Colouring Complexes of Planar Triangulations and the Line Graphs of Cubic Graphs

by

Nathan Singer

M.Sc., Simon Fraser University, 2011
B.Sc., Dalhousie University, 2009

Thesis Submitted in Partial Fulfillment of the
Requirements for the Degree of
Doctor of Philosophy

in the
Department of Mathematics
Faculty of Science

SIMON FRASER UNIVERSITY
Summer 2020

Declaration of Committee

Name:	**Nathan Singer**
Degree:	**Doctor of Philosophy (Mathematics)**
Thesis Title:	**Colouring Complexes of Planar Triangulations and the Line Graphs of Cubic Graphs**
Committee:	**Chair:** Petr Lisonek Professor, Mathematics
	Bojan Mohar Supervisor Professor, Mathematics
	Matt DeVos Committee Member Associate Professor, Mathematics
	Ladislav Stacho Examiner Professor, Mathematics Department of Mathematics
	Penny Haxell External Examiner Professor, Combinatorics and Optimization University of Waterloo

Abstract

In order to study the parity of a k-colouring, Tutte introduced the notion of a k-colouring complex in 1969. Given a k-colourable graph X, the k-colouring complex $B_k(X)$ is the graph which has all the independent sets which are colour classes of X as its vertices and two vertices A and B in $V(B_k(X))$ joined by an edge if the colour classes A and B appear together in a k-colouring of X. Subsequently, Fisk proved that the graph $B_k(X)$ is k-colourable and discovered infinite families of graphs for which $B_k(B_k(X))$ is isomorphic to X.

In this thesis, we resolve a question Tutte posed about the 4-colouring complex at one of his final public lectures in 1999. He asked whether the 4-colouring complex of a planar triangulation could have two components in which all colourings have the same parity. In response, we construct triangulations of the plane whose 4-colouring complexes have arbitrarily many even and odd components. Furthermore, we exhibit an infinite family of 4-connected triangulations of the plane whose 4-colouring complexes have an arbitrarily large number of even components, as well as a number of 5-connected triangulations of the plane whose 4-colouring complexes have at least two components in which all colourings have the same parity.

In the later chapters of this thesis, we continue our study of the k-colouring complex, discovering many new infinite families of graphs X for which $B_k(B_k(X))$ is isomorphic X. We call these graphs reflexive graphs. Most notably, if G is a 3-edge-colourable, connected, cubic (possibly including half-edges) outerplanar graph, we prove that $L(G)$ is reflexive if and only if G is triangle-free. In order to establish this result, we show how to reduce questions about the reflexivity of a connected graph to questions about the reflexivity of its 2-edge-connected components. Then we determine conditions under which subdividing an edge preserves reflexivity. These two novel theorems are of independent interest. In particular, we apply the latter theorem to prove that theta graphs have reflexive line graphs.

Keywords: colouring complex; graph colouring; edge-colouring; triangulation

Acknowledgements

I would like to thank my supervisor, Bojan Mohar, for introducing me to the work of William T. Tutte and Steve Fisk from which this thesis developed, as well as his valuable guidance throughout my career as a graduate student. His help in refining many of my ideas, and the new perspectives he offered, proved invaluable throughout my time at Simon Fraser University.

Many more people have helped make my time at SFU productive, valuable and memorable. Among these numerous, talented colleagues and friends, I would particularly like to thank Fiachra Knox, with whom I shared many interesting discussions about reflexive colouring complexes, as well as Matthew DeVos, whose comments on this dissertation proved very valuable.

Finally, I would like to thank my parents and family for their love and support throughout my entire academic career.

Table of Contents

List of Tables

List of Figures

Index of Definitions

Chapter 1

Introduction

A *graph homomorphism* $f : G \to H$ is a map from the vertex set of a graph G to the vertex set of a graph H such that, for any $u, v \in V(G)$, if $u \sim v$, then $f(u) \sim f(v)$. Whenever there exists a graph homomorphism $f : G \to K_k$ (where we label the vertices of the complete graph K_k with distinct elements from $\{1, 2, \ldots, k\}$) we say that the graph G is *k-colourable*. Such a map f is commonly referred to as a k-colouring of G. The sets $f^{-1}(i)$ $(1 \leq i \leq k)$ are then termed the colour classes of f.

However, in this dissertation, we find it convenient to introduce non-standard definitions for k-colourings and colour classes. For the purposes of the forthcoming definitions, we permit a partition of a set to contain $\emptyset$, so we say that a partition of a set U is a collection of pairwise disjoint subsets of U with union equal to U. With this clarification, we define a *k-colouring* of a graph G to be a partition $\mathcal{C}$ of $V(G)$ with $|\mathcal{C}| \leq k$, so that every member of $\mathcal{C}$ is an independent set in G. Additionally, we require the technical condition that $\emptyset \in \mathcal{C}$ whenever $|\mathcal{C}| < k$. The independent sets $C \in \mathcal{C}$ are termed the *colour classes* of the k-colouring.

Observe that, for k-colourable graphs containing a $(k-1)$-clique, these two interpretations of a k-colouring can be reconciled in a natural way. In this case, a graph homomorphism $f : X \to K_k$ induces a partition of $V(X)$ into the k independent sets $\{f^{-1}(1), f^{-1}(2), \ldots, f^{-1}(k)\}$. Meanwhile, given a partition $\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$ of $V(G)$ into k independent sets (where G contains a $(k-1)$-clique) we can construct a graph homomorphism $f : G \to K_k$ by taking $f^{-1}(1) = C_1, f^{-1}(2) = C_2, \ldots, f^{-1}(k) = C_k$. In this thesis, one reason why we will often concern ourselves only with k-colourable graphs which contain a $(k-1)$-clique is because we want our definition of a k-colouring to easily reconcile with the usual definition in this way.

Now, given a fixed k-colouring f of a k-colourable graph G (whose colour classes are $\{A_1, A_2, \ldots, A_k\}$), we say that an *$A_iA_j$-Kempe chain* of f $(i, j \in [k])$ is a connected component of $G[A_i \cup A_j]$. Swapping the two colours on a Kempe chain transforms one colouring into a different colouring. This operation is called a *Kempe change*.

By the late 19th century, some mathematicians suspected that a proof of the, at that time, Four Colour Conjecture was likely to be found in determining the circumstances under which a triangulation of the plane could be recoloured using Kempe changes [14, 38]. In 1879, Kempe presented an attempted proof of the Four Colour Theorem using Kempe changes [28], which was thought to be correct until 1890, when Heawood found a planar graph on 25 vertices for which Kempe's argument did not work [26]. Heawood subsequently used Kempe changes in order to establish the Five Colour Theorem.

Even long after the Four Colour Theorem was resolved in [1], recolouring methods using Kempe changes remained an important and powerful tool in the study of colouring problems [40]. In more recent times, building upon work of Mohar regarding the equivalence classes of a graph's colourings under the Kempe change relation [32], a large body of work on Kempe equivalence classes has been completed [4, 5, 7, 9, 16, 30]. In this context, two colourings f and g are said to be *Kempe equivalent* if g can be obtained from f through a sequence of Kempe changes. Related work regarding reconfiguration graphs has concerned itself with studying the graphs formed by colourings when one takes Kempe change as the adjacency relation [36]. In this thesis, we examine an older, related construction, which W.T. Tutte introduced in 1969 in order to study the recolourings of a graph [46].

The reader should note that the purpose of this chapter is to summarize our results and review some of the history of the problems we are studying. Consequently, we will not prove any results in this introduction, and many of the definitions presented will be restated more formally in Chapters 2 and 3. In the event of any confusion this may cause, we refer the reader to the Index of Definitions on page x, where all the basic definitions used in this dissertation are referenced.

1.1 Parity of 4-Colourings

Tutte was interested in studying the recolourings of the faces of cubic graphs embedded in the plane, as part of his investigation into the Four Colour Conjecture. We define a *plane graph* to be a graph H equipped with an embedding in the plane. Every such plane graph has a set of faces which we denote by $F(H)$. A *4-colouring* of the faces of a cubic plane graph H is a partition of the faces of H into four disjoint sets A, B, C and D (called *colour classes*) so that no two faces in the same colour class have an edge in common.

In [46], Tutte defined the *colour complex* of H to be the pure simplicial 3-complex $K(H)$ whose 3-simplices are the 4-colourings of the faces of H. In this context, an AB-Kempe chain of H is a maximal, connected region of H whose faces are coloured with A and B only. He denoted the number of such AB-Kempe chains by $p(A, B)$, in order to define (for any 4-colouring of faces f)

$$J_A(f) = p(A, B) + p(A, C) + p(A, D) - p(B, C) - p(B, D) - p(C, D). \tag{1.1}$$

Observing that, for a given 4-colouring of faces f, the parity of $J_A(f)$ equals the parity of $J_X(f)$ (for each $X \in \{B, C, D\}$), Tutte chose to call a 4-colouring f an *even 4-colouring* when $J_A(f)$ was even and called f an *odd 4-colouring* when $J_A(f)$ was odd.

Two 4-colourings in the same connected component of $K(H)$ are easily seen to have the same parity. Thus, parity could serve as an invariant, allowing Tutte to distinguish between 4-colourings of faces which could not be obtained from one another through a sequence of recolourings, each of which fixed at least one colour class. However, the construction also naturally motivated another question which would interest Tutte for much of his life. As we find it more straightforward to study vertex colourings of planar triangulations, rather than face colourings of cubic graphs, we will restate Tutte's question in this form.

At first glance, there does not appear to be any reason why all colourings of the same parity should lie in the same component of the colour complex, and yet, examining many examples over a number of years, Tutte was unable to find any triangulation of the plane whose colour complex had two components of the same parity. However, he was reluctant to make this a conjecture, as, in his view, "the data are too few to justify a Conjecture." Nonetheless, he asked: "If anyone knows of any case of two components of the same parity, I would be glad to hear of it" [47].

1.2 Reflexive Graphs

Roughly a decade after Tutte first introduced the colouring complex, Fisk further elaborated upon the simplicial complex's structure in [20] (and later, at greater length, in [21, 22]). Generalizing Tutte's construction to k-colourings of pure $(k-1)$-complexes, Fisk defined the *colouring functor* B on the category of pure $(k-1)$-complexes. As we prefer to avoid a lengthy digression into category theory, and find it more convenient to examine graphs, rather than pure $(k-1)$-complexes, we will restate Fisk's results here in terms of graphs and graph homomorphisms.

Given a k-colourable graph G which contains a $(k-1)$-clique, we say that the *k-colouring complex* $B_k(G)$ is the graph which has the colour classes in all k-colourings of G as its vertices. Two vertices $A, B \in V(B_k(G))$ are joined by an edge if A and B appear together in a k-colouring of G. Adjacency in the k-colouring complex generalizes the notion of Kempe equivalence, as two colourings f and g intersect in a vertex, if they share a common colour class. See Chapter 2 for a more complete introduction to k-colouring complexes and their properties.

In [20], Fisk showed that the k-colouring complex $B_k(G)$ of a k-chromatic graph G, where every edge of G lies in some k-clique, and G has no isolated vertices, is k-colourable. As $B_k(G)$ is k-colourable in this case, it is natural to consider the graph $B_k(B_k(G))$, which we will denote by $B_k^2(G)$.

In his attempts to resolve the questions posed by Tutte, Fisk discovered a graph homomorphism $\phi_G : G \to B^2_k(G)$. He defined $\phi_G(v)$ to be the set of all colour classes of G containing the vertex v, and he showed that this function ϕ_G is a well-defined graph homomorphism whenever every edge of G lies in some k-clique, and G has no isolated vertices. Graphs for which the homomorphism ϕ_G was an isomorphism he called *reflexive*.

The reader should note that the homomorphism ϕ_G (and $B^2_k(G)$) can be defined for a broader class of graphs. These formal definitions will be presented in Chapter 2. However, as our purpose in this introduction is to discuss the history of the field and motivation for our work, we will define terms more narrowly in this section.

In this setting, the 3-edge-colourings of planar cubic graphs arose rather naturally by considering the inner duals of triangulations of the plane (inner duals are formally defined in Section 4.3). We regard a *3-edge-colouring* f of a graph G (with at least one edge) as a partition $\{E_1, E_2, E_3\}$ of $E(G)$ into three independent edge sets. Tutte had examined the 4-colourings of planar triangulations, and, by the well-known colouring-flow duality [37, 41, 45], the 4-colourings of planar triangulations are in one-to-one correspondence with the 3-edge-colourings of their duals.

In [20, 21], Fisk showed that the line graphs of cubic trees and cubic cycles are reflexive. By a *cubic tree* (*cubic cycle*) we refer to the cubic graph obtained by adding half-edges to a tree (cycle) until all of its vertices have degree three. This result of Fisk appears to be an interesting phenomenon, and one of our goals is to understand why reflexive graphs exist.

After resolving Tutte's speculations regarding the 4-colouring complex, this dissertation continues the work started by Fisk. Indeed, while it might appear surprising that reflexive graphs exist at all, our research exhibits several, new, non-trivial, infinite families of reflexive graphs.

Even so, we may only be looking at the tip of the iceberg. The real question that remains open is why there are so many reflexive graphs (and why there are any at all). We do not have an answer to this question, but we believe that this thesis takes some important, early steps towards an answer.

As we will devote considerable energy to studying reflexive line graphs of cubic graphs, we will introduce some new terminology for the sake of brevity. We say that a 3-edge-colourable graph G is *edge-reflexive* if $L(G)$ is a reflexive graph with respect to 3-colourings. So, for example, we would say Fisk proved that the cubic trees and cubic cycles are edge-reflexive.

1.3 Summary of Results

In this dissertation, we prove several new results about the structure of the 3-colouring complex $B_3(X)$, where $X = L(G)$ is the line graph of a cubic graph, as well as the 4-colouring complex $B_4(T)$, where T is a near-triangulation of the plane. Our main contributions are contained in Chapters 3 and 5, where we resolve Tutte's speculations regarding the 4-colouring complex and extend Fisk's work on 3-colouring complexes in a number of ways.

The second chapter of this thesis is a gentle introduction to k-colouring complexes and a review of the literature on this subject. We formally define notions related to k-colouring complexes, and we also establish some important, basic properties of k-colouring complexes.

