Circular Flow and Circular Chromatic Number in the Matroid Context

by

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Abstract

This thesis considers circular flow-type and circular chromatic-type parameters (ϕ and χ, respectively) for matroids. In particular we focus on *orientable* matroids and $\sqrt[6]{1}$-*matroids.* These parameters are obtained via two approaches: algebraic and orientation-based.

The general questions we discuss are: bounds for flow number; characterizations of Eulerian and bipartite matroids; and possible connections between the two possible extensions of ϕ: algebraic and orientation.

In the case of orientable matroids, we obtain characterizations of bipartite rank-3 matroids and Eulerian uniform, rank-3 matroids; an asymptotic result regarding the flow number of uniform matroids; and an improvement on the known bound for flow number of matroids of arbitrary rank. This bound is further improved for the uniform case.

For $\sqrt[6]{1}$-matroids, we examine an algebraic extension of the parameters χ and ϕ. We also introduce a notion of orientation and the corresponding flow and chromatic numbers applicable to this class. We investigate the possibility of a connection between the algebraic and orientation-based parameters, akin to that established for regular matroids by *Hoffman's Circulation Lemma* and we obtain a partial connection. We extend the notion of "Eulerian" to $\sqrt[6]{1}$-matroids. We call such matroids

hex-Eulerian. We show that every maximum-sized $\sqrt[6]{1}$-matroid, of fixed rank, is hex-Eulerian. We also show that a regular matroid is hex-Eulerian if and only if it admits a nowhere-zero 3-flow. We include an extension of Tutte's *chain groups*, which characterize regular matroids, to what we term *hex-chain modules* which describe $\sqrt[6]{1}$-matroids.

Dedication

A mis padres; Beatríz y Efraín.

A mis hermanos y hermanas.

A la memoria de Guadalupe Lomelí Herrera.

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Last, but not least, I wish to thank all the friends and family members who, through their support and company, even from a distance, have helped with this project. In particular, to my nephews and nieces who help me keep in touch with home and always have something fun to say; something new to chat about; something non-mathematical to bring to my life.

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Chapter 1

Introduction

> 'Oh,' said Tiffany, cheering up. 'Really? Well, then... there's our door, everyone!'
>
> 'Right,' said Rob Anybody. 'Now show us the way through.' [1]

The *circular chromatic number* of a graph, $\chi_c(G)$ has been studied now for quite some time and there is a large body of beautiful results on the subject. One of the reasons for such wide interest in this parameter is the fact that it can be defined in so many ways, which adds to the richness of the theory. Another graph parameter closely related to $\chi_c(G)$, is the *circular flow number*, $\phi_c(G)$. In this thesis, we generalize these concepts to matroids, thus obtaining circular chromatic and circular flow number-like parameters for particular classes of matroids.

The flow-colouring link in the previous paragraph, is a duality relation originally described by Tutte in [Tut54]. In the abstract of this paper he writes:

> "It is observed that the theory of spanning trees now links the theory of graph colourings to that of electrical networks".

As he described this link, he found an equivalent statement to the 4-colour conjecture in terms of what is now known as the *flow polynomial*, introduced a two variable

[1]Reproduced with permission of the author. Terry Prattchet, [Pra04]

polynomial, now known as the *Tutte polynomial* and stated his *5-flow conjecture* all of which has motivated considerable research in this area, including this thesis.

Generalizing parameters that can be defined in several different ways, such as $\chi_c(G)$ and $\phi_c(G)$, results in many possible extensions. Out of all of these, the focus of this thesis is on two extensions; one based on orientations and the other on linear algebra. These apply to two particular matroid classes: *orientable* and *sixth root of unity*, resulting in three different parameters that apply to specific classes of matroids. Fig. 1.1 shows some containment relations between these two classes and some other well-known matroid classes.

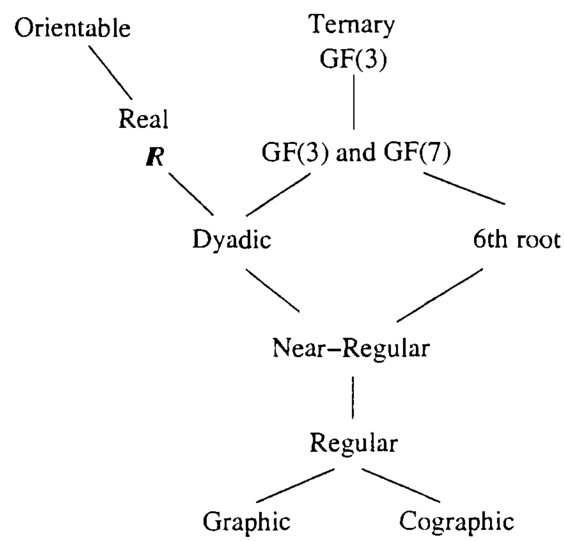


Figure 1.1: A hierarchy of matroid classes. Fields in the diagram ($\mathbb{R}$, $GF(3), GF(7)$) stand for the class of matroids representable over that field. Dyadic, sixth root and near-regular matroids are all representable over transcendental extensions $\mathbb{Q}(\alpha)$.

While some of these containment relations are well-known (eg. all graphic matroids are regular) proofs of all these containments are found in a variety of papers. We direct an interested reader to [LS99] for the proof that a matroid is ternary and orientable if and only if it is dyadic. Containment relations of the ternary sub-classes: dyadic, sixth-root and near-regular, are due to Whittle [Whi97], who also showed that a ternary matroid is representable over $GF(7)$ if and only if it can be obtained taking direct sums and 2-sums of dyadic and $\sqrt[6]{1}$-matroids.

Defining chromatic and flow numbers for matroids is a natural question and indeed it has been addressed before. An approach via the Tutte polynomial, leading to the concept of *critical index* for matroids representable over finite fields can be found in [BO92].

The decision to extend $\chi_c(G)$ and $\phi_c(G)$, rather than other chromatic and flow numbers, is also a natural one, as some of the definitions of $\chi_c(G)$ and $\phi_c(G)$ are already known to generalize to *regular matroids.* The first result in this direction is the work, appearing in [GTZ98], of Goddyn, Tarsi and Zhang, which extended definitions of these parameters, based on orientations, to regular matroids ($\phi_o(M)$, $\chi_o(M)$).

Whenever it is possible to define both an algebraic and an orientation-based parameter, for a matroid class, a central question is whether the parameters thus obtained satisfy some relation of the type of *Hoffman's Circulation Lemma.* That is, are the parameters obtained through these generalizations equivalent, as in the graphic case, or do they differ and if so, how widely.

In all cases, we are interested in bounds for flow numbers. In graph theory, we have the striking result [Sey81b], that $\phi(G) \leq 6$ for any coloop-free graph. Ideally we would hope for some upper bounds for other natural classes of matroids to follow directly via our generalization.

Finding bounds on chromatic numbers is always interesting. Finding the chromatic number of a graph is an $\mathcal{NP}$-hard problem, thus a bound can be taken as an approximation. Furthermore, graph colouring is a problem with well documented applications (eg. scheduling) hence finding or approximating the chromatic number of a graph has implications beyond theoretical concerns. Circular chromatic number has also been helpful with some applications (eg. flow of traffic patterns) and the appeal of finding bounds of the circular chromatic number is further enhanced by the fact that it approximates the chromatic number so closely. It is known that: $\lceil \chi_c(G) \rceil = \chi(G)$. Finally, it should be noted that circular chromatic number has connections to other topics of mathematics and these applications are perhaps of greater interest than those to industry. Notably, the relation between $\chi_c(G)$ and the study of homomorphisms between graphs. For example, it is known that a graph G admits a homomorphism onto C_{2k+1}, $G \rightarrow C_{2k+1}$ if and only if $\chi_c(C) \leq 2 + \frac{1}{k}$.

Graphs with $\phi_c(G) = 2$ and $\chi_c(G) = 2$ (minimal values for these parameters) have simple characterizations; these are *Eulerian* and *bipartite* graphs respectively. Accordingly, we devote some effort to study whether the classes of matroids attaining minimal values of ϕ and χ also admit simple characterizations. We define Eulerian and bipartite matroids in terms of our generalized parameters and find characterizations of such matroids, within certain subclasses.

Efforts to define Eulerian matroids started with Welsh [Wel69] who defined a binary matroid to be Eulerian if its element set can be partitioned into circuits. This resulted in a number of characterizations for this class. This thesis deals with matroids which are not necessarily binary, so Welsh's result may not apply. However, the definition of an Eulerian matroid as one with $\phi(M) = 2$ coincides with Welsh's definition, when the matroid is binary.

The questions just described are those at the center of this thesis.

Finding a *partition* or a *cover* by circuits of the edge set of a graph is a well known technique to research flow number ([Jae79], [JT92]). While the present work was not conceived as a study of circuit covers of matroids, this technique is ultimately a recurring theme of this research and will be mentioned throughout.

Other techniques used here range from elementary linear algebra to application of probabilistic arguments. An important tool to deal with orientations is the topological representation of oriented matroids.

We assume that the reader is familiar with the basics of graphs and matroids.

Chapter 2

Preliminaries

2.1 Beginnings in Graph Theory

We begin with a review of the graph theory concepts we aim to generalize. An excellent reference on the topic of circular chromatic number of graphs is the survey paper of X. Zhu [Zhu01]. We include only a minimum of information here and direct an interested reader to this reference for a broader review. An introductory view of nowhere-zero flows on graphs can be found in Seymour [Sey95].

2.1.1 Orientation Circular Flow and Chromatic numbers

The first definition of circular chromatic and circular flow number we introduce is the one shown to extend to regular matroids by Goddyn, Tarsi and Zhang in [GTZ98]. In 2002, Hell and Brewster [BH02] provided another proof of this result, using homomorphisms and avoiding Hoffman's lemma. The connection between orientations and flow/chromatic numbers, in the graphic case, was first noted by Minty [Min62].

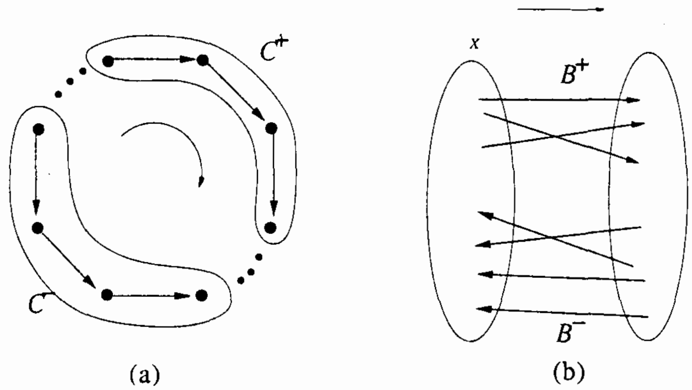


Figure 2.1: Partitions of a circuit (a) and cocircuit (b) imposed by an orientation.

Given any oriented graph $\vec{G}$, for each oriented circuit $\vec{C}$ consider a reference orientation of C defined by "traversing" the edges "forward" in cyclic order (if G is plane, this orientation can be "clockwise" or "counterclockwise") and let C^+ (resp. C^-) denote the set of edges of C whose orientation agrees (resp. disagrees) with the reference orientation (Fig. 2.1(a)). Note that there are two possible reference orientations which yield the same partition of $E(C)$ but swap the roles of C^+ and C^-. Similarly, for each oriented cocircuit $\vec{B} = \delta(X)$, we can define a partition B^+, B^- induced by a reference orientation; all edges "out" (or all edges "in") of $X \subset V$ (Fig. 2.1(b)). Again, the choice of reference orientation does not change the partition B^+, B^-, it simply swaps the roles of the parts.

Definition 2.1 *Let G be a graph and denote by $\mathcal{C}$ the family of circuits of G. Given any orientation $\vec{G}$, of G, the* imbalance *of a circuit $C \in \mathcal{C}$ is defined as:*

$$imbal(C) = \max\left\{\frac{|C|}{|C^-|}, \frac{|C|}{|C^+|}\right\},$$

we write $imbal(C) = \infty$ if $C^+ = \emptyset$ or $C^- = \emptyset$.

Similarly, if $\mathcal{C}^*$ denotes the family of cocircuits of a graph G, the imbalance of a *cocircuit* $B \in \mathcal{C}^*$ *is*

$$imbal(B) = \max\left\{\frac{|B|}{|B^-|}, \frac{|B|}{|B^+|}\right\}.$$

Again, $imbal(B) = \infty$ *if* $B^+ = \emptyset$ *or* $B^- = \emptyset$*.*

A circuit C (cocircuit B) for which either C^+ (B^+) or C^- (B^-) is empty, is called *totally oriented.*

Definition 2.2 *The* circular chromatic number, $\chi_c(G)$, *of a loop-free graph* G, *is:*

$$\chi_c(G) = \min_{\vec{G}}\left\{\max_{C\in\mathcal{C}}\{imbal(C)\}\right\},$$

where the minimum is taken over all orientations $\vec{G}$ *of* G*.*

Definition 2.3 *Similarly, the* circular flow number, $\phi_c(G)$, *of a coloop free graph* G *is:*

$$\phi_c(G) = \min_{\vec{G}}\left\{\max_{B\in\mathcal{C}^*}\{imbal(B)\ \}\right\}.$$

The following list of examples, includes some well known graphs and their circular flow and chromatic numbers.

Example 2.4 *Flow and chromatic numbers of small examples.*

1. Complete graphs.

   Complete graphs are example of graphs for which $\chi(G) = \chi_c(G)$,

   $$\chi(K_n) = n, \qquad \chi_c(K_n) = n.$$

   Note that only transitive tournaments give an orientation of K_n with bounded imbalance of circuits. Since every permutation of a subset of vertices of K_n yields a circuit, for any such tournament we can find circuits C, of every length, with imbalance $|C|$. Thus $\chi_c(K_n) = n$.

$$\phi_c(K_n) = \begin{cases} 2 & \text{if } n = 2k+1 \\ 2 + \frac{1}{k-1} & \text{if } n = 2k \end{cases}$$

This is not hard to show since K_n decomposes into cycles if n is odd and into cycles and a matching if n is even.

2. Odd cycles.

   The circular chromatic number is considered a refinement of the chromatic number. The reason for this is illustrated by odd cycles since, while all odd cycles have equal chromatic number, the circular chromatic number of an odd cycle depends on its length. Thus circular chromatic number is sensitive to the difference in structure of odd cycles of varying length, which are treated the same by chromatic number.

$$\chi(C_{2k+1}) = 3, \qquad \chi_c(C_{2k+1}) = 2 + \frac{1}{k}.$$

   On the other hand, all cycles are Eulerian, so $\phi(C_n) = 2 = \phi_c(C_n)$.

3. Plane dual of odd cycles.

   Let C_n^* denote the plane dual graph of C_n. Then

$$\phi(C_{2k+1}^*) = 3, \qquad \phi_c(C_{2k+1}^*) = 2 + \frac{1}{k}.$$

   These equalities will become clearer after introduction of Definition 2.7 and Theorem 2.8.

□

Before going any further, let us note the features at the core of these definitions. Firstly, there is no mention whatsoever of vertices. These parameters depend entirely on the structural properties of oriented circuits and cocircuits in the graph. Oriented circuits and cocircuits are regarded as sets of edges with a partition induced by orientation or "sign". These properties are precisely captured by the notion of an *oriented matroid.* The orientation in an orientable matroid partitions each circuit and cocircuit into two parts that we call *negative* and *positive* parts. This assignment of "positive" and "negative" labels, in the graphic case, is somewhat arbitrary in the sense that the parts are interchangeable, however, the partition is unique and this uniqueness, up to *total reversal* is also present in arbitrary orientable matroids.

As we mentioned before, finding bounds for circular flow numbers and circular chromatic numbers is of particular interest. Mostly we are concerned with upper bounds but we include here the *odd girth bound (ogb)* and *odd cogirth bound (ocg)* which are generally known lower bounds.

Definition 2.5 *The* odd-girth *of a graph, G, denoted $og(G)$, is the length of the shortest odd-length cycle, C, in G, and infinity, if G has no cycles of odd length. Similarly, the cardinality of the shortest odd-size cocircuit of G is called the* odd cogirth *of G and denoted $oc(G)$.*

Proposition 2.6 *If $og(G) = 2k+1$ then $\chi_c(G) \geq 2 + \frac{1}{k}$. If $oc(G) = 2k+1$, then $\phi_c(G) \geq 2 + \frac{1}{k}$.*

Proof: Let C be a circuit of G with length $2k+1$. Clearly the most that an orientation of G can balance C, is partitioning it into sets, C^+, C^-, of sizes k and $k+1$. This yields an imbalance of $2 + \frac{1}{k}$, which immediately provides the required bound for $\chi_c(G)$. We refer to this lower bound as the *odd girth bound $ogb(G)$.*

Applying the same analysis to cocircuits, we obtain the *odd-cogirth bound* for circular flow number: $\phi_c(G) \geq 2 + \frac{1}{k}$, when the odd-cogirth of G is $2k+1$.

□

This bound is attained for chromatic numbers by odd cycles ($\chi_c(C_{2k+1}) = 2 + \frac{1}{k}$) and for flow numbers by complete graphs of even order ($\phi_c(K_{2k}) = 2 + \frac{1}{k-1}$) as seen in Example 2.4.

The duality between flows and colorings described by Tutte, and mentioned in the introduction of this thesis, can also be seen in Example 2.4. Note that $\phi_c(C^*_{2n+1}) = \chi_c(C_{2n+1})$.

Formally, this link is often introduced in terms of *tensions* and *potentials*, as follows. Let $G = (V, E)$ be a connected, loop-free, directed graph. Given any walk $W = v_1e_1v_2e_2 \ldots e_kv_{k+1}$ of G, the orientation of the edges of W can be captured by a coefficient as follows:

$$\sigma(e_i) = \begin{cases} 1 & \text{if } e_i = (v_i, v_{i+1}) \\ -1 & \text{if } e_i = (v_{i+1}, v_i) \end{cases}$$

Let Γ be an Abelian group and $\Psi : E(G) \to \Gamma$ a map. Ψ is a *tension* if for every closed walk $W = v_1e_1v_2e_2 \ldots e_kv_1$ of G,

$$\sum_{i=1}^{k} \sigma(e_i)\Psi(e_i) = 0.$$

The tension Ψ is *nowhere-zero* if $0 \neq \Psi(e)$ for all $e \in E(G)$.

An integer coloring ρ of a graph G can be interpreted as a potential function on the vertices. This function induces a tension, also called a *potential difference* function on the edges of the graph as follows. Consider an arbitrary orientation of G, we obtain the potential difference function Ψ by taking, for each arc $e = (u, v)$, the difference of the colouring (or "potential values") on the end points of the edge (head minus tail) $\Psi(e) = \rho(v) - \rho(u)$. These potential difference values add to 0 over any circuit and the assignment is nowhere-zero if and only if the potential on V was a proper coloring. A function on the edges of a connected graph is a tension if and only if fixing the value of the potential function on any one vertex, determines uniquely the potential on all other vertices. It is this way that we can discuss vertex-colorings in terms of the edge set of a graph.

Definition 2.7 *Let G be an oriented graph, and for each vertex $v \in V(G)$, let $\delta^+(v)$ denote the set of edges of G whose tail is at v and $\delta^-(v)$ denote the set of edges of G whose head is at v. Given an Abelian group Γ, a Γ-valued* flow, *F on G is an assignment $F : E(G) \to \Gamma$ of elements of Γ to the edges of G, satisfying, at all vertices v, the following* flow conservation equation*:*

$$\sum_{e \in \delta^+(v)} F(e) - \sum_{e \in \delta^-(v)} F(e) = 0. \tag{2.1}$$

A consequence of this definition is that the flow conservation equation (Equation 2.1) holds for all cocircuits $\{\delta^+(X), \delta^-(X)\}$ which, as we shall see, makes it possible to extend this definition to regular matroids.

Thus a tension (which arises from a coloring or potential) adds to zero over circuits, while a flow adds to zero over cocircuits. This connection coloring/flow is the duality circuit/cocircuit that has prompted the consideration of both parameters, χ and ϕ, in this thesis. Furthermore, it motivates interest in researching circular flow number.

The following theorem appears in Jaeger [Jae79]. It summarizes a number of results that have been proven in separate papers. Most of the equivalences stated there are consequences of Tutte's work.

Theorem 2.8 *Let G be a graph. For every integer $\alpha \geq 2$, the following conditions are equivalent:*

1. *There exists a nowhere-zero $\mathbb{Z}_\alpha$-flow in G.*
2. *For any Abelian group Γ of order α, there exists a nowhere-zero Γ-flow in G.*
3. *There exists a nowhere-zero $\mathbb{Z}$-flow in G taking values in $\{1-\alpha, \ldots, \alpha-1\}$.*

This result allows us to talk simply about existence of a nowhere-zero α-flow, which is taken to be an integer-valued flow where all values are bounded $-(\alpha-1) \leq F(e) \leq \alpha-1$. The flow number of G, $\phi(G)$ is the least α for which G has a nowhere-zero α-flow. The duality between flows and colourings is more clearly seen in this case, where flows can be considered as tensions and yield integer colourings. Tutte's paper [Tut54], where he first discussed this duality, stated the following conjecture which remains un-proven to date.

Conjecture 2.9 (Tutte) *Every coloop-free graph G admits a nowhere zero 5-flow.*

While the conjecture has not been proven, it has been shown that the flow number of a coloop-free graph is indeed bounded by a constant Seymour [Sey81b].

