\frac{t-1}{2}}, a_{t-5})$. This edge belongs to $A_{\frac{t-9}{2}}$.

P_7 is a path with $V(P_7) = \{a_{\frac{t+9}{2}}, a_{\frac{t+11}{2}}, a_{\frac{t+13}{2}}, \ldots, a_{t-1}\}$. The edge-orbits intersected are $A_1, A_2, A_3, \ldots, A_{\frac{t-11}{2}}$ and one of the endpoints is a_{t-5}. The existence of such a path is equivalent to the existence of a graceful labeling of a path on $\frac{t-9}{2}$ vertices, such that one of the endpoints is assigned label 5. Therefore, such a path exists by Lemma 2.2.1 (see Figure 3.4).

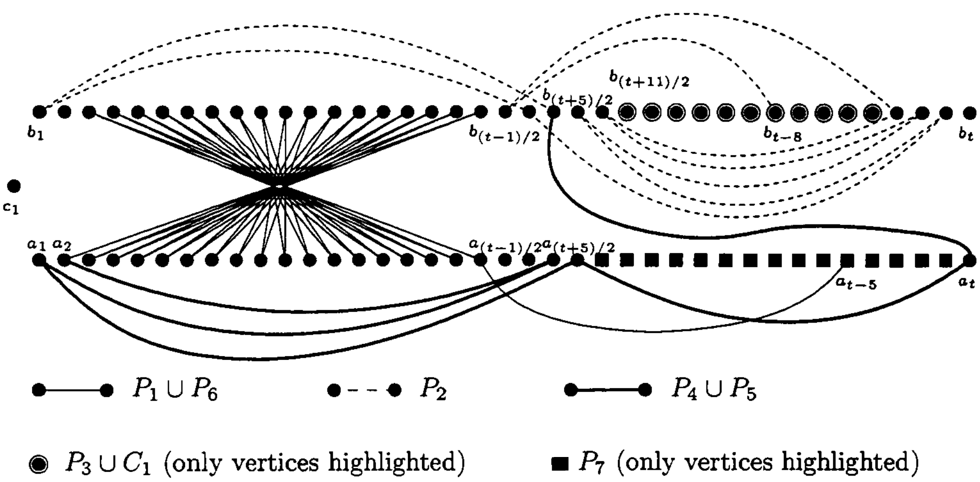

Figure 3.4: Construction of P and C_1 for $OP(3,4,72)$

C_2 is the cycle $a_{\frac{t+1}{2}}, b_t, a_{\frac{t+3}{2}}, b_2$ (and thus intersects the orbits $AB_{\frac{t-3}{2}}$, $AB_{\frac{t-1}{2}}$, $AB_{\frac{t+1}{2}}$, $AB_{\frac{t+3}{2}}$).

We let P be the union of P_1 through P_7. Then P is a path of length $n-9$, with one endpoint in A, and the other in B. If we connect c_1 to these endpoints, we obtain a cycle C_3.

It may be checked that the union of C_1, C_2 and C_3 intersects each edge-orbit in at most one edge. Since the lengths of C_1, C_2, and C_3 are $3, 4$ and $n-7$, respectively, the 2-factor comprising these cycles gives rise to an $OP(3,4,n-7)$. ∎

Theorem 3.1.5 combines Lemmas 3.1.1, 3.1.2 and 3.1.4 , and covers the remaining small cases through explicit construction.

Theorem 3.1.5 *Let $n \geq 10$, $n \equiv 0, 2,$ or $3 \pmod 4$. Then $OP(3,4,n-7)$ has a solution.*

Proof. Lemmas 3.1.1, 3.1.2, and 3.1.4 cover all but finitely many cases. We will now construct the 2-factor F yielding the desired $OP(3,4,n-7)$ in each of these cases.

When $n \equiv 0 \pmod 4$, the only unresolved case is $n = 12$. In this case, the 2-factor $F = (a_1, b_2, b_1), (a_3, a_4, b_3, b_5), (a_2, a_5, c_2, b_4, c_1)$ yields $OP(3,4,5)$, as desired.

When $n \equiv 2 \pmod 4$, the generating 2-factors for the remaining unresolved cases are listed in the following table:

Table 3.1: $OP(3,4,(4k+2)-7)$

n	OP	cycles in F
10	$OP(3,4,3)$	$(a_1, b_2, b_1), (a_2, a_3, c_2, b_4), (a_4, b_3, c_1)$
14	$OP(3,4,7)$	$(a_1, b_2, b_1), (a_5, a_6, b_5, b_3), (a_2, a_4, c_2, b_6, a_3, c_1, b_4)$
18	$OP(3,4,11)$	$(a_1, a_6, a_8), (a_7, c_2, b_8, c_1), (a_2, b_4, a_3, b_3, a_4, b_2, a_5, b_1, b_6, b_7, b_5)$
22	$OP(3,4,15)$	$(a_9, c_2, b_4), (b_1, b_3, b_2, b_5), (a_1, a_3, a_4, c_1, b_6, a_{10}, b_7, a_8, b_8, a_7, b_9, a_6, b_{10}, a_2, a_5)$
26	$OP(3,4,19)$	$(a_1, b_5, b_1), (a_9, b_2, b_4, c_2), (a_2, a_6, a_3, a_5, a_4, c_1, b_3, b_{12}, b_{11}, b_6, a_{12}, b_7, a_{11}, b_8, a_{10}, b_9, a_8, b_{10}, a_7)$
30	$OP(3,4,23)$	$(a_1, b_1, c_1), (b_2, b_4, b_5, b_{11}), (a_2, a_7, a_3, a_6, a_4, a_5, c_2, b_6, b_3, b_7, a_{14}, b_8, a_{13}, b_9, a_{12}, b_{10}, a_{11}, b_{12}, a_{10}, b_{13}, a_9, b_{14}, a_8)$
34	$OP(3,4,27)$	$(a_7, b_7, c_1), (a_1, a_2, a_{12}, a_5), (a_3, a_6, a_4, c_2, b_2, b_5, b_3, b_4, b_{16}, b_6, b_1, b_8, a_{16}, b_9, a_{15}, b_{10}, a_{14}, b_{11}, a_{13}, b_{12}, a_{11}, b_{13}, a_{10}, b_{14}, a_9, b_{15}, a_8)$
38	$OP(3,4,31)$	$(a_1, b_1, c_1), (b_3, b_4, b_6, b_{14}), (a_2, a_9, a_3, a_8, a_4, a_7, a_5, a_6, c_2, b_7, b_2, b_8, b_5, b_9, a_{18}, b_{10}, a_{17}, b_{11}, a_{16}, b_{12}, a_{15}, b_{13}, a_{14}, b_{15}, a_{13}, b_{16}, a_{12}, b_{17}, a_{11}, b_{18}, a_{10})$

n	OP	cycles in F
42	$OP(3,4,35)$	$(a_7, b_7, c_1), (a_3, a_8, a_6, a_{15}), (a_1, a_2, a_9, a_5, c_2, b_3, b_2, b_5, b_9, b_4, b_6, b_{20}, b_8, b_1, b_{10}, a_{20}, b_{11}, a_{19}, b_{12}, a_{18}, b_{13}, a_{17}, b_{14}, a_{16}, b_{15}, a_{14}, b_{16}, a_{13}, b_{17}, a_{12}, b_{18}, a_{11}, b_{19}, a_{10}, a_4)$
46	$OP(3,4,39)$	$(a_2, b_2, c_1), (b_5, b_8, b_{17}, b_{10}), (a_1, a_8, a_6, a_7, a_4, a_{10}, a_5, a_9, c_2, b_4, b_6, b_7, b_3, b_9, b_1, b_{11}, a_{22}, b_{12}, a_{21}, b_{13}, a_{20}, b_{14}, a_{19}, b_{15}, a_{18}, b_{16}, a_{17}, b_{18}, a_{16}, b_{19}, a_{15}, b_{20}, a_{14}, b_{21}, a_{13}, b_{22}, a_{12}, a_3, a_{11})$
50	$OP(3,4,43)$	$(a_1, b_1, c_1), (a_6, a_7, a_{18}, a_{10}), (a_{12}, a_2, a_{11}, a_4, a_9, a_3, a_5, a_8, c_2, b_6, b_5, b_{10}, b_3, b_9, b_7, b_4, b_8, b_{24}, b_{11}, b_2, b_{12}, a_{24}, b_{13}, a_{23}, b_{14}, a_{22}, b_{15}, a_{21}, b_{16}, a_{20}, b_{17}, a_{19}, b_{18}, a_{17}, b_{19}, a_{16}, b_{20}, a_{15}, b_{21}, a_{14}, b_{22}, a_{13}, b_{23})$
54	$OP(3,4,47)$	$(a_5, b_5, c_1), (b_7, b_8, b_{20}, b_{11}), (a_1, a_6, a_9, a_8, a_{10}, c_2, b_9, b_6, b_1, b_3, b_{10}, b_4, b_{12}, b_2, b_{13}, a_{26}, b_{14}, a_{25}, b_{15}, a_{24}, b_{16}, a_{23}, b_{17}, a_{22}, b_{18}, a_{21}, b_{19}, a_{20}, b_{21}, a_{19}, b_{22}, a_{18}, b_{23}, a_{17}, b_{24}, a_{16}, b_{25}, a_{15}, b_{26}, a_{14}, a_2, a_{13}, a_3, a_{12}, a_4, a_{11}, a_7)$
58	$OP(3,4,51)$	$(a_7, b_7, c_1), (a_8, a_{11}, a_{21}, a_{12}), (a_9, a_3, a_{10}, a_2, a_{13}, a_1, a_{14}, b_{27}, a_{15}, b_{26}, a_{16}, b_{25}, a_{17}, b_{24}, a_{18}, b_{23}, a_{19}, b_{22}, a_{20}, b_{21}, a_{22}, b_{20}, a_{23}, b_{19}, a_{24}, b_{18}, a_{25}, b_{17}, a_{26}, b_{16}, a_{27}, b_{15}, a_{28}, b_{14}, b_3, b_{13}, b_{28}, b_{12}, b_5, b_1, b_6, b_4, b_{10}, b_2, b_{11}, b_8, b_9, c_2, a_5, a_6, a_4)$
62	$OP(3,4,55)$	$(a_1, b_1, c_1), (b_8, b_9, b_{23}, b_{13}), (a_2, a_{15}, a_3, a_{14}, a_4, a_{13}, a_5, a_{12}, a_6, a_{11}, a_7, a_{10}, a_8, a_9, c_2, b_6, b_{10}, b_7, b_5, b_{11}, b_4, b_{12}, b_3, b_{14}, b_2, b_{15}, a_{30}, b_{16}, a_{29}, b_{17}, a_{28}, b_{18}, a_{27}, b_{19}, a_{26}, b_{20}, a_{25}, b_{21}, a_{24}, b_{22}, a_{23}, b_{24}, a_{22}, b_{25}, a_{21}, b_{26}, a_{20}, b_{27}, a_{19}, b_{28}, a_{18}, b_{29}, a_{17}, b_{30}, a_{16})$

When $n \equiv 3 \pmod 4$, the generating 2-factors are as follows:

Table 3.2: $OP(3,4,(4k+3)-7)$

n	OP	cycles in F
11	$OP(3,4,4)$	$(a_3,b_2,c_1),(a_1,a_2,a_5,b_3),(a_4,b_5,b_1,b_4)$
15	$OP(3,4,8)$	$(a_1,a_3,a_4),(a_2,b_2,b_6,c_1),(a_5,b_7,a_6,b_5,a_7,b_4,b_3,b_1)$
19	$OP(3,4,12)$	$(a_3,b_3,c_1),(b_1,b_2,b_9,b_4),(a_1,a_7,a_2,a_4,a_5,b_8,a_6,b_7,a_8,b_6,a_9,b_5)$
23	$OP(3,4,16)$	$(a_4,b_4,b_9),(b_1,b_3,b_2,b_5),(a_1,a_2,a_6,a_3,a_5,c_1,b_6,a_{11},b_7,a_{10},b_8,a_9,b_{10},a_8,b_{11},a_7)$
27	$OP(3,4,20)$	$(a_3,a_{10},b_3),(b_1,b_2,b_6,b_4),(a_1,a_4,a_2,a_7,b_{12},a_8,b_{11},a_9,b_{10},a_{11},b_9,a_{12},b_8,a_{13},b_7,b_{13},b_5,c_1,a_6,a_5)$
31	$OP(3,4,24)$	$(a_5,b_5,b_{12}),(a_3,a_4,a_8,a_6),(a_1,a_7,a_2,c_1,b_3,b_7,b_4,b_6,b_1,b_2,b_8,a_{15},b_9,a_{14},b_{10},a_{13},b_{11},a_{12},b_{13},a_{11},b_{14},a_{10},b_{15},a_9)$
35	$OP(3,4,28)$	$(a_4,b_4,a_{13}),(b_1,b_7,b_3,b_8),(a_1,a_5,a_7,a_6,a_3,a_8,a_2,a_9,b_{16},a_{10},b_{15},a_{11},b_{14},a_{12},b_{13},a_{14},b_{12},a_{15},b_{11},a_{16},b_{10},a_{17},b_9,b_{17},b_2,b_5,b_6,c_1)$
39	$OP(3,4,32)$	$(a_6,b_6,b_{15}),(a_4,a_5,a_8,a_{10}),(a_1,a_9,a_2,a_7,a_3,c_1,b_4,b_1,b_5,b_7,b_8,b_3,b_9,b_2,b_{10},a_{19},b_{11},a_{18},b_{12},a_{17},b_{13},a_{16},b_{14},a_{15},b_{16},a_{14},b_{17},a_{13},b_{18},a_{12},b_{19},a_{11})$
43	$OP(3,4,36)$	$(a_5,b_5,a_{16}),(b_1,b_8,b_2,b_{10}),(a_1,a_4,a_8,a_7,a_9,a_3,a_{10},a_2,a_{11},b_{20},a_{12},b_{19},a_{13},b_{18},a_{14},b_{17},a_{15},b_{16},a_{17},b_{15},a_{18},b_{14},a_{19},b_{13},a_{20},b_{12},a_{21},b_{11},b_{21},b_4,b_9,b_7,b_6,b_3,c_1,a_6)$
47	$OP(3,4,40)$	$(a_7,b_7,b_{18}),(b_1,b_{10},b_3,b_{11}),(a_1,a_5,c_1,b_5,b_8,b_9,b_4,b_2,b_6,b_{12},a_{23},b_{13},a_{22},b_{14},a_{21},b_{15},a_{20},b_{16},a_{19},b_{17},a_{18},b_{19},a_{17},b_{20},a_{16},b_{21},a_{15},b_{22},a_{14},b_{23},a_{13},a_2,a_{12},a_3,a_{11},a_4,a_{10},a_8,a_9,a_6)$

n	OP	cycles in F
51	$OP(3,4,44)$	$(a_6, a_{19}, b_6), (a_1, a_{10}, a_2, a_{12}), (a_3, a_5, a_8, a_9, a_4, a_{11}, a_7, a_{13}, b_{24}, a_{14}, b_{23}, a_{15}, b_{22}, a_{16}, b_{21}, a_{17}, b_{20}, a_{18}, b_{19}, a_{20}, b_{18}, a_{21}, b_{17}, a_{22}, b_{16}, a_{23}, b_{15}, a_{24}, b_{14}, a_{25}, b_{13}, b_1, b_{12}, b_2, b_{11}, b_3, b_{10}, b_4, b_{25}, b_5, b_8, b_7, b_9, c_1)$
55	$OP(3,4,48)$	$(a_8, b_8, b_{21}), (b_1, b_{12}, b_3, b_{13}), (a_1, a_6, c_1, b_{10}, b_4, b_5, b_2, b_9, b_7, b_{11}, b_6, b_{14}, a_{27}, b_{15}, a_{26}, b_{16}, a_{25}, b_{17}, a_{24}, b_{18}, a_{23}, b_{19}, a_{22}, b_{20}, a_{21}, b_{22}, a_{20}, b_{23}, a_{19}, b_{24}, a_{18}, b_{25}, a_{17}, b_{26}, a_{16}, b_{27}, a_{15}, a_2, a_{14}, a_3, a_{13}, a_4, a_{12}, a_5, a_9, a_{11}, a_{10}, a_7)$
59	$OP(3,4,52)$	$(a_7, a_{22}, b_7), (a_1, a_{13}, a_3, a_{14}), (a_2, a_9, a_{11}, a_6, a_{15}, b_{28}, a_{16}, b_{27}, a_{17}, b_{26}, a_{18}, b_{25}, a_{19}, b_{24}, a_{20}, b_{23}, a_{21}, b_{22}, a_{23}, b_{21}, a_{24}, b_{20}, a_{25}, b_{19}, a_{26}, b_{18}, a_{27}, b_{17}, a_{28}, b_{16}, a_{29}, b_{15}, b_1, b_{14}, b_2, b_{13}, b_3, b_{12}, b_4, b_{11}, b_8, b_{10}, b_9, b_5, b_{29}, b_6, c_1, a_{12}, a_8, a_5, a_4, a_{10})$

∎

3.2 Construction of $OP(3, 6, n-9)$

In Lemma 3.1.4 we used a gracefully labeled union of a path and a 3-cycle. In this and the next section, we will use graceful labelings of unions of a path and a longer cycle. Lemmas 3.2.1–3.2.4 provide us with such graceful labelings.

Lemma 3.2.1 *Let $n = 2k+1 \geq 3$ and let G be the graph consisting of an n-cycle C and a path P on k vertices. Then G has a graceful labeling in which one of the endpoints of P is assigned label* 2.

Proof. Let $V(G) = \{1, 2, 3, \ldots, 3k+1\}$. We will construct paths P_1, P_2 and P as follows:

P_1 :

k odd		k even	
edge	$\|u-v\|$	edge	$\|u-v\|$
$(1,3k)$	$3k-1$	$(1,3k)$	$3k-1$
$(3k,4)$	$3(k-1)-1$	$(3k,4)$	$3(k-1)-1$
$(4,3k-3)$	$3(k-2)-1$	$(4,3k-3)$	$3(k-2)-1$
$(3k-3,7)$	$3(k-3)-1$	$(3k-3,7)$	$3(k-3)-1$
$(7,3k-6)$	$3(k-4)-1$	$(7,3k-6)$	$3(k-4)-1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(\frac{3k-1}{2},\frac{3k+3}{2})$	2	$(\frac{3k+6}{2},\frac{3k+2}{2})$	2

P_2 :

k odd		k even	
edge	$\|u-v\|$	edge	$\|u-v\|$
$(3k+1,3)$	$3(k-1)+1$	$(3k+1,3)$	$3(k-1)+1$
$(3,3k-2)$	$3(k-2)+1$	$(3,3k-2)$	$3(k-2)+1$
$(3k-2,6)$	$3(k-3)+1$	$(3k-2,6)$	$3(k-3)+1$
$(6,3k-5)$	$3(k-4)+1$	$(6,3k-5)$	$3(k-4)+1$
$(3k-5,9)$	$3(k-5)+1$	$(3k-5,9)$	$3(k-5)+1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(\frac{3k+5}{2},\frac{3k+3}{2})$	1	$(\frac{3k}{2},\frac{3k+2}{2})$	1

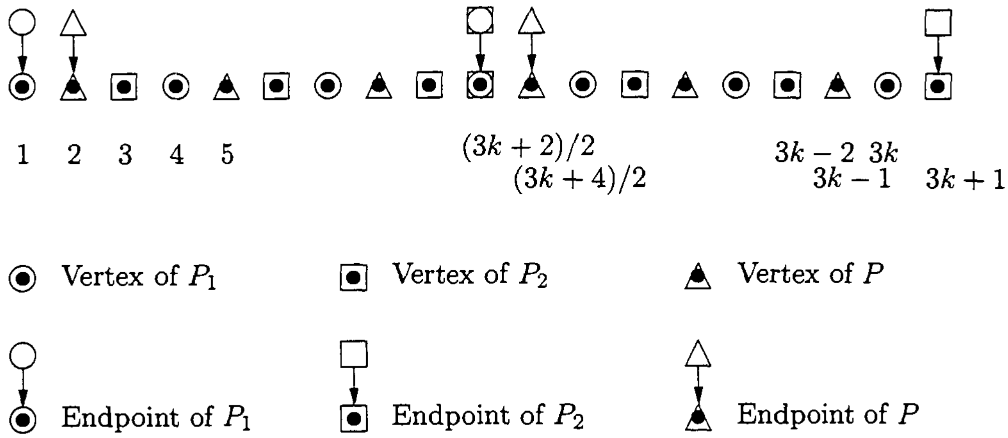

Figure 3.5: Vertex pattern for P_1, P_2 and P_3 in Lemma 3.2.1

P :

k odd		**k even**	
edge	$\|u-v\|$	edge	$\|u-v\|$
$(2, 3k-1)$	$3(k-1)$	$(2, 3k-1)$	$3(k-1)$
$(3k-1, 5)$	$3(k-2)$	$(3k-1, 5)$	$3(k-2)$
$(5, 3k-4)$	$3(k-3)$	$(5, 3k-4)$	$3(k-3)$
$(3k-4, 8)$	$3(k-4)$	$(3k-4, 8)$	$3(k-4)$
$(8, 3k-7)$	$3(k-5)$	$(8, 3k-7)$	$3(k-5)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(\frac{3k+7}{2}, \frac{3k+1}{2})$	3	$(\frac{3k-2}{2}, \frac{3k+4}{2})$	3

Figure 3.5 illustrates the pattern for even k. One observes that the pattern is very simple — with vertices alternating between the three paths except for possible small adjustments around the middle of the vertex range, i.e., around $\frac{3k+2}{2}$. One may also observe that the union of P_1 and P_2 is a path of length $n-1$, with the vertices 1 and $3k+1$ being its endpoints. Therefore, by appending the edge $(1, 3k+1)$ to this path, we obtain a cycle of length n. Also, P is a path of length $k-1$ with vertex 2 as one of its endpoints. It is somewhat tedious, but otherwise easy, to check that the union of P and C is as required. ■

Lemma 3.2.2 *Let $n = 2k + 2 \geq 4$, and let G be the graph consisting of an n-cycle C and a path P on k vertices. Then G has a graceful labeling in which one of the endpoints of P is assigned label 2.*

Proof. Now, $V(G) = \{1, 2, 3, \ldots, 3k + 2\}$. Paths P_1, P_2 and P are constructed as follows.