In the third chapter, we first reprove Tutte's results on the parity of 4-colourings (in particular, Tutte's Second Parity Theorem, from [47]) in the context of vertex-colourings of planar triangulations. We also define a homological version of parity due to Fisk [18], and prove an analogue of Tutte's Second Parity Theorem in this setting. Theorem 3.2.2 resolves Tutte's speculations regarding the parity of a 4-colouring complex's components by exhibiting an infinite family of triangulations of the plane whose 4-colouring complexes each have arbitrarily many components with even parity and arbitrarily many components with odd parity. Subsequently, we strengthen this result by constructing an infinite family of 4-connected triangulations whose 4-colouring complexes have arbitrarily many even components (as well as one odd component) in Theorem 3.3.1. Finally, we address the 5-connected case, giving examples of 5-connected triangulations of the plane whose 4-colouring complexes have at least three components. We first published these results in [34].

In Chapter 4, we focus our attention on identifying a number of non-trivial, infinite families of reflexive graphs. We begin by reproving Fisk's classical result that the cubic trees are edge-reflexive. Then we show precisely which ladders, circular ladders and Möbius ladders are edge-reflexive. Finally, it is established that the suspension of an even cycle has a reflexive 4-colouring complex, as do the members of another class of graphs closely related to the wheel graphs.

Our main result in Chapter 5 is concerned with connected, cubic outerplanar graphs. Note that a cubic outerplanar graph will always contain half-edges.

Theorem 1.3.1. *Let G be a connected, cubic, outerplanar graph. Then G is edge-reflexive if and only if it is triangle-free.*

Additionally, the edge-reflexive theta graphs are classified, and the reflexive, non-planar graph $L(K_{3,3})$ is discussed, along with how subdividing edges of $K_{3,3}$ impacts edge-reflexivity. In the process of establishing these results, new lemmas are proven, showing how to reduce questions about the edge-reflexivity of a connected graph G to questions about the connected graphs obtained from G by cutting all of its cut-edges, as well as conditions under which subdividing an edge preserves edge-reflexivity. The results of this chapter were first submitted for publication in [29].

Examining 3-colouring complexes of the line graphs of cubic graphs provides us with useful insights into the realm of 4-colouring complexes in two different ways. Firstly, as there is a bijection between the 4-colourings of planar triangulations and the 3-edge-colourings of their associated dual cubic graphs (and they also have a similar Kempe structure) [41] we gain some insight into problems involving the 4-colouring complexes of planar triangulations by studying the 3-colouring complexes of the line graphs of cubic graphs. Indeed, we originally conjectured that the reflexive 4-colouring complexes we discuss in Chapter 4 are reflexive for this reason. A number of other structural results also generalize (with some minor modifications) from the 3-colouring complexes of the line graphs of cubic graphs to 4-colouring complexes of planar triangulations.

Secondly, two 3-colourings of a graph are adjacent if and only if they are Kempe equivalent. By contrast, two 4-colourings of a graph can also be adjacent in cases where they only have one colour class in common. Thus, we can acquire some insight into the differences between the Kempe equivalence of 4-colourings and Tutte's more general notion of 4-colouring adjacency by comparing the properties of these complexes in cases (such as the 3-edge-colourings of cubic graphs and 4-colourings of triangulations of the plane) where their Kempe structures are very similar.

Our sixth chapter introduces the novel notion of very colourful graphs, which illustrates the approach we discussed in the preceding paragraph. Very colourful graphs are computationally and theoretically useful in studying the 3-colouring complex of the line graph of a cubic graph G. We show that the graph homomorphism ϕ_X is injective whenever $X = L(G)$ is very colourful, and we prove that all 3-colouring complexes of the line graphs of cubic graphs are very colourful. This new idea also helps us to easily prove that cubic graphs (in which each vertex is incident with at most one half-edge) with 1, 2 or 3 half-edges cannot be edge-reflexive. However, not all of these results still apply in the context of the 4-colourings of planar triangulations. The remainder of Chapter 6 is spent exploring the extent to which properties of the very colourful line graphs of cubic graphs generalize to the 4-colouring complexes of planar triangulations.

We conclude this thesis with a chapter summarizing a large number of open questions and conjectures our inquiries into these relatively unexplored areas have uncovered.

Chapter 2

Colouring Complexes

This chapter is aimed at gently introducing the reader to k-colouring complexes. We present definitions and basic results from the literature in our new notation, accompanied by simple examples which are intended to clarify these concepts.

In the first section of this chapter, we formally define the k-colouring complex $B_k(X)$ of a graph X, explain the circumstances under which $B_k(X)$ is k-colourable and discuss some relevant notation and conventions. The second section of the chapter establishes when the graph homomorphism $\phi_X : X \to B_k^2(X)$ (introduced by Fisk in [20]) exists. In particular, we prove that ϕ_X is well-defined for near-triangulations of the plane, the line graphs of 3-edge-colourable cubic graphs and k-colouring complexes. We also discuss the related "hat" notation (originally due to Fisk) in this section, as well as basic definitions involving half-edges.

Frequently, we can show that ϕ_X is a graph isomorphism. In this case, we say that the graph X is reflexive. In the third section of this chapter, we introduce a number of basic definitions and properties of ϕ_X, which are useful in proving whether this map is a graph isomorphism or not. Subsequently, the final section of this chapter introduces a number of less straightforward definitions and properties of the k-colouring complex, which we will employ in proofs later in this dissertation.

Throughout the remainder of this chapter (and, indeed, this whole dissertation) we will employ the standard terminology of graph theory, for which we refer the reader to Bondy and Murty's classic textbook [10]. We deviate from the standard terminology in two significant ways. Firstly, when we refer to a graph, we will invariably mean a finite graph without either loops or multi-edges, but we will often allow graphs to contain half-edges. Secondly, we use a non-standard definition of a k-colouring. We will provide a detailed explanation of these deviations in the next section.

2.1 Colouring Complexes

For the purposes of the forthcoming definitions, we permit a partition of a set to contain $\emptyset$, so we say that a partition of a set U is a collection of pairwise disjoint subsets of U with union equal to U. With this clarification, we define a *k-colouring* of a graph X to be a partition $\mathcal{C}$ of $V(X)$ with $|\mathcal{C}| \leq k$, so that every member of $\mathcal{C}$ is an independent set in X. Additionally, we require the technical condition that $\emptyset \in \mathcal{C}$ whenever $|\mathcal{C}| < k$. The independent sets $C \in \mathcal{C}$ are termed the *colour classes* of the k-colouring.

Definition 2.1.1. The *k-colouring complex* $B_k(X)$ is the graph whose vertices are the colour classes in all k-colourings of X. Two vertices $C, D \in V(B_k(X))$ are joined by an edge if C and D appear together in a k-colouring of X.

At times (in particular, in chapters 3 and 6) we will also find it useful to think of a k-colouring as a graph homomorphism $f : X \to K_k$, where we label the vertices of the complete graph K_k with distinct elements from $\{1, 2, \ldots, k\}$. Recall that a *graph homomorphism* $f : X \to Y$ is a map from the vertex set of a graph X to the vertex set of a graph Y such that, for any $u, v \in V(X)$, if $u \sim v$, then $f(u) \sim f(v)$. The graph homomorphism $f : X \to K_k$ induces a partition of $V(X)$ into the k independent sets $\{f^{-1}(1), f^{-1}(2), \ldots, f^{-1}(k)\}$. This partition reconciles this view of a k-colouring with our original definition.

Throughout this dissertation, in order to avoid confusion, we will refer to vertices and edges of a graph X with lower case letters, such as u or v. Sets of vertices and edges in X (in particular, colour classes of X) will be identified with capital letters at the beginning of the alphabet like A, B and C. Colourings (or edge-colourings) of X will usually be denoted by f or g. Sets of vertices in $B_k(X)$ (in particular, colour classes of $B_k(X)$) will be represented by capital letters at the beginning of the alphabet, written in script. For example, we might denote a colour class of $B_k(X)$ by $\mathcal{A}$. Colourings of the k-colouring complex $B_k(X)$ will be denoted by lower case Greek letters (usually χ). As the vertices of $B_k(X)$ are colour classes of X, we represent the colour class $A \subseteq V(X)$ with the same symbol as the vertex $A \in V(B_k(X))$. Similarly, we will frequently use the symbol $\mathcal{A} \subseteq V(B_k(X))$ to also represent a vertex $\mathcal{A} \in V(B_k^2(X))$.

Finally, we will sometimes find it convenient to discuss when two k-colourings f and g have a colour class in common. In this case, we will say that the two colourings f and g are *adjacent* ($f \sim g$). Taking the transitive closure of this adjacency relation, we say that two k-colourings are *connected* if they lie in the same equivalence class of the transitive closure. The reader should observe that the adjacency of colourings generalizes the notion of a Kempe change, while the notion of two colourings being connected generalizes the idea (introduced by Mohar in [32]) of Kempe equivalence. Of course, we also observe that connected colourings lie in the same connected component of the k-colouring complex.

Now, it is easy to see that the graph $B_k(X)$ is k-colourable under a fairly simple condition.

Lemma 2.1.2 (Fisk [18])**.** *Let X be a k-colourable graph which contains some $(k-1)$-clique. Then $B_k(X)$ is k-colourable.*

Proof. Let Q be some $(k-1)$-clique of X which has vertices $\{q_1, q_2, \ldots, q_{k-1}\}$, and let $\mathcal{C}_{q_i}$ be the set of all the colour classes in k-colourings of X which contain the vertex q_i. We claim that $\{\mathcal{C}_{q_i} : i \in [k-1]\} \cup \{(V(B_k(X)) \setminus \cup_{i=1}^{k-1} \mathcal{C}_{q_i})\}$ is a k-colouring of $B_k(X)$. Each vertex of $B_k(X)$ must be a colour class of X containing either exactly one of the vertices q_i or none of them, so the aforementioned set partitions the vertices of $B_k(X)$. Moreover, no two colour classes of X in $\mathcal{C}_{q_i}$ may be adjacent in $B_k(X)$ (since each contains q_i) and no two colour classes of X in $\{(V(B_k(X)) \setminus \cup_{i=1}^{k-1} \mathcal{C}_{q_i})\}$ may be adjacent in $B_k(X)$ (since every k-colouring of X contains at most one such colour class). $\square$

As an aside, we note that the condition in Lemma 2.1.2 that X must contain a $(k-1)$-clique is necessary, as the 4-colouring complex of the 2-regular 4-cycle is not 4-colourable.

Now, let us discuss a few classes of k-colouring complexes which will be relevant to our work. As we mentioned earlier, we allow for the possibility that a graph may contain half-edges. *Half-edges* are edges which are only incident with one vertex and contribute one to the degree of this vertex. In particular, a *cubic graph G* is a graph all of whose vertices have degree three. Notice that we can treat every graph of maximum degree three as a cubic graph by adding half-edges to the vertices of smaller degrees. With this understanding, we will in particular speak of *cubic paths*, *cubic cycles* and *cubic trees*. We refer to Figure 2.1 for some examples.

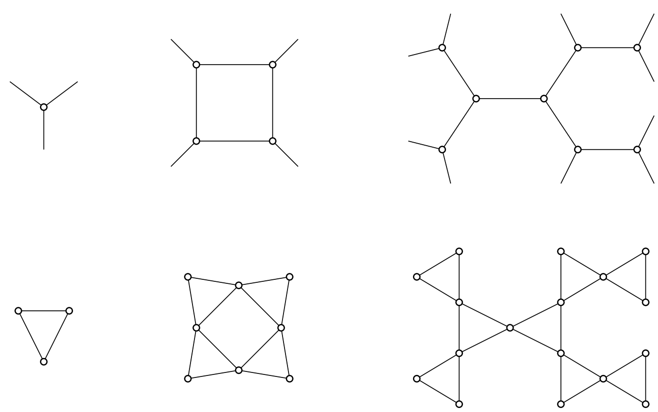


Figure 2.1: Cubic vertex, cubic 4-cycle and a cubic tree with their line graphs.

Additionally, we will often discuss the line graph of a graph G which contains half-edges. Thus, we define the *line graph* $L(G)$ of G as the graph which has as its vertex set the edge set of G (this includes all the half-edges of G). Two vertices of $L(G)$ are adjacent in $L(G)$ if and only if their corresponding edges (in G) are incident with a common vertex.

Finally, when we say that T is a *planar near-triangulation*, we mean that T is a triangulation of some disk embedded in the plane.

When X is a 3-chromatic line graph of a cubic graph (which may contain half-edges) or a 3-chromatic 3-colouring complex, X contains a 2-clique, so the graph $B_3(X)$ is 3-colourable by Lemma 2.1.2. Similarly, if X is either a planar near-triangulation or a 4-chromatic 4-colouring complex, then X contains a 3-clique, so the graph $B_4(X)$ is 4-colourable. Consequently, in these cases, as $B_k(X)$ is k-colourable, we can consider its k-colouring complex $B_k^2(X) := B_k(B_k(X))$.

2.2 The Homomorphism ϕ_X

Whenever the graph X contains a $(k-1)$-clique, we can define the k-colouring complex $B_k^2(X) := B_k(B_k(X))$ by Lemma 2.1.2. Surprisingly, Fisk established in [20] that there exists a graph homomorphism $\phi_X : X \to B_k^2(X)$ under only slightly stricter conditions.

Definition 2.2.1. Let X be a k-colourable graph which contains a $(k-1)$-clique. Then the mapping $\phi_X : V(X) \to V(B_k^2(X))$ is defined as follows:

$$\phi_X(v) = \{C \in V(B_k(X)) \mid v \in C\} \quad (v \in V(X)).$$

Lemma 2.2.2 (Fisk [20])**.** *Let X be a k-colourable graph without isolated vertices in which each edge is contained in a k-clique. Then the map $\phi_X : X \to B_k^2(X)$ is a graph homomorphism.*

Proof. The argument that $B_k(X)$ is k-colourable employed in the proof of Lemma 2.1.2 also establishes that ϕ_X maps k-cliques to k-cliques. Consequently, as each edge of X is contained in a k-clique, the map ϕ_X is a graph homomorphism. □

In fact, we can establish slightly more. Let X be a k-colourable graph in which every edge of X is contained in a $(k-1)$-clique. We can transform X into a k-colourable graph in which every edge of X is contained in a k-clique in the following way. For each $(k-1)$-clique Q of X which is not contained in a k-clique, add a new vertex u_Q to the clique Q in order to make it a k-clique. Observe that this new graph has precisely the same k-colourings as the original graph X.

Fisk defined the graph $\hat{X}$ as the graph obtained from applying the operation described above to the graph X. Furthermore, he observed that $B_k(X) \cong B_k(\hat{X})$. The isomorphism $\psi : B_k(\hat{X}) \to B_k(X)$ takes each vertex of $B_k(\hat{X})$ (colour class of $\hat{X}$) to its restriction to X. The corollary below follows immediately from said observation.