Theorem 2.10 (Seymour) *For any coloop-free graph G, $\phi(G) \leq 6$.*

Prior to this result of Seymour, Jaeger [Jae79] proved that for any 2-edge connected graph, G, $\phi(G) \leq 8$. His proof considers group-valued flows, on the group of residues modulo 2, that is to say Γ-flows where $\Gamma = \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$. It uses the fact that the support of a $\mathbb{Z}_2$-flow is a disjoint union of circuits, known as an *even subgraph*, to obtain a nowhere zero flow as a sum of flows on 3 even subgraphs, over $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Adding flows over $\mathbb{Z}_2$ in a graph, is equivalent to taking symmetric difference of even subgraphs, which results in an even subgraph. This is the technique briefly mentioned in the introduction.

There are a number of conjectures of the same type of Conjecture 2.9, among them the following, which are all listed in [Sey95].

Conjecture 2.11 (Tutte) *Every coloop-free graph G with no Petersen graph minor admits a nowhere zero 4-flow.*

Conjecture 2.12 (Tutte) *Every 4-edge connected graph G admits a nowhere zero 3-flow.*

2.1.2 Algebraic Definitions

The second definition considered here is algebraic in nature. It relies on the representation of a graph given by a *Totally Unimodular Matrix*, that is, a $\{0, \pm 1\}$-valued matrix, having all subdeterminants (determinants of all square submatrices) also in the set $\{0, \pm 1\}$.

Example 2.13 *Graphic matroids.*

Let G be an oriented graph of order n, with m edges, and A its $n \times m$ incidence matrix. To show that A is a TUM note that each column contains exactly one 1 and one -1. This implies [Sch86, p.274] that A is totally unimodular.

□

Given a matrix A, the *null space* and the *row space* of A are denoted by: $Nullsp(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}$ and $Rowsp(A) = \{y^T A \mid y \in \mathbb{R}^m\}$, respectively.

Definition 2.14 *If G is an oriented graph with incidence matrix A, we write $G = G[A]$ and define the* circular chromatic number *of G as:*

$$\chi_c(G) = min\{\alpha \in \mathbb{R} : \exists\ f \in\ Rowsp(A),\ 1 \leq |f_e| \leq \alpha - 1\ \forall e \in E\}.$$

Similarly, define the circular flow number *of G as:*

$$\phi_c(G) = min\{\alpha \in \mathbb{R} : \exists\ f \in\ Nullsp(A),\ 1 \leq |f_e| \leq \alpha - 1\ \forall e \in E\}.$$

Although Definitions 2.14 are different in form from Definitions 2.2 and 2.2 they are equivalent, as shown by Hoffman's circulation lemma.

Proposition 2.15 *The parameters $\chi_c(G)$ and $\phi_c(G)$ are invariant under the following* elementary unimodular operations:

1. *Permuting rows or columns of A*

2. *Multiplying a column of A by* -1

3. Pivoting *on a non-zero element* $a_{k,l}$ *of A which is achieved through the following sequence of elementary row operations:*

   - *For each* i *in* $\{1, 2, \ldots, k-1, k+1, \ldots, m\}$, *replace row* i *of A by:* *row* $i - (a_{i,l}/a_{k,l})$ *row* k.
   - *Multiply row* k *by* $1/a_{k,l}$

At the core of these definitions is the existence of a representation of any graph by a totally unimodular matrix, which is unique up to operations (1) - (3) above, and the existence of a *norm* on $\mathbb{R}$, to bound f_e. There is another matroid class allowing for representations that share these features namely the class of $\sqrt[6]{1}$-matroids. In Section -2.3 it will be shown how these properties can be turned into definitions of flow and chromatic number-like parameters for certain classes of matroids.

Totally unimodular matrices have many nice properties, in particular, there is a $\{0, \pm 1\}$-valued vector representing any circuit and cocircuit, which is unique up to constant multiples. This property will be the basis for an alternative definition of χ and ϕ for $\sqrt[6]{1}$-matroids, akin to the orientation-based definition, which will be introduced in Chapter 4.

The well known *Hoffman's circulation lemma* [Hof76] shows that the parameters in Definition 2.14 coincide with those obtained through imbalance (Definitions 2.2 and 2.3). Indeed Hoffman's proof applies to totally unimodular matrices, thus the result holds true for matroids representable by a TUM, not just graphic matroids. These form the class of *regular matroids*. The precise statement of this result follows below. The reference included here is one of several papers where this result was published, but this seems to be the only one where it appears in its most general form.

Given an $m \times n$ matrix A, consider the polytope:

$$P = P(A; a, b, l, u) = \{x \mid a \leq Ax \leq b \ ; \ l \leq x \leq u\},$$

defined for numbers $\{a_i, b_i\}$; $i = 1 \ldots m$ and $\{l_j, u_j\}$; $j = 1 \ldots n$, satisfying $a_i \leq b_i$ for all i and $l_j \leq u_j$ for all j. Call such collections of numbers *data*.

Condition 2.16 (Hall) *For any two* $\{0, \pm 1\}-$*valued vectors,* w *and* v*, satisfying* $w^T A = v^T$*, we have:*

$$\sum_{i:w_i=-1} a_i + \sum_{j:v_j=1} l_j \leq \sum_{i:w_i=1} b_i + \sum_{j:v_j=-1} u_j,$$

Theorem 2.17 (Hoffman) *Let* A *be any* m *by* n *matrix, and let* P *be the associated polytope. Then the following statements about* A *are equivalent:*

1. *A is totally unimodular.*

2. *For all data, Hall's condition (2.16) implies that P is non-empty.*

3. *For all integral data, if P is not empty, then it contains an integral point. Indeed every vertex of P is integral.*

Hoffman's circulation lemma for graphs establishes the equivalence of Definitions 2.2 and 2.14. It can be stated as follows.

Theorem 2.18 *Let* G *be an oriented graph with incidence matrix* A*. Then there exists a vector* f *satisfying:* $Af = 0$ *and* $1 \leq |f_e| \leq \alpha - 1$ *if and only if* G *admits an orientation* $(\vec{G})$ *such that each cocircuit* (bond) *of* $(\vec{G})$*,* $B = B^+ \cup B^-$ *satisfies:*

$$\max\left\{\frac{|B|}{|B^+|}, \frac{|B|}{|B^-|}\right\} \leq \alpha. \tag{2.2}$$

Proof: Let $G = G[A]$ be an oriented graph for which there is a vector f satisfying $Af = 0$ and $1 \leq |f_e| \leq \alpha - 1$ (a nowhere zero *α-flow*). Reorient G by changing the orientation of any edge e with $f_e < 0$. Note that there is a vector f', which is positive in all coordinates and which is a NWZ-α-flow of the resulting oriented graph $\vec{G}$.

We claim that this orientation $\vec{G}$ satisfies the required condition, namely: for each cocircuit $B = B^+ \cup B^-$, $imbal(B) \leq \alpha$. Given a cocircuit $B = B^+ \cup B^-$, $Af = 0$ implies:

$$|B^+| \leq \sum_{e \in B^+} f_e = \sum_{e \in B^-} f_e \leq (\alpha - 1)|B^-|,$$

which implies $imbal(B) \leq \alpha$.

Conversely, if $\vec{G} = G[A]$ is an orientation of G with $imbal(B) \leq \alpha$ for all cocircuits B, consider the following data: $a_j = 0 = b_j,\ l_i = 1,\ u_i = \alpha - 1$ for all i and j. Thus for each cocircuit B, there are $\{0, \pm 1\}-$valued vectors, w and v, satisfying $w^T A = v^T$ and we have:

$$|B^+| = \sum_{w_i = -1} 0 + \sum_{v_j = 1} 1 \leq \sum_{w_i = 1} 0 + \sum_{v_j = -1} (\alpha - 1) = (\alpha - 1)|B^-|,$$

which is Hall's condition with the data set defined above. Thus, by Theorem 2.17, since A is a TUM, there exists a vector f satisfying $Af = 0$ and $1 \leq |f_e| \leq \alpha - 1$.

□

A similar statement about chromatic number is obtained by taking the matrix A^* representing the matroid $M^*[G] = M[G^*]$ and applying Theorem 2.17, as above, with data set: $a_i = 0 = b_i;\ l_j = 1;\ u_j = \alpha - 1$ for all i and j.

2.2 Oriented Matroids

This section includes those definitions and results from the theory of oriented matroids which are most relevant to this work. A valuable reference on this subject is [BLVS+99]. As a general rule, we follow the notation in this book so it can be consulted for more details.

The following definition of *oriented matroid* is attributed to Bland and Las Vergnas [BLVS+99, p. 118].

A *signing* of a set X is an unordered partition $\vec{X} = \{X^+, X^-\}$ of $X = X^+ \cup X^-$, where either part may be empty. It is sometimes convenient to view X as two ordered partitions $\{(X^+, X^-), (X^-, X^+)\}$. A pair $(\vec{C}, \vec{B})$ of signed sets is *orthogonal* if

$$(C^+ \cap B^+) \cup (C^- \cap B^-) = \emptyset \iff (C^+ \cap B^-) \cup (C^- \cap B^+) = \emptyset. \tag{2.3}$$

Definition 2.19 *An* oriented matroid *on the ground set E is a triple $\vec{\mathcal{O}} = (M, \vec{\mathcal{C}}, \vec{\mathcal{B}})$ where:*

1. *M is a matroid with ground set E, and collections $\mathcal{C}$ of circuits, and $\mathcal{B}$ of cocircuits.*
2. *$\vec{\mathcal{C}} = \{\vec{C} \mid C \in \mathcal{C}\}$ and $\vec{\mathcal{B}} = \{\vec{B} \mid B \in \mathcal{B}\}$ are signings of the circuits and cocircuits such that each pair in $\vec{\mathcal{C}} \times \vec{\mathcal{B}}$ is orthogonal.*

The oriented matroid $\vec{\mathcal{O}} = (M, \vec{\mathcal{C}}, \vec{B})$ is called an *orientation* of M. M is said to be *orientable* and is referred to as *the underlying matroid of $\mathcal{O}$*. This mirrors the situation with orientations of a graph. Indeed, oriented graphic matroids correspond to orientations of graphs.

To reverse the orientation of a set F of elements of $\vec{\mathcal{O}}$ is to replace each signed circuit and cocircuit (X^+, X^-) with $(X^+ \triangle F, X^- \triangle F)$ where $\triangle$ denotes symmetric difference. This results in another orientation of M, denoted $\mathcal{O}_F$. Every connected oriented matroid $\vec{\mathcal{O}}$ of order $n \geq 2$ has exactly 2^{n-1} reorientations, since reversing the orientation on E leaves $\mathcal{O}$ unchanged. The set of reorientations of $\vec{\mathcal{O}}$ is called the *reorientation class* of M and is denoted $\mathcal{O}$. In this text, we some times prefer the simpler term *orientation class*.

This definition of oriented matroid is concise and underlines the duality relation of circuits and cocircuits given by orthogonality. Matroids representable over the reals provide insightful examples of orientable matroids. A representation of such a matroid M, by an $r \times n$ matrix A, induces an orientation of M. Each element of M is represented by a column of A. A circuit C corresponds to a minimally linearly dependent set of such columns, which can be encoded by a non-zero vector x, indexed by E, such that $Ax^T = 0$. Here C is the support of x. The vector x is unique, up to a constant factor, thus the sign pattern of these coefficients is unique, up to total reversal. That is, x induces a signing $\{C^+, C^-\}$ where the roles of C^+ and C^- can be reversed (multiplying all coefficients by -1) but no other partition is possible. Similar statements can be made of cocircuits. Each cocircuit B is the support of a non-zero member yA of the row space of A. That the sign pattern of yA is unique up to total sign reversal follows from the minimality of the support of yA among non-zero vectors in the row space of A. We have that in any oriented matroid $\mathcal{O}$, for every circuit $C = (C^+, C^-)$ and cocircuit $B = (B^+, B^-)$, there is a circuit $-C = (C^-, C^+)$ and a cocircuit $-B = (B^-, B^+)$, called their *negatives*, also in $\mathcal{O}$.

2.2.1 Topological Representation

The example of representable matroids is also a good venue to introduce another description of oriented matroids which is central to this thesis. We first describe it for the rank-3 case. Let A be a $3 \times n$ matrix over $\mathbb{R}$of rank 3, representing an oriented matroid $\mathcal{O}$. Each column v of A can be seen as a vector in $\mathbb{R}^3$ and each such vector is normal to a plane through the origin. This plane defines two half-spaces in $\mathbb{R}^3$, one of which corresponds to points $x \in \mathbb{R}^3$ such that $v \cdot x > 0$ and is naturally distinguished as positive. This arrangement of planes, and choice of positive half-spaces, characterizes the oriented matroid represented by A. Different representations A of M may yield different arrangements of hyperplanes. Thus M may admit many different orientations

$\vec{\mathcal{O}}$. In particular, multiplying a subset of columns by -1 has the effect of switching the positive and negative half spaces corresponding to these elements. This is an alternative view of the notion of reorientation of an oriented matroid. Furthermore, some of these representations are indeed combinatorially different and it is not possible to obtain one from the other by reorientation as defined above. Thus reorientation defines an equivalence relation on the set of all orientations $\vec{\mathcal{O}}$ of a matroid M and the equivalence classes induced by this relation are the reorientation classes $\mathcal{O}$ mentioned above.

Alternatively, if we consider the intersection of these planes with the unit 2-sphere, S^2, we obtain a collection of great circles (1-spheres), on the 2-sphere also known as a *sphere complex*. When the planes are in general position, every point of intersection of great circles is the intersection of exactly two great circles (Fig. 2.2). Such an arrangement is a representation of the *uniform matroid* $U_{3,n}$, whose independent sets are all sets of cardinality at most 3. This collection of 1-spheres induces a partition of S^2 into cells of dimensions 0, 1 and 2. Each of these cells can be described uniquely indicating, for each 1-sphere, S, whether the cell lies in the positive side of S, the negative side of S or is contained within S itself. Thus a sphere complex, with a choice of positive half-sphere, for each 1-sphere in the complex, represents a given rank-3 oriented matroid.

A central result of the theory of oriented matroids is the topological representation theorem of Folkman and Lawrence and Edmonds and Mandel (see [BLVS$^+$99]). In the rank-3 case, it states that there is a one to one correspondence between reorientation classes of simple, rank-3 orientable matroids and arrangements of *1-pseudospheres* or *Pseudosphere complexes* (PSC). These pseudospheres are roughly defined as a relaxation of a sphere (see Definition 2.20). Like in the case of sphere complexes, these arrangements partition the sphere into cells of dimension $0, 1$ and 2, and each such cell corresponds to a unique $\{0, +, -\}$-valued sign pattern determined by the choices

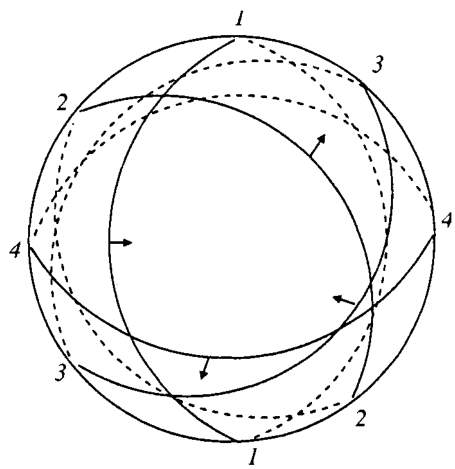


Figure 2.2: An arrangement of 1-spheres representing $U_{3,4}$.

of positive sides in the PSC. If the elements of $\vec{\mathcal{O}}$ are ordered, each cell in the decomposition is associated with a unique "signed characteristic vector", known as a *covector*. The sign vectors corresponding to 2-cells in this configuration are called *topes*, we extend the use of this term to the cells themselves. Note that every pair of great circles in a sphere complex must intersect in exactly two antipodal points on the sphere. This property of sphere complexes generalizes to pseudosphere complexes, and will be used repeatedly.

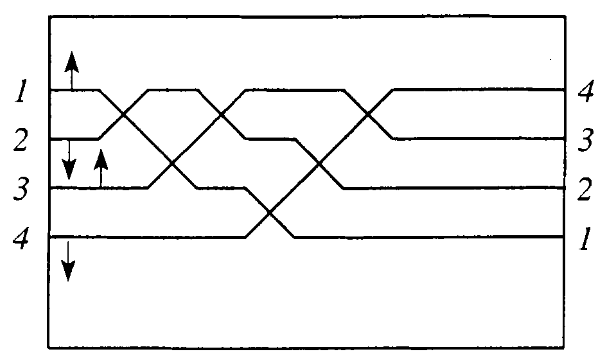


Figure 2.3: A wiring diagram representing $U_{3,4}$.

A rank-3 oriented matroid is also described as a *wiring diagram* [BLVS+99, p. 260] which is a special presentation of an arrangement of lines in the projective plane. This (projective) arrangement can be obtained from our PSC by identifying antipodal points on the sphere. Figure 2.3 shows a wiring diagram obtained from the PSC in Fig. 2.2, by identifying antipodal points and drawing the resulting lines on a Möebius strip. Thus this figure also encodes an orientation class of $U_{3,4}$. Note that we can define covectors for the points of intersection of lines (0-cells). This alone encodes the oriented matroid.

Definition 2.20 *A pseudosphere in S^d (a $(d-1)$-*pseudosphere*) is a subset S of the d-sphere S^d if $S = h(S^{d-1})$ for some homomorphism $h : S^d \longrightarrow S^d$ where*

$$S^{d-1} = \{x \in S^d : x_{d+1} = 0\}.$$

That is, a pseudosphere is a subset of a sphere of higher dimension, which is *topologically equivalent* to a sphere and thus, has two sides; a positive one, denoted S^+, and a negative one, S^-.

Definition 2.21 *A finite multiset $\mathcal{A} = (S_e)_{e \in E}$ of pseudospheres in S^d is called an* arrangement of pseudospheres *if the following conditions hold:*

1. *Every non-empty intersection $S_A = \cap_{e \in A} S_e$ is homeomorphic to a sphere of some dimension for all $A \subseteq E$.*

2. *If $S_A \not\subseteq S_e$, for $A \subseteq E$, $e \in E$ and S_e^+, S_e^- are the sides of S_e, then the intersection $S_A \cap S_e$ is a pseudosphere in S_A with sides $S_A \cap S_e^+$ and $S_A \cap S_e^-$. (If $S_a \cap S_e = S^{-1} = \emptyset$ is the empty sphere in a zero sphere $S_A \cong S^0$, then the sides of this empty sphere are the two points of S_A.)*

3. *The intersection of an arbitrary collection of closed sides is either a sphere or a ball.*

Axiom (3) in Definition 2.21 was shown to be redundant by Edmonds and Mandel (see [BLVS+99, p. 227]).

The collection of covectors of a PSC precisely encodes the oriented matroid. In particular, the supports of covectors of 0-cells (also called *vertices*) in the arrangement precisely correspond to signed cocircuits in $\vec{\mathcal{B}}$. The sets of elements corresponding to $+$ entries and $-$ entries in such a covector, form the sets B^+ and B^- respectively. This alone, describes the oriented matroid and will be particularly useful in the study of flow numbers. Note that the covectors of any antipodal pairs of vertices in a PSC representing a matroid $\vec{\mathcal{O}}$ are negative of each other, so there is a natural correspondence between an opposing pair of signed cocircuits $B = (B^+, B^-)$ and $-B = (B^-, B^+)$ of $\vec{\mathcal{O}}$ and an antipodal pair of vertices in the PSC.

Observation 2.22 *Note that any representation of the uniform matroid $U_{r,n}$, of rank r on n elements, will correspond with an arrangement of pseudospheres where every vertex is contained in exactly $r-1$ pseudospheres, that is, an arrangement of pseudospheres in* general position. *Conversely, any arrangement of n pseudospheres in S^{r-1}, in general position, represents an orientation class of the uniform matroid $U_{r,n}$.*

It should be mentioned that a matroid which is not *uniquely representable* is the underlying matroid of more than one combinatorially distinct configuration of pseudospheres, as is the case with most uniform matroids (see Obs. 2.22). In this case, not all possible reorientations are obtained from a PSC, by taking different choices of positive side for each pseudosphere. That is, reorientation of any given PSC, as we have defined it, will not produce all orientations of a matroid unless it is uniquely representable.

2.2.2 Extension to Orientable Matroids

Orientation classes of matroids have those features discussed following Definitions 2.2 and 2.3 and the definitions for $\chi_c(G)$ and $\phi_c(G)$ found in [GTZ98] extend not only to regular matroids, but also to reorientation classes of orientable matroids, denoted $\mathcal{O}$, as follows:

Definition 2.23 *Given an oriented matroid $\vec{\mathcal{O}}$, for any circuit C, in $\vec{\mathcal{O}}$ define the* imbalance *of C as:*

$$imbal(C) = \max\left\{\frac{|C|}{|C^-|}, \frac{|C|}{|C^+|}\right\}$$

Similarly, for any cocircuit, B, in $\vec{\mathcal{O}}$ the imbalance *of B is:*

$$imbal(B) = \max\left\{\frac{|B|}{|B^-|}, \frac{|B|}{|B^+|}\right\}.$$

Here, (C^+, C^-) and (B^+, B^-) are the signings of C and B imposed by $\vec{\mathcal{O}}$.

Definition 2.24 *The* orientation flow number *of the orientation class $\mathcal{O}$ is*

$$\phi_o(\mathcal{O}) = \min_{\vec{\mathcal{O}}}\left\{\max_{B\in\mathcal{C}^*}\{imbal(B)\ \}\right\}.$$

Dually, the orientation chromatic number *of $\mathcal{O}$ is*

$$\chi_o(\mathcal{O}) = \min_{\vec{\mathcal{O}}}\left\{\max_{C\in\mathcal{C}}\{imbal(C)\}\right\},$$

where both minima are taken over all reorientations in $\mathcal{O}$ and $\mathcal{C}$ and $\mathcal{C}^*$ are the families of signed circuits and cocircuits of $\vec{\mathcal{O}}$, respectively.

