GENERALIZATIONS AND EXTENSIONS OF THE CONSTANT TRAVELLING SALESMAN PROBLEM

by

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B.Math., University of Waterloo, 2005

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A THESIS SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE
in the Department
of
Mathematics

SIMON FRASER UNIVERSITY
Spring 2007

APPROVAL

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Title of thesis: Generalizations and Extensions of the Constant Travelling Salesman Problem

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Date Approved: March 30, 2007

Abstract

This thesis gives a characterization of cost matrices associated with the complete directed graph that have at most three distinct values for linear spanning 2-forests. Alternative characterizations of cost matrices where all Hamiltonian paths and Hamiltonian cycles have at most two distinct values are also given. The only known characterization for three values of Hamiltonian paths and cycles is for skew-symmetric matrices. Similar results are obtained with different restrictions on the structure of the cost matrix.

Furthermore, this thesis identifies generalizations of the constant-TSP for arbitrary graphs whose associated cost matrix is well-structured. The SC-Hamiltonicity of various classes of undirected and directed graphs is determined. A complete characterization of SC-Hamiltonicity in terms of strong Hamiltonicity is given for undirected graphs. In addition, interesting classifications are found which contradict previous claims regarding SC-Hamiltonian graphs.

This thesis is dedicated to
all those who have asked
"How's your research going?"

The most exciting phrase to hear in science,
the one that heralds the most discoveries,
is not "Eureka!", but "That's funny..."
— Isaac Asimov

Acknowledgments

I would like to thank my family for their love, support and eagerness to have me close once again. Thank you for taking the time to make me feel as if I were still there.

I would like to thank my supervisor for suggesting such an interesting topic and whose guidance encouraged me to excel.

Finally, I owe a debt of gratitude to my loving wife-to-be for allowing me to babble-on about things that no person outside of academia should ever be expected to comprehend. Your unlimited tolerance in this matter was indispensable. Thank you for being there for me through every step of this adventure.

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Chapter 1

Introduction

Given a set of cities and a cost to travel between any pair within the set, the Travelling Salesman Problem (TSP) arises when attempting to find the shortest valued route visiting each city exactly once and returning to the starting city. Although the TSP can be stated with such simplicity, applications of the model arise in numerous real world situations. Applications include printed circuit board design, flexible manufacturing systems, DNA analysis and warehouse material handling.

There are several different representations that can be used to describe the TSP. Mathematical programming models, graph theoretic models and permutation models are a selection of the representations that are available. Throughout this thesis the graph theoretic model will be utilized to represent the TSP.

Let $G = (V, E)$ be a (directed or undirected) graph with node set $V(G) = \{1, 2, \ldots, n\}$ and edge set $E(G)$. A cost c_{ij} is prescribed for each edge $ij \in E(G)$. Hence, there is a $|V(G)| \times |V(G)|$ matrix $C = (c_{ij})$ called a cost matrix associated with the graph G. Set $c_{ij} = \infty$ if $ij \notin E(G)$. Given any Hamiltonian cycle (tour) T the cost (value) of the tour is denoted by $C(T) = \sum_{ij \in E(T)} c_{ij}$. The TSP is to find a tour T^* in G such that $C(T^*)$ is minimized over all tours. An instance of the TSP is completely defined by the associated cost matrix C.

Another problem closely related to the TSP is the Wandering Salesman Problem (WSP) [8]. The WSP arises when attempting to find the shortest valued route in some set of cities visiting each city exactly once. It should be noted that the WSP differs from the TSP in that the salesman does not need to return to the starting city. Informally, the WSP considers information about Hamiltonian paths rather than considering information about

Hamiltonian cycles.

The TSP is an classical, well-studied topic in combinatorial optimization. Although deceptively simple to state, it is well-known that the TSP is **NP**-hard (and when restated in a decisive form is **NP**-complete). Thus, the TSP is a representative of a much larger class of problems. Finding an efficient algorithm for solving an arbitrary instance of the TSP would answer the open question of whether **P** and **NP** are the same complexity class. Even though the TSP is **NP**-hard, there are special cases of the problem that are known to be solvable in polynomial time. For the state-of-the-art status of work performed on such cases of the TSP, the reader is referred to [2], [5] and [9].

Perhaps the simplest of all polynomially solvable cases of the TSP is the Constant Travelling Salesman Problem (constant-TSP). An instance of the TSP is called *constant-TSP* if and only if all tours in graph G have the same cost with respect to cost matrix C. Note that if cost matrix $C = (c_{ij}) = (a_i + b_j)\ \forall ij \in E(G)$ then all tours in G with associated cost matrix C have the same value. Gabovich [4] proved that this condition is both necessary and sufficient for complete graphs. That is, all tours in K_n with respect to cost matrix $C = (c_{ij})$ have the same cost if and only if there exists constants $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ such that $c_{ij} = a_i + b_j\ \forall ij \in E(K_n)$. This provides an elegant characterization of cost matrices for the TSP where all tours have the same value. Alternative proofs of this characterization have been provided by Leont'ev [14], Rublinetskii [16], Berenguer [1], Lenstra and Rinnoy Kan [13], Gilmore, Lawler and Shmoys [6], Chandrasekaran [3], Queyranne and Wang [15], Kryński [12], Kabadi and Punnen [11], and Jones, Kayll, Mohar and Wallis [7].

This raises an interesting question. What is the structure of cost matrices associated with graph G such that the distinct values of all tours in G is at most k (for $k \in \mathbb{Z}^+$)? Tarasov [17] gave a complete characterization of cost matrices to this question when $k = 2$. Tarasov also gave characterizations of the 1-value bipartite matching problem, 2-value bipartite matching problem and 3-value bipartite matching problem. Later, Kabadi and Punnen [10] gave an alternative characterization for $k = 2$ with a much simpler proof. For $k = 3$, Kabadi and Punnen [10] gave a characterization with the assumption that the associated cost matrix is skew-symmetric. Hence, a polynomially testable characterization of general cost matrices with the property that all associated Hamiltonian cycles have three distinct values remained an open question. In this thesis, new classes of cost matrices where all Hamiltonian cycles have exactly three cost values are discovered. As a by-product, we also obtain a polynomially testable characterization of cost matrices (with some additional restrictions) such that all

Hamiltonian paths have at most three distinct values.

The aforementioned results are achieved by extending the proof techniques used by Kabadi and Punnen in [10] and introducing the concept of linear spanning 2-forests. A *linear forest* is a graph where every component is a path (and isolated vertices are permitted). A *linear spanning forest* is a spanning subgraph of a graph G where every component of the subgraph is a path. A *linear spanning 2-forest* (LS2F) of G is a linear spanning forest of G with precisely two components. The cost (value) of a linear spanning 2-forest L in G with respect to cost matrix C is given by $\sum_{e \in E(L)} c_e$. Furthermore, C is a k *distinct linear spanning 2-forest* cost matrix, denoted LS2F(k), if and only if there exist exactly k distinct values for all LS2Fs in G with associated cost matrix C.

Another direction of research related to the constant-TSP is to attempt to extend the characterization of cost matrices for constant-TSP to arbitrary graphs, rather than just considering the complete graph. Obliviously, such a characterization is hard since it would resolve a Hamiltonian cycle detection problem. This raises a second intriguing question. For what classes of graphs with all tours having constant cost does the associated cost matrix $C = (c_{ij})$ have the form $c_{ij} = a_i + b_j \; \forall ij \in E(G)$? Such a class of graphs is called separable constant Hamiltonian [11]. Graph G with $|V(G)| = n$ is *separable constant Hamiltonian* (*SC-Hamiltonian*) if and only if G is Hamiltonian and for any cost matrix $C = (c_{ij})$ associated with G where all tours have the same value, there exist constants $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ such that $c_{ij} = a_i + b_j \; \forall ij \in E(G)$. Moreover, if G is undirected then $a_i = b_i \; \forall i$.

In [12], Kryński observed that for undirected graphs, odd cycles are SC-Hamiltonian but even cycles are not. In that same paper, Kryński claimed that strong Hamiltonicity is a necessary requirement for SC-Hamiltonicity. This was disproved by Kabadi and Punnen in [11] who gave a counter-example to the claim. Interestingly, we show that for all graphs excluding those which have the structure provided in the aforementioned counter-example, strong Hamiltonicity is a necessary requirement for SC-Hamiltonicity.

Other SC-Hamiltonian graphs were identified by Kabadi and Punnen [11] including complete bipartite graphs, 1-extensions of Hamiltonian graphs and various classes of directed graphs. This thesis extends these results further and identifies new classes of both undirected and directed graphs as having the SC-Hamiltonian property.

In [11], Kabadi and Punnen conjectured that a symmetric digraph is SC-Hamiltonian only if it's underlying undirected graph is SC-Hamiltonian. A counter-example to this conjecture is presented. Furthermore, in that same paper it is claimed that subdividing an edge of any undirected SC-Hamiltonian graph twice creates a new SC-Hamiltonian graph. We will show that this claim is not necessarily true.

The major contributions of this thesis are summarized below.

1. A polynomially testable characterization of cost matrices such that all tours in an instance of the TSP will have three distinct values is given for matrices with some restricted properties. Such a characterization was known for skew-symmetric matrices but this identifies various other classes. Furthermore, we also provide characterizations of cost matrices where linear spanning 2-forests have at most three distinct values.

2. A complete characterization of undirected graphs where strong Hamiltonicity is a necessary condition for SC-Hamiltonicity. It was conjectured that strong Hamiltonicity is a necessary condition for SC-Hamiltonicity for all undirected graphs. This conjecture was later disproved by counter-example. Our result establishes that for all classes of graphs not excluded by the counter-example, strong Hamiltonicity is a necessary condition for SC-Hamiltonicity.

3. New classes of both undirected and directed SC-Hamiltonian graphs are identified. We also provide a counter-example to the claim that subdividing an edge of an undirected SC-Hamiltonian graph twice preserves SC-Hamiltonicity.

The remainder of this thesis is organized as follows.

- Chapter 2 introduces notations, definitions and preliminary results.

- Chapter 3 gives a closed form characterization of linear spanning 2-forests with either one or two distinct values. This chapter also provides alternative characterizations of matrices with either one or two distinct values for both Hamiltonian cycles and Hamiltonian paths.

- Chapter 4 gives a closed form characterization of cost matrices where all linear spanning 2-forests have three distinct values. In addition, Chapter 4 gives a characterization of cost matrices where all Hamiltonian paths and Hamiltonian cycles have three distinct values in certain restricted matrices.

- Chapter 5 investigates the SC-Hamiltonicity of general graphs and digraphs.

- Finally, Chapter 6 provides concluding remarks, points out future research avenues and identifies open questions.

Chapter 2

Notations and Basic Results

Let C be a cost matrix associated with the complete graph for an instance of the TSP. When we refer to the elements of C it will be assumed that only the finite elements of the matrix are considered (this excludes all entries on the main diagonal of C). We sometimes use the terminology G *described by* C when G is a graph and C is the associated cost matrix. Furthermore, we sometimes use the terminology *an edge* (*arc*) *of* C when the edge (arc) is in graph G with associated cost matrix C.

Cost matrix C associated with graph G is a k *distinct tour-cost* matrix, denoted DTC(k), if and only if there exist exactly k distinct tour values in G described by C [11]. Thus, the TSP on a DTC(1) matrix associated with the complete graph is an instance of the constant-TSP. Cost matrix C associated with graph G is a k *distinct path-cost* matrix, denoted DPC(k), if and only if there exist exactly k distinct Hamiltonian path values in G described by C [11]. Since an instance of the TSP in complete directed graph G is completely defined by the associated cost matrix C, we often refer to a LS2F, path or tour of C when the LS2F, path or tour is in G with associated cost matrix C.

For any cost matrix C associated with graph G and any $v_1 \in V(G)$ let $a_{v_1} = 0$, $b_{v_1} = 0$, $a_i = c_{iv_1}$ $\forall i \neq v_1$ and $b_j = c_{v_1 j}$ $\forall j \neq v_1$. Define the $n \times n$ v_1*-reduced matrix of* C to be $\hat{C}_{v_1} = (\hat{c}_{ij})$ where $\hat{c}_{ij} = c_{ij} - a_i - b_j$. Notice that $\hat{c}_{v_1 j} = \hat{c}_{iv_1} = 0$ $\forall i$ $\forall j$.

The $(n-1) \times (n-1)$ submatrix formed by deleting row v_1 and column v_1 of any v_1-reduced cost matrix $\hat{C}_{v_1}$ is denoted $\hat{C}^{r,v_1} = \hat{C}_{v_1} - v_1 = (\hat{c}_{ij}^{r,v_1})$. In graphical terms, this reduction of v_1 is equivalent to deleting node v_1 from the associated graph G. When there is no ambiguity, we will omit the reduction index of $\hat{C}^{r,v_1}$ and represent the matrix by $\hat{C}^r$ to simplify notation.

Example 2.0.1. Let $G \cong \vec{K}_3$, $v_1 = 1$ and $C = \begin{pmatrix} \infty & 1 & -4 \\ 7 & \infty & 2 \\ 10 & 0 & \infty \end{pmatrix}$.

Then, $\hat{C}_1 = \begin{pmatrix} \infty & 1-1 & -4+4 \\ 7-7 & \infty & 2+4-7 \\ 10-10 & 0-1-10 & \infty \end{pmatrix} = \begin{pmatrix} \infty & 0 & 0 \\ 0 & \infty & -1 \\ 0 & -11 & \infty \end{pmatrix}$.

Thus, $\hat{C}^{r,1} = \begin{pmatrix} \infty & -1 \\ -11 & \infty \end{pmatrix}$.

For any v_1-reduced cost submatrix $\hat{C}^r$ associated with graph G and any $v_2 \in V(G)\backslash\{v_1\}$ define $a_{v_2} = 0$, $b_{v_2} = 0$, $a_i = \hat{c}^r_{iv_2}$ for $i \neq v_2$ and $b_j = \hat{c}^r_{v_2 j}$ for $j \neq v_2$. Define the $(n-1)\times(n-1)$ *v_2-reduced matrix of $\hat{C}^r$* as $\overline{C} = (\hat{C}_{v_1} - v_1)_{v_2} = (\overline{c}_{ij})$ where $\overline{c}_{ij} = \hat{c}^r_{ij} - a_i - b_j$. Notice that $\overline{c}_{v_2 j} = \overline{c}_{iv_2} = 0$. The $(n-2)\times(n-2)$ submatrix formed by deleting row v_2 and column v_2 of any v_2-reduced cost matrix $\overline{C}$ is denoted $\overline{C}^r = (\overline{c}^r_{ij})$.

Throughout Chapter 3 and Chapter 4 we use v_1 as standard notation to denote the first reduction index and v_2 is used to denote the second reduction index.

2.1 Preliminary results

The following preliminary results will be useful in obtaining characterizations of DTC(k) matrices for $k \in \mathbb{Z}^+$.

Observation 2.1.1. *For any $k \in \mathbb{Z}^+$, if a constant is subtracted from every entry in any row or column of a DTC(k) matrix then the resulting matrix is also DTC(k).*

Proof. Let C be a cost matrix associated with graph G and α be some fixed constant. Suppose α is subtracted from every entry in row r of C. This is equivalent to reducing the cost of all arcs entering node r by α. Any tour in G must enter node r exactly once, so the value of all tours has been reduced by α. Thus, the number of distinct tour costs remains unaffected. The argument is analogous when performing the reduction on a column of C by considering out-going arcs of node r. □

Observation 2.1.2. *Let C be a cost matrix associated with graph G. For any $k \in \mathbb{Z}^+$ and any $v_1 \in V(G)$, C is a DTC(k) matrix if and only if the v_1-reduced matrix $\hat{C}_{v_1}$ is a DTC(k) matrix.*

Proof. This result follows by repeatedly invoking Observation 2.1.1 to reduce the cost of every entry in row i by c_{iv_1} for $i \in \{1, \dots, n\}$ and the cost of every entry in column j by $c_{v_1 j}$ for $j \in \{1, \dots, n\}$. $\square$

Theorem 2.1.3. *[10] Let $k \in \mathbb{Z}^+$. If C is a DTC(k) matrix then it's v_1-reduced submatrix $\hat{C}^r$ is a DPC(k) matrix for every $v_1 \in \{1, \dots, n\}$. Conversely, if there exists $v_1 \in \{1, \dots, n\}$ such that the v_1-reduced submatrix $\hat{C}^r$ is a DPC(k) matrix, then C is a DTC(k) matrix.*

Proof. Let C be the cost matrix associated with graph G and $v_1 \in \{1, \dots, n\}$. Notice that row v_1 and column v_1 of $\hat{C}_{v_1}$ have all zero entries. Any Hamiltonian path P in $\hat{G}^r$ can be extended to a Hamiltonian cycle T in G such that $\hat{C}^r(P) = \hat{C}(T)$. Conversely, for any Hamiltonian cycle T in G the Hamiltonian path P in $\hat{C}^r$ obtained by deleting node v_1 from G has cost $\hat{C}(T) = \hat{C}^r(P)$. Hence, $\hat{C}_{v_1}$ is a DTC(k) matrix if and only if $\hat{C}^r$ is a DPC(k) matrix. By Observation 2.1.2, C is a DTC(k) matrix if and only if $\hat{C}_{v_1}$ is a DTC(k) matrix. $\square$

Lemma 2.1.4. *Let $k \in \mathbb{Z}^+$. If cost matrix C associated with graph G is a DPC(k) matrix and T is any tour in G, then arcs of T have at most k distinct costs.*

Proof. Let T be any tour that contains arcs of $k+1$ distinct costs $c_1, c_2, \dots, c_{k+1}$. By deleting these arcs one at a time from T we get $k+1$ distinct path values of costs $C(T) - c_1, C(T) - c_2, \dots, C(T) - c_{k+1}$. $\square$

2.2 Characterizations of DTC(k) and DPC(k) matrices

Kabadi and Punnen [10] succeeded in obtaining explicit characterizations of DPC(1) matrices, DPC(2) matrices and DPC(3) skew-symmetric matrices. By Theorem 2.1.3, this provides corresponding characterizations for DTC(1) matrices, DTC(2) matrices and DTC(3) skew-symmetric matrices.

It appears that attempting to follow the approach used in [10] to obtain a characterization of DPC(3) and DTC(3) matrices without the skew-symmetric restriction is difficult. Hence, to characterize such matrices a new approach is considered. Interestingly, we show that a polynomially testable characterization of DPC(k) and DTC(k) matrices with some additional properties can be obtained if we have polynomially testable characterizations of

LS2F(l) matrices ($\forall l \leq k$). We now introduce various classes of matrices where a characterization of LS2F(k) matrices can be used to obtain the desired characterization of DTC(k) matrices.

Definition 2.2.1. An $n \times n$ cost matrix C is *k-index constant, IC(k)*, if and only if it has k distinct indices $x_1, \ldots, x_k \in \{1, \ldots, n\}$ such that $c_{x_j i} = c_{i x_j} = \alpha$ for all $i \neq x_j$ and $j = 1, \ldots, k$. That is, all entries of C in rows and columns $x_1, \ldots, x_k$ have constant cost α.

Theorem 2.2.2. *An IC(3) cost matrix C is DPC(k) if and only if the reduced submatrices $\hat{C}^{r,1}$, $\hat{C}^{r,2}$ and $\hat{C}^{r,3}$ have all LS2Fs of costs z^i_m for $m = 1, \ldots, k_i$ with $k_i \leq k$, $i \in \{1, 2, 3\}$ and $|S| = k$ where $S = \{z^i_m + (2n-4)\alpha \mid m = 1, \ldots, k_i,\ i = 1, 2, 3\}$.*

Proof. Let C be an IC(3) cost matrix with constant rows and columns for indices 1, 2 and 3. Furthermore, let H denote the set of Hamiltonian paths in C and H_i denote the set of all Hamiltonian paths in C with node i as an interior node, for $i \in \{1, 2, 3\}$. Note that $H = H_1 \cup H_2 \cup H_3$. Otherwise, there exists some Hamiltonian path $P \in H \backslash \{H_1 \cup H_2 \cup H_3\}$. $P \notin H_i$ implies that node i is not an interior node of P for $i \in \{1, 2, 3\}$. But any Hamiltonian path has only two nodes that are not interior nodes. Hence, $H = H_1 \cup H_2 \cup H_3$.

Any LS2F, L, in $\hat{C}^{r,i}$ can be extended to a Hamiltonian path, P, such that $\hat{C}^{r,i}(L) = \hat{C}_i(P)$ and $C(P) = \hat{C}_i(P) + (2n-4)\alpha$. This is done by first re-introducing node i and all arcs (with zero cost) incident with it. This extends any LS2F into a Hamiltonian path with node i as an interior node. Then the cost of the Hamiltonian path with respect to the original matrix is obtained by adding back the reduction constants from each row and column. But all entries in row and column i of C have some fixed cost α, so all the reduction constants have cost α. Since every LS2F has $(n-3)$ arcs in the reduced submatrix, it follows that the value of the Hamiltonian path in C is precisely $[2(n-3)]\alpha + 2\alpha$ more than it's cost in $\hat{C}_i$.

Thus, C is DPC(k) precisely when $|S| = k$ and each $\hat{C}^{r,i}$ contains at most k distinct LS2F values, for $i \in \{1, 2, 3\}$. $\square$

We now consider a slightly more general class of matrices where it is possible to determine if all Hamiltonian cycles have at most three distinct values given a characterization of LS2F(l) matrices for $l \in \{1, 2, 3\}$.

Definition 2.2.3. Cost matrix C is *LS2F(3)-reconcilable* if C is:

(a) DPC(k) for $k \in \{1, 2\}$, or

(b) IC(1) with all entries in row and column i of constant cost.

Moreover, if (b) holds then deleting row and column i from C yields a LS2F(3)-reconcilable submatrix.

Let $f(k, C)$ denote the function which tests whether all LS2Fs described by cost matrix C have at most k distinct values. If cost matrix C contains at most k distinct values then f outputs a list of the distinct LS2F cost values $W = \{w^1, \ldots, w^j\}$ for $j \leq k$. Otherwise, f outputs an indicator element (such as a null element). Thus,

$$f(k, C) = \begin{cases} W & \text{if there are at most } k \text{ distinct LS2F cost values in } C, \\ null & \text{otherwise.} \end{cases}$$

Theorem 2.2.4. *If C is LS2F(3)-reconcilable then it is possible to determine if C is DPC(k) for $k \in \{1, 2, 3\}$.*

Proof. If C is DPC(k) for $k \in \{1, 2\}$ then we are done. Set $i = 1$. Since C is LS2F(3)-reconcilable there exists some index x_i such that all entries in both row and column x_i have a constant cost α_i. Moreover, deleting this row and column yields a LS2F(3)-reconcilable submatrix. Denote this submatrix D_i. If D_i is not DPC(k) for $k \in \{1, 2\}$ then take $i \leftarrow i+1$ and re-iterate the deletion process. Otherwise, there are Hamiltonian path values v_1 and v_2 (whose uniqueness is dependent on the value of k). Notice that since there are a finite number of indices, some submatrix of C must be DPC(k) for $k \in \{1, 2\}$.

Test $f(3, D_i)$. If $f(3, D_i) = null$ then there are at least four LS2F values in D_i. Denote four of these values w^1, w^2, w^3 and w^4. But notice that every LS2F in D_i can be extended to a Hamiltonian path in D_{i-1} with i as an internal node. Hence, there exist Hamiltonian paths in D_i of at least four distinct costs $w^j + 2\alpha_i$ for $j \in \{1, 2, 3, 4\}$. If $D_{i-1} = C$ then there exist more than three distinct Hamiltonian path values in C, so we are done. Otherwise, work backward to create C from D_{i-1} by adding row and column x_{i-1} with all elements of cost α_{i-1} then setting $i \leftarrow i - 1$. Any Hamiltonian path in D_{i-1} can be extended to a Hamiltonian path in C by adding back these indices one at a time. This is simply a linear shift (of cost $\alpha_{i-1} + \alpha_{i-2} + \ldots + \alpha_1$) of the distinct Hamiltonian path values found in D_{i-1}. Hence, C contains more than three distinct Hamiltonian path values.

Otherwise, the function f returns a list $W_i = \{w_i^1, w_i^2, w_i^3\}$ of at most three distinct values. Notice that any Hamiltonian path in D_{i-1} with i as an internal node contains some LS2F of D_i. Furthermore, any Hamiltonian path in D_{i-1} with i as an end node contains some Hamiltonian path of D_i. Conversely, every LS2F and Hamiltonian path in D_i can be extended to a Hamiltonian path in D_{i-1}. It now follows that D_{i-1} is DPC(3) if and only if $|S_{i-1}| = 3$ where $S_{i-1} = \{w_i^1 + 2\alpha_i, w_i^2 + 2\alpha_i, w_i^3 + 2\alpha_i, v_1 + \alpha_i, v_2 + \alpha_i\}$. (Since D_{i-1} was tested to see if it contained at most two distinct Hamiltonian path values, at this stage there exist at least three distinct Hamiltonian path values.)

Now re-iterate the process by testing $f(3, D_{i-1})$ and obtaining a list of values W_{i-1}. Let $W_{i-1}^{2\alpha_{i-1}}$ denote the set of values obtained by adding $2\alpha_{i-1}$ to each element in W_{i-1}. Furthermore, let $S_{i-1}^{\alpha_{i-1}}$ denotes the set of values obtained by adding α_{i-1} to each element in S_{i-1}. Then D_{i-2} is DPC(3) if and only if $|S_{i-2}| = 3$ where $S_{i-2} = \{W_{i-1}^{2\alpha_{i-1}}, S_{i-1}^{\alpha_{i-1}}\}$. Repeat this process (by decrementing i) until the number of distinct Hamiltonian path values in C is determined. $\square$

By considering $\hat{C}^r$ rather than the original matrix C the true potential of Theorem 2.2.4 may be exploited. Theorem 2.1.3 provides a characterization of DTC(k) matrices based on a characterization of DPC(k) matrices. Theorem 2.2.2 and Theorem 2.2.4 provide characterizations of DPC(k) matrices based on a characterization of LS2F(l) matrices ($\forall l \leq k$). Hence, we now shift our focus to determining the structure of matrices in terms of LS2Fs.