P_1 :

k odd		**k even**	
edge	$\|u - v\|$	edge	$\|u - v\|$
$(1, 3k + 1)$	$3k$	$(1, 3k + 1)$	$3k$
$(3k + 1, 4)$	$3(k - 1)$	$(3k + 1, 4)$	$3(k - 1)$
$(4, 3k - 2)$	$3(k - 2)$	$(4, 3k - 2)$	$3(k - 2)$
$(3k - 2, 7)$	$3(k - 3)$	$(3k - 2, 7)$	$3(k - 3)$
$(7, 3k - 5)$	$3(k - 4)$	$(7, 3k - 5)$	$3(k - 4)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(\frac{3k-1}{2}, \frac{3k+5}{2})$	3	$(\frac{3k+8}{2}, \frac{3k+2}{2})$	3
$(\frac{3k+5}{2}, \frac{3k+3}{2})$	1	$(\frac{3k+2}{2}, \frac{3k+4}{2})$	1

P_2 :

k odd		**k even**	
edge	$\|u - v\|$	edge	$\|u - v\|$
$(3k + 2, 3)$	$3k - 1$	$(3k + 2, 3)$	$3k - 1$
$(3, 3k - 1)$	$3(k - 1) - 1$	$(3, 3k - 1)$	$3(k - 1) - 1$
$(3k - 1, 6)$	$3(k - 2) - 1$	$(3k - 1, 6)$	$3(k - 2) - 1$
$(6, 3k - 4)$	$3(k - 3) - 1$	$(6, 3k - 4)$	$3(k - 3) - 1$
$(3k - 4, 9)$	$3(k - 4) - 1$	$(3k - 4, 9)$	$3(k - 4) - 1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(\frac{3k+7}{2}, \frac{3k+3}{2})$	2	$(\frac{3k}{2}, \frac{3k+4}{2})$	2

P :

k odd		k even	
edge	$\lvert u-v\rvert$	edge	$\lvert u-v\rvert$
$(2,3k)$	$3(k-1)+1$	$(2,3k)$	$3(k-1)+1$
$(3k,5)$	$3(k-2)+1$	$(3k,5)$	$3(k-2)+1$
$(5,3k-3)$	$3(k-3)+1$	$(5,3k-3)$	$3(k-3)+1$
$(3k-3,8)$	$3(k-4)+1$	$(3k-3,8)$	$3(k-4)+1$
$(8,3k-6)$	$3(k-5)+1$	$(8,3k-6)$	$3(k-5)+1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(\frac{3k+9}{2},\frac{3k+1}{2})$	4	$(\frac{3k-2}{2},\frac{3k+6}{2})$	4

The patterns are similar to the previous lemma; the union of P_1 and P_2 is a path on n vertices, with 1 and $3k+2$ being its endpoints. By adding the edge $(1, 3k+2)$ to this path, we obtain the cycle C. Again, one may check that if G is the union of P and C, then G satisfies the conditions of this lemma. ■

We will denote the disjoint union of an s-cycle and a path on t vertices by C_sP_t. Unless stated otherwise, we will always assume that $V(C_sP_t) = \{1, 2, 3, \ldots, s+t\}$.

Lemma 3.2.3 *If there is a graceful labeling of C_sP_t in which one of the endpoints of the path is assigned label 2, then there are also such graceful labelings of the graphs C_sP_{t+3} and C_sP_{t+4}.*

Proof. Consider first C_sP_{t+3}. Let P_1 consist of the edges $(2, s+t+3), (s+t+3, 1)$ and $(1, s+t+1)$. If we can show that there is a graph H isomorphic to C_sP_t, such that

(i) $V(H) = \{3, 4, 5, \ldots, t+s+2\}$,

(ii) $\{|u-v| : (u,v) \in E(H)\} = \{1, 2, 3, \ldots, s+t-1\}$, and

(iii) one of the endpoints of the path in H is $s+t+1$,

then we are done, because the union of H and P_1 will be as required (see Figure 3.6). But the existence of H is easily seen to be equivalent to the existence of a graceful labeling of C_sP_t in which one of the endpoints of the path is assigned label 2, and as

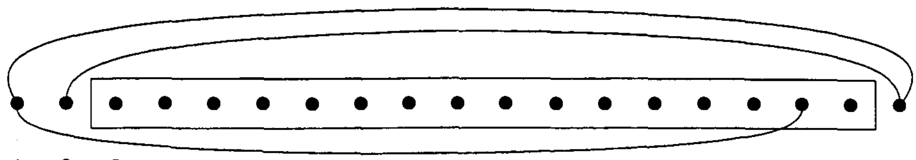

Figure 3.6: Inductive construction of C_sP_{t+3}

such is guaranteed by the assumption. Thus we conclude that the required graceful labeling of C_sP_{t+3} exists.

Now, consider C_sP_{t+4}. The idea is the same as for C_sP_{t+3}. Let P_1 consist of the edges $(2, s+t+3), (s+t+3, 1), (1, s+t+4)$ and $(s+t+4, 4)$. Now, we need to find a graph H isomorphic to C_sP_t, such that

(i) $V(H) = \{3, 4, 5, \ldots, s+t+2\}$,

(ii) $\{|u-v| : (u,v) \in E(H)\} = \{1, 2, 3, \ldots, s+t-1\}$, and

(iii) one of the endpoints of the path in H is 4.

The rest of the proof goes through the same way as in the first part. ∎

Lemma 3.2.4 *Let $s \geq 3$ be given and let t be such that $t \geq \frac{s}{2} + 5$. Then there is a graceful labeling of C_sP_t in which one of the endpoints of the path is assigned label 2.*

Proof. It follows from Lemmas 3.2.2 and 3.2.1 that, for $s \geq 3$, C_sP_t has the required graceful labeling when $t = \lceil s/2 - 1 \rceil$.

From Lemma 3.2.3, it follows that C_sP_t has the required labeling also when $t = \lceil s/2 + 2 \rceil$ and $t = \lceil s/2 + 3 \rceil$. Applying Lemma 3.2.3, starting with these two values of t, we conclude that the labeling also exists when $t = \lceil s/2+5 \rceil, \lceil s/2+6 \rceil$ and $\lceil s/2+7 \rceil$. Starting with these three values, it is immediately seen that applying Lemma 3.2.3 inductively proves that the labeling exists for all values $t \geq s/2+5$, which proves our lemma. ∎

Lemma 3.2.5 *Let $n \geq 60$, $n \equiv 0 \pmod 4$. Then $OP(3, 6, n-9)$ has a solution.*

Proof. $n = 2t + 2$, where t is odd. We construct the following paths and cycles:

P_1 :			
$a_{\frac{t+3}{2}}$	$AB_{\frac{t-3}{2}}$	$\vdots$	AB_1
b_t	$AB_{\frac{t-5}{2}}$	$\vdots$	AB_0
$a_{\frac{t+5}{2}}$	$AB_{\frac{t-7}{2}}$	$\vdots$	AB_{t-1}
b_{t-1}	$AB_{\frac{t-9}{2}}$	$\vdots$	$\vdots$
$a_{\frac{t+7}{2}}$	$AB_{\frac{t-11}{2}}$	a_t	$AB_{\frac{t+3}{2}}$
$\vdots$	$\vdots$	$b_{\frac{t+3}{2}}$	

P_2: $V(P_2) = \{a_2, a_3, a_4, \ldots, a_{\frac{t+3}{2}}\}$, one of the endpoints of P_2 is $a_{\frac{t+3}{2}}$, and P_2 meets each of the orbits $A_1, A_2, A_3, \ldots, A_{\frac{t-1}{2}}$ exactly once. The existence of this P_2 is equivalent to the existence of a graceful labeling of a path on $\frac{t+1}{2}$ vertices, with one endpoint assigned label 1, and thus is guaranteed by Lemma 2.2.1.

P_3 and C_1: $V(P_3 \cup C_1) = \{b_1, b_2, b_3, \ldots, b_{\frac{t-1}{2}}\}$, C_1 is a 6-cycle, P_3 is a path of length $\frac{t-15}{2}$, one endpoint of P_3 is b_2, and $E(P_3) \cup E(C_1)$ intersects each of the orbits $B_1, B_2, B_3, \ldots, B_{\frac{t-3}{2}}$ exactly once. The existence of such P_3 and C_1 is seen to be equivalent to the existence of a graceful labeling of $C_6P_{\frac{t-13}{2}}$, in which one of the endpoints of the path is assigned label 2. Since $n \geq 60$, we have $t \geq 29$ and $\frac{t-13}{2} \geq 8$. The existence of the graceful labeling now follows by Lemma 3.2.4.

C_2 is the 3-cycle $a_1, b_{\frac{t+1}{2}}, c_1$ (and thus intersects the orbits $AB_{\frac{t-1}{2}}, AC_1$ and BC_1).

P_4 consists of the single edge $(b_2, b_{\frac{t+3}{2}})$ (and thus intersects the orbit $B_{\frac{t-1}{2}}$).

Now, $P = P_1 \cup P_2 \cup P_3 \cup P_4$ is a path of length $n - 11$, with one endpoint in A, and the other in B. We extend P to a cycle C by connecting its endpoints to c_2. See Figure 3.7 for a visual description of the construction.

It is easily checked that the 2-factor F formed by C_1, C_2 and C intersects each of the edge-orbits in at most one edge. Since the lengths of C_1, C_2 and C_3 are $6, 3$ and $n-9$, respectively, F gives rise to an $OP(3, 6, n-9)$. ■

Lemma 3.2.6 *Let $n \geq 62$, $n \equiv 2 \pmod 4$. Then $OP(3, 6, n-9)$ has a solution.*

Proof. $n = 2t + 2$, where t is even. We will construct the following paths and cycles:

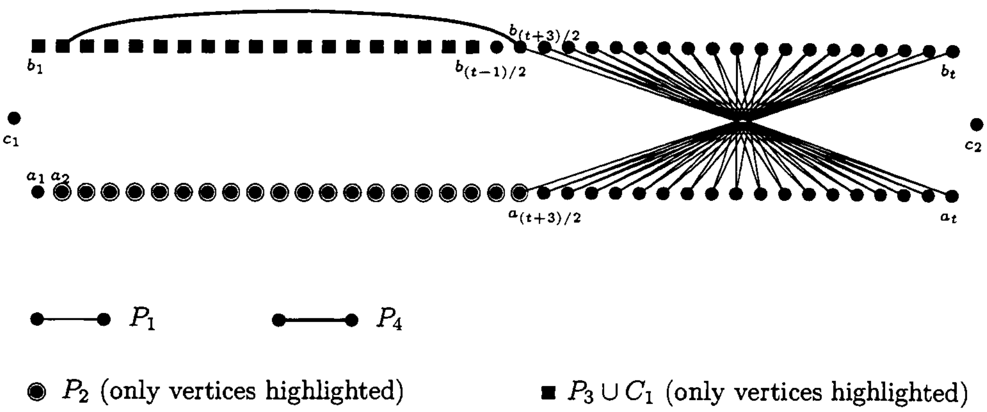

Figure 3.7: Construction of P and C_1 for $OP(3,6,71)$

P_1:			
$a_{\frac{t}{2}}$	$AB_{\frac{t}{2}}$	$\vdots$	$\vdots$
b_t	$AB_{\frac{t-4}{2}}$	$\vdots$	AB_1
$a_{\frac{t+4}{2}}$	$AB_{\frac{t-6}{2}}$	$\vdots$	AB_0
b_{t-1}	$AB_{\frac{t-8}{2}}$	$\vdots$	AB_{t-1}
$a_{\frac{t+6}{2}}$	$AB_{\frac{t-10}{2}}$	$\vdots$	$\vdots$
b_{t-2}	$AB_{\frac{t-12}{2}}$	a_t	$AB_{\frac{t+2}{2}}$
$a_{\frac{t+8}{2}}$	$AB_{\frac{t-14}{2}}$	$b_{\frac{t+2}{2}}$	

P_2: $V(P_2) = \{a_2, a_3, a_4, \ldots, a_{\frac{t+2}{2}}\}$, one of the endpoints of P_2 is $a_{\frac{t}{2}}$, and the edge-orbits intersected by P_2 are $A_1, A_2, A_3, \ldots, A_{\frac{t-2}{2}}$. The existence of such a P_2 is equivalent to the existence of a graceful labeling of a path on $\frac{t}{2}$ vertices, such that one of the endpoints is assigned label 2, and as such is guaranteed by Lemma 2.2.1.

P_3 and C_1: $V(P_3) \cup V(C_1) = \{b_1, b_2, b_3, \ldots, b_{\frac{t-2}{2}}\}$, C_1 is a 6-cycle, P_3 is a path on $\frac{t-14}{2}$ vertices, with b_2 being one of the endpoints. The edge-orbits intersected by $E(P_3) \cup E(C_1)$ are $B_1, B_2, B_3, \ldots, B_{\frac{t-4}{2}}$. Similarly as in Lemma 3.2.5, the existence of P_3 and C_1 is guaranteed by Lemma 3.2.4 (here we need $n \geq 62$).

C_2 is the 3-cycle $a_1, b_{\frac{t}{2}}, c_1$ (the edge-orbits intersected are $AB_{\frac{t-2}{2}}, AC_1$ and BC_1).

P_4 consists of the single edge $(b_2, b_{\frac{t+2}{2}})$ (intersecting the orbit $B_{\frac{t-2}{2}}$).

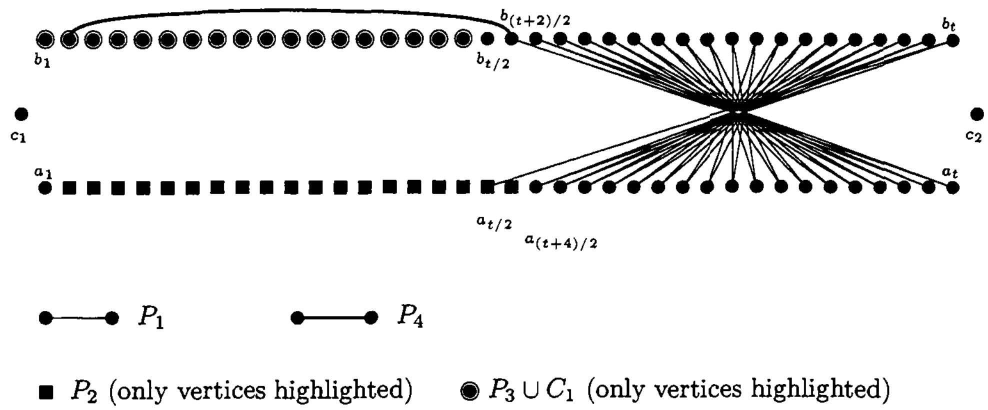

Figure 3.8: Construction of P and C_1 for $OP(3,6,69)$

The union P of P_1 through P_4 is a path of length $n-11$, with one of its endpoints in A, and the other in B (see Figure 3.8). We obtain an $(n-9)$-cycle C by joining the endpoints of P to c_2.

The 2-factor F comprising C_1, C_2 and C_3 intersects each edge-orbit in at most one edge, does not intersect any of the orbits with size less than t (i.e., orbits $A_{\frac{t}{2}}, B_{\frac{t}{2}}$ and C_0), and thus gives rise to an $OP(3,6,n-9)$. ∎

Lemma 3.2.7 *Let* $n \geq 55$, $n \equiv 3 \pmod 4$. *Then there is an* $OP(3,6,n-9)$.

Proof. $n = 2t+1$, where t is odd. We will construct the following paths and cycles.

P_1 :			
a_2	$AB_{\frac{t-1}{2}}$	$\vdots$	AB_0
$b_{\frac{t+3}{2}}$	$AB_{\frac{t+3}{2}}$	$\vdots$	AB_1
a_t	$AB_{\frac{t+5}{2}}$	$\vdots$	$\vdots$
$b_{\frac{t+5}{2}}$	$AB_{\frac{t+7}{2}}$	b_{t-1}	$AB_{\frac{t-7}{2}}$
a_{t-1}	$AB_{\frac{t+9}{2}}$	$a_{\frac{t+5}{2}}$	$AB_{\frac{t-5}{2}}$
$b_{\frac{t+7}{2}}$	$AB_{\frac{t+11}{2}}$	b_t	$AB_{\frac{t-3}{2}}$
a_{t-2}	$AB_{\frac{t+13}{2}}$	$a_{\frac{t+3}{2}}$	$AB_{\frac{t+1}{2}}$
$\vdots$	$\vdots$	b_2	
$\vdots$	AB_{t-1}		

P_2 and C_1: $V(P_2) \cup V(C_1) = \{a_1, a_2, a_3, \ldots, a_{\frac{t+1}{2}}\}$, C_1 is a 6-cycle, P_2 is a path on $\frac{t-11}{2}$ vertices, with a_2 being one of its endpoints, and P_2 and C_1 are such that the edge-orbits intersected by $E(P_2) \cup E(C_1)$ are the orbits $A_1, A_2, A_3, \ldots, A_{\frac{t-1}{2}}$. The existence of these P_2 and C_1 is equivalent to the existence of a graceful labeling of $C_6 P_{\frac{t-11}{2}}$ in which one of the endpoints of the path is given label 2. Such a labeling is guaranteed by Lemma 3.2.4 (here we need $\frac{t-11}{2} \geq 8$, but this is guaranteed by $n \geq 55$).

P_3 and C_2: $V(P_3) \cup V(C_2) = \{b_1, b_2, b_3, \ldots, b_{\frac{t+1}{2}}\}$, where C_2 is a 3-cycle and P_3 is a path on $\frac{t-5}{2}$ vertices. One of the endpoints of P_3 is b_2, and the edge-orbits intersected by the edges of C_2 and P_3 are the orbits $B_1, B_2, B_3, \ldots, B_{\frac{t-1}{2}}$. As in the case of P_2 and C_1, Lemma 3.2.4 guarantees the existence of P_3 and C_2. Figure 3.9 illustrates this construction.