It is an easy observation that the existence of a loop in M, the underlying matroid of $\mathcal{O}$, implies $\chi_o(\mathcal{O}) = \infty$, likewise, the existence of a coloop implies $\phi_o(\mathcal{O}) = \infty$, therefore, from now on, all matroids considered will be *loop-free* and *coloop-free.*

This extension first appeared in [GHH06], where the authors show that the flow number of an orientation class is always bounded by a quantity that depends only on the rank of the matroid. The main result in that paper is:

Theorem 2.25 (Goddyn, Hliněný, Hochstättler) *If M is a rank $r \geq 2$, orientable matroid, then $\phi_o(\mathcal{O}) \leq 14r^2 \ln r$, for any orientation class $\mathcal{O}$ of M.*

The proof of this result uses a probabilistic argument. In this thesis, we include an improvement on this result (Theorem 3.17).

Further work in the area produced an improved bound for the rank-3 case, due to Edmonds and McNulty, [EM04].

Theorem 2.26 (Edmonds, McNulty) *Let M be an orientable rank-3 matroid, then for any orientation class $\mathcal{O}$ of M we have: $\phi_o(\mathcal{O}) \leq 4$.*

The proof of this result relies on a connection between the notion of imbalance and that of *discrepancy*, which is defined next. Here we regard a signed circuit (and signed cocircuit) as an ordered pair (B^+, B^-) ((C^+, C^-), respectively).

Definition 2.27 *For any signed circuit $C = (C^+, C^-)$ and cocircuit $B = (B^+, B^-)$ in an oriented matroid, the* discrepancies *of C and B, are $\delta(C) = |C^+| - |C^-|$ and $\delta(B) = |B^+| - |B^-|$, respectively.*

It is important to note that this definition of discrepancy differs slightly from the standard, which is taken to be the absolute value of what is defined here as discrepancy. Note that $\delta(C) = -\delta(-C)$.

Another important observation is that discrepancy and imbalance can be defined, in an obvious way, for any covector of an oriented matroid. Thus it is possible to consider the discrepancy of topes ($(r-1)$-cells) or indeed cells of any other dimension in a PSC. Furthermore, the discrepancy of any cell in a PSC can be determined by the discrepancies of the cells incident with it. The discrepancy of a vertex v (a 0-cell) in a PSC is the average of the discrepancies of the 1-cells incident with v, or indeed, the average of the k-cells incident with it, for any fixed $1 \leq k \leq r-1$.

Thus, the main advantage in considering δ instead of *imbal* is that δ has an "additive quality" to it. That quality goes a little further, consider an arbitrary oriented matroid $\vec{\mathcal{O}}$ associated with a given PSC and suppose we wish to evaluate the discrepancy of a tope, T, of $\mathcal{O}$. One way to calculate this discrepancy is to partition the elements in the PSC into subsets $S_1, \ldots S_k$ and calculate the discrepancies of the topes induced by each subset. Then $\delta(T) = \sum_{S_i} \delta(T_{S_i})$ where $\delta(T_{S_i})$ is the discrepancy of the tope, T_{S_i}, containing T in the sub-PSC induced by the elements in S_i.

The connection between discrepancy and imbalance, mentioned above is the following:

$$imbal(B) = 2\left(1 + \frac{|\delta(B)|}{|B| - |\delta(B)|}\right).$$

Hence, the study of any one of these parameters will produce information on both.

The proof of Theorem 2.26 combines the ideas just discussed and a connection between the discrepancy of topes and that of vertices in any PSC. This link is most clearly seen in the case of uniform matroids, as is shown by the following lemma.

Lemma 2.28 (Goddyn) *Let $\vec{\mathcal{O}}$ be a rank-r oriented matroid with n elements and for simplicity, denote the associated PSC also by $\vec{\mathcal{O}}$. For any tope T and any vertex v on the boundary of T, we have: $|\delta(T)| \leq |\delta(v)| + |n - |B||$, where B is the cocircuit corresponding to v. Furthermore, if the underlying matroid M is $U_{r,n}$, then this bound is attained. That is, for each tope T there is a vertex, v, incident with T such that $\delta(T) = |\delta(v)| + r - 1$.*

Proof: Let v be a vertex in $\vec{\mathcal{O}}$, incident with a tope T. First note that each pseudosphere S_e contributes ± 1 to the discrepancy of any tope $\delta(T)$ and $0, \pm 1$ to the discrepancy $\delta(v)$ of any vertex v, depending on whether v is contained in S_e or not. A vertex v corresponding to a cocircuit B is in the intersection of $n - |B|$ pseudospheres, which gives the inequality: $|\delta(T)| \leq |\delta(v)| + |n - |B||$.

If the underlying matroid of $\mathcal{O}$ is $U_{r,n}$ then any vertex v in $\vec{\mathcal{O}}$ is one of the two points in the intersection of $r-1$ elements of $\vec{\mathcal{O}}$, spanning a subconfiguration which determines 2^r topes. Thus any orientation of these elements, produces all possible sign patterns in the topes it induces. In this sense, these elements act like coordinate hyperplanes. We refer to these topes as *orthants*.

Given a vertex v, let T be the unique tope of $\vec{\mathcal{O}}$ which is adjacent to v and contained in the positive orthant of v. Then, $\delta(T) = \delta(v) + r - 1$, which shows that the above inequality is attained.

□

2.3 Sixth root of unity matroids

It is known that Definition 2.14 extends to regular matroids. It is not hard to see that it does not generalize to matroids representable by arbitrary matrices over the real numbers. This definition is based on finding a nowhere-zero vector, in the null space or row space of a matrix A, with all entries as "close to each other in magnitude" as possible. The flow number is then obtained taking a minimum over all possible representations of M by a TUM A. That the resulting parameter is well defined, follows from:

Theorem 2.29 *Let $M = M[A]$ be a regular matroid represented by the TUM A, if $M[A] = M[A']$ then:*

$$\min\{\alpha \in \mathbb{R} : \exists\ f \in Rowsp(A),\ 1 \le |f_e| \le \alpha - 1\ \forall e \in E\}$$
$$= \min\{\alpha \in \mathbb{R} : \exists\ f \in Rowsp(A'),\ 1 \le |f_e| \le \alpha - 1\ \forall e \in E\}$$

and

$$\min\{\alpha \in \mathbb{R} : \exists\ f \in Nullsp(A),\ 1 \leq |f_e| \leq \alpha - 1\ \forall e \in E\}$$
$$= \min\{\alpha \in \mathbb{R} : \exists\ f \in Nullsp(A'),\ 1 \leq |f_e| \leq \alpha - 1\ \forall e \in E\}.$$

Proof: It suffices to note that if A and A' represent the same matroid, then there is a sequence of elementary unimodular operations (listed in 2.1.2) that transform A into A'. It was noted before that χ and ϕ are invariant under such operations.

□

However, a matroid representable over the reals admits an infinite number of equivalent representations. If the matrix A, with columns $[A_1 \ldots A_n]$, is a representation of a matroid M, which admits a nowhere zero flow $(f_1, \ldots, f_n)^T$, then we have:

$$\begin{bmatrix} A_1 & A_2 & \cdots & A_n \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} = 0.$$

However, we can find another representation of the same matroid, and a corresponding flow, as follows:

$$\begin{bmatrix} f_1A_1 & f_2A_2 & \cdots & f_nA_n \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = 0.$$

This would indicate that M is Eulerian, in the sense discussed in the introduction ($\phi_a(M) = 2$). Thus any matroid admitting a nowhere-zero-flow, would be Eulerian under this definition.

The special property of totally unimodular matrices (hence regular matroids) which allows the algebraic definitions of circular chromatic and circular flow numbers to work is the following:

Properties 2.30

Each circuit (cocircuit) of a regular matroid $M = M[A]$, is the support of a $\{0, \pm 1\}$-valued vector in the column space (row space) of A. These vectors are unique *up to multiplication by -1.*

This property is shared by the following class of matroids, first defined in [Whi97].

Definition 2.31 *Consider the primitive sixth root of unity $w = \frac{1}{2} + \frac{i\sqrt{3}}{2}$. The matrix A is a* sixth root of unity matrix *if all its non-zero subdeterminants are of the form w^k. A matroid M is a* $\sqrt[6]{1}$-matroid *if it can be represented by a $\sqrt[6]{1}$-matrix.*

Example 2.32 *The matroid $U_{2,4}$ is a $\sqrt[6]{1}$-matroid which can be represented by the following matrix A.*

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & w \end{bmatrix} \qquad A' = \begin{bmatrix} -w & 0 & w^2 & 1 \\ 0 & w^4 & w^2 & w \end{bmatrix}$$

□

The matrix A' is also a $\sqrt[6]{1}$-representation of $U_{2,4}$. Note that A' can be obtained from A by multiplying each column of A by some w^k. In this sense, these representations are "resignings" in analogy to scaling columns of a TUM by -1. Indeed, $\sqrt[6]{1}$-matroids are uniquely representable, up to field automorphism and application of the following elementary operations on A.

1. Multiplying a row or column by a $\sqrt[6]{1}$.
2. Interchange or two rows or two columns.
3. Pivoting on a non-zero entry.

The definition of $\sqrt[6]{1}$-matroids greatly resembles that of regular matroids (which are representable by a totally unimodular matrix) and it is this similarity that makes them relevant to this work. It suggests the possibility of extending ideas from regular matroids to $\sqrt[6]{1}$-matroids. This fact allows us to sensibly define algebraic chromatic and flow numbers for $\sqrt[6]{1}$-matroids. We pursue this idea in the next section.

2.3.1 Algebraic Extension

Definition 2.33 *Given a $\sqrt[6]{1}$-matroid $M = M[A]$, define the* algebraic flow number *of M as:*

$$\phi_a(M) = \inf\left\{\alpha \in \mathbb{R} : \exists\ f \in \ \mathit{Nullsp}(A), 1 \leq \|f_e\| \leq \alpha - 1\ \forall e \in E\right\}.$$

Here $\|\cdot\|$ is the complex norm, thus we seek a vector with entries in an annulus of width $\alpha - 2$ in the complex plane. We note that $\phi_a(M)$ is well defined since every representation A of any $\sqrt[6]{1}$-matroid, satisfies properties (1)-(3) stated at the end of the previous section. Further note that a slightly different definition can be obtained by taking a different norm in the above definition. For instance, one might choose a norm that results in a "polygonal annulus". Having straight lines as constraints for our flow values suggests considering a linear programming approach.

It is possible to define other variations of ϕ for $\sqrt[6]{1}$-matroids. To introduce the next one we return to our starting point with flows in graphs. Recall that if all arc capacities in a network (that is, an oriented graph with given capacity on each edge) are integer, then there is an integer-valued, optimal circulation. This follows from the fact that the incidence matrix of any directed graph is totally unimodular. The matrix B in the standard linear programming formulation of the network flow problem is therefore totally unimodular. A basic feasible solution x to this problem satisfies $Bx = b$, where b is the vector of capacities. Since all determinants are ± 1 and b is integer valued, x is an integer valued feasible circulation. Thus it is natural to consider a notion of flow number where flow values are restricted to a complex equivalent of $\mathbb{Z}$. We do this using *Eisenstein integers*, which we denote $\mathbb{E}$, for our parallel with $\mathbb{Z}$. These are obtained by taking the closure of the six complex sixth roots of unity, under sum. Hence all Eisenstein integers can be written as a sum of integer multiples, of powers of w and can be identified with the elements of $\mathbb{Z}[w]$.

These numbers were defined by Eisenstein, who considered them to be the complex equivalent of real integer numbers. They form an *Euclidean domain*. Those with argument in the interval $(-\pi/3, \pi/3]$ are generally considered *positive* and this convention seems quite apposite to our purposes. Some references for further information on Eisenstein integers are [CG96] and [HW79].

Definition 2.34 *Given a $\sqrt[6]{1}$-matroid $M = M[A]$, define the* Eisenstein flow number *and* Eisenstein chromatic number *of M as:*

$$\phi_{\mathbb{E}}(M) = \min \{\alpha \in \mathbb{R} : \exists\ f \in \ \mathit{Nullsp}(A),\ 1 \leq \|f_e\| \leq \alpha - 1, f_e \in \mathbb{E}\ \forall e \in E\},$$

$$\chi_{\mathbb{E}}(M) = \min \{\alpha \in \mathbb{R} : \exists\ f \in \ \mathit{Rowsp}(A),\ 1 \leq \|f_e\| \leq \alpha - 1, f_e \in \mathbb{E}\ \forall e \in E\},$$

where $\|\cdot\|$ is the complex norm.

Again, these parameters are well defined by the same properties of $\sqrt[6]{1}$-matroids which made it possible to define $\phi_a(M)$.

2.3.2 Partial Fields and Orientation Numbers

Further work of Semple and Whittle on what they termed *partial fields* (see [SW96]), paves the way to define an analogue of orientation numbers for $\sqrt[6]{1}$-matroids. A first requirement to define these parameters for $\sqrt[6]{1}$-matroids is showing that such a matroid M satisfies a property similar to 2.30. That is, that every circuit and cocircuit is associated with a $\{0, \sqrt[6]{1}\}$-valued vector, unique up to a constant $\sqrt[6]{1}$-factor. This avenue of research is detailed in Chapter 3, however, we include next some results from [SW96] to lay the foundations for that chapter.

A *partial binary operation*, $+$ on a set S is a function $+ : A \to S$ defined on a subset A of $S \times S$.

Definition 2.35 *Let* $\mathbf{P}$ *be a set with a distinguished element called* 0, *and let* $\mathbf{P}^* = \mathbf{P} \setminus \{0\}$. *Let also,* $\circ$ *be a binary operation defined on* $\mathbf{P}$, *and* $+$ *a partial binary operation defined on* $\mathbf{P}$. *Then* $\mathbf{P}$ *is a* partial field *if the following properties are satisfied:*

1. $\mathbf{P}^*$ *is an Abelian group under* $\circ$.
2. *For all* $a \in \mathbf{P}$, $a + 0 = a$.
3. *For all* $a \in \mathbf{P}$, *there exists an element* $-a \in \mathbf{P}$ *with the property that* $a + (-a) = 0$.
4. *For all* $a, b \in \mathbf{P}$, *if* $a + b$ *is defined, then* $b + a$ *is defined and* $a + b = b + a$.
5. *For all* $a, b, c \in \mathbf{P}$, $a \circ (b + c)$ *is defined if and only if* $a \circ b + a \circ c$ *is defined; in which case,* $a \circ (b + c) = (a \circ b) + (a \circ c)$.
6. *The associative law holds for* $+$.

Examples of partial fields can be obtained from fields as follows. Let $\mathbb{F}$ be a field and G be a subgroup of the multiplicative group $\mathbb{F}^*$, such that for any $g \in G$, it must be that also $-g \in G$. Then a partial field, denoted $\mathbf{P} = (G, \mathbb{F})$, is obtained by considering $G \cup \{0\}$ with the operations induced by restricting the field operations to $G \cup \{0\}$. In this thesis, we are only concerned with the partial field obtained by taking $\mathbb{F} = \mathbb{C}$ and $G = \{w^i; i = 0 \ldots 5\}$, the group of sixth roots of unity.

It is possible to define vectors and matrices with entries in a partial field, where the sum of two vectors (or matrices) is defined if and only if all the sums, coordinate by coordinate, are defined. Having defined matrices in partial fields, the question is how to proceed to define a matroid. The standard way uses linear dependence and independence of columns, which is in turn defined through linear combinations. However, linear combinations involve sums which may not be defined in a partial field. Restricting to matrices over a partial field $\mathbf{P}$ having all partial subdeterminants also in $\mathbf{P}$, called $\mathbf{P}$-matrices, Semple and Whittle define independence of a set via subdeterminants. In this setting, an empty set of columns of an $m \times n$ $\mathbf{P}$-matrix is always considered independent and a non-empty set of columns indexed by the set $i_1, \ldots i_k$ is independent if $k \leq m$ and at least one of the $k \times k$ submatrices of A with columns indexed by $i_1, \ldots i_k$, has non-zero determinant. This yields the following:

Proposition 2.36 (Semple, Whittle) *Let A be a $\mathbf{P}$-matrix with columns indexed by a set S. Then the supports of the independent subsets of columns of A are the family of independent sets of a matroid on S.*

Furthermore, they show that if $\mathbf{P}$ is embedded in a field, then Proposition 2.36 results in exactly the same matroids that would be obtained by the standard definition when working over a field (in terms or linear combinations).

Using the results and concepts in that publication, it is possible to prove the following useful proposition:

Proposition 2.37 (Chávez Lomelí) *Let $M = M[A]$ be a rank r matroid represented by a full rank matrix A over a partial field $\mathbf{P} = (G, F)$, then the circuits (cocircuits) of M are the minimal non-empty supports of vectors in the null space (row space) of A, taken over the partial field $\mathbf{P}$.*

Proof: It suffices to show that if the set $C = \{v_1, \ldots, v_k\}$ of columns of A corresponds to a circuit, that is, C is a minimally dependent set of vectors, then there is a linear dependence $\alpha_1 v_1 + \alpha_2 v_2 + \cdots \alpha_k v_k = 0$ where not all $\alpha_i = 0$ and all α_i are elements of G.

To find the coefficients α_i above we need to solve the system of linear equations described by the *augmented* matrix $[A'|B]$ with columns:

$$[v_1\ v_2 \ldots v_{k-1}|v_k]$$

If the rank of this matrix is r, then since all subdeterminants of A are in $\mathbf{P}$, applying Cramer's rule concludes the proof. If the rank is less than r then using results from [SW96] we can perform elementary operations (interchanges of rows or columns, taking of multiples of rows or columns by a non-zero element in $\mathbf{P}$ and performing pivots) in the partial field, to reduce the matrix to a square matrix, thus reducing this case to the previous situation.

The class of $\mathbf{P}$-matroids is closed under duality. Thus the above proof shows that the result holds for cocircuits.

□

The key observation in the proof of Proposition 2.37 is the fact, noted in [SW96], that performing pivots on a matrix can be expressed using subdeterminants. These, by definition, are in the partial field and the resulting matrix will also be in the class. Performing a pivot on the non zero element a_{ij} of a $\sqrt[6]{1}$-matrix A, is obtained by multiplying row i by $1/a_{ij}$ and, for all $k \neq i$, replacing a_{kl} by

$$a_{ij}^{-1} \begin{vmatrix} a_{ij} & a_{il} \\ a_{kj} & a_{kl} \end{vmatrix}.$$

Proposition 2.37 ensures that each circuit and each cocircuit of a $\sqrt[6]{1}$-matroid, is represented by a $\{0, w^i\}$-valued vector. Furthermore, since all circuits and cocircuits are minimal, that vector is unique up to constant factors, drawing an exact parallel with the regular case, where each circuit and cocircuit is associated with a $\{0, \pm 1\}$-valued vector that is unique up to *sign reversal*. The notion of considering multiples of such a vector by a factor of the form w^k as a *re-signing* already suggests a way of defining orientation numbers for this class.

For matroids admitting such representations, denoted as $M = M[A]$, the correspondence between circuits $C \in \mathcal{C}$ and cocircuits $B \in \mathcal{C}^*$ and the supports of vectors v_C, v_B in the row space and null space of A can be used to define the discrepancy $\delta(B)$ and $\delta(C)$.

Definition 2.38 *For any circuit C and cocircuit B of a $\sqrt[6]{1}$-matroid M, the* discrepancy, δ, *is:*

$$\delta(C) = \sum_{e \in E} v_C(e), \quad \delta(B) = \sum_{e \in E} v_B(e),$$

where $v_C(e)$ and $v_B(e)$ denote the entry for e in v_C and v_B, respectively.

Thus $\sqrt[6]{1}$-matroids display the features required to define orientation flow and chromatic numbers. The details of such definitions will be discussed in chapter 4. However, we note that having the potential to define chromatic and flow numbers in two ways, for the matroids in this class, opens up a number of interesting questions. In particular, whether these definitions yield the same parameter or not.

Chapter 3

Oriented Matroids

In this chapter we report our study of χ_o and ϕ_o, as defined for orientation classes of orientable matroids. Most of the results in this chapter rely on the topological representation of oriented matroids discussed in Chapter 2. We often use discrepancy as a tool to approach imbalance, but we also include some comments on discrepancy as an invariant of the covectors of topes in an oriented matroid.

First we consider the rank 3 case. For uniform matroids, we provide a characterization of bipartite orientation classes and a characterization of Eulerian orientation classes. These characterizations are given in terms of the PSC representing the orientation class. We describe a local transformation that produces one Eulerian orientation class from another. While repeated application of this transformation does not generate all possible Eulerian orientations from any starting configuration, it provides a super-exponential lower bound on the number of Eulerian orientation classes of $U_{3,n}$. These results appear in [CLG07].

Continuing with the uniform case of arbitrary fixed rank, we show an asymptotic result regarding flow number of orientation classes. We show that it approaches 2 as the number of elements increases. It does so rapidly enough that a random orientation provides a proof of this result.

Our third result, improves to $3r$ a previously known upper bound (of $O(r^2 \log r)$) on the flow number of a coloop-free matroid of arbitrary rank, r, on n elements.