Chapter 3

LS2F(1) and LS2F(2) Matrices

Let G be a complete directed graph and C be a cost matrix associated with G. We first focus on the problem of characterizing all cost matrices that are LS2F(1) and then focus on characterizing all cost matrices that are LS2F(2).

3.1 LS2F(1)

If $n = 2$ then cost matrix C is a 2×2 matrix. So every LS2F is composed of two paths of zero length. Thus, it is vacuously true that if C is a 2×2 matrix then all LS2Fs have a single cost.

Two cost matrices C and D associated with graph G are *equivalent* if and only if $C(T) - D(T) = \alpha$ for some constant α and every tour T in G.

Observation 3.1.1. *Every DTC(1) matrix is equivalent to a matrix with tour cost zero.*

Proof. Suppose every tour in G described by C has cost α. Subtract α from every entry in row i of C to create cost matrix D. By Observation 2.1.1, the new cost matrix D is DTC(1) and the cost of each tour has been reduced by α. Hence, all tours described by D have cost 0. $\square$

Lemma 3.1.2. *Let $n > 2$. If C is an LS2F(1) cost matrix associated with graph G then any tour in G contains arcs of a single cost.*

Proof. Let C be an LS2F(1) cost matrix associated with graph G. Furthermore, let T be any tour in G with arcs of distinct costs α and β. Since $n > 2$, without loss of generality,

T contains another arc of cost α. Then there exist LS2Fs of costs $C(T) - (\alpha + \beta)$ and $C(T) - (2\alpha)$ in G. This contradicts the fact that C is LS2F(1). $\square$

Theorem 3.1.3. *Cost matrix C is LS2F(1) if and only if:*
(a) $n = 2$ or
(b) all entries of C have constant cost.

Proof. Note that both the sufficiency and necessity of (a) is trivially established.

Suppose C is an LS2F(1) cost matrix associated with graph G. Let T_1 and T_2 be any two tours in G. Furthermore, suppose T_1 has an arc of cost α and T_2 has an arc of cost $\beta \neq \alpha$. Then $C(T_1) = n\alpha$ and $C(T_2) = n\beta$ by Lemma 3.1.2. So there exist LS2Fs of costs $(n-2)\alpha$ and $(n-2)\beta$. Since G is LS2F(1), $(n-2)\alpha = (n-2)\beta \Rightarrow \alpha = \beta$ which gives a contradiction.

Suppose that all entries of C have a constant cost, α. Then every tour in G has cost $n\alpha$. Since every tour contains only arcs of cost α, all LS2Fs have cost $n\alpha - 2\alpha = (n-2)\alpha$. Thus, C is LS2F(1). $\square$

Using Theorem 3.1.3, we can give an alternative characterization of DTC(1) and DPC(1) matrices as provided in the corollaries below.

Corollary 3.1.4. *Cost matrix C is LS2F(1) if and only if it is DPC(1).*

Proof. This follows directly from the characterization of DPC(1) matrices provided in [10] and Theorem 3.1.3. $\square$

Corollary 3.1.5. *Let $n > 2$. Cost matrix C is DTC(1) if and only if $\hat{C}^r$ is LS2F(1).*

Proof. From Observation 2.1.2 and Theorem 2.1.3, C is DTC(1) if and only if $\hat{C}^r$ is DPC(1). This result now follows from Corollary 3.1.4. $\square$

The following theorem gives another relationship between an LS2F(1) matrix and an associated DTC(1) matrix.

Theorem 3.1.6. *If cost matrix C is DTC(1) then $\overline{C}^r$ is LS2F(1).*

Proof. Suppose C is an $n \times n$ DTC(1) matrix. By Theorem 2.1.3, $\hat{C}^r$ is a DPC(1) matrix (associated with graph G). By Lemma 2.1.4, all tours in G described by $\hat{C}^r$ contain arcs of a single cost. Let T_1 and T_2 be two tours of G. Tour T_1 has cost $(n-1)\alpha$ using only arcs of cost α. Suppose tour T_2 has an arc of cost β then T_2 has cost $(n-1)\beta$. Thus, $\alpha = \beta$ implies that all cost elements of $\hat{C}^r$ are a single constant α.

Choose $v_2 \in V(G)\backslash\{v_1\}$ and form $\overline{C}$ by reducing all elements of both row and column v_2 to zero. This reduces all elements that are not in row or column v_2 by 2α. Eliminating row and column v_2 from $\overline{C}$ yields $\overline{C}^r$, a constant matrix of with all entries of cost $\alpha - 2\alpha = -\alpha$. By Theorem 3.1.3, $\overline{C}^r$ is LS2F(1). □

Interestingly, if $\overline{C}^r$ is an LS2F(1) cost matrix then it is not necessarily true that C is DTC(1). The failure occurs when forming $\hat{C}^r$ from $\overline{C}$. The number of Hamiltonian paths is not preserved during this reduction except in very limited cases. The following examples will illustrate this remark.

Example 3.1.7. Let $\overline{C}^r$ be LS2F(1) with all entries having cost α. If $\hat{C}^r$ is created from $\overline{C}$ by setting $\hat{c}^r_{ij} = a_i + b_j + \bar{c}_{ij}$ with $a_{v_2} = b_{v_2} = 0$ and $a_i = b_j = -\alpha \ \forall\ i \neq v_2 \ \forall\ j \neq v_2$ then C will be DTC(1).

Proof. Given $\bar{c}^r_{ij} = \alpha \ \forall i \neq j$ notice that $\bar{c}_{iv_2} = \bar{c}_{v_2j} = 0$ and $\bar{c}_{ij} = \alpha \ \forall\ i \neq v_2 \ \forall\ j \neq v_2$. So

$\hat{c}^r_{iv_2} = a_i + b_{v_2} + \bar{c}_{iv_2} = -\alpha + 0 + 0 = -\alpha,$

$\hat{c}^r_{v_2j} = a_{v_2} + b_j + \bar{c}_{v_2j} = 0 - \alpha + 0 = -\alpha$ and

$\hat{c}^r_{ij} = a_i + b_j + \bar{c}_{ij} = -\alpha - \alpha + \alpha = -\alpha$ otherwise.

Hence, $\hat{C}^r$ is a constant matrix with all entries having cost $-\alpha$. It is clear that $\hat{C}^r$ is DPC(1) as any tour has cost $-n\alpha$ and every Hamiltonian path has cost $-(n-1)\alpha$. By Theorem 2.1.3, C is DTC(1). □

Example 3.1.8. Given $\overline{C}^r$ as shown below perform the following operations:

(a) create $\overline{C}$ by inserting row and column $v_2 = 1$,

(b) create $\hat{C}^r$ by setting $\hat{c}^r_{ij} = a_i + b_j + \bar{c}_{ij}$ with $a_i = b_i = 1 \ \forall\ i = 2, \dots, 4$ and $a_1 = b_1 = 0$.

(c) create $\hat{C}$ by inserting row and column $v_1 = 1$, then

(d) create C by setting $c_{ij} = a_i + b_j + \hat{c}_{ij}$ with $a_i = b_i = 0 \ \forall\ i = 1, \dots, 4$.

$$\overline{C}^r = \begin{pmatrix} \infty & 1 & 1 \\ 1 & \infty & 1 \\ 1 & 1 & \infty \end{pmatrix}$$

$$C = \begin{pmatrix} \infty & 0 & 0 & 0 & 0 \\ 0 & \infty & 1 & 1 & 1 \\ 0 & 1 & \infty & 3 & 3 \\ 0 & 1 & 3 & \infty & 3 \\ 0 & 1 & 3 & 3 & \infty \end{pmatrix}$$

From Theorem 3.1.3, $\overline{C}^T$ is LS2F(1). However, C contains tours $T_1 = (1,2,3,4,5)$ and $T_2 = (1,5,3,2,4)$ with $C(T_1) = 7$ and $C(T_2) = 5$. Hence, C is not DTC(1).

Thus, the relationship between a DTC(1) matrix and an associated DPC(1) matrix almost extends directly to the relationship between a DTC(1) matrix and an associated LS2F(1) matrix with minor exceptions. We now explore the structure of LS2F(2) matrices.

3.2 LS2F(2)

From Theorem 3.1.3, it is known that if cost matrix C is LS2F(2) then C contains at least two elements with distinct costs α and β. Clearly, a 2×2 cost matrix can not be LS2F(2). If cost matrix C associated with graph G is 3×3 then every tour in G is composed of three arcs. Since G contains arcs of costs α and β and each tour is a 3-cycle, there exist LS2Fs of costs α and β. If C is LS2F(2) then all LS2Fs must have either cost α or β. Hence, all arcs of G have either cost α or β. Conversely, suppose C is a 3×3 cost matrix containing only elements of distinct costs α and β. Any LS2F will contain only a single arc since any tour contains precisely three arcs. Hence, all LS2Fs have either cost α or β, which implies that C is LS2F(2). Thus, we assume that $n \geq 4$.

Lemma 3.2.1. *Let $n \geq 4$. If cost matrix C is LS2F(2) then no tour in graph G contains arcs with more than two distinct costs.*

Proof. Let T be a tour in G containing arcs of distinct costs α, β and γ. There exist LS2Fs of costs $C(T) - (\alpha + \beta)$, $C(T) - (\alpha + \gamma)$ and $C(T) - (\beta + \gamma)$ in G. All of these LS2F values must be distinct since α, β and γ are all distinct. $\square$

The following constructions are used in various proofs. For compactness they are summarized below.

Construction 3.2.2. ***Ordered 3-Exchange:*** *Take a tour $T_1 = (u_1, u_2, \ldots, u_n)$ in graph $G = (V, E)$. Let $pq \in E(G) \backslash E(T_1)$. Without loss of generality, let $u_1 = p$ and $u_r = q$ for* *some $2 < r < n$. Also take integer l such that $r \leq l \leq n$. Construct a new tour with the following structure $T_2 = (u_1, u_r, u_{r+1}, \ldots, u_l, u_2, u_3, u_{r-1}, u_{l+1}, u_{l+2}, u_n)$. See Figure 3.1 for an illustration.*

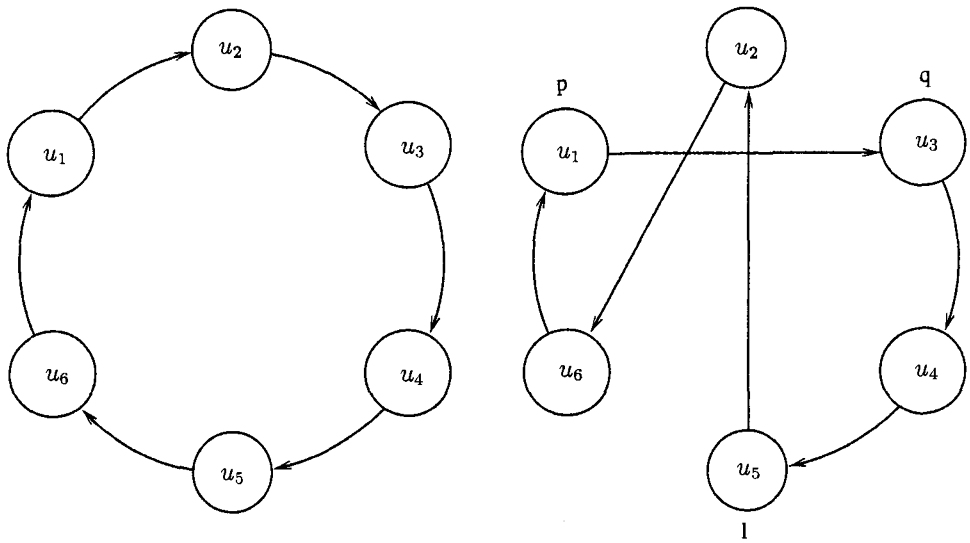

Figure 3.1: Ordered 3-exchange procedure with $n = 6$, $p = u_1$, $q = u_r = u_3$ and $l = u_5$

Construction 3.2.3. ***Arc Reversal:*** *Take tour $T_1 = (u_1, u_2, \ldots, u_n)$ in graph $G = (V, E)$. Let $pq \in E(T_1)$. Without loss of generality, let $u_1 = p$ and $u_2 = q$. Construct a new tour by taking $T_2 = (u_2, u_1, u_3, \ldots, u_n)$. See Figure 3.2 for an illustration.*

Lemma 3.2.4. *Let α and β be distinct constants and $n \geq 4$. If there exists a tour T_1 using only arcs of cost α and a tour T_2 using only arcs of cost β in graph G described by cost matrix C then there exists a tour using arcs with both costs α and β.*

Proof. Take tour T_1 in G and perform an ordered 3-exchange or arc reversal (see Constructions 3.2.2 and 3.2.3) on T_1 by introducing any arc of T_2. This construction only removes three arcs from T_1. But G contains at least four nodes so any tour will contain at least four arcs. Hence, the newly formed tour has arcs of both costs α and β. □

Lemma 3.2.5. *Let α and β be distinct constants and $n \geq 4$. If cost matrix C is LS2F(2) then no tour with costs described by C uses both multiple arcs of cost α and multiple arcs of cost β.*

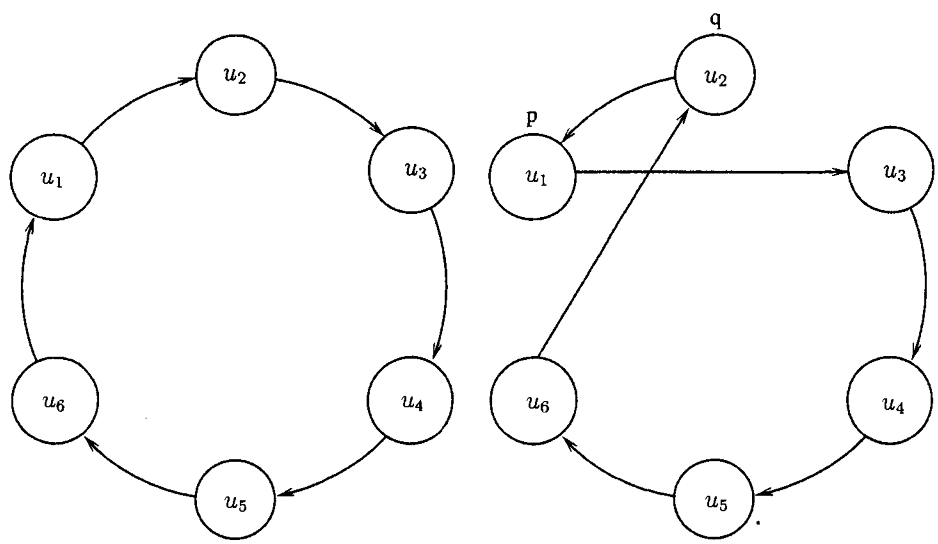

Figure 3.2: Arc reversal procedure with $n = 6$, $p = u_1$ and $q = u_2$

Proof. Suppose there exists a tour T with costs described by C using at least two arcs of cost α and at least two arcs of cost $\beta \neq \alpha$. Then there exist LS2Fs of costs $C(T) - 2\alpha$, $C(T) - 2\beta$ and $C(T) - (\alpha + \beta)$ which are all distinct since $\beta \neq \alpha$. □

We now determine the number of distinct cost elements that any LS2F(2) cost matrix may contain.

Lemma 3.2.6. *Let $n \geq 4$. If cost matrix C is LS2F(2) then C contains exactly two distinct cost elements.*

Proof. Let C be a cost cost matrix associated with graph G. By Theorem 3.1.3, C must contain at least two distinct cost elements. Denote these distinct elements by α and β. From Lemma 3.2.4, there exists a tour T_1 in G using arcs of both costs α and β. Without loss of generality, Lemma 3.2.5 guarantees that T_1 has a single arc of cost α and all other arcs of cost β. If C contains only elements of costs α and β then the result follows. So assume, if possible, that C also contains an arc of cost $\gamma \notin \{\alpha, \beta\}$.

Perform an ordered 3-exchange or arc reversal (see Constructions 3.2.2 and 3.2.3) on T_1 to include an arc of cost γ to create T_γ. Suppose that the arc of cost α does not appear in T_γ. Then T_γ contains an arc of cost γ and all remaining arcs have cost β (from Lemma

3.2.5). Thus, T_γ emits an LS2F of cost $(n-3)\beta+\gamma$. But T_1 also emits LS2Fs of costs $(n-2)\beta$ and $(n-3)\beta+\alpha$. Hence, C contains more than three distinct LS2F values.

Thus, it must be the case that the arc of cost α in T_1 is also in T_γ. If $n>4$ then T_γ contains arcs of distinct costs α, β and γ which contradicts Lemma 3.2.1. So $n=4$ and there exist LS2Fs of costs $\alpha+\beta$, $\alpha+\gamma$ and 2β. Notice that T_γ must also emits an LS2F of either cost 2α or 2γ. In either case, there exist more than two distinct LS2F values in C. □

Lemma 3.2.7. *Let α and β be distinct constants and $n \geq 4$. If C is an LS2F(2) cost matrix with a tour T_1 containing multiple arcs of cost β and an arc of cost α then no tour in C contains multiple arcs of cost α.*

Proof. Suppose T_1 is a tour containing multiple arcs of cost β and at least one arc of cost α. From Lemma 3.2.5, T_1 does not contain multiple arcs of cost α. Suppose that tour T_2 contains multiple arcs of cost α. From Lemma 3.2.5, T_2 can contain at most a single arc not having cost α. By Lemma 3.2.6, C contains exactly two distinct costs α and β. Thus, either (i) $C(T_2)=n\alpha$ implies that there exist LS2Fs of cost $(n-2)\alpha$, or (ii) $C(T_2)=(n-1)\alpha+\beta$ implies that there exist LS2Fs of costs $(n-2)\alpha$ and $(n-1)\alpha+\beta$. Notice that in both cases there exist LS2Fs of cost $(n-2)\alpha$.

Tour T_1 yields LS2Fs of unique costs $(n-2)\beta$ and $\alpha+(n-3)\beta$. Since C is LS2F(2) either $(n-2)\alpha=(n-2)\beta$ or $(n-2)\alpha=\alpha+(n-3)\beta$. If $(n-2)\alpha=(n-2)\beta \Rightarrow \alpha=\beta$ which gives a contradiction. If $(n-2)\alpha=\alpha+(n-3)\beta \Rightarrow (n-3)\alpha=(n-3)\beta \Rightarrow \alpha=\beta$ which gives a contradiction.

Therefore, if there exists a tour containing multiple arcs of cost β and an arc of cost α in LS2F(2) cost matrix C then no tour contains multiple arcs of cost α. ■

We now give a complete characterization of cost matrices where all linear spanning 2-forests have exactly two distinct values.

Theorem 3.2.8. *Cost matrix C is LS2F(2) if and only if:*
(a) $n=3$ and C contains only elements of distinct costs α and β, or
all entries of C have cost $\beta \neq \alpha$ except for
(b) a single row containing at least one element of cost α, or
(c) a single column containing at least one element of cost α, or
(d) $c_{pq}=c_{qp}=\alpha$ for exactly two fixed nodes p and q.

Proof. First note that both the sufficiency and necessity of (a) is trivially established. Assume $n \geq 4$.

Suppose C is LS2F(2). By Lemma 3.2.6, since $n \geq 4$, it follows that C contains exactly two distinct cost elements $\alpha \neq \beta$. Without loss of generality, assume that there are at least as many elements of cost β in C as there are elements of cost α. By Theorem 3.1.3, Lemma 3.2.4 and Lemma 3.2.5, there exists a tour T_1 using multiple arcs of cost β and a single arc of cost α. Denote the arc of T_1 with cost α as pq for fixed nodes $p \neq q$ (i.e. $c_{pq} = \alpha$). Notice that $c_{ij} = \beta$ if (i) i, j, p and q are all distinct, or (ii) $i \neq q$ and $j = p$, or (iii) $i = q$ and $j \neq p$. Otherwise, a tour using (at least) two arcs of cost α can be easily found which contradicts Lemma 3.2.7.

Suppose $c_{pj} = \alpha$ for $j \neq q$. Notice that if either $c_{qp} = \alpha$ or $c_{iq} = \alpha$ for $i \neq p$ then it is trivial to find a tour using multiple arcs of cost α, which once again contradicts Lemma 3.2.7. Hence, all arcs of cost α are contained in row p.

Suppose $c_{iq} = \alpha$ for $i \neq p$. Notice that if either $c_{qp} = \alpha$ or $c_{pj} = \alpha$ for $j \neq q$ then it is trivial to find a tour using multiple arcs of cost α, which gives a contradiction. Hence, all arcs of cost α are contained in column q.

Suppose $c_{qp} = \alpha$. Notice that if either $c_{pj} = \alpha$ for $j \neq q$ or $c_{iq} = \alpha$ for $i \neq p$ then it is trivial to find a tour using multiple arcs of cost α, which gives a contradiction. Hence, only pq and qp have cost α.

Conversely, if (b) is satisfied then any tour in G described by C uses at most a single arc of cost α, as all such arcs are entering a single node. If (c) is satisfied then any tour in C uses at most a single arc of cost α, as all such arcs are leaving a single node. If (d) is satisfied then any tour can use at most one arc of cost α as no tour can utilize both pq and qp. It should be noted that a tour could use both these arcs if G contains only two nodes, but recall that G contains at least four nodes (as all smaller cases have been exhausted). Since all cases (b)-(d) allow tours to contain at most a single arc of cost α, all tours have cost $n\beta$ or $(n-1)\beta + \alpha$. Moreover, since C contains an element of cost α there must be a tour of cost $(n-1)\beta + \alpha$. Hence, all LS2Fs in G have cost $(n-2)\beta$ or $(n-3)\beta + \alpha$. Therefore, C is LS2F(2). $\square$

An immediate consequence of Theorem 3.2.8 is that we are able to give a characterization of symmetric cost matrices that are LS2F(2). This result will prove to be helpful when considering the TSP with an associated symmetric cost matrix.

Corollary 3.2.9. *Symmetric cost matrix C is LS2F(2) if and only if all entries of C have cost $\beta \neq \alpha$ except for exactly two elements with $c_{pq} = c_{qp} = \alpha$ for fixed nodes p and q.*

3.3 Structure of DTC(2) matrices

Let us now investigate the relationship between an LS2F(2) matrix and an associated DTC(2) matrix.

Theorem 3.3.1. *Let $n \geq 4$ denote the number of rows in the cost matrix $\hat{C}^r$. If $\hat{C}^r$ is LS2F(2) then C is DTC(2).*

Proof. Suppose $\hat{C}^r$ is LS2F(2). The structure of $\hat{C}^r$ is given explicitly in Theorem 3.2.8. Let α and β be distinct constants contained in $\hat{C}^r$. The tours in $\hat{C}^r$ have values $\alpha + (n-1)\beta$ ans $n\beta$. It should be noted that there may not be a tour of cost $n\beta$ described by $\hat{C}^r$ but this does not affect the analysis. Notice that every Hamiltonian path will have either cost $\alpha + (n-2)\beta$ or $(n-1)\beta$. Thus, $\hat{C}^r$ is a DPC(2) matrix. By Theorem 2.1.3, C is DTC(2). □

Note that the restriction $n \geq 4$ in Theorem 3.3.1 is necessary as illustrated by the following example.

Example 3.3.2. Let $\alpha \neq \beta$ and

$$\hat{C}^r = \begin{pmatrix} \infty & \alpha & \beta \\ \alpha & \infty & \alpha \\ \beta & \beta & \infty \end{pmatrix}.$$

Form cost matrix C by inserting row and column 1 then adding back zero cost to every row and column of $\hat{C}$. Thus,

$$C = \begin{pmatrix} \infty & 0 & 0 & 0 \\ 0 & \infty & \alpha & \beta \\ 0 & \alpha & \infty & \alpha \\ 0 & \beta & \beta & \infty \end{pmatrix}.$$

Since $\hat{C}^r$ is 3×3 and contains exactly two distinct cost elements, it follows from Theorem 3.2.8 that it is LS2F(2). However, C contains tours $T_1 = (1,2,3,4)$, $T_2 = (1,3,4,2)$ and $T_3 = (1,2,4,3)$ of costs 2α, $\alpha + \beta$ and 2β respectively. Hence, C is not DTC(2).

Also note that the converse of Theorem 3.3.1 does not necessarily hold. This is established in Example 3.3.3.

Example 3.3.3. Suppose C is DTC(2). By Theorem 2.1.3, $\hat{C}^r$ is DPC(2). If the converse of Theorem 3.3.1 were true then we need only show that if $\hat{C}^r$ is DPC(2) then $\hat{C}^r$ is LS2F(2).

Let

$$\hat{C}^r = \begin{pmatrix} \infty & \beta & \beta & \beta \\ \beta & \infty & \beta & \beta \\ \alpha & \alpha & \infty & \alpha \\ \alpha & \alpha & \alpha & \infty \end{pmatrix}.$$

Suppose $\hat{C}^r$ is associated with graph G. Notice that $\hat{C}^r$ is DTC(1) as every arc entering nodes 1 and 2 has cost β and every arc entering nodes 3 and 4 has cost α. Thus, the cost of every tour in G is $2\alpha + 2\beta$. Any Hamiltonian path will have cost $\alpha + 2\beta$ or $2\alpha + \beta$. So $\hat{C}^r$ is DPC(2). But there exist LS2Fs of costs 2α, 2β and $\alpha + \beta$. Hence, $\hat{C}^r$ is LS2F(3) not LS2F(2).

The previous examples illustrate some potential difficulties in obtaining an explicit characterization of DTC(2) matrices from our characterization of LS2F(2) matrices. However, the discussions in Section 2.2 illustrate that determining if a arbitrary cost matrix is DTC(2) can be done in polynomial time using the characterizations of LS2F(1) and LS2F(2) matrices in Theorem 3.1.3 and Theorem 3.2.8. This process will also determine if the given matrix is DPC(2).

Chapter 4

LS2F(3) Matrices

In this chapter, we provide a closed form characterization of LS2F(3) matrices. As may be anticipated, obtaining such a characterization is more complex than the arguments needed for the LS2F(2) counter-part. Thus, we first formulate several simple results which will be used to establish our main résult.