Lemma 2.2.3. *Let X be a k-colourable graph without isolated vertices in which each edge is contained in a $(k-1)$-clique. Then the map $\phi_X : X \to B_k^2(X)$ is a graph homomorphism.*

In fact, this is best possible. To see that Lemma 2.2.3 cannot be improved, consider the graph X drawn in Figure 2.2 and its 4-colouring complex $B_4(X)$.

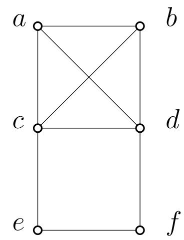


Figure 2.2: A graph X for which $\phi_X(e)$ and $\phi_X(f)$ do not appear together as colour classes in a 4-colouring of $B_4(X)$.

Observe that $\{a\}, \{c\}, \{d\} \in V(B_4(X))$. Moreover, the following 4-colourings of X appear as copies of K_4 in $B_4(X)$: $\{\{c\}, \{d\}, \{a, f\}, \{b, e\}\}$, $\{\{c\}, \{a\}, \{b, f\}, \{d, e\}\}$ and $\{\{d\}, \{a\}, \{b, e\}, \{c, f\}\}$. Thus, a triangle $\{a\}\{c\}\{d\}$ exists in the complement of $\phi_X(e) \cup \phi_X(f)$, so $\phi_X(e)$ and $\phi_X(f)$ are not adjacent in $B_4^2(X)$.

For our purposes, it is important to note here that (by Lemma 2.2.3) the map $\phi_X : X \to B_k^2(X)$ is a well-defined graph homomorphism whenever X is the line graph of a 3-edge-colourable cubic graph and $k = 3$. Similarly, ϕ_X is a well-defined homomorphism when X is a near-triangulation of the plane and $k = 4$. As we will employ these facts throughout this dissertation, we will state them formally as corollaries of Lemma 2.2.3.

Corollary 2.2.4. *Let X be the line graph of a 3-edge-colourable cubic graph. Then the map $\phi_X : X \to B_3^2(X)$ is a graph homomorphism.*

Corollary 2.2.5. *Let T be a near-triangulation of the plane, and consider the 4-colourings of T. Then the map $\phi_T : T \to B_4^2(T)$ is a graph homomorphism.*

The following example illustrates some of the concepts which we have discussed in this section. In particular, we emphasize how the 'hat' construction which was used in order to establish Lemma 2.2.3 applies to 4-colourings of near-triangulations of the plane.

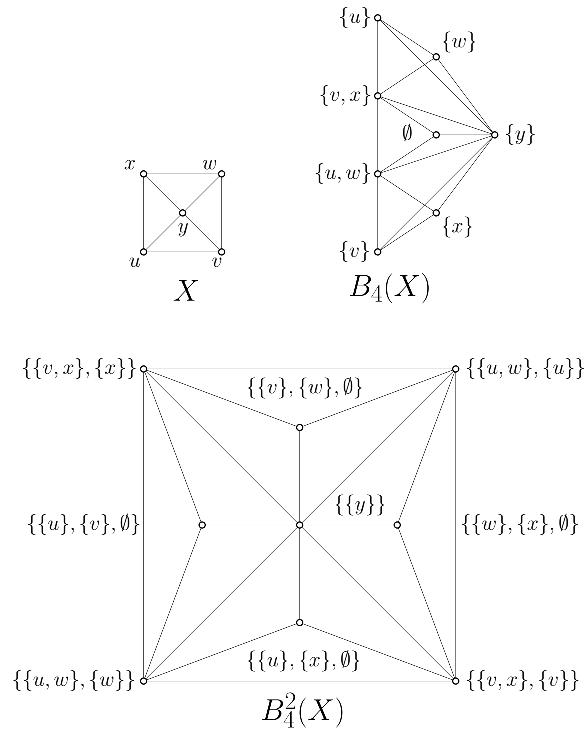


Figure 2.3: Graphs X, $B_4(X)$ and $B_4^2(X)$. The graph $B_4^2(\hat{X})$ is isomorphic to $B_4^2(X)$, and so $\hat{X} \cong B_4^2(\hat{X})$.

Recall from earlier in this section that $B_4(\hat{X}) \cong B_4(X)$, and observe that $\hat{X} \cong B_4^2(X)$ for this example. Thus, $\hat{X} \cong B_4^2(\hat{X})$. Indeed, it is easy to see that this isomorphism is given by $\phi_{\hat{X}}$.

First, notice that (as Corollary 2.2.3 established) since every edge of X is contained in a triangle, the map $\phi_X : X \to B_4^2(X)$ is a graph homomorphism which maps the triangle uvy to the triangle $\phi_X(u)\phi_X(v)\phi_X(y)$ in $B_4^2(X)$. This triangle is contained in the 4-clique $\{\{\{u,w\},\{u\}\},\{\{v,x\},\{v\}\},\{\{y\}\},\{\{w\},\{x\},\emptyset\}$ in $B_4^2(X)$. This 4-clique of $B_4^2(X)$ is also a 4-colouring of $B_4(X)$, so we can apply the inverse of the isomorphism $\psi : B_4(\hat{X}) \to B_4(X)$ to these colour classes in order to obtain a 4-colouring of $B_4(\hat{X})$. The isomorphism ψ takes each vertex of $B_4(\hat{X})$ (colour class of $\hat{X}$) to its restriction to X. Consequently, $\phi_{\hat{X}}(u)$, $\phi_{\hat{X}}(v)$ and $\phi_{\hat{X}}(y)$ are three colour classes in a 4-colouring of $B_4(\hat{X})$, together with the colour class of $B_4(\hat{X})$ which consists of all colour classes of $\hat{X}$ (vertices of $B_4(\hat{X})$) which do not contain any of u, v or y. Now, observe that the colour classes of $\hat{X}$ which do not contain any of u, v or y are precisely the colour classes of $\hat{X}$ containing the vertex z, which is added by the 'hat' operation to the triangle uvy. Thus, the remaining colour class of $B_4(\hat{X})$ is $\phi_{\hat{X}}(z)$. So, the 4-colouring of $B_4(\hat{X})$ obtained by applying ψ^{-1} to $\phi_X(u)$, $\phi_X(v)$, $\phi_X(y)$ and the colour class of $B_4(X)$ which consists of all colour classes of X (vertices of $B_4(X)$) which do not contain any of u, v or y is of the form $\{\phi_{\hat{X}}(u), \phi_{\hat{X}}(v), \phi_{\hat{X}}(y), \phi_{\hat{X}}(z)\}$. By symmetry, it follows that each of the four 4-cliques of $B_4^2(\hat{X})$ may be obtained by applying $\phi_{\hat{X}}$ to each of the four 4-cliques in $\hat{X}$. Therefore, the isomorphism between $\hat{X}$ and $B_4^2(\hat{X})$ is given by $\phi_{\hat{X}}$.

Of course, while this example illustrates how the homomorphism ϕ_X can be an isomorphism and the intricacies of the 'hat' operation, it does not suggest an approach to proving that ϕ_X is an isomorphism for larger classes of graphs (which is our real interest). We will discuss how to begin developing such an approach in the next section.

2.3 Colourful and Reflexive Graphs

We say that a graph X is *colourful* (for k-colourings) if, for any two vertices x and y, there exists a k-colouring which has x and y in different colour classes. Furthermore, we say that a k-edge-colourable graph G is *edge-colourful* if $X = L(G)$ is a colourful graph.

So, for example, if we consider the 3-edge-colourings of any 2-regular cycle C_n, we can easily see that C_n is edge-colourful. In order to prove that C_n is edge-colourful, it suffices to show that, for each pair of edges $a, b \in E(C_n)$, there exists a 3-edge-colouring f of C_n such that $f(a) \neq f(b)$. So, colour the edge a with the colour 3 and all other edges with colours 1 and 2. This is a colouring f of C_n such that $f(a) \neq f(b)$. As we chose a and b arbitrarily, it follows that C_n is edge-colourful.

That X must be colourful is a necessary and sufficient condition for the map ϕ_X to be an injection.

Observation 2.3.1. *Let X be a k-colourable graph which contains some $(k-1)$-clique. Then X is colourful for k-colourings if and only if the mapping ϕ_X is injective.*

Also, in order for X to be a 3-colouring complex or 4-colouring complex (which is necessary for a 3-chromatic graph X to satisfy $X \cong B_3^2(X)$ or for a 4-chromatic graph X to satisfy $X \cong B_4^2(X)$) the following structural conditions must be satisfied.

Lemma 2.3.2. *Let X be a 3-chromatic graph with no isolated vertices. Then any triangle in $B_3(X)$ must represent a 3-colouring of X. Consequently, each edge of $B_3(X)$ is contained in precisely one triangle.*

Proof. Suppose, for a contradiction, that $B_3(X)$ contains a triangle $C_1C_2C_3$ which does not represent a 3-colouring of X. Each edge of $C_1C_2C_3$ must be in a 3-colouring of X. For each $i \neq j$ in $\{1, 2, 3\}$, let $C_{ij} = V(X) \setminus (C_i \cup C_j)$ be the third colour class in a 3-colouring of X which contains C_i and C_j.

Now, $C_1 \cap C_2 = C_2 \cap C_3 = C_3 \cap C_1 = \emptyset$. Hence, $C_1 \subseteq C_{23}$, $C_2 \subseteq C_{31}$ and $C_3 \subseteq C_{12}$. So, let $H = V(X) \setminus (C_1 \cup C_2 \cup C_3)$. Then $C_{12} = V(X) \setminus (C_1 \cup C_2) = C_3 \cup H$, $C_{23} = C_1 \cup H$ and $C_{31} = C_2 \cup H$. If $H = \emptyset$, then $C_{12} = C_3$ and $C_1C_2C_3$ is a 3-colouring of X, which is a contradiction.

Consequently, we may assume that $H \neq \emptyset$. Then there exists a vertex $v \in H$, which has some neighbour $u \notin H$ (since X has no isolated vertices, and H is an independent set).

Without loss of generality, suppose that $u \in C_1$. Then, as $C_1 \subseteq C_{23}$, $u, v \in C_{23}$. However, this is a contradiction, since u and v are neighbours, while C_{23} is an independent set. $\square$

Lemma 2.3.3. *Let Y be a 4-chromatic graph, and suppose that every vertex of Y is contained in a triangle. Then any subgraph of $B_4(Y)$ which is isomorphic to K_4 must represent a 4-colouring of Y.*

Proof. For a contradiction, suppose that $B_4(Y)$ contains a 4-clique $\{C_1, C_2, C_3, C_4\}$ which is isomorphic to K_4 and does not represent a 4-colouring of Y. Observe that each edge of $B_4(Y)$ must be in some 4-clique Z of $B_4(Y)$ which corresponds to a 4-colouring of Y, as all the edges of $B_4(Y)$ are derived from 4-colourings of Y. If the subgraph Z contains exactly three vertices C_i, C_j and C_k from $\{C_1, C_2, C_3, C_4\}$, we call its remaining vertex A_{ijk}. Meanwhile, when Z contains only two vertices C_i and C_j from $\{C_1, C_2, C_3, C_4\}$, we call its remaining two vertices A_{ij} and B_{ij}.

Let $l, m \in \{1, 2, 3, 4\}$. Any vertex A_{ijk} such that $l \notin \{i, j, k\}$ satisfies $C_l \subseteq A_{ijk}$. Similarly, any pair of vertices A_{ij} and B_{ij} such that $l, m \notin \{i, j\}$ satisfy $C_l \cup C_m \subseteq A_{ij} \cup B_{ij}$. Now, let $H = (C_1 \cup C_2 \cup C_3 \cup C_4)^c$. Then (whenever these sets are defined) $A_{ijk} = V(Y) \setminus (C_i \cup C_j \cup C_k) = C_l \cup H$, where $\{l\} = \{1, 2, 3, 4\} \setminus \{i, j, k\}$. Likewise, $A_{ij} \cup B_{ij} = V(Y) \setminus (C_i \cup C_j) = C_l \cup C_m \cup H$, where $\{l, m\} = \{1, 2, 3, 4\} \setminus \{i, j\}$.

Now, if $H = \emptyset$, then $\{C_1, C_2, C_3, C_4\}$ is a 4-colouring of Y, contradicting our initial assumption. Consequently, we may assume that $H \neq \emptyset$. Thus, we may assume that there exists some vertex $v \in H$. Moreover, v is in some triangle uvw of Y.

We now consider two cases.

Case 1: there is a 4-colouring of Y of the form $\{C_i, C_j, C_k, A_{ijk}\}$. We are given that $v \in H$, and that v is in some triangle uvw of Y. Thus, $u, w \notin C_l \subseteq A_{ijk}$, where $\{l\} = \{1, 2, 3, 4\} \setminus \{i, j, k\}$. Without loss of generality, suppose that $u \in C_i$. Then either $w \in C_j$ or $w \in C_k$. So, without loss of generality, suppose that $w \in C_j$. However, as the edge C_lC_k of $B_4(Y)$ must be in some 4-clique Z of $B_4(Y)$ which corresponds to a 4-colouring of Y, either $C_j \subseteq A_{ikl}$, $C_i \subseteq A_{jkl}$ or $C_i \cup C_j \subseteq A_{kl} \cup B_{kl}$, where $\{l\} = \{1, 2, 3, 4\} \setminus \{i, j, k\}$. In the first case, $v, w \in A_{ikl}$, contradicting the fact that A_{ikl} is an independent set. In the second case, $u, v \in A_{jkl}$, contradicting the fact that A_{jkl} is an independent set. In the third case, $u, v, w \in A_{kl} \cup B_{kl}$, which is impossible, as a triangle cannot be contained in the union of two independent sets.