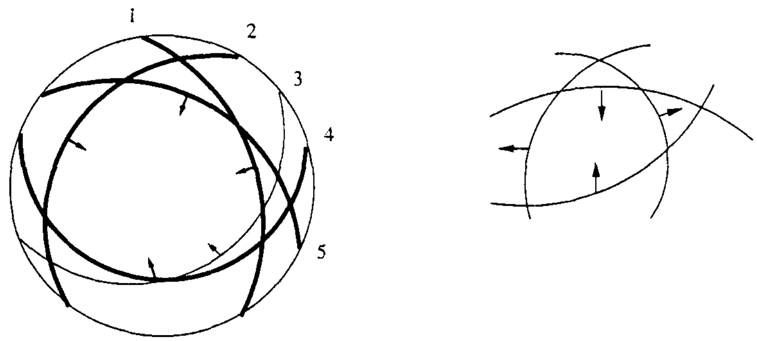
3.1 Bipartite and Eulerian Characterizations

Recall that an orientation class $\mathcal{O}$ is *bipartite* if $\chi_o(\mathcal{O}) = 2$ and *Eulerian* if $\phi_o(\mathcal{O}) = 2$. In this section we establish characterizations of bipartite oriented matroids and Eulerian classes of uniform matroids of rank three. A *configuration* is an arrangement of pseudospheres, representing a rank-3 orientation class. Each pseudosphere is a pseudocircle which is naturally the union of 1-balls called *edges*.

3.1.1 Bipartite Rank 3 Oriented Matroids

All circuits in a rank-r matroid M have size at most $r+1$. If a rank-3 matroid M contains a circuit C of size 3, the odd girth bound (ogb) implies that $\chi_o(M) \geq 3$. Thus to characterize simple rank 3 bipartite matroids, we need only consider matroids of girth 4, which are orientations of the uniform matroid $U_{3,n}$. A 4-circuit in such a matroid is a set of 4 pseudolines and the partition C^+, C^- is encoded in the arrangement as follows. In Fig. 3.1(a), the bold edges induce the signed circuit $(+,+,0,-,-)$ (or $(-,-,0,+,+)$) in $U_{3,5}$. The orientation is determined by comparison with the reference orientation indicated in Fig. 3.1(b). A graph theorist may be reminded of orientations of circuits, where edges are positively or negatively oriented depending on whether their orientation agrees, or disagrees, with a reference orientation, namely clockwise or anticlockwise.

Recall that we say a tope is *big* if its boundary intersects every pseudosphere in an edge. Such an edge is a side of the tope. A tope is *all-positive* if its covector is



(a) (b)

Figure 3.1: (a) A 4-circuit in an oriented, uniform matroid. (b) the orientation of reference for such a 4-circuit.

$(+, +, \ldots, +)$.

Proposition 3.1 *A 4-circuit C in a rank 3, oriented matroid $\vec{\mathcal{O}}$, is balanced if and only if the restriction $\vec{\mathcal{O}}|_C$ has an all-positive big tope (of size 4).*

To prove the bipartite characterization theorem (Theorem 3.4) we will make use of the following oriented matroid concepts [BLVS+99, §9.1, §9.2]. Theorem 3.3 is introduced in [BLVS+99, §9.2] as an oriented matroid equivalent of Weyl's Theorem.

Definition 3.2 *Let M be an acyclic oriented matroid on a set E. Given any cocircuit B, we say that the set B^+ is an* open half-space *of M. The set $B^+ \cup \overline{B}$ is called a* closed half space *of M. Here $\overline{B}$ is the complement of B.*

The convex hull *of a set A is denoted $conv_M(A)$ and defined as:*

$$conv_M(A) = \{A \cup \{x\} \in E \setminus A : \exists\ C \in \mathcal{C} \text{ with } C^- = \{x\}, C^+ \subseteq A\}$$

Theorem 3.3 *[BLVS+99, p. 382] Let $\mathcal{M}$ be an acyclic oriented matroid of rank r on E, and let $A \subseteq E$. Then $conv_{\mathcal{M}}(A)$ is equal to the intersection of the closed half-spaces of $\mathcal{M}$ containing A.*

Theorem 3.4 (Chávez Lomelí) Bipartite Characterization Theorem

Let $\mathcal{O}$ be an orientation class (wiring diagram) of a simple rank 3 matroid on n elements. Then $\chi_o(\mathcal{O}) = 2$ if and only if the underlying matroid is uniform and the pseudosphere arrangement representing $\mathcal{O}$ has a big tope.

Proof: First note, by our earlier remarks, that the underlying matroid of $\mathcal{O}$ must be uniform.

Suppose that $\mathcal{O}$ has a big tope, then orient all the elements so that this tope is all-positive, that is, its covector is $(+, +, \cdots, +)$. Any 4 lines define a circuit, and the all-positive region they form is 4-sided. By Proposition 3.1, these circuits are balanced, thus $\chi_o(\mathcal{O}) = 2$.

Conversely, suppose that $\chi_o(\mathcal{O}) = 2$. Take an orientation $\vec{\mathcal{O}}$ of $\mathcal{O}$ for which the imbalance is 2.

Again, we know that the underlying matroid of $\mathcal{O}$ is uniform. If $n = 4$, then the all positive region is 4-sided, and the result holds.

We may assume that $n > 4$. Since $\chi_o(\vec{\mathcal{O}}) < \infty$, $\vec{\mathcal{O}}$ is *acyclic* (has no totally oriented circuit). Thus $\vec{\mathcal{O}}$ contains an all-positive tope, T [BLVS+99, p. 123]. It remains to show that this tope is bounded by n lines.

Suppose, towards a contradiction, that there is at least one line e not incident with T. We will show that there is a set of 4 elements, containing e, forming a 4-circuit C with a triangular all positive tope. By Proposition 3.1 such a circuit is not perfectly balanced. The circuit C will have a negative sign on e and positive sign on all other elements. This is equivalent to $e \in conv_{\mathcal{O}}(A)$.

Thus we require a set of 3 elements $A = \{e_1, e_2, e_3\}$ such that the all-positive tope T_1 defined by e_1, e_2, e_3 is completely contained in the positive side of e.

Now note that T is contained in the positive side of e. This implies that e is in the intersection of the closed half-spaces of $E \setminus \{e\}$. By Theorem 3.3 $e \in conv_{\mathcal{O}}(E \setminus \{e\})$. This, by Definition 3.2 implies the existence of the circuit $\{e, e_1, e_2, e_3\}$.

□

Unfortunately, Theorem 3.4 does not directly generalize to higher rank cases. Suppose that $\mathcal{O}$ is a representation of $U_{5,n}$ with a big tope. A circuit C, of $\mathcal{O}$ is formed by any set of 6 elements and, in the restriction of $\mathcal{O}$ to this circuit, any big tope is a 4-polytope which is isomorphic to the prism $T \times [0, 1]$ over a tetrahedron T [Zie95, p. 10]. The signing of such a circuit induces a partition $\{C^+, C^-\}$ with sizes 4 and 2.

There exist, however, bipartite, higher rank, uniform, orientation classes. For example, consider *alternating matroids* denoted $C^{n,r}$, of odd rank r and order n (see [BLVS+99, §8.2 and §9.4]). These are characterized by the fact that their element set can be ordered in such a way that all bases are positively oriented. This, in turn, implies that the sign pattern on any circuit alternates with respect to this order (see [BLVS+99, p. 124]). Since these matroids are uniform, their circuits have size $r + 1$, which is even. Thus alternating matroids of odd rank are bipartite. If $r > 3$, $C^{n,r}$ is realizable and represented by a Vandermonde matrix,

$$\begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ t_1 & t_2 & \cdots & t_{n-1} & t_n \\ t_1^2 & t_2^2 & \cdots & t_{n-1}^2 & t_n^2 \\ \vdots & \vdots & & \vdots & \vdots \\ t_1^{r-1} & t_2^{r-1} & \cdots & t_{n-1}^{r-1} & t_n^{r-1} \end{pmatrix}$$

where $t_1 < t_2 < \ldots < t_n$. All submatrices are Vandermonde and hence have positive determinants.

3.1.2 Eulerian Rank-3 Uniform Matroids

Recall that an orientation class $\mathcal{O}$ is Eulerian if $\phi_o(\mathcal{O}) = 2$. That is, if it admits an orientation where all cocircuits are perfectly balanced.

Let $\mathcal{O}$ be an Eulerian orientation class of rank 3. Consider a tope T in a PSC representing $\mathcal{O}$. This orientation of $\mathcal{O}$ is such that each tope T is bounded by a set of elements oriented in one of two possible ways, namely:

1. Alternating towards and away from T *(alternating).*
2. All pointing towards or all away from T *(consistent).*

To see this, recall that the orientation balances all vertices perfectly, so an exploration of adjacent vertices along a cell T shows that, if two consecutive lines along the border of T are pointed consistently (both toward or both away from T) then all other lines around T are also oriented consistently toward or away from T, while if they differ, all other lines must alternate.

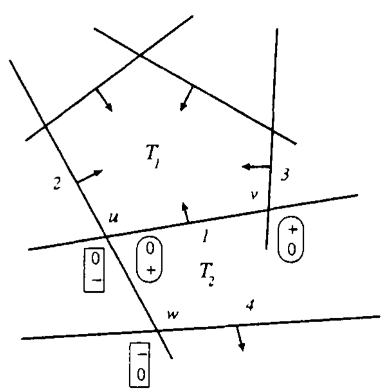


Figure 3.2: A local exploration of the changes in sign vectors on adjacent vertices.

In Fig. 3.2 vertices u and v lie on the border of a (consistent) tope, T_1, thus the only covector entries that differ between these two vertices correspond to the elements

2 and 3. If 1 and 2 are oriented consistently, then 3 must also be consistent, so that the changing covector entries are 0,+ and +,0 as indicated. Vertices u and w are incident with a neighboring tope, T_2. We must argue that T_2 is alternating. In this case, the covector entries that differ between u and w, are those corresponding to elements 1 and 4. Thus the orientations of 1 and 2 force the orientation of 4 so that we get covector entries $0, -$ and $-, 0$ as indicated. Furthermore, note that two *adjacent* cells T_1, T_2 (separated by one element of $\mathcal{O}$) cannot be of the same type. Thus the 2-cells of such pseudosphere arrangements are properly 2-colored, according to their type, as above.

Indeed, if we regard a PSC representing $U_{3,n}$, as an embedded, planar graph, (vertices, the 0-dimensional cells, edges corresponding to 1-dimensional cells and faces, the 2-dimensional cells), we have a 4-regular graph X which can be face 2-colored. We can construct two graphs out of these arrangements, having vertices corresponding to the cells T with color 1 and color 2, respectively. Two vertices are adjacent if the corresponding cells are incident with a common vertex in the arrangement.

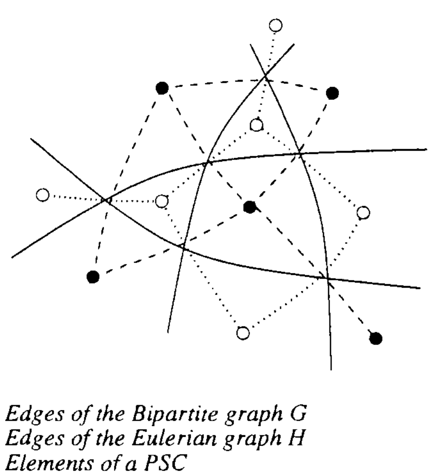


Figure 3.3: A small portion of the graphs G, H and the associated PSC (X).

For an Eulerian orientation class of $U_{3,n}$ one of these graphs, which we call G is bipartite and the other one, called H, is Eulerian. In Figure 3.3 we can see the portion of the graphs G and H obtained from two pairs of crossing pseudolines. As noted before G and H are a dual pair of plane graphs, furthermore, G and H have X as their common *medial graph* (see for example [GR01] and [Wel93]).

Example 3.5 *The bipartite orientation class of $U_{3,8}$ (one with a big tope) is Eulerian.*

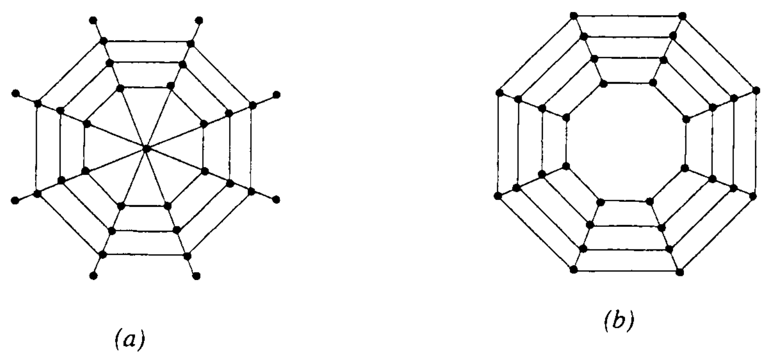


Figure 3.4: The dual pair (H, G) of plane graphs obtained from an Eulerian orientation of $U_{3,8}$. Identify vertices of degree 1 in H (a).

An Eulerian orientation is obtained by making the big tope alternating. One of the graphs associated to such a configuration is a *spiderweb* with $n = 8$ "spokes" and $n/2 = 4$ "levels" (see Fig. 3.4(a) and identify all the pendant vertices in the outside face) and the other one is a cylindrical grid of width $n = 8$ and with $n/4$ levels (see Fig. 3.4(b)).

□

Theorem 3.6 (Chávez Lomelí, Goddyn) Characterization of Eulerian Uniform

An orientation class $\mathcal{O}$ of the matroid $U_{3,n}$ is Eulerian if and only if n is even and the dual pair of graphs H and G described above, form an Eulerian-bipartite pair.

Proof: First assume that $\mathcal{O}$ is an Eulerian orientation class of $U_{3,n}$, we must show that n is even and that (G, H) is a bipartite-Eulerian pair.

All cocircuits of $U_{3,n}$ have cardinality $n-2$ thus if $\mathcal{O}$ is Eulerian, it follows that n must be even. Since $\mathcal{O}$ admits an Eulerian orientation, let $\vec{\mathcal{O}}$ be such an orientation. From the discussion preceeding this theorem, it follows that every tope of $\vec{\mathcal{O}}$ is either alternating or consistent. Furthermore, this partition defines the vertex set of a dual pair of graphs H and G. Since any alternating tope must be of even cardinality, the graph with vertex set corresponding to alternating topes is Eulerian. It follows that the other graph is bipartite.

Conversely, suppose that $\mathcal{O}$ is a representation of $U_{3,n}$ whose topes induce an Eulerian-bipartite pair of graphs. We must show that there is an orientation $\vec{\mathcal{O}}$ of $\mathcal{O}$, which balances all vertices of the configuration.

To describe this orientation, consider a proper two coloring of the vertices of the bipartite graph and orient all elements incident with topes T corresponding to vertices of color one, so that T is in their positive side or *in*, and those of color 2 so that T is in their negative side or *out*. This will induce an orientation for all elements in the matroid, provided we are able to show that it is well defined. That is, that once an element e is oriented one way, because of incidence with some tope T_1, there is no other tope T_2 forcing a different orientation on e. This follows from the fact that the graph defining the orientation is bipartite and such a contradiction would imply the existence of an odd circuit in a bipartite graph.

Finally, it must be shown that all vertices have zero discrepancy. It suffices to show that all vertices have equal discrepancy, since we know that antipodal vertices have discrepancies negative of each other. This equality follows from the same analysis of adjacent vertices done before. These vertices appear consecutively around the border of some consistent tope and the covectors of these vertices differ in exactly two entries which are zero in one vector and both $+$ or both $-$ in the other (see Fig. 3.2).

The observation that all vertices in the arrangement are incident with one such tope completes the proof.

□

Thus the configurations representing $U_{3,n}$ correspond to certain 4-regular graphs, embedded on the sphere, which is reminiscent of the theory of knots. By analogy with Reidemeister moves, there is a local transformation that changes one Eulerian orientation of $U_{3,n}$ into another Eulerian orientation of $U_{3,n}$ in a different reorientation class.

The transformation, which we call an *e-move*, involves four pseudocircles bounding a quadrilateral tope Q. An e-move at Q is defined if and only if Q is adjacent to two triangular topes, T_1, T_2. An e-move is depicted in Fig. 3.5(a). Since antipodal symmetry of a pseudosphere arrangement must be preserved, an e-move is simultaneously applied to the antipodal quadrilateral tope.

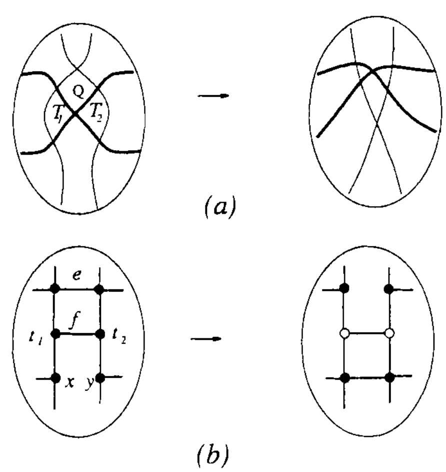


Figure 3.5: An Eulerian orientation preserving transformation on the bipartite graph G.

We consider the effect of an e-move on the associated plane bipartite graph G. This is illustrated in Fig. 3.5(b). Here G contains adjacent vertices t_1 and t_2 of degree 3, which correspond to the triangular topes T_1 and T_2. The edge $f = t_1t_2$ is incident to a face Q' of length 4 which corresponds to the quadrilateral tope Q. Let e be the opposite edge to f in this quadrilateral face. Since two pseudocircles cross exactly twice, one easily sees that the other face of G incident with e has length at least 6. Let x, y be the neighbors of t_1, t_2 which lie on the boundary of the bigger face. The effect of performing an *e-move* at e is to delete e and add the edge xy. One might recognize that an e-move can be realized as a combination of four Δ−Y operations on the graph G (and, dually, on the Eulerian graph H).

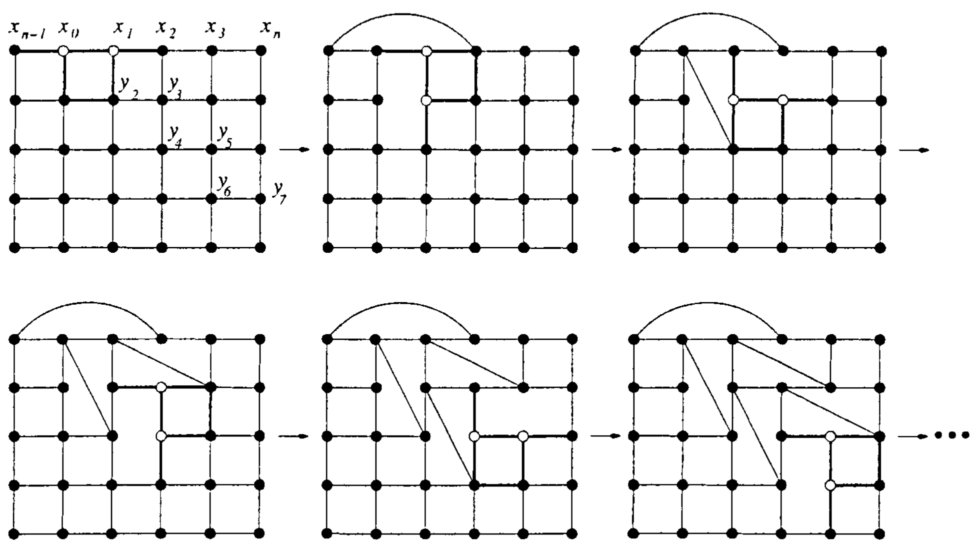


Figure 3.6: A sequence of transformations in a subgraph of G.

Let $n \geq 6$ be an even integer, and let $\vec{\mathcal{O}}$ be an Eulerian orientation of the reorientation class of $U_{3,n}$ with a big tope. The plane bipartite graph G associated with $\vec{\mathcal{O}}$ is depicted in Fig. 3.4(b) (for $n = 8$). Let $x_0x_1x_2 \ldots x_{n-1}x_0$ be the cycle in G which bounds one of the two faces of length n. For some positive integer $k_0 \leq \frac{n-2}{2}$, let y_2, $y_3 \ldots y_{k_0}$ a sequence of vertices in G as illustrated in Fig. 3.6. In that diagram we illustrate a sequence of e-moves, performed successively on the edges

$$x_0x_1, \quad x_1y_2, \quad y_2y_3, \quad y_3y_4, \quad \ldots, \quad y_{k_0-1}y_{k_0}.$$

These moves are, of course, mirrored on the opposite side of the sphere. The final graph, G', in this sequence is the result of *growing a ladder of length* k_0 *starting from* x_0x_1.

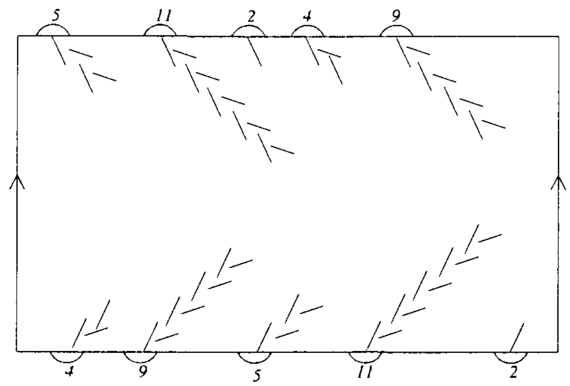


Figure 3.7: A schematic look at the graph G associated with a reorientation class produced by a sequence of switches.