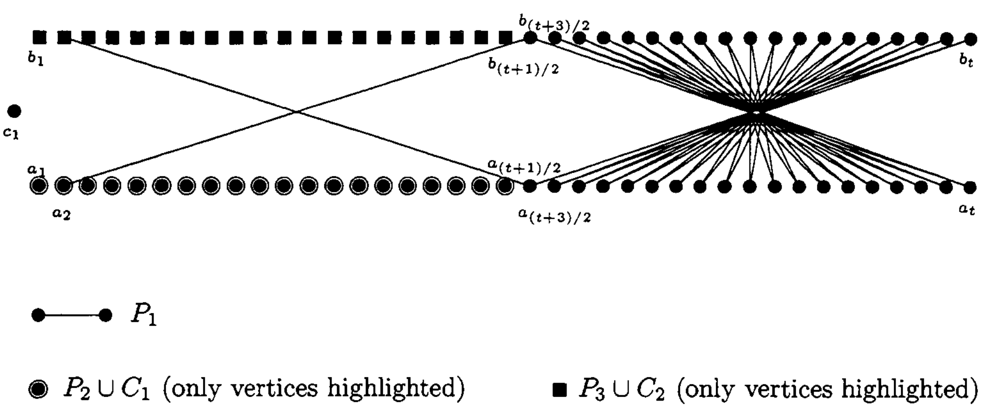

Figure 3.9: Construction of P, C_1 and C_2 for $OP(3, 6, 70)$

The union P of P_1, P_2 and P_3 is a path of length $n - 11$, with one endpoint in A, and the other in B. We extend it to an $(n-9)$-cycle C by connecting its endpoints to c_1. The union F of C, C_1 and C_2 is a 2-factor intersecting each of the edge-orbits in at most one edge. Since the lengths of the three cycles C, C_1 and C_2 are $n - 9, 6$ and 3, respectively, F gives rise to an $OP(3, 6, n-9)$. ■

Theorem 3.2.8 *Let* $n \geq 12$, $n \equiv 0, 2$ *or* $3 \pmod 4$. *Then* $OP(3, 6, n-9)$ *has a solution.*

Proof. Lemmas 3.2.5, 3.2.6, and 3.2.7 cover all but finitely many cases. We will now construct the 2-factor F yielding the desired $OP(3, 6, n-9)$ in each of these cases.

When $n \equiv 0 \pmod 4$, the 2-factors are listed in the following table:

Table 3.3: $OP(3,6,4k-9)$

n	OP	cycles in F
12	$OP(3,6,3)$	$(a_2,b_4,c_2),(a_1,a_4,a_5,b_5,b_3,b_2),(a_3,b_1,c_1)$
16	$OP(3,6,7)$	$(b_1,b_2,b_4),(a_1,a_2,a_5,a_3,c_1,b_3),(a_4,b_7,a_6,b_6,a_7,$ $b_5,c_2)$
20	$OP(3,6,11)$	$(b_1,b_3,b_4),(a_1,a_3,a_6,a_2,b_5,c_1),(a_4,a_5,b_9,a_7,b_8,$ $a_8,b_7,a_9,b_6,b_2,c_2)$
24	$OP(3,6,15)$	$(b_1,b_3,b_4),(a_1,a_4,a_3,a_5,c_1,b_5),(a_2,a_6,b_{11},a_8,b_{10},$ $a_9,b_9,a_{10},b_8,a_{11},b_7,b_2,b_6,c_2,a_7)$
28	$OP(3,6,19)$	$(b_3,b_4,b_6),(a_2,a_5,a_3,a_4,c_1,b_2),(a_1,a_6,a_{10},c_2,b_1,$ $b_5,b_{13},b_7,a_{13},b_8,a_{12},b_9,a_{11},b_{10},a_9,b_{11},a_8,b_{12},a_7)$
32	$OP(3,6,23)$	$(a_1,a_7,a_8),(b_2,b_{12},b_6,b_5,a_5,c_1),(a_2,a_4,a_9,b_{15},$ $a_{10},b_{14},a_{11},b_{13},a_{12},b_{11},a_{13},b_{10},a_{14},b_9,a_{15},b_8,b_1,$ $b_3,b_7,b_4,c_2,a_3,a_6)$
36	$OP(3,6,27)$	$(b_3,b_6,b_8),(a_1,a_3,a_8,a_2,b_2,c_1),(a_4,a_7,a_6,a_{13},a_5,$ $a_9,b_{16},a_{10},b_{15},a_{11},b_{14},a_{12},b_{13},a_{14},b_{12},a_{15},b_{11},a_{16},$ $b_{10},a_{17},b_9,b_{17},b_7,b_1,b_5,b_4,c_2)$
40	$OP(3,6,31)$	$(a_1,a_5,a_6),(a_8,b_8,b_{15},b_7,b_2,c_1),(a_2,a_{10},a_3,a_9,a_7,$ $a_4,c_2,b_9,b_3,b_6,b_4,b_5,b_1,b_{10},a_{19},b_{11},a_{18},b_{12},a_{17},$ $b_{13},a_{16},b_{14},a_{15},b_{16},a_{14},b_{17},a_{13},b_{18},a_{12},b_{19},a_{11})$
44	$OP(3,6,35)$	$(b_4,b_5,b_{21}),(a_2,a_3,a_5,c_1,b_7,a_7),(a_1,a_{10},a_4,a_8,$ $a_{16},a_9,a_6,c_2,b_8,b_6,b_3,b_9,b_2,b_{10},b_1,b_{11},a_{21},b_{12},$ $a_{20},b_{13},a_{19},b_{14},a_{18},b_{15},a_{17},b_{16},a_{15},b_{17},a_{14},b_{18},$ $a_{13},b_{19},a_{12},b_{20},a_{11})$
48	$OP(3,6,39)$	$(a_1,a_5,a_6),(a_8,b_8,b_2,b_5,b_3,c_1),(a_2,a_{12},a_3,a_{11},a_4,$ $a_{10},a_7,a_9,c_2,b_7,b_6,b_{10},b_{18},b_9,b_4,b_{11},b_1,b_{12},a_{23},$ $b_{13},a_{22},b_{14},a_{21},b_{15},a_{20},b_{16},a_{19},b_{17},a_{18},b_{19},a_{17},$ $b_{20},a_{16},b_{21},a_{15},b_{22},a_{14},b_{23},a_{13})$

n	OP	cycles in F
52	$OP(3,6,43)$	$(b_5, b_6, b_{25}), (a_4, a_5, c_1, b_9, a_9, a_6), (a_1, a_{12}, a_2, a_8, a_3, a_{10}, a_{19}, a_{11}, a_7, c_2, b_7, b_4, b_8, b_{10}, b_3, b_{11}, b_2, b_{12}, b_1, b_{13}, a_{25}, b_{14}, a_{24}, b_{15}, a_{23}, b_{16}, a_{22}, b_{17}, a_{21}, b_{18}, a_{20}, b_{19}, a_{18}, b_{20}, a_{17}, b_{21}, a_{16}, b_{22}, a_{15}, b_{23}, a_{14}, b_{24}, a_{13})$
56	$OP(3,6,47)$	$(a_1, a_6, a_7), (b_4, b_5, b_{10}, a_{10}, c_1, b_7), (a_2, a_{14}, a_3, a_{13}, a_4, a_{12}, a_5, a_9, a_{11}, a_8, c_2, b_6, b_8, b_{12}, b_{21}, b_{11}, b_3, b_9, b_2, b_{13}, b_1, b_{14}, a_{27}, b_{15}, a_{26}, b_{16}, a_{25}, b_{17}, a_{24}, b_{18}, a_{23}, b_{19}, a_{22}, b_{20}, a_{21}, b_{22}, a_{20}, b_{23}, a_{19}, b_{24}, a_{18}, b_{25}, a_{17}, b_{26}, a_{16}, b_{27}, a_{15})$

When $n \equiv 2 \pmod 4$, the 2-factors are:

Table 3.4: $OP(3, 6, (4k + 2) - 9)$

n	OP	cycles in F
14	$OP(3,6,5)$	$(a_4, b_3, c_1), (a_1, b_1, a_3, a_5, a_6, b_2), (a_2, b_5, b_6, b_4, c_2)$
18	$OP(3,6,9)$	$(b_1, b_3, b_8), (a_1, a_2, b_2, c_1, a_6, a_3), (a_4, b_7, a_5, b_6, a_7, b_5, a_8, b_4, c_2)$
22	$OP(3,6,13)$	$(a_1, a_2, a_5), (a_3, c_1, b_1, b_3, b_2, c_2), (a_4, a_6, b_{10}, a_7, b_9, a_8, b_7, a_9, b_6, a_{10}, b_5, b_8, b_4)$
26	$OP(3,6,17)$	$(a_1, a_3, a_4), (a_2, a_9, a_5, c_1, b_3, b_2), (a_6, b_{11}, a_7, b_{10}, a_8, b_9, a_{10}, b_8, a_{11}, b_7, a_{12}, b_6, b_4, b_1, b_5, b_{12}, c_2)$
30	$OP(3,6,21)$	$(b_2, b_5, b_{11}), (a_1, a_7, a_2, a_6, c_1, b_1), (a_3, a_4, c_2, b_6, b_4, b_3, b_7, a_{14}, b_8, a_{13}, b_9, a_{12}, b_{10}, a_{11}, b_{12}, a_{10}, b_{13}, a_9, b_{14}, a_8, a_5)$
34	$OP(3,6,25)$	$(a_7, b_7, c_1), (a_1, a_3, a_2, a_5, a_{12}, a_6), (a_4, a_8, b_{15}, a_9, b_{14}, a_{10}, b_{13}, a_{11}, b_{12}, a_{13}, b_{11}, a_{14}, b_{10}, a_{15}, b_9, a_{16}, b_8, b_1, b_5, b_{16}, b_6, b_3, b_4, b_2, c_2)$

n	OP	cycles in F
38	$OP(3,6,29)$	$(a_2,b_2,c_1),(b_1,b_5,b_8,b_6,b_{14},b_7),(a_1,a_9,a_3,a_{10},b_{18},$ $a_{11},b_{17},a_{12},b_{16},a_{13},b_{15},a_{14},b_{13},a_{15},b_{12},a_{16},b_{11},$ $a_{17},b_{10},a_{18},b_9,b_4,b_3,c_2,a_4,a_8,a_5,a_7,a_6)$
42	$OP(3,6,33)$	$(a_8,b_8,c_1),(a_1,a_2,a_4,a_7,a_{15},a_6),(a_3,a_9,a_5,c_2,b_5,$ $b_4,b_6,b_{20},b_3,b_7,b_2,b_9,b_1,b_{10},a_{20},b_{11},a_{19},b_{12},a_{18},$ $b_{13},a_{17},b_{14},a_{16},b_{15},a_{14},b_{16},a_{13},b_{17},a_{12},b_{18},a_{11},$ $b_{19},a_{10})$
46	$OP(3,6,37)$	$(a_1,b_1,c_1),(b_2,b_6,b_9,b_7,b_{17},b_8),(a_2,a_{11},a_3,a_{10},a_4,$ $a_9,a_5,a_8,a_6,a_7,c_2,b_4,b_5,b_{10},b_3,b_{11},a_{22},b_{12},a_{21},$ $b_{13},a_{20},b_{14},a_{19},b_{15},a_{18},b_{16},a_{17},b_{18},a_{16},b_{19},a_{15},$ $b_{20},a_{14},b_{21},a_{13},b_{22},a_{12})$
50	$OP(3,6,41)$	$(a_9,b_9,c_1),(a_1,a_6,a_{10},a_7,a_{18},a_8),(a_2,a_4,a_5,a_{11},$ $a_3,a_{12},b_{23},a_{13},b_{22},a_{14},b_{21},a_{15},b_{20},a_{16},b_{19},a_{17},b_{18},$ $a_{19},b_{17},a_{20},b_{16},a_{21},b_{15},a_{22},b_{14},a_{23},b_{13},a_{24},b_{12},b_1,$ $b_{11},b_2,b_{10},b_5,b_8,b_6,b_{24},b_7,b_3,b_4,c_2)$
54	$OP(3,6,45)$	$(a_{11},b_{11},c_1),(b_1,b_4,b_2,b_8,b_{20},b_9),(a_1,a_5,a_{10},a_4,$ $a_{12},a_3,a_{13},a_2,a_{14},b_{26},a_{15},b_{25},a_{16},b_{24},a_{17},b_{23},a_{18},$ $b_{22},a_{19},b_{21},a_{20},b_{19},a_{21},b_{18},a_{22},b_{17},a_{23},b_{16},a_{24},$ $b_{15},a_{25},b_{14},a_{26},b_{13},b_3,b_{12},b_5,b_{10},b_6,b_7,c_2,a_6,a_9,$ $a_7,a_8)$
58	$OP(3,6,49)$	$(a_2,b_2,c_1),(a_1,a_6,a_{12},a_8,a_{21},a_9),(a_3,a_{13},a_4,a_{11},$ $a_{10},a_7,a_5,c_2,b_7,b_9,b_8,b_3,b_6,b_{10},b_{28},b_{11},b_5,b_{12},b_4,$ $b_{13},b_1,b_{14},a_{28},b_{15},a_{27},b_{16},a_{26},b_{17},a_{25},b_{18},a_{24},b_{19},$ $a_{23},b_{20},a_{22},b_{21},a_{20},b_{22},a_{19},b_{23},a_{18},b_{24},a_{17},b_{25},$ $a_{16},b_{26},a_{15},b_{27},a_{14})$

When $n \equiv 3 \pmod 4$, the 2-factors are:

Table 3.5: $OP(3,6,(4k+3)-9)$

n	OP	cycles in F
15	$OP(3,6,6)$	$(a_2,b_3,c_1),(a_1,a_4,a_3,a_5,b_7,b_1),(a_6,b_2,b_6,b_4,a_7,b_5)$
19	$OP(3,6,10)$	$(a_1,a_4,b_4),(a_2,a_3,a_5,c_1,b_2,a_7),(a_6,b_7,a_8,b_6,a_9,b_5,b_1,b_3,b_9,b_8)$
23	$OP(3,6,14)$	$(a_2,a_5,a_6),(a_4,b_4,b_3,b_5,b_2,b_9),(a_1,a_3,c_1,b_1,b_6,a_{11},b_7,a_{10},b_8,a_9,b_{10},a_8,b_{11},a_7)$
27	$OP(3,6,18)$	$(a_2,a_3,a_6),(a_4,a_{10},a_5,a_7,c_1,b_4),(a_1,b_6,b_2,b_1,b_3,b_{13},b_5,b_{12},a_8,b_{11},a_9,b_{10},a_{11},b_9,a_{12},b_8,a_{13},b_7)$
31	$OP(3,6,22)$	$(a_7,b_7,c_1),(b_1,b_3,b_2,b_5,b_{12},b_6),(a_1,a_3,a_8,b_{15},a_9,b_{14},a_{10},b_{13},a_{12},b_{11},a_{13},b_{10},a_{14},b_9,a_{15},b_8,b_4,a_2,a_6,a_5,a_{11},a_4)$
35	$OP(3,6,26)$	$(a_2,b_2,c_1),(b_1,b_5,b_8,b_6,b_{14},b_7),(a_1,a_5,a_{13},a_6,a_4,a_7,a_8,a_3,a_9,b_{17},a_{10},b_{16},a_{11},b_{15},a_{12},b_{13},a_{14},b_{12},a_{15},b_{11},a_{16},b_{10},a_{17},b_9,b_4,b_3)$
39	$OP(3,6,30)$	$(a_2,b_2,c_1),(b_1,b_5,b_8,b_6,b_{15},b_7),(a_1,a_4,a_6,a_{14},a_5,a_9,a_8,a_3,a_{10},b_{19},a_{11},b_{18},a_{12},b_{17},a_{13},b_{16},a_{15},b_{14},a_{16},b_{13},a_{17},b_{12},a_{18},b_{11},a_{19},b_{10},b_3,b_4,b_9,a_7)$
43	$OP(3,6,34)$	$(a_1,b_1,c_1),(b_2,b_5,b_9,b_8,b_{17},b_7),(a_2,a_7,a_{16},a_6,a_8,a_{11},b_{21},a_{12},b_{20},a_{13},b_{19},a_{14},b_{18},a_{15},b_{16},a_{17},b_{15},a_{18},b_{14},a_{19},b_{13},a_{20},b_{12},a_{21},b_{11},b_3,b_{10},b_4,b_6,a_4,a_5,a_9,a_3,a_{10})$
47	$OP(3,6,38)$	$(a_7,a_9,b_9),(b_1,b_6,b_{10},b_7,b_{18},b_8),(a_1,a_2,a_{11},a_3,a_{10},a_5,a_8,a_4,a_{17},a_6,a_{12},b_{23},a_{13},b_{22},a_{14},b_{21},a_{15},b_{20},a_{16},b_{19},a_{18},b_{17},a_{19},b_{16},a_{20},b_{15},a_{21},b_{14},a_{22},b_{13},a_{23},b_{12},b_3,b_{11},b_5,b_4,b_2,c_1)$

n	OP	cycles in F
51	$OP(3,6,42)$	$(a_9, a_{11}, b_{11}), (b_1, b_4, b_2, b_8, b_{20}, b_9), (a_1, a_8, a_{19}, a_7, a_2, a_5, a_6, a_{10}, a_4, a_{12}, a_3, a_{13}, b_{25}, a_{14}, b_{24}, a_{15}, b_{23}, a_{16}, b_{22}, a_{17}, b_{21}, a_{18}, b_{19}, a_{20}, b_{18}, a_{21}, b_{17}, a_{22}, b_{16}, a_{23}, b_{15}, a_{24}, b_{14}, a_{25}, b_{13}, b_3, b_{12}, b_5, b_{10}, b_6, b_7, c_1)$

■

3.3 Construction of $OP(3, a, n-3-a)$

For graphs $G_1, G_2, \ldots, G_n$, we will denote by $G_1 \cup G_2 \cup \cdots \cup G_n$ any vertex-disjoint union of graphs isomorphic to $G_1, G_2, \ldots, G_n$. Also, we will denote by C_n the cycle on n vertices. Finally, in Lemmas 3.3.1 and 3.3.2 only, we will use P_n to denote the path on n vertices.

We start this section with two more graceful labeling results.

Lemma 3.3.1 *Let $m \geq 13$ and let $G = P_3 \cup C_3 \cup P_{m-6}$. Then G has a graceful labeling in which one of the endpoints of P_3 has label 1 and one of the endpoints of P_{m-6} has label m.*

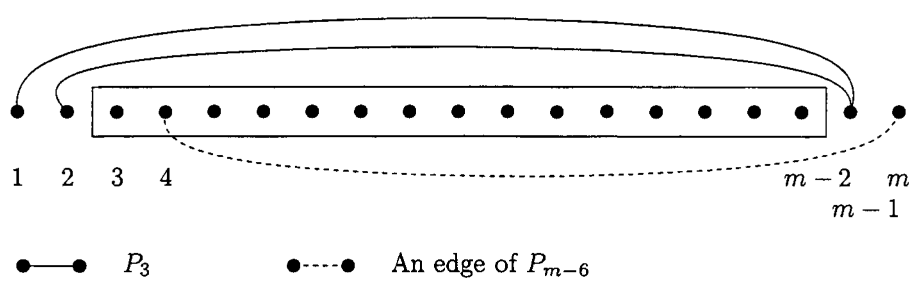

Figure 3.10: Lemma 3.3.1

Proof. For $m > 13$, P_3 will be the path $1, m-1, 2$. P_{m-6} will contain the edge $(m, 4)$. The rest follows by Lemma 3.2.4 (see Figure 3.10).

For $m = 13$, we construct the graceful labeling explicitly: $P_3 = 1, 12, 7$, $C_3 = 4, 6, 10$ and $P_7 = 13, 3, 11, 2, 9, 8, 5$. ■

Lemma 3.3.2 *Let $m \geq 13$ and let $G = P_2 \cup C_3 \cup P_{m-5}$. Then G has a graceful labeling in which one of the endpoints of P_2 has label 1, and one of the endpoints of P_{m-5} has label m.*

Proof. P_2 will be the path $1, m-1$. P_{m-5} will contain the edge $(m, 3)$. The rest follows by Lemma 3.2.4. ■

Lemma 3.3.3 *Let $n \geq 48$, $n \equiv 0 \pmod 4$ and let $a < n-3-a$. Then $OP(3, a, n-3-a)$ has a solution whenever $a > 3$.*

Proof. We write $n = 2t+2$. Let $s = \lfloor \frac{a-2}{2} \rfloor$. Construct P_1, P_2 and P_3 as follows.