Case 2: there is no 4-colouring of Y of the form $\{C_i, C_j, C_k, A_{ijk}\}$. Thus, all 4-colourings of Y are of the form $\{C_i, C_j, A_{ij}, B_{ij}\}$, for some choice of $i, j \in \{1, 2, 3, 4\}$. Fix some choice of $\{i, j, k, l\} = \{1, 2, 3, 4\}$. As $v \in H$ and uvw is a triangle of Y, we know that either u or w must be contained in $C_i \cup C_j$. Without loss of generality, suppose that $u \in C_i$. Note that $C_i \subseteq A_{jk} \cup B_{jk}$, $C_i \subseteq A_{jl} \cup B_{jl}$ and $C_i \subseteq A_{kl} \cup B_{kl}$. Thus, uvw is a triangle in $A_{jk} \cup B_{jk}$ (a contradiction) unless $w \notin A_{jk} \cup B_{jk}$. Similarly, $w \notin A_{jl} \cup B_{jl}$ and $w \notin A_{kl} \cup B_{kl}$. Hence, $w \notin (A_{jk} \cup B_{jk}) \cup (A_{jl} \cup B_{jl}) \cup (A_{kl} \cup B_{kl}) = C_1 \cup C_2 \cup C_3 \cup C_4 \cup H$, a contradiction. $\square$

When the graph homomorphism $\phi_X : X \to B_k^2(X)$ is an isomorphism, we say that the graph X is *reflexive* with respect to k-colourings (or simply reflexive, if k is clear from the context). Furthermore, we say that a k-edge-colourable graph G is *edge-reflexive* with respect to k-edge-colourings if $X = L(G)$ is a reflexive graph with respect to k-colourings. In the event that G is a cubic graph, we can derive a simple necessary condition from Observation 2.3.1.

Lemma 2.3.4. *A cubic graph G containing a triangle is not edge-colourful with respect to 3-edge-colourings. Consequently, it is not edge-reflexive with respect to 3-edge-colourings.*

Proof. Let $\{a, b, c\}$ be the vertex set of a triangle in G, and let e be the third edge incident with the vertex a. Then the edges e and bc have the same colour in every 3-edge-colouring of G. Consequently, by Observation 2.3.1, the map $\phi_{L(G)} : L(G) \to B_3^2(L(G))$ is not injective. □

The trivial direction of Theorem 1.3.1 follows from Lemma 2.3.4. In order to establish the non-trivial direction of this theorem (and similar results about reflexivity) we have invested considerable effort in devising tools which can be used to prove that graphs are reflexive. The next section of this chapter will define and explain many of these tools.

2.4 Tools for Proving Reflexivity

Throughout the remainder of this dissertation, we will frequently discuss vertices of $B_k^2(X)$, which are also colour classes of $B_k(X)$. In order to distinguish these vertices from vertices of X (which are denoted by lower case letters) or vertices of $B_k(X)$ (which are denoted by upper case letters) we will refer to these vertices with upper case letters in script font. For example, we might refer to a triangle in $B_k^2(X)$ as the triangle $\mathcal{ABC}$.

Now, in order to establish that the graph homomorphism ϕ_X is an isomorphism, we will find it convenient to first show that ϕ_X is an injective homomorphism. Then we count the number of k-cliques of $B_k^2(X)$ (for k-colourings). The following lemmas establish that this approach is acceptable for 3-colourings of the line graphs of cubic graphs and 4-colourings of near-triangulations of the plane (in which every maximal clique has been made into a 4-clique using the "hat" operation).

Lemma 2.4.1. *Let X and Y be k-chromatic graphs such that every edge of X or Y lies in some k-clique, and so that X and Y have no isolated vertices. Furthermore, suppose that $\phi : X \to Y$ is an injective homomorphism, and that X and Y contain the same number of k-cliques. Then ϕ is an isomorphism.*

Proof. By the injectivity of the graph homomorphism ϕ, the set $\{\phi(Q) : Q$ is a k-clique of $X\}$ is a set of k-cliques in Y that is the same size as the set of k-cliques in X. If there is a vertex or edge of Y which is not contained in $\{\phi(Q) : Q$ is a k-clique of $X\}$, then, as each edge of Y lies in some k-clique, and Y has no isolated vertices, there is some k-clique of Y which is not contained in $\{\phi(Q) : Q \text{ is a } k\text{-clique of } X\}$. However, this contradicts the fact that X and Y have the same number of k-cliques, so ϕ must be surjective.

It remains to establish that $\phi(x) \sim \phi(y) \Rightarrow x \sim y$, for any $\phi(x), \phi(y) \in V(Y)$ (since ϕ is surjective). If $\phi(x) \sim \phi(y)$, then $\phi(x)\phi(y)$ is an edge of Y in some k-clique Q' of Y. Now, as X and Y have the same number of k-cliques, and ϕ is an injective graph homomorphism, ϕ maps k-cliques to k-cliques. As a consequence of this fact, $xy \in E(X)$, as required. □

Of course, there are considerable practical challenges in determining the number of k-cliques in $B_k^2(X)$, but we have found that a significant amount of progress can be made by working in this direction. We organize our approach using the following two lemmas.

Lemma 2.4.2. *Let G be a 3-edge-colourable, cubic, edge-colourful graph of order n, and let $X = L(G)$. The following statements are equivalent:*

(i) *G is edge-reflexive with respect to 3-edge-colourings.*

(ii) *$B_3(X)$ has precisely n 3-colourings.*

(iii) *For every 3-colouring $\{\mathcal{A}, \mathcal{B}, \mathcal{C}\}$ of $B_3(X)$, there is a vertex in G with incident edges e, f, g such that $\{\mathcal{A}, \mathcal{B}, \mathcal{C}\} = \{\phi_X(e), \phi_X(f), \phi_X(g)\}$.*

Proof. As argued in the proof of Lemma 2.1.2, all partitions of $V(B_3(X))$ of the form $\{\phi_X(e), \phi_X(f), \phi_X(g)\}$ are 3-colourings of $B_3(X)$. This shows that (ii) and (iii) are equivalent.

Next, by Observation 2.3.1, $\phi_X : X \to B_3^2(X)$ is injective, and every triangle in X corresponds to a vertex of G by Lemma 2.3.4. Thus X has precisely n triangles. As $B_3(X)$ has no isolated vertices, every triangle in $B_3^2(X)$ must represent a 3-colouring of $B_3(X)$ by Lemma 2.3.2, so (ii) is equivalent to the statement that $B_3^2(X)$ has precisely n triangles. By Lemma 2.4.1, this yields the equivalence of (i) and (ii). □

Lemma 2.4.3. *Let X be a 4-colourable, colourful near-triangulation of the plane such that all maximal cliques of X are 4-cliques. Furthermore, suppose that X contains n 4-cliques. Then the following statements are equivalent:*

(i) *X is reflexive with respect to 4-colourings.*

(ii) *$B_4(X)$ has precisely n 4-colourings.*

(iii) *For every 4-colouring $\{\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}\}$ of $B_4(X)$, there is a 4-clique in X with vertices a, b, c, d such that $\{\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}\} = \{\phi_X(a), \phi_X(b), \phi_X(c), \phi_X(d)\}$.*

Proof. As argued in the proof of Lemma 2.1.2, whenever $\{a, b, c, d\}$ is a clique of X, $\{\phi_X(a), \phi_X(b), \phi_X(c), \phi_X(d)\}$ is a 4-colouring of $B_4(X)$. This shows that (ii) and (iii) are equivalent.

Next, by Observation 2.3.1, $\phi_X : X \to B_4^2(X)$ is injective. As every vertex of $B_4(X)$ is contained in a triangle, every 4-clique in $B_4^2(X)$ must represent a 4-colouring of $B_4(X)$ by Lemma 2.3.3, so (ii) is equivalent to the statement that $B_4^2(X)$ has precisely n 4-cliques. By Lemma 2.4.1, this yields the equivalence of (i) and (ii). □

It remains to explain how we plan on counting the colourings of the graph $B_k(X)$. Our main tool in this endeavour is the notion of clustering, which was inspired by Fisk's proof that cubic trees are edge-reflexive in [21]. Clustering the vertices of the graph $B_k(X)$ involves partitioning the vertices of $B_k(X)$. We denote by $\mathcal{C}_{x_1x_2\ldots x_t}$ the set of colour classes of X (vertices of $B_k(X)$) which contain the vertices $x_1, x_2, \ldots, x_t$. In other words,

$$\mathcal{C}_{x_1x_2\ldots x_t} = \bigcap_{i=1}^{t} \phi_X(x_i).$$

These sets $\mathcal{C}_{x_1x_2\ldots x_t}$ will be referred to as *clusters*.

For example, suppose that G is the cubic path on five consecutive vertices u, v, w, x and y (as shown in Figure 2.4). Let the three edges incident with x be denoted by $a = wx$, b, $c = xy$, and let the three edges incident with y be denoted by $c = xy$, d, e. Finally, let $X = L(G)$.

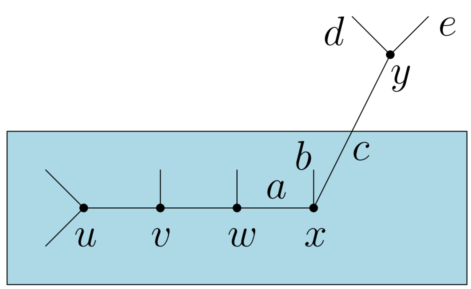


Figure 2.4: The cubic path on five vertices

Now, we can partition the vertices of $B_3(X)$ into the five clusters $\mathcal{C}_{ad}$, $\mathcal{C}_{be}$, $\mathcal{C}_{c}$, $\mathcal{C}_{ae}$ and $\mathcal{C}_{bd}$. Each cluster corresponds to one of the five colour classes which occur among all the 3-edge-colourings of the cubic edge on the vertices x and y.

In these terms, observe that the 3-colourings of X fall into two types, as indicated in Figure 2.5. We will say that a 3-colouring of X in which a and d are in the same colour class is a *type 1* colouring of X. Similarly, a 3-colouring of X in which a and d are in different colour classes will be called a *type 2* colouring of X. The subgraph of $B_3(X)$ consisting of all the triangles of $B_3(X)$ (3-colourings of X) of type $i \in \{1, 2\}$ will be denoted by $\mathcal{T}_i$. Notice that $\mathcal{T}_i$ contains only those vertices of $B_3(X)$ that appear as colour classes in $\mathcal{T}_i$. Thus, $\mathcal{T}_i$ is obtained from the induced subgraph on the three clusters by removing the isolated vertices (which must participate in triangles of some neighbouring $\mathcal{T}_j$, but not in $\mathcal{T}_i$ itself). We say that $\mathcal{T}_i$ and $\mathcal{T}_j$ are *neighbouring subgraphs* when $\mathcal{T}_i$ and $\mathcal{T}_j$ have some cluster in common.

We will also denote the subgraph of $\mathcal{T}_i$ induced by the union of the two adjacent clusters $C_{x_1\ldots x_t}$ and $C_{w_1\ldots w_s}$ by $C_{x_1\ldots x_t}C_{w_1\ldots w_s}$.

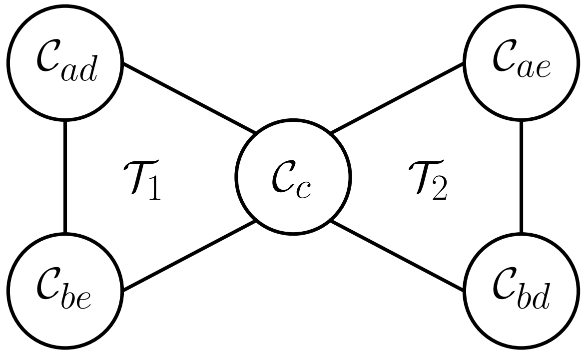


Figure 2.5: The 3-colouring complex $B_3(X)$

Each cluster is an independent set in $B_3(X)$, and there is a natural homomorphism from $B_3(X)$ to $B_3(X[\{a,b,c,d,e\}])$ given by restricting X to the vertex set $\{a,b,c,d,e\}$ of X (ie. each colour class C of X is mapped to $C \cap \{a,b,c,d,e\}$). Moreover, it is not hard to see that each subgraph $\mathcal{T}_i$ is isomorphic to $B_3(L(G[\{u,v,w,x\}]))$. We will prove this observation formally in Chapter 4. Consequently, if we know the number of 3-colourings of $B_3(L(G[\{u,v,w,x\}]))$ (as is the case, for example, when we know that $G[\{u,v,w,x\}]$ is edge-reflexive) then we can count the number of 3-colourings of $B_3(X)$ by counting the extensions of 3-colourings of $B_3(L(G[\{u,v,w,x\}]))$ using the cluster graph.

In order to count these extensions of colourings, we require a few useful lemmas. The remainder of this section is focused on outlining these results.

First, however, we will require one more set of definitions. Suppose that $S \subsetneq V(X)$, and that we have a 3-colouring $f = \{A,B,C\}$ of $X[S]$. A colouring $\{A',B',C'\}$ of X is an *extension* of the colouring f if $A \subseteq A'$, $B \subseteq B'$, and $C \subseteq C'$ (or some permutation of the colour classes satisfies the same inclusion relations). If v is a vertex and for every extension of f, v is in the colour class containing the same colour class in f, then we say that *the colour of* v *is determined* by f (or that the colouring f *determines the colour* of v). If the colours of all vertices are determined by f and f has an extension, we say that f *determines* the extension (or that f *extends uniquely* to X).

Lemma 2.4.4. *Let F be a graph with a 3-colouring whose colour classes are A, B and C. Suppose that F has no isolated vertices, that every edge of F is contained in exactly one triangle, and that $F[B \cup C]$ is connected. Then any 3-colouring of F that uses at least two colours on A is determined by its restriction to A.*

Proof. Let x_0 and y be two vertices of A which have different colours, and let $v, w \in B$ be neighbours of x_0 and y, respectively. Let $P = v_1v_2\ldots v_n$ be a path in $F[B \cup C]$ with $v_1 = v$ and $v_n = w$. For each $i \in [n-1]$, let $x_i \in A$ be the common neighbour of v_i and v_{i+1} (which exists, since every edge of F is contained in a unique triangle). Relabel y as x_n. Let $j \in [n]$ be such that x_j and x_{j-1} have different colours. Such an index exists because x_0 and x_n have different colours. Then v_j is adjacent to both of these vertices, and hence its colour is determined. Relabel v_j as z.

Now let u be any vertex in $B \cup C$. Let $Q = u_1 u_2 \dots u_m$ be a path in $F[B \cup C]$ with $z = u_1$ and $u = u_m$. The colour of u_1 is determined, and whenever the colour of u_i is determined, so is the colour of u_{i+1} (since it is in a triangle with u_i and a vertex of A, both of whose colours are determined). By induction the colour of $u = u_m$ is determined. Since u was arbitrary, the entire colouring is determined. □

So, for example, if we could prove that the subgraph $\mathcal{C}_{ae}\mathcal{C}_{bd}$ of $B_3(X)$ was connected, then any 3-colouring of $\mathcal{T}_1 \cup \mathcal{T}_2$ which used at least two colours on $\mathcal{C}_c \cap \mathcal{T}_2$ would be determined by its restriction to $\mathcal{T}_1 \cup (\mathcal{C}_c \cap \mathcal{T}_2)$. In cases where one colour was used on $\mathcal{C}_c \cap \mathcal{T}_2$, a 3-colouring of $\mathcal{T}_1 \cup (\mathcal{C}_c \cap \mathcal{T}_2)$ would extend in two ways to $\mathcal{T}_1 \cup \mathcal{T}_2$.