We now select another integer $1 \leq k_1 \leq \frac{n-2}{2}$. Starting with G', we grow a ladder of length k_1 starting from x_3x_4. The two ladders we have grown do not interfere with each other, nor with the corresponding ladders on the other side of the sphere. Continuing in this way we see that, for any sequence of positive integers $k_0, k_1, \ldots k_\ell$ where $\ell \leq \lfloor n/3 \rfloor$ and with each $k_i \leq \frac{n-2}{2}$, we may sequentially grow ladders of length k_i starting from $x_{3i}x_{3i+i}$. In Fig. 3.7 we present a schematic illustration of the result of this construction for some $n \geq 24$ where $(k_0, k_1, k_2, k_3, k_4) = (5, 11, 2, 4, 9)$.

Clearly, two sequences (k_i) and (k_i') will result in non-isomorphic graphs provided (k_i') is not a rotation or reflection of the sequence (k_i). Each such graph represents a distinct Eulerian reorientation class of $U_{3,n}$. Therefore there exist at least $\frac{1}{2}(\frac{n}{2})^{\lfloor n/3 \rfloor - 1}$ distinct Eulerian reorientation classes for $U_{3,n}$.

An interesting example is the matroid $U_{3,6}$. There are 4 reorientation classes of this matroid, only one of which is Eulerian. The graphs H and G associated with these classes are double covers of graphs in the Petersen family. That is, identifying

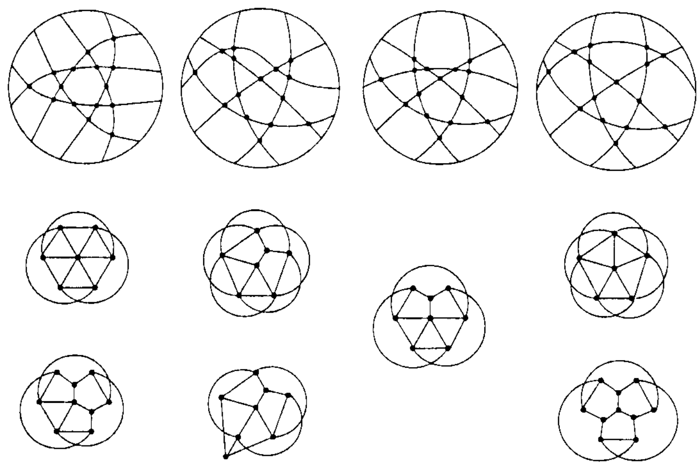


Figure 3.8: The orientation classes of $U_{3,6}$ and the corresponding graphs in the Petersen family.

antipodal vertices in G and H produces projective embeddings of these graphs (see Fig. 3.8). It is surprising, on the basis of small examples such as this one, that the number of Eulerian classes should grow super-exponentially.

These are not the only Eulerian classes for $U_{3,n}$. Others can be derived from the configuration with the big tope using e-moves. For example, some ladders can be grown toward the South-West as well as toward the South-East. There is also a construction due to D. Archdeacon, which results in a large family of Eulerian reorientation classes of $U_{3,n}$, none of which can be obtained by applying e-moves to the configuration with the big tope.

Consider a disc and take two diameters, x, y to be pseudolines. This cuts the perimeter of the disc in four intervals; North, South, East and West. Now add, repeatedly, pairs of pseudolines which run from North to South, but are "almost parallel" to x and to y, respectively. Each successive added pair is made to intersect either above (a) or below (b) all existing lines. In this way we construct one wiring diagram for each sequence of "a" and "b". In Fig. 3.9 we see the diagram corresponding to

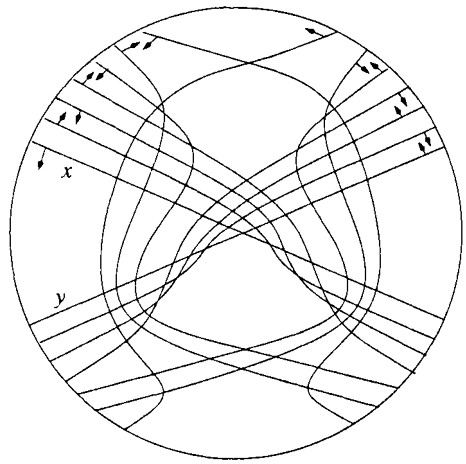


Figure 3.9: The configuration associated with the sequence aaabba.

the sequence aaabba. To see that each resulting configuration is Eulerian, we orient x and y so that the West interval is on the positive side of both pseudolines. The remaining pseudolines are alternatingly oriented starting from x and y as illustrated. It is not hard to see that this is an Eulerian orientation of the wiring diagram. It is also not hard to see that this construction yields an exponential number of pairwise distinct reorientation classes for $U_{3,n}$, for even integers n.

3.1.3 The Behaviour of $\phi(U_{r,n})$ for Large n

In this section, we present a new asymptotic result. In broad terms it shows that, given fixed r, the flow number of orientation classes of $U_{r,n}$ approaches 2 as n gets larger. One might say that uniform matroids are *closer* to being Eulerian, the larger the number of elements is. We show that, with positive probability, there is an orientation of $U_{r,n}$, with imbalance bounded by $2 + \varepsilon$, where $\varepsilon \to 0$ as $n \to \infty$. The precise statement of this result is given in Theorem 3.9, but first, we introduce two necessary results.

The first one is a concentration bound from Molloy and Reed [MR02]. We paraphrase their explanation that it formalizes the intuitive notion, from investment portfolios, which says that, if a portfolio is sufficiently diversified and, changes in any given investment will not change dramatically the value of the entire portfolio, then profit will tend to be quite close to its expected value.

Theorem 3.7 (Molloy, Reed) *Let X be a random variable determined by n independent trials $T_1, \ldots, T_n$ and satisfying the following: Changing the outcome of any one trial can affect X by at most c. Then*

$$Pr(|X - E(X)| > t) \leq 2e^{-\frac{t^2}{2c^2n}}.$$

□

The other result establishes that the number of topes in a pseudohyperplane arrangement (hence in an oriented matroid) is given, up to sign, by the Tutte polynomial of the underlying matroid. This is a well known theorem regarding evaluations of the Tutte polynomial $T(M; x, y)$. It makes explicit the relation between acyclic orientations and the number of topes of a matroid. This result is attributed to Zaslavsky and Las Vergnas, who proved it independently, in [BLVS+99]. [1]

Theorem 3.8 (Las Vergnas, Zaslavsky) *The number of topes in an arrangement of pseudohyperplanes, associated with a matroid M, of rank r is:*

$$(-1)^r T(M; 2, 0).$$

□

With these results, our proof of Theorem 3.9 shows that, with positive probability, there is an orientation of any PSC representing $U_{r,n}$, with bounded discrepancy of all

[1] Zaslavsky's proofs appear in his Doctoral thesis, which was also published in [Zas75].

its topes (rather than its vertices). This bound then implies a bound on discrepancy of vertices and therefore, a bound on imbalance and flow number.

Theorem 3.9 (Chávez Lomelí, Goddyn) Asymptotic Behaviour of $\phi(\mathbf{U_{r,n}})$

Let r be a fixed, positive integer. Then, for each $\varepsilon > 0$ there is a natural number n_0 such that if $n > n_0$, then $U_{r,n}$ admits an orientation satisfying :

$$max\left\{\frac{|B|}{|B^-|}, \frac{|B|}{|B^+|}\right\} \leq 2+\varepsilon.$$

Thus $\phi_o(U_{r,n}) \leq 2+\varepsilon$.

Proof: Recall that, by Lemma 2.28, for uniform matroids: $\max|\delta(T)| = \max|\delta(B)| + (r-1)$, and that $\text{imbal}(B) \leq b$ if and only if $|\delta(B)| \leq \frac{b-2}{b}|B|$. Thus $\phi_o(U_{r,n}) \leq b$ if and only if

$$|\delta(T)| \leq \frac{(b-2)n + 2(r-1)}{b}$$

for every tope T. Taking $b = 2+\varepsilon$, we require

$$|\delta(T)| \leq \frac{\varepsilon n + 2(r-1)}{2+\varepsilon}.$$

Given $U_{r,n}$ we orient the elements independently at random. For a given tope T, the discrepancy of T, $\delta(T)$ is a random variable determined by the outcome of the n independently oriented elements. Let X_T denote this variable and note that the expected value of all these variables is 0, since for any orientation, with $X_T = d$ there is another orientation for which $X_T = -d$.

We must show that, with positive probability, all the variables X_T are (simultaneously) bounded

$$X_T \leq \frac{\varepsilon n + 2(r-1)}{2+\varepsilon}. \tag{3.1}$$

That is, we must show that $Pr(\cap_T A_T) > 0$, where A_T is the event that the discrepancy of a tope T is bounded by expression (3.1). We show, equivalently, that:

$$Pr(\overline{\cap_T A_T}) = Pr(\cup_T \overline{A_T}) \leq \sum_T Pr(\overline{A_T}) < 1.$$

This is accomplished in two stages. First we show that the number of terms in the previous sum is not too large. Then we show that each term is not too large.

Applying Theorem 3.8 to the matroid $U_{r,n}$, we obtain:

$$T(U_{r,n};2,0) = \sum_{X\subseteq E}(-1)^{|X|}(-1)^{r(M)-r(X)}$$

$$\begin{aligned} T(U_{r,n};2,0) &= \sum_{i=0}^{r}\binom{n}{i}(-1)^{i}(-1)^{r-i}+\sum_{i=r+1}^{n}\binom{n}{i}(-1)^{i} \\ &= \sum_{i=0}^{r}\binom{n}{i}(-1)^{r}+\sum_{i=r+1}^{n}\binom{n}{i}(-1)^{i} \\ &= \sum_{i=0}^{r-1}\binom{n}{i}[(-1)^{r}+(-1)^{i+1}], \end{aligned}$$

which is bounded above by a polynomial in n of degree $r-1$.

It remains to show $Pr(\overline{A_T}) < \frac{1}{n^r}$ for any tope T. Notice that changing the choice of orientation for any element, changes the value of X_T by at most 2, thus we can apply Theorem 3.7 with $c=2$, and n trials and obtain

$$Pr(|X_T| > \frac{\varepsilon n + 2(r-1)}{2+\varepsilon}) \leq 2e^{-\frac{[\varepsilon n+2(r-1)]^2}{8n(2+\varepsilon)^2}}.$$

We now require that, for sufficiently large n,

$$2e^{-\frac{[\varepsilon n+2(r-1)]^2}{8n(2+\varepsilon)^2}} \leq \frac{1}{n^r},$$

which holds for fixed r since exponentials grow faster than any polynomial.

□

An interesting observation is that the odd cogirth bound for matroids in this class is also approaching 2, so this result is consistent with Jaeger's flow conjecture [Jae84].

Conjecture 3.10 (Jaeger) *If G is a $4k-$edge connected graph, then $\phi_c(G) \leq 2+\frac{1}{k}$.*

One might hope to extend Theorem 3.9 to other classes of matroids with large cogirth. However, this is not possible in general. There exist graphs with arbitrarily high girth and chromatic number. This implies the existence of *cographic matroids* (also orientable, of course) with arbitrarily large cogirth and flow number.

3.2 Bounds and discrepancy

As mentioned before, the connection between discrepancy and imbalance was successfully used by Edmonds and McNulty [EM04] to show that the orientation flow number of a rank-3 oriented matroid is bounded by 4. The strategy used in their paper can be generalized to higher ranks with some success. Again, we consider discrepancy of topes and, from this, derive a bound on imbalance of cocircuits.

The solution of the rank-3 case, required case-by-case analysis of matroids with fewer than 12 elements. This approach is not feasible in the general rank case, as the number of basis configurations to be analyzed soon becomes too large. This is one of several problems encountered when generalizing their ideas to higher rank. Nevertheless, in this section, we pursue this technique which does yield a bound for the case of uniform matroids of arbitrary rank. Using the uniform case as a basis we can find an upper bound for the discrepancy of topes in arbitrary matroids.

3.2.1 Uniform Matroids

Lemma 3.11 *The matroid $U_{r,r+1}$ admits an orientation such that $|\delta(T)| \leq r - 1$ for every tope T.*

Proof: First note that the $r+1$ elements of $U_{r,r+1}$ form a circuit. Thus, this matroid is uniquely representable (it is, in fact, graphic) and it has only one orientation class. Thus, viewed as a circuit, C, these elements can be oriented (cyclically) so that $\vec{C}$

is $(+, +, \ldots, +)$. This orientation of the corresponding PSC is such that the union of all positive half-spaces covers the entire space. Thus no tope F', gets covector $(+, +, \ldots, +)$ or $(-, -, \ldots, -)$, and max $|\delta(F')| \leq r - 1$, as required.

□

This lemma can be used to find an orientation of $U_{r,n}$, for any r and n, where the discrepancy of all topes is bounded.

Lemma 3.12 *Any orientation class of $U_{r,n}$ admits an orientation such that, for every tope T, $|\delta(T)| \leq k(r-1) + \rho$ where k and ρ are integers satisfying $n = k(r+1) + \rho$, with $k > 0$ and $0 \leq \rho \leq r$.*

Proof: Let $\mathcal{O}$ be an arrangement of pseudo hyperplanes representing $U_{r,n}$ with $n = k(r+1) + \rho$, for some integers k and $0 \leq \rho \leq r$. The elements of $\mathcal{O}$ can be partitioned into k circuits, $C_1, \ldots, C_k$ (all of size $r+1$) and an independent set of size ρ. According to Proposition 3.11, each such circuit can be oriented so that the discrepancy of every tope in the subconfiguration induced by the elements in the circuit satisfies: $|\delta(T)| \leq r - 1$. We have oriented the set $S = \cup_{i=1}^{k} C_i$.

Since the discrepancy of a tope in the matroid is the sum of the discrepancies of those topes containing it, in the sub-configurations determined by any partition, this bounds the discrepancy of any tope in $\mathcal{O}|_S$, by $k(r-1)$. The remaining elements can be oriented arbitrarily adding at most ρ to the discrepancy of any tope, and giving a total bound of $k(r-1) + \rho$.

□

This bound on the discrepancy of topes, results in the following bound on the flow number.

Theorem 3.13 (Chávez Lomelí, Goddyn, McNulty) *For any rank r and number of elements n,* $\phi_o(U_{r,n}) \leq r+2$.

Proof: By Lemma 3.12, we can produce an orientation of any PSC representing $U_{r,n}$ satisfying $|\delta(T)| \leq k(r-1)+\rho$ for all topes T. Lemma 2.28 then implies that $|\delta(v)| \leq k(r-1)+\rho-r+1 = n-2k-r+1 = |B|-2k$.

Finally, the relation between discrepancy and imbalance:

$$imbal(B) = 2 + \frac{2|\delta(B)|}{|B|-|\delta(B)|},$$

gives:

$$imbal(B) \leq 2 + \frac{2(|B|-2k)}{|B|-(|B|-2k)} = \frac{|B|}{k}.$$

Since $|B| = n-r+1$ and $\rho \leq r$, we have that:

$$imbal(B) \leq r+1+\frac{1}{k},$$

and we conclude that $\phi_o(U_{r,n}) \leq r+2$.

□

3.2.2 A Bound on Flow Number

Theorem 3.13 is another instance of obtaining a bound for flow number by orienting circuits cyclically. A careful choice of the circuits we orient produces the following bound which applies to arbitrary matroids and improves on the previously known bound of $14r^2 \log r$ given in [GHH06]. The argument mostly relies on matroid concepts, rather than oriented matroid theory, and is surprisingly short. It uses the following result on "removable circuits" from [LO99]. A circuit C in a connected matroid M is called *removable* if $M \setminus C$ is connected.

Definition 3.14 *The* circumference *of a matroid* M *is the length of the largest circuit in* M *and is denoted* $c(M)$.

Theorem 3.15 (Lemos,Oxley) *Let M be a connected, rank-r matroid with at least two elements and let C' be a largest circuit of M. If $|E(M)| \geq 3(r+1) - c(M)$, then M has a removable circuit C that is disjoint from C' such that rank$(M \setminus C) = r$.*

In the graphic case, when a circuit C is removed from a graph G, for any cocircuit B, the set $B \setminus C$ is an edge-cut in the resulting graph $G' = G \setminus C$. The set $B \setminus C$ may not be a minimal cut (ie a cocircuit in the matroid sense) but it partitions into cocircuits. In the general matroid case, we require the following lemma:

Lemma 3.16 *Let M be an orientable matroid, B a cocircuit and C a circuit of M. Let $M' = M \setminus C$. Then $B \setminus C$ contains a cocircuit of M'.*

Proof: Let $\mathcal{O}$ be a PSC representing M and v_B be the vertex of $\mathcal{O}$ corresponding to a cocircuit B. Let $\mathcal{O}'$ be the PSC obtained after deleting a circuit C from $\mathcal{O}$. Thus there is a cell R of $\mathcal{O}'$ that contains the point v_B. If R is a vertex, it corresponds to a cocircuit of M' contained in $B \setminus C$, as required. Otherwise, let v be a vertex incident with R. This vertex corresponds to a cocircuit in M'. Since v is in R, every element in $\mathcal{O}'$ incident with v is also incident with v_B in $\mathcal{O}$. This means that the cocircuit corresponding with v, which is formed by the elements *not incident* with v is contained in $B \setminus C$.

□

Theorem 3.17 (Chávez Lomelí, Goddyn, Hochstättler) Bound on Flow Number

Let M be a simple, connected, orientable, rank-r matroid, on n elements and $\mathcal{O}$ a reorientation class of M. Then $\phi_o(\mathcal{O}) \leq 3r$.

Proof: First note that a bound for $\phi_o(\mathcal{O})$ is the size of the largest cocircuit, $\max|B|$, which is at most $n - r + 1$. This follows from the assumption that $\mathcal{O}$ is connected and the fact that it is always possible to find a *totally cyclic orientation* [BLVS+99, p. 122]. Such an orientation satisfies $imbal(B) < \infty$, for all cocircuits B. Thus $\phi_o(\mathcal{O}) \leq n - r + 1$ and if $n < 3r$ we have $\phi_o(\mathcal{O}) \leq 2r$. Thus we need only consider matroids with at least $3r$ elements.

If $n \geq 3r$ then, since M is simple, $c(M) \geq 3$ and we have $n \geq 3r + 3 - c(M)$ as required in Theorem 3.15. Thus M contains a removable circuit, C, which can be oriented cyclically and removed from M, leaving a connected matroid M', with fewer elements than M. This process can be repeated until we have a collection of circuits $C_1, \ldots, C_k$ and a remaining, connected submatroid L with fewer than $3r$ elements, which can be oriented so that all its cocircuits have imbalance at most $2r$.

It remains to show that the cyclic orientation on the C_i extends the orientation of L to an orientation of M with imbalance of all cocircuits of M bounded by $3r$ also. The elements of a cocircuit B of M are partitioned into sets $B_1, \ldots, B_k, B_L$, where $B_i = B \cap C_i$, which also partitions B^+ and B^- in the obvious way. Thus we can talk about the imbalance of B_i; $i = 1, \ldots, k, L$ and we have that $imbal(B) \leq \max\{imbal(B_1), \ldots imbal(B_k), imbal(B_L)\}$. Thus it suffices to show that the imbalance of each of the B_i is bounded by $3r$.

Any cocircuit $B \in \mathcal{C}^*(M)$ intersecting a circuit C_i, must do so in at least two elements with agreeing and opposing orientation, due to orthogonality. Thus the imbalance of each $B_i; i = 1 \ldots k$ is bounded by: $|C| \leq r + 1$. If B_L is non-empty, by Lemma 3.16, B_L contains a cocircuit of L, and thus B_L has finite imbalance and $imbal(B_L) \leq |B_L| < 3r$.

□

Thus, provided we can construct removable circuits, we have a method that will orient any orientation class of a matroid with bounded imbalance of cocircuits. This method partitions the element set into circuits and a "left-over" set, very much like the orientation of uniform matroids in Theorem 3.13. As mentioned before, this has developed into a recurring theme in this thesis, even though the research was never conceived in these terms.

Chapter 4

Sixth Root of Unity Matroids

In the Chapter 2, we defined a measure of discrepancy, akin to that of regular matroids, for the $\sqrt[6]{1}$-matroids class (Definition 2.38). This rests on the existence of a unique $\sqrt[6]{1}$-vector (up to scaling) associated with each circuit and cocircuit. Thus if B is a cocircuit of a $\sqrt[6]{1}$-matroid, represented by a $\sqrt[6]{1}$-valued vector v, the discrepancy of B, is defined as $\delta(B) = \sum_{e \in E(M)} v(e)$. By this definition, discrepancy is a complex number. It would seem appropriate to have a real number as a measure of imbalance. This can be accomplished considering the norm of $\delta(B)$, however, in the spirit of preserving the parallels and in order to remain consistent, we keep this as a definition. Moreover, a definition of δ as $\| \sum_{e \in E(M)} v(e) \|$ cannot be *additive*, as it was the case with orientation numbers.

Alternatively, we can *normalize* $\delta(B)$, dividing by the number of elements in B, thus obtaining an average or, in a more physical interpretation, a measure of *center of mass* for each cocircuit or circuit in the matroid. Formally:

Definition 4.1 *For each cocircuit B in a $\sqrt[6]{1}$-matroid M, the* center of mass *of B is defined as:*

$$\mu(B) = \frac{1}{|B|}\sum_{e\in B} B_e$$

A reorientation of a regular matroid $M[A]$ is obtained by multiplying a set of columns of A by -1 (and all others by 1). We can consider a *reorientation* of a $\sqrt[6]{1}$-matroid $M[A]$, to be obtained by multiplying each column of A by some power of $w = e^{i\pi/3}$. Thus the two matrices in Example 2.32 represent different orientations of $U_{2,4}$. The columns of A are multiplied by $w^4, w^4, w^2, 1$, respectively to obtain A'. We may write $A' = AD$ where $D = diag(w^4, w^4, w^2, 1)$.