Observation 4.0.1. *Let C be an $n \times n$ cost matrix and C^ϕ be a cost matrix obtained by subtracting a constant ϕ from every entry in C. The matrix C is LS2F(k) if and only if C^ϕ is LS2F(k).*

Proof. Take any tour T in graph G described by cost matrix C. LS2Fs have costs of the form $C(T) - (c_e + c_f)$ for some arcs e and f in $E(T)$. Subtract a constant ϕ from every entry of C to obtain C^ϕ. The corresponding LS2F in the new matrix C^ϕ has cost $[C(T) - n\phi] - [(c_e - \phi) + (c_f - \phi)] = [C(T) - (c_e + c_f)] - [(n+2)\phi]$. Notice that $(n+2)\phi$ is a constant independent of tour T and arcs e and f. Hence, the result follows. □

Using Observation 4.0.1, it may be assumed that, without loss of generality, any cost matrix we consider contains at least one element of cost zero.

Observation 4.0.2. *Let $n \geq 4$ and T be the tour containing the highest number of distinct arc costs in graph G described by cost matrix C. If G contains at least two arcs of distinct costs then all arcs of T can not have the same cost.*

Proof. If all arcs of T have cost α, then all other tours only use arcs of a single cost as well (by the definition of T). Since C contains at least two distinct arc costs, α and β, perform an ordered 3-exchange or arc reversal (see Constructions 3.2.2 and 3.2.3) on T to include an arc of cost β. Notice that $n \geq 4$ implies that the newly formed tour retains an arc of cost α since only three arcs of T are effected by either construction technique. Hence, the newly formed tour contains arcs of both costs α and β, contradicting the choice of T. $\square$

Lemma 4.0.3. *If C is an LS2F(3) cost matrix associated with graph G then any tour in G contains arcs of at most three distinct costs.*

Proof. Suppose T is a tour in G using arcs of distinct costs α, β, γ and δ. Then there exist LS2Fs of costs $C(T) - (\alpha + \delta)$, $C(T) - (\beta + \delta)$ and $C(T) - (\gamma + \delta)$ which are all distinct. But there are also LS2Fs of costs $C(T) - (\alpha + \beta)$, $C(T) - (\alpha + \gamma)$ and $C(T) - (\beta + \gamma)$. Since C is LS2F(3) and α, β, γ and δ are all distinct constants, it follows that $C(T) - (\alpha + \beta) = C(T) - (\gamma + \delta)$, $C(T) - (\alpha + \gamma) = C(T) - (\beta + \delta)$ and $C(T) - (\beta + \gamma) = C(T) - (\alpha + \delta)$. But this system has no solution with distinct constants α, β, γ and δ. $\square$

Lemma 4.0.4. *If C is an LS2F(3) matrix associated with graph G that has a tour T containing arcs of three distinct costs then T does not contain multiple arcs of two distinct costs.*

Proof. Without loss of generality, assume T contains multiple arcs of cost α, multiple arcs of cost β and (at least) one arc of cost γ, where α, β and γ are all distinct. Then there exist LS2Fs of costs $C(T) - 2\alpha$, $C(T) - 2\beta$, $C(T) - (\alpha + \beta)$, $C(T) - (\alpha + \gamma)$ and $C(T) - (\beta + \gamma)$. Since α, β and γ are all distinct, $C(T) - (\alpha + \beta)$, $C(T) - (\alpha + \gamma)$ and $C(T) - (\beta + \gamma)$ are all distinct. Furthermore, $C(T) - 2\beta \notin \{C(T) - (\alpha + \beta), C(T) - (\beta + \gamma)\} \Rightarrow$ (1) $C(T) - 2\beta = C(T) - (\alpha + \gamma)$. Similarly, (2) $C(T) - 2\alpha = C(T) - (\beta + \gamma)$. The difference of (1) and (2) yields $-2\beta + \alpha = -(\alpha + \gamma) + (\beta + \gamma) \Rightarrow \alpha = \beta$ which gives a contradiction. $\square$

Lemma 4.0.5. *If C is an LS2F(3) matrix associated with graph G that has a tour containing arcs of distinct costs α, β and multiple arcs of cost γ then $2\gamma = \alpha + \beta$. In particular, if $\gamma = 0$ then $\alpha = -\beta$.*

Proof. Given the conditions of the lemma, there exist LS2Fs of costs $C(T) - 2\gamma$, $C(T) - (\alpha + \beta)$, $C(T) - (\alpha + \gamma)$ and $C(T) - (\beta + \gamma)$ for some tour T in G described by C. Since α, β and γ are all distinct and C is LS2F(3) it follows that $C(T) - 2\gamma = C(T) - (\alpha + \beta) \Rightarrow 2\gamma = \alpha + \beta$. $\square$

Let $n = 2$. Then all LS2Fs have zero cost and any associated cost matrix C can not be LS2F(3). Let $n = 3$. Then every LS2F is only comprised of a single arc as every tour is a 3-cycle. Hence, C is LS2F(3) if and only if there exist arcs of precisely three distinct costs in C. For the remainder of the chapter, it will be assumed that $n \geq 4$.

Observation 4.0.6. *Let T be a tour in G with costs described by LS2F(3) cost matrix C and $n \geq 4$. It is possible to subtract a constant from all elements of C in such a way that the tour T will use multiple arcs of cost 0.*

Proof. Suppose that tour T uses multiple arcs of cost $\alpha \neq 0$. Then by Observation 4.0.1, reduce every element of C by α to obtain the desired result. Otherwise, T does not contain multiple arcs of the same cost. By Lemma 4.0.3, T can not contain more than three distinct arc costs. It must be the case that $n \leq 3$ which gives a contradiction. □

By Theorem 3.1.3, it is known that if cost matrix C contains only arcs of a single cost then C is an LS2F(1) matrix. Hence, all LS2F(3) cost matrices can be grouped into three distinct classifications. These classifications will now be considered:

- C contains at least four distinct costs (see Section 4.1),
- C contains exactly three distinct costs (see Section 4.2) and
- C contains exactly two distinct costs (see Section 4.3).

The sections to come examine the permissible structures of LS2F(3) cost matrices for each of the three classifications.

4.1 Cost matrices containing at least four cost values

The main result of this section is to establish that if cost matrix C is LS2F(3) then C contains at most three distinct arc-costs. By Observation 4.0.1, one of the costs of C may be forced to 0 by subtracting ϕ from every element in C to create C^{ϕ}. Furthermore, by Observation 4.0.1 C is LS2F(k) if and only if C^{ϕ} is LS2F(k). Hence, to simplify arguments we consider the matrix C^{ϕ} in the analysis to follow. Recall that Lemma 4.0.3 ensures that no tour can contain more than three distinct arc costs. Also, recall that we are only considering matrices on $n \geq 4$. We now proceed by first establishing some necessary results.

Lemma 4.1.1. *Let C^Φ be a cost matrix associated with graph G that contains at least four distinct costs and $n \geq 4$. If C^Φ is LS2F(3) then there does not exist a tour using three distinct arc costs in G described by C^Φ.*

Proof. Without loss of generality, assume cost matrix C^Φ contains elements of at least four distinct costs. Denote four of these costs by 0, α, β and γ. Let T_1 be a tour in G that contains arcs of costs 0, α and β. From Observation 4.0.6, T_1 has multiple arcs of cost 0. From Lemma 4.0.4, T_1 contains a single arc of cost α and a single arc of cost β. Thus, $C(T_1) = \alpha + \beta$ and there exist LS2Fs of costs 0, α and β. Also, by Lemma 4.0.5, $\alpha = -\beta$. So C^Φ describes LS2Fs of costs 0, α and $-\alpha$. Perform an ordered 3-exchange or arc reversal (see Constructions 3.2.2 and 3.2.3) on T_1 to include an arc of cost γ creating tour T_γ. We now consider the LS2F in T_γ created by deleting the two arcs of unknown costs.

If both the arc of cost α and the arc of cost β from T_1 do not appear in T_γ then T_γ emits an LS2F of cost γ (since all arcs of known cost in T_γ have cost 0). But $\gamma \notin \{0, \alpha, \beta\}$ implies that there are more than three distinct LS2F costs in C^Φ.

If both the arc of cost α and the arc of cost β from T_1 appear in T_γ then T_γ has arcs of costs 0, α, β and γ unless $n = 5$ and all arcs of cost 0 are removed during the construction of T_γ. Assume that $n = 5$. The LS2F in T_γ formed by deleting the two arcs of unknown cost has cost $\alpha + \beta + \gamma = \alpha + (-\alpha) + \gamma = \gamma$. But $\gamma \notin \{0, \alpha, \beta\}$ implies that there are more than three distinct LS2F costs in C^Φ.

If the arc of cost α does not appear in T_γ but the arc of β does appear then we consider the value of n.

- Suppose $n > 5$ then T_γ contains arcs of costs β, γ and multiple arcs of cost 0. From Lemma 4.0.4 and Lemma 4.0.5, it follows that $\beta = -\gamma$. But $\beta = -\alpha$ yields a contradiction since 0, α, β and γ are all distinct.

- Suppose $n = 5$ then T_γ contains an arc of costs 0, β, γ and two arcs of unknown costs. From Lemma 4.0.3, T_γ can only contain these three arc costs. If one of the arcs of unknown cost has cost β then there exists an LS2F of cost 2β which gives more than three distinct LS2F values. Hence, there are only three cases to consider, the arcs of unknown costs are: (a) both 0, (b) both γ or (c) 0 and γ. In (a) and (c), T_γ emits an LS2F of cost γ which gives more than three distinct LS2F values. In (b), T_γ emits LS2Fs of costs 3γ, 2γ, $2\gamma + \beta$ and $\gamma + \beta$. Simple analysis shows that C^Φ has more than three distinct LS2F cost values.

- Suppose $n = 4$ then T_γ contains an arc of cost β, an arc of cost γ and two arcs of unknown costs. If one of these arcs of unknown cost has cost 0 then there exists an LS2F of cost γ. If one of these arcs of unknown cost has cost α then there exists an LS2F of cost $\alpha+\gamma$. If one of these arcs of unknown cost has cost β then there exists an LS2F of cost 2β. If one of these arcs of unknown cost has cost γ then there exists an LS2F of cost 2γ. All four of these cases yield contradictions as there would exist more than three distinct LS2F values in C^ϕ. From Lemma 4.0.3, T_γ can contain arcs of at most three distinct costs. Hence, both arcs of unknown cost have cost $\delta \notin \{0, \alpha, \beta, \gamma\}$. Thus, T_γ emits LS2Fs of costs 2δ, $\delta+\beta$, $\delta+\gamma$ and $\beta+\gamma$. Once again, simple analysis shows that C^ϕ has more than three distinct LS2F values.

If the arc of cost β does not appear in T_γ but the arc of α does appear then apply the same logic used in the previous analysis to achieve a contradiction. $\square$

The following three lemmas consider cost matrices on $n = 4$. These results will prove to be useful in obtaining a more general result (Lemma 4.1.5).

Lemma 4.1.2. *Let cost matrix C^ϕ associated with graph G on $n = 4$ contain arcs of at least four distinct costs, two of which are 0 and α. If G contains a tour $T_1 = (1, 2, 3, 4)$ with $c_{12} = c_{34} = \alpha$ and $c_{23} = c_{41} = 0$ then C^ϕ is not LS2F(3).*

Proof. Let 0, α, β and γ be distinct constants contained in C^ϕ. Tour T_1 yields LS2Fs of costs 2α, α and 0. Denote the remaining five tours of G as follows: $T_2 = (1, 2, 4, 3)$, $T_3 = (1, 3, 4, 2)$, $T_4 = (1, 3, 2, s)$, $T_5 = (1, 4, 2, 3)$ and $T_6 = (1, 4, 3, 2)$.

Notice that tours T_2 and T_3 yield LS2Fs of costs $\alpha + \delta \in \{2\alpha, \alpha, 0\} \Rightarrow \delta \in \{\alpha, 0, -\alpha\}$. Notice that tours T_4 and T_5 yield LS2Fs of costs $0 + \varepsilon \in \{2\alpha, \alpha, 0\} \Rightarrow \varepsilon \in \{2\alpha, \alpha, 0\}$. This implies that $c_{13}, c_{31}, c_{24}, c_{42} \in \{0, \alpha\} = \{\alpha, 0, -\alpha\} \cap \{2\alpha, \alpha, 0\}$.

Also, it is known that C^ϕ contains entries of distinct costs 0, α, β and γ. So without loss of generality, $c_{32} = \beta = 2\alpha$ and $\gamma = -\alpha$ with $\gamma \in \{c_{21}, c_{43}\}$. Consider the arcs of tour T_6. Notice that $c_{14} \notin \{2\alpha, \alpha\}$ or there would exist an LS2F of either cost 4α or 3α using arcs 14 and 32. Hence, $c_{14} = 0$. But recall that 21 and 43 are the only arcs that possibly have cost γ. In either case, it is possible to obtain an LS2F of cost $\gamma = -\alpha$ from T_6 using arc 14 and the arc of cost γ. This obtains a contradiction as C^ϕ is LS2F(3). $\square$

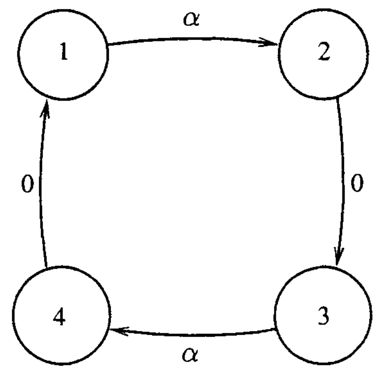

Figure 4.1: Tour $T_1 = (1, 2, 3, 4)$ given $c_{12} = c_{34} = \alpha$ and $c_{23} = c_{41} = 0$

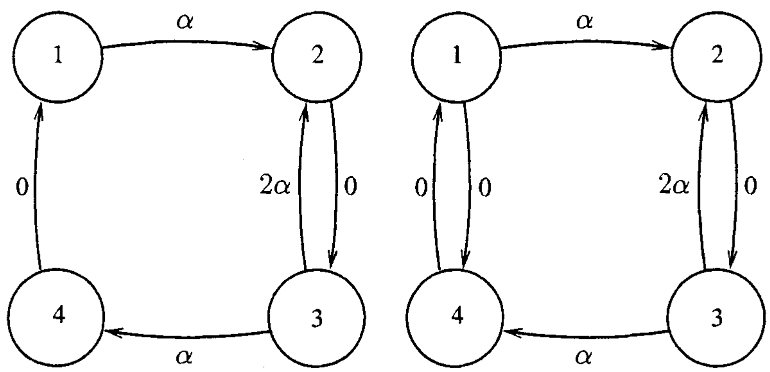

Figure 4.2: Further structure of T_1 given $c_{12} = c_{34} = \alpha$ and $c_{23} = c_{41} = 0$

Lemma 4.1.3. *Let cost matrix C^Φ associated with graph G on $n = 4$ contain arcs of at least four distinct costs, two of which are 0 and α. If G contains a tour $T_1 = (1, 2, 3, 4)$ with $c_{12} = c_{23} = \alpha$ and $c_{34} = c_{41} = 0$ then C^Φ is not LS2F(3).*

Proof. Let 0, α, β and γ be distinct constants contained in C^Φ. Take $T_1 = (1, 2, 3, 4)$ with $c_{12} = c_{41} = \alpha$ and $c_{23} = c_{34} = 0$. There exist LS2Fs of costs 2α, α and 0 from T_1. Tours $(1, 4, 2, 3)$ and $(1, 3, 4, 2)$ yield LS2Fs of cost $0 + \delta$ so $\delta \in \{2\alpha, \alpha, 0\}$. Tours $(1, 3, 2, 4)$ and $(1, 2, 4, 3)$ yield LS2Fs of cost $\alpha + \varepsilon$ so $\varepsilon \in \{\alpha, 0, -\alpha\}$. Since all arcs of G whose cost is not determined are covered by these four tours and C^Φ contains elements of at least four distinct costs it follows (without loss of generality) that $\beta = -\alpha$ and $\gamma = 2\alpha$. Thus, C^Φ can not contain more than four distinct costs. Furthermore, arcs 13 and 31 have

$c_{13}, c_{31} \in \{0, \alpha\} = \{2\alpha, \alpha, 0\} \cap \{\alpha, 0, -\alpha\}$.

If two of the arcs 14, 42 or 21 have cost γ then there would exist an LS2F of cost $\gamma + \gamma = (2\alpha) + (2\alpha) = 4\alpha$ which gives a contradiction. Assume $c_{42} \neq \gamma$. Since C^{Φ} contains four distinct cost values $\{c_{14}, c_{21}\} = \{0, \gamma\}$. Thus, arcs 14, 43, 32 and 21 do not have cost α or there would be an LS2F of cost $\alpha + \gamma = 3\alpha$. Furthermore, c_{43},$c_{32} \neq \beta$ or there would exist an LS2F of cost $\beta = -\alpha$ using tour $(1, 4, 3, 2)$. Hence, $c_{24} = \beta$. But now no cost can be attributed to c_{43} without arriving at a contradiction. So it must be the case that $c_{42} = \gamma$ where arc 42 is the only arc of cost γ in G. Similar arguments show that $c_{24} = \beta$ with only arc 24 having cost β in G. But tour $(1, 2, 4, 3)$ implies $c_{31} \neq 0$ or there would be an LS2F of cost $\beta = -\alpha$. Similarly, tour $(1, 4, 2, 3)$ implies $c_{31} \neq \alpha$ or there would be an LS2F of cost $-\alpha$. Hence, no cost can be attributed to c_{31} without arriving at a contradiction. □

Lemma 4.1.4. *Let cost matrix C^{Φ} associated with graph G on $n = 4$ contain arcs of at least four distinct costs, two of which are 0 and α. If G contains a tour $T_1 = (1, 2, 3, 4)$ with $c_{12} = \alpha$ and $c_{23} = c_{34} = c_{41} = 0$ then C^{Φ} is not LS2F(3).*

Proof. Let 0, α, β and γ be distinct constants. Take tour $T_1 = (1, 2, 3, 4)$ with $c_{12} = \alpha$ and $c_{23} = c_{34} = c_{41} = 0$. There exist LS2Fs of costs 0 and α from T_1. Of the remaining arcs there must be at least one arc of cost β and at least one arc of cost γ. Tour $(1, 4, 2, 3)$ uses arcs 14, 42, 31 and 23 with $c_{23} = 0$. Tour $(1, 3, 4, 2)$ uses arcs 13, 42, 21 and 34 with $c_{34} = 0$. Tour $(1, 3, 2, 4)$ uses arcs 14, 32, 24 and 41 with $c_{41} = 0$.

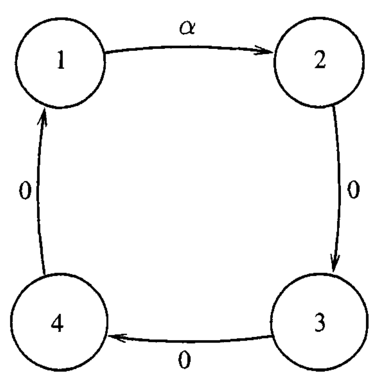

Figure 4.3: Tour structure of T_1 given $c_{12} = \alpha$ and $c_{23} = c_{34} = c_{41} = 0$

Suppose arcs of costs β and γ were associated with two of the arcs 14, 42, 31, 13, 21, 32 or 24 then it would be possible to find LS2Fs of costs β and γ as all arcs in this list have been shown to be contained in a tour using an arc of cost β or γ, and an arc of cost 0. Thus, there can not both be an arc of cost β and an arc of cost γ in this list. Notice that the only arc of unknown cost not contained in the list is the arc 43. Without loss of generality, let $c_{43} = \gamma$ and C^Φ can not contain more than four distinct costs. Since an arc of cost β must be placed in G, arc 43 is the only arc with cost $c_{43} = \gamma$ in G. Furthermore, there exists an LS2F of cost $\alpha + \gamma$ formed by using the tour $(1, 2, 4, 3)$ and deleting the arcs 31 and 24. There is also an LS2F of cost β as previously observed. Hence, there are LS2Fs of costs 0, α, β and $\alpha + \gamma$. Notice tour $(1, 2, 4, 3)$ uses costs $c_{12} = \alpha$ and $c_{43} = \gamma$ and tours can use at most two distinct arc costs. So the other two arcs of this tour have costs $c_{24}, c_{31} \in \{\alpha, \gamma\}$. But C^Φ only contains a single arc of cost γ so $c_{24}, c_{31} = \alpha$. Using tour $(1, 2, 4, 3)$ there is an LS2F of cost 2α with $c_{24}, c_{31} = \alpha$. Tour $(1, 4, 2, 3)$ uses costs $c_{23} = 0$ and $c_{31} = \alpha \Rightarrow c_{14}, c_{42} \in \{0, \alpha\}$. Tour $(1, 3, 2, 4)$ uses costs $c_{41} = 0$ and $c_{24} = \alpha \Rightarrow c_{13}, c_{32} \in \{0, \alpha\}$. This leaves arc 21 as the only candidate to have cost β. But tour $(1, 4, 3, 2)$ uses at least three arc costs with $c_{21} = \beta$, $c_{43} = \gamma$ and $c_{14} \in \{0, \alpha\}$. □

The previous three lemmas are now used to extinguish the case where there exists a tour using two distinct arc costs in a cost matrix containing at least four distinct values.

Lemma 4.1.5. *Let C^Φ be a cost matrix associated with graph G that contains at least four distinct costs and $n \geq 4$. If C^Φ is LS2F(3) then there does not exist a tour using two distinct arc costs in G described by C^Φ.*

Proof. Suppose cost matrix C^Φ contains elements of at least four distinct costs. Denote four of these costs by 0, α, β and γ. Let T_1 be a tour in G that contains arcs of both costs 0 and α. From Observation 4.0.6, T_1 has multiple arcs of cost 0. Force T_1 to contain at least as many arcs of cost 0 as arcs of cost α (which is done by reducing all elements of C^Φ appropriately). So $C(T_1) = k\alpha$ for $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$. Perform an ordered 3-exchange or arc reversal (see Constructions 3.2.2 and 3.2.3) on T_1 to include an arc of cost β creating tour T_β. Repeat this process on T_1 to include an arc of cost γ creating tour T_γ.

If both an arc of cost 0 and an arc of cost α from T_1 appear in T_β then this tour contains (at least) three distinct arc costs. This contradicts Lemma 4.1.1. Thus, either all the arcs of cost 0 or all the arcs of cost α do not appear in the constructed tours (T_β and T_γ). Recall that only three arcs are affected during an ordered 3-exchange or an arc reversal. Since T_1 contains more arcs of cost 0 than α, there can be at most three arcs of cost α in T_1. We now consider the number of arcs of cost α in T_1.

- Suppose $k = 3$ and consider the cases $n > 6$ and $n = 6$ separately.

  Assume $n > 6$. Then no arc of cost α appears in T_β or there would exist arcs of (at least) three distinct costs. Consider the subgraph formed by removing all arcs of cost α from T_1. Notice that this subgraph consists of three disjoint paths. Connect these three paths to create a tour which is different than T_1. But these three paths may only be connected in a single unique way without recreating T_1. Since both T_β and T_γ are formed by deleting the three arcs of cost α from T_1 to create a new tour, it follows that $T_\beta = T_\gamma$. Hence, this tour contains arcs of costs 0, β and γ which contradicts Lemma 4.1.1.

  Assume $n = 6$. Then it is possible that no arc of cost 0 appears in T_β or T_γ. In fact, by the same logic presented when $n > 6$ it follows that at least one of these tours has no arcs of cost 0 from T_1 appearing in it. Without loss of generality, assume T_β is such a tour. Hence, T_β is composed of three arcs of cost α, one arc of cost β and two arcs of unknown cost. If one of the arcs of unknown cost has cost α then there exist LS2Fs of cost 4α from T_β and LS2Fs of costs 3α, 2α and α from T_1. Hence, there exist more than three LS2F values in C^ϕ. Since T_β can contain at most two distinct values, the two arcs of unknown costs both have cost β. So C^ϕ contains LS2Fs of costs 3α, 2α, α, $\alpha + 3\beta$, $2\alpha + 2\beta$ and $3\alpha + \beta$. But this also implies that there are more than three distinct LS2F values in C^ϕ.

- Suppose $k = 2$ and consider the cases $n > 5$, $n = 5$ and $n = 4$ separately. There are LS2Fs of costs 0, α and 2α in T_1.

  Assume $n > 5$. Then no arc of cost α appears in T_β or there would exist arcs of (at least) three distinct costs. Thus, T_β is comprised of one arc of cost β, two arcs of unknown costs and all remaining arcs of cost 0. Hence, T_β yields LS2Fs of cost β (by deleting the two arcs of unknown costs). Similarly, T_γ yields LS2Fs of cost γ. Since 0, α, β and γ are all distinct there exist more than three LS2F values in C^ϕ.

  Assume $n = 5$. Then it is possible that no arc of cost 0 appears in T_β or T_γ. In fact, by the same logic presented when $n > 5$ it follows that at least one of these tours has no arcs of cost 0 appearing in it. Without loss of generality, assume T_β is such a tour. Hence, T_β is composed of two arcs of cost α, one arc of cost β and two arcs of unknown cost. Since any tour can contain arcs of at most two distinct values, the arcs of unknown costs must (a) both have cost β, (b) both have cost α or (c) have one arc of cost α and the other with cost β. In both (b) and (c), T_β yields an LS2F of cost 3α. In (a), T_β yields LS2Fs of costs 3β, $\alpha + 2\beta$ and $2\alpha + \beta$. But in all cases, these LS2F values combined with those from T_1 give more than three distinct values.

Assume $n = 4$. This case is exhausted by Lemma 4.1.2 and Lemma 4.1.3.

- Suppose $k = 1$ and consider the cases $n > 4$ and $n = 4$ separately.

  Assume $n > 4$. Then the arc of cost α does not appear in T_β or there would exist arcs of (at least) three distinct values. Thus, T_β is comprised of one arc of cost β, two arcs of unknown costs and all remaining arcs of cost 0. Hence, T_β yields LS2Fs of cost β (by deleting the two arcs of unknown costs). Similarly, T_γ yields LS2Fs of cost γ. Since 0, α, β and γ are all distinct there exist more than three LS2F cost values in C^ϕ.