Consequently, provided we can prove that $\mathcal{C}_{ae}\mathcal{C}_{bd}$ is connected, Lemma 2.4.4 is a reliable tool which will help us count extensions of colourings. We can use the following lemma to show $\mathcal{C}_{ae}\mathcal{C}_{bd}$ is connected.

Lemma 2.4.5. *Let X be a graph that is reflexive for 3-colourings. If v is a vertex of degree d in X, then $d = 2^t$, where t is the number of components of the bipartite graph $B_3(X) - \phi_X(v)$.*

Proof. As X is reflexive, the set $\phi_X(v)$ is a colour class in a 3-colouring of $B_3(X)$. Since $B_3(B_3(X))$ is isomorphic to X, the number of 3-colourings of $B_3(X)$ that have $\phi_X(v)$ as one of their colour classes is equal to $d/2$ (each colouring contributes 2 towards the degree by Lemma 2.3.2). Thus, $d/2$ is equal to the number of 2-colourings of $B_3(X) - \phi_X(v)$. In particular, this subgraph is bipartite, and it is clear that the number of 2-colourings is equal to 2^{t-1}. Thus, $d = 2 \cdot 2^{t-1} = 2^t$. □

Of course, a similar result also holds for 4-colourings.

Lemma 2.4.6. *Let X be a graph that is reflexive for 4-colourings. If uv is an edge whose endpoints have d common neighbours in X (excluding u and v), then $d = 2^t$, where t is the number of components of the bipartite graph $B_4(X) - (\phi_X(u) \cup \phi_X(v))$.*

Proof. Since X is reflexive, the set $\phi_X(u) \cup \phi_X(v)$ is the union of a pair of colour classes in a 4-colouring of $B_4(X)$. Since $B_4(B_4(X))$ is isomorphic to X, the number of 4-colourings of $B_4(X)$ that have $\phi_X(u)$ and $\phi_X(v)$ as two of their colour classes is equal to $d/2$ (each colouring contributes 2 towards the number of neighbours by Lemma 2.3.3). Thus, $d/2$ is equal to the number of 2-colourings of $B_4(X) - (\phi_X(u) \cup \phi_X(v))$. In particular, this subgraph is bipartite, and it is clear that the number of 2-colourings is equal to 2^{t-1}. Thus, $d = 2 \cdot 2^{t-1} = 2^t$. □

With these results in hand, we are now in a much better position to study reflexive colouring complexes. If the graph $G[\{u, v, w, x\}]$ is an edge-reflexive graph, then the vertex c of $L(G[\{u, v, w, x\}])$ is a vertex of degree $2 = 2^1$ in a reflexive graph for 3-colourings. Thus, by Lemma 2.4.5 (since $\mathcal{T}_1$ and $\mathcal{T}_2$ are each isomorphic copies of $B_3(L(G[\{u, v, w, x\}])))$ both $\mathcal{C}_{ae}\mathcal{C}_{bd}$ and $\mathcal{C}_{ad}\mathcal{C}_{be}$ must be bipartite and connected subgraphs of $B_3(X)$.

More generally, we can now easily establish results like Observation 2.4.7. This easy result is intended to illustrate how we use will use the results we established earlier in order to prove more difficult theorems in chapters 4, 5, and 6.

Observation 2.4.7. *Let G be a 3-edge-colourable, edge-reflexive, cubic graph, which contains at least one half-edge. Then the 3-colouring complex $B_3(L(G))$ is connected.*

Proof. Let $X := L(G)$, and let $e \in E(G)$ be some half-edge in G. The vertex $e \in V(X)$ has degree 2. Thus, as X is reflexive, by Lemma 2.4.5 (which applies, since G is a 3-edge-colourable, cubic graph) it follows that $B_3(X) - \phi_X(e)$ consists of exactly one bipartite component. Now, since $B_3(X) - \phi_X(e)$ is connected, $\phi_X(e)$ is an independent set of $B_3(X)$, and each vertex of $B_3(X)$ is in some triangle (by definition of $B_3(X)$) it follows that $B_3(X)$ must be connected. □

2.5 Concluding Remarks

In this chapter, we have reviewed much of the literature which dealt with colouring complexes, taking care to rigorously redefine the notions Fisk and Tutte introduced for simplicial complexes in terms of graphs. The colouring complex $B_k(X)$ has been defined for a graph X, as has been the special homomorphism $\phi_X : X \to B_k^2(X)$ which Fisk introduced. We have also explained when these constructions are well-defined.

Finally, we discussed some properties of colouring complexes and reflexive colouring complexes (those for which ϕ_X is a graph isomorphism). In particular, we introduced the idea of partitioning $B_k(X)$ into clusters, established lemmas which are used to interpret the structure of $B_k(X)$ in terms of clusters, and discussed how studying extensions of colourings using the cluster graph allows us to count the colourings of $B_k(X)$. These are the foundational tools used in almost all our proofs of reflexivity. Consequently, we will make considerable use of these ideas in chapters 4, 5 and 6 of this dissertation.

Chapter 3

Even and Odd Colourings

In [46], Tutte introduced even and odd 4-colourings of faces. Subsequently, in [47], he reformulated this definition in terms of 4-colourings of the vertices of planar triangulations. As we noted in the introduction, we find this second definition more useful for our purposes, so we dispense with discussing 4-colourings of faces. Moreover, as Tutte's colour complex $K(T)$ is a special case of the 4-colouring complex $B_4(T)$, which we defined in Chapter 2, we find it productive to restate his work in terms of 4-colouring complexes.

In [46], Tutte established that all 4-colourings in a particular connected component of the 4-colouring complex $B_4(T)$ (where T is a triangulation of the plane) must have the same parity, and he devised a way to efficiently compute this parity. However, in his long study of parity and the 4-colouring complex, Tutte never encountered a triangulation of the plane whose 4-colouring complex had two components of the same parity. This experience motivated him to ask whether such triangulations could be found at all [47]. In this chapter, we answer Tutte's question in the affirmative.

In Section 3.1, we review results of Tutte and Fisk related to the parity of 4-colourings and introduce some relevant terminology. In particular, we reprove Tutte's Second Parity Theorem (and Fisk's variant of this theorem for triangulations of surfaces) which provide a more efficient way of computing the parity of a 4-colouring. In Section 3.2, we prove that there are infinitely many triangulations of the plane T such that $B_4(T)$ has more than k components of even parity and more than k components of odd parity, where $k \in \mathbb{N}$. This argument not only resolves Tutte's question, but goes considerably further. In the third section of this chapter, we consider the 4-connected version of Tutte's question. Here, we find a 4-connected triangulation of the plane which has 2^k even components, for each $k \in \mathbb{N}$. Finally, in Section 3.4, we answer the 5-connected variant of Tutte's question through computation, though, in that case, we only find examples of 5-connected triangulations of the plane with three or four connected components. We also introduce a new conjecture based upon our results, suggesting a direction for future work.

3.1 Parity Theorems

Two similar notions of a colouring's parity have been introduced (the first by Tutte in [46] and the second by Fisk in [18]). In this section, we will show how each of these quantities may be efficiently computed (using parity theorems) and how to reconcile the two definitions. As we noted in the introduction, we will reinterpret Tutte's original results in terms of vertex colourings of planar triangulations (rather than face-colourings of cubic graphs) since we find this formulation more convenient.

We denote by $p(A, B)$ the number of AB-Kempe chains in a planar triangulation T. Observe that one can derive $2^{p(A,B)-1} - 1$ new unlabelled colourings of T from a 4-colouring f by taking an arbitrary subset of AB-Kempe chains and performing a Kempe change on each AB-Kempe chain in the subset.

Given this definition, Tutte defined

$$J_A(f) = p(A, B) + p(A, C) + p(A, D) - p(B, C) - p(B, D) - p(C, D). \tag{3.1}$$

For a given 4-colouring f, the parity of $J_A(f)$ equals the parity of $J_X(f)$, for each $X \in \{B, C, D\}$. This observation motivated Tutte to call a 4-colouring f an *even 4-colouring* if $J_A(f)$ is even and to call f an *odd 4-colouring* if $J_A(f)$ is odd.

Tutte's Second Parity Theorem (which he first referred to by this name in [47]) gives us another way of computing the parity of a 4-colouring, which is often more convenient. Moreover, it establishes that every colouring in the same connected component of $B_4(T)$ has the same parity whenever T is a triangulation of the plane. In order to state this result, we will denote by $\deg(A)$ the sum of the degrees of all the vertices in the colour class A.

Theorem 3.1.1 (Tutte [46]). *Let T be a triangulation of the plane with n vertices and A a colour class of its* 4*-colouring f. Then*

$$J_A(f) = 2|A| - \deg(A) + n - 3. \tag{3.2}$$

Proof. Euler's Formula establishes that T has $t = 2n - 4$ triangular faces. Additionally, we observe that the edges of the dual graph T^* of T which correspond to AB-edges and CD-edges form a perfect matching in T^*. Thus,

$$e(A, B) + e(C, D) = \frac{t}{2} = n - 2. \tag{3.3}$$

Now, let us consider the subgraph $T[A \cup B]$ and its faces in the plane. Since it has $|A|+|B|$ vertices, $e(A, B)$ edges and $p(A, B)$ components, we can obtain from Euler's Formula that it has precisely $e(A, B) - (|A| + |B| - p(A, B)) + 1$ faces. Each of these faces contains exactly one CD-Kempe chain (as T is a triangulation) so we conclude that $p(C, D) = e(A, B) - (|A| + |B| - p(A, B)) + 1$. By exchanging the roles of AB and CD, we obtain the expression $p(A,B) = e(C,D) - (|C| + |D| - p(C,D)) + 1$. Combining these two equations, using the fact that $|A| + |B| + |C| + |D| = n$, and using (3.3), we then find that

$$p(A,B) - p(C,D) = |A| + |B| - e(A,B) - 1. \tag{3.4}$$

Finally, by using the same equations for $p(A,C) - p(B,D)$ and $p(A,D) - p(B,C)$, and then taking their sum, we obtain

$$\begin{aligned} J_A(f) &= 3|A| + |B| + |C| + |D| - (e(A,B) + e(A,C) + e(A,D)) - 3 \\ &= 2|A| - \deg(A) + n - 3. \end{aligned}$$

This completes the proof. □

The following corollary then immediately follows from the equation (3.2).

Corollary 3.1.2. *If f and g are two 4-colourings of T with a common colour class A, then $J_A(f) = J_A(g)$.*

For triangulations of general surfaces, we do not obtain quite so strong a result as Corollary 3.1.2. However, Fisk's analogous parity theorem still establishes that all 4-colourings within the same component of $B_4(T)$ will have the same parity. A good reference for those unfamiliar with the study of graphs on surfaces is [35].

Given a triangulation T of an orientable surface with one of its orientations fixed, we can view a 4-colouring f of T as a simplicial mapping onto the boundary of the tetrahedron, which will be denoted by K_4. For each triangle $T_{i,j,l}$ of K_4, we consider the facial triangles in T that are mapped onto $T_{i,j,l}$ in such a way that their orientation is preserved and those facial triangles whose orientation is reversed. Let t^+_{ijl} (t^-_{ijl}) be the number of triangular faces in T which are mapped onto $T_{i,j,l}$ with positive (negative) orientation. Then the value

$$\deg(f) = t^+_{ijl} - t^-_{ijl}$$

is independent of the choice of the i, j and l and is called the *(homology) degree of the mapping* $f : T \to K_4$. Of course, modulo 2, $\deg(f)$ is equal to the $t^+_{ijl} + t^-_{ijl}$, which is the number of triangular faces of T with colours i, j and l.

If T is a triangulation of a non-orientable surface, then we cannot define the values of t^+_{ijl} and t^-_{ijl}, but we can still define $\deg(f)$. In this case, we will say that $\deg(f)$ is defined to be the number of triangular faces of T with colours i, j and l modulo 2. This value is also independent of the choice of i, j and l.

Now, reinterpreting Tutte in the context of higher genus surfaces, we first define the *Euler genus* of a surface triangulation T to be equal to twice the genus, if it is orientable, and to be equal to the crosscap number of the surface if it is non-orientable. We then define

$$J_A(f) = 2|A| - \deg(A) + n - 3 + g,$$

where $n = |V(T)|$ and g is the Euler genus of T. This definition leads to the following theorem, which reconciles Tutte's notion of parity with the more recent definition due to Fisk.

Theorem 3.1.3. *Let T be a triangulation of a surface of Euler genus g with $n = |V(T)|$ vertices. If f is a 4-colouring of T and A is one of its colour classes, then*

$$J_A(f) \equiv \deg(f) + n - 3 + g \pmod 2.$$

Consequently, the homology degree $\deg(f)$ *has the same parity as* $\deg(A)$.

Proof. We may assume that the colour class A corresponds to the colour 4. We let T_{123} correspond to the triangle of K_4 representing the other three colours of the 4-colouring. Then the number of triangles of T coloured with these three colours $t_{123} \equiv \deg(f) \pmod 2$. On the other hand,

$$\deg(A) = \sum_{v \in A} \deg(v) = t_{124} + t_{134} + t_{234} = |F(T)| - t_{123},$$

where $F(T)$ denotes the set of triangular faces of T. Since T is a triangulation, Euler's Formula implies that $|F(T)| = \frac{2}{3}|E(T)| = 2n - 4 - 2g$. Thus,

$$\begin{aligned} J_A(f) &= 2|A| - \deg(A) + n - 3 + g \\ &= 2|A| + t_{123} - |F(T)| + n - 3 + g \\ &\equiv t_{123} + n - 3 + g \pmod 2 \\ &\equiv \deg(f) + n - 3 + g \pmod 2. \end{aligned}$$

□

In order to illustrate these definitions and parity theorems, we present the following example.

We can compute the parity of the triangulation's 4-colourings in three different ways.

In Figure 3.2, we have computed the 4-colouring complex $B_4(T)$ of the triangulation T in Figure 3.1. See Definition 2.1.1 for the formal definition of the 4-colouring complex. Computing the 4-colouring complex in this way allows us to see the quantities $p(A, B)$ (where A and B are colour classes in some 4-colouring of T) at a glance.