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & w \end{bmatrix} \qquad A' = \begin{bmatrix} -w & 0 & w^2 & 1 \\ 0 & w^4 & w^2 & w \end{bmatrix} \tag{4.1}$$

Definition 4.2 *Under the above definition of re-orientation, let the* orientation flow number, *denoted $\phi_\mu(M)$ of a $\sqrt[6]{1}$-matroid be:*

$$\phi_\mu(M) = \min_{\vec{A}}\left\{\max_{B\in\mathcal{C}^*}\{\|\mu(B)\|\}\right\},$$

where the minimum is taken over all reorientations $\vec{A}$ of $M[A]$.

Note that different measures may be obtained simply taking alternatives to the norm of $\mu(B)$. Note also that ϕ_μ takes values in the interval $[0,1]$; 0 corresponding to what will be called an Eulerian orientation and 1 for an orientation containing a *totally unbalanced cocircuit.*

Thus for $\sqrt[6]{1}$-matroids not only do we have an algebraic flow number, but also an orientation flow number. In this chapter, we discuss these concepts. We obtain a partial connection between the algebraic and orientation parameters (Theorems 4.3 and 4.4). We define "Eulerian" matroids for this class, called *hex-Eulerian.* Under this definition, we obtain a characterization of regular hex-Eulerian matroids and show that maximum-sized $\sqrt[6]{1}$-matroids are hex-Eulerian (Theorems 4.9 and 4.8). Finally we describe an extension of Tutte's regular chain groups that describes $\sqrt[6]{1}$-matroids(Theorem 4.23).

4.1 Flows on Sixth Root of Unity Matroids

Because of Definitions 2.33 and 4.2, the class of $\sqrt[6]{1}$-matroids becomes a very interesting one for this thesis. Whenever it is possible to define an algebraic and an orientation flow number, a question of primary interest is whether we can generalize Hoffman's circulation lemma (Theorem 2.17) which states that the algebraic and orientation based parameters coincide for regular matroids.

The following result provides a first glimpse into this issue. It establishes that it is not possible for one of these parameters to be *"unbounded"* while the other is *"bounded"*. In the case of ϕ_μ, by bounded we mean $\phi_\mu < 1$, while unbounded corresponds to the case $\phi_\mu = 1$.

Theorem 4.3 (Chávez Lomelí) *Let $M = M[A]$ be a $\sqrt[6]{1}$-matroid represented by the $\sqrt[6]{1}$-matrix A. The following statements are equivalent:*

1. *M is coloop-free.*
2. *$\phi_\mu(M) < 1$, that is, M admits an orientation with no totally unbalanced cocircuit.*
3. *$\phi_a(M) < \infty$, that is, M admits a nowhere-zero algebraic flow.*
4. *$\phi_{\mathbb{E}}(M) < \infty$, that is M admits an Eisenstein-integer-valued nowhere zero flow.*

Proof: The implications (2) $\Rightarrow$ (1) (and (3) $\Rightarrow$ (1) and (4) $\Rightarrow$ (1)) are immediate since the existence of a coloop bars all possibility of balancing orientations (or of finding a nowhere zero vector in the null space of A). The implication (4) $\Rightarrow$ (3) is trivial. Thus we prove the result by verifying the implications (1) $\Rightarrow$ (4) and (3) $\Rightarrow$ (2). This shows:

$$\begin{array}{ccc} (1) & \Rightarrow & (4) \\ \Uparrow & & \Downarrow \\ (2) & \Leftarrow & (3) \end{array}$$

(1) $\Rightarrow$ (4) Let $M = M[A]$ be a coloop free $\sqrt[6]{1}$- matroid. It must be shown that M admits a nowhere zero Eisenstein flow, that is, that there exists a nowhere zero vector in the kernel of A having all coordinates in $\mathbb{E}$. A basis for the kernel of A can be easily obtained through Gauss elimination. Indeed, since matrix pivoting can be done within a partial field [SW96], we may assume that A is of the form:

$$\left(\begin{array}{c|c} I_r & B \end{array} \right).$$

Thus $Ker(A)$ is the span of the vectors:

$$\left\{ \begin{bmatrix} -B^1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} -B^2 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} -B^{n-r} \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right\},$$

where B^j is the $j-th$ column of B. If we denote the vectors above by $u_1, \ldots, u_{n-r}$, then f is a flow if and only if it is of the form: $f = t_1u_1 + \cdots t_{n-r}u_{n-r}$ for some choice of $t_1, \ldots, t_{n-r}$. Now f is nowhere zero if and only if $t_j \neq 0$ for all j and the $(n-r)$-tuple $(t_1, \ldots, t_{n-r})$ is *not* a solution to any of the equations:

$$B_{i1}t_1 + B_{i2}t_2 + \cdots + B_{in-r}t_{n-r} = 0.\ i = 1, \ldots, r. \tag{4.2}$$

We describe how to find non-zero values for t_j; $j = 1, \ldots, n-r$ which are not solutions to any of the equations (4.2) above. The set of equations (4.2), can be partitioned into $n-r$ classes: $[1], [2], \ldots, [n-r]$. Equation i is in class $[j]$ if and only if $B_{i,j}$ is the first non-zero entry in row i of B. Note that B has no zero row, because M is coloop-free. Now, we can iteratively assign Eisenstein integer (or indeed real integer) values to the coefficients t_j, in class $[n-r], [n-r-1], \ldots, [1]$ in this order. Letting $t_{n-r} = 1$ ensures that the equations in class $[n-r]$ are all violated. Continuing then to set values for t_j so that, at each step, all equations in class $[j]$ are violated is not only possible, but straight forward. Equations in classes $[k]$; $k \geq j$ are not affected by our choices of t_j, by our construction of these classes. By our choice of coefficients, all these flow values are Eisenstein integers, which concludes the proof.

(3) $\Rightarrow$ (2) Let f be a nowhere-zero α-flow of $M[A]$, that is, f is a vector satisfying $Af = 0$ and $1 \leq \|f_e\| \leq \alpha - 1$ for all $e \in E(M)$. We must show that there exists a reorientation of M, ie a $\sqrt[6]{1}$-matrix A' representing the same matroid and such that, for each cocircuit B, $\|\mu(B)\| < 1$.

Suppose, towards a contradiction, that this is not the case, that is M admits an algebraic nowhere-zero α-flow f, but for all reorientations A' of A, there exists a cocircuit B such that $\|\mu(B)\| = 1$. Consider now that reorientation of M obtained by multiplying each column j of A by the power of w, w^k, whose argument satisfies $|arg(f_e) - arg(w^k)| \leq \pi/6$. The resulting orientation admits an algebraic flow f' with the property that all entries f'_e have argument between $-\frac{\pi}{6}$ and $\frac{\pi}{6}$.

According to our hypothesis, there must be a cocircuit B which is *totally unbalanced* that is, all non-zero entries in the vector, v_B, representing it are equal. We may assume that they are all 1. Thus:

$$f \cdot v_B = \sum_{e \in B} f_e.$$

The left hand side in the equation above must be 0 if f is a flow, however, the right hand side cannot be 0 as it is the sum of complex numbers, with norm at least 1 and satisfying:

$$-\frac{\pi}{6} \leq arg(f_e) \leq \frac{\pi}{6}.$$

This sum can be interpreted as the sum of 2 dimensional, real vectors of norm at least 1, in a positive cone with vertex angle $\frac{\pi}{3}$. Clearly their sum must lie in the intersection of the cone with the set $\|z\| \geq 1$ and hence, cannot be zero.

□

The equivalence between conditions (1) and (3) in Theorem 4.3 extends to matroids represented over all "large enough" fields or partial fields. However it does not necessarily hold in the case of "small" finite fields or finite partial fields. For example the matroid $U_{2,4}$, is represented over $GF(3)$ by the matrix:

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{pmatrix}.$$

The flow space is spanned by the vectors:

$$\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Since the only possible coefficients in any linear combination are $0, \pm 1$, the entire flow space is formed by the following vectors and their negatives:

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\},$$

a set which contains no nowhere zero vector.

4.2 Toward a Hoffman Lemma

Hoffman's Circulation Lemma states that there is a nowhere-zero flow, f on M, with $1 \leq |f_e| \leq \alpha - 1$ for all e in M, if and only if there is an orientation $\vec{M}$, with $\left\{\frac{|B|}{|B^-|}, \frac{|B|}{|B^+|}\right\} \leq \alpha$, for all cocircuits B of M. In particular, we have that $\phi_a(M) = \phi_c(M)$ for regular matroids. We would like to show that there is an analogous connection between two similarly defined flow numbers for $\sqrt[6]{1}$-matroids, M. That is, that there ought to be a nowhere-zero complex valued flow on a $\sqrt[6]{1}$-matroid $M[A]$, with $1 \leq \|f_e\| \leq \alpha - 1$ if and only if the $\sqrt[6]{1}$-vectors associated with the cocircuits of M, v_B are "not too unbalanced" and vice versa.

The proof that a regular matroid admitting an algebraic flow with bounded entries must have an orientation with bounded imbalance on all cocircuits constructs such an orientation from a flow, f, with positive entries (the proof of Theorem 2.17 in Chapter 2 shows this argument for the graphic case). This orientation is then shown to have bounded imbalance by analyzing each of the terms in the flow conservation equation:

$$\sum_e v_B(e) f(e) = 0, \tag{4.3}$$

grouped according to their orientation (or sign) in $\vec{B}$. That is, splitting (4.3) into:

$$\sum_{e:v_B(e)=1} f(e) - \sum_{e:v_B(e)=-1} f(e) = 0, \tag{4.4}$$

the bounds on $|f|$ conclude the argument. Thus, the proof hinges on having a clear understanding of how flow values affect Equation (4.3) under the orientation of each cocircuit B. We would like to do this kind of analysis for the $\sqrt[6]{1}$–case.

Recall that the imbalance of a cocircuit B in a regular matroid $M[A]$ is given by the greater of the ratios $|B|/|B^+|$ or $|B|/|B^-|$. A totally oriented cocircuit will have one of the sets B^+, B^-, empty and "infinite imbalance". In the $\sqrt[6]{1}$-case, a cocircuit B, associated with a $\sqrt[6]{1}$-vector v_B, can be partitioned into six classes; $B^0, B^1, \ldots, B^5$, indeed $(B^0, B^1, \ldots, B^5)$ is an ordered partition, and this order is unique up to cyclic permutation. These correspond to six possible values that can appear in a $\sqrt[6]{1}$-vector representing an oriented cocircuit $\vec{B}$, namely $1, w, \ldots, w^5$. The notion of imbalance resulting from this argument is, naturally, more involved that for regular matroids. A cocircuit B in a $\sqrt[6]{1}$-matroid may be perfectly balanced, in the sense that $\sum v_B(e) = 0$ and yet, one, or more of the classes B^i may be empty. In the regular case, empty classes only appear on totally oriented cocircuits.

If a $\sqrt[6]{1}$-matroid admits a nowhere-zero flow, with all coordinates satisfying $1 \leq \|f(e)\| \leq \alpha - 1$, $|arg(f(e)| \leq \pi/6$, then an entry in v_B with value 1, for example, results in a contribution to the sum (4.3), a complex number lying in the shaded region illustrated in Fig. 4.1. By analogy, we say that a flow with $|arg(f(e)| \leq \pi/6$ is a *positive complex flow*. Any $\sqrt[6]{1}$-flow can be converted to a positive one by reorienting its representation as shown for $U_{2,4}$ (matrices 4.1) in the opening of this chapter.

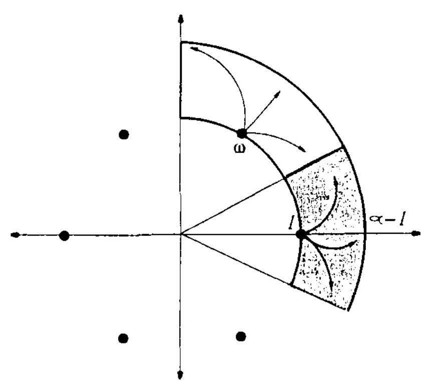


Figure 4.1: Arrows indicate some possible transformations of the form:
$v_B(e) \rightarrow f(e)v_B(e)$ where f is a positive bounded flow.

Having determined the possible effect of a bounded positive flow value on an oriented cocircuit, we can continue our analysis of the sum in Equation 4.3. We partition its terms according to the orientation of each cocircuit, as in the regular case.

We consider the real part and the imaginary part of complex numbers separately, starting with the real part. The corresponding orthogonality condition here, says that if the number of entries with negative real part is large, in proportion to those with positive real part, then a positive complex flow must take values with large modulus on the positive entries, resulting in a term of (4.3) with large, positive real part. Similarly for the imaginary part. Fig. 4.2 shows how a bounded, positive flow can best compensate an orientation with many more elements with negative real part (oriented w^2, w^3, w^4) than with positive real part. A flow must "shift" the elements of any cocircuit to be symmetrically placed about zero. Both the real and imaginary parts of the sum must be zero.

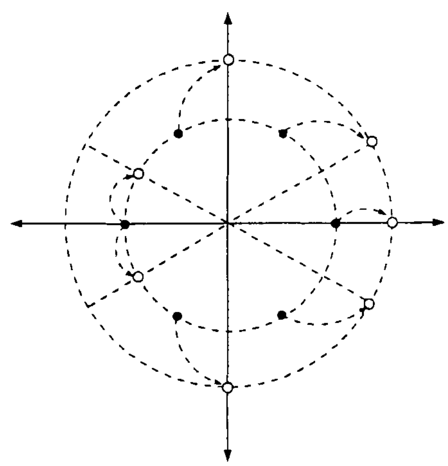


Figure 4.2: The most a bounded positive flow can balance a cocircuit.

The real part of a sixth root of unity is one of $-1, -\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 1$. We partition the set $W = \{w^i | i = 0, \ldots, 5\}$ into classes, accordingly.

$$W^+ = \{1\},\ W^{\oplus} = \{w, w^5\},\ W^{\ominus} = \{w^2, w^4\},\ W^- = \{-1\}, \tag{4.5}$$

Thus a cocircuit B, corresponding to a $\sqrt[6]{1}$-vector v_B, is partitioned into sets $B^+ \cup B^{\oplus} \cup B^{\ominus} \cup B^-$, defined as:

$$B^{\odot} = \{e \in B \mid v_B(e) \in W^{\odot}\}, \tag{4.6}$$

where $^{\odot}$ denotes any of $^{+}$, $^{-}$, $^{\oplus}$, $^{\ominus}$.

Given a $\sqrt[6]{1}$-matroid $M = M[A]$, admitting a flow f with $1 \leq \|f_e\| \leq \alpha - 1$ and $|arg(f_e)| \leq \pi/6$, for each $e \in E$, for any cocircuit, B, we have that $\sum_{e \in E} v_B(e) f_e = 0$, and we know that $v_B(e)$ can only take values in W. Thus we can evaluate the contribution of each term to the real part of this sum, and this contribution depends only on membership into the classes $B^{\odot}$. Denoting the real part of a complex number z by z', and splitting this sum according to our partition of B, it must be the case that:

$$\sum_{e \in B^+ \cup B^{\oplus}} (f_e v_B(e))' \leq \sum_{e \in B^+} \|f_e\| + \frac{\sqrt{3}}{2} \sum_{e \in B^{\oplus}} \|f_e\|$$

and, likewise:

$$\sum_{e \in B^- \cup B^{\ominus}} (f_e v_B(e))' \leq -\frac{\sqrt{3}}{2} \sum_{e \in B^{\ominus}} \|f_e\|.$$

Adding the above inequalities, we obtain:

$$0 \leq \sum_{e \in B^+} \|f_e\| + \frac{\sqrt{3}}{2} \sum_{e \in B^{\oplus}} \|f_e\| - \frac{\sqrt{3}}{2} \sum_{e \in B^{\ominus}} \|f_e\|.$$

Thus

$$|B^{\ominus}| \leq \sum_{e \in B^{\ominus}} \|f_e\| \leq \frac{2}{\sqrt{3}} \sum_{e \in B^+} \|f_e\| + \sum_{e \in B^{\oplus}} \|f_e\| \leq (\alpha - 1)\left(\frac{2}{\sqrt{3}}|B^+| + |B^{\oplus}|\right),$$

$$\alpha - 1 \geq \frac{|B^{\ominus}|}{\frac{2}{\sqrt{3}}|B^+| + |B^{\oplus}|}$$

$$\alpha - 1 \geq \frac{\sqrt{3}|B^{\ominus}|}{2|B^+| + \sqrt{3}|B^{\oplus}|}.$$

The analysis regarding the imaginary part, results in the following partition of $W = W^- \cup W^0 \cup W^+$, with:

$$W^+ = \{w, w^2\},\ W^0 = \{1, -1\},\ W^- = \{w^4, w^5\}. \tag{4.7}$$

If z'' denotes the imaginary part of z and B denotes a cocircuit, then we have:

$$\sum_{e \in B^+ \cup B^0} (f_e v_B(e))'' \leq \sum_{e \in B^+} \|f_e\| + \frac{1}{2} \sum_{e \in B^0} \|f_e\|$$

and similarly:

$$\sum_{e \in B^-} (f_e v_B(e))'' \leq -\frac{1}{2} \sum_{e \in B^-} \|f_e\|.$$

Adding the above inequalities, we obtain:

$$0 \leq \sum_{e \in B^+} \|f_e\| + \frac{1}{2} \sum_{e \in B^0} \|f_e\| - \frac{1}{2} \sum_{e \in B^-} \|f_e\|.$$

Thus

$$|B^-| \leq \sum_{e \in B^-} \|f_e\| \leq 2 \sum_{e \in B^+} \|f_e\| + \sum_{e \in B^0} \|f_e\| \leq (\alpha - 1)(2|B^+| + |B^0|).$$

$$\alpha - 1 \geq \frac{|B^-|}{2|B^+| + |B^0|}.$$

The above arguments show the following:

Theorem 4.4 *Given a $\sqrt[6]{1}$-matroid $M = M[A]$, if there exist a real number $\alpha \geq 1$, and a nowhere zero flow $f \in \Gamma$ satisfying $1 \leq \|f\| \leq \alpha - 1$ with $|arg f(e)| < \pi/3$, for all $e \in E$, then for every cocircuit B, represented by the $\sqrt[6]{1}$-vector v_B of M, we have:*

$$\frac{|B^-|}{2|B^+| + |B^0|} \leq \alpha - 1$$

where B^+, B^0 and B^- are defined according with (4.7) and:

$$\frac{\sqrt{3}|B^{\ominus}|}{2|B^+| + \sqrt{3}|B^{\oplus}|} \leq \alpha - 1$$

where $B^-, B^{\ominus}, B^{\oplus}$ and B^+ are defined according to (4.5). Furthermore, these two inequalities must be satisfied by all six signings $w^k v_B$; $k = 0, \ldots, 5$ of B.

□

Conjecturally a converse to Theorem 4.4 also holds. Theorem 4.4 shows a connection between the algebraic flow number of $\sqrt[6]{1}$-matroids and *some* notion of imbalance.

What we would like to find is a strong enough notion of imbalance under which a converse Hoffman-like statement holds. We would like a result of the form: *If in some orientation, every cocircuit of M is well balanced, then $\phi_a(M)$ is small.* The question is, what invariant (if any) of a six-tuple $(B^0, \ldots, B^5)$ should play the role that $\max\left\{\frac{|B|}{|B^+|}, \frac{|B|}{|B^-|}\right\}$ does for regular matroids? Furthermore, we would like to know how accurately can we quantify "well balanced" and "small $\phi_a(M)$" here.

Unfortunately, we do not have answers to these questions. However, in our search for an answer, we considered an extension of *regular chain groups* to $\sqrt[6]{1}$-matroids, which appears to be interesting on its own right. As such, it is included at the end of this chapter, with such insights into these questions as we have gained in the process.

4.3 Hex-Eulerian Matroids

As in the case of orientation numbers, it is also possible to define *Eulerian* matroids in terms of ϕ_a as follows:

Definition 4.5 *A $\sqrt[6]{1}$-matroid, $M[A]$ is* hex-Eulerian *if there is a $\sqrt[6]{1}$-valued vector f in the null space of A, and we call such a flow, f, a* unit flow *on M.*

Equivalently, we say that M is hex-Eulerian if it admits a hex-orientation A' with the property that the vector $f = (1, \ldots, 1)^T$ is in the kernel of A'.

Note that hex-Eulerian can also be defined in terms of $\phi_\mu(M)$. A matroid is hex-Eulerian if and only if $\phi_\mu(M) = 0$.

4.3.1 Dense Hex-Eulerian Matroids

Examples of hex-Eulerian matroids include those described by Oxley, Vertigan and Whittle [OVW98] as *maximum-sized* among simple $\sqrt[6]{1}$-matroids of a given rank,

$r \geq 3$. A graph theorist may find a parallel to the definition of maximum-sized in the case of graphic matroids where the matroid of the complete graph on n vertices has the most elements amongst all graphic matroids of rank $n-1$. In the $\sqrt[6]{1}$-case, these maximum-sized matroids are $AG(2,3)$ and those denoted $T_r, r \geq 1$ which we describe shortly.