P_1 :	
$a_{\frac{t+2s+1}{2}}$	$AB_{\frac{t-4s+1}{2}}$
b_{t+1-s}	$AB_{\frac{t-4s-1}{2}}$
$a_{\frac{t+2s+3}{2}}$	$AB_{\frac{t-4s-3}{2}}$
b_{t-s}	$AB_{\frac{t-4s-5}{2}}$
$a_{\frac{t+2s+5}{2}}$	$AB_{\frac{t-4s-7}{2}}$
$\vdots$	$\vdots$
$\vdots$	$\vdots$
a_t	$AB_{\frac{t+3}{2}}$
$b_{\frac{t+3}{2}}$	

P_2 :	
a_s	$AB_{\frac{t-4s+3}{2}}$
$b_{\frac{t-2s+3}{2}}$	$AB_{\frac{t-4s+5}{2}}$
a_{s-1}	$AB_{\frac{t-4s+7}{2}}$
$b_{\frac{t-2s+5}{2}}$	$AB_{\frac{t-4s+9}{2}}$
a_{s-2}	$AB_{\frac{t-4s+11}{2}}$
$b_{\frac{t-2s+7}{2}}$	$AB_{\frac{t-4s+13}{2}}$
$\vdots$	$\vdots$
a_1	$AB_{\frac{t-1}{2}}$
$b_{\frac{t+1}{2}}$	

P_3: Vertices of P_3 are the vertices of B which are not used by P_1 and P_2 plus the vertex $b_{\frac{t+3}{2}}$. This vertex is one of the endpoints of P_3. The edge-orbits intersected by P_3 are the orbits $B_1, B_2, \ldots, B_{\frac{t-1}{2}}$. The existence of such a path follows immediately from Lemma 2.2.1 (connect $b_{\frac{t+3}{2}}$ to $b_{\frac{t+3}{2}-\frac{t-1}{2}}$ and use this latter vertex as an endpoint of a path on $\{b_{t+2-s}, b_{t+3-s}, \ldots, b_t, b_1, b_2, \ldots, b_{\frac{t-2s+1}{2}}\}$).

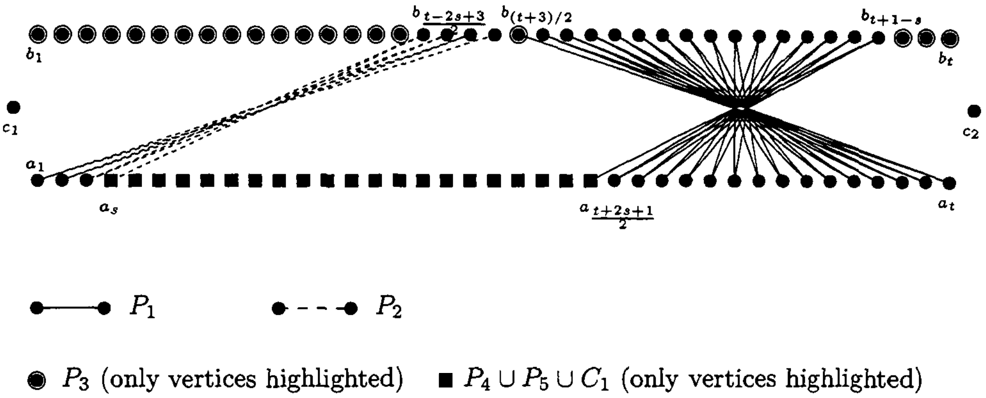

Figure 3.11: Example: $n = 80$, $a = 11$, $s = 4$

P_4, P_5 and C_1: $V(P_4 \cup P_5 \cup C_1) = \{a_s, a_{s+1}, \ldots, a_{\frac{t+2s+1}{2}}\}$. C_1 is a 3-cycle, P_4 is a path of length 1 (if a is even) or 2 (if a is odd), with one of its endpoints being a_s, and P_5 is a path of length $\frac{t-9}{2}$ (if a is even) or $\frac{t-11}{2}$ (if a is odd), with one of its endpoints being $a_{\frac{t+2s+1}{2}}$. The orbits intersected by $E(P_4 \cup P_5 \cup C_1)$ are the orbits $A_1, A_2, A_3, \ldots, A_{\frac{t-1}{2}}$. The existence of P_4, P_5 and C_1 is guaranteed by Lemma 3.3.1 (when a is odd) or 3.3.2 (when a is even). Note that here we need $n \geq 48$. Figure 3.11 depicts the construction when $n = 80$ and when $a = 11$.

The union of P_2 and P_4 is a path of length $a - 2$ with one of its endpoints in A and the other in B. The a-cycle C_2 is obtained by joining the endpoints of this path to c_1.

The union of P_1, P_3 and P_5 is a path of length $n - 5 - a$ with one of the endpoints in A and the other in B. The $(n-3-a)$-cycle C_3 is obtained by joining the endpoints of this path to c_2.

The 2-factor F comprising C_1, C_2 and C_3 intersects each edge-orbit in at most one edge and gives rise to an $OP(3, a, n-3-a)$. ∎

Lemma 3.3.4 *Let $n < 48$, $n \equiv 0 \pmod 4$ and let a be such that $3 \leq a < n-3-a$. Then $OP(3, a, n-3-a)$ has a solution.*

Proof. We will construct the desired $OP(3, a, n-3-a)$ for all pairs n and a. For each $n > 16$, the outline of the construction is described below. Cases where $n \leq 16$ are then handled independently.

Outline of the construction: For each $n > 16$, we will construct a 3-cycle C_1 and two paths P_1 and P_1', each on $n-5$ vertices and each disjoint from C_1, such that each of the unions $C_1 \cup P_1$ and $C_1 \cup P_1'$ satisfies the following two conditions:

(i) The vertex set of the union is $V(K_n) \setminus \{c_1, c_2\}$.

(ii) The edges of the union intersect each of the orbits of the form A_i, B_i and AB_i exactly once and, of course, they do not intersect the orbits AC_1, AC_2, BC_1, BC_2 and C_0.

Furthermore, (P_1')'s second, third, fourth, ..., $(n/2)$th vertices will be the same as (P_1)'s first, second, third, ..., $(n/2-1)$st vertices, respectively. We will denote by $e_1, e_2, e_3, \ldots, e_{n/2-2}$ the first $n/2-2$ edges of P_1. We will construct paths P_1 and P_1' such that they satisfy the following. If we

(i) remove any of the edges $e_2, e_4, e_6, \ldots, e_{n/2-4}$ from P_1 or from P_1',

(ii) complete thus created two paths into cycles by attaching their endpoints to c_1 and c_2, and

(iii) denote the resulting 2-factor of K_n as F, then

F intersects each edge-orbit except that it avoids the orbit C_0 and one of the orbits of the form AB_i.

The action of α on this 2-factor will then result in an $OP(3, a, n-a-3)$.

Though somewhat tedious, it is easy to check that the construction indeed results in solutions to $OP(3, a, n-a-3)$ as described. It is then easy to see that the 2-factorizations resulting from the removal of $e_2, e_4, e_6, \ldots, e_{n/2-4}$ from P_1 will result in solutions to $OP(3,3,n-6), OP(3,5,n-8), OP(3,7,n-10), \ldots, OP(3,n/2-3,n/2)$, respectively. Similarly, the removal of e_2, e_4, $e_6, \ldots$, $e_{n/2-4}$ from P_1' will result in solutions to $OP(3,4,n-7)$, $OP(3,6,n-9)$, $OP(3,8,n-11), \ldots$, $OP(3,n/2-2,n/2-1)$. respectively. These will then combine to cover all a for a given n.

We will look at $n = 24$ in more detail. All other cases are handled in a similar manner. Figure 3.12 illustrates construction of P_1 and C_1 for $n = 24$ (P_1 and C_1 are written out explicitly on page 60). P_1 and C_1 give rise to the solutions of the corresponding Oberwolfach Problems as follows:

$OP(3,3,18)$ is obtained by breaking the path P_1 into two subpaths through the removal of the edge (a_6, b_{10}). One subpath, namely (b_{11}, a_6), consists of a single edge.

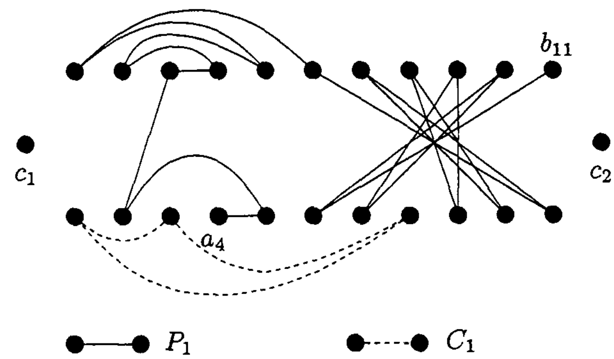

Figure 3.12: Construction of P_1 and C_1 for $n = 24$

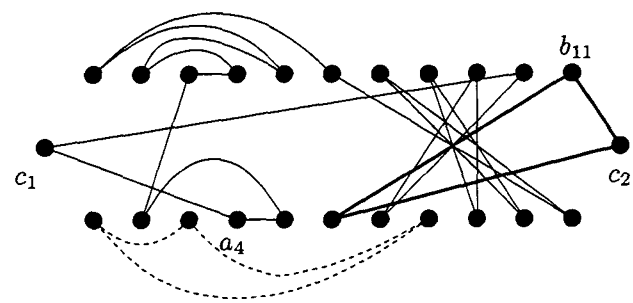

Figure 3.13: Base 2-factor for $OP(3, 3, 18)$

By attaching the vertex c_2 to this path, a 3-cycle is created. The other subpath has 16 edges and its endpoints are b_{10} and a_4. By attaching the vertex c_1 to these endpoints, an 18-cycle is created. Thus, we have constructed three cycles of the required lengths (see Figure 3.13). The action of α then creates a solution of $OP(3, 3, 18)$.

$OP(3, 5, 16)$ is obtained in exactly the same manner except that we break the path P_1 into 2 subpaths by removing the edge (a_7, b_9). $OP(3, 7, 14)$ is obtained with the removal of the edge (a_9, b_8) from P_1. Finally, $OP(3, 9, 12)$ is obtained by the removal of (a_{10}, b_7).

Note that for the construction to work, path P_1 has to be broken into two subpaths such that each of the subpaths has one vertex in A and the other vertex in B. Therefore, for example, we cannot construct $OP(3, 4, 17)$ in this manner.

We use P_1' to construct $OP(3, 4, 17)$, $OP(3, 6, 15)$, $OP(3, 8, 13)$ and $OP(3, 10, 11)$ in the same way in which we used P_1. In fact, we will be removing the same edges from P_1' that we removed from P_1.

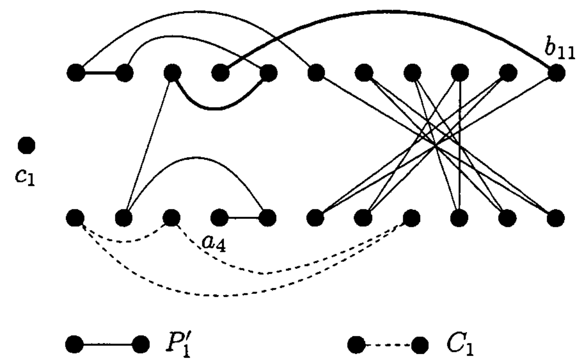

Figure 3.14: Construction of P_1' and C_1 for $n = 24$

For this to work, we construct P_1' from P_1 by making a couple of adjustments. First, we append an extra vertex to b_{11} so that b_{11} is not an endpoint of P_1'. Second, we will, in general, have to adjust the subpath of P_1 that is in B to make room for the vertex attached to b_{11}.

Figure 3.14 shows how P_1' can be constructed for $n = 24$. The edges that differ from the edges in P_1 are highlighted in bold.

We will now construct C_1, P_1 and P_1' for each $n > 16$:

$n = 44$:

$C_1 = (a_2, a_9, a_{10})$

$P_1 = (b_{21}, a_{11}, b_{20}, a_{12}, b_{19}, a_{13}, b_{18}, a_{14}, b_{17}, a_{15}, b_{16}, a_{17}, b_{15}, a_{18}, b_{14}, a_{19}, b_{13}, a_{20}, b_{12}, a_{21}, b_{11}, b_1, b_{10}, b_2, b_9, b_3, b_8, b_4, b_7, b_5, b_6, a_6, a_{16}, a_7, a_1, a_4, a_8, a_3, a_5)$

P_1' is obtained from P_1 by inserting the edge (b_{10}, b_{21}) as its first edge and by replacing the subpath $(b_{11}, \ldots, b_6)$ of P_1 by the path $(b_{11}, b_2, b_9, b_1, b_7, b_3, b_8, b_5, b_4, b_6)$.

$n = 40$:

$C_1 = (a_1, a_2, a_8)$

$P_1 = (b_{19}, a_{10}, b_{18}, a_{11}, b_{17}, a_{12}, b_{16}, a_{13}, b_{15}, a_{15}, b_{14}, a_{16}, b_{13}, a_{17}, b_{12}, a_{18}, b_{11}, a_{19}, b_{10}, b_1, b_9, b_2, b_8, b_3, b_7, b_4, b_6, b_5, a_4, a_9, a_7, a_3, a_6, a_{14}, a_5)$

P_1' is obtained from P_1 by inserting (b_8, b_{19}) as its first edge and by replacing $(b_{10}, \ldots, b_5)$ with $(b_{10}, b_1, b_6, b_2, b_9, b_3, b_4, b_7, b_5)$.

$n = 36$:

$C_1 = (a_1, a_2, a_7)$

$P_1 = (b_{17}, a_9, b_{16}, a_{10}, b_{15}, a_{11}, b_{14}, a_{12}, b_{13}, a_{14}, b_{12}, a_{15}, b_{11}, a_{16}, b_{10}, a_{17}, b_9, b_1, b_8, b_2, b_7, b_3, b_6, b_4, b_5, a_5, a_{13}, a_3, a_6, a_4, a_8)$

P_1' is obtained from P_1 by inserting (b_8, b_{17}) as its first edge and by replacing $(b_9, \ldots, b_5)$ with $(b_9, b_2, b_6, b_1, b_7, b_4, b_3, b_5)$.

$n = 32$:

$C_1 = (a_4, a_5, a_7)$

$P_1 = (b_{15}, a_8, b_{14}, a_9, b_{13}, a_{10}, b_{12}, a_{12}, b_{11}, a_{13}, b_{10}, a_{14}, b_9, a_{15}, b_8, b_1, b_7, b_2, b_6, b_3, b_5, b_4, a_3, a_{11}, a_2, a_6, a_1)$

P_1' is obtained from P_1 by inserting (b_6, b_{15}) as its first edge and by replacing $(b_8, \ldots, b_4)$ with $(b_8, b_1, b_3, b_7, b_2, b_5, b_4)$.

$n = 28$:

$C_1 = (a_3, a_5, a_{10})$

$P_1 = (b_{13}, a_7, b_{12}, a_8, b_{11}, a_9, b_{10}, a_{11}, b_9, a_{12}, b_8, a_{13}, b_7, b_1, b_6, b_2, b_5, b_3, b_4, a_4, a_1, a_2, a_6)$

P_1' is obtained from P_1 by inserting (b_6, b_{13}) as its first edge and by replacing $(b_7, \ldots, b_4)$ with $(b_7, b_2, b_3, b_5, b_1, b_4)$.

$n = 24$:

$C_1 = (a_1, a_3, a_8)$

$P_1 = (b_{11}, a_6, b_{10}, a_7, b_9, a_9, b_8, a_{10}, b_7, a_{11}, b_6, b_1, b_5, b_2, b_4, b_3, a_2, a_5, a_4)$

P_1' is obtained from P_1 by inserting (b_4, b_{11}) as its first edge and by replacing $(b_6, \ldots, b_3)$ with $(b_6, b_1, b_2, b_5, b_3)$.

$n = 20$:

$C_1 = (a_1, a_2, a_4)$

$P_1 = (b_9, a_5, b_8, a_6, b_7, a_8, b_6, a_9, b_5, b_1, b_4, b_2, b_3, a_3, a_7)$

P_1' is obtained from P_1 by inserting (b_4, b_9) as its first edge and by replacing $(b_5, \ldots, b_3)$ with (b_5, b_2, b_1, b_3).

$n = 16$:

$$C_1 = (a_2, a_3, a_5)$$
$$P_1 = (b_7, a_4, b_6, a_6, b_5, a_7, b_4, b_1, b_3, b_2, a_1)$$

For $n = 16$ we do not construct P_1'. Therefore, our construction only gives rise to $OP(3,3,10)$ and to $OP(3,5,8)$. However, the remaining two cases, $OP(3,4,9)$ and $OP(3,6,7)$ had previously been constructed in Lemma 3.1.1 and Theorem 3.2.8.

$n = 12$: Both $OP(3,3,6)$ and $OP(3,4,5)$ had previously been constructed (in Theorems 3.2.8 and 3.1.5).

Since there are no a's satisfying conditions of this lemma for $n < 12$, the proof is completed. ■

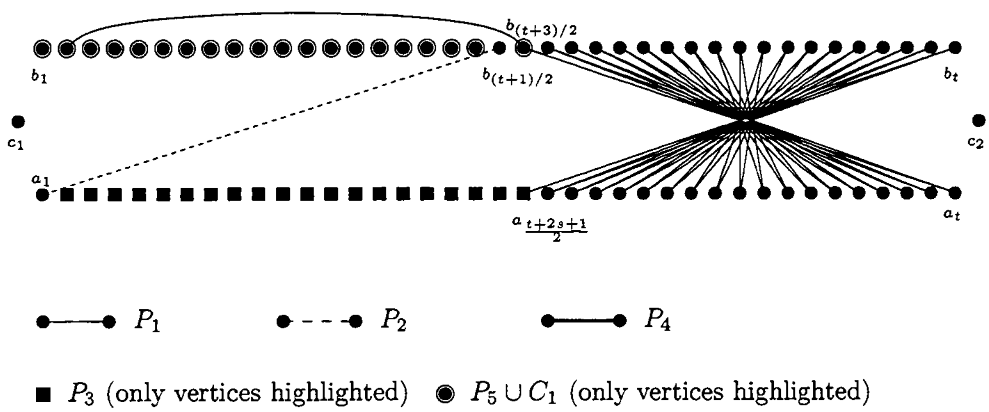

Figure 3.15: Example: $OP(3,3,74)$

Lemma 3.3.5 *Let $n \geq 44$, $n \equiv 0 \pmod 4$. Then $OP(3,3,n-6)$ has a solution.*

Proof. Again, $n = 2t+2$. Paths P_1 and P_2 are constructed as in Lemma 3.3.3 except that we use $s = 1$.

P_3: $V(P_3) = \{a_2, a_3, \ldots, a_{\frac{t+3}{2}}\}$. One of the endpoints of P_3 is $a_{\frac{t+3}{2}}$, and the edge-orbits intersected by $E(P_3)$ are the orbits $A_1, A_2, \ldots, A_{\frac{t-1}{2}}$. The existence of this path is guaranteed by Lemma 2.2.1.

P_4: P_4 is a path containing a single edge, namely, the edge $(b_2, b_{\frac{t+3}{2}})$.

P_5 and C_1: $V(P_5 \cup C_1) = \{b_1, b_2, \ldots, b_{\frac{t-1}{2}}\}$. C_1 is a 3-cycle and P_5 is a path, with one of the endpoints being b_2. The edge-orbits intersected by $E(P_5 \cup C_1)$ are the orbits

$B_1, B_2, \ldots B_{\frac{t-3}{2}}$. The existence of P_5 and C_1 is guaranteed by Lemma 3.2.4 (here we need $n \geq 44$). Figure 3.15 illustrates the construction.

C_2 will be the 3-cycle constructed by connecting the endpoints of P_2 to c_1.

Finally, C_3 will be the cycle constructed by connecting the endpoints of the path $P_1 \cup P_3 \cup P_4 \cup P_5$ to c_2.