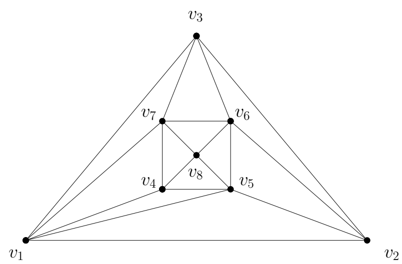


Figure 3.1: A triangulation of the plane T

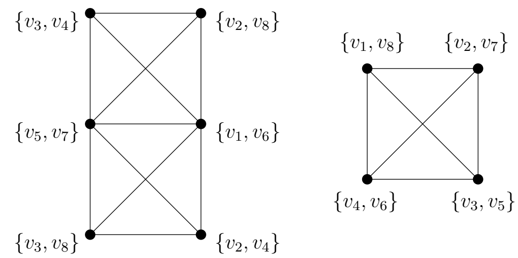


Figure 3.2: Its 4-colouring complex $B_4(T)$

Firstly, for the 4-colouring f given by $\{\{v_1, v_6\}, \{v_5, v_7\}, \{v_3, v_4\}, \{v_2, v_8\}\}$, Tutte's original definition of the invariant $J_{\{v_1,v_6\}}(f)$ gives that

$$\begin{aligned} J_{\{v_1,v_6\}}(f) &= p(\{v_1, v_6\}, \{v_5, v_7\}) + p(\{v_1, v_6\}, \{v_3, v_4\}) + p(\{v_1, v_6\}, \{v_2, v_8\}) \\ &\quad - p(\{v_5, v_7\}, \{v_3, v_4\}) - p(\{v_5, v_7\}, \{v_2, v_8\}) - p(\{v_3, v_4\}, \{v_2, v_8\}) \\ &= 1 + 1 + 1 - 1 - 1 - 2 \\ &= -1. \end{aligned}$$

Thus, f is an odd colouring. Through similar computations, we find that the 4-colouring in the rightmost component of $B_4(T)$ is an even colouring, agreeing with Tutte's observation that most small examples seem to have $B_4(T)$ with either a single component or two components of differing parities.

A less cumbersome approach is to use the Second Parity Theorem. Using the colour class $A = \{\{v_1, v_6\}\}$ again, we compute

$$\begin{aligned} J_A(f) &= 2|A| - \deg(A) + n - 3 \\ &= 2 \times 2 - (5 + 5) + 8 - 3 \\ &= -1. \end{aligned}$$

As expected, this agrees with our earlier computation.

Fisk's approach merely requires that we count the number of triangular faces of T coloured with three particular colours (say colours 1, 2 and 3). So, for example, using the 4-colouring represented by the rightmost tetrahedron in our drawing of $B_4(T)$ to colour T, we obtain the colouring illustrated in Figure 3.3.

There are 3 triangular faces coloured with 1, 2 and 3 (each ordered counterclockwise) so $\deg(f) = 3$. Consequently,

$$\begin{aligned} J_A(f) &\equiv 3 + 8 - 3 + 0 \pmod 2 \\ &\equiv 0. \end{aligned}$$

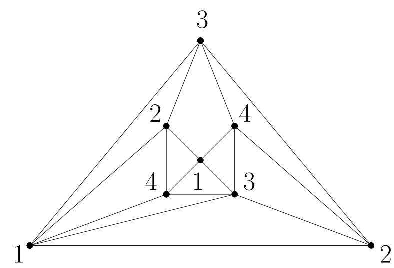


Figure 3.3: A 4-coloured
triangulation of the plane

Thus, as expected, this 4-colouring has even parity, while the other two colourings have odd parity by similar computations.

3.2 3-Connected Triangulations of the Plane

As any triangulation T of the plane such that $B_4(T)$ has at least three components would resolve Tutte's question, it might seem surprising that he struggled with this problem. After all, there are many well known triangulations of the plane with many Kempe components. For example, the icosahedron, pictured in Figure 3.4, has ten 4-colourings each of which cannot be obtained from any other through Kempe changes. However, each of these 4-colourings can be obtained from any other by a sequence of changes in the 4-colouring of T, each of which fixes a single colour class. Thus, $B_4(T)$ is actually connected when we take T to be the icosahedron.

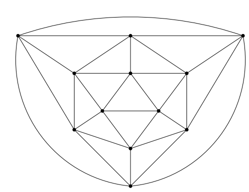
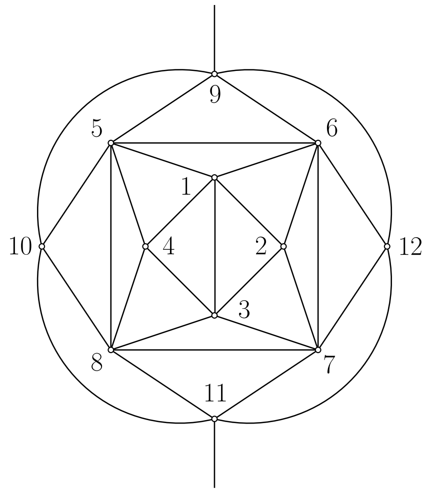


Figure 3.4: The icosahedron

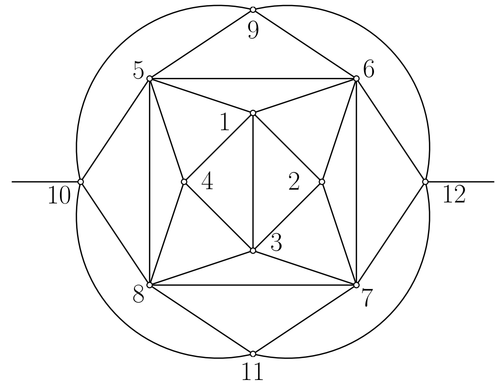
Computation shows that there are no triangulations of the plane T on eleven or fewer vertices such that $B_4(T)$ has two components of the same parity. Consequently, the smallest examples answering Tutte's question in the affirmative are two triangulations of the plane on twelve vertices. Their 4-colouring complexes each have three components, from which it follows that two of these components must have the same parity. These triangulations are drawn in Figure 3.5.

We will now determine the 4-colouring complex of Example 1, and show that it has three connected components (the argument for Example 2 is similar).

Theorem 3.2.1. *The 4-colouring complex of the first example T given in Figure 3.5(a) is the graph drawn in Figure 3.6.*



(a) Example 1



(b) Example 2

Figure 3.5: Triangulations whose 4-colouring complexes have three components. The half-edges sticking out form the edges 9-11 and 10-12, respectively.

Proof. We need to determine all the 4-colourings f of T. We start by precolouring the 4-cycle with vertices labelled 5, 6, 7 and 8. Up to relabelling, this 4-cycle can be precoloured in 4 ways:

(a) $f(5) = 1, f(6) = 2, f(7) = 3, f(8) = 4$.
(b) $f(5) = 1, f(6) = 2, f(7) = 1, f(8) = 2$.
(c) $f(5) = 1, f(6) = 2, f(7) = 1, f(8) = 3$.
(d) $f(5) = 1, f(6) = 2, f(7) = 3, f(8) = 2$.

We consider these cases separately.

(a) Suppose that $f(1) = 3$. Then $f(4) = 2$, $f(3) = 1$ and $f(2) = 4$. Alternatively, if $f(1) = 4$, then $f(2) = 1$, $f(3) = 2$ and $f(4) = 3$. Now, observe that T is symmetric about the precoloured 4-cycle. Thus, there are exactly two ways of colouring the exterior of this 4-cycle. Consequently, we find that there are exactly four colourings satisfying (a). These 4-colourings are given by:
$\{\{3, 5, 11\}, \{4, 6, 10\}, \{1, 7, 9\}, \{2, 8, 12\}\}$, $\{\{3, 5, 12\}, \{4, 6, 11\}, \{1, 7, 10\}, \{2, 8, 9\}\}$,
$\{\{2, 5, 11\}, \{3, 6, 10\}, \{4, 7, 9\}, \{1, 8, 12\}\}$, $\{\{2, 5, 12\}, \{3, 6, 11\}, \{4, 7, 10\}, \{1, 8, 9\}\}$.

(b) We may assume that $f(1) = 3$. In this case, $f(2) = 4$, and so vertex 3 has no available colour. Consequently, no 4-colourings of T satisfy (b).

(c) If $f(1) = 3$, then $f(2) = 4$, and so $f(3) = 2$. Thus, $f(4) = 4$. On the other hand, when $f(1) = 4$, then $f(2) = 3$. Consequently, $f(3) = 2$, and vertex 4 cannot be coloured. Hence, this colouring does not occur. Thus, we find that there is exactly one 4-colouring satisfying (c): $\{\{5, 7\}, \{3, 6, 11\}, \{1, 8, 9\}, \{2, 4, 10, 12\}\}$.

(d) This case is mirror-symmetric to (c). Thus, we obtain exactly one 4-colouring satisfying (d): $\{\{3, 5, 11\}, \{6, 8\}, \{1, 7, 9\}, \{2, 4, 10, 12\}\}$.

The six colourings of T clearly result in the colouring complex exhibited in Figure 3.6. $\square$

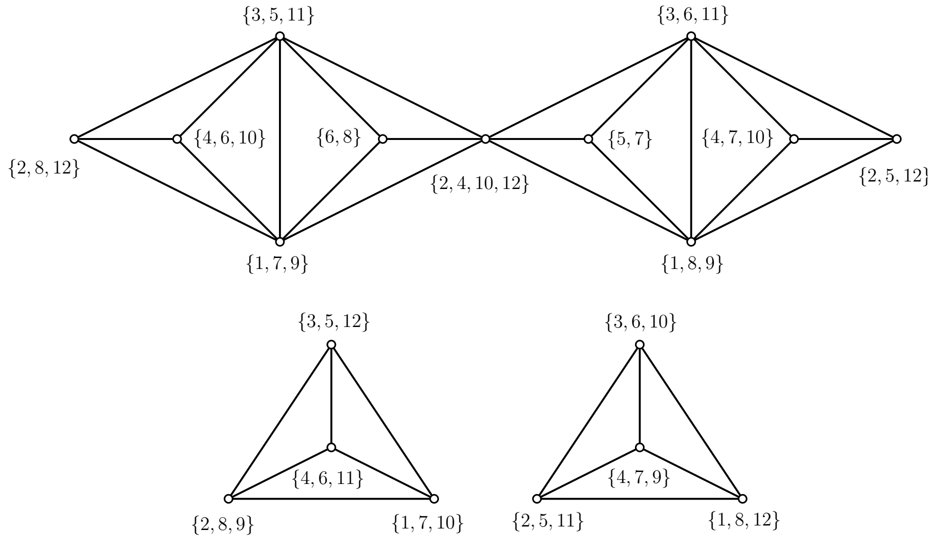


Figure 3.6: The colouring complex $B_4(T)$
for Example 1

The 4-colouring complex $B_4(T)$ from Theorem 3.2.1 certainly has three components, two of which have even parity, while the third one has odd parity. This provides the affirmative answer to Tutte's question we sought. The triangulation shown in Figure 3.5(b) is not isomorphic to the previous example (as the triangulation given in Example 1 has two degree 5 vertices with two common neighbours of degree 6, while the triangulation presented in Example 2 does not) but their colouring complexes are isomorphic.

Using the triangulation T in Example 1, we can now easily construct infinitely many examples by successively adding vertices to a triangle in T. For example, consider the triangle 123 in T. If we add vertex 123 in the centre of this triangle, and join it to vertices 1, 2 and 3 by edges, then we obtain a new triangulation T' of the plane such that $B_4(T) \cong B_4(T')$. As we can repeat this procedure indefinitely, this approach yields infinitely many examples of graphs which have two components with even parity.

Indeed, with a bit more effort, we can obtain a stronger result in this direction.

Theorem 3.2.2. *Let $k \in \mathbb{N}$. Then there are infinitely many triangulations T of the plane such that $B_4(T)$ has more than k components with even colourings and more than k components with odd colourings.*

This theorem follows directly from Theorem 3.2.1 and Lemma 3.2.3. However, stating Lemma 3.2.3 requires some additional notation.

Let T and T' be triangulations of surfaces of Euler genus g_1 and g_2, respectively. Let $T_1 = xyz$ ($T_2 = x'y'z'$) be a face in T (T'). By identifying x with x', y with y', and z with

z', we obtain a new triangulation $T \triangle T'$ with $|V(T)| + |V(T')| - 3$ vertices and with Euler genus $g_1 + g_2$. Any 4-colouring of $T \triangle T'$ induces a 4-colouring f on T and a 4-colouring f' on T'. We then use $f \triangle f'$ to denote the colouring of $T \triangle T'$. This yields a bijection between colourings of $T \triangle T'$ and pairs (f, f') of colourings of T and T'.

Lemma 3.2.3. *(a) If two 4-colourings $f \triangle f'$ and $h \triangle h'$ of $T \triangle T'$ are adjacent in $B_4(T \triangle T')$, then $f \sim h$ in $B_4(T)$ and $f' \sim h'$ in $B_4(T')$. Conversely, if $f \sim h$ in $B_4(T)$ and $f' \sim h'$ in $B_4(T')$, then $f \triangle f' \sim h \triangle f' \sim h \triangle h'$ in $B_4(T \triangle T')$.*
(b) If A is a colour class, then $J_A(f \triangle f') = J_{A \cap V(T)}(f) + J_{A \cap V(T')}(f')$.

Proof. (a) By definition, $f \triangle f' \sim h \triangle h'$ means that $f \triangle f'$ and $h \triangle h'$ have a colour class in common, say A. Thus, $A \cap V(T)$ is a colour class that is common to f and h, so $f \sim h$. Similarly, $f' \sim h'$.

Conversely, if $f \sim h$ and $f' \sim h'$, then f shares some colour class A with h, so $f \triangle f' \sim h \triangle f'$. By a similar argument, $h \triangle f' \sim h \triangle h'$, as required.