Thus these maximum-sized matroids are hex-Eulerian, the exceptional cases being ranks $r = 1, 2$; with $T_1 = U_{1,1}$ and $T_2 = U_{2,4}$, which are not hex-Eulerian. For $r = 3$ the maximum sized $\sqrt[6]{1}$-matroid is $AG(2,3)$ on 9 elements, whereas T_3 contains only 8 elements. However, T_3 is also hex-Eulerian. The following hex-orientations of $AG(2,3)$ and T_3 are clearly hex-Eulerian, as the vector $f = (1, 1, \ldots, 1)^T$ is a flow. An interested reader can easily verify that all row sums are 0. $\vec{A}'_3$ is a representation of $AG(2,3)$ and $\vec{A}_3$ of the matroid T_3. The columns, in this order, correspond to the elements labeled as in Fig. 4.3.

$$\vec{A}'_3 = \begin{bmatrix} w & 0 & 0 & -1 & w & 0 & -1 & w^5 & w^5 \\ 0 & 1 & 0 & -1 & 0 & -1 & -1 & 1 & 1 \\ 0 & 0 & w^5 & 0 & w^2 & w^2 & -1 & 1 & w^5 \end{bmatrix}$$

$$\vec{A}_3 = \begin{bmatrix} w^2 & 0 & 0 & 1 & -1 & 1 & -w & 0 \\ 0 & w^2 & 0 & 1 & 0 & w^5 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 & -1 & 1 \end{bmatrix}$$

These hex-orientations were obtained via a nowhere zero hex-flow, which was constructed as a sum of flows on circuits, that is, flows with minimal support. This brings to mind the graph theoretical notion that a *circuit decomposition* results in a nowhere-zero unit flow. More precisely, that a graph is Eulerian if and only if its edge set can be partitioned into circuits. In the case of $\sqrt[6]{1}$-matroids, such a partition of the element set is not necessary. As we shall see, a *cover* by circuits, allowing "some limited" intersections may suffice. With such a cover, a nowhere zero flow is obtained

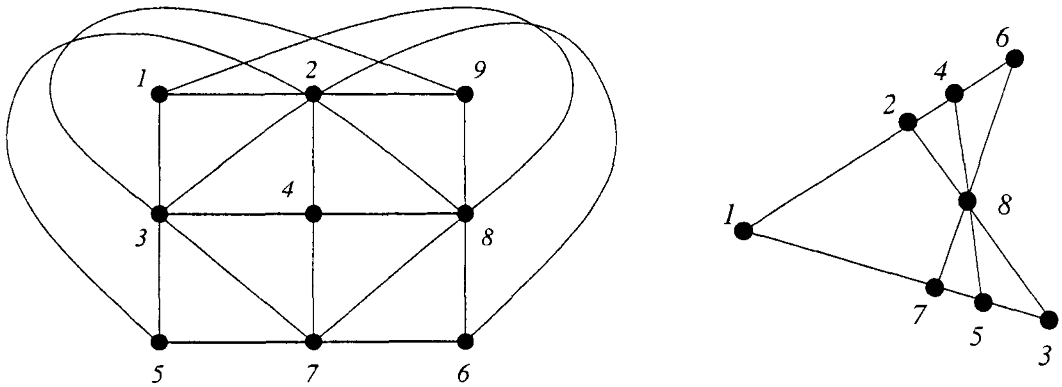


Figure 4.3: The matroids $AG(2,3)$ and T_3.

by adding flows with values in w^k; $k = 0, \dots, 5$ on these circuits. More precisely, we have the following lemma.

Lemma 4.6 *A $\sqrt[6]{1}$-matroid $M = (E, \mathcal{B})$ is hex-Eulerian if there is a family of circuits $C_1, \dots, C_p$ satisfying:*

1. $E = \cup_{i=1}^{p} C_i$.
2. *Each circuit C_i has a fixed element e_i such that for any $j \neq i$, $C_i \cap C_j \subsetneq \{e_i\}$.*
3. *The set of elements $\{e_1, \dots, e_p\}$ is independent.*

Proof: For each element e_i, consider the circuits $\mathcal{C}'_i = \{C_j : e_i \in C_j\}$. These circuits are represented by $\sqrt[6]{1}$-vectors $v_{C_j}, C_j \in \mathcal{C}'_j$ and we can assign a $\sqrt[6]{1}$-valued flow f_j on each of them, so that $\sum_{\mathcal{C}'} v_{C_j}(e_i) f_j(e_i) \neq 0$. Since e_i is the only element shared by C_i and C_j; $j \neq i$, the sum of these flows has support $\cup_{\mathcal{C}'} E(C_j)$. This can be accomplished independently for each e_i and the flow obtained as a sum of all these flows on circuits, is a nowhere-zero $\sqrt[6]{1}$-flow on M since the set $\{e_1, \dots, e_p\}$ is independent in M.

□

Note that Lemma 4.6 provides a sufficient (but by no means necessary) condition for a $\sqrt[6]{1}$-matroid M to be Eulerian in terms of the existence of a family of circuits in M.

The matrix representing T_r is defined as follows: Let D_r denote the $r \times \binom{r}{2}$ matrix whose columns are all $\{0, \pm 1\}$-valued vectors, of length r with exactly two non-zero entries, the first one being 1 and the second one -1. This is, of course, a representation of an orientation of the complete graph K_r, that is, a *tournament*. Then, the matroid T_r is represented by the matrix:

$$A_r = \begin{array}{c} \\ \\ \end{array} \begin{array}{c|cccc|cccc|cccc|cccc} \alpha & & \beta & & & & \Gamma & & & & \Delta & & & & \Lambda & & \\ \hline 1 & 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & w & w & \cdots & w & 0 & 0 & \cdots & 0 \\ \hline 0 & & & & & & & & & & & & & & & & \\ \vdots & & I_{r-1} & & & & I_{r-1} & & & & I_{r-1} & & & & D_{r-1} & & \\ 0 & & & & & & & & & & & & & & & & \end{array}$$

Thus T_r contains K_r as a minor, spanned by the elements represented by columns of A_r with first coordinate equal to 0. The elements in the last block of the matrix (Λ) correspond to a K_{r-1} and those in the second block (β) correspond to a star centered at a new vertex.

Theorem 4.7 (Oxley, Vertigan and Whittle) *Let M be a simple $\sqrt[6]{1}$-matroid of rank r. Then*

$$|E(M)| \leq \begin{cases} \frac{(r+2)(r+1)}{2} & \text{if} \quad r \neq 3; \\ 9 & \text{if} \quad r = 3. \end{cases}$$

Moreover, equality is attained in this bound if and only if $M \cong T_r$ when $r \neq 3$, or $M \cong AG(2,3)$ when $r = 3$.

Now we can show the following:

Theorem 4.8 (Chávez Lomelí, Goddyn) *For all $r \geq 3$, every max-sized $\sqrt[6]{1}$-matroid is hex-Eulerian.*

Proof: To show that T_r is hex-Eulerian, for all r, it suffices to produce a partition of the elements of T_r into sets E_1, E_2 such that E_1 is the support of a nowhere zero hex-flow, with values w^k, and the restriction of T_r to E_2 is the circuit matroid of an Eulerian graph which, as such, supports a nowhere zero, ± 1-valued flow. The set E_1 will be constructed by a set of circuits as in Lemma 4.6.

Case 1: r is odd. E_1 will be formed by the elements in blocks α, Γ, Δ and an even subgraph of K_r.

The elements in Λ span the matroid K_{r-1} which contains two edge disjoint perfect matchings. Let one of these matchings be the set of matroid elements in Λ with consecutive non-zero entries and having a positive entry in an even row of A_r. Each edge in this set, forms a circuit with two consecutive elements in block Γ. These circuits are pairwise disjoint and cover all of Γ. Following the same process, with the elements in the second matching, we can cover all of block β. Finally, take the circuit C formed by the first elements in blocks α, Γ and Δ, respectively. This circuit intersects one of those previously considered, call it C'. The corresponding *oriented* circuits are, up to taking w^k multiples,

$$\begin{aligned} C &= (1|0, \cdots, 0|w^4, 0, \ldots, 0|w, 0, \ldots, 0|0, \ldots, 0) \\ C' &= (0|0, \ldots, 0|1, -1, 0, \cdots, 0|0, \ldots, 0|-1, 0, \ldots, 0) \end{aligned}$$

The sum, $C + C' = (1|0, \ldots, 0|w^5, -1, 0, \cdots, 0|w, 0, \ldots, 0|-1, 0, \ldots, 0)$, of these two vectors, is a unit flow with *supp*$(C + C') =$ *supp*$(C) \cup$ *supp*(C'). The set E_1 contains all of Γ, Δ and the edges of two disjoint matchings in K_{r-1}. Thus if we let $E_2 = \overline{E_1}$, the *restriction* $M|_{E_3}$ is the cycle matroid of an Eulerian graph. The sum of a ± 1-flow on E_2 and F_1 is a nowhere zero unit flow on T_r.

Case 2: r is even. Here we rely on a maximum matching on K_{r-1} and construct a set E_1 covering this matching, α, Δ and Γ.

Let β_i denote the i−th column of block β, and extend this notation to blocks

Γ and Δ in the matrix. Let S be the set of elements in block Λ, with consecutive non-zero entries and positive entry in rows $2k+1$, with $k \geq 1$ (ie S is a maximum matching in K_{r-1}). Each one of these elements forms a circuit with two consecutive elements in blocks Γ and Δ. Thus we can construct a family of circuits such that every element in S is in exactly two of them. Since S is an independent set we can apply Lemma 4.6 to obtain a flow which is non-zero on the union of all these circuits. That is all Γ_k and Δ_k with $k \geq 2$. However, note that the set $\alpha, \Gamma_1, \Delta_1$ supports a circuit. Thus, with the addition of this circuit to our family (putting a flow of 1 on it) we have covered all of α, Δ and Γ.

The elements not yet covered do not span an Eulerian graph. The second row in the matrix A_r has an odd number of non-zero entries, corresponding to an odd degree vertex in a subgraph of K_r. Thus we require at least one more circuit to cover one element represented by a column of A_r with non-zero entry in the second coordinate. Take the pairs $\{\beta_1, \beta_2\}$ and $\{\beta_1, \beta_r\}$. Each of these pairs forms a circuit with a unique element from block Λ, not yet covered by circuits. Furthermore, this element, seen in the K_r-minor of T_r, is incident with a vertex not saturated by our maximum matching, thus it has odd degree. These circuits intersect only on the element β_1, thus we can apply Lemma 4.6 as before.

Define E_1 to be the union of the supports of all the circuits considered in the previous two paragraphs, and let $E_2 = \overline{E}_1$. Then E_1 contains all of Γ, Δ and an odd number of edges incident with each vertex of the K_r-minor of T_r, thus $M|_{E_2}$ is the graphic matroid of an even subgraph, as required.

□

Thus these matroids, which are the most dense in their class, are Eulerian. The idea that very dense matroids should have small flow numbers is appealing, as it would seem to suggest a possible connection between high density, larger cogirth and low flow number. In this case, without the constraints of parity, we have hex-Eulerian matroids, while in the graphic case, complete graphs are densest and are Eulerian only when they have odd order. Once again, this is consistent with Jaeger's Conjecture 3.10.

4.3.2 Regular Hex-Eulerian Matroids

As mentioned before, the class of $\sqrt[6]{1}$–matroids contains the class of regular matroids. The examples T_r and $AG(2,3)$ are not regular but there are, of course, hex-Eulerian regular matroids. We provide a characterization.

Theorem 4.9 (Chávez Lomelí, Goddyn) *A regular matroid M is hex-Eulerian if and only if M has a nowhere zero 3-flow.*

Proof: Let $M = M[A]$ be a regular matroid represented by the totally unimodular matrix A. If M is hex-Eulerian, then we can find a $\sqrt[6]{1}$-matrix $A' = AD$, with D a diagonal matrix, whose diagonal entries are all sixth roots of unity, and such that $f = (1, 1, \ldots, 1)^T$ is a flow, ie $A'f = 0$.

Consider the vector f' with entries in $\mathbb{Z}_3$ defined as follows: For each element $e \in M$ let $D_{e,e} = w^k$, $k = 0, \ldots, 5$ be the *weight* of e, then $f'_e = 1$ if k is even and 2 otherwise. We claim that this is a nowhere zero $\mathbb{Z}_3$-flow of $M = M[A]$.

Since $A'f = 0$, we know that

$$\sum_j A'_{i,j} = 0.$$

It is not hard to show inductively that for each row, we can partition the elements of M into parts of sizes 2 and 3 with the property that the entries in this row of the matrix corresponding to the elements in any one part, add to zero. These entries are given by the weights of the elements of M and they are of the form $X = (w^i, w^{i+3})$, $Y = (w^i, w^{i+2}, w^{i+4})$, where the sum in the exponent is taken mod 6.

Since the powers of w in X have different parity, the sum of flow values f' for the elements in this row is $1 + 2 = 0 \pmod 3$ similarly, the sum over Y is either $1+1+1 = 0 \pmod 3$ or $2+2+2 = 0 \pmod 3$, showing that we have a $\mathbb{Z}_3$-flow. It is straightforward that this flow is nowhere zero. By Theorem 2.8, M has a nowhere-zero 3-flow.

Conversely, let M be a regular matroid with flow number at most three. Since M has a NWZ-3-flow $\phi : \vec{E} \rightarrow \{\pm 1, \pm 2\}$, the restriction of M to the set $D = \phi^{-1}$ is Eulerian thus it supports a 2-flow ψ. Now consider $\{-\psi, \frac{\phi+\psi}{2}, \frac{\psi-\phi}{2}\}$. These are 2-flows which add to 0 and form a 2-cover of $E(M)$. Following standard methodology we take the flow $f = 0\psi + w\frac{\phi+\psi}{2} + w^2\frac{\psi-\phi}{2}$. Since each element appears in exactly 2 cycles and $-\psi + \frac{\phi+\psi}{2} + \frac{\psi-\phi}{2} = 0$, f takes values: $\pm w, \pm w^2, \pm 1$. We can now define the diagonal matrix D required, using these weights, as: $D_{e,e} = f(e)$. It follows that the sum over each row of A' is 0. Thus the all 1 vector is a flow and M is hex-Eulerian as required.

□

4.4 Eisenstein Flows and Tutte's Chain Groups

The idea of restricting flow values, for this class, to Eisenstein integers was already introduced in Theorem 4.3. In this section we explore this possibility further. We restrict our attention to such flows in connection with a definition of *hex-chain modules,* which generalizes Tutte's *regular chain groups* to sixth root of unity matroids. Regular chain groups are used by Seymour [Sey81a] as the basis of a proof of Hoffman's circulation lemma which does not use powerful tools from polyhedral theory. Thus it seems a promising avenue to try to further connect some idea of imbalance such as ϕ_μ with $\phi_{\mathbb{E}}$.

In the graphic case, Hoffman's Circulation Lemma further states that, if all edge capacities on an oriented graph are integer-valued, then it is possible to find an integer-valued optimal flow. For $\sqrt[6]{1}$-matroids, we study *Eisenstein-integer flows*, which leads us to consider $\phi_{\mathbb{E}}$ instead of ϕ_a.

Hopefully this puts prior references to Eisenstein integers into a broader context, in which the idea of restricting to Eisenstein flows is of greater interest. This section follows Tutte's work in [Tut56] closely, both in methods and structure. To the best of our knowledge this treatment of $\sqrt[6]{1}$-matroids is new. We are not aware of work extending Tutte's chain groups to other classes of matroids.

Definition 4.10 *Given a set E, a* hex-chain *on E is a mapping $f : E \longrightarrow \mathbb{Z}[w]$ from E to the Eisenstein integers. The set of hex-chains on any set E forms a group under sum, defined as $(f + g)(e) = f(e) + g(e)$ and any subgroup of this group is a* hex-chain group. *Given any hex-chain group G, consider the $\mathbb{E}$-module obtained taking Eisenstein multiples of chains in G. We call such a module a* hex-chain module.

Definition 4.11 *The support of a hex-chain f, denoted supp(f), is the set $\{e \in E : f(e) \neq 0\}$. A hex-chain f is* elementary *if it is non-zero and has minimal support (minimal with respect to set containment). It is* unitary *if $f(e) \in \{0, w^k\}, k = 0, \dots 5$ for all e. It is* primitive *if it is elementary and unitary.*

Definition 4.12 *A hex-chain module is* hex-regular *if for any elementary hex-chain f there exists a primitive g with equal support.*

Example 4.13 *Flows and tensions.*

Consider the hex-chain modules Γ (resp. Γ^*) defined on the set E of elements on a $\sqrt[6]{1}$-matroid $M[A]$, by the vectors in the Null space of A, (resp. the Row space of A) with values in the Eisenstein integers. That is, a hex-chain f is in Γ if, seen as a vector, it is in the null space of A and it is in Γ^* if, as a vector, it is in the row space of A.

Lemma 4.14 *Both Γ and Γ^* are hex-regular.*

Proof: Let f be an elementary hex-chain in Γ, we must show that there exist a unitary hex-chain $g \in \Gamma$ with equal support. But f has minimal support if and only if its support is a circuit of $M = M[A]$. We know that if C is a a circuit of M, there is a vector g with support C and whose non-zero entries are all in $w^k \ : \ k = 0, \ldots, 5$. This vector provides the required hex-chain.

Finally, note that given a matroid M, their hex-chain modules $\Gamma$and Γ^* denoted $\Gamma(M)$ and $\Gamma^*(M)$ satisfy: $\Gamma^*(M) = \Gamma(M^*)$.

□

As it will be shown later, all hex-regular chain modules correspond to $\Gamma(M)$ for some $\sqrt[6]{1}$-matroid.

A *dendroid* in a chain group is a subset D of E such that D intersects the support of every non-zero chain in the group (a *blocking set*) and it is minimal with this property. We extend this definition to hex-chain modules. Note that if the chain module in question is Γ, as defined in example 4.13, then a dendroid is a *cobasis* of the matroid. If it is Γ^*, then a dendroid is a *basis* of the matroid.

Given a dendroid D of a hex-chain module $\mathcal{G}$, and an element $e \in D$, there exists a non-zero hex-chain $f \in \mathcal{G}$ which intersects D exactly in e, that is, such that $supp(f) \cap D = \{e\}$. Clearly $f(e) \neq 0$ and furthermore, we may take f such that $f(e) = k \in \mathbb{Z}^+$ since the Eisenstein multiple of f, $\overline{f(e)}f$ is also a chain in $\mathcal{G}$. Of all chains which intersect D exactly in e, denote by $J_D(e)$ the hex-chain f with $k = f(e) \in \mathbb{Z}^+$, as small as possible. This choice of $J_D(e)$ is unique for each e, otherwise there is a chain g, with equal support to f and $f(e) = g(e)$ which implies the existence of the non-zero chain $(f - g)$, whose support avoids D, which is a contradiction. Note that $J_D(e)$ is describing a generalization of the notion of *elementary cuts* of a graph, with respect to a tree.

Lemma 4.15 *$J_D(e)$ is an elementary hex-chain.*

Proof: Let $g = J_D(e)$ and suppose it is not elementary. Then there is another chain f, satisfying $supp(f) \cap D = \{e\}$ and with $supp(f) \subset supp(g)$. Let $z = g(e)$ and $y = f(e)$, both non-zero, and consider the chain $h = zf - yg$ which is zero on e and thus does not intersect D. Hence h is zero which implies that f and g have equal support (are equal, up to an Eisenstein integer factor).

□

Lemma 4.16 *If $\mathcal{G}$ is hex-regular, then $J_D(e)$ is primitive.*

Proof: By Lemma 4.15 $J_D(e)$ is elementary and because $\mathcal{G}$ is hex-regular, there is a primitive chain f in $\mathcal{G}$ with $supp(f) = supp(J_D(e))$. It follows immediately that $J_D(e)$ is primitive by definition and uniqueness of $J_D(e)$ (it is elementary, with smallest positive real value on e)

□

Lemma 4.17 *If $\mathcal{G}$ is a regular hex-chain module with a non-null dendroid D, any hex-chain $g \in \mathcal{G}$ can be written as the sum $g = g_1 + \cdots + g_k$ where all g_i are elementary.*

Proof: Indeed, each g can be written as: $g = \sum_{e \in D} g(e) J_D(e)$. Since the support of $g' = g - \sum_{e \in D} g(e) J_D(e)$ does not intersect D, thus g' is the zero hex-chain.

□

This lemma relates to comments made during our discussion on hex-Eulerian matroids regarding the decomposition of any flow into flows on circuits.

Corollary 4.18 *Every Eisenstein hex-flow on a $\sqrt[6]{1}$-matroid M can be decomposed as the sum of Eisenstein hex-flows on circuits of M.*

Until this point it was not clear that this would be the case for Eisenstein flows on $\sqrt[6]{1}$-matroids.

Definition 4.19 *Given a non-null hex-chain module $\mathcal{G}$, and a dendroid D, in $\mathcal{G}$, consider an ordering of the elements of $\mathcal{G}$, $E = \{e_1, e_2, \cdots, e_n\}$. Define, for each hex-chain $f \in \mathcal{G}$, a* representative vector, *with respect to the given enumeration, as the vector with entries $(f(e_1), \cdots, f(e_n))$.*

With this representation of a chain, we can interpret the rows of a $\sqrt[6]{1}$-matrix A of full row rank, as hex-chains and consider the hex-chain module $\mathcal{G}$, formed by all the linear combinations, with Eisenstein coefficients, of the rows of A. We say that A is the *representative matrix* of $\mathcal{G}$. This is the hex-chain module Γ^* we described in Example 4.13.

Our next goal is to show (Theorem 4.23) that, like in the regular case, every hex-regular chain group $\mathcal{G}$, has a representative matrix A, which is a $\sqrt[6]{1}$-matrix. We construct $A = A[\mathcal{G}]$ as follows. Consider a dendroid D of $\mathcal{G}$ and define the representative matrix A whose rows are the representative vectors of the chains $J_D(e)$. By Lemma 4.17, every hex-chain in $\mathcal{G}$ is represented by a vector which is a linear combination, with Eisenstein integer coefficients, of the rows of A. It remains to be shown, however, that A is a $\sqrt[6]{1}$-matrix.