  Assume $n = 4$. This case is exhausted by Lemma 4.1.4. □

We now provide the proof to the claim made at the beginning of the chapter.

Theorem 4.1.6. *If cost matrix C is LS2F(3) then C contains at most three distinct arc costs.*

Proof. Suppose C^ϕ contains at least three distinct values. By Lemma 4.0.3, any tour can contain at most three distinct arc costs. By Lemma 4.1.1, any tour can contain at most two distinct arc costs. By Lemma 4.1.5, any tour can contain at most one distinct arc cost. By Observation 4.0.2, there exists a tour using arcs of at least two distinct costs. Hence, if C^ϕ is LS2F(3) then C^ϕ contains at most three distinct arc-costs. By Observation 4.0.1, C is LS2F(k) if and only if C^ϕ is LS2F(k). □

4.2 Cost matrices containing three cost values

Let C be an LS2F(3) cost matrix containing exactly three distinct values. By Observation 4.0.2, there exists a tour in C using arcs of at least two distinct costs. To determine the structure of C we consider three cases.

- C contains a tour using three distinct arc costs (Lemma 4.2.1).
- All tours in C use at most two distinct costs and there exists a tour using multiple arcs of two costs (Lemma 4.2.3).

- All tours in C use at most two distinct costs and there does not exist a tour using multiple arcs of two costs (Lemma 4.2.4).

The following definitions are introduced to condense the arguments regarding the placement of elements within a cost matrix. A *line* of a matrix is either a row or column of the matrix. A *line cover* of cost α in cost matrix C is the minimum set of lines needed to contain all elements of cost α.

Lemma 4.2.1. *Let α, β and γ be distinct constants contained in cost matrix C with $n \geq 4$. The matrix C is an LS2F(3) matrix with a tour containing all three costs if and only if the following conditions are satisfied.*
(a) $2\gamma = \alpha + \beta$.
(b) If there are more than two elements of cost α then α has a line cover of size one.
(c) If there are exactly two elements of cost α then either α has a line cover of size one or there exists nodes p and q such that $c_{pq} = c_{qp} = \alpha$.
(d) Properties (b) and (c) hold for the elements of cost β.
(e) If there is only one element of cost α and only one element of cost β then it is not the case that $c_{pq} = \alpha$ and $c_{qp} = \beta$.
(f) There does not exist a line containing all elements of both costs α and β.

Proof. Suppose that C is an LS2F(3) cost matrix associated with graph G. Furthermore, suppose that G contains a tour T that uses distinct costs α, β and γ. By Lemma 4.0.5, $2\gamma = \alpha + \beta$ where T uses multiple arcs of cost γ. The choice of γ is uniquely defined since $2\beta = \alpha + \gamma$ or $2\alpha = \beta + \gamma$ can not have solutions with α, β and γ all distinct. Thus, every tour that contains arcs of all three distinct costs must contain multiple arcs of cost γ. By Lemma 4.0.4, every tour containing all three distinct costs must contain exactly one arc of cost α and exactly one arc of cost β. This can only be achieved if the required conditions are satisfied.

Conversely, given the conditions stated in the preamble, it is possible to find a tour using at most a single arc of cost α and at most a single arc of cost β. Furthermore, it is possible to find a tour using exactly one arc of each cost as guaranteed by the conditions. Hence, all tours in C have cost $n\gamma$, $(n-1)\gamma + \alpha$, $(n-1)\gamma + \beta$ or $(n-2)\gamma + \alpha + \beta$. Thus, C is LS2F(3) as there exist LS2Fs of costs $(n-2)\gamma$, $(n-3)\gamma + \alpha$, $(n-3)\gamma + \beta$ and $(n-4)\gamma + \alpha + \beta = (n-4)\gamma + (2\gamma) = (n-2)\gamma$ since $2\gamma = \alpha + \beta$. $\square$

Lemma 4.2.1 establishes the structure of LS2F(3) cost matrices containing exactly three distinct cost elements and a tour that uses all three distinct costs. From Lemma 4.0.3, no tour uses arcs of more than three distinct costs. Observation 4.0.2 guarantees that not all tours use arcs of a single cost. Hence, we need only consider cost matrices where all tours use arcs of at most two distinct costs. We first establish a helpful result.

Lemma 4.2.2. *Let α, β and γ be distinct constants contained in an LS2F(3) matrix C. If there exists a tour using multiple arcs of cost α and multiple arcs of cost γ, but no tour using arcs of all three distinct costs then:*
(a) there does not exist a tour using multiple arcs of cost β, and
(b) there does not exist a tour using arcs of both costs α and β, given that there also exists a tour using arcs of both costs γ and β.

Proof. We first show (a). Denote the tour using multiple arcs of both costs α and γ as T. Notice $C(T) = k\alpha + (n-k)\gamma$ for $2 \leq k \leq n-2$ and T yields LS2Fs of costs $k\alpha + (n-k-2)\gamma$, $(k-1)\alpha + (n-k-1)\gamma$ and $(k-2)\alpha + (n-k)\gamma$. Suppose there exists a tour T_1 containing multiple arcs of cost β and multiple arcs of (without loss of generality) cost α. This tour generates LS2Fs of costs $l\alpha + (n-l-2)\beta$, $(l-1)\alpha + (n-l-1)\beta$ and $(l-2)\alpha + (n-l)\beta$ for some $2 \leq l \leq n-2$. It can be verified that these six LS2Fs have at least four distinct values. This gives a contradiction.

Now suppose that there exists a tour T_2 using multiple arcs of cost β and (without loss of generality) a single arc of cost α. Then $C(T_2)$ yields LS2Fs of costs $(n-2)\beta$ and $(n-3)\beta+\alpha$. If $n > 4$ then perform an ordered 3-exchange or arc reversal (see Constructions 3.2.2 and 3.2.3) using T_2 and an arc of cost γ to create T_γ. Since there does not exist a tour using arcs of all three distinct costs, the arc of cost α does not appear in T_2. By deleting the two arcs of unknown cost in T_γ an LS2F of cost $(n-3)\beta + \gamma$ is obtained. But the number of distinct LS2F values from T, T_2 and T_γ exceeds three. This obtains a contradiction. If $n = 4$ then there exist LS2Fs of costs 2β and $\alpha + \beta$ from T_2. There also exist LS2Fs of costs 2α, $\alpha + \gamma$ and 2γ from T. This again yields more than three distinct LS2F values.

Lastly, suppose that there exists a tour T_2 using only arcs of costs β. Since there exists an arc of cost γ in C it is trivial to find a tour using multiple arcs of cost β and an arc of cost γ unless $n = 4$. But then it is possible to find an LS2F of cost $\alpha + \beta$. This combined with the LS2Fs of costs 2β from T_2 and 2α, $\alpha + \gamma$ and 2γ from T gives a contradiction.

The remainder of the lemma, (b), is established by noting that if there exists a tour using arcs of both costs γ and β then there exists LS2Fs of cost $(n-3)\gamma+\beta$ and $(n-2)\gamma$. Similarly, if there exists a tour using arcs of both costs α and β then there exists an LS2F of cost $(n-3)\alpha+\beta$ and $(n-2)\gamma$. But these four LS2F values combined with those obtained from T give more than three distinct LS2F values. This obtains a contradiction. $\square$

We will now use this lemma to determine the structure of LS2F(3) cost matrices where all tours use at most two distinct costs and there exists a tour using multiple arcs of two distinct costs.

Lemma 4.2.3. *Let α, β and γ be distinct constants contained in cost matrix $C = (c_{ij})$ associated with graph G and $n \geq 4$. Furthermore, let $p, q, r \in V(G)$ be distinct nodes and identify the following sets of arcs $S_1 = \{pj \mid j \neq q\}$, $S_2 = \{iq \mid i \neq p\}$ and $S_3 = \{qp\}$. The matrix C is LS2F(3) with a tour containing multiple arcs of cost α and multiple arcs of cost γ, but no tour containing all three distinct costs if and only if:*
(a) $2\alpha = \beta + \gamma$,
(b) at least two of the sets S_1, S_2 and S_3 contain an arc of cost α,
(c) $pr \in S_1$, $rq \in S_2$ and $qp \in S_3$ are the only arcs of cost α in G if S_1, S_2 and S_3 all contain arcs of cost α, and
(d) C has the following structure:

$$c_{ij} \in \begin{cases} \{\infty\} & \text{if } i = j, \\ \{\beta\} & \text{if } ij = pq, \\ \{\gamma, \alpha\} & \text{if } ij \in S_1 \cup S_2 \cup S_3 \\ \{\gamma\} & \text{otherwise.} \end{cases}$$

Proof. Suppose C is an LS2F(3) matrix containing a tour T with multiple arcs of both costs α and γ. By Lemma 4.2.2, there does not exist a tour using multiple arcs of cost β. Since C contains an element of cost β, denote one such element as pq (so $c_{pq} = \beta$). Without loss of generality, suppose that pq is contained in a tour T_γ that also uses an arc of cost γ. Since there does not exist a tour using all three distinct costs and there does not exist a tour using multiple arcs of cost β (from Lemma 4.2.2) it follows that all arcs of T_γ other than pq have cost γ. Furthermore, from Lemma 4.2.2 no tour contains arcs of both costs α and β.

Suppose that there exists more than a single element of cost β in C. From Lemma 4.2.2, no tour contains multiple arcs of cost β. Thus, either all entries of cost β are contained within a single line or the only entries of cost β are pq and qp. Since T contains multiple arcs of cost α it is trivial to find a tour using arcs of both costs α and β. This contradicts Lemma 4.2.2. This implies that there is a single element (pq) of cost β and no tour may contain arcs of both costs α and β. Hence, the only arcs which are candidates to have cost α are arcs of the form $S_1 = \{pj \mid j \neq q\}$, $S_2 = \{iq \mid i \neq p\}$ and $S_3 = \{qp\}$.

Since C contains a tour using multiple arcs of cost α, at least two of the three sets (S_1, S_2 and S_3) must contain an arc of cost α. Suppose that all three sets of arcs contain an element of cost α. Let arc $pu_1 \in S_1$, $u_2q \in S_2$ and $qp \in S_3$ all have cost α. If $u_1 \neq u_2$ then it is trivial to find a tour with three arcs of cost α (using the subpath $u_2 - q - p - u_1$). This gives a contradiction as there would exist more than three distinct LS2F values in C. Otherwise, $u_1 = u_2$. If either S_1 or S_2 contains multiple arcs of cost α then again it is trivial to find a tour using three arcs of cost α. Hence, for the case where all three sets (S_1, S_2 and S_3) contain an arc of cost α, there exist only four arcs with cost not equal to γ in C (three of cost α and one of cost β).

Lastly, it has been shown that there exist tours of costs $2\alpha + (n-2)\gamma$ and $\beta + (n-1)\gamma$. Hence, there exist LS2Fs of costs $2\alpha + (n-4)\gamma$, $\alpha + (n-3)\gamma$, $(n-2)\gamma$ and $\beta + (n-3)\gamma$. Since C is LS2F(3) this implies that $2\alpha + (n-4)\gamma = \beta + (n-3)\gamma \Rightarrow 2\alpha = \beta + \gamma$.

Conversely, it is given that $2\alpha = \beta + \gamma$. First, suppose that S_1, S_2 and S_3 all contain an arc of cost α. Then $c_{pr} = c_{rq} = c_{qp} = \alpha$ and $c_{pq} = \beta$. Since $n \geq 4$, any tour in G contains at most two arcs from this 3-cycle of α cost arcs. However, there does exist a tour using two arcs of cost α (using pq and an arc from either S_1 or S_2). Second, suppose that C contains arcs of cost α from exactly two of the three possible sets S_1, S_2 and S_3. For all three pairs of arc sets containing arcs of cost α (namely, S_1S_2, S_2S_3 and S_1S_3) it is trivial to find a tour using two arcs of cost α (and all other arcs of cost γ).

Notice that no tour uses both an arc of cost α and the arc of cost β. Otherwise, suppose that there exists a tour using both pq and $pu_1 \in S_1$. But both arcs are leaving node p so no tour can use both arcs, which gives a contradiction. Suppose there exists a tour using both pq and $u_2q \in S_2$. But both arcs are entering node q which gives a contradiction. Lastly, suppose there exists a tour using both pq and qp. Since $n \geq 4$ this gives a contradiction.

Hence, any tour containing an arc of cost α has cost $\alpha + (n-1)\gamma$ or $2\alpha + (n-2)\gamma$. Also, there exists a tour using the arc of cost β and all other arcs of cost γ. Thus, there exist LS2Fs of costs $2\alpha + (n-4)\gamma$, $\alpha + (n-3)\gamma$, $(n-2)\gamma$ and $\beta + (n-3)\gamma = 2\alpha + (n-4)\gamma$ since $2\alpha = \beta + \gamma$. □

Lemma 4.2.3 establishes the structure of LS2F(3) cost matrices containing exactly three distinct cost elements where all tours contain at most two distinct costs and there exists a tour that uses multiple arcs of two distinct costs. To finish off this section, we need only consider matrices containing exactly three distinct cost elements where all tours contain at most two distinct costs and there does not exist a tour that uses multiple arcs of two distinct costs.

Lemma 4.2.4. *Let α, β and γ be distinct constants contained in cost matrix $C = (c_{ij})$ associated with graph G and $n \geq 4$. The matrix C is LS2F(3) with no tour containing multiple arcs of cost α or β if and only if C has:*
(a) a single line containing all the elements of costs α and β, or
(b) only two elements that do not have cost γ, $c_{pq} = \alpha$ and $c_{qp} = \beta$ for fixed nodes p and q.

Proof. Cost matrix C is structured such that every tour will use either precisely one arc of cost $\delta \in \{\alpha, \beta\}$, or will use all arcs of cost γ. The same logic presented in Theorem 3.2.8 can be used to describes the structure of a matrix such that every tour uses at most a single arc of cost δ. Thus, the same structure is used here applying the additional requirement that there must be at least one arc of cost α and at least one arc of cost β in C.

Conversely, it is given that C contains the distinct costs α, β and γ. In (a), suppose that C has a single line containing at least one arc of cost α and at least one arc of cost β. All arcs of these costs are either entering or leaving a single node in G (depending on whether the line is a column or row, respectively). Any tour can use at most one of these arcs. Thus, there exist tours of costs $\alpha + (n-1)\gamma$ and $\beta + (n-1)\gamma$. (It may be noted that if the line in question contains an element of cost γ then there will also be a tour of cost $n\gamma$, but such a tour does not affect this analysis.) Hence, there exist LS2Fs with distinct costs $(n-2)\gamma$, $\alpha + (n-3)\gamma$ and $\beta + (n-3)\gamma$ since $n \geq 4$. In (b), suppose pq and qp are the only elements in C that do not have cost γ. It is clear that any tour can use either pq or qp, but not both. Once again, it is found that these tours provide LS2Fs of distinct costs $(n-2)\gamma$, $\alpha + (n-3)\gamma$ and $\beta + (n-3)\gamma$. Thus, in both (a) and (b), C is LS2F(3). $\square$

Lemma 4.2.1, Lemma 4.2.3 and Lemma 4.2.4 determine the structure of LS2F(3) cost matrices containing exactly three distinct values. We now focus on establishing a characterization of cost matrices containing elements of precisely two distinct costs.

4.3 Cost matrices containing two cost values

From Section 4.1, there does not exist an LS2F(3) matrix containing more than three distinct values. Section 4.2 established the structure of LS2F(3) matrices containing exactly three distinct costs. Hence, we now consider cost matrices containing only two distinct elements. Let C be such a matrix. By Observation 4.0.1, one of these two costs may be forced to be 0 by subtracting ϕ from every element in C to create C^{ϕ}. Denote the other cost in C^{ϕ} as $\alpha \neq 0$. Throughout this section, let T_1 be the tour in C^{ϕ} containing the most arcs of cost α over all tours using arcs of both costs 0 and α. By Lemma 4.0.2, such a tour exists.

Lemma 4.3.1. *Let 0 and α be distinct constants contained in cost matrix C^{ϕ} associated with graph G and $n \geq 4$. The matrix C^{ϕ} is LS2F(3) and all tours containing arcs of both cost 0 and α contain at most one arc of cost α if and only if $n = 4$ and there are exactly four arcs of cost α which form a 4-cycle.*

Proof. If T_1 contains only a single arc of cost α then all other tours in G are either constant or contain exactly one arc of cost α. Suppose no tour in C^{ϕ} contains multiple arcs of cost α. From Theorem 3.2.8, C^{ϕ} is LS2F(2). Thus, there exists some tour using multiple arcs of cost α. Since all tours using arcs of both costs 0 and α contain at most a single arc of cost α there exists a tour T_2 using only arcs of cost α. Perform an ordered 3-exchange or arc reversal (see Constructions 3.2.2 and 3.2.3) using T_2 and any arc of cost 0. This newly created tour has arcs of both costs but uses more arcs of cost α than T_1, unless $n = 4$ and all arcs swapped in have cost 0. Repeat this argument for every arc of T_2 to see that all arcs not in T_2 must have cost 0.

Conversely, suppose $n = 4$ and the only arcs of cost α form a 4-cycle. The cost matrix C^{ϕ} describes tours in G of costs 0, α and 4α. Hence, there exist LS2Fs of costs 0, α and 2α. Thus, C^{ϕ} is LS2F(3). □

It may now be assumed that C^{ϕ} contains a tour using arcs of cost 0 and multiple arcs of cost α. In particular, $C(T_1) = k\alpha$ for $2 \leq k \leq n-1$. Also, notice that T_1 yields LS2Fs of costs $k\alpha$, $(k-1)\alpha$ and $(k-2)\alpha$. We now proceed to establish a characterization for all matrices not verified by Lemma 4.3.1. To do this, we differentiate two cases $n > 4$ and $n = 4$, in Lemma 4.3.2 and Lemma 4.3.3 respectively.

Lemma 4.3.2. *Let 0 and α be distinct constants contained in cost matrix C^{ϕ} associated with graph G and $n > 4$. The matrix C^{ϕ} is LS2F(3) and contains a tour using arcs of cost*

0 and multiple arcs of cost α if and only if:
(a) C^{ϕ} is DTC(1) with all tours of cost $k\alpha$ for $2 \leq k \leq n-2$, or
(b) all tours have cost 2α, α or 0 with at least one tour of cost 2α, or
(c) all tours have cost $n\alpha$, $(n-1)\alpha$ or $(n-2)\alpha$ with at least one tour of cost $(n-2)\alpha$.

Proof. Suppose cost matrix C^{ϕ} is LS2F(3) and tour T_1 uses arcs of cost 0 and multiple arcs of cost α with $C(T_1) = k\alpha$ for $2 \leq k \leq n-1$. Take any other tour T_2 in G with $C(T_2) = l\alpha$ for $0 \leq l \leq n$.

If $3 \leq k \leq n-3$ then there exist LS2Fs of costs $k\alpha$, $(k-1)\alpha$ and $(k-2)\alpha$. Notice that if $l \neq k$ then there are more than three distinct LS2F values. It follows that $l = k$. Hence, any tour in G has cost $k\alpha$ implying that C^{ϕ} is DTC(1). This illustrates (a) for all $k \notin \{2, n-2\}$.

If $k = 2$ then T_1 yields LS2Fs of costs 0, α and 2α. It is possible to have tours in G of costs 2α, α and 0 and yield only three distinct LS2F values. This gives (b) and a subcase of (a).

If $k = n-2$ then T_1 yields LS2Fs of costs $(n-2)\alpha$, $(n-3)\alpha$ and $(n-4)\alpha$. It is possible to have tours in G of costs $n\alpha$, $(n-1)\alpha$ and $(n-2)\alpha$. This gives (c) and a subcase of (a).

If $k = n-1$ then it could be the case that all tours have costs $n\alpha$, $(n-1)\alpha$ and $(n-2)\alpha$ as in (c). Suppose this is not the case. If $1 \leq l \leq n-3$ then there are more than four distinct LS2F values in G described by C^{ϕ}. Since not all tours have costs $n\alpha$, $(n-1)\alpha$ and $(n-2)\alpha$ it follows that T_2 yields only a single LS2F value. Hence, $l = 0$. So $C(T_2) = 0$ and T_2 is composed of all arcs having cost 0. Now perform an ordered 3-exchange or arc reversal (see Constructions 3.2.2 and 3.2.3) using T_2 and any arc of cost α in G. The LS2F formed by deleting the two arcs of unknown cost in the newly formed tour has cost $C(T_2) - 3(0) + \alpha = \alpha$. But T_1 yields LS2Fs of costs $(n-2)\alpha$ and $(n-3)\alpha$ and T_2 yields LS2Fs of cost 0. Since C^{ϕ} is LS2F(3), this yields a contradiction.

Conversely, suppose (a) that C^{ϕ} is a DTC(1) matrix with all tours having cost $k\alpha$ for $2 \leq k \leq n-2$. Since all tours have cost $k\alpha$ and there exist LS2Fs of costs $k\alpha$, $(k-1)\alpha$ and $(k-2)\alpha$. Hence, C^{ϕ} is LS2F(3). Suppose (b) that all tours in G have cost 2α, α or 0 and there exists a tour of cost 2α. Thus, there must exist a tour using multiple arcs of cost α. This implies that there are LS2Fs of costs 2α, α and 0 in C^{ϕ}. Suppose (c). Then the result follows immediately from (b) by reversing the roles of α and 0. $\square$

Lemma 4.3.3. *Let 0 and α be distinct constants contained in cost matrix C^{ϕ} associated with graph G and $n = 4$. The matrix C^{ϕ} is LS2F(3) and contains a tour using arcs of cost 0 and multiple arcs of cost α if and only if:*

(a) there is a tour of cost 2α, or

(b) there is a tour of cost 3α and a tour of cost α or 0, or

(c) there is a tour of cost 4α and a tour of cost α.

Proof. Suppose that $n = 4$, then it is possible to have tours of costs 0, α, 2α, 3α and 4α. A tour of cost 4α yields LS2Fs of only cost 2α. A tour of cost 3α yields LS2Fs of costs 2α and α. A tour of cost 2α yields LS2Fs of costs 2α, α and 0. A tour of cost α yields LS2Fs of costs α and 0. A tour of cost 0 only yields LS2Fs of cost 0. If all tours in G have a single value then it must be 2α. Suppose G contains at least two distinct tour values. If G contains a tour of cost 2α then it may contain any other tour value and remain LS2F(3) which gives (a). Otherwise, G does not contain a tour of cost 2α. Suppose G contains a tour of cost 3α then there are LS2Fs of costs 2α and α. Either a tour of cost α or 0 will produce only three LS2F costs, which gives (b). Suppose G contains a tour of cost 4α then there are LS2Fs of cost 2α. Since there do not exist LS2Fs of costs 2α or 3α for C^{ϕ} to be LS2F(3) there must exist a tour of cost α, which gives (c).

Conversely, suppose (a) that $n = 4$ and there is a tour of cost 2α. Then there are LS2Fs of costs 0, α and 2α. Any other tour cost in G yields LS2Fs with one of these three costs. Hence, C^{ϕ} is LS2F(3). Suppose (b) that $n = 4$ and there is a tour of cost 3α and a tour of cost α or 0. The tour of cost 3α yields LS2Fs of costs 2α and α. The tour of cost α yields LS2Fs of costs α and 0 which would yield three LS2F values. The tour of cost 0 yields LS2Fs of cost 0 which would again yield three LS2F values. In either case, C^{ϕ} is LS2F(3). Suppose (c) that $n = 4$ and there is a tour of cost 4α and a tour of cost α. The tour of cost 4α yields LS2Fs of cost 2α and the tour of cost α yields LS2Fs of costs α and 0. Hence, C^{ϕ} is LS2F(3). □

Lemma 4.3.2 and Lemma 4.3.3 give a complete characterization of all LS2F(3) matrices containing exactly two distinct costs that are not excluded by Lemma 4.3.1. Currently, the characterization requires enumerating all tours to determine if a given cost matrix has a specified list of tour values. This process is not feasible for large cases. (If $n = 4$ then there are only six tours, so determining every tour cost is trivial.) Since the structure of the elements in a DTC(1) cost matrix has already been established, we must only consider the structure of cost matrices that satisfy (b) and (c) in Lemma 4.3.2. Thus, we now explore the structure of a matrix where T_1 contains precisely two arcs of cost α. This exhausts (b) in Lemma 4.3.2. (The logic is analogous for (c) in the same lemma.)

The following observations have been previously established and are re-iterated to provide compactness to Lemma 4.3.7.

Observation 4.3.4. *Any tour can use at most one arc entering or leaving a single node.*

Observation 4.3.5. *Let $n \geq 4$. Also, let p and q be distinct nodes. Any tour can use at most one of pq and qp.*

Observation 4.3.6. *Let $n \geq 4$. Also, let p, q and r be distinct nodes. Any tour can use at most two of pq, qp, pr, rp, qr and rq.*

Lemma 4.3.7. *Let 0 and α be distinct constants contained in cost matrix $C^{\phi} = (c_{ij}^{\phi})$ associated with graph G and $n \geq 4$. Furthermore, let C^{ϕ} contain a tour that uses two arcs of cost α. All tours in C^{ϕ} have costs 2α, α or 0 if and only if:*
(a) C^{ϕ} has a line cover of size two, or
(b) the indices of C^{ϕ} can be reordered such that the elements of cost α are contained in a 2×2 submatrix and a single line, or
(c) the indices of C^{ϕ} can be reordered such that the elements of cost α are contained in a 3×3 submatrix, or
(d) all entries of C^{ϕ} are zero except for some permutation of the indices of the submatrix $S \in \{S_1, S_2, S_3\}$ where

$$S_1 = \begin{pmatrix} \infty & \alpha & 0 & 0 \\ \alpha & \infty & 0 & 0 \\ 0 & 0 & \infty & \alpha \\ 0 & 0 & \alpha & \infty \end{pmatrix}, \; S_2 = \begin{pmatrix} \infty & \alpha & 0 & \alpha \\ \alpha & \infty & 0 & 0 \\ \alpha & 0 & \infty & \alpha \\ 0 & 0 & 0 & \infty \end{pmatrix} \; \textit{and} \; S_3 = \begin{pmatrix} \infty & \alpha & 0 & 0 \\ \alpha & \infty & 0 & \alpha \\ 0 & \alpha & \infty & \alpha \\ 0 & 0 & 0 & \infty \end{pmatrix}.$$

Proof. Suppose that all tours have cost 2α, α or 0, and that there exists a tour T using two arcs of cost α. Denote these two arcs pq and rs. Note that all four nodes need not be distinct. If these are the only two elements of cost α in C^{ϕ} then there is a line cover of size two. By performing arc contractions on pq and rs it follows that all entries of C^{ϕ} not contained in row p, row r, column q, column s, qp or sr must have cost 0. Suppose $c_{pt}^{\phi} = \alpha$ for some $t \notin \{p, q, r, s\}$. Clearly, $c_{qp}^{\phi} \neq \alpha$ or subpath $r - s - q - p - t$ uses three arcs of cost

α, which yields a contradiction (as no tour can have more than cost 2α). Since any tour can use a single arc of cost α not leaving p, the only candidate arcs are those entering s, leaving r or sr. But the first two cases can be covered with two lines and the latter case can be covered with a 2×2 submatrix and line. By symmetry, we can assume that the only elements of cost α reside within the submatrix on nodes p, q, r and s.