The 2-factor $C_1 \cup C_2 \cup C_3$ uses at most one edge from each edge-orbit, and gives rise to an $OP(3, 3, n-6)$. ∎

Theorem 3.3.6 *Let* $n \equiv 0 \pmod 4$ *and let* a *be such that* $3 \leq a < n-3-a$. *Then* $OP(3, a, n-3-a)$ *has a solution.*

Proof. This result is an immediate consequence of Lemmas 3.3.3, 3.3.4 and 3.3.5.∎

Chapter 4

Two Cycle Lengths - Many Cycles

In this Chapter, we prove that $OP(a^n, b^n)$ has a solution for all odd a, b and n with $5 \leq a < b$ and $n \neq 7, 11$.

In order to prove this result, we decompose $K_{n(a+b)} - I$ into graphs isomorphic to $K_{a+b} - I$, $C_3 \wr \overline{K}_{a+b}$ and $C_5 \wr \overline{K}_{a+b}$. Furthermore, this decomposition will be such that the graphs of the form $C_3 \wr \overline{K}_{a+b}$ and $C_5 \wr \overline{K}_{a+b}$ can be grouped into a number of $2(a+b)$-factors of $K_{n(a+b)}$. It then will be seen that if $OP(K_{a+b} - I; a, b)$, $OP(C_3 \wr \overline{K}_{a+b}; a^3, b^3)$ and $OP(C_5 \wr \overline{K}_{a+b}; a^5, b^5)$ all exist, then $OP(a^n, b^n)$ also exists.

$OP(K_{a+b} - I; a, b) = OP(a, b)$ was shown to exist for all odd a and b in Theorem 2.3.9. We will prove the existence of $OP(C_3 \wr \overline{K}_{a+b}; a^3, b^3)$ in Section 4.1 and the existence of $OP(C_5 \wr \overline{K}_{a+b}; a^5, b^5)$ in Section 4.2.

The proof will be similar in both cases. We start with a number of paths (6 for $OP(C_3 \wr \overline{K}_{a+b}; a^3, b^3)$ and 10 for $OP(C_5 \wr \overline{K}_{a+b}; a^5, b^5)$) in a subgraph isomorphic to $K_2 \wr \overline{K}_{a+b} = K_{a+b,a+b}$. We impose certain invariant conditions on these paths and show that

(i) paths satisfying the invariant conditions exist, and

(ii) such paths can be inductively extended to longer paths satisfying the same invariant conditions until a solution of $OP(C_3 \wr \overline{K}_{a+b}; a^3, b^3)$ (or $OP(C_5 \wr \overline{K}_{a+b}; a^5, b^5)$) is reached.

4.1 Construction of $OP(C_3 \wr \overline{K}_n; a^3, b^3)$

Let $n > 0$ and let $K_{n,n}$ be the complete bipartite graph with each part having n vertices. Let the vertices in each part be denoted as $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$, respectively. Let permutation α be as in the previous chapters, i.e., $\alpha = (a_1, a_2, \ldots, a_n)(b_1, b_2, \ldots, b_n)$. Further, for a given m, let $s_1, s_2, \ldots, s_m$ be positive integers and $t_1, t_2, \ldots, t_m$ be non-negative integers such that $\sum_{i=1}^{m} s_i \cdot t_i = n$. We will then denote by $P(s_1^{t_1}, s_2^{t_2}, \ldots, s_m^{t_m})$ the set of graphs G satisfying the following properties:

1. G is a subgraph of $K_{n,n}$ and $V(G) = V(K_{n,n})$.

2. G is a vertex-disjoint union of $\sum_{i=1}^{m} t_i$ paths, with exactly t_i paths of length s_i for each $i, 1 \le i \le m$, and of a number of isolated vertices.

3. $E(G)$ intersects each of the edge-orbits of α in exactly one edge.

Figure 4.1 depicts a graph that belongs to $P(1^2, 3^1)$.

We will use graphs in $P(s_1^{t_1}, s_2^{t_2}, \ldots, s_m^{t_m})$ as a basis for an inductive construction of $OP(C_3 \wr \overline{K}_n; a^3, b^3)$. Lemmas 4.1.1–4.1.4 guarantee the existence of the graphs we will need for the construction.

Lemma 4.1.1 *$P(1^3, s^t, (s+4)^{(3-t)})$ is non-empty for all odd $s \ge 3$ and all $t, 0 \le t \le 3$, as well as for $s = 1$ and all t, $0 \le t \le 2$.*

Proof. We will construct a $G \in P(1^3, s^t, (s+4)^{(3-t)})$. Note that G has $30 + 6s - 8t$ vertices and is a factor of $K_{15+3s-4t,15+3s-4t}$. Let $u_1 = u_2 = \cdots = u_{3-t} = s + 4$ and let $u_{4-t} = \cdots = u_3 = s$. Further, let $n = 15 + 3s - 4t$. Note that $u_1 + u_2 + u_3 = n - 3$. We construct three paths P_1, P_2 and P_3 as follows:

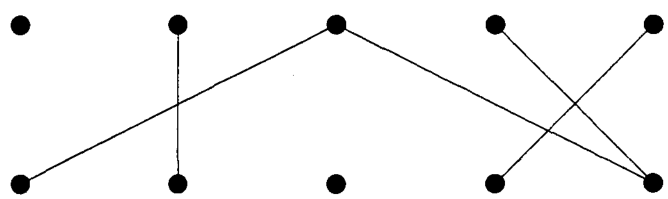

Figure 4.1: A graph in $P(1^2, 3^1)$

P_1 :		P_2 :		P_3 :	
$a_{\frac{n}{2}}$	AB_0	$a_{\frac{n}{2}-\frac{u_3-1}{2}-1}$	AB_{u_3+1}	$a_{\frac{n}{2}-\frac{u_2+u_3}{2}-1}$	$AB_{u_2+u_3+2}$
$b_{\frac{n}{2}}$	AB_1	$b_{\frac{n}{2}+\frac{u_3-1}{2}+1}$	AB_{u_3+2}	$b_{\frac{n}{2}+\frac{u_2+u_3}{2}+1}$	$AB_{u_2+u_3+3}$
$a_{\frac{n}{2}-1}$	AB_2	$a_{\frac{n}{2}-\frac{u_3-1}{2}-2}$	AB_{u_3+3}	$a_{\frac{n}{2}-\frac{u_2+u_3}{2}-2}$	$AB_{u_2+u_3+4}$
$b_{\frac{n}{2}+1}$	AB_3	$b_{\frac{n}{2}+\frac{u_3-1}{2}+2}$	AB_{u_3+4}	$b_{\frac{n}{2}+\frac{u_2+u_3}{2}+2}$	$AB_{u_2+u_3+5}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$a_{\frac{n}{2}-\frac{u_3-1}{2}}$	AB_{u_3-1}	$a_{\frac{n}{2}-\frac{u_2+u_3}{2}}$	$AB_{u_2+u_3}$	a_3	AB_{n-6}
$b_{\frac{n}{2}+\frac{u_3-1}{2}}$		$b_{\frac{n}{2}+\frac{u_2+u_3}{2}}$		b_{n-3}	AB_{n-5}
				a_2	AB_{n-4}
				b_{n-2}	AB_{n-3}
				a_1	AB_{n-1}
				b_n	

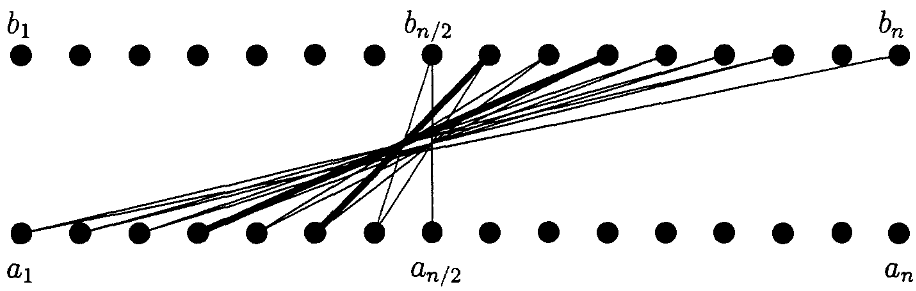

Figure 4.2: $n = 16$, $s = 3$, $t = 2$, P with deleted edges highlighted in bold

Figures 4.2 and 4.3 illustrates how paths P_1, P_2 and P_3 can be viewed as obtained from a path P by deleting two edges. Paths P_1, P_2 and P_3 are vertex-disjoint and they have lengths u_3, u_2 and u_1, respectively. They also intersect each orbit of α in exactly one edge except that they do not intersect orbits $AB_{u_3}, AB_{u_2+u_3+1}$ and $AB_{u_1+u_2+u_3+1} = AB_{n-2}$. To finish the proof of the lemma, we need to find three edges in G that belong to one of these orbits each. Further, these edges must be mutually vertex-disjoint and vertex-disjoint from paths P_1, P_2 and P_3. It can be seen that the edges

$(a_{\frac{n+2}{2}}, b_{\frac{n-2}{2}})$,

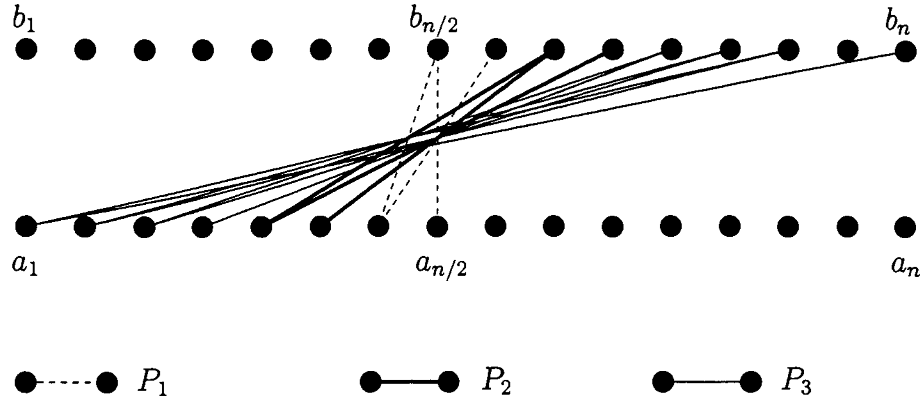

Figure 4.3: P_1, P_2 and P_3 after deletion of two edges from P

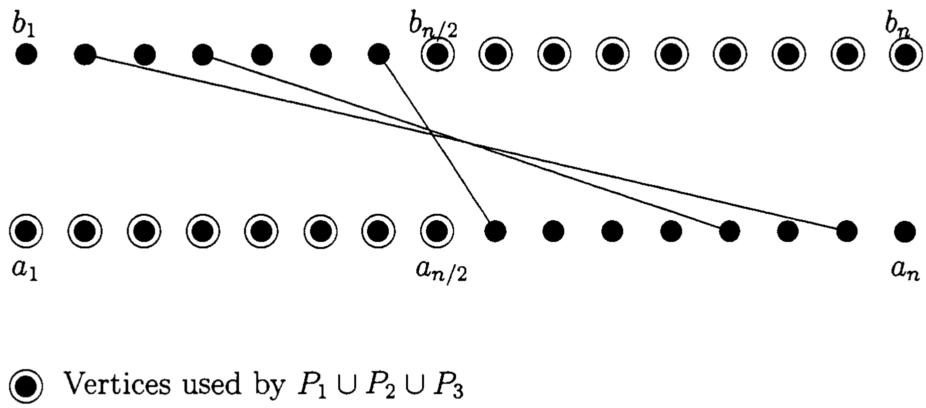

Figure 4.4: Insertion of three additional edges

$(a_{\frac{n+u_1+3}{2}}, b_{\frac{n-u_1-1}{2}})$, and
$(a_{\frac{n+u_1+u_2+4}{2}}, b_{\frac{n-u_1-u_2-2}{2}})$
will work. Note that here we need to exclude the case $s = 1, t = 3$. In that case, $u_1 = u_2 = u_3 = 1$ and $b_{\frac{n-2}{2}} = b_{\frac{n-u_1-1}{2}}$, so the three edges are not mutually vertex-disjoint. Figure 4.4 illustrates their construction. ■

Lemma 4.1.2 *$P(3^3, s^t, (s+4)^{(3-t)})$ is non-empty for all odd $s \geq 3$ and all $t, 0 \leq t \leq 3$.*

Proof. The construction is very similar to the previous lemma. We will construct a $G \in P(3^3, s^t, (s+4)^{3-t})$. Note that G is a factor of $K_{n,n}$ with $n = 21 + 3s - 4t$. Let $u_1 = u_2 = \cdots = u_{3-t} = s + 4$ and let $u_{4-t} = \cdots = u_3 = s$. This time, $u_1 + u_2 + u_3 = n - 9$. We will define paths P_1, P_2 and P_3 as follows:

P_1:	
$a_{\frac{n}{2}-1}$	AB_0
$b_{\frac{n}{2}-1}$	AB_1
$a_{\frac{n}{2}-2}$	AB_2
$b_{\frac{n}{2}}$	AB_3
$a_{\frac{n}{2}-3}$	AB_4
$b_{\frac{n}{2}+1}$	AB_5
$\vdots$	$\vdots$
$a_{\frac{n}{2}-\frac{u_3-1}{2}-1}$	AB_{u_3-1}
$b_{\frac{n}{2}+\frac{u_3-1}{2}-1}$	

P_2:	
$a_{\frac{n}{2}-\frac{u_3-1}{2}-3}$	AB_{u_3+3}
$b_{\frac{n}{2}+\frac{u_3-1}{2}+1}$	AB_{u_3+4}
$a_{\frac{n}{2}-\frac{u_3-1}{2}-4}$	AB_{u_3+5}
$b_{\frac{n}{2}+\frac{u_3-1}{2}+2}$	AB_{u_3+6}
$a_{\frac{n}{2}-\frac{u_3-1}{2}-5}$	AB_{u_3+7}
$b_{\frac{n}{2}+\frac{u_3-1}{2}+3}$	AB_{u_3+8}
$\vdots$	$\vdots$
$a_{\frac{n}{2}-\frac{u_2+u_3}{2}-2}$	$AB_{u_2+u_3+2}$
$b_{\frac{n}{2}+\frac{u_2+u_3}{2}}$	

P_3:	
$a_{\frac{n}{2}-\frac{u_2+u_3}{2}-4}$	$AB_{u_2+u_3+6}$
$b_{\frac{n}{2}+\frac{u_2+u_3}{2}+2}$	$AB_{u_2+u_3+7}$
$a_{\frac{n}{2}-\frac{u_2+u_3}{2}-5}$	$AB_{u_2+u_3+8}$
$b_{\frac{n}{2}+\frac{u_2+u_3}{2}+3}$	$AB_{u_2+u_3+9}$
$a_{\frac{n}{2}-\frac{u_2+u_3}{2}-6}$	$AB_{u_2+u_3+10}$
$\vdots$	$\vdots$
$\vdots$	$\vdots$
b_{n-5}	AB_{n-7}
a_2	AB_{n-6}
b_{n-4}	AB_{n-5}
a_1	AB_{n-1}
b_n	

Paths P_1, P_2 and P_3 have lengths u_3, u_2 and u_1, respectively. They are vertex-disjoint and intersect each orbit of α in exactly one edge except that they do not intersect the following 9 orbits: AB_{u_3}, AB_{u_3+1}, AB_{u_3+2}, $AB_{u_2+u_3+3}$, $AB_{u_2+u_3+4}$, $AB_{u_2+u_3+5}$, $AB_{u_1+u_2+u_3+5} = AB_{n-4}$, $AB_{u_1+u_2+u_3+6} = AB_{n-3}$, $AB_{u_1+u_2+u_3+7} = AB_{n-2}$.

We will conclude the proof by constructing 3 paths P_4, P_5 and P_6 each of length 3 which intersect these 9 orbits:

$P_4 = (a_{\frac{n}{2}}, b_{\frac{n-4}{2}}, a_{\frac{n+2}{2}}, b_{\frac{n-6}{2}})$,
$P_5 = (a_{\frac{n+u_1+3}{2}}, b_{\frac{n-u_1-5}{2}}, a_{\frac{n+u_1+5}{2}}, b_{\frac{n-u_1-7}{2}})$,
$P_6 = (a_{\frac{n+u_1+u_2+6}{2}}, b_{\frac{n-u_1-u_2-8}{2}}, a_{\frac{n+u_1+u_2+8}{2}}, b_{\frac{n-u_1-u_2-10}{2}})$. ■

Lemma 4.1.3 *$P(3^3, 1^t, 5^{3-t})$ is non-empty for all $t, 0 \le t \le 3$.*

Proof. For $t = 3$ the result follows from Lemma 4.1.1. For $t = 0$, it follows by Lemma 4.1.2. For $t = 1$ and $t = 2$, we will construct the required paths. Paths P_1 through P_4 will be as in Lemma 4.1.2 (with $s = 1$). We need to adjust paths P_5 and P_6 because $b_{\frac{n-6}{2}} = b_{\frac{n-u_1-5}{2}}$, so the construction in Lemma 4.1.2 does not work here. Paths P_5 and P_6 will be as follows:

$t = 2$. Now $P_5 = (a_{13}, b_4, a_{14}, b_3)$ and $P_6 = (a_{15}, b_2, a_{16}, b_1)$.
$t = 1$. Now $P_5 = (a_{15}, b_6, a_{16}, b_5)$ and $P_6 = (a_{19}, b_2, a_{20}, b_1)$.
It is straightforward to check that the paths as constructed here will work. ∎

Lemma 4.1.4 *$P(1^3, s^t, (s+4)^{3-t})$ and $P(3^3, s^t, (s+4)^{3-t})$ are non-empty for all odd $s \geq 1$ and all $t, 0 \leq t \leq 3$ except that $P(1^3, 1^3, 5^0) = P(1^6)$ may be empty.*

Proof., This lemma follows directly from Lemmas 4.1.1,4.1.2 and 4.1.3. ∎

The next lemma is used for the induction step in our inductive construction.

Lemma 4.1.5 *Let $V = \{1, 2, 3\} \times \{1, 2, \ldots, n\}$ be the vertex set of $C_3 \wr \overline{K}_n$ and let β be the permutation of V defined by*

$$\beta = ((1,1), (1,2), \ldots, (1,n))((3,1), (3,2), \ldots, (3,n)).$$

Let $n_1, n_2, \ldots, n_6 \geq 3$ be odd integers whose sum is $3n$. Further, let $P_1, P_2, \ldots, P_6$ be a collection of 6 vertex-disjoint paths or cycles with vertices in V, where c of $P_1, P_2, \ldots, P_6$ are cycles. Finally, let m_i be the cardinality of $\{(i,1), (i,2), \ldots, (i,n)\} \cap (V(P_1) \cup V(P_2) \cup \ldots \cup V(P_6))$ for $1 \leq i \leq 3$. Assume that the following conditions are met:

1. *$|E(P_i)| \leq n_i$ for all i,*
2. *P_i is a cycle if and only if $|E(P_i)| = n_i$,*
3. *the union of all $E(P_i)$ intersects each edge-orbit of β in at most one edge. In addition, it intersects each of the edge-orbits of β connecting vertices of $\{(1,1), (1,2), \ldots, (1,n)\}$ with vertices of $\{(3,1), (3,2), \ldots, (3,n)\}$ in exactly one edge,*
4. *$|E(P_i)| \equiv n_i \pmod 4$ whenever P_i is a cycle and $|E(P_i)| \equiv n_i + 2 \pmod 4$ otherwise,*
5. *each P_i that is not a cycle has one of its endpoints in $\{(1,1), (1,2), (1,3), \ldots, (1,n)\}$ and the other in $\{(3,1), (3,2), (3,3), \ldots, (3,n)\}$, and*

6. $m_2 = m_1 + m_3 - n - (6 - c)$ *and* $m_1 = m_3$.

Then paths $P_1, P_2, \ldots, P_6$ *can be extended to a set of* 6 *vertex-disjoint cycles* C_1, $C_2, \ldots, C_6$ *such that*

7. $|E(C_i)| = n_i$ *for each* i, *and*

8. *the union of all* C_i *intersects each edge-orbit of* β *in exactly one edge.*

Proof. The proof of this lemma is analogous to the proof of Lemma 4.2.5. The proof of Lemma 4.2.5 will be given in Section 4.2. ■

Lemma 4.1.6 shows why we were interested in Lemma 4.1.5.