(b) Consider the colour class A of $f \triangle f'$ which is disjoint from the new triangle in T we created by identifying vertices in T and T'. We write A_T for $A \cap V(T)$ and $A_{T'}$ for $A \cap V(T')$. Then:

$$J_{A_T}(f) = 2|A_T| - \deg_T(A_T) + n_1 - 3 + g_1,$$

$$J_{A_{T'}}(f') = 2|A_{T'}| - \deg_{T'}(A_{T'}) + n_2 - 3 + g_2.$$

Now, as A_T and $A_{T'}$ are disjoint, $\deg(A) = \deg_T(A_T) + \deg_{T'}(A_{T'})$. Consequently, we can compute $J_A(f \triangle f')$:

$$\begin{aligned} J_A(f \triangle f') &= 2|A| - \deg(A) + (n_1 + n_2 - 3) - 3 + (g_1 + g_2) \\ &= 2(|A_T| + |A_{T'}|) - \deg_T(A_T) - \deg_{T'}(A_{T'}) + (n_1 - 3 + g_1) + (n_2 - 3 + g_2) \\ &= J_{A_T}(f) + J_{A_{T'}}(f'). \end{aligned}$$

When the colour class A contains one of the identified vertices, the calculation is the same, except that $|A| = |A_T| + |A_{T'}| - 1$ and $\deg(A) = \deg_T(A_T) + \deg_{T'}(A_{T'}) - 2$, which has the same result. $\square$

3.3 4-Connected Triangulations of the Plane

In the previous section, we found small triangulations of the plane whose 4-colouring complexes have three components, two of which have even parity. We then established in Lemma 3.2.3 that, using repeated identification over a triangle, we can construct triangulations of the plane which have an arbitrarily large number of even components and an arbitrarily large number of odd components. However, identification over a triangle produces triangulations with 3-separators. Thus, one might reasonably ask whether or not

4-connected triangulations of the plane may be found whose 4-colouring complexes have an arbitrarily large number of components of the same parity. In this section, we answer this question by constructing 4-connected triangulations of the plane with an arbitrarily large number of even components.

Our construction begins by considering the two triangulations Q_1 and Q'_1 in Figure 3.5. These are the smallest examples in two infinite families of 4-connected triangulations of the plane. We define $Q_0, Q_1, Q_2, \ldots$ as follows. To obtain Q_k we take $k+2$ nested 4-cycles $D_i = a_i b_i c_i d_i$ $(i = 0, 1, \ldots, k+1)$. We connect D_i with D_{i+1} $(i = 0, 1, \ldots, k)$ as shown in Figure 3.7. To obtain Q_k, we add the edges a_0c_0 and $a_{k+1}c_{k+1}$. We also define Q'_k, which is obtained in the same way, except that we add the edge $b_{k+1}d_{k+1}$ instead of $a_{k+1}c_{k+1}$.

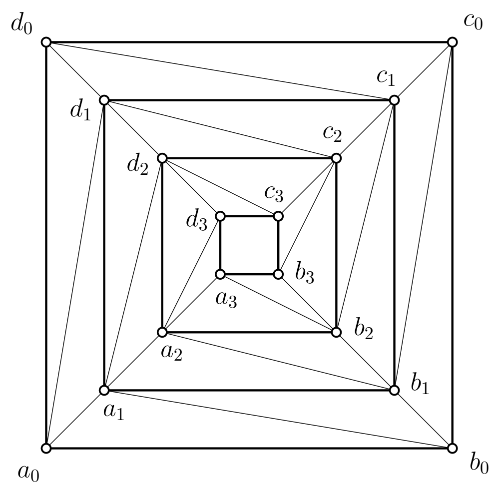


Figure 3.7: Nested 4-cycles D_0, D_1, D_2, D_3. To obtain Q_2, we add the edges a_0c_0 and a_3c_3.

Theorem 3.3.1. *The* 4*-colouring complex of the* 4*-connected triangulation* Q_k *(and that of* Q'_k*) has* $2^k + 1$ *connected components. One of them corresponds to* 2^{k+1} *odd colourings, and each of the other* 2^k *components corresponds to a single even* 4*-colouring.*

Proof. We say that a 4-colouring of Q_k is of *type I* if it uses all four colours on D_0 and of *type II* if it uses only 3 colours on D_0. As $a_0b_0c_0$ is a triangle in Q_k, any type II colouring f must satisfy $f(b_0) = f(d_0)$ (and $f(a_0) \neq f(c_0)$). As we saw in the proof of Theorem 3.2.1, if D_i is 4-coloured with four distinct colours, then this 4-colouring extends to D_{i+1} in two different ways, and both extensions use four distinct colours on D_{i+1}. Similarly, we observe that if D_i is 3-coloured, then this 3-colouring extends in two different ways to a 3-colouring of D_{i+1}. We can also see this by directly examining the extensions of colourings from D_i to D_{i+1}, as shown in Figure 3.8. Of course, as we note in Figure 3.8, when $i = k$, only one 3-colouring of D_i will extend to a 3-colouring of D_{i+1}. This implies that Q_k has precisely 2^{k+1} 4-colourings of type I and has precisely 2^k 4-colourings of type II.

Now, consider two type I 4-colourings f and g of Q_k which have a common colour class. In this case, the 4-colouring of D_{i+1} is uniquely determined by the 4-colouring of D_i for

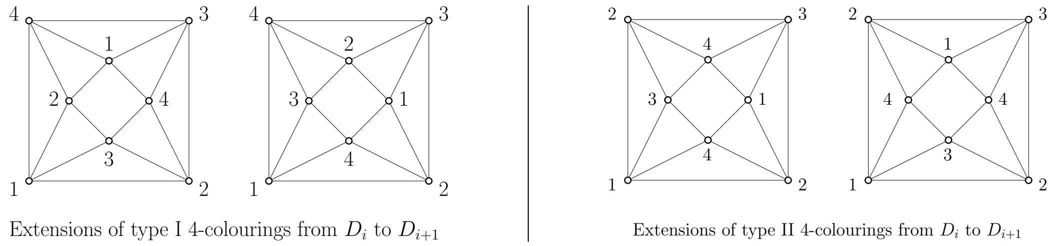


Figure 3.8: The extensions of partial colourings from D_i to D_{i+1}. Observe that when two opposite vertices in D_{i+1} are joined by an edge, only one of the type II extensions exists.

each i, so f and g must be identical colourings. In 4-colourings of type I, all colour classes have a vertex in each D_i. A 4-colouring of Q_k of type II has two colour classes that meet every D_i, while the other two colour classes have two vertices in every second 4-cycle D_i. Hence, if a colour class of a type II 4-colouring coincides with a colour class of a type I 4-colouring, then the colour class corresponds to exactly one 4-colouring of type I and one 4-colouring of type II. In this way, each 4-colouring of type II has two colour classes that coincide with two colour classes of a 4-colouring of type I, and this is a 1-1 correspondence. The 4-colourings f of type I under this correspondence are precisely those for which either $\{f(b_1), f(d_1)\} = \{f(b_{k+1}), f(d_{k+1})\}$ (if k is even) or $\{f(b_0), f(d_0)\} = \{f(b_{k+1}), f(d_{k+1})\}$ (if k is odd). All of the colourings fulfilling this condition are of odd parity and of even degree $\deg(f)$ (by Theorem 3.1.1, since the only vertices of odd degree are a_0, c_0, a_{k+1} and c_{k+1}).

There are 2^k 4-colourings of type I which fulfill the condition we described in the previous paragraph. Consequently, 2^k 4-colourings of type I remain which do not fulfill this condition. We will establish that each such colouring forms a separate component in $B_4(Q_k)$. In particular, deleting any colour class C which contains exactly one vertex from each of the nested 4-cycles D_i results in a subgraph R_k of Q_k which can be conveniently described. Upon deletion of a colour class from a type I 4-colouring, the 4-cycles D_i and D_{i+1} must be coloured in one of the two ways shown in Figure 3.9 (up to relabelling).

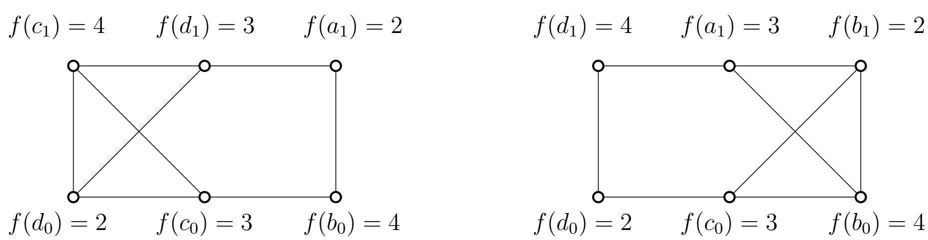


Figure 3.9: D_0 and D_1 after deleting a colour class from a type I colouring (up to relabelling)

So, each successive subgraph of R_k induced on D_i and D_{i+1} can be thought of as one additional rung of a three vertex ladder, as well as a cross on either the left part of the ladder or the right part of the ladder. Thus, in order to construct R_k, we begin with a three vertex wide ladder, as drawn in Figure 3.10. Then we add either the pair of edges $u_i v_{i+1}$ and $u_{i+1} v_i$ or $v_{i+1} w_i$ and $v_i w_{i+1}$ for each $i \in [k]$. Finally, we may choose to add $u_0 w_0$ or $u_{k+1} w_{k+1}$ to complete R_k. Now, if we added at least one of $u_0 w_0$ and $u_{k+1} w_{k+1}$ in completing R_k, then R_k is uniquely 3-colourable. Otherwise, R_k has two 3-colourings. Observe that there are 2^{k+1} ways of choosing an R_k with two 3-colourings (at each level, we may choose whether to have a left or right cross) representing the 2^{k+1} colour classes of type I which are also of type II. Thus, excluding the colour classes in these 2^k colourings of type I, which are Kempe-equivalent to colourings of type II, there remain 2^{k+2} colour classes of type I which have uniquely 3-colourable complements in Q_k. Having uniquely 3-colourable complements in Q_k implies that these colour classes only appear in one 4-colouring of Q_k. Consequently, excluding the 2^{k+2} colour classes in 4-colourings which are Kempe equivalent to colourings of type II, we obtain 2^{k+2} colour classes in the 2^k remaining 4-colourings of type I, each of which then forms a separate component in the colouring complex $B_4(Q_k)$. Observe that all such colourings have even parity (odd homology degree).

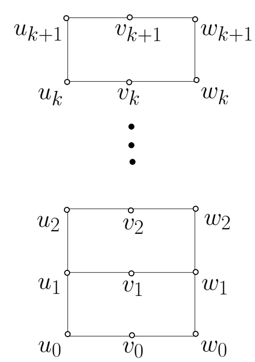


Figure 3.10: A three vertex wide ladder. To obtain R_k, we add either $u_i v_{i+1}$ and $u_{i+1} v_i$ or $v_{i+1} w_i$ and $v_i w_{i+1}$ for each i. Then we may add $u_0 w_0$ or $u_{k+1} w_{k+1}$.

Finally, we prove that all colourings of type II (together with their mates of type I) form a single component in $B_4(Q_k)$. To see this, observe that two colourings of type II with a common colour class that intersects each 4-cycle D_i must be identical. However, if they have a common colour class containing b_0 and d_0, then there are two ways to colour each D_i with i odd, so there are 2^t colourings of type II with the same colour class, where $t = \lfloor (k+1)/2 \rfloor$. Meanwhile, if they share a colour class containing a_1 and c_1, or b_1 and d_1, then the same holds with $t = \lfloor k/2 \rfloor$.

In conclusion, $B_4(Q_k)$ has one large component containing all 2^{k+1} odd 4-colourings, as well as 2^k other components, each of which corresponds to a single even 4-colouring that is of type I.

A similar argument applies for Q'_k. $\square$

3.4 Concluding Remarks

At present, we have not found a 5-connected triangulation of the plane whose 4-colouring complex has a large number of components. However, we have found some 5-connected examples whose 4-colouring complexes have more than one component of the same parity. The three smallest triangulations of this kind are illustrated in Figure 3.11. These are triangulations of the plane on 20 vertices, and their minimality follows from a lengthy computation we performed on 5-connected triangulations with at most 23 vertices.

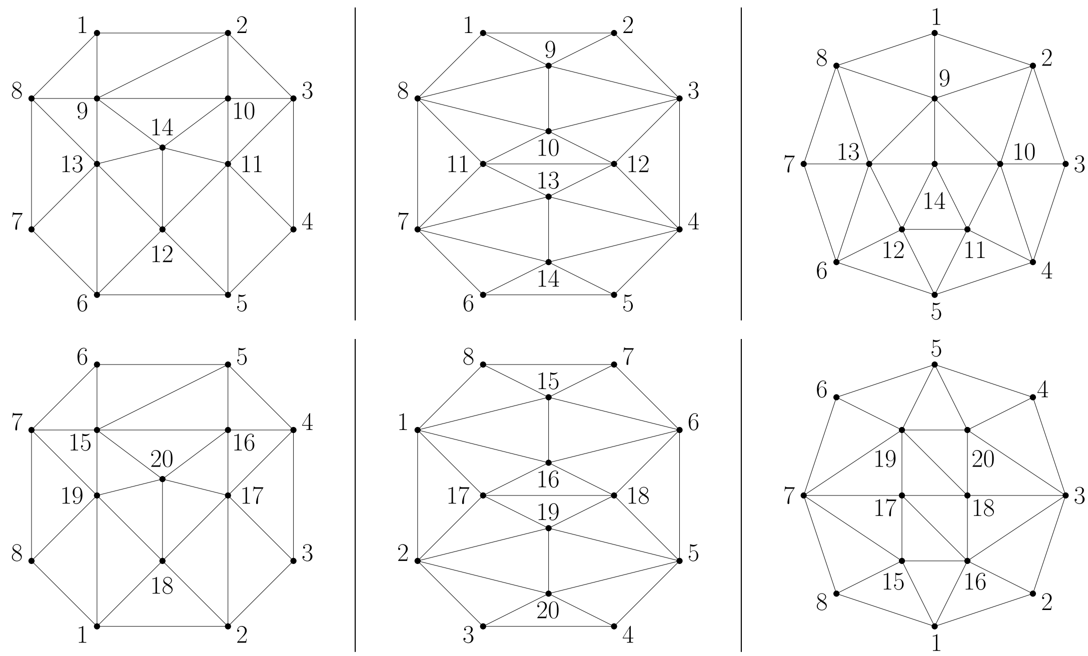


Figure 3.11: Three 5-connected triangulations on 20 vertices. Each one is represented by two triangulated octagons. In order to obtain the full graph from each pair of subgraphs, identify the vertices on the outer cycle according to their labelling.

The 4-colouring complex of our first example has three components, two odd and one even. The second example's 4-colouring complex has three even components and one odd component. Our final example's 4-colouring complex has two even components and one odd component. There are also 6 non-isomorphic 5-connected examples on 21 vertices, 33 on 22 vertices and 66 on 23 vertices. Unfortunately, we do not have an infinite family. Of course, by Euler's Formula and the handshake lemma, we cannot have a 6-connected triangulation of the plane, preventing us from continuing further in searching for examples with higher connectivity.