If not all four nodes are distinct then all elements of cost α can be covered by a 3×3 submatrix. Otherwise, we consider the subgraph on the four nodes corresponding to the 4×4 submatrix with indices p, q, r and s. First, note that $c^{\phi}_{qr} = c^{\phi}_{sp} = 0$ or subpaths $p - q - r - s$ and $r - s - p - q$ would use three arcs of cost α. If $c^{\phi}_{qp} = c^{\phi}_{sr} = \alpha$ then all other elements of C^{ϕ} must have cost 0. This yields submatrix S_1. Hence, it can be assumed (without loss of generality) that $c^{\phi}_{sr} = 0$. To determine the structure of the rest of the submatrix, enumeration is used. To simplify notation, define the arcs of unknown cost with endpoints p, q, r and s numerically as shown in Figure 4.4.

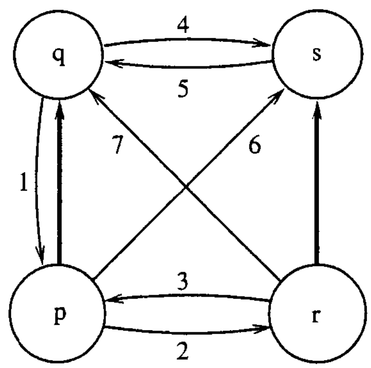

Figure 4.4: Definition of arcs given that pq and rs have cost α

We consider combinations of these seven arcs having cost α in conjunction with the two known arcs of cost α, namely pq and rs. We call a combination *invalid* if it is possible to find a tour of cost at least 3α using the arcs in the combination. If a combination of arcs emits a path using three (or more) arcs of cost α then it may be discarded from future analysis, as all tours in G use at most two arcs of cost α. A combination is called *valid* if it is not invalid.

First, consider each of the seven arcs on it's own with pq and rs. Clearly, each combination is valid (or it would have already been given cost 0). Second, consider combinations involving two arcs of cost α with pq and rs. Notice that the combinations $\{1, 2\}$, $\{1, 5\}$,

Table 4.1: Invalid and valid combinations of arcs of cost α

Invalid combinations		Valid combinations			
1,2	1,6,7	1	1,3	**1,3,6**	**2,3,6,7**
1,5	2,4,7	2	1,4	**1,3,7**	**4,5,6,7**
2,5	3,5,6	3	1,6	**1,4,6**	
3,4		4	1,7	**1,4,7**	
		5	2,3	2,3,6	
		6	2,4	2,3,7	
		7	2,6	**2,4,6**	
			2,7	2,6,7	
			3,5	**3,5,7**	
			3,6	3,6,7	
			3,7	4,5,6	
			4,5	4,5,7	
			4,6	4,6,7	
			4,7	5,6,7	
			5,6		
			5,7		
			6,7		

$\{2,5\}$ and $\{3,4\}$ may be discarded by considering paths $q-p-r-s$, $r-s-q-p$, $p-r-s-q$ and $r-p-q-s$, respectively. Third, consider combinations involving three arcs of cost α with pq and rs. Notice that the combinations $\{1,6,7\}$, $\{2,4,7\}$ and $\{3,5,6\}$ may be discarded by considering paths $r-q-p-s$, $p-r-q-s$ and $r-p-s-q$, respectively. Fourth, notice that remaining combinations are valid.

Table 4.1 shows that $\{1,3,6\}$, $\{1,3,7\}$, $\{1,4,6\}$, $\{1,4,7\}$, $\{2,4,6\}$, $\{3,5,7\}$, $\{2,3,6,7\}$ and $\{4,5,6,7\}$ are all valid combinations of arcs with cost α that are not contained in any larger combination (as shown in bold). All other valid combinations are contained within a larger valid combination.

Combination $\{1,3,6\}$ yields the structure of submatrix S_2. Combination $\{1,3,7\}$ can be covered by row r, pq and qp. Combination $\{1,4,6\}$ can be covered by column s, pq and qp. Combination $\{1,4,7\}$ yields the structure of submatrix S_3. Combination $\{2,4,6\}$ can be covered by row p and column s. Combination $\{3,5,7\}$ can be covered by row r and column q. Combination $\{2,3,6,7\}$ can be covered by row p and row r. Combination $\{4,5,6,7\}$ can be covered by column q and column s.

Conversely, Observations 4.3.4, 4.3.5 and 4.3.6 guarantee that any tour in a matrix satisfying (a), (b), (c) or (d) with the submatrix S_1, uses at most two arcs of cost α. Let T be some tour in G. Suppose C^ϕ has the structure in (d) with the submatrix S_2. If T uses the arc pq then T can not use the arcs qp or ps. All remaining arcs of cost α are leaving node r. If T uses the arc qp then T can not use the arcs pq or rp. All remaining arcs of cost α are entering node s. If T does not uses the arcs pq or qp then all remaining arcs of cost α are either entering node s or leaving node r. Suppose C^ϕ has the structure in (d) with the submatrix S_3. If T uses the arc pq then T can not use the arcs qp or rq. All remaining arcs of cost α are entering node s. If T uses the arc qp then T can not use the arcs pq or qs. All remaining arcs of cost α are leaving node r. If T does not uses the arcs pq or qp then all remaining arcs of cost α are either entering node s or leaving node r. Hence, for matrices with the structure described by (d) with either submatrix S_2 or S_3, the fact that any tour uses at most two arcs of cost α follows from Observations 4.3.4, 4.3.5 and 4.3.6.

Since there exists a tour using consecutive arcs of cost α in C^ϕ, all LS2Fs have costs 2α, α or 0. Hence, C^ϕ is LS2F(3). □

Currently, all the results of this section are in terms of the matrix C^ϕ. But Observation 4.0.1 guarantees that C^ϕ is LS2F(3) if and only if C is LS2F(3). Since we created C^ϕ by choosing ϕ to be any element of the original matrix C, it is a simple matter to translate our results to obtain a characterization of the original matrix.

Before stating the main result of this section, we eliminate any redundancies that occur between the conclusions reached thus far. First, notice that Lemma 4.3.1 describes cost matrices containing two distinct elements (0 and α) with $n = 4$ containing a single tour using all arcs of cost α and all other arcs in G having cost 0. But Lemma 4.3.3 describes cost matrices with $n = 4$ containing a tour of cost 4α and a tour of cost α. Hence, Lemma 4.3.3 contains Lemma 4.3.1 as a subcase since it does not place restrictions on the cost of the elements of the matrix except for these two tours.

Second, Lemma 4.3.3 describes matrices with $n = 4$ containing tours of cost 3α and a tour of cost α or 0, in addition to those matrices where there is a tour of cost 4α and a tour of cost α. By adding a constant to every entry of C^ϕ, it may be seen that the first conclusion contains the second.

Theorem 4.3.8. *Let α and β be the only distinct constants contained in C and $n \geq 4$. Cost matrix C is LS2F(3) if and only if:*
(a) C is DTC(1) with all tours of cost $k\alpha + (n-k)\beta$ for $2 \leq k \leq n-2$ with $n > 4$, or

(b) $n = 4$ *and* G *contains a tour of cost* $2\alpha + 2\beta$*, or*

(c) $n = 4$ *and* G *contains a tour of cost* $3\alpha + \beta$ *and a tour of cost* $\alpha + 3\beta$ *or* 4β*, or*

(d) all arcs of cost α *are covered by two lines and there exists a tour in* G *using two arcs of cost* α*, or*

(e) the indices of C *can be re-arranged such that the arcs of cost* α *are contained in a* 3×3 *submatrix and* G *contains a tour using two arcs of cost* α*, or*

(f) the indices of C *can be re-arranged such that the arcs of cost* α *are contained in a* 2×2 *submatrix and a single line, and* G *contains a tour using two arcs of cost* α*, or*

(g) all entries of C *have cost* β *except for some permutation of the indices of the submatrix* S *where*

$$S = \begin{pmatrix} \infty & \alpha & \beta & \beta \\ \alpha & \infty & \beta & \beta \\ \beta & \beta & \infty & \alpha \\ \beta & \beta & \alpha & \infty \end{pmatrix}, \begin{pmatrix} \infty & \alpha & \beta & \alpha \\ \alpha & \infty & \beta & \beta \\ \alpha & \beta & \infty & \alpha \\ \beta & \beta & \beta & \infty \end{pmatrix} \text{ or } \begin{pmatrix} \infty & \alpha & \beta & \beta \\ \alpha & \infty & \beta & \alpha \\ \beta & \alpha & \infty & \alpha \\ \beta & \beta & \beta & \infty \end{pmatrix}.$$

Proof. Since both the necessity and sufficiency of all the results has already been established in previous lemmas, we only discuss the translation from C^{ϕ} to C.

Recall that C contains only two cost elements α and β. (a) Lemma 4.3.2 states that all tours have k arcs of non-zero cost. After the translation, there are k arcs of one cost and the remaining arcs all have a distinct cost. (b) Lemma 4.3.3 states that $n = 4$ and there exists a tour using two arcs of non-zero costs. After the translation, there are two arcs of each distinct cost. (c) Lemma 4.3.3 states that $n = 4$ and there exists a tour using three arcs of non-zero costs and either a tour using one arc of non-zero cost or a tour using all arcs of zero cost. This translates into the existence of a tour using three arcs of cost α and either a tour using one arc of cost α or a tour using no arcs of cost α. (d)-(e)-(f) These appear as stated in Lemma 4.3.7. (g) This translation is evident from the statement of Lemma 4.3.7. □

We now have a characterization of all LS2F(3) matrices containing exactly two distinct costs in terms of it's elements (for all large cases). The next section will combine the results shown throughout this chapter to establish a concise conclusion regarding the structure of LS2F(3) matrices.

4.4 Structure of LS2F(3) matrices

As done in Section 4.3, redundancies between the various conclusions reached in each section will be eliminated for compactness. Notice that Lemma 4.2.1 and Lemma 4.2.4 may be combined to relax the restriction that there exists a tour using all three distinct arc costs. This provides a generalization to Lemma 4.2.1.

Also, notice that Lemma 4.2.3 places a requirement on the number of arcs of cost α within each of the specified sets of arcs (S_1, S_2 and S_3). However, if this restriction is dropped then either two or three of these sets may not contain an element of cost α. In either case, there exists a line cover of size at most one. Such a case is permissible by Lemma 4.2.1 and Lemma 4.2.4.

We now formulate the final conclusion regarding LS2Fs of three distinct values from the all work executed throughout this chapter.

Theorem 4.4.1. *Let α, β and γ be distinct constants and $n \geq 4$ unless specified otherwise. Also, let p, q, r and s be distinct nodes in G. Cost matrix $C = (c_{ij})$ associated with G is LS2F(3) if and only if:*
G contains arcs of cost α, β and γ with:
(a) $n = 3$, or
(b) $c_{pq} = \alpha$, $c_{qp} = \beta$ and all other entries have cost γ, or
(c) a single line containing all elements of both costs α and β, or
(d) $2\gamma = \alpha + \beta$, all entries of cost α are either contained in a single line or $c_{pq} = c_{qp} = \alpha$, and all entries of cost β are either contained in a single line or $c_{rs} = c_{sr} = \beta$, or
(e) $\beta = 2\alpha - \gamma$, $S_1 = \{pj \mid j \neq q\}$, $S_2 = \{iq \mid i \neq p\}$, $S_3 = \{qp\}$, C has the following structure:

$$c_{ij} \in \begin{cases} \{\infty\} & \text{if } i = j, \\ \{\beta\} & \text{if } ij = pq, \\ \{\gamma, \alpha\} & \text{if } ij \in S_1 \cup S_2 \cup S_3 \\ \{\gamma\} & \text{otherwise.} \end{cases}$$

and if S_1, S_2 and S_3 all contain arcs of cost α then $pr \in S_1$, $rq \in S_2$ and $qp \in S_3$ are the only arcs of cost α in G, or
G contains only arcs of costs α and β with:
(f) all tours of cost $k\alpha + (n-k)\beta$ for $2 \leq k \leq n-2$ and $n > 4$, or
(g) $n = 4$ and G contains a tour of cost $2\alpha + 2\beta$, or

(h) $n = 4$ and G contains a tour of cost $3\alpha + \beta$ and a tour of cost $\alpha + 3\beta$ or 4β, or

(i) all arcs of cost α are covered by two lines and there exists a tour in G using two arcs of cost α, or

(j) the indices of C can be reordered such that the arcs of cost α are contained in a 3×3 submatrix and G contains a tour using two arcs of cost α, or

(k) the indices of C can be reordered such that the arcs of cost α are contained in a 2×2 submatrix and a single line, and G contains a tour using two arcs of cost α, or

(l) all entries of C have cost β except for some permutation of the indices of the submatrix S where

$$S = \begin{pmatrix} \infty & \alpha & \beta & \beta \\ \alpha & \infty & \beta & \beta \\ \beta & \beta & \infty & \alpha \\ \beta & \beta & \alpha & \infty \end{pmatrix}, \begin{pmatrix} \infty & \alpha & \beta & \alpha \\ \alpha & \infty & \beta & \beta \\ \alpha & \beta & \infty & \alpha \\ \beta & \beta & \beta & \infty \end{pmatrix} \text{ or } \begin{pmatrix} \infty & \alpha & \beta & \beta \\ \alpha & \infty & \beta & \alpha \\ \beta & \alpha & \infty & \alpha \\ \beta & \beta & \beta & \infty \end{pmatrix}.$$

Proof. The result follows directly from: (a) discussions at the beginning of Chapter 4, (b) Lemma 4.2.4, (c) Lemma 4.2.4, (d) Lemma 4.2.1 and Lemma 4.2.4, (e) Lemma 4.2.3, (f) Lemma 4.3.2 and Theorem 4.3.8, (g) Lemma 4.3.3 and Theorem 4.3.8, (h) Lemma 4.3.3 and Theorem 4.3.8, (and as a subcase Lemma 4.3.1), (i) Lemma 4.3.7 and Theorem 4.3.8, (j) Lemma 4.3.7 and Theorem 4.3.8, (k) Lemma 4.3.7, (l) Lemma 4.3.7 and Theorem 4.3.8. □

An immediate consequence of Theorem 4.4.1 is that we are able to give a characterization of symmetric cost matrices that are LS2F(3). This result will prove to be helpful when considering the TSP with an associated symmetric cost matrix.

Corollary 4.4.2. *Let α, β and γ be distinct constants and $n \geq 4$ unless specified otherwise. Also, let p, q, r and s be distinct nodes in G. Symmetric cost matrix C is LS2F(3) if and only if:*

G contains arcs of cost α, β and γ with:

(a) $n = 3$, or

(b) $2\gamma = \alpha + \beta$ where $c_{pq} = c_{qp} = \alpha$ and $c_{rs} = c_{sr} = \beta$ are the only entries not of cost γ, or

G contains only arcs of costs α and β with:

(c) all tours of cost $k\alpha + (n-k)\beta$ for $2 \leq k \leq n-2$ and $n > 4$, or

(d) $n = 4$ and G contains a tour of cost $2\alpha + 2\beta$, or

(e) all arcs of cost α are contained in row p and column p, or

(f) the indices of C can be reordered such that the arcs of cost α are contained in a 3×3 submatrix and G contains a tour using two arcs of cost α,
(g) the only arcs of cost α are pq, qp, rs and sr.

Unlike the results shown in Chapter 3, we can not use the characterization of LS2F(k) matrices (for $k = 1, 2, 3$) to obtain a complete characterization of both DTC(3) and DPC(3) matrices. However, by applying Theorem 2.2.2 a new class of both DPC(3) and DTC(3) matrices with the additional restriction that they are IC(3) is discovered. Hence, we now have a polynomially testable characterization of DTC(3) and DPC(3) IC(3) matrices given our closed-form characterization of LS2F(3) matrices.

Chapter 5

SC-Hamiltonicity

Let $G = (V, E)$ be an undirected graph with $V(G) = \{1, \ldots, n\}$ and edge set $E(G)$. Furthermore, let $C = (c_{ij})$ be a cost matrix associated with G where $c_{ij} = \infty$ if $ij \notin E(G)$. Graph G is *Hamiltonian* if and only if it contains a Hamiltonian cycle. A Hamiltonian cycle is commonly referred to as a tour. Throughout this chapter, all graphs are assumed to be Hamiltonian. Graph G is *strongly Hamiltonian* if and only if for every edge $uv \in E(G)$ there exists a Hamiltonian path from v to u. A direct consequence of a graph being strongly Hamiltonian is that every edge is contained in some Hamiltonian cycle.

For convenience, the following notations are reiterated. Cost matrix C is a k *distinct tour cost* matrix, denoted DTC(k), if and only if there exist exactly k distinct tour values in G described by C. In particular, the cost matrix C is DTC(1) if and only if all tours in G have a single cost. It may be noted that given a graph G and an associated DTC(1) matrix C, the TSP is trivially solved by finding any tour in G. This follows from the fact that all tours in G have the same cost, so any tour is optimal. Undirected graph G with $|V(G)| = n$ is *separable constant Hamiltonian* (*SC-Hamiltonian*) if and only if G is Hamiltonian and for any DTC(1) matrix $C = (c_{ij})$ associated with it there exist constants $a_1, \ldots, a_n$ such that $c_{ij} = a_i + a_j \ \forall ij \in E(G)$.

Originally, the problem of classifying graphs as SC-Hamiltonian was only considered for the complete graph. As discussed in Chapter 1, this problem has been well studied by numerous authors.

Theorem 5.0.1. *Both the complete graph and complete digraph are SC-Hamiltonian.*

This chapter characterizes classes of graphs (that are not complete) based on the property of SC-Hamiltonicity.

5.1 Undirected graphs

Kryński was the first author to consider graphs that are not complete. In an attempt to extend the classification of graphs that are SC-Hamiltonian we begin by considering the results obtained by previous authors to gain insight into the problem.

5.1.1 Previous results

This section illustrates the difficulty of classifying graphs as SC-Hamiltonian by providing a brief survey of past results.

Lemma 5.1.1. *[12] Let graph G be a cycle on n vertices. G is SC-Hamiltonian if and only if n is odd.*

Proof. Let n be even. Take $G \cong C_4$ with cycle $(1,2,3,4)$. Construct the cost matrix $C = (c_{ij})$ with $c_{12} = 1$ and $c_{ij} = 0$ otherwise. Since there is only a single tour in G, C is DTC(1). Suppose G SC-Hamiltonian, so there exist constants such that $c_{ij} = a_i + a_j$ $\forall ij \in E(G)$. Then $1 = c_{12} - c_{23} + c_{34} - c_{41} = (a_1 + a_2) - (a_2 + a_3) + (a_3 + a_4) - (a_4 + a_1)$ $= 0$.

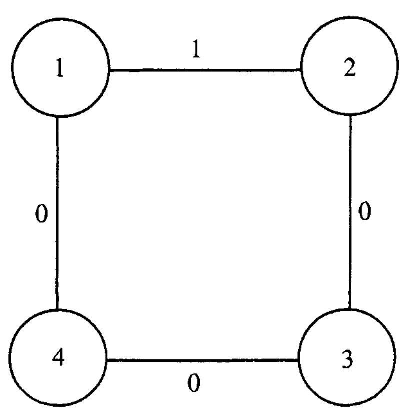

Figure 5.1: Even cycles are not SC-Hamiltonian

Let n be odd and $T = (1, \ldots, n)$ be a tour in G. Take $a_i = \frac{1}{2}\sum_{j=1}^{n}(-1)^{j+1}c_{(i+j-1)(i+j)}$ *mod* n. Hence, $a_u + a_{u+1} = \frac{1}{2}\sum_{j=1}^{n}(-1)^{j+1}c_{(u+j-1)(u+j)} + \frac{1}{2}\sum_{j=1}^{n}(-1)^{j+1}c_{(u+j)(u+j+1)}$

$mod\ n = \frac{1}{2}\left(2c_{u(u+1)}\right) = c_{u(u+1)}$. □

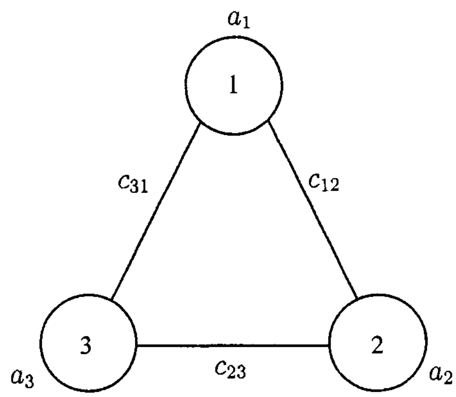

Figure 5.2: Odd cycles are SC-Hamiltonian

The original proof of Lemma 5.1.1 may be found in [12] as presented by Kryński. Furthermore, it is claimed in [12] that strong Hamiltonicity is a necessary condition for a graph to be SC-Hamiltonian. However, Kabadi and Punnen [11] provide a counter-example to this claim.

Let G be an SC-Hamiltonian bipartite graph. Denote the bipartitions of $V(G)$ as $V_1(G)$ and $V_2(G)$. For some u and v in $V_1(G)$, let G^* be the graph with $V(G^*) = V(G)$ and $E(G^*) = E(G) \cup \{uv\}$. Any graph of this form belongs to the *bipartite graph with fraudulent edge* (*BGFE*) class of graphs. It should be noted that if G is a BGFE then edge uv is not contained in any tour of G.

Lemma 5.1.2. *[11] If G is a BGFE then G is SC-Hamiltonian.*

Proof. Let $C = (c_{ij})$ be a DTC(1) matrix associated with BGFE G. Furthermore, denote the endpoints of the fraudulent edge as $u, v \in V_1(G)$. Since $G - \{uv\}$ is SC-Hamiltonian there exist constants such that $c_{ij} = a_i + a_j\ \forall ij \in E(G) - \{uv\}$. Let $x = c_{uv} - a_u - a_v$ and define $b_i = a_i + \frac{x}{2}\ \forall i \in V_1(G)$ and $b_i = a_i - \frac{x}{2}\ \forall i \in V_2$. Then $c_{ij} = b_i + b_j\ \forall ij \in E(G)$. □

This counter-example raises the question: For what classes of graphs is strong Hamiltonicity a requirement for SC-Hamiltonicity? We will show that this counter-example forms a more general class of graphs in the section to follow.

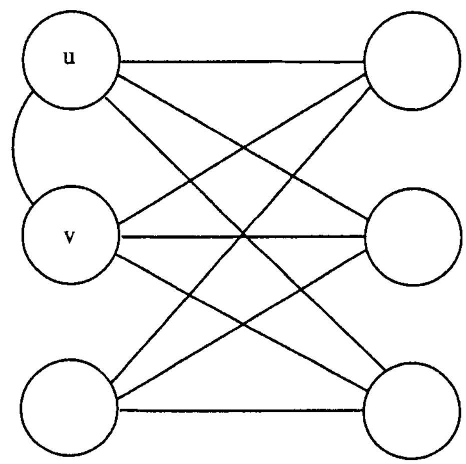

Figure 5.3: Graphs in the BGFE class are SC-Hamiltonian

5.1.2 Graphs that are not strongly Hamiltonian

Theorem 5.1.3. *Let G be an undirected graph that is not strongly Hamiltonian. G is SC-Hamiltonian if and only if G is a BGFE.*

We proceed by first establishing some necessary lemmas.

Lemma 5.1.4. *Let n be odd. If graph G is not strongly Hamiltonian then G is not SC-Hamiltonian.*

Proof. Let every edge contained in a Hamiltonian cycle of G have cost 0 and all other edges have cost 1. Notice that since G is not strongly Hamiltonian, there exists at least a single edge uv with cost 1. Denote this resulting cost matrix $C = (c_{ij})$. Clearly, the cost of every tour in G is 0. Hence, C is a DTC(1) matrix associated with G. Suppose G is SC-Hamiltonian. Then there exist constants such that $c_{ij} = a_i + a_j \; \forall ij \in E(G)$. If necessary, relabel $V(G)$ to take tour $T = (1, 2, \ldots, n)$. Consider the sum $\sum_{i=1}^{n}(-1)^i c_{i(i+1)}$ with subscripts taken modulo n. (Henceforth, this sum will be referred to as an alternating sum traversing tour T starting at vertex 1). Notice that $c_{i(i+1)} = 0$ for every edge of T. So,
$c_{12} - c_{23} + \ldots + c_{n1} = 0$
$\Rightarrow (a_1 + a_2) - (a_2 + a_3) + \ldots + (a_n + a_1) = 0$
$\Rightarrow 2a_1 = 0$

$\Rightarrow a_1 = 0$
$\Rightarrow a_i = 0 \; \forall i \in \{1, \dots, n\}$.