Lemma 4.1.6 *Suppose the assumptions of Lemma 4.1.5 are satisfied. Then* $OP(C_3 \wr \overline{K}_n, n_1, n_2, n_3, \ldots, n_6)$ *has a solution.*

Proof. The lemma follows as $C_1, C_2, \ldots, C_6$ provided by Lemma 4.1.5 form a 2-factor of $C_3 \wr \overline{K}_n$ whose cycle lengths are $n_1, n_2, \ldots, n_6$ and the action of β on this 2-factor yields the desired 2-factorization. ■

Lemma 4.1.7 *Let* $3 \le a < b$ *be odd integers, let* $n = a+b$ *and let* $G = C_3 \wr \overline{K}_n$. *Then* $OP(G; a^3, b^3)$ *has a solution except possibly when* $(a, b) = (3, 5)$ *or* $(a, b) = (5, 9)$.

Proof. The proof is analogous to that of Lemma 4.2.7 which will be provided in Section 4.2.■

Lemma 4.1.8 $OP(C_3 \wr \overline{K}_8; 3^3, 5^3)$ *has a solution.*

Proof. Let

$$\begin{aligned} F = \{ & ((3,8),(1,1),(2,6)),((3,3),(1,5),(2,2)),((3,5),(1,3),(2,3)), \\ & ((3,2),(1,6),(3,1),(1,8),(2,4)),((3,7),(1,2),(2,5),(3,6), \\ & (2,8)),((3,4),(1,4),(2,7),(1,7),(2,1))\} \end{aligned}$$

be a 2-factor of $C_3 \wr \overline{K}_n$ consisting of 3 3-cycles and 3 5-cycles. Note that 2-factors $F, F\beta, F\beta^2, \ldots, F\beta^7$ contain each of the edges of $C_3 \wr \overline{K}_n$ exactly once except that the edges of the form $((1,i),(2,7))$ and $((3,i),(2,8))$ are contained twice each, and the edges of the form $((3,i),(2,7))$ and $((1,i),(2,8))$ are not contained at all.

Let F' be obtained from F by "swapping" vertices $(2,7)$ and $(2,8)$. Explicitly,

$$\begin{aligned} F' = \{&((3,8),(1,1),(2,6)),((3,3),(1,5),(2,2)),((3,5),(1,3),(2,3)),\\ &((3,2),(1,6),(3,1),(1,8),(2,4)),((3,7),(1,2),(2,5),(3,6),\\ &(2,7)),((3,4),(1,4),(2,8),(1,7),(2,1))\}. \end{aligned}$$

Note that then $F', F'\beta, F'\beta^2, \ldots, F'\beta^7$ contain each of the edges of $C_3 \wr \overline{K}_n$ exactly once except that the edges of the form $((3,i),(2,7))$ and $((1,i),(2,8))$ are contained twice each, and the edges of the form $((1,i),(2,7))$ and $((3,i),(2,8))$ are not contained at all.

Using these observations and some more simple checking, one can easily see that graphs $F, F\beta^2, F\beta^4, F\beta^6, F'\beta, F'\beta^3, F'\beta^5$ and $F'\beta^7$ form the desired 2-factorization. ■

Lemma 4.1.9 *$OP(C_3 \wr \overline{K}_{14}; 5^3, 9^3)$ has a solution.*

Proof. Let

$$\begin{aligned} F = \{&((1,7),(3,7),(1,8),(3,6),(2,1)),((3,5),(1,9),(3,4),(1,10),\\ &(2,2)),((3,3),(1,11),(3,2),(1,1),(2,3)),((3,8),(1,6),(3,9),\\ &(1,5),(2,5),(3,1),(2,6),(1,2),(2,7)),((3,10),(1,3),(2,4),\\ &(1,4),(2,8),(3,13),(2,9),(1,12),(2,10)),((3,11),(1,14),(2,11),\\ &(3,14),(2,12),(1,13),(2,14),(3,12),(2,13))\}. \end{aligned}$$

Similar to Lemma 4.1.8, we can see that F is a 2-factor of $C_3 \wr \overline{K}_{14}$ consisting of three 5-cycles and three 9-cycles. Its edge set intersects each orbit of β in exactly one edge except that the edges of the form $((1,i),(2,4))$ and $((3,i),(2,13))$ are contained twice each, and the edges of the form $((3,i),(2,4))$ and $((1,i),(2,13))$ are not contained at all.

Similar to Lemma 4.1.8, we can also see that if F' is obtained from F by "swapping" vertices $(2,4)$ and $(2,13)$, then F, $F\beta^2$, $F\beta^4, \ldots, F\beta^{12}$, $F'\beta$, $F'\beta^3$, $F'\beta^5, \ldots, F'\beta^{13}$ form a solution to $OP(C_3 \wr \overline{K}_{14}; 5^3, 9^3)$. ∎

Thus, we arrive at the following lemma:

Lemma 4.1.10 *Let $3 \leq a < b$ be odd integers, let $n = a + b$ and let $G = C_3 \wr \overline{K}_n$. Then $OP(G; a^3, b^3)$ has a solution.*∎

4.2 Construction of $OP(C_5 \wr \overline{K}_n; a^5, b^5)$

We proceed using an approach similar to that of Section 4.1. We will prove that $P(1^5, s_1^{t_1}, (s_1+4)^{5-t_1})$ and $P(3^5, s_1^{t_1}, (s_1+4)^{5-t_1})$ are non-empty for all $s_1 \geq 1$ and all $t_1, 0 \leq t_1 \leq 5$ except for $P(1^5, 1^5)$ (which can be seen to be empty).

We will prove this through a series of lemmas.

Lemma 4.2.1 *$P(1^5, s^t, (s+4)^{5-t})$ is non-empty for all odd $s \geq 3$ and all $t, 0 \leq t \leq 5$, as well as for $s = 1$ and all $t, 0 \leq t \leq 4$.*

Proof. We will construct a $G \in P(1^5, s^t, (s+4)^{5-t})$. Note that G has $50 + 10s - 8t$ vertices and is a factor of $K_{25+5s-4t,25+5s-4t}$. Let $u_1 = u_2 = \cdots = u_{5-t} = s+4$ and let $u_{6-t} = \cdots = u_5 = s$. Further, let $n = 25 + 5s - 4t$. Note that $u_1 + u_2 + u_3 + u_4 + u_5 = n - 5$. We construct five paths $P_1, P_2, \ldots, P_5$ as follows:

P_1:		P_2:		P_3:	
$a_{\frac{n}{2}}$	AB_0	$a_{\frac{n}{2}-\frac{u_5-1}{2}-1}$	AB_{u_5+1}	$a_{\frac{n}{2}-\frac{u_4+u_5}{2}-1}$	$AB_{u_4+u_5+2}$
$b_{\frac{n}{2}}$	AB_1	$b_{\frac{n}{2}+\frac{u_5-1}{2}+1}$	AB_{u_5+2}	$b_{\frac{n}{2}+\frac{u_4+u_5}{2}+1}$	$AB_{u_4+u_5+3}$
$a_{\frac{n}{2}-1}$	AB_2	$a_{\frac{n}{2}-\frac{u_5-1}{2}-2}$	AB_{u_5+3}	$a_{\frac{n}{2}-\frac{u_4+u_5}{2}-2}$	$AB_{u_4+u_5+4}$
$b_{\frac{n}{2}+1}$	AB_3	$b_{\frac{n}{2}+\frac{u_5-1}{2}+2}$	AB_{u_5+4}	$b_{\frac{n}{2}+\frac{u_4+u_5}{2}+2}$	$AB_{u_4+u_5+5}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$a_{\frac{n}{2}-\frac{u_5-1}{2}}$	AB_{u_5-1}	$a_{\frac{n}{2}-\frac{u_4+u_5}{2}}$	$AB_{u_4+u_5}$	$a_{\frac{n}{2}-\frac{u_3+u_4+u_5+1}{2}}$	$AB_{u_3+u_4+u_5+1}$
$b_{\frac{n}{2}+\frac{u_5-1}{2}}$		$b_{\frac{n}{2}+\frac{u_4+u_5}{2}}$		$b_{\frac{n}{2}+\frac{u_3+u_4+u_5+1}{2}}$	

P_4:	
$a_{\frac{n}{2}-\frac{u_3+u_4+u_5+1}{2}-1}$	$AB_{u_3+u_4+u_5+3}$
$b_{\frac{n}{2}+\frac{u_3+u_4+u_5+1}{2}+1}$	$AB_{u_3+u_4+u_5+4}$
$a_{\frac{n}{2}-\frac{u_3+u_4+u_5+1}{2}-2}$	$AB_{u_3+u_4+u_5+5}$
$b_{\frac{n}{2}+\frac{u_3+u_4+u_5+1}{2}+2}$	$AB_{u_3+u_4+u_5+6}$
$\vdots$	$\vdots$
$a_{\frac{n}{2}-\frac{u_2+u_3+u_4+u_5+2}{2}}$	$AB_{u_2+u_3+u_4+u_5+2}$
$b_{\frac{n}{2}+\frac{u_2+u_3+u_4+u_5+2}{2}}$	

P_5:	
$a_{\frac{n}{2}-\frac{u_2+u_3+u_4+u_5+2}{2}-1}$	$AB_{u_2+u_3+u_4+u_5+4}$
$b_{\frac{n}{2}+\frac{u_2+u_3+u_4+u_5+2}{2}+1}$	$AB_{u_2+u_3+u_4+u_5+5}$
$a_{\frac{n}{2}-\frac{u_2+u_3+u_4+u_5+2}{2}-2}$	$AB_{u_2+u_3+u_4+u_5+6}$
$b_{\frac{n}{2}+\frac{u_2+u_3+u_4+u_5+2}{2}+2}$	$AB_{u_2+u_3+u_4+u_5+7}$
$\vdots$	$\vdots$
a_3	AB_{n-6}
b_{n-3}	AB_{n-5}
a_2	AB_{n-4}
b_{n-2}	AB_{n-3}
a_1	AB_{n-1}
b_n	

Paths $P_1, P_2, \ldots, P_5$ are vertex-disjoint and have lengths u_5, u_4, u_3, u_2 and u_1, respectively. Also, they intersect each orbit of α in exactly one edge. The only exceptions are the following orbits, which are not intersected at all: $AB_{u_5}, AB_{u_4+u_5+1}$, $AB_{u_3+u_4+u_5+2}, AB_{u_2+u_3+u_4+u_5+3}$ and $AB_{u_1+u_2+u_3+u_4+u_5+3} = AB_{n-2}$. To finish the proof of the lemma, we need to find five edges in G which belong to one of these orbits each. Further, these edges must be mutually vertex-disjoint and vertex-disjoint from paths $P_1, P_2, \ldots, P_5$. It can be seen that the edges

$(a_{\frac{n+2}{2}}, b_{\frac{n-2}{2}})$,
$(a_{\frac{n+u_1+3}{2}}, b_{\frac{n-u_1-1}{2}})$,
$(a_{\frac{n+u_1+u_2+4}{2}}, b_{\frac{n-u_1-u_2-2}{2}})$,
$(a_{\frac{n+u_1+u_2+u_3+5}{2}}, b_{\frac{n-u_1-u_2-u_3-3}{2}})$,
$(a_{\frac{n+u_1+u_2+u_3+u_4+6}{2}}, b_{\frac{n-u_1-u_2-u_3-u_4-4}{2}})$

will work (here we require $b_{\frac{n-2}{2}} \neq b_{\frac{n-u_1-1}{2}}$, i.e., $u_1 \neq 1$, which excludes the case $s = 1$ and $t = 5$ from the proof). ■

Lemma 4.2.2 *$P(3^5, s^t, (s+4)^{5-t})$ is non-empty for all odd $s \geq 3$ and all $t, 0 \leq t \leq 5$.*

Proof. We will construct a $G \in P(3^5, s^t, (s+4)^{5-t})$. Note that G is a factor of $K_{35+5s-4t,35+5s-4t}$. Let $u_1 = u_2 = \cdots = u_{5-t} = s+4$ and let $u_{6-t} = \cdots = u_5 = s$. Further, let $n = 35 + 5s - 4t$. This time $u_1 + u_2 + u_3 + u_4 + u_5 = n - 15$. Paths $P_1, P_2, \ldots, P_5$ will now be as follows:

P_1 :		P_2 :		P_3 :	
$a_{\frac{n}{2}-1}$	AB_0	$a_{\frac{n}{2}-\frac{u_5-1}{2}-3}$	AB_{u_5+3}	$a_{\frac{n}{2}-\frac{u_4+u_5}{2}-4}$	$AB_{u_4+u_5+6}$
$b_{\frac{n}{2}-1}$	AB_1	$b_{\frac{n}{2}+\frac{u_5-1}{2}+1}$	AB_{u_5+4}	$b_{\frac{n}{2}+\frac{u_4+u_5}{2}+2}$	$AB_{u_4+u_5+7}$
$a_{\frac{n}{2}-2}$	AB_2	$a_{\frac{n}{2}-\frac{u_5-1}{2}-4}$	AB_{u_5+5}	$a_{\frac{n}{2}-\frac{u_4+u_5}{2}-5}$	$AB_{u_4+u_5+8}$
$b_{\frac{n}{2}}$	AB_3	$b_{\frac{n}{2}+\frac{u_5-1}{2}+2}$	AB_{u_5+6}	$b_{\frac{n}{2}+\frac{u_4+u_5}{2}+3}$	$AB_{u_4+u_5+9}$
$a_{\frac{n}{2}-3}$	AB_4	$a_{\frac{n}{2}-\frac{u_5-1}{2}-5}$	AB_{u_5+7}	$a_{\frac{n}{2}-\frac{u_4+u_5}{2}-6}$	$AB_{u_4+u_5+10}$
$b_{\frac{n}{2}+1}$	AB_5	$b_{\frac{n}{2}+\frac{u_5-1}{2}+3}$	AB_{u_5+8}	$b_{\frac{n}{2}+\frac{u_4+u_5}{2}+4}$	$AB_{u_4+u_5+11}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$a_{\frac{n}{2}-\frac{u_5-1}{2}-1}$	AB_{u_5-1}	$a_{\frac{n}{2}-\frac{u_4+u_5}{2}-2}$	$AB_{u_4+u_5+2}$	$a_{\frac{n}{2}-\frac{u_3+u_4+u_5-1}{2}-4}$	$AB_{u_3+u_4+u_5+5}$
$b_{\frac{n}{2}+\frac{u_5-1}{2}-1}$		$b_{\frac{n}{2}+\frac{u_4+u_5}{2}}$		$b_{\frac{n}{2}+\frac{u_3+u_4+u_5-1}{2}+2}$	

P_4 :		P_5 :	
$a_{\frac{n}{2}-\frac{u_3+u_4+u_5-1}{2}-6}$	$AB_{u_3+u_4+u_5+9}$	$a_{\frac{n}{2}-\frac{u_2+u_3+u_4+u_5}{2}-7}$	$AB_{u_2+u_3+u_4+u_5+12}$
$b_{\frac{n}{2}+\frac{u_3+u_4+u_5-1}{2}+4}$	$AB_{u_3+u_4+u_5+10}$	$b_{\frac{n}{2}+\frac{u_2+u_3+u_4+u_5}{2}+5}$	$AB_{u_2+u_3+u_4+u_5+13}$
$a_{\frac{n}{2}-\frac{u_3+u_4+u_5-1}{2}-7}$	$AB_{u_3+u_4+u_5+11}$	$a_{\frac{n}{2}-\frac{u_2+u_3+u_4+u_5}{2}-8}$	$AB_{u_2+u_3+u_4+u_5+14}$
$b_{\frac{n}{2}+\frac{u_3+u_4+u_5-1}{2}+5}$	$AB_{u_3+u_4+u_5+12}$	$b_{\frac{n}{2}+\frac{u_2+u_3+u_4+u_5}{2}+6}$	$AB_{u_2+u_3+u_4+u_5+15}$
$a_{\frac{n}{2}-\frac{u_3+u_4+u_5-1}{2}-8}$	$AB_{u_3+u_4+u_5+13}$	$a_{\frac{n}{2}-\frac{u_2+u_3+u_4+u_5}{2}-9}$	$AB_{u_2+u_3+u_4+u_5+16}$
$b_{\frac{n}{2}+\frac{u_3+u_4+u_5-1}{2}+6}$	$AB_{u_3+u_4+u_5+14}$	$b_{\frac{n}{2}+\frac{u_2+u_3+u_4+u_5}{2}+7}$	$AB_{u_2+u_3+u_4+u_5+17}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$a_{\frac{n}{2}-\frac{u_2+u_3+u_4+u_5}{2}-5}$	$AB_{u_2+u_3+u_4+u_5+8}$	a_3	AB_{n-8}
$b_{\frac{n}{2}+\frac{u_2+u_3+u_4+u_5}{2}+3}$		b_{n-5}	AB_{n-7}
		a_2	AB_{n-6}
		b_{n-4}	AB_{n-5}
		a_1	AB_{n-1}
		b_n	

Similar to the previous lemma, paths P_1, P_2, P_3, P_4 and P_5 are vertex-disjoint, have lengths u_5, u_4, u_3, u_2 and u_1, and intersect each orbit of α in exactly one edge except that they do not intersect the following 15 orbits:

$$AB_{u_5}, AB_{u_5+1}, AB_{u_5+2}, AB_{u_4+u_5+3}, AB_{u_4+u_5+4}, AB_{u_4+u_5+5},$$

$$AB_{u_3+u_4+u_5+6}, AB_{u_3+u_4+u_5+7}, AB_{u_3+u_4+u_5+8},$$

$$AB_{u_2+u_3+u_4+u_5+9}, AB_{u_2+u_3+u_4+u_5+10}, AB_{u_2+u_3+u_4+u_5+11},$$

$$AB_{u_1+u_2+u_3+u_4+u_5+11} = AB_{n-4},$$

$$AB_{u_1+u_2+u_3+u_4+u_5+12} = AB_{n-3},$$

$$AB_{u_1+u_2+u_3+u_4+u_5+13} = AB_{n-2}.$$

We will complete the proof by constructing five paths $P_6, P_7, \ldots, P_{10}$, each of length 3 which intersect these 15 orbits:

$P_6 = (a_{\frac{n}{2}}, b_{\frac{n-4}{2}}, a_{\frac{n+2}{2}}, b_{\frac{n-6}{2}})$,
$P_7 = (a_{\frac{n+u_1+3}{2}}, b_{\frac{n-u_1-5}{2}}, a_{\frac{n+u_1+5}{2}}, b_{\frac{n-u_1-7}{2}})$,
$P_8 = (a_{\frac{n+u_1+u_2+6}{2}}, b_{\frac{n-u_1-u_2-8}{2}}, a_{\frac{n+u_1+u_2+8}{2}}, b_{\frac{n-u_1-u_2-10}{2}})$,
$P_9 = (a_{\frac{n+u_1+u_2+u_3+9}{2}}, b_{\frac{n-u_1-u_2-u_3-11}{2}}, a_{\frac{n+u_1+u_2+u_3+11}{2}}, b_{\frac{n-u_1-u_2-u_3-13}{2}})$, and
$P_{10} = (a_{\frac{n+u_1+u_2+u_3+u_4+12}{2}}, b_{\frac{n-u_1-u_2-u_3-u_4-14}{2}}, a_{\frac{n+u_1+u_2+u_3+u_4+14}{2}}, b_{\frac{n-u_1-u_2-u_3-u_4-16}{2}})$. ■

Lemma 4.2.3 *$P(3^5, 1^t, 5^{5-t})$ is non-empty for all $t, 0 \le t \le 5$.*

Proof. For $t = 5$ the result follows from Lemma 4.2.1. For $t = 0$ it follows from Lemma 4.2.2. For the remaining t, we will construct the required paths. In all cases, paths P_1 through P_6 are as in Lemma 4.2.2. The remaining paths are "shifted" by 1. Explicitly, they are as follows:

Case $t = 1$. Now $n = 36$ and

$$P_7 = (a_{23}, b_{14}, a_{24}, b_{13}), \quad P_8 = (a_{27}, b_{10}, a_{28}, b_9),$$
$$P_9 = (a_{31}, b_6, a_{32}, b_5), \quad P_{10} = (a_{35}, b_2, a_{36}, b_1).$$

Case $t = 2$. Now $n = 32$ and

$$P_7 = (a_{21}, b_{12}, a_{22}, b_{11}), \quad P_8 = (a_{25}, b_8, a_{26}, b_7),$$
$$P_9 = (a_{29}, b_4, a_{30}, b_3), \quad P_{10} = (a_{31}, b_2, a_{32}, b_1).$$

Case $t = 3$. Now $n = 28$ and

$$P_7 = (a_{19}, b_{10}, a_{20}, b_9), \quad P_8 = (a_{23}, b_6, a_{24}, b_5),$$
$$P_9 = (a_{25}, b_4, a_{26}, b_3), \quad P_{10} = (a_{27}, b_2, a_{28}, b_1).$$

Case $t = 4$. Now $n = 24$ and

$$P_7 = (a_{17}, b_8, a_{18}, b_7), \quad P_8 = (a_{19}, b_6, a_{20}, b_5),$$
$$P_9 = (a_{21}, b_4, a_{22}, b_3), \quad P_{10} = (a_{23}, b_2, a_{24}, b_1).$$

It is somewhat tedious but otherwise easy to check that these constructions work.■

Lemma 4.2.4 *$P(1^5, s^t, (s+4)^{5-t})$ and $P(3^5, s^t, (s+4)^{5-t})$ are non-empty for all odd $s \geq 1$ and all $t, 0 \leq t \leq 5$, except that $P(1^5, 1^5, 5^0) = P(1^{10})$ may be empty.*

Proof. This lemma is an immediate consequence of Lemmas 4.2.1, 4.2.2 and 4.2.3.■

The following Lemma 4.2.5, together with its counterpart, Lemma 4.1.5, provide crucial building blocks for $OP(a^n, b^n)$ with odd a and b. The idea is to construct a 2-factor of $C_5 \wr \overline{K}_n$ that, under the action of a permutation β, gives rise to a 2-factorization of $C_5 \wr \overline{K}_n$. To do so, we start with paths whose existence is guaranteed by Lemma 4.2.4 and expand them, four vertices at a time, into the required 2-factor.