Lemma 4.20 *Let A be a representative matrix of a hex-chain module $\mathcal{G}$. Then D is a dendroid in $\mathcal{G}$ if and only if $|D| = r$ and the submatrix formed by the columns of A, corresponding the elements of D, has non-zero determinant.*

Proof: Consider a set S in $\mathcal{G}$ and denote by $A(S)$ the submatrix of A formed by the columns indexed by S. Then if $rank(A(S)) < r$, there is some linear combination of the rows of A which has only 0 entries for the elements in S. This corresponds to a hex-chain that does not intersect S which implies that S is not a dendroid. Hence, if S is a dendroid then $rank(A(S)) = r$ and $|S| \geq r$.

Conversely, if $rank(A(S)) = r$ then there is some subset D of elements of S corresponding to a square submatrix of A with rank r, hence non-zero determinant. Since the rows of $A(D)$ are linearly independent, D meets the support of all hex-chains in $\mathcal{G}$ and thus D contains a dendroid. This dendroid must be D since a dendroid must have at least r elements.

□

To prove the next result, we need to define the restriction of a hex-chain module:

Definition 4.21 *Given a hex-chain module $\mathcal{G}$ on the element set E, the restriction of a chain $f \in \mathcal{G}$ to $S \subseteq E$, denoted $f|_S$ is the hex-chain on S with coefficient $f|_S(e) = f(e)$ for each $e \in S$. The restrictions of hex-chains in $\mathcal{G}$ to a set S form a hex-chain module on S denoted $\mathcal{G}|_S$.*

Lemma 4.22 *Let A be a rank r, $r \times n$ matrix with Eisenstein integer entries. Let E be a set of n elements. Then A is a representative matrix of a hex-regular chain module, $\mathcal{G}$, on E if and only if the determinants of its square submatrices of order r are restricted to w^k values.*

Proof: First let A be the representative matrix of a hex-regular chain module $\mathcal{G}$. A submatrix of A, say $A(S)$, of order r, has non-zero determinant if and only if the elements in S, indexing its columns, correspond to a dendroid. In this case, consider a matrix, A', representing $\mathcal{G}$ under the same enumeration of the elements, but with respect to S, thus the columns of S in this matrix form an identity. Since this is another representation of $\mathcal{G}$, the rows of A are linear combinations, with Eisenstein integer coefficients, of the rows of A'. Thus there is an Eisenstein integer valued matrix

P such that $A' = PA$. Thus $A'(S) = PA(S)$ and $\pm 1 = det(A'(S)) = det(P)det(A(S))$. All these numbers are Eisenstein integers hence the only way their product is ± 1 is if all their moduli are 1, thus they are all powers of w. This shows that all submatrices of rank r have determinant in $\{0, 1, w, \dots, w^5\}$, as required.

Now consider a matrix A with subdeterminants restricted to 0 and powers of w. Fix an enumeration of a set of elements E, corresponding to the columns of A, and consider a hex-chain module $\mathcal{G}$ on E whose representative matrix is A.

Let f be an elementary hex-chain of $\mathcal{G}$. We must show that f is an Eisenstein multiple of some primitive hex-chain. Let e be an element in the support of f, with $\|f(e)\|$ as large as possible, and consider a dendroid H of $\mathcal{G}|_{(E-supp(f))}$. Any hex-chain g of $\mathcal{G}$ whose support does not intersect $H \cup \{e\}$ must have support a subset of $supp(f) - \{e\}$. Since f is elementary, then g must be the zero hex-chain. Thus $H \cup \{e\}$ intersects every hex-chain in $\mathcal{G}$ which implies that some subset $D \subseteq H \cup \{e\}$ is a dendroid in $\mathcal{G}$. Finally $D \cap supp(f) = \{e\}$.

Now, the submatrix A_D corresponding to the elements in D, must have non-zero determinant, and by our restriction on A, $det(A_D)$ must be a power of w. Furthermore, the inverse of A_D exists and all its entries are Eisenstein integers.

Consider $A' = A_D^{-1}A$. The rows of A' represent hex-chains of $\mathcal{G}$, since they are Eisenstein combinations of the rows of A. But A'_D is an identity matrix, thus there is a chain g in $\mathcal{G}$ such that $g(e) = 1$ and $D \cap supp(g) = \{e\}$. Thus $f = f(e)g$ since $f - f(e)g$ must be the zero hex-chain. It suffices now to show that g is unitary, but this follows from the choice of e ($\|f(e)\|$ is largest possible) and the fact that $g(e) = 1$.

□

Theorem 4.23 (Chávez Lomelí) *Let A be a rank r, $r \times n$ matrix with Eisenstein integer entries, in which some r columns constitute an identity matrix. Denote the submatrix formed by the remaining columns, B. Let E be a set of n elements, then A*

is a representative matrix for a hex-regular chain module on E if and only if the determinants of the square submatrices of B are restricted to values $0, w^k;\ k = 0, \ldots, 5$.

Proof: It suffices to note that there is a one-to-one correspondence between the determinants of $r \times r$ submatrices of A and the determinants of square submatrices of B. This correspondence shows that the $r \times r$ subdeterminants of A are restricted to $0, w^k;\ k = 0, \ldots, 5$, if and only if *all* the square subdeterminants of B are also restricted to $0, w^k;\ k = 0, \ldots, 5$.

The result now follows from Lemma 4.22.

□

Theorem 4.23 states that the relation between hex-regular chain groups and $\sqrt[6]{1}$-matroids is analogous to that between regular chain groups and regular matroids. Accordingly, it is also possible to consider duals.

Definition 4.24 *The dual of a hex-chain module $\mathcal{G}$ on the set E is the hex-chain module $\mathcal{G}^*$, containing all chains f on E such that $\sum_{e \in E} f(e)g(e) = 0$ for all $g \in \mathcal{G}$.*

Proposition 4.25 $(\mathcal{G}^*)^* = \mathcal{G}$.

Proof: $f \in (\mathcal{G}^*)^*$ if and only if $\sum_{e \in E} f(e)g(e) = 0$ for all $g \in \mathcal{G}^*$. This happens if and only if $f \in \mathcal{G}$.

□

Lemma 4.26 *If the group $\mathcal{G}$ is hex-regular then $\mathcal{G}^*$ is also hex-regular. In this case, the dendroids of $\mathcal{G}^*$ are the complements of the dendroids of $\mathcal{G}$ in E.*

Corollary 4.27 *Γ and Γ^* are a dual pair of hex-regular, hex-chain modules.*

In spite of all the parallels between regular and hex-chains, Seymour's proof of Hoffman's lemma does not extend directly. In the case of chains over the integers (Tutte's original chains) the notion of conformal chains is central to Seymour's proof.

Definition 4.28 *Let f and g be chains in a regular chain group $\mathcal{G}$. Then f is said to be* conformal *with g if, whenever $f(e)$ is non-zero, $g(e)$ is also non-zero and both $f(e)$ and $g(e)$ have equal sign.*

To extend Hoffman's Lemma to hex-chains, we need an analog of the following extension of Lemma 4.17.

Lemma 4.29 *If $\mathcal{G}$ is a regular chain group with a non-null dendroid D, any chain $g \in \mathcal{G}$ can be written as the sum $g = g_1 + \cdots + g_k$ where each g_i, $i = 1, \ldots, k$ is elementary and conformal with g.*

As commonly done in network flows, we abbreviate

$$l(E^-(g)) = \sum_{g(e)<0} l(e) \quad \text{and} \quad u(E^+(g)) = \sum_{g(e)>0} u(e).$$

The precise statement of Hoffman's Lemma, in terms of chains is the following:

Theorem 4.30 *Let Γ be a regular chain-group and let $l, u : E \to \mathbb{Z}$ satisfy $u \geq l$. Then the following are equivalent:*

1. *There is a chain f of Γ with $l \leq f \leq u$;*

2. *For each unitary chain g of Γ^*, $l(E^-(g)) \leq u(E^+(g))$.*

The proof that (1) implies (2) is not much different but easier than that of Theorem 4.4. Seymour's proof of the reverse implication uses the construction of a flow with bounded entries, which is done by induction, shifting the lower and upper bounds, l

and u, one unit at a time, until $l(e) = u(e)$ for all e. At this point, $f(e) = l(e)$ is a flow. Thus, the induction is on $\sum u(e) - l(e)$, and an inductive step takes an element e_0 such that $l(e_0) < u(e_0)$ and considers:

$$u'(e) = \begin{cases} u(e) - 1 & \text{if } e = e_0 \\ u(e) & \text{otherwise} \end{cases} \qquad l'(e) = \begin{cases} l(e) + 1 & \text{if } e = e_0 \\ l(e) & \text{otherwise} \end{cases}$$

Now Seymour shows that either l', u or l, u' satisfies the conditions on imbalance of cocircuits (statement 2 above). This is accomplished by contradiction and this is where conformal decompositions come into play.

In the hex-chain case, it is not even clear how to define "conformal". A direct generalization of this concept would require that the entries in two chains had *equal orientation*, which in this context means that they were positive, real multiples of the same w^k. Accordingly, a unitary hex-chain and a hex-chain with a value, say $w + 1$ could not even be compared.

We propose that the definition of conformal should be stated in terms of the argument of the elements of the chain. Precisely, one should make a choice of parameter k in the following definition:

Definition 4.31 *Let $k \geq 0$. The hex chain f is k-*conformal *to g if whenever $f(e)$ is non-zero, $g(e)$ is also non-zero and $|arg(f(e)) - arg(g(e))| \leq k\pi/3$.*

Different choices of k run the whole spectrum of such definitions of conformal. $k = 0$ restricts conformal hex-chains to $c \cdot w^t$-valued hex-chains, $c \in \mathbb{R}$. As we require to compare all possible hex-chains, this means restricting the investigation to $\mathbb{Z}$-valued flows. $k > 3$ makes all chains conformal, which is clearly not what we wish to do.

Intuitively, it is not hard to see that restricting to real integer valued flows (defining conformal with $k = 0$, above) may lead to a situation where many matroids do not admit nowhere zero hex-flows. Even the small example of $U_{2,4}$ explored before, does not admit a real integer valued hex-flow.

Example 4.32 *Hex-flows in $U_{2,4}$*

The matrix A below, represents an orientation of $U_{2,4}$ which admits the Eisenstein integer, nowhere zero flow $f = (w^4, w^5 + w^4, w^2, 1)$.

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & w \end{pmatrix}$$

The flow space of the matroid $M[A]$ is spanned by the vectors $f_1 = (-1, -1, 1, 0)^T$ and $f_2 = (-1, -w, 0, 1)^T$. It is not hard to see that a real valued, no-where zero flow of the form $F = sf_1 + tf_2$, cannot be obtained. Note first that any real-valued flow f of M is a vector f taking values of the form cw^k with $c \in \mathbb{R}$ and $k \in \{0, \ldots, 5\}$, satisfying $Af = 0$. Furthermore, the sum $\alpha w^i + \beta w^j$ is again a multiple of w^k, for some k, only if $\alpha w^i + \beta w^j = 0$, (so $|i - j| = 3$) or $|i - j| = 2$ and $\alpha = \beta$, where the difference $i - j$ is taken modulo 6.

Then, if we insist on restricting flow values to those of the form cw^k, with c $\in \mathbb{R}$, $k \in \{0, \ldots, 5\}$, each coordinate in F imposes a constraint and we have the following conditions on the coefficients s and t:

1. Both s and t must, themselves, be non zero, real multiples of a power of w. This is imposed by the constraints on the last two coordinates of f_1 and f_2.

2. Considering the first coordinate in f_1 and f_2, we require s and t which further satisfy: $-(s + t)$ is of the form cw^k, $c \neq 0$, $c \in \mathbb{R}$.

3. Finally, the last coordinate of F, $-(s + wt)$ must be of the form cw^k, $c \neq 0$.

Due to all the "rotational symmetries" on the sixth roots of unity, we may assume that $s \in \mathbb{R}^+$ and that $t = cw^i$ for some $i \in \{0, \ldots, 5\}$ and $c \in \mathbb{R}^+$. Condition (2) implies that $t = cw^2$, $t = cw^4$ or $t = cw^3 = -c$. Using now "reflectional symmetries" it is enough to consider the cases $t = cw^2$ and $t = cw^3$. In the first case, if condition

(2) is satisfied, then we can conclude that $s = c$ and condition (3) is not satisfied since $s + wt = s - s = 0$. In the second case, if condition (2) is satisfied then $s - c \neq 0$ and $c \neq s$, but this implies that condition (3), on $s + wt = s + cw^4$, cannot be satisfied.

Therfore $U_{2,4}$ does not admit a real-valued hex-flow.

□

Note also, that the flow in Example 4.32, $f = (w^4, w^5 - w, w^2, 1)^T$ can be written as: $f = w^2 f_1 + f_2 = (w^5, w^5, w^2, 0)^T + (w^3, w^4, 0, 1)^T$. This is a decomposition of f into flows with minimal support and it is not hard to see that whenever e is in the support of f_i for any $i = 1, 2$, $|arg(f(e)) - arg(f_i(e))| \leq \pi/3$. That is, both f_1 and f_2 are 1-conformal with f and we would have a 1-conformal decomposition of f, even if f_1 and f_2 are not 1-conformal with each other.

For the $\sqrt[6]{1}$-case, we propose the following analog of Hoffman. We abbreviate:

$$l(E^-(g)) = \sum_{(g(e))'<0} l(e)g(e), \quad u(E^+(g)) = \sum_{(g(e))'>0} u(e)g(e),$$

and denote the real part of a complex number z by z', as before.

Conjecture 4.33 *Let Γ be a hex-regular hex-chain module and let $l, u : E \to \mathbb{Z}^+$ satisfy $u \geq l$. Then if for each unitary chain g of Γ^*, we have $(l(E^-(G)))' \leq h(u)(u(E^+(g)))'$, then there is a chain f of Γ with $l \leq ||f|| \leq u$, where $h(u)$ is some fixed function of u.*

That is, even if the algebraic and imbalance flow numbers are not equal, we would like to bound how widely they differ. Ideally, one could hope for h to be a positive real constant.

To extend Seymour's proof from the regular case, there is also the question of constructing, when $l(e) = u(e)$, an Eisenstein flow having values with the given norms. Thus we are interested in the following questions:

Question 4.34 *Let* Γ *be a hex-regular hex-chain module and let* $l : E \rightarrow \mathbb{Z}^+$. *If for each unitary chain* g *of* Γ^*, $(l(E^-(G)))' = h(u)(l(E^+(g)))'$, *then is there a chain* f *of* Γ *with* $\|f(e)\| = l(e)$ *for all* $e \in E$*?*

Question 4.35 *Can every chain* f, *in a hex-regular hex-chain module be decomposed as the sum* $f = f_1 + \ldots + f_k$ *where all* f_i *are 1-conformal with* f*?*

Questions 4.34 and 4.35, if answered in affirmative, would establish the equivalent of the inductive basis and the core of the inductive step in Seymour's proof. This observation was the main motivator to develop the material in this section.

Chapter 5

Conclusion

> "That was how it worked. No magic at all. But that time it had been magic. And it didn't stop being magic just because you found out how it was done..." [1]

The main results in this thesis are the following:

For orientation classes of orientable matroids we proved a characterization of bipartite and Eulerian uniform rank-3 classes (Theorems 3.4 and 3.6), an asymptotic result concerning the orientation flow number $\phi_o(\mathcal{O})$ of uniform matroids of fixed rank (Theorem 3.9) and two bounds on $\phi_o(\mathcal{O})$; Theorem 3.17 for arbitrary matroids and Theorem 3.13 concerning uniform matroids.

For $\sqrt[6]{1}$-matroids we defined hex-Eulerian, provided a characterization of hex-Eulerian regular matroids (Theorem 4.9) and showed that max-sized $\sqrt[6]{1}$-matroids of rank $r > 2$ are hex-Eulerian (Theorem 4.8). We also showed some connections between $\phi_a(M)$ and $\phi_\mu(M)$, as defined for $\sqrt[6]{1}$-matroids (Theorems 4.3 and 4.4). Finally we defined hex-chain modules, which describe $\sqrt[6]{1}$-matroids in terms of their Eisenstein integer flows (Theorem 4.23).

Looking through this thesis, the use of circuit covers and partitions is so widespread

[1] Reproduced with permission of the author. Terry Prattchet, [Pra04]

that it would seem that the study of flows is really a study of circuit covers and partitions. This impression is the strongest while looking at orientation numbers however it is not entirely accurate. A difference between these concepts was noted earlier. Simply stated, $\phi_o(\mathcal{O})$ is a parameter associated with orientation classes or PSCs, while circuit covers or partitions are defined in terms of the underlying matroid. Even if some orientation class of a matroid M has $\phi_o(\mathcal{O}) = 2$, it does not follow that M admits a partition into circuits, as can be seen in $U_{3,6}$. I would be interested in further understanding this connection between cycle covers and flow numbers.

There are, of course, a number of concepts related to flow and chromatic numbers, which have not been addressed by this thesis. Here are some examples that I find appealing. To my knowledge, at least in the case of $\sqrt[6]{1}$-matroids, none of these has been explored at all.

In the case of regular matroids, Hoffman's Theorem can be used to produce an integer valued flow from a rational valued flow f (just set $l(e) = \lfloor f(e) \rfloor$, $u(e) = \lceil f(e) \rceil$). This result can be re-stated in the following terms: If f is a $\mathbb{Q}$-valued flow of M, then there is a $\mathbb{Z}$-valued flow f' with the property that $\|f - f'\|_\infty < 1$.

Is there an equivalent of this concept in the complex case? First one must find a suitable analog of the bound $\|f - f'\|_\infty < 1$.

Question 5.1 *Is it the case that given a complex-valued flow, f, of a $\sqrt[6]{1}$-matroid, there is always an Eisenstein valued flow, f' satisfying $\|f - f'\|_\infty < c$?*

An even weaker version is:

Question 5.2 *Is it the case that given a complex-valued flow, f, of a $\sqrt[6]{1}$-matroid, there is always an Eisenstein valued flow, f' satisfying $|\|f(e)\| - \|f'(e)\|| < c$ for all $e \in E(M)$?*

The natural candidate for c is 1, since there is always at least one Eisenstein integer at distance 1 from any complex number. This would show that the difference between ϕ_a and $\phi_{\mathbb{E}}$ is bounded by 1 for all $\sqrt[6]{1}$-matroids M. Furthermore, this might result in an alternative approach to prove the Hoffman-type result in 4.33.

Another concept we have not investigated is the idea of a *group-valued flow* or Γ-flow on $\sqrt[6]{1}$-matroids. This approach is very productive in the regular case. It allows application of the Tutte polynomial, which in turn provides a proof that the number of Γ-flows on a regular matroid depends only on $|\Gamma|$. This renders a definition of flow number of a matroid M, as the fewest number of elements in a group Γ, such that M has a Γ-flow. Furthermore, it was also used by Jaeger [Jae79] who combined this notion with *coarboricity* in a very successful way.

In the graphic case, orientations result in vectors with values ± 1. It is also the case that each element in any group has an inverse, $-a$. The choice of a and $-a$ assigned as a flow value depends on the ± 1 orientation, as specified by a TUM. It would be interesting to examine the following question.

Question 5.3 *Can we define group valued flows for $\sqrt[6]{1}$-matroids via an Abelian group Γ containing a subgroup G isomorphic to $\mathbb{Z}_6$?*

The idea would be to associate the elements $g_i \in G$ with the sixth roots of unity, w^i, so that a group valued flow on a $\sqrt[6]{1}$-matroid satisfied the conservation equation:

$$\sum_B g_e f(e) = 0,$$

for all cocircuits B, where g_e is the group element in G corresponding to the e-th entry in the vector v_B, which encodes the hex-orientation of B. The flow number could be defined in terms of the smallest possible order of the quotient group Γ/G.

We close this thesis with a problem that we considered early in this research and had not mentioned so far.

A graph G is Eulerian if and only if $E(G)$ is the disjoint union of circuits. Circuits are, in some sense, minimal Eulerian graphs thus we might say that G is Eulerian if and only if it is the disjoint union of minimal Eulerian graphs. When regarded algebraically, circuits are represented by vectors in the null space of the matrix representing G, with minimal support. In an Eulerian graph, the sum of a collection of such vectors produces a nowhere zero ± 1-valued vector.

A graphic matroid (or indeed a regular matroid) $M = M[A]$ is also binary and taking a sum of vectors in the null space of A (circuits) in a binary matroid corresponds to taking the symmetric difference of their supports. Thus there seems to be a strong connection between an algebraic operation (sum of vectors) and a combinatorial operation (disjoint union) on the circuits of an Eulerian matroid.

Theorem 4.9 shows that this is not the case for regular hex-Eulerian matroids. A matroid $M = M[A]$ with flow number $\phi(M) = 3$ is not Eulerian, hence is not the disjoint union of circuits, but it is hex-Eulerian. One might ask whether circuits are no longer the minimal hex-Eulerian matroids or whether disjoint union is not the right operation for this class.

Certainly circuits are hex-Eulerian and minimal with this property. Furthermore, in algebraic terms, it is still the case that any hex-Eulerian $\sqrt[6]{1}$-matroid admits an orientation for which the vector $(1, 1, \ldots, 1)$ is a flow and this flow can be decomposed into *unit flows* as was shown using hex-chain modules. Thus it seems that the real question is what is the corresponding combinatorial operation in this case?

I believe that an answer to this question would provide great insight into flows and circuit covers on $\sqrt[6]{1}$-matroids.

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