But $c_{uv} = 1$ and $c_{uv} = a_u + a_v = 0$, which gives a contradiction. □

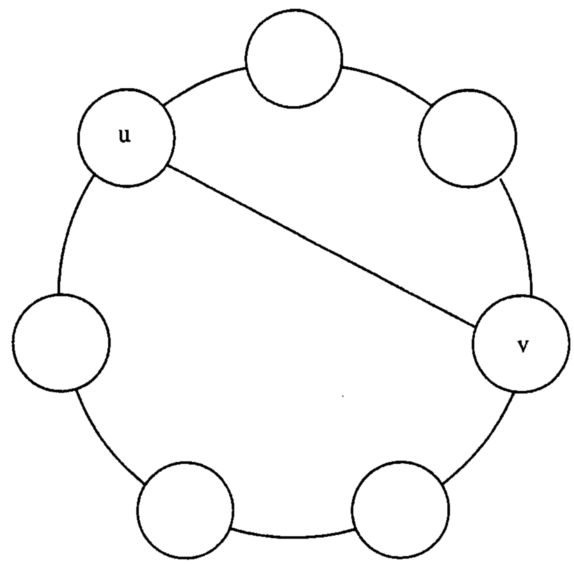

Figure 5.4: A graph that is not strongly Hamiltonian with n odd

Unfortunately, classifying SC-Hamiltonian graphs that are not strongly Hamiltonian on an even number of vertices is more elusive.

Lemma 5.1.5. *Let n be even. If graph G contains an edge not utilized by any tour but that partitions some tour into two even length cycles then G is not SC-Hamiltonian.*

Proof. Let all edges contained in some tour of G have cost 0 and all other edges have cost 1. Denote the resulting cost matrix $C = (c_{ij})$. Suppose there exists constants such that $c_{ij} = a_i + a_j \; \forall ij \in E(G)$. Since all edges contained in some tour have cost 0 it follows that every tour has cost 0. Hence, C is a DTC(1) matrix associated with G. Suppose uv is an edge of G not contained in any tour but which partitions some tour T into two even length cycles. If necessary relabel $V(G)$ so that $T = (1, 2, \dots, n)$. Notice $c_{12} = a_1 + a_2 = 0 \Rightarrow a_1 = -a_2$. Since every edge of T has cost 0, repeating this argument yields $a_1 = -a_2 = a_3 = \dots = -a_n$. Since uv partitions T into two even length cycles, (without loss of generality) a_u is associated with vertex u where u is even and a_v is associated with vertex v where v is odd. Thus,

$c_{uv} = a_u + a_v = a_u + (-a_u) = 0$ but $c_{uv} = 1$ since uv is not contained in any tour. This gives a contradiction. $\square$

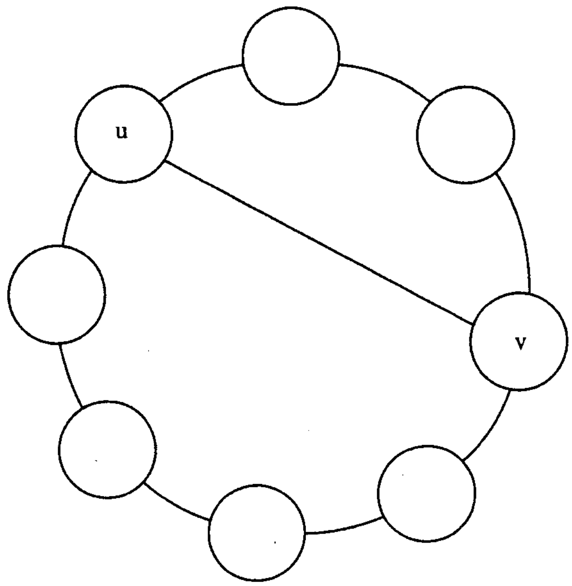

Figure 5.5: A graph that is not strongly Hamiltonian with n even

Lemma 5.1.5 implies that for a graph on an even number of vertices to be SC-Hamiltonian, an edge not contained in any tour must partition every tour into two odd length cycles.

Lemma 5.1.6. *Let n be even. If graph G is composed of a single tour and a chord partitioning that tour into two odd length cycles then G is not SC-Hamiltonian.*

Proof. Denote the tour of G as T and the edge partitioning T into two odd length cycles as uv. Let $C = (c_{ij})$ be a cost matrix associated with G where all edges of G have cost 0 except for a single edge of T. Thus, all tours have cost 1 which implies that C is a DTC(1) matrix associated with G. Suppose G is SC-Hamiltonian. Then there exist constants such that $c_{ij} = a_i + a_j \; \forall ij \in E(G)$. An alternating sum traversing T (staring at any vertex) has 0 cost when considering the constants a_i. However, since only a single edge of T has cost 1 and all others edges have cost 0, this alternating sum will not equal 0. Hence, G is not SC-Hamiltonian. $\square$

Only Hamiltonian graphs that are not strongly Hamiltonian on an even number of vertices containing at least $(n+2)$ edges must be considered.

Lemma 5.1.7. *Let n be even. If graph G has (at least) two edges not contained in any tour then G is not SC-Hamiltonian.*

Proof. Let $T = (1, 2, \ldots, n)$ be some tour in G. Denote the edges not contained in any tour of G as uv and wx (with u,v,w,$x \in \{1, \ldots, n\}$). By Lemma 5.1.5, both uv and wx partition T into two odd length cycles. Hence, u and v have the same parity. Similarly, w and x have the same parity. Let $C = (c_{ij})$ denote the cost matrix with $c_{uv} = 2$, $c_{wx} = 0$ and all other edges of G having cost 0. Since all edges contained in a tour have cost 0, it follows that C is a DTC(1) matrix associated with G. Suppose G is SC-Hamiltonian, then there exist constants such that $c_{ij} = a_i + a_j \; \forall ij \in E(G)$. Considering the edges of T,

$$c_{12} = a_1 + a_2 = 0 \Rightarrow a_1 = -a_2$$
$$c_{23} = a_2 + a_3 = 0 \Rightarrow a_2 = -a_3$$
$$\vdots$$
$$c_{n1} = a_n + a_1 = 0 \Rightarrow a_n = -a_1$$

If u and v are both odd then $c_{uv} = a_1 + a_1 = 2a_1 = 2 \Rightarrow a_1 = 1$. If u and v are both even then $c_{uv} = -a_1 - a_1 = -2a_1 = 2 \Rightarrow a_1 = -1$.

If w and x are both odd then $c_{wx} = a_1 + a_1 = 2a_1 = 0 \Rightarrow a_1 = 0$. If w and x are both even then $c_{wx} = -a_1 - a_1 = -2a_1 = 0 \Rightarrow a_1 = 0$.

The value of a_1 found by considering uv and the value of a_1 found by considering wx are inconsistent. Thus, if G contains (at least) two edges not utilized by any tour then G is not SC-Hamiltonian. □

Observation 5.1.8. *The statement of Lemma 5.1.2 is equivalent to the statement: Let n be even. If $G - \{uv\}$ is an SC-Hamiltonian, bipartite subgraph of G and uv partitions every tour into two odd length cycles then G is SC-Hamiltonian.*

Proof. First, notice that a Hamiltonian bipartite graph has an even number of vertices. Second, notice that any edge added within a partition splits every tour into two odd length cycles. □

In order to classify all undirected graphs on an even number of vertices that are not strongly Hamiltonian only graphs G with subgraphs $H = G - \{uv\}$ that are not SC-Hamiltonian or not bipartite must be considered.

Lemma 5.1.9. *Let n be even and uv be an edge not contained in any tour of graph G. If subgraph $H = G - \{uv\}$ is not SC-Hamiltonian then graph G is not SC-Hamiltonian.*

Proof. Suppose G is SC-Hamiltonian. Then for any DTC(1) matrix $C = (c_{ij})$ there exist constants such that $c_{ij} = a_i + a_j \ \forall ij \in E(G)$. In particular, these constants hold for every edge of H. Since uv is not contained in any tour, all tours still have the same value. Thus, H is SC-Hamiltonian. □

Lemma 5.1.10. *Let n be even. If graph G is not bipartite but is SC-Hamiltonian then the constants used to describe a DTC(1) matrix associated with G are unique.*

Proof. Let G be a graph that is SC-Hamiltonian but not bipartite. Let $C = (c_{ij})$ be a DTC(1) matrix associated with G. Since G is SC-Hamiltonian there exist constants such that $c_{ij} = a_i + a_j \ \forall ij \in E(G)$. Suppose there also exist constants such that $c_{ij} = b_i + b_j \ \forall ij \in E(G)$ where not all $a_i = b_i$ for $i \in \{1, \ldots, n\}$. If necessary, relabel $V(G)$ to take tour $T = (1, 2, \ldots, n)$. Without loss of generality, let $a_1 \neq b_1 \Rightarrow a_1 = b_1 + k$ for $k \neq 0$. Thus,

$c_{12} = a_1 + a_2 = (b_1 + k) + a_2$ and $c_{12} = b_1 + b_2$ so $a_2 = b_2 - k$.

$c_{23} = a_2 + a_3 = (b_2 - k) + a_3$ and $c_{23} = b_2 + b_3$ so $a_3 = b_3 + k$.

$\vdots$

$c_{(n-1)n} = a_{n-1} + a_n = (b_{n-1} + k) + a_n$ and $c_{(n-1)n} = b_{n-1} + b_n$ so $a_n = b_n - k$.

Since G is SC-Hamiltonian, Lemma 5.1.1 implies that it can not simply be an even cycle (so $G \ncong T$). Hence, there exists some edge $pq \in E(G) \backslash E(T)$. Moreover, since G is not bipartite (and T is a bipartite subgraph), p and q may be chosen to have the same parity.

Suppose p and q are both odd. Then $c_{pq} = a_p + a_q = (b_p + k) + (b_q + k)$ and $c_{pq} = b_p + b_q$. But $2k = 0 \Rightarrow k = 0$ gives a contradiction.

Suppose p and q are both even. Then $c_{pq} = a_p + a_q = (b_p - k) + (b_q - k)$ and $c_{pq} = b_p + b_q$. But $-2k = 0 \Rightarrow k = 0$ gives a contradiction.

Thus, if G is not bipartite but is SC-Hamiltonian then the constants used to describe a DTC(1) matrix associated with G are unique. □

Corollary 5.1.11. *Let n be even and H be a Hamiltonian subgraph of graph G on n vertices. Furthermore, let $C = (c_{ij})$ be a DTC(1) matrix associated with G and $D = (d_{ij})$ be a DTC(1) matrix associated with H where $d_{ij} = c_{ij} \ \forall ij \in E(H)$ and $d_{ij} = \infty$ otherwise. If H is an SC-Hamiltonian graph with constants $a_1, \ldots, a_n$ such that $d_{ij} = a_i + a_j \ \forall ij \in E(H)$ then $c_{ij} = a_i + a_j \ \forall ij \in E(G)$.*

Proof. Take SC-Hamiltonian subgraph H of G and constants $a_1, \ldots, a_n$. Suppose SC-Hamiltonian graph G has constants $b_1, \ldots, b_n$. Then the constants b_i are also applied to the subgraph H. The result now follows from Lemma 5.1.10. ∎

We now proceed to establish the theorem stated at the onset of this section.

Theorem 5.1.3. *Let G be an undirected graph that is not strongly Hamiltonian. G is SC-Hamiltonian if and only if G is a BGFE.*

Proof. Let G be a graph that is not strongly Hamiltonian with $uv \in E(G)$ not in any tour of G. Furthermore, let $H = G - \{uv\}$. Lemma 5.1.2, Lemma 5.1.4, Lemma 5.1.5, Lemma 5.1.7 and Lemma 5.1.9 show that a graph that is not strongly Hamiltonian is not SC-Hamiltonian for all graphs except those where H is not bipartite but is SC-Hamiltonian. Notice that since H is not bipartite, G can not be bipartite.

Suppose that G is also SC-Hamiltonian. Then take DTC(1) matrix $C = (c_{ij})$ associated with G and DTC(1) matrix $D = (d_{ij})$ associated with H where $d_{uv} = \infty$ and $d_{ij} = c_{ij}$ otherwise. Since H is SC-Hamiltonian there exists constants $a_i, \ldots, a_n$ such that $d_{ij} = a_i + a_j$ $\forall ij \in E(H)$. By Lemma 5.1.10, these constants uniquely describe D. By Corollary 5.1.11, $c_{ij} = a_i + a_j$ $\forall ij \in E(G)$ (where the a_i are the constants obtained from D). Create $C' = (c'_{ij})$ where $c'_{ij} = c_{ij}$ $\forall ij \neq uv$. Notice that C' is a DTC(1) matrix associated with G (since uv is not in any tour). Once again, by Corollary 5.1.11, $c'_{ij} = a_i + a_j$. This achieves a contradiction since $c'_{uv} = a_u + a_v = c_{uv}$ but $c'_{uv} \neq c_{uv}$. Hence, if G is not strongly Hamiltonian and not a BGFE then G is not SC-Hamiltonian.

The converse follows from Lemma 5.1.2. ∎

5.1.3 Undirected graphs with n odd

Section 5.1.2 gives a complete characterization of SC-Hamiltonicity in terms of strong Hamiltonicity for all undirected graphs. Lemma 5.1.1 suggests that graphs be differentiated based on the size of the vertex set. This section will consider the SC-Hamiltonicity of undirected graphs that are strongly Hamiltonian on an odd number of vertices.

Lemma 5.1.12. *Let n be odd. If G is an SC-Hamiltonian graph then the constants used to describe any DTC(1) matrix associated with G are unique.*

Proof. Take DTC(1) matrix $C = (c_{ij})$ associated with SC-Hamiltonian graph G. There exist constants such that $c_{ij} = a_i + a_j \; \forall ij \in E(G)$. Notice that once a single constant a_i is set the constants on all other vertices are established (this follows since G is Hamiltonian). Suppose there is another set of constants such that $c_{ij} = b_i + b_j \; \forall ij \in E(G)$ where not all a_i and b_i are equal. Without loss of generality, suppose $a_1 \neq b_1$ then $a_1 = b_1 + k$ for some non-zero constant k. If necessary, relabel $V(G)$ and consider tour $T = (1, 2, \ldots, n)$. Then

$c_{12} = a_1 + a_2 = (b_1 + k) + a_2$ but $c_{12} = b_1 + b_2$ so $a_2 = b_2 - k$.

$c_{23} = a_2 + a_3 = (b_2 - k) + a_3$ and $c_{23} = b_2 + b_3$ so $a_3 = b_3 + k$.

$\vdots$

$c_{(n-1)n} = a_{n-1} + a_n = (b_{n-1} - k) + a_n$ and $c_{12} = b_{n-1} + b_n$ so $a_n = b_n + k$.

Thus, $c_{n1} = a_n + a_1 = (b_n + k) + (b_1 + k) = b_n + b_1 + 2k$ but $c_{n1} = b_n + b_1$ so $2k = 0$. Hence, $k = 0$ obtains a contradiction. □

Corollary 5.1.13. *Let n be odd and H be a Hamiltonian subgraph of graph G on n vertices. Furthermore, let $C = (c_{ij})$ be a DTC(1) matrix associated with G and $D = (d_{ij})$ be a DTC(1) matrix associated with H where $d_{ij} = c_{ij} \; \forall ij \in E(H)$ and $d_{ij} = \infty$ otherwise. If H is an SC-Hamiltonian graph with constants $a_1, \ldots, a_n$ such that $d_{ij} = a_i + a_j \; \forall ij \in E(H)$ then $c_{ij} = a_i + a_j \; \forall ij \in E(G)$.*

Proof. Take SC-Hamiltonian graph H and constants $a_1, \ldots, a_n$. Suppose SC-Hamiltonian graph G has constants $b_1, \ldots, b_n$. Then the constants b_i are applied to the subgraph H. The result now follows from Lemma 5.1.12. □

Lemma 5.1.14. *Let n be odd and G be a strongly Hamiltonian graph that is not a cycle. If there exist two edges in some tour that only appear in a single tour then G is not SC-Hamiltonian.*

Proof. Suppose $e_1 = uv$ and e_2 are two edges of tour T_1 which only appear in this single tour. Let e_1 have cost 1, e_2 have cost -1 and all other edges of G have cost 0. Let $C = (c_{ij})$ denote the matrix describing these costs. Notice that all tours of G have cost 0 since only e_1 and e_2 have non-zero cost. Hence, C is a DTC(1) matrix associated with G. Since G is not a cycle there exists an edge of G not in T_1. Graph G being strongly connected implies that this edge lies in some tour T_2 with every edge having cost 0. Suppose that there exists constants such that $c_{ij} = a_i + a_j \; \forall ij \in E(G)$. Relabel $V(G)$ such that $T_2 = (1, 2, \ldots, n)$. Notice $c_{12} - c_{23} + \ldots + c_{n1} = 0$ but $c_{12} - c_{23} + \ldots + c_{n1} = (a_1 + a_2) - (a_2 + a_3) + \ldots + (a_n + a_1) = 2a_1$.

Thus, $a_1 = 0$ implies $a_i = 0$ for all $i \in \{1, \ldots, n\}$ since every edge of T_2 has cost 0. But $c_{e_1} = c_{uv} = 1 \neq a_u + a_v = 0$ obtains a contradiction. ☐

Lemma 5.1.15. *Let n be odd and G be a strongly Hamiltonian graph that is not a cycle. If G contains two edges such that they either both appear in a tour or neither does, and there exists (at least) one tour not using either edge then G is not SC-Hamiltonian.*

Proof. The proof is analogous to that given in Lemma 5.1.14. ☐

Lemma 5.1.16. *Let n be odd. If there exists two edges in every tour of G then either G is not SC-Hamiltonian or G is composed of two bipartite graphs $G_1 = (V_1 \cup W_1, E_1)$ and $G_2 = (V_2 \cup W_2, E_2)$ that are connected by an edge from W_1 to V_2 and an edge from W_1 to W_2.*

Proof. Denote the two edges in every tour of G as e_1 and e_2. If necessary, relabel $V(G)$ to take tour $T = (1, 2, \ldots, n)$ with $e_1 = 1n$ and $e_2 = u(u+1)$, $u \in \{1, \ldots, n-1\}$. Let $c_{e_1} = \alpha$, $c_{e_2} = \beta$ and $c_{ij} = 0$ otherwise. Since e_1 and e_2 are in every tour of G the cost of every tour is $\alpha + \beta$. Hence, C is DTC(1). Take the alternating sums around T starting at vertices 1 and n. If u is odd then these sums have cost $\alpha + \beta$ and $\alpha - \beta$ respectively. If u is even then these sums have costs $\alpha - \beta$ and $\alpha + \beta$ respectively.

Suppose G is SC-Hamiltonian then when considering the constants associated with each vertex, these alternating sums have costs $2a_1$ and $2a_n$ respectively. The value of all other constants may be obtained in the same manner to find,

$$a_i = \begin{cases} a_1 & \text{if } i \in \{1, \ldots, u\} \text{ and } i \text{ is odd,} \\ -a_1 & \text{if } i \in \{1, \ldots, u\} \text{ and } i \text{ is even,} \\ a_n & \text{if } i \in \{u+1, \ldots, n\} \text{ and } i \text{ is even,} \\ -a_n & \text{if } i \in \{u+1, \ldots, n\} \text{ and } i \text{ is odd.} \end{cases}$$

The edges of G other than e_1 and e_2 must be of the form ij with $i, j \in \{1, \ldots, u\}$ or $i, j \in \{u+1, \ldots, n\}$ where the parity of i and j is different. This forms the desired bipartite subgraphs connected by edges e_1 and e_2. ☐

A complete characterization in terms of SC-Hamiltonicity of all graphs not excluded by previous results is currently unknown. It may be noted that graphs formed from 1-extensions

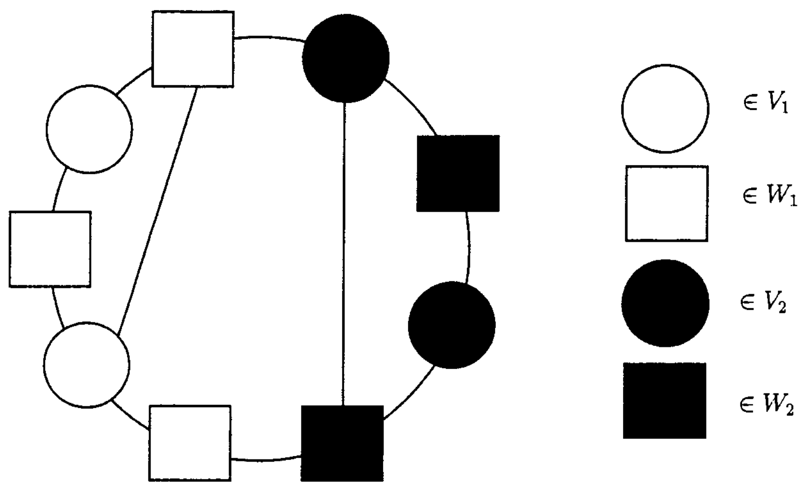

Figure 5.6: A graph with two edges in every tour and n odd

of Hamiltonian graphs on an even number of vertices are included within this set. However, it was shown in [11] by Kabadi and Punnen that such graphs are SC-Hamiltonian.

5.1.4 Undirected graphs with n even

This section will now consider the SC-Hamiltonicity of undirected graphs that are strongly Hamiltonian on an even number of vertices.

Lemma 5.1.17. *Let n be even. If graph G contains two edges whose deletion turns some tour into two paths of even length and for every tour either both edges appear or neither does then G is not SC-Hamiltonian.*

Proof. Suppose G is SC-Hamiltonian. Let e_1 and e_2 denote edges of G that either both appear in any tour or neither does. Furthermore, let $C = (c_{ij})$ be a DTC(1) matrix associated with G. Then there exist constants such that $c_{ij} = a_i + a_j \ \forall ij \in E(G)$. Candidate classes of SC-Hamiltonian graphs can have at most one edge not appearing in any tour (by Lemma 5.1.7), so both e_1 and e_2 appear in some tour $T = (1, 2, \ldots, n)$. Consider the alternating sum traversing T starting at vertex 1, keeping in mind that e_1 and e_2 must have different parity.

$c_{12} - c_{23} - \ldots - c_{n1} = (a_1 + a_2) - (a_2 + a_3) + \ldots - (a_6 + a_1) = 0$

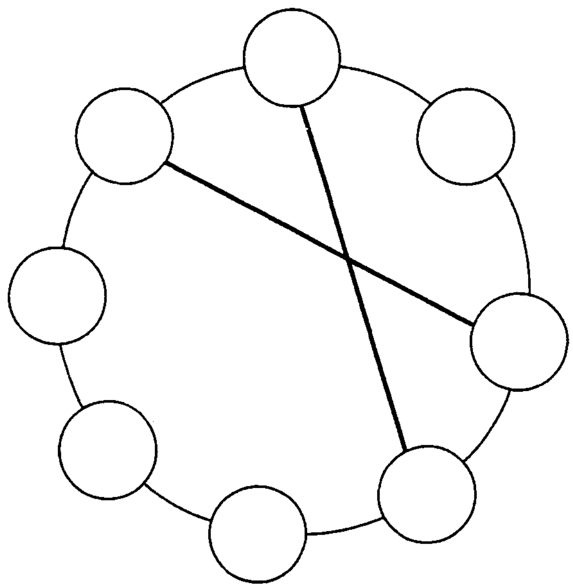

Figure 5.7: A graph with n even that has two edges whose deletion turns some tour into two paths of even length and either both or neither edge appears in every tour

Suppose $c_{e_1} \neq c_{e_2}$. Then let D be the matrix formed by setting $d_{e_1} = c_{e_2}$, $d_{e_2} = c_{e_1}$ and $d_{ij} = c_{ij}$ otherwise. Notice that D is a DTC(1) matrix since e_1 and e_2 either both appear in a tour or neither does. Furthermore, repeating the alternating sum traversing T argument with the constants associated with D, once again yields a zero sum. Consider the difference of the two resulting alternating sum equations from C and D.

$c_{12} - \ldots + c_{e_1} - \ldots - c_{e_2} + \ldots - c_{n1} = 0$

$d_{12} - \ldots + d_{e_1} - \ldots - d_{e_2} + \ldots - d_{n1} = 0$

$\Rightarrow c_{e_1} - d_{e_1} - c_{e_2} + d_{e_2} = 0$

$\Rightarrow c_{e_1} - c_{e_2} - c_{e_2} + c_{e_1} = 0$

$\Rightarrow c_{e_1} = c_{e_2}$ which gives a contradiction.

Suppose $c_{e_1} = c_{e_2}$. Then let D be the matrix formed by setting $d_{e_1} = c_{e_1} + \alpha$, $d_{e_2} = c_{e_2} - \alpha$ (where $\alpha \neq 0$) and $d_{ij} = c_{ij}$ otherwise. Notice that D is a DTC(1) matrix since e_1 and e_2 either both appear in a tour or neither does. Furthermore, repeating the alternating sum traversing T argument with the constants associated with D once again yields a zero sum. Now consider the difference of the two resulting alternating sum equations from C and D.

$c_{12} - \ldots + c_{e_1} - \ldots - c_{e_2} + \ldots - c_{n1} = 0$

$d_{12} - \ldots + d_{e_1} - \ldots - d_{e_2} + \ldots - d_{n1} = 0$
$\Rightarrow c_{e_1} - d_{e_1} - c_{e_2} + d_{e_2} = 0$
$\Rightarrow c_{e_1} - (c_{e_1} + \alpha) - c_{e_2} + (c_{e_2} - \alpha) = 0$
$\Rightarrow \alpha = 0$ which gives a contradiction.

Thus, it follows that if graph G has an even number of vertices containing two edges whose deletion turns some tour into two paths of even length and for every tour either both edges appear or neither does then G is not SC-Hamiltonian. □

Corollary 5.1.18. *Let n be even. If graph G contains a vertex of degree two then G is not SC-Hamiltonian.*

Proof. Take tour $T = (1, \ldots, n)$ in G. Suppose that vertex 2 has degree two. Notice that any tour must use both edges 12 and 23. Also notice that the deletion of 12 and 23 creates a path of length 0 and a path of length $(n-2)$ which are both even. Hence, the result follows from Lemma 5.1.17. □

Observation 5.1.19. *Let n be even. If G is a bipartite, SC-Hamiltonian graph then the constants used to represent any DTC(1) matrix associated with G are not necessarily unique.*

Example 5.1.20. Kabadi and Punnen [11] showed that any complete bipartite graph is SC-Hamiltonian. Consider graph $K_{3,3}$ and cost matrix $C = (c_{ij})$ with $c_{ij} = a_i + a_j = 0$ $\forall ij \in E(G)$. Take tour $T = (1, \ldots, n)$ in G. It is clear that $a_i = 0$ $\forall i \in \{1, 2, \ldots, n\}$ is a valid set of constants. However, the set $a_i = k$ if $2|i$ and $a_i = -k$ otherwise, is also a valid set of constants. Choosing $k \neq 0$ yields multiple valid sets of constants. See Figure 5.8 for an illustration.