The conditions of Lemma 4.2.5 appear somewhat convoluted, but after a short observation, they are quite natural.

Conditions 1, 2 and the first part of Condition 3 are obvious. Condition 4 is needed to ensure that we can complete the construction by adding four edges at a time. The second part of Condition 3 and Conditions 5 and 6 are invariants that are a by-product of the particular construction that we use to construct the 2-factors. We need them to ensure that the construction does not "stray" from where we want it to go.

Lemma 4.2.5 *Let* $V = \{1,2,3,4,5\} \times \{1,2,\ldots,n\}$ *be the vertex set of* $C_5 \wr \overline{K}_n$ *and let* β *be the permutation of* V *defined by*

$$\beta = ((1,1),(1,2),(1,3),\ldots,(1,n))((3,1),(3,2),(3,3),\ldots,$$

$$(3,n))((5,1),(5,2),(5,3),\ldots,(5,n)).$$

Let $n_1, n_2, n_3, \ldots, n_{10} \geq 5$ *be odd integers whose sum is* $5n$. *Further, let* $P_1, P_2, P_3, \ldots, P_{10}$ *be a collection of* 10 *disjoint paths or cycles with vertices in* V, *where* c *of* $P_1, P_2, \ldots, P_{10}$ *are cycles. Finally, let* m_i *be the cardinality of* $\{(i,1),(i,2),\ldots,(i,n)\} \cap (V(P_1) \cup V(P_2) \cup \ldots \cup V(P_{10}))$ *for* $1 \leq i \leq 5$. *Assume that the following conditions are met:*

1. $|E(P_i)| \leq n_i$ *for all* i,
2. P_i *is a cycle if and only if* $|E(P_i)| = n_i$,
3. *the union of all* $E(P_i)$ *intersects each edge-orbit of* β *in at most one edge. In addition, it intersects each of the edge-orbits of* β *connecting vertices of* $\{(1,1),(1,2),\ldots,(1,n)\}$ *with vertices of* $\{(5,1),(5,2),\ldots,(5,n)\}$ *in exactly one edge,*
4. $|E(P_i)| \equiv n_i \pmod 4$ *for all* i,
5. *each* P_i *that is not a cycle has one of its endpoints in* $\{(1,1),(1,2),(1,3),\ldots,(1,n)\}$ *and the other in* $\{(5,1),(5,2),(5,3),\ldots,(5,n)\}$, *and*

6. $m_2 + m_4 = 2m_3$, $|m_2 - m_4| \leq 2$, $m_2 = 2(m_1 - n/2 - 5) + c$ *and* $m_4 = 2(m_5 - n/2 - 5) + c$.

Then paths $P_1, P_2, \ldots, P_{10}$ *can be extended to a set of 10 disjoint cycles* $C_1, C_2, C_3, \ldots, C_{10}$ *such that*

7. $|E(C_i)| = n_i$ *for each* i, *and*

8. *the union of all* C_i *intersects each edge-orbit of* β *in exactly one edge.*

Proof. We will prove the lemma by contradiction. Suppose there is a counterexample to the Lemma for a given n. Select a counterexample in which the total number of edges in $P_1, P_2, \ldots, P_{10}$ is maximal. We will note that there is an i such that $|E(P_i)| < n_i$. Otherwise, Condition 1 would imply $|E(P_i)| = n_i$ for all i (i.e., Condition 7 is satisfied). Then by Condition 2, each P_i would be a cycle. Also, the total number of edges in all P_i would be $n_1 + n_2 + \cdots + n_{10} = 5n$. Since there are $5n$ orbits of β each containing exactly n edges, Condition 3 would imply Condition 8. However, then $P_1, P_2, \ldots, P_{10}$ would be 10 disjoint cycles satisfying Conditions 7 and 8.

Therefore, we may assume that $|E(P_1)| < n_1$. We may further assume that for all $i \neq 1$, either $|E(P_i)| = n_i$ or $n_i - |E(P_i)| \geq n_1 - |E(P_1)|$. From Condition 4 it follows that $|E(P_1)| \leq n_1 - 4$. We now proceed in two cases.

Case 1. $|E(P_1)| = n_1 - 4$. We will first show that m_2, m_3 and m_4 are all less than n. If $m_2 = n$, then Condition 6 implies that $m_1 = n + (10 - c)/2$. However, since $|E(P_1)| < n_1$, Condition 2 implies that $c \leq 9$. Therefore $m_1 > n$, which is impossible. Similarly, $m_4 < n$. However, if m_2 and m_4 are both less than n, then so is m_3 by Condition 6. Therefore, there are vertices $(2, v_2), (3, v_3)$ and $(4, v_4)$ not yet used by any of the paths. We connect the two endpoints of P_1, $(1, v_1)$ and $(5, v_5)$ with the path consisting of these three vertices to create a cycle P_1'. Towards contradiction, we only need to prove that $P_1', P_2, P_3, P_4, \ldots, P_{10}$ satisfy Conditions 1 through 6. All conditions except for Condition 3 are trivially satisfied. To see that Condition 3 is also satisfied one only needs to observe that the orbits of β (other than those that involve edges of the form $((1,i),(5,j))$) are of the form $\{((i,j),(k,l)) : 1 \leq l \leq n\}$ for $i \in \{2,4\}$, $1 \leq j \leq n$ and $k = i \pm 1$.

Case 2. $|E(P_1)| < n_1 - 4$. Therefore $|E(P_1)| \leq n_1 - 8$. Without loss of generality we may assume that $m_2 \leq m_4$. Then also $m_2 \leq m_3$ by Condition 6. Also by Condition 6, $m_1 = (n + 10 - c + m_2)/2$ and $m_5 = (n + 10 - c + m_4)/2$. Since $m_4 \geq m_2$, we have $m_5 \geq m_1$. Therefore,

$$m_1 + m_2 + m_3 + m_4 + m_5 \geq \frac{n + 10 - c + m_2}{2} + m_2 + m_2 + m_2 + \frac{n + 10 - c + m_2}{2}$$

$$= 4m_2 + n + 10 - c.$$

On the other hand,

$$m_1 + m_2 + m_3 + m_4 + m_5 = |V(P_1)| + |V(P_2)| + \cdots + |V(P_{10})| =$$

$$|E(P_1)| + |E(P_2)| + \cdots + |E(P_{10})| + 10 - c \leq$$

$$n_1 + n_2 \cdots + n_{10} - 8(10 - c) + 10 - c = 5n - 70 + 7c.$$

Combining the above two inequalities gives

$$4m_2 + n + 10 - c \leq 5n - 70 + 7c,$$

or, equivalently, $m_2 \leq n - 20 + 2c$. From Condition 6 we then also get $m_1 \leq n - (10 - c)/2$. Since $c \leq 9$, we get $m_2 \leq n - 2$ and $m_1 < n$, i.e., $m_1 \leq n - 1$. Furthermore, by Condition 6, $2m_3 = m_2 + m_4 \leq (n - 2) + n = 2n - 2$ and, thus, $m_3 \leq n - 1$. Therefore, one can find four distinct vertices $(1, v_1), (2, v_2), (2, v_2')$ and $(3, v_3)$ which are not used by paths P_1 through P_{10}. We can now extend path P_1 to a path P_1' by adding these vertices in the following order: $(2, v_2), (3, v_3), (2, v_2'), (1, v_1)$. We add them to the endpoint of P_1 whose first coordinate is 1. As in Case 1, it can be seen that Conditions 1-6 remain satisfied.

We conclude that in both cases we can extend the set of paths to another set of paths satisfying conditions of the lemma. However, since we originally selected the set of paths to have the maximum number of edges from among the sets that do not extend to cycles C_1 through C_{10} satisfying Conditions 7 and 8, we know that this new set of paths extends to such cycles. However, it means that paths $P_1, P_2, \ldots, P_{10}$ extend also, which gives a contradiction. ■

Lemma 4.2.6 *Suppose the assumptions of Lemma 4.2.5 are satisfied. Then* $OP(C_5 \wr \overline{K}_n, n_1, n_2, n_3, \ldots, n_{10})$ *has a solution.*

Proof. It is easily seen that $C_1, C_2, \ldots, C_{10}$ from Lemma 4.2.5 form a 2-factor of $C_5 \wr \overline{K}_n$ whose cycle lengths are $n_1, n_2, \ldots, n_{10}$ and that - since each orbit under β contains exactly n edges - the action of β on this 2-factor yields the desired 2-factorization.■

Lemma 4.2.7 *Let* $5 \leq a < b$ *be odd integers, let* $n = a + b$ *and let* $G = C_5 \wr \overline{K}_n$. *Then* $OP(G; a^5, b^5)$ *has a solution except possibly for the following pairs* (a, b)*:* $(5,7), (5,11), (7,9), (7,11), (7,15), (7,19)$, *and* $(11,15)$.

Proof. Set $n_1 = n_2 = \cdots = n_5 = a$ and $n_6 = n_7 = \cdots = n_{10} = b$. By Lemma 4.2.6 we only need to find 10 paths $P_1, P_2, \ldots, P_{10}$ that will satisfy the conditions of Lemma 4.2.5. We will use Lemma 4.2.4 to prove the existence of such paths. We will divide the proof into four cases depending on the congruence classes of a and b modulo 4.

Case 1. $a \equiv b \equiv 1 \pmod 4$. Since $5 \leq a < b$, we have $n \geq 14$. Note that $n - 5 \equiv 1 \pmod 4$. Let s be the integer satisfying $s \equiv 1 \pmod 4$ and $s \leq (n-5)/5 < s+4$. Then there is a unique quintuple $t_1, t_2, \ldots, t_5 \in \{s, s+4\}$ satisfying $t_1 \leq t_2 \leq t_3 \leq t_4 \leq t_5$ and $t_1 + t_2 + t_3 + t_4 + t_5 = n - 5$. Since not all of t_i are equal to 1 (as $n \geq 14$), $P(1^5, t_1, t_2, t_3, t_4, t_5)$ is non-empty by Lemma 4.2.4. Let $P_1, P_2, P_3, \ldots, P_{10}$ be a collection of paths comprising a graph in $P(1^5, t_1, t_2, t_3, t_4, t_5)$ with $|E(P_1)| = |E(P_2)| = \cdots |E(P_5)| = 1$ and with $|E(P_{5+i})| = t_i$ for $1 \leq i \leq 5$. We may assume that the vertex set of this collection is $\{(1,i), (5,i) : 1 \leq i \leq n\}$. We will show that these 10 paths satisfy the conditions of Lemma 4.2.5. Thus we have $c = 0$, $m_1 = m_5 = \frac{a+b+10}{2}$, and $m_2 = m_3 = m_4 = 0$.

Conditions 3, 4, 5 and 6 are obviously met. Since no P_i is a cycle, Conditions 1 and 2 will be shown to be met if we can prove that $|E(P_i)| < n_i$ for all i. This is obvious for $1 \leq i \leq 5$. Since $n_6 = n_7 = \cdots = n_{10} = b$ and since $t_1 \leq t_2 \leq \cdots \leq t_5$, it suffices to show that $t_5 < b$. However, $b > n/2$ while

$$2t_5 \leq t_4 + t_5 + 4 = (n-5) - t_1 - t_2 - t_3 + 4 \leq (n-5) + 1 = n - 4,$$

and thus $t_5 < n/2$. This concludes the proof for Case 1.

Case 2. $a \equiv 1 \pmod 4, b \equiv 3 \pmod 4$. Assume further that $n = a + b \geq 20$. The proof is identical to Case 1 with the following adjustment: Now we require that $s \equiv 3 \pmod 4$. Note that here we require $n \geq 20$ to guarantee that $s > 0$.

Case 3. $a \equiv 3 \pmod 4$ and $b \equiv 1 \pmod 4$. Once again, we require $n \geq 20$. Again, the proof is similar to Case 1. First, let s be the positive integer satisfying $s \equiv 1 \pmod 4$ and $s \leq (n-15)/5 < s+4$ (here we require that $n \geq 20$). Let t_1, t_2, t_3, t_4, t_5 be the unique quintuple with $t_1 \leq t_2 \leq t_3 \leq t_4 \leq t_5$ and $t_1 + t_2 + t_3 + t_4 + t_5 = n - 15$. By Lemma 4.2.4, $P(3^5, t_1, t_2, t_3, t_4, t_5)$ is non-empty. Let a graph in this set be comprised of paths $P_1, P_2, \ldots, P_{10}$ with $|E(P_1)| = |E(P_2)| \cdots = |E(P_5)| = 3$ and with $|E(P_{5+i})| = t_i$ for $1 \leq i \leq 5$. As in Case 1., we will show that these paths satisfy conditions of Lemma 4.2.5. Again, as before, we only need to show that $t_5 < b$. However, the same argument as in Case 1. applies here.

Case 4. $a \equiv b \equiv 3 \pmod 4$. Assume further that $n \geq 30$. The proof is identical to Case 3. except that we require that $s \equiv 3 \pmod 4$. To guarantee that $s > 0$, we need $n \geq 30$.

Combining Cases 1–4 covers all pairs (a, b) with the exception of $(5,7), (5,11), (7,9), (7,11), (7,15), (7,19)$, and $(11,15)$.■

Lemma 4.2.8 *Let $5 \leq a < b$ be odd integers, let $n = a+b$ and let $G = C_5 \wr \overline{K}_n$. Then $OP(G; a^5, b^5)$ has a solution for the following pairs (a,b):* $(5,7), (5,11), (7,9), (7,11), (7,15), (7,19)$ *and* $(11,15)$.

Proof. For each pair (a, b), we will construct an $OP(G; a^5, b^5)$.

Case 1. $(a,b) = (5,7)$. F will contain the following 10 cycles:

$$C_1 = ((5,1),(4,1),(5,12),(4,12),(3,12),(2,12),(1,12))$$
$$C_2 = ((5,3),(1,1),(2,1),(3,1),(4,3))$$

$$
\begin{aligned}
C_3 &= ((5,5),(1,2),(2,2),(3,2),(4,5))\\
C_4 &= ((5,7),(1,7),(5,2),(4,2),(3,3),(2,3),(1,3))\\
C_5 &= ((5,9),(1,4),(2,4),(3,4),(4,9))\\
C_6 &= ((5,11),(1,5),(2,5),(3,5),(4,11))\\
C_7 &= ((5,4),(1,8),(2,8),(3,8),(4,4))\\
C_8 &= ((5,6),(1,9),(2,6),(1,6),(2,7),(3,7),(4,6))\\
C_9 &= ((5,8),(1,10),(2,10),(3,10),(4,10),(3,9),(4,8))\\
C_{10} &= ((5,10),(1,11),(2,11),(3,11),(2,9),(3,6),(4,7)).
\end{aligned}
$$

Let F' be obtained from F by "swapping" vertex $(4,1)$ with vertex $(4,10)$ and vertex $(2,6)$ with vertex $(2,9)$. Similar to Lemma 4.1.9, one will observe that $F, F\beta^2, F\beta^4, \ldots, F\beta^{10}, F'\beta, F'\beta^3, F'\beta^5, \ldots, F'\beta^{11}$ form the desired 2-factorization.

The proof in the remaining cases will be along the same lines. In each case, we will construct two 2-factors F and F' (with F' being obtained from F by swapping a few pairs of vertices), such that F, $F\beta^2$, $F\beta^4, \ldots$, $F\beta^{n-2}$, $F'\beta$, $F'\beta^3$, $F'\beta^5, \ldots$, $F'\beta^{n-1}$ form the required 2-factorization. Although it is rather tedious to check in each case that the 2-factorization is as required, it is essentially trivial. In what follows, we will list the pair F, F' in each case. The reader may verify that they are indeed as required.

Case 2. $(a,b) = (5,11)$. Let F contain the following 10 cycles:

$$
\begin{aligned}
C_1 &= ((5,7),(1,3),(5,1),(1,16),(2,6),(3,6),(2,7),(1,14),(2,10),\\
&\quad (3,7),(4,6))\\
C_2 &= ((5,9),(1,4),(5,3),(1,1),(2,11),(3,8),(4,7),(5,12),(4,8),\\
&\quad (3,9),(4,9))\\
C_3 &= ((5,11),(1,5),(5,5),(1,2),(2,14),(3,10),(4,10),(5,14),(4,11),\\
&\quad (3,11),(4,12))\\
C_4 &= ((5,13),(1,6),(2,1),(3,1),(4,1))\\
C_5 &= ((5,15),(1,7),(2,2),(3,2),(4,2))\\
C_6 &= ((5,2),(1,9),(2,8),(1,8),(2,9),(3,14),(4,13),(5,16),(4,14),\\
&\quad (3,15),(4,15))\\
C_7 &= ((5,4),(1,10),(2,13),(3,13),(2,12),(3,12),(2,15),(1,15),\\
&\quad (2,16),(3,16),(4,16))
\end{aligned}
$$

$$C_8 = ((5,6),(1,11),(2,3),(3,3),(4,3))$$
$$C_9 = ((5,8),(1,12),(2,4),(3,4),(4,4))$$
$$C_{10} = ((5,10),(1,13),(2,5),(3,5),(4,5)).$$

F' is obtained from F by swapping vertex $(2,8)$ with vertex $(2,12)$.

Case 3. $(a,b) = (7,9)$. Let F contain the following cycles:

$$C_1 = ((5,7),(1,3),(5,1),(1,16),(2,1),(3,1),(4,1))$$
$$C_2 = ((5,9),(1,4),(5,3),(1,1),(2,2),(3,2),(4,2))$$
$$C_3 = ((5,11),(1,5),(5,5),(1,2),(2,3),(3,3),(4,3))$$
$$C_4 = ((5,13),(1,6),(2,4),(3,5),(4,6),(5,12),(4,7),(3,6),(4,8))$$
$$C_5 = ((5,15),(1,7),(2,5),(3,7),(4,9),(5,14),(4,10),(3,8),(4,11))$$
$$C_6 = ((5,2),(1,9),(2,8),(1,8),(2,9),(3,4),(4,4))$$
$$C_7 = ((5,4),(1,10),(2,13),(3,13),(2,12),(3,12),(4,5))$$
$$C_8 = ((5,6),(1,11),(2,6),(3,9),(2,7),(1,14),(2,10),(3,10),(4,12))$$
$$C_9 = ((5,8),(1,12),(2,11),(3,11),(4,13),(5,16),(4,14),(3,14),(4,15))$$
$$C_{10} = ((5,10),(1,13),(2,14),(3,15),(2,15),(1,15),(2,16),(3,16),(4,16)).$$

F' is again obtained from F by swapping vertex $(2,8)$ with vertex $(2,12)$.

Case 4. $(a,b) = (7,11)$. F will contain these cycles:

$$C_1 = ((5,9),(1,9),(5,8),(1,10),(2,1),(3,1),(4,1))$$
$$C_2 = ((5,6),(1,12),(5,5),(1,13),(2,3),(3,2),(4,2))$$
$$C_3 = ((5,3),(1,15),(5,2),(1,16),(2,6),(3,3),(4,3))$$
$$C_4 = ((5,18),(1,17),(5,1),(1,11),(2,10),(3,9),(2,11),(1,5),(2,12),(3,10),(4,6))$$
$$C_5 = ((5,15),(1,18),(2,18),(1,7),(2,9),(3,6),(4,4))$$
$$C_6 = ((5,16),(1,2),(2,2),(1,3),(2,13),(3,13),(4,7),(5,4),(4,8),(3,14),(4,9))$$
$$C_7 = ((5,17),(1,4),(2,4),(3,4),(2,5),(3,5),(4,5))$$
$$C_8 = ((5,10),(1,1),(2,7),(3,7),(2,8),(3,8),(4,10),(5,7),(4,13),(3,15),(4,15))$$

$$C_9 = ((5,14),(4,14),(5,13),(1,6),(2,14),(3,16),(2,15),(1,14),$$
$$(2,16),(3,17),(4,16))$$
$$C_{10} = ((3,11),(4,11),(3,12),(4,12),(5,11),(1,8),(2,17),(3,18),$$
$$(4,17),(5,12),(4,18)).$$

F' is obtained from F by swapping the following three pairs of vertices: $(2,18)$ with $(2,5)$, $(2,8)$ with $(2,2)$ and $(4,14)$ with $(4,11)$.

Case 5. $(a,b) = (7,15)$. Let F contain these cycles:

$$C_1 = ((5,11),(1,11),(5,10),(1,12),(2,1),(3,1),(4,1))$$
$$C_2 = ((5,8),(1,14),(5,7),(1,15),(2,2),(3,2),(4,2))$$
$$C_3 = ((5,5),(1,17),(5,4),(1,18),(2,3),(3,3),(4,3))$$
$$C_4 = ((5,2),(1,20),(5,1),(1,21),(2,4),(3,4),(4,4))$$
$$C_5 = ((5,13),(1,16),(5,12),(1,22),(2,8),(3,5),(4,5))$$
$$C_6 = ((5,3),(1,19),(5,14),(1,13),(2,12),(3,8),(4,6),(5,6),(4,7),$$
$$(3,9),(4,8),(5,9),(4,9),(3,12),(4,10))$$
$$C_7 = ((5,15),(1,10),(2,10),(3,10),(2,11),(3,11),(4,11),(5,19),$$
$$(4,12),(3,13),(2,13),(1,1),(2,14),(3,14),(4,13))$$
$$C_8 = ((5,16),(1,9),(2,9),(1,8),(2,15),(3,15),(4,14),(5,20),(4,15),$$
$$(3,16),(2,16),(1,2),(2,17),(3,17),(4,16))$$
$$C_9 = ((5,17),(1,6),(2,6),(3,6),(2,7),(3,7),(4,17),(5,21),(4,18),$$
$$(3,18),(2,18),(1,3),(2,19),(3,19),(4,19))$$
$$C_{10} = ((5,18),(1,5),(2,5),(1,4),(2,20),(3,20),(4,20),(5,22),(4,21),$$
$$(3,21),(2,21),(1,7),(2,22),(3,22),(4,22)).$$

F' is obtained from F by swapping vertex $(2,11)$ with vertex $(2,9)$ and vertex $(2,7)$ with vertex $(2,5)$.

Case 6. $(a,b) = (7,19)$. Let F contain these cycles:

$$C_1 = ((5,13),(1,13),(5,12),(1,14),(2,1),(3,1),(4,1))$$
$$C_2 = ((5,10),(1,16),(5,9),(1,17),(2,2),(3,2),(4,2))$$
$$C_3 = ((5,7),(1,19),(5,6),(1,20),(2,3),(3,3),(4,3))$$
$$C_4 = ((5,4),(1,22),(5,3),(1,23),(2,4),(3,4),(4,4))$$
$$C_5 = ((5,2),(1,25),(5,1),(1,26),(2,5),(3,5),(4,5))$$

$$
\begin{aligned}
C_6 &= ((5,14),(1,5),(5,15),(1,4),(2,6),(3,6),(4,6),(5,5),(4,7),\\
&\quad (3,7),(2,7),(1,1),(2,8),(3,8),(4,8),(5,8),(4,9),(3,9),(4,10))\\
C_7 &= ((5,24),(1,3),(5,25),(1,2),(2,9),(3,10),(2,10),(1,8),(2,11),\\
&\quad (3,11),(4,11),(5,11),(4,12),(3,12),(2,12),(1,9),(2,13),\\
&\quad (3,13),(4,13))\\
C_8 &= ((5,23),(1,6),(5,22),(1,7),(2,14),(3,14),(4,14),(5,16),\\
&\quad (4,15),(3,17),(2,17),(1,10),(2,18),(3,18),(4,16),(5,17),\\
&\quad (4,17),(3,19),(4,18))\\
C_9 &= ((5,26),(1,21),(2,21),(1,18),(2,19),(3,20),(2,20),(1,11),\\
&\quad (2,22),(3,21),(4,19),(5,18),(4,20),(3,22),(2,23),(1,12),\\
&\quad (2,24),(3,23),(4,21))\\
C_{10} &= ((5,19),(1,15),(2,15),(3,15),(2,16),(3,16),(4,22),(5,20),\\
&\quad (4,23),(3,24),(2,25),(1,24),(2,26),(3,25),(4,24),(5,21),\\
&\quad (4,25),(3,26),(4,26))
\end{aligned}
$$

F' is obtained from F by swapping vertex $(2,21)$ with vertex $(2,16)$.

Case 7. $(a,b)=(11,15)$. F will consist of the following 10 cycles:

$$
\begin{aligned}
C_1 &= ((5,13),(1,13),(5,12),(1,14),(2,1),(3,1),(4,1),(5,5),(4,2),\\
&\quad (3,2),(2,2),(1,1),(2,3),(3,3),(4,3))\\
C_2 &= ((5,10),(1,16),(5,9),(1,17),(2,4),(3,4),(4,4),(5,8),(4,5),\\
&\quad (3,5),(2,5),(1,8),(2,6),(3,6),(4,6))\\
C_3 &= ((5,7),(1,19),(5,6),(1,20),(2,7),(3,7),(4,7),(5,11),(4,8),\\
&\quad (3,8),(2,8),(1,9),(2,9),(3,9),(4,9))\\
C_4 &= ((5,4),(1,22),(5,3),(1,23),(2,10),(3,10),(4,10),(5,16),\\
&\quad (4,11),(3,11),(2,11),(1,10),(2,12),(3,12),(4,12))\\
C_5 &= ((5,2),(1,25),(5,1),(1,26),(2,13),(3,13),(4,13),(5,17),\\
&\quad (4,14),(3,14),(2,14),(1,11),(2,17),(3,17),(4,15))\\
C_6 &= ((5,14),(1,5),(5,15),(1,4),(2,18),(3,18),(4,16),(5,18),\\
&\quad (4,17),(3,19),(4,18))\\
C_7 &= ((5,24),(1,3),(5,25),(1,2),(2,19),(3,20),(4,19),(5,20),\\
&\quad (4,20),(3,21),(4,21))
\end{aligned}
$$

$$C_8 = ((5,23),(1,6),(5,22),(1,7),(2,20),(3,22),(2,22),(1,12),$$
$$(2,23),(3,23),(4,22))$$
$$C_9 = ((5,26),(1,21),(2,21),(1,18),(2,24),(3,24),(4,23),(5,21),$$
$$(4,24),(3,25),(4,25))$$
$$C_{10} = ((5,19),(1,15),(2,15),(3,15),(2,16),(3,16),(2,25),(1,24),$$
$$(2,26),(3,26),(4,26)).$$

F' is obtained from F by swapping vertex $(2,21)$ with vertex $(2,16)$. ■

Combining Lemmas 4.2.7 and 4.2.8, we obtain the following:

Lemma 4.2.9 *Let $5 \leq a < b$ be odd integers, let $n = a + b$ and let $G = C_5 \wr \overline{K}_n$. Then $OP(G; a^5, b^5)$ has a solution.*■

4.3 Construction of $OP(a^n, b^n)$

The following lemma, which was proven in [7], will help us decompose $K_{n(a+b)}$ into the desired component graphs.

Lemma 4.3.1 *Let n be a positive integer other than 3 or 5. Then K_{2n+1} has a 2-factorization in which each 2-factor consists of 3-cycles and 5-cycles only.* ■

Theorem 4.3.2 *Let $n \neq 7, 11$ be odd. Further, let $5 \leq a < b$ be odd integers. Then $OP(a^n, b^n)$ has a solution.*

Proof. For $n = 1$, the result follows from Theorem 2.3.9. For $n > 1$, we need to prove that $K_{n(a+b)} - I$ has a 2-factorization in which each 2-factor consists of n a-cycles and n b-cycles. $K_{n(a+b)} - I$ is a vertex-disjoint union of n copies of $K_{a+b} - I$, plus a copy of $K_n \wr \overline{K}_{a+b}$. Each copy of $K_{a+b} - I$ has an $OP(a, b)$ by Theorem 2.3.9. Therefore, the vertex-disjoint union of these $K_{a+b} - I$ has a 2-factorization in which each 2-factor consists of n a-cycles and n b-cycles. It remains to prove that $K_n \wr \overline{K}_{a+b}$ also has such a 2-factorization. To this end, note that, by Lemma 4.3.1, K_n has a 2-factorization in which each cycle of every 2-factor is a 3-cycle or a 5-cycle (note that for Lemma 4.3.1 to apply, we need $n \neq 7, 11$). This 2-factorization induces a division of $K_n \wr \overline{K}_{a+b}$ into a collection of $(n-1)/2$ $2(a+b)-$ factors, each of which consists exclusively of disjoint copies of $C_3 \wr \overline{K}_{a+b}$ and $C_5 \wr \overline{K}_{a+b}$. One can easily see that the proof will be completed if we can show that $C_3 \wr \overline{K}_{a+b}$ has a 2-factorization in which each 2-factor consists of three a-cycles and three b-cycles and that $C_5 \wr \overline{K}_{a+b}$ has a 2-factorization in which each 2-factor consists of five a-cycles and five b-cycles. However, this is true by Lemmas 4.1.10 and 4.2.9.■

Chapter 5

Final Remarks

5.1 Possible Extensions of Our Results

In the thesis, we solved the Oberwolfach Problem in case $OP(a, b)$ where a and b are odd. We believe that our methods can be used to prove all possible cases $OP(a, b)$ except when $a + b$ is congruent to one modulo 4. The conjecture below, if true, would give us more freedom in selecting the starting 2-factor F and, we believe, would go a long way towards settling the remaining cases of the form $OP(a, b)$.

When $n = a + b$ is congruent to one modulo 4, the methods presented in the thesis do not work without modification. The reason for this is that, for the methods to work, each of the edges that we use has to be in an edge-orbit of cardinality $t = \lfloor \frac{n-1}{2} \rfloor$. However, when t is even, our construction produces two orbits of cardinality $t/2$ each. When n is even, we are decomposing $K_n - I$, and we are able to avoid the edges in these orbits, which then form part of the one-factor I. However, when n is odd, we are decomposing K_n, so we cannot avoid any edges. Therefore, in the case when n is odd and t is even (i.e., the case n congruent to one modulo 4), the construction does not work.

Conjecture 5.1.1 *Let $n > 1$ be an integer. Then there is a graceful path on n vertices with endpoints a and b if and only if all of the following conditions hold true:*

(i) $a \neq b$,

(ii) $1 \leq a, b \leq n$,

(iii) $|a - b|$ *has the same parity as* $\lfloor n/2 \rfloor$,

(iv) $|a - b| \leq n/2$, *and*

(v) $(n+3)/2 \leq a + b \leq (3n+1)/2$.

We will prove that these conditions are necessary for the existence of the desired graceful labeling.

To this end, we define the *label length* of a path P with the vertex set $\{1, 2, 3, \ldots, n\}$ (denoted by $ll(P)$) as

$$ll(P) = \sum_{(u,v),(u,v) \in E(P)} |u - v|.$$

For a given n and $1 \leq a, b \leq n$, we define $ll_n(a, b)$ to be the maximum label length over all paths on n vertices with one endpoint a and the other b. A path that attains the maximum is a *maximum label length* path.

The following lemma provides us with insight into structural properties of paths with maximum label length.

Lemma 5.1.2 *Let P be a maximum label length path for given n, a and b. Then there is no pair of edges (u, u') and (v, v') in P such that* $\max(u, u') < \min(v, v')$.

Proof. Suppose the lemma is false. Choose a path P, and edges (u, u') and (v, v') that contradict the lemma. Then P can be broken down into five subpaths $P_1, (u, u'), P_2, (v, v'), P_3$ such that each pair of consecutive paths intersect in a single vertex (without loss of generality, we may assume that u, u', v and v' appear in this order). We allow degenerate paths P_1, P_2 and P_3 consisting of a single vertex.

Let P_2' denote the path P_2 traversed backwards and consider the path $P' = P_1, (u, v), P_2', (u', v'), P_3$. Then

$$ll(P') = ll(P) - |u - u'| - |v - v'| + |u - v| + |u' - v'| > ll(P).$$

Since P' has the same endpoints a and b as P, we reach a contradiction.■

Lemma 5.1.3 calculates $ll_n(a, b)$ for some n, a and b.

Lemma 5.1.3 *Let n, a and b be such that $a < (n+1)/2$ and $b > (n+1)/2$. Then $ll_n(a,b) = \lfloor n^2/2 \rfloor - (b-a)$.*

Proof. Let P be a maximum label length path with endpoints a and b. We will divide the proof into two cases.

Case 1. n=2k. P contains as many vertices less than or equal to k as vertices greater than k. Therefore, if P contains an edge with both endpoints less than or equal to k, then P must also contain an edge with both endpoints at least $k+1$ (and vice versa). However, this would contradict Lemma 5.1.2. Therefore, each edge of P must have one endpoint less than or equal to k and the other endpoint at least $k+1$. Then

$$ll_n(a,b) = ll(P) = \sum_{(u,v),(u,v) \in E(P)} |u-v| =$$

$$((\sum_{i=k+1}^{2k} 2i) - b) - ((\sum_{i=1}^{k} 2i) - a) = 2k^2 - (b-a) = \lfloor n^2/2 \rfloor - (b-a).$$

Case 2. n=2k+1. Similar reasoning as in the even case shows that P does not contain any edges with both endpoints either less than $k+1$ or greater then $k+1$, and that one of the edges with the endpoint $k+1$ has the other endpoint less then $k+1$ while the other one has the other endpoint greater than $k+1$. Then

$$ll_n(a,b) = ll(P) = \sum_{(u,v),(u,v) \in E(P)} |u-v| =$$

$$((\sum_{i=k+2}^{2k+1} 2i) - b) - ((\sum_{i=1}^{k} 2i) - a) = 2k^2 + 2k - (b-a) = \lfloor n^2/2 \rfloor - (b-a). \blacksquare$$

Lemma 5.1.4 provides upper bounds on some $ll_n(a,b)$ not addressed by Lemma 5.1.3.

Lemma 5.1.4 *Let n, a and b be such that $a < (n+1)/2$ and $b < (n+1)/2$. Then $ll_n(a,b) \le \lfloor \frac{(n-1)^2-4}{2} \rfloor + (a+b)$.*

Proof. As in Lemma 5.1.3, let P be a maximum label length path with endpoints a and b. Again, we will divide the proof into the same two cases.

Case 1. n=2k. An application of Lemma 5.1.2 now implies that P contains exactly one edge (w, w') with both endpoints $\geq k+1$ and that all other edges (u, u') in P satisfy $\min(u, u') \leq k$ and $\max(u, u') \geq k+1$. Further, we may assume that $w < w'$. Therefore,

$$ll_n(a,b) = ll(P) = \sum_{(u,v),(u,v) \in E(P)} |u-v| =$$

$$((\sum_{i=k+1}^{2k} 2i) - w) - ((\sum_{i=1}^{k} 2i) - a - b + w) = 2k^2 - 2w + (a+b) \leq$$

$$2k^2 - 2(k+1) + (a+b) = \left\lfloor \frac{(n-1)^2 - 4}{2} \right\rfloor + (a+b).$$

Case 2. n=2k+1. As before, we argue that all edges (u, u') in P must satisfy $\min(u, u') < k+1$ and $\max(u, u') > k+1$ except that there may be an edge e with both endpoints greater than $k+1$. If there is such an e, then one of the two edges with $k+1$ as an endpoint has the other endpoint less than $k+1$, while the other one has the other endpoint greater than $k+1$. If e does not exist, then both edges with one endpoint equal to $k+1$ have the other endpoint greater than $k+1$. We then get

$$ll_n(a,b) = ll(P) = \sum_{(u,v),(u,v) \in E(P)} |u-v| \leq$$

$$(\sum_{i=k+2}^{2k+1} 2i) - ((\sum_{i=1}^{k+1} 2i) - a - b) = 2k^2 - 2 + (a+b) =$$

$$\left\lfloor \frac{(n-1)^2 - 4}{2} \right\rfloor + (a+b). \blacksquare$$

We can now prove the necessity of the conditions in Conjecture 5.1.1.

Theorem 5.1.5 *Let P be a graceful path on n vertices, with endpoints a and b. Then the conditions (i)-(v) of Conjecture 5.1.1 are satisfied.*

Proof. Conditions (i) and (ii) are obvious. For condition (iii), note that if the vertices along the path are labeled consecutively as $a = s_1, s_2, s_3, \ldots, s_n = b$, then

$$|a-b| \equiv |s_1 - s_2| + |s_2 - s_3| + \cdots + |s_{n-1} - s_n| \equiv 1 + 2 + 3 + \cdots + (n-1) \equiv$$

$$\frac{(n-1)n}{2} \equiv \left\lfloor \frac{n}{2} \right\rfloor \pmod 2.$$

Also note that $ll(P) = \sum_{i=1}^{n-1} i = n(n-1)/2$.

To prove condition (iv), note that it is satisfied whenever either $\max(a,b) \leq (n+1)/2$ or $\min(a,b) \geq (n+1)/2$. Therefore, we may assume that the conditions of Lemma 5.1.3 are satisfied. Then

$$\frac{n(n-1)}{2} = ll(P) \leq ll_n(a,b) = \lfloor n^2/2 \rfloor - |b-a|.$$

However, this implies condition (iv).

For condition (v), we note that the first inequality in the condition is satisfied whenever $\max(a,b) \geq (n+1)/2$. When $\max(a,b) < (n+1)/2$, we may apply Lemma 5.1.4 to get

$$\frac{n(n-1)}{2} = ll(P) \leq ll_n(a,b) \leq \left\lfloor \frac{(n-1)^2 - 4}{2} \right\rfloor + (a+b).$$

Again, this implies the first inequality in condition (v). The second inequality is equivalent to $(n+3)/2 \leq (n-a+1) + (n-b+1)$, and is thus symmetrical to the first inequality.■

It is far less obvious that conditions (i)-(v) are sufficient for the existence of the desired graceful labeling. However, we strongly suspect that the conjecture is true. In practice, one will find the following:

When one of a and b is close to either 1 or to n, the graceful labelings are few but they are easy to construct. When a and b are both further away from the endpoints, it becomes difficult to find a general construction. However, in this case, there are usually many graceful labelings with the required endpoints.

We have tested the conjecture numerically by searching through all graceful labelings for paths with up to 20 vertices. Table below shows typical output. In this case, the path has 16 vertices and the entry in the ath row and the bth column shows the number of labelings in which the first vertex is labeled a and the last vertex is labeled b.

.	.	.	.	.	.	.	.	1	.	.	.	.	.	.	.
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Although we believe that the conjecture is very likely true, we were unable to prove it.

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