Lemma 5.1.21. *Let n be even. If there exists an edge that appears in every tour of graph G then G is not SC-Hamiltonian.*

Proof. Let $uv \in E(G)$ be an edge that is used by every tour in G. Suppose G is SC-Hamiltonian and take any tour T in G. An alternating sum around T staring at any vertex will be 0 when considering the constants attributed to each vertex. Let C be a cost matrix associated with G such that $c_{uv} \neq 0$ and $c_{ij} = 0$ $\forall ij \in E(G) - \{uv\}$. Since uv is an edge of every tour, it follows that C is DTC(1) with all tours of cost c_{uv}. Furthermore, since uv is the only non-zero edge in G, an alternating sum around any tour can not have cost 0. This forms a contradiction. □

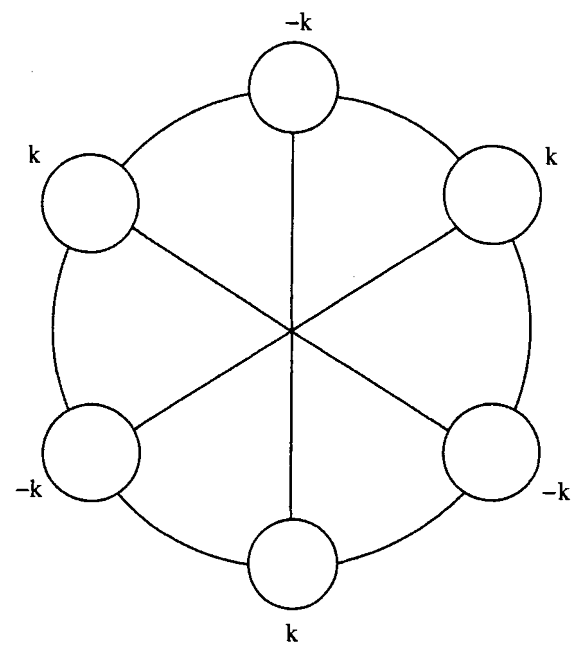

Figure 5.8: Bipartite SC-Hamiltonian graph with $C = (0)$

Lemma 5.1.22. *Let n be even. If every tour in G contains precisely one edge from the nonempty set $S \subset E(G)$ then G is not SC-Hamiltonian.*

Proof. This is simply a generalization of Lemma 5.1.21 which follows by setting $c_s = \alpha \neq 0$ $\forall s \in S \subset E(G)$ and $c_{ij} = 0$ otherwise. Thus, every tour in G will have cost α. □

5.1.5 Structured classes

Sections 5.1.3 and 5.1.4 considered the size of the vertex set as a determining factor for investigating the SC-Hamiltonicity of various classes of graphs. In this section, the SC-Hamiltonicity of undirected graphs is considered independent of the size of the vertex set. The classes examined in this section have some nice underlying structure that will be exploited.

Edge subdivision

When attempting to find large classes of SC-Hamiltonian graphs the idea of performing edge subdivisions was proposed as a likely source by Kabadi and Punnen in [11]. Let G^k denote the graph formed by subdividing some edge of G exactly k times. The following question is considered: If G is SC-Hamiltonian, is G^k SC-Hamiltonian?

The result is immediate if $k = 1$ by letting $G \cong C_{2n-1}$, the cycle of length $2n-1$. Since G is simply an odd cycle, Lemma 5.1.1 guarantees that G is SC-Hamiltonian. However, $G^1 \cong C_{2n}$ is an even cycle which has been shown not to be SC-Hamiltonian. Thus, subdividing an edge once does not preserve SC-Hamiltonicity. Similarly, G^{2l+1} is not SC-Hamiltonian for any positive integer l.

This easy counter example will not hold if an edge is subdivided twice as cycles of odd length remain odd and cycles of even length remain even. However, contrary to the proposition put forth in [11] it can be shown that the result does not hold for $k = 2$. Let $G \cong K_4$ with $V(G) = \{1, 2, 3, 4\}$. The complete graph K_n has been shown to be SC-Hamiltonian in Theorem 5.0.1.

Consider G^2 formed by replacing edge 41 with the path $4 - 5 - 6 - 1$. Associate the following cost matrix with G^2.

$$C = (c_{ij}) = \begin{pmatrix} \infty & 1 & 2 & \infty & \infty & 0 \\ 1 & \infty & 0 & 0 & \infty & \infty \\ 2 & 0 & \infty & 1 & \infty & \infty \\ \infty & 0 & 1 & \infty & 0 & \infty \\ \infty & \infty & \infty & 0 & \infty & 0 \\ 0 & \infty & \infty & \infty & 0 & \infty \end{pmatrix}$$

Notice that C is DTC(1) with $C(T_1) = C(T_2) = 2$ for tour $T_1 = (1, 2, 3, 4, 5, 6)$ and $T_2 = (1, 3, 2, 4, 5, 6)$. Suppose that G^2 is SC-Hamiltonian. Then there exist constants such that $c_{ij} = a_i + a_j \; \forall ij \in E(G^2)$. Now take an alternating sum traversing T_1 starting at vertex 1.

$c_{12} - c_{23} + c_{34} - c_{45} + c_{56} - c_{61}$
$= (a_1 + a_2) - (a_2 + a_3) + (a_3 + a_4) - (a_4 + a_5) + (a_5 + a_6) - (a_6 + a_1)$
$= 0$

But

$c_{12} - c_{23} + c_{34} - c_{45} + c_{56} - c_{61}$
$= 1 - 0 + 1 - 0 + 0 - 0$
$= 2$

This clearly yields a contradiction. Hence, subdividing an edge twice does not preserve

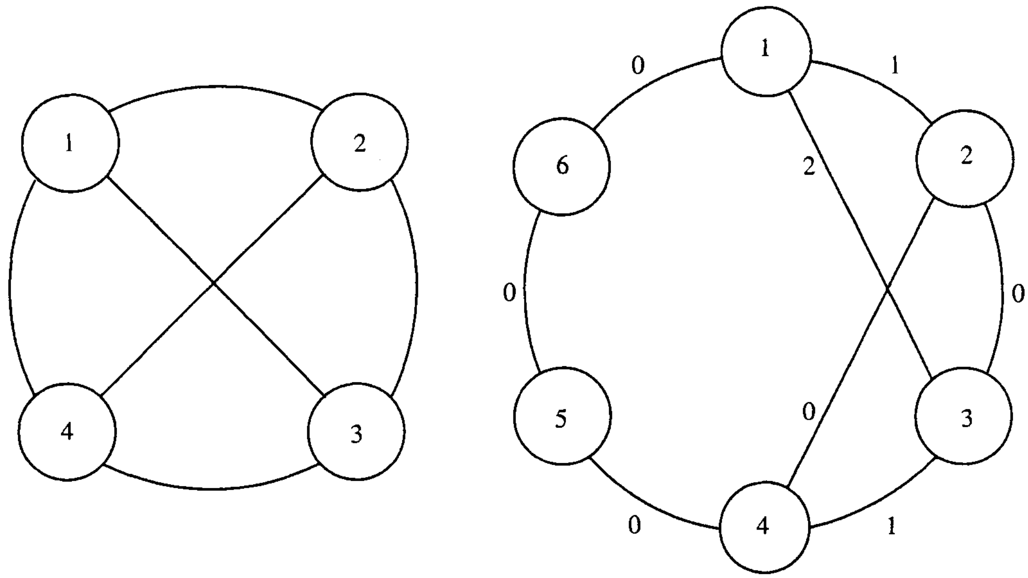

Figure 5.9: Counter-example to double edge subdivision claim

SC-Hamiltonicity. In fact, Lemma 5.1.18 shows that subdividing an edge any number of times does not necessarily preserve SC-Hamiltonicity.

Observation 5.1.23. *Let G be an SC-Hamiltonian graph. Subdividing edges of G does not necessarily preserve SC-Hamiltonicity.*

Incomplete 1-extensions

Definition 5.1.24. Given graphs G and H their *join* denoted $G+H$ is formed by adding an edge ij for every $i \in V(G)$ and $j \in V(H)$. If $|V(H)| = 1$ then $G+H$ is called a *1-extension* of G.

Given a 1-extension $G+H$ of graph G, an *incomplete 1-extension* is formed by deleting any edge from the 1-extension. That is, deleting any edge with one end incident to a vertex of G and the other end incident to a vertex in H. In [11], Kabadi and Punnen showed that 1-extensions of Hamiltonian graphs were SC-Hamiltonian.

We now investigate whether adding less than $|V(G)|$ edges in the 1-extension of Hamiltonian graph G creates an SC-Hamiltonian graph. If this incomplete 1-extension fails to be SC-Hamiltonian then, in some sense, this construction technique is best possible.

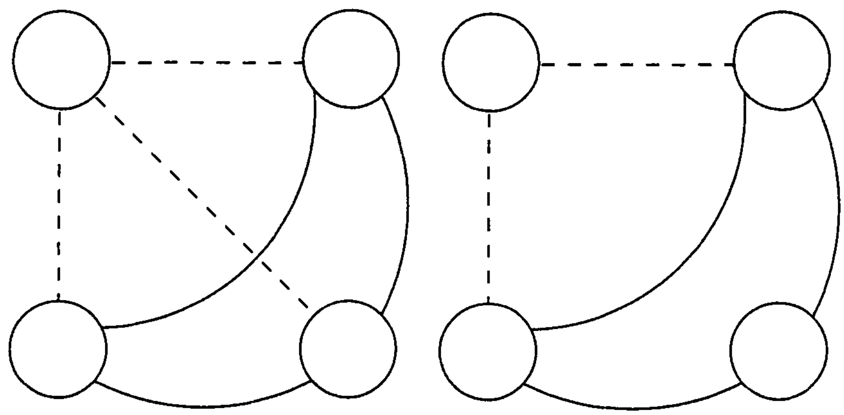

Figure 5.10: Incomplete 1-extension with $H \cong K_1$ and $G \cong C_3$

Observation 5.1.25. *Let G and H be disjoint graphs. An incomplete 1-extension of G is not necessarily SC-Hamiltonian.*

Proof. Let $G \cong C_5$ and $H \cong K_1$. It is clear that G is Hamiltonian. Furthermore, by Lemma 5.1.1, G is SC-Hamiltonian. Construct an incomplete 1-extension I between H and G by adding all edges from G to H except for a single edge. Denote the vertices of G cyclically as $(1, 2, 3, 4, 5)$ and the vertex of H as 6. Without loss of generality, let vertex 1 be the vertex of degree two in I. By Corollary 5.1.18, I is not SC-Hamiltonian. $\square$

1^k-extensions

A new proof technique first utilized by Kabadi and Punnen [11] will be used to discover a new class of SC-Hamiltonian graphs.

Let C and D be $n \times n$ cost matrices. Define $\chi(x) = x$ if x is finite and $\chi(\infty) = 0$. Cost matrix $D = (d_{ij})$ is a *reduced matrix* of $C = (c_{ij})$ if there exist finite constants a_i and b_i for $i = 1, \ldots, n$ such that $d_{ij} = c_{ij} - a_i - b_j$ $\forall i \forall j$. Furthermore, if C is a symmetric matrix then $a_i = b_i$ $\forall i$. For any vertex $u \in \{1, \ldots, n\}$ D is called a *u-reduced matrix of C* if it is a reduced matrix of C and $\chi(d_{iu}) = \chi(d_{ui}) = 0$ $\forall i$. For any distinct vertices r, s and u in $\{1, \ldots, n\}$ D is an *(r,s,u)-reduced matrix of C* if it is a u-reduced matrix of C and $d_{rs} = 0$.

Proofs for the following Lemmas may be found in [11].

Lemma 5.1.26. *Let $C = (c_{ij})$ be an $n \times n$ cost matrix and $c_{rs} \neq \infty$ with $r, s, u \in \{1, \dots, n\}$. There exists an (r,s,u)-reduced matrix of C.*

Lemma 5.1.27. *Let D be a reduced matrix of a matrix C. D is a DTC(1) matrix if and only if C is a DTC(1) matrix.*

Lemma 5.1.28. *A graph is SC-Hamiltonian if and only if for any DTC(1) matrix C associated with it there exists a reduced matrix D of C such that every entry of D is either 0 or ∞.*

For k-partite graph G let $V_1, V_2, \dots, V_k$ denote the partition sets of vertices. Let P_k denote a path of length k with $P_k = 1 - 2 - \cdots - k$. Given k-partite graphs $G = (V, E)$ with $V = V_1, V_2, \dots, V_k$ and $H \cong P_k$ the graph $G +_k H$ is called a 1^k-*extension* formed by adding edge ij for every $i \in V(H)$ and $j \notin V_i(G)$.

Theorem 5.1.29. *A 1^k-extension of a Hamiltonian k-partite graph is SC-Hamiltonian.*

Proof. The result has been established for $k = 1$ and $k = 2$ in [11]. Assume $k \geq 3$. Let G be a Hamiltonian k-partite graph with vertex partition $V_1(G)$, ..., $V_k(G)$. Let G^* be a 1^k-extension of G and $P_k = 1 - 2 - \cdots - k$ such that $V_i(G^*) = V_i(G) \cup \{i\}$ for $i = 1, \dots, k$. Notice that G^* is Hamiltonian.

Let C be a DTC(1) matrix associated with G^* and D be a $(1, 2, \dots, k)$-reduced matrix of C. (Since G is k-partite, such a matrix can be constructed in k steps by setting $C = D^0$ and finding $D^x = (d^x_{ij})$ for $x = 1, \dots, k$ by subtracting d^{x-1}_{ix} from every row i and column i of D^x. Then setting $D = D^k$.). Since G is Hamiltonian, let T denote some tour in G. Without loss of generality, consider edge $u_1u_2 \in E(T)$ with $u_i \in V_i(G^*)$ for $i = 1, 2$. Let $u_1 - u_2 - u_3$ denote a subpath of length two using edge u_1u_2 in T. (Notice that it might be the case that $u_3 \in V_1(G^*)$.) Construct $T_1 = T\backslash\{u_1u_2\} \cup \{u_21, P_k, ku_1\}$ and $T_2 = T\backslash\{u_1u_2, u_2u_3, 23\} \cup \{u_13, P_k, ku_2, u_21, 2u_3\}$.

These are valid tours since $u_1 \in V_1(G^*)$, $u_2 \in V_2(G^*)$ and $u_3 \notin V_2(G^*)$ as $u_2u_3 \in E(G)$. Notice that since D is a $(1, 2, \dots, k)$-reduced matrix $D(T_1) = D(T_2) + d_{u_2u_3}$ as every edge incident with a vertex of P_k has cost 0. Since D is DTC(1) it follows that $d_{u_2u_3} = 0$. But recall that u_1u_2 (and hence u_2u_3) is an arbitrary edge of T. Thus, every edge of T has 0 cost implies $D(T) = 0$.

Now consider an arbitrary edge u_pu_q of G not in T. Without loss of generality, let $u_p \in V_p(G^*)$, $u_q \in V_q(G^*)$ and $p < q$. Let $w - u_p - x$ and $y - u_q - z$ be subpaths of T. So

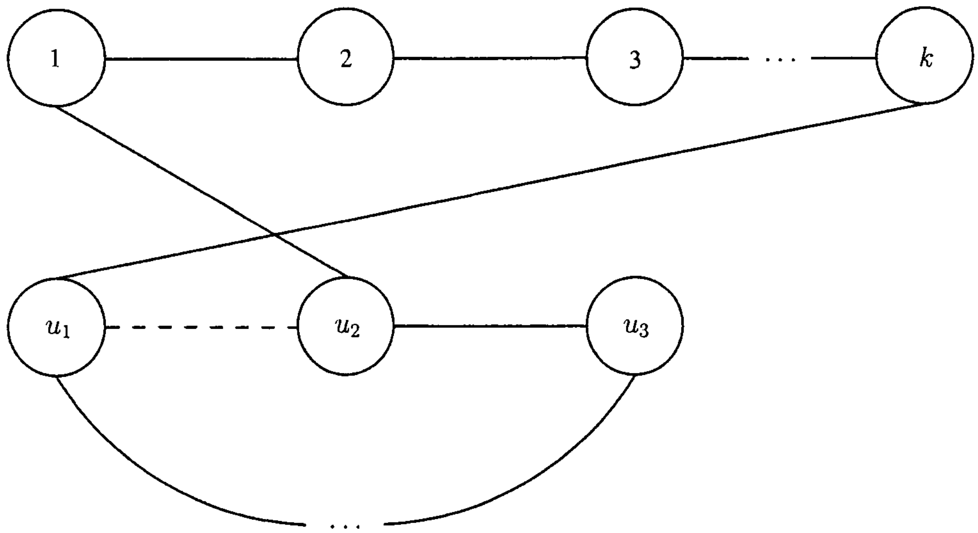

Figure 5.11: Tour T_1 in 1^k-extension G^*

$w, x \notin V_p(G^*)$ and $y, z \notin V_q(G^*)$. Notice that since $u_pu_q \notin E(T)$ there is at least one vertex separating p from q along either path around T. Also, notice that since $k \geq 3$ we can chose $V_1 \notin \{V_p, V_q\}$. (A consequence is that this may force $q = p+1$.) Suppose $w \notin V_1$ and $y \notin V_{p+1}$ then construct $T_3 = T\backslash\{wu_p, u_px, yu_q, p(p+1)\} \cup \{w1, P_k, px, y(p+1), ku_p\}$. Suppose $w \notin V_1$ and $y \in V_{p+1}$ then construct $T_4 = T\backslash\{wu_p, yu_q\} \cup \{w1, P_k, ky\}$. Suppose $w \in V_1$ and $y \notin V_{p+1}$ then construct $T_5 = T\backslash\{wu_p, u_px, yu_q, p(p+1)\} \cup \{u_p1, P_k, px, y(p+1), kw\}$. Suppose $w \in V_1$ and $y \in V_{p+1}$ then construct $T_6 = T\backslash\{wu_p, yu_q\} \cup \{y1, P_k, kw\}$.

Given the vertex restrictions stated, it follows that T_3, T_4, T_5 and T_6 are tours in G^*. Then $D(T_3) = D(T_4) = D(T_5) = D(T_6) = d_{u_pu_q}$ since every edge incident to a vertex in P_k has cost 0 and any edge in T has cost 0. But D is DTC(1) with all tours of cost 0 (since $D(T_1) = D(T_2) = 0$). Hence, $d_{u_pu_q} = 0$ and since u_pu_q was chosen arbitrarily it follows that $d_{ij} = 0$ for any edge $ij \notin E(T)$. Thus, all finite entries of D are 0. The result now follows from Lemma 5.1.28. □

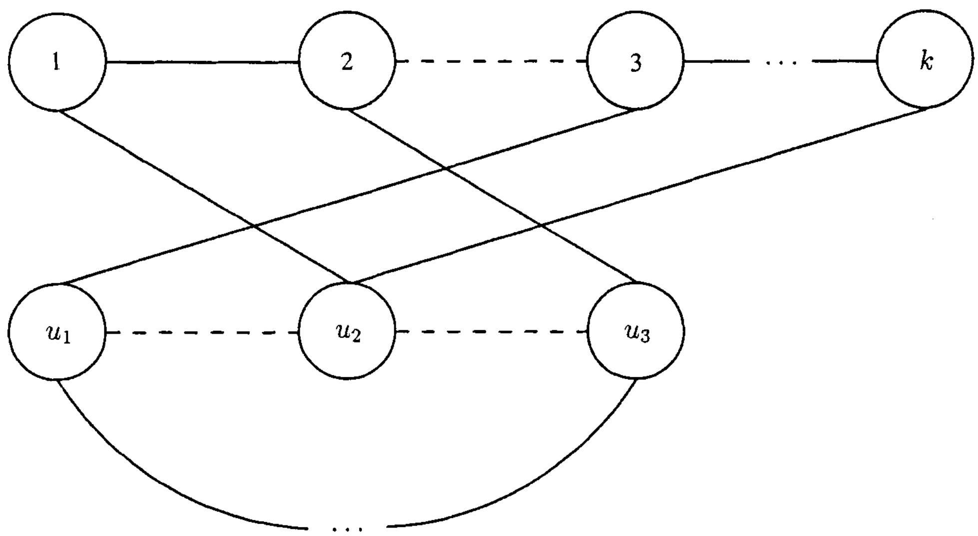

Figure 5.12: Tour T_2 in 1^k-extension G^*

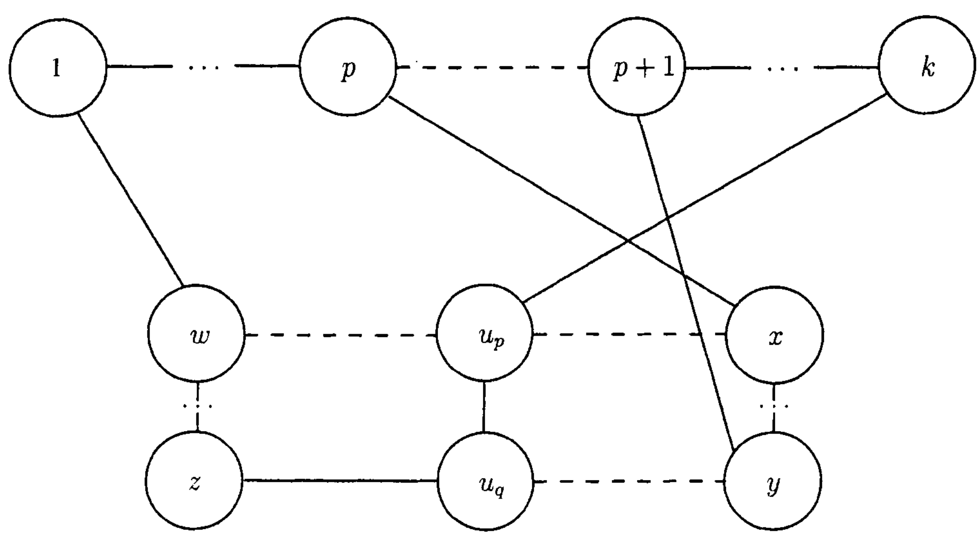

Figure 5.13: Tour T_3 in 1^k-extension G^*

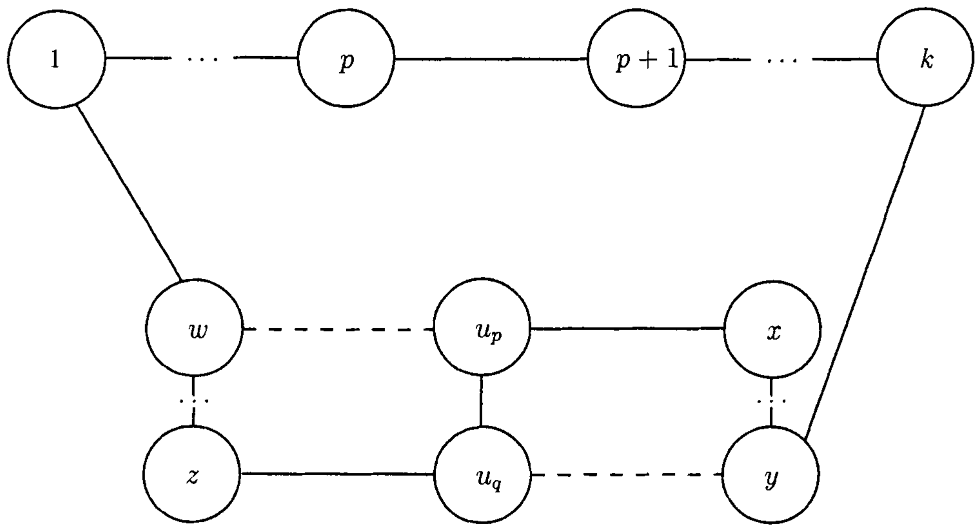

Figure 5.14: Tour T_4 in 1^k-extension G^*

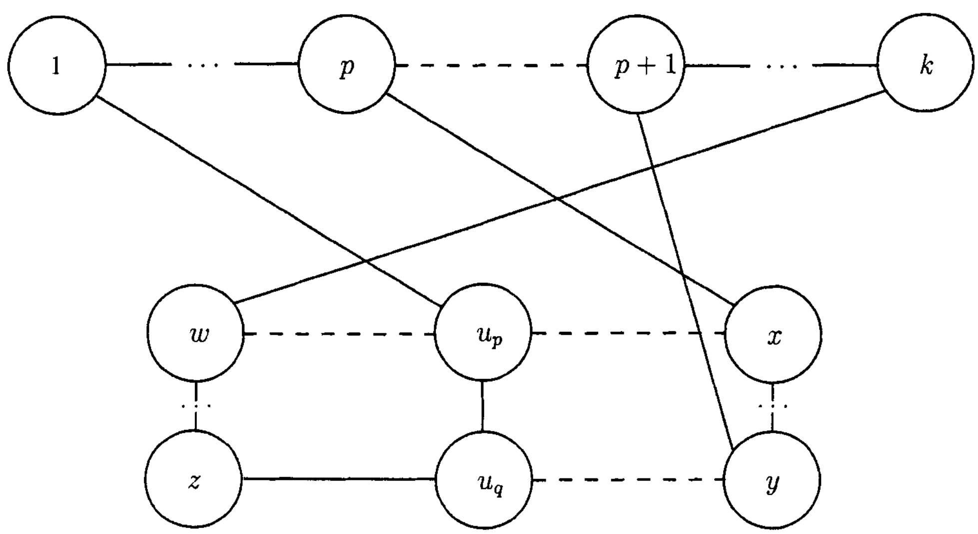

Figure 5.15: Tour T_5 in 1^k-extension G^*

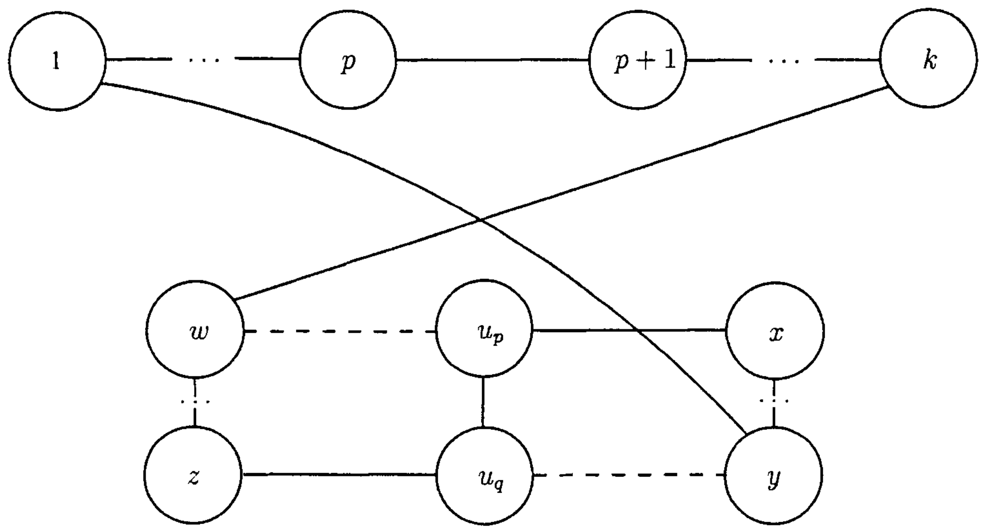

Figure 5.16: Tour T_6 in 1^k-extension G^*

5.2 Directed graphs

Many of the concepts introduced for undirected graphs in Section 5.1 have analogs for the directed case. These ideas will be applied to discover the SC-Hamiltonicity of various classes of directed graphs.

Let $\vec{G}$ be a digraph. The *underlying undirected graph of* $\vec{G}$ denoted by G with $V(G) = V(\vec{G})$ and edge set $E(G)$ is formed by ignoring the directions on the arcs of $\vec{G}$ and eliminating any multiple edges. Digraph $\vec{G}$ is *Hamiltonian* if and only if it contains a directed Hamiltonian cycle. Digraph $\vec{G}$ is *strongly Hamiltonian* if and only if for every edge uv there exists a directed Hamiltonian path from v to u. A digraph $\vec{G}$ with $|V(\vec{G})| = n$ is *separable constant Hamiltonian (SC-Hamiltonian)* if and only if $\vec{G}$ is Hamiltonian and for any DTC(1) matrix $C = (c_{ij})$ associated with it there exists constants $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ such that $c_{ij} = a_i + b_j$ $\forall ij \in E(\vec{G})$.

5.2.1 Symmetric digraphs

By Theorem 5.0.1, the complete directed graph $\vec{K}_n$ is SC-Hamiltonian.

Lemma 5.2.1. *Let $n > 2$ and $\vec{G}$ be a symmetric digraph with underlying undirected graph G. If $\vec{G}$ is SC-Hamiltonian then G is SC-Hamiltonian.*

Proof. Let $C = (c_{ij})$ be a symmetric DTC(1) matrix associated with G. Notice that C may also be associated with $\vec{G}$ (because it is symmetric). Since $\vec{G}$ is SC-Hamiltonian there exist constants such that $c_{ij} = a_i + b_j$ $\forall ij \in E(\vec{G})$. Let $x_i = \frac{a_i+b_i}{2}$ $\forall i \in \{1, \dots, n\}$. Then $c_{ij} = \frac{2c_{ij}}{2} = \frac{c_{ij}+c_{ji}}{2} = \frac{(a_i+b_j)+(a_j+b_i)}{2} = \frac{a_i+b_i}{2} + \frac{a_j+b_j}{2} = x_i + x_j$ follows as $c_{ij} = c_{ji}$. Thus, G is SC-Hamiltonian. □

In [11], Kabadi and Punnen conjectured that the converse of Lemma 5.2.1 was also true. However, the following example shows that this is not the case.

Example 5.2.2. Let $V(\vec{K}_{3,3}) = \{1, \dots, 6\}$ have vertex bipartition $\{1, 3, 5\}$ and $\{2, 4, 6\}$ and $\vec{G} \cong \vec{K}_{3,3} \cup \{13, 31\}$. By Lemma 5.1.2, the underlying undirected graph of $\vec{G}$ is SC-Hamiltonian. Let $C = (c_{ij})$ denote the cost matrix with $c_{13} = 1$, $c_{31} = 2$ and $c_{ij} = 0$ for every other arc of $\vec{G}$ (as shown in Figure 5.17). Since neither arc 13 nor arc 31 is contained in any tour of $\vec{G}$ it follows that all tours have cost 0. Hence, C is a DTC(1) matrix associated with $\vec{G}$. Suppose $\vec{G}$ is SC-Hamiltonian. Then there exist constants such that $c_{ij} = a_i + b_j$ $\forall ij \in E(\vec{G})$. Take directed tours $T_1 = (1, 2, 3, 4, 5, 6)$ and $T_2 = (6, 5, 4, 3, 2, 1)$ in $\vec{G}$. The following are obtained from T_1 and T_2 respectively,

$$
\begin{array}{ll}
0 = c_{12} = a_1 + b_2 \Rightarrow a_1 = -b_2 & 0 = c_{16} = a_1 + b_6 \Rightarrow a_1 = -b_6 \\
0 = c_{23} = a_2 + b_3 \Rightarrow a_2 = -b_3 & 0 = c_{21} = a_2 + b_1 \Rightarrow a_2 = -b_1 \\
0 = c_{34} = a_3 + b_4 \Rightarrow a_3 = -b_4 & 0 = c_{32} = a_3 + b_2 \Rightarrow a_3 = -b_2 \\
0 = c_{45} = a_4 + b_5 \Rightarrow a_4 = -b_5 & 0 = c_{43} = a_4 + b_3 \Rightarrow a_4 = -b_3 \\
0 = c_{56} = a_5 + b_6 \Rightarrow a_5 = -b_6 & 0 = c_{54} = a_5 + b_4 \Rightarrow a_5 = -b_4 \\
0 = c_{61} = a_6 + b_1 \Rightarrow a_6 = -b_1 & 0 = c_{65} = a_6 + b_5 \Rightarrow a_6 = -b_5
\end{array}
$$

Thus, $a_1 = a_3 = a_5 = -b_2 = -b_4 = -b_6$ and $a_2 = a_4 = a_6 = -b_1 = -b_3 = -b_5$. So $1 = c_{13} = a_1 + b_3 = (a_3) + (b_1) = c_{31} = 2$, yields a contradiction.

It is clear that the argument in the Example 5.2.2 can be repeated for any graph that is bipartite with a single symmetric pair of arcs within a bipartition whose underlying undirected graph is in the BGFE class of graphs. However, is it true that if G is SC-Hamiltonian then $\vec{G}$ is SC-Hamiltonian for other classes of graphs?

Definition 5.2.3. A *bi-cycle* is the graph formed by taking an undirected Hamiltonian cycle and replacing each edge ij by arcs ij and ji.

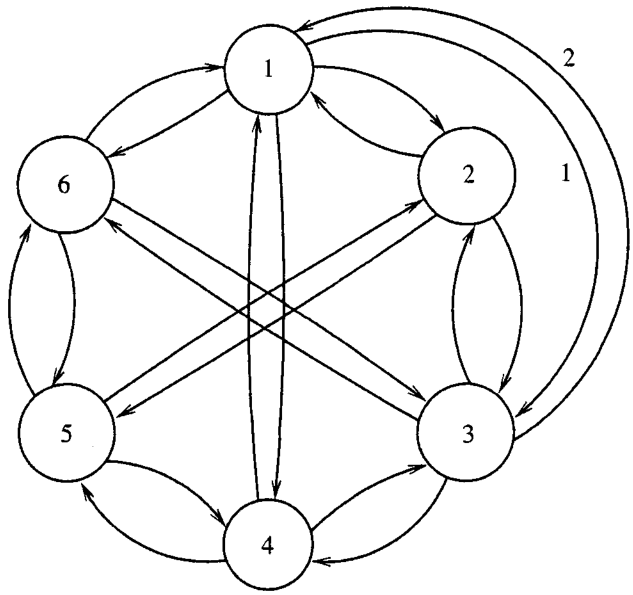

Figure 5.17: $\vec{G}$ is not SC-Hamiltonian but G is SC-Hamiltonian

Lemma 5.2.4. *Bi-cycles are SC-Hamiltonian if and only if n is odd.*

Proof. Let $\vec{G}$ be a bi-cycle and $C = (c_{ij})$ be a DTC(1) matrix associated with $\vec{G}$.

Suppose n is odd. Let $a_i = c_{i(i-1)} - b_{i-1}$ and $b_i = c_{(i-1)i} - a_{i-1}$. Then $a_i + b_{i+1} = a_i + (c_{i(i+1)} - a_i) = c_{i(i+1)}$ and $a_{i+1} + b_i = (c_{(i+1)i} - b_i) + b_i = c_{(i+1)i}$. Since all edges of $\vec{G}$ are of the form $i(i+1)$ or $(i+1)i$ it follows that $c_{ij} = a_i + b_j$ for every arc in $\vec{G}$. Hence, $\vec{G}$ is SC-Hamiltonian.

Suppose n is even. Assume bi-cycle $\vec{G}$ is SC-Hamiltonian. Then for any DTC(1) matrix $C = (c_{ij})$ there exist constants such that $c_{ij} = a_i + b_j$ $\forall ij \in E(\vec{G})$. In particular, take $c_{12} = c_{21} = 1$ and all other entries of cost 0. Since $\vec{G}$ is composed of only two disjoint cycles, all tours have cost one. Take tour $T_1 = (1, 2, \ldots, n)$ in $\vec{G}$ and it's reversal $T_2 = (n, \ldots, 2, 1)$. Considering alternating sums starting at vertex 1 around both tours. T_1 yields $c_{12} - c_{23} + \ldots - c_{n1}$ and T_2 yields $c_{1n} - c_{n(n-1)} + \ldots - c_{21}$. The difference of these two equations is 2. But when considering the constants a_i and b_i notice that the difference of these two equations is 0. This gives a contradiction. □

Observation 5.2.5. *If $\vec{G}$ is an SC-Hamiltonian digraph then the constants associated with* *any DTC(1) matrix are not unique.*

Proof. Let $C = (c_{ij}) = (a_i + b_j)$ be a DTC(1) matrix associated with graph $\vec{G}$. Let k be some non-zero constant. Then do the following: add k to all a_i and subtract k from all b_j. Thus, $\forall ij \in E(\vec{G})$ we have,

$$(a_i + k) + (b_j - k) = a_i + b_j = c_{ij}. \qquad \square$$

Example 5.2.6. Let digraph $\vec{G}$ be a bi-cycle on an odd number of nodes. By Lemma 5.2.4, odd bi-cycles are SC-Hamiltonian. Let every arc of $\vec{G}$ have cost 0 and $C = (c_{ij}) = (a_i + b_j)$ $\forall ij \in E(\vec{G})$ be the matrix which describes these costs. Clearly, C is DTC(1) as every tour has 0 cost. Let $a_i = \alpha$ and $b_i = -\alpha$ $\forall i \in V(\vec{G})$. Thus, $c_{ij} = a_i + b_j = (\alpha) + (-\alpha) = 0$ for any choice of α.

A direct consequence of Observation 5.2.5 is that a single constant associated with the vertex set may be forced to be any value. In particular, it will often be useful to force this constant to have cost 0.

Definition 5.2.7. Let $\vec{G}$ be a digraph containing tour $T = (1, 2, \ldots, n)$. Furthermore, let C be a cost matrix associated with $\vec{G}$. A *reversing alternating sum* from 1 to $u \in V(\vec{G})\backslash\{1\}$ around T is defined to be the sum $c_{12} - c_{32} + c_{34} - c_{54} + \ldots - c_{u(u-1)}$ if u is odd and $c_{12} - c_{32} + c_{34} - c_{54} + \ldots + c_{(u-1)u}$ if u is even.

Observation 5.2.8. *A reversing alternating sum from i to j of odd length has cost $a_i + b_j$ in an SC-Hamiltonian digraph. In particular, a reversing alternating sum around any odd length cycle starting at node i has cost $a_i + b_i$ in an SC-Hamiltonian digraph.*

Observation 5.2.9. *A reversing alternating sum from i to $j \neq i$ of even length has cost $a_i - a_j$ in an SC-Hamiltonian digraph. In particular, a reversing alternating sum around any even length cycle starting at any node has cost 0 in an SC-Hamiltonian graph.*

Lemma 5.2.10. *Let n be odd. If symmetric digraph $\vec{G}$ is not strongly Hamiltonian then $\vec{G}$ is not SC-Hamiltonian.*

Proof. Suppose there is a pair of arcs $1v$ and $v1$ that are not contained in any tour of $\vec{G}$. Let $c_{1v} = c_{v1} = 1$ and every other arc of $\vec{G}$ have cost 0. Since $1v$ and $v1$ are not contained in any tour, all tours in $\vec{G}$ have 0 cost. Hence, $C = (c_{ij})$ is DTC(1). Suppose $\vec{G}$ is SC-Hamiltonian. Then there exist constants such that $c_{ij} = a_i + b_j$ $\forall ij \in E(\vec{G})$.

Since n is odd $1v$ and $v1$ partition any tour of $\vec{G}$ into an odd length cycle and an even length cycle. Let $T = (1, 2, \ldots, n)$ be a tour of $\vec{G}$ where the cyclic ordering of the node set around T is taken in the direction corresponding to the odd length path from 1 to v. This forces v to have even parity with respect to the labelling of tour T. Take a reversing alternating sum from 1 to v around T along the odd length path. This sum yields $c_{12} - c_{32} + \ldots + c_{(v-1)v} = a_1 + b_2 - a_3 - b_2 + \ldots + a_{v-1} + b_v = a_1 + b_v = c_{1v} = 1$. But $c_{12} - c_{32} + \ldots + c_{(v-1)v} = 0$ as all arcs of T have cost 0. This yields a contradiction. □

Lemma 5.2.11. *Let n be even. If symmetric digraph $\vec{G}$ has a pair of arcs not contained in any tour but that partitions some tour into two even length cycles then $\vec{G}$ is not SC-Hamiltonian.*

Proof. The proof is analogous to that of Lemma 5.2.10. □

Lemma 5.2.12. *Let n be even. If symmetric digraph $\vec{G}$ has a pair of arcs not contained in any tour but that partitions some tour into two odd length cycles then $\vec{G}$ is not SC-Hamiltonian.*

Proof. Suppose there is a pair of arcs $1v$ and $v1$ that are not contained in any tour of $\vec{G}$. Let $c_{1v} = 1$, $c_{v1} = 2$ and every other arc of $\vec{G}$ have cost 0. Since $1v$ and $v1$ are not contained in any tour, all tours have 0 cost. Hence, C is DTC(1). Suppose $\vec{G}$ is SC-Hamiltonian. Then there exists constants such that $c_{ij} = a_i + b_j \; \forall ij \in E(\vec{G})$. Let $T = (1, 2, \ldots, n)$ be a tour of $\vec{G}$. Lemma 5.2.11 suggests that $1v$ and $v1$ must partition T into two odd length cycles. Denote either one of these cycles of odd length as K. Observation 5.2.8 shows that a reversing alternating sum taken around K starting at 1 will have cost $a_1 + b_1$ when considering the constants associated with each vertex. Notice that reversing alternating sum around K taken in the direction taken along $1v$ has cost 1 (as $c_{1v} = 1$). But the reversing alternating sum around K in the opposite direction has cost 2 (as $c_{v1} = 2$). Thus, $a_1 + b_1 = 2 = 1$ yields a contradiction. □

Theorem 5.2.13. *Let $\vec{G}$ be a symmetric digraph. If $\vec{G}$ is not strongly Hamiltonian then $\vec{G}$ is not SC-Hamiltonian.*

Proof. This result follows directly from Lemma 5.2.10, Lemma 5.2.11 and Lemma 5.2.12. □

5.2.2 General digraphs

The following lemma was provided by Kabadi and Punnen in [11] without proof.

Lemma 5.2.14. *Directed cycles are SC-Hamiltonian.*

Proof. Let $T = (1, 2, \ldots, n)$ be a directed tour in $\vec{G}$. For any DTC(1) matrix C take $a_i = c_{i(i+1)}$ and $b_i = 0$. Since every arc lies in T it follows that $c_{ij} = a_i + b_j$ $\forall ij \in E(\vec{G})$. $\square$

Definition 5.2.15. Let $\vec{G}$ be a digraph and let $K = (1, 2, \ldots, k)$ be a cycle in the underlying undirected graph of $\vec{G}$ with k even. A *reversing cycle of* K is present in $\vec{G}$ if the arcs $12, 32, 34, 54, \ldots, (k-1)k$ and $1k$ are in $\vec{G}$.

Lemma 5.2.16. *Let $\vec{G}$ be a digraph that contains a reversing cycle of cycle K. If K contains an edge that is not utilized by any tour in $\vec{G}$ then $\vec{G}$ is not SC-Hamiltonian.*

Proof. Let $\vec{G}$ contain a reversing cycle of cycle $K = (1, 2, \ldots, k)$ where arc $1k$ is not contained in any tour of $\vec{G}$. Furthermore, let $c_{1k} = 1$ and $c_{ij} = 0$ $\forall ij \in E(\vec{G})\backslash\{1k\}$. Every tour has cost 0 implies that $C = (c_{ij})$ is DTC(1). Suppose $\vec{G}$ is SC-Hamiltonian. Then there exist constants such that $c_{ij} = a_i + b_j$ $\forall ij \in E(\vec{G})$. By Observation 5.2.5, it is possible to set $a_1 = 0$. This forces $b_2 = 0$, which forces $a_3 = 0$, …, which forces $b_k = 0$. So $1 = c_{1k} = a_1 + b_k = 0 + 0 = 0$ which yields a contradiction. $\square$

Lemma 5.2.16 identifies several different classes of digraphs that are not SC-Hamiltonian. Bi-cycles on an odd number of vertices with a single other arc and bi-cycles on an even number of vertices less a single edge are two such classes. Furthermore, the proof in Lemma 5.2.16 can easily be extended to obtain the following results.

Lemma 5.2.17. *Let $\vec{G}$ be a digraph that contains a reversing cycle K'. Furthermore, let E^* be a set of arcs such that every tour in $\vec{G}$ contains exactly $\varepsilon \in \mathbb{Z}^+$ arcs from E^*. If K' contains exactly one arc from E^* then $\vec{G}$ is not SC-Hamiltonian.*

Lemma 5.2.18. *Let $\vec{G}$ be a digraph that contains a reversing cycle K' of cycle K in the underlying undirected graph G. Furthermore, let E^* be a set of arcs such that every tour in $\vec{G}$ contains exactly $\varepsilon \in \mathbb{Z}^+$ arcs from E^*. If K' contains multiple arcs from E^* where the deletion of two such arcs creates a path of odd length in the traversal of K (in G) then $\vec{G}$ is not SC-Hamiltonian.*

Lemma 5.2.19. *Any Hamiltonian subgraph of a bi-cycle on an odd number of vertices is SC-Hamiltonian.*

Proof. The proof follows directly from Lemma 5.2.4 as the constants described in the proof also hold for any subgraph. It can be observed that Lemma 5.2.14 is a subcase of this lemma. □

A complete characterization of SC-Hamiltonicity of all undirected graphs that are not strongly Hamiltonian has been established. However, the same results do not extend for the directed case. In particular, consider the digraphs in Figure 5.18.

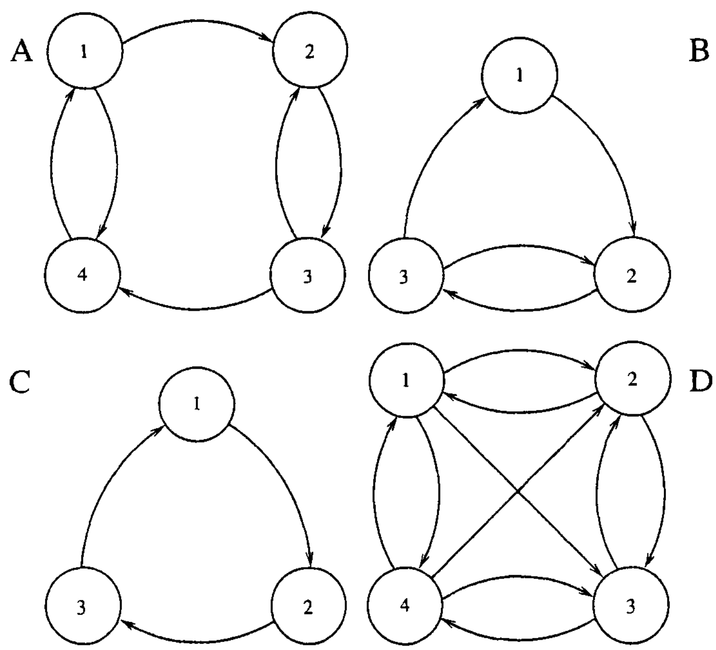

Figure 5.18: Examples illustrating the difficulty of classifying the SC-Hamiltonicity of digraphs in terms of strong Hamiltonicity

Denote the graphs in Figure 5.18 as $\vec{G}_A$, $\vec{G}_B$, $\vec{G}_C$ and $\vec{G}_D$ for Examples A, B, C and D respectively.

In Example A, notice $\vec{G}_A$ contains a reversing cycle so by Lemma 5.2.16 this digraph is not SC-Hamiltonian. Moreover, since arcs 14 and 32 do not reside in any tour of $\vec{G}_A$ it is not strongly Hamiltonian. Example B illustrates a digraph that is SC-Hamiltonian but not strongly Hamiltonian as guaranteed by Lemma 5.2.16. Example C is strongly Hamiltonian since $\vec{G}_C$ is simply an odd directed cycle. Lemma 5.2.14 suggests $\vec{G}_C$ is also SC-Hamiltonian.

The digraph in Example D has three tours $T_1 = (1,2,3,4)$, $T_2 = (1,4,3,2)$ and $T_3 = (1,3,4,2)$. Hence, $\vec{G}_D$ is strongly Hamiltonian and every arc of $\vec{G}_D$ is contained in some tour. Associate cost matrix C with $\vec{G}_D$ with $c_{12} = c_{13} = 2$, $c_{14} = c_{43} = 1$ and $c_{ij} = 0$ for all other arcs of $\vec{G}_D$. Then $C(T_1) = C(T_2) = C(T_3) = 2$. Hence, C is a DTC(1) matrix associated with $\vec{G}_D$. Now suppose that $\vec{G}_D$ is SC-Hamiltonian with constants such that $c_{ij} = a_i + b_j$ for every arc of $\vec{G}_D$. By Observation 5.2.5, take $a_1 = 0$. This forces $b_2 = 2$, $b_3 = 2$ and $b_4 = 1$. Arc 42 suggests $a_4 = -2$ but arc 43 suggests $a_4 = -1$. Thus, $\vec{G}_D$ is strongly Hamiltonian but not SC-Hamiltonian.

These examples illustrate that it is possible to find strongly Hamiltonian digraphs that are either SC-Hamiltonian or not SC-Hamiltonian, and that it is also possible to find digraphs that are not strongly Hamiltonian but are either SC-Hamiltonian or not SC-Hamiltonian. Thus, it appears that there is not a strong correlation between the SC-Hamiltonicity of a digraph $\vec{G}$ and the SC-Hamiltonicity of it's underlying undirected graph G.

Chapter 6

Concluding Remarks

Chapter 2 relates the problem of testing the number of distinct Hamiltonian cycle values into the problem of testing the number of distinct Hamiltonian path values (for the complete directed graph). This chapter also gave a reduction theorem that converts the problem of testing the number of distinct Hamiltonian cycle values into the problem of testing the number of distinct linear spanning 2-forest values for classes of matrices with some additional restrictions.

Chapter 3 gave a characterization of both LS2F(1) and LS2F(2) cost matrices. This provides a characterization of DTC(1) and DTC(2) matrices. As a by-product, a testable characterization of DPC(1) and DPC(2) matrices was also established.

In Chapter 4, a characterization of cost matrices with some additional restrictions where all linear spanning 2-forests have three distinct values is given. As a consequence, this gives a characterization of cost matrices where all Hamiltonian paths and Hamiltonian cycles have three distinct values in these restricted cases.

Hence, characterizing all cost matrices such that there exist at most three distinct Hamiltonian cycle and Hamiltonian path values remains an open question. Resolving this question would give a new solvable class of the TSP.

Based on these conclusions, an immediate question may be posed. What is the structure of DTC(k) matrices for $k \geq 4$? Since the direct approach to this question seems difficult, two alternative approaches have been established. These approaches are to either give a characterization of DPC(k) matrices or to give a characterization of LS2F(l) matrices $\forall l \leq k$. However, as the value of k increases, both of these tasks becomes exceedingly difficult. We suggest extending the concepts presented in this thesis to consider the number of distinct linear spanning p-forest values in an effort to reduce the complexity of the procedure.

A *linear spanning p-forest* (LSpF) of G is a linear spanning forest of G with precisely p components. Furthermore, C is a k *distinct linear spanning p-forest* cost matrix, denoted LSpF(k), if and only if there exist exactly k distinct LSpF values in G described by C.

Although considering these structures will provide fruitful results for small values of k, there are limitations. In particular, the procedure for determining the number of distinct LSpFs will fail to run in polynomial time if either k or the number of conditions that must be checked (to determine if a matrix is LSpF) is too large. Hence, this topic remains open.

Chapter 5 was successful in characterizing the SC-Hamiltonicity of all Hamiltonian undirected graphs that are not strongly Hamiltonian. However, it still remains an open question to determine the SC-Hamiltonicity of all strongly Hamiltonian undirected graphs. It is hoped that the many classes considered in this thesis will provide some insight into completing this characterization.

Furthermore, the same question remains open for directed graphs. In this case, a characterization of graphs that are not strongly Hamiltonian was not achieved. Thus, there is still much work to be performed in this field.

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