However, there is another interesting question about parity we can ask. If the 4-colouring complex of a triangulation of the plane has at least two components, must it have one component of even parity and one component of odd parity? Based on a large number of computations, we formulate the following conjecture.

Conjecture 3.4.1. *Suppose that T is a triangulation of the plane, and that its 4-colouring complex $B_4(T)$ has at least two components. Then $B_4(T)$ has a component of even parity and a component of odd parity.*

These remaining questions inspired by Tutte, together with our interest in the structure of the colourings of a graph, serve as motivation for our further examination of the k-colouring complex. We pursue this topic at more length in the next three chapters.

Chapter 4

Reflexive Graphs

While Tutte motivated and defined the 4-colouring complex [46], it was Fisk who first examined the structure of the more general k-colouring complex in [20], and then, later, greatly expanded on this work in [21]. In Chapter 2, we reiterated Fisk's argument that the k-colouring complex of a graph X must be k-colourable (whenever X contains a $(k-1)$-clique) in the proof of Lemma 2.1.2. A similar argument in the proof of Lemma 2.2.3 showed that we could recursively define $B_k^i(X)$ as the k-colourable graph $B_k(B_k^{i-1}(X))$ (where $i \geq 2$) and define a graph homomorphism $\phi_X : X \to B_k^2(X)$ (whenever each edge of X is contained in a $(k-1)$-clique and X has no isolated vertices). The graph homomorphism ϕ_X maps a vertex $v \in V(X)$ to $\{C \in V(B_k(X)) \mid v \in C\}$. Recall that we call graphs for which this homomorphism is injective colourful graphs (equivalently, a graph X is colourful if, for any two vertices $x, y \in V(X)$, there exists a k-colouring of X which colours x and y differently), and we say that a graph X is reflexive (for k-colourings) whenever ϕ_X is an isomorphism.

In this chapter, we will focus our attention on identifying a number of non-trivial families of 4-chromatic, reflexive, near-triangulations of the plane, and 3-edge-colourable, edge-reflexive, cubic graphs. Our proofs will rely heavily on the methods we discussed in Chapter 2, and we will extend these methods even further in Chapter 5.

In Section 4.1, we reprove Fisk's result that the cubic trees are edge-reflexive. This proof also serves as an introduction to using clustering in order to prove reflexivity results (see Section 2.4 for the definition of clustering), a method which we will employ regularly in Chapter 5. Then, in Section 4.2, we use our new insight into cubic trees (and cubic cycles) to quickly determine the edge-reflexive ladders, circular ladders and Möbius ladders, identifying two new classes of reflexive graphs. These classes of graphs are formally defined in Section 4.2. It is particularly notable that the even circular ladders are edge-reflexive, as this is a class of cubic graphs without any half-edges. Finally, in Section 4.3, we consider the edge-reflexive graphs we found in Section 4.2 as the inner duals of near-triangulations of the plane. As a result, we discover two new classes of reflexive, 4-chromatic, planar near-triangulations. We conclude this section with a brief discussion of the relation between 3-edge-colouring cubic graphs and 4-colouring near-triangulations of the plane, which gives some insight into why results regarding the 3-colouring complexes of the line graphs of cubic graphs sometimes generalize to the 4-colouring complexes of planar, near-triangulations.

4.1 Cubic Trees

In this section, we will always assume that a graph G is a cubic graph which may contain half-edges.

Our main focus will be on reiterating Fisk's argument (presented in [21]) that the cubic trees are edge-reflexive with respect to 3-edge-colourings. We will be somewhat pedantic in giving this argument, as we are employing clustering in a proof for the first time here, and we would like the reader to understand this approach, in preparation for more difficult clustering arguments in Chapter 5.

Theorem 4.1.1 (Fisk [21])**.** *Suppose that G is a cubic tree. Then G is edge-reflexive with respect to 3-edge-colourings.*

Proof. Let $X := L(G)$. We wish to show that a graph G of order n is edge-reflexive (where $n \in \mathbb{N}$). By Lemma 2.4.2, in order to accomplish this goal, it suffices to establish that G is edge-colourful, and that $B_3(X)$ has precisely n 3-colourings.

We will prove both that G is edge-colourful and that $B_3(X)$ has exactly n 3-colourings by separate induction arguments. In each case, the underlying graphs on which we perform induction are the same.

A cubic tree on one vertex is a single vertex with three half-edges. Its one 3-colouring colours all three edges of G with different colours, so G is edge-colourful for $n = 1$. The line graph of the cubic tree on one vertex is the complete graph on three vertices. As $B_3(K_3)$ is also the complete graph on three vertices, $B_3(K_3)$ has only one 3-colouring. This establishes that $B_3(X)$ has precisely n 3-colourings for $n = 1$.

For $n \geq 2$, the graph obtained from G by removing the half-edges from $E(G)$ is a tree on at least two vertices. Thus, this graph has a leaf vertex v with a unique neighbour u. Let e be the edge of G which joins u and v, and let f_1 and f_2 be the other two edges of G which are incident with v. The result of deleting v from G and then adding a half-edge e incident with the vertex u is a cubic tree on $n-1$ vertices. We will call this graph H, and we also denote the line graph of H by Y. By our induction hypotheses, H is edge-colourful, and $B_3(Y)$ has precisely $n-1$ 3-colourings. The following claim completes the first of our two induction arguments.

<u>Claim 1:</u> If H is edge-colourful, then G is edge-colourful. Thus, by induction, G is edge-colourful for all $n \geq 1$.

<u>Proof:</u> As H is edge-colourful, it immediately follows that for any pair of distinct edges $a, b \in V(G) \setminus \{e, f_1, f_2\}$, there exists a 3-edge-colouring of G in which a and b are coloured differently. Similarly, as H is edge-colourful, for any $a \in V(G) \setminus \{e, f_1, f_2\}$, there exists a

3-edge-colouring of G which colours a and e differently. Finally, by performing a Kempe change on the vertices f_1 and f_2 (if needed) we can arrange that any $a, b \in V(G) \setminus \{e, f_1, f_2\}$ is coloured differently from f_1 (or, respectively, f_2). This establishes our claim.

It remains to prove that $B_3(X)$ has precisely n 3-colourings whenever $B_3(Y)$ has precisely $n-1$ 3-colourings. We employ clustering in order to establish this claim.

Notice that we can partition the vertices of $B_3(X)$ into the five clusters shown in Figure 4.1. In this figure, we denote the neighbours of e which were present in $L(H)$ by e_1 and e_2. Recall from Section 2.4 that we denote by $\mathcal{C}_{x_1x_2\ldots x_t}$ the set of colour classes of X which contain the vertices $x_1, x_2 \ldots x_t \in \{e, f_1, f_2, e_1, e_2\}$. In other words,

$$\mathcal{C}_{x_1x_2\ldots x_t} = \bigcap_{i=1}^{t} \phi_X(x_i).$$

Observe that, for every 3-edge-colouring of G, we can perform a Kempe change on $\{f_1, f_2\}$ that swaps the colours on f_1 and f_2, while fixing the colour of all other edges. Immediately, it follows that every colour class of X (vertex of $B_3(X)$) in $\mathcal{C}_e$ will have an adjacent colour class of X (vertex of $B_3(X)$) in each of the other four clusters. Thus, as in Section 2.4, we may observe that the 3-colourings of X fall into two types, as indicated in Figure 4.1.

The subgraph of $B_3(X)$ consisting of all the triangles of $B_3(X)$ (3-colourings of X) of type $i \in \{1, 2\}$ will be denoted by $\mathcal{T}_i$. We will also denote the subgraph of $\mathcal{T}_i$ induced by the union of the two adjacent clusters $C_{x_1\ldots x_t}$ and $C_{w_1\ldots w_s}$ by $C_{x_1\ldots x_t}C_{w_1\ldots w_s}$.

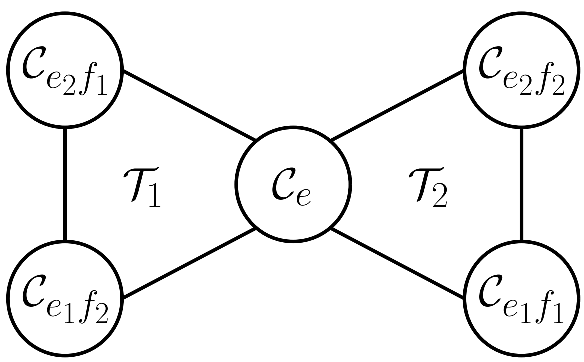


Figure 4.1: Partitioning $B_3(X)$ for a cubic tree

<u>Claim 2:</u> $\mathcal{T}_1$ and $\mathcal{T}_2$ are each isomorphic to $B_3(Y)$. Under this isomorphism, $\mathcal{C}_e$ is mapped onto $\phi_Y(e)$.

<u>Proof:</u> Consider the map $\psi : \mathcal{T}_1 \to B_3(Y)$ which takes a colour class C of X to its intersection with $V(Y)$. As every colouring c of Y extends uniquely to a colouring c' of X with $c'(e_1) = c'(f_2)$, ψ is a bijection. Moreover, this unique extension also establishes that ψ is a graph homomorphism, since it follows that ABC is a triangle in $\mathcal{T}_1$ if and only if $\psi(A)\psi(B)\psi(C)$ is

a 3-colouring of Y. The argument showing that the map $\psi : \mathcal{T}_2 \to B_3(Y)$ is an isomorphism is similar. Thus, Claim 2 is proven.

As the vertex e of Y has degree 2, and Y is reflexive, it follows from Lemma 2.4.5 that $B_3(Y) - \phi_Y(e)$ has precisely one bipartite component. Thus, by Claim 2, $\mathcal{C}_{e_1f_2}\mathcal{C}_{e_2f_1}$ is a bipartite and connected subgraph of $\mathcal{T}_1$, while $\mathcal{C}_{e_1f_1}\mathcal{C}_{e_2f_2}$ is a bipartite and connected subgraph of $\mathcal{T}_2$.

Now, let χ be a 3-colouring of $B_3(X)$. We will say that a colouring χ is *constant* on a set S of vertices if S is contained in a colour class of the colouring. We consider two cases.

Case 1: χ is constant on $\mathcal{C}_e \cap \mathcal{T}_1$. We claim that χ is constant on the whole cluster $\mathcal{C}_e$. So, consider a triangle ABC in $\mathcal{T}_2$. This triangle represents a 3-colouring c of X in which $c(e_1) = c(f_1)$ and $c(e_2) = c(f_2)$. By performing a Kempe change on $\{f_1, f_2\}$, the 3-colouring ABC in $\mathcal{T}_2$ is transformed into a 3-colouring $A'B'C'$ in $\mathcal{T}_1$ in which the colour class of $A'B'C'$ containing e is unchanged. Thus, as each colour class $A \in \mathcal{C}_e \cap \mathcal{T}_2$ is also a colour class $A' \in \mathcal{C}_e \cap \mathcal{T}_1$, χ is constant on the whole cluster $\mathcal{C}_e$.

As χ is constant on $\mathcal{C}_e$, $\mathcal{C}_{e_1f_2}\mathcal{C}_{e_2f_1}$ is a bipartite and connected subgraph of $\mathcal{T}_1$ and $\mathcal{C}_{e_1f_1}\mathcal{C}_{e_2f_2}$ is a bipartite and connected subgraph of $\mathcal{T}_2$, it immediately follows that χ is constant on each of the five clusters of $B_3(X)$. There are two such colourings of $B_3(X)$, compared to one such colouring of $B_3(Y)$.

Case 2: χ is not constant on $\mathcal{C}_e \cap \mathcal{T}_1$. In this case, by Lemma 2.4.4, χ is completely determined on $\mathcal{T}_1$ by its restriction to $\mathcal{C}_e \cap \mathcal{T}_1$. As performing a Kempe change on $\{f_1, f_2\}$ transforms any 3-colouring of X in $\mathcal{T}_2$ into a 3-colouring in $\mathcal{T}_1$, χ is also determined and non-constant on $\mathcal{C}_e \cap \mathcal{T}_2$. Thus, by Lemma 2.4.4, χ is completely determined on $\mathcal{T}_2$.

Consequently, the colouring χ of $B_3(X)$ is completely determined by its restriction to $\mathcal{T}_1$, which is isomorphic to $B_3(Y)$ by Claim 2. Each such colouring of $B_3(X)$ is uniquely determined by a colouring of $B_3(Y)$ which is not constant on some cluster of $B_3(X)$. Since one colouring of $B_3(Y)$ corresponds to Case 1, we get one fewer colourings than in Case 1, so we obtain one less colouring than the number of colourings of $B_3(Y)$ in this case.

Thus, in total, $B_3(X)$ has one more 3-colouring than $B_3(Y)$ had. As $B_3(Y)$ had $(n-1)$ 3-colourings by our induction hypothesis, it follows that $B_3(X)$ has n 3-colourings, as required. Hence, by Lemma 2.4.2, G is edge-reflexive. $\square$

Using a similar approach, Fisk also proved that the cubic cycles are edge-reflexive in [20]. Rather than reiterating his proof here, we will prove a stronger result in Chapter 5 (Lemma 5.3.4) from which this result follows as a Corollary.

Corollary 4.1.2 (Fisk [20]). *Suppose that C_n is a cubic cycle on n vertices, where $n \geq 4$. Then C_n is edge-reflexive with respect to 3-edge-colourings.*

4.2 More Classes of Edge-Reflexive Graphs

Of course, the edge-reflexive graphs would not be a very interesting object of study if the cubic trees and cubic cycles were the only classes of edge-reflexive graphs. Fortunately, it is easy to construct additional, non-trivial classes. The ladders and even circular ladders are two such classes of edge-reflexive cubic graphs.

The *n-ladder* L_n is the cubic graph of order $2n$ which is obtained by adding half-edges to the Cartesian product of a path on n vertices (not a cubic path) and K_2 until each vertex of L_n has degree three. The *circular n-ladder* CL_n is the cubic graph of order $2n$ which is obtained by taking the Cartesian product of a 2-regular n-cycle and K_2.

Theorem 4.2.1. *For every $n \geq 2$, the ladder L_n is edge-reflexive with respect to 3-edge-colourings.*

Proof. Let A_0 and B_0 be the colour classes in a 2-edge-colouring of the first 2-regular n-path P_0 in L_n, and let A_1 and B_1 be the corresponding colour classes in a 2-edge-colouring of the second 2-regular n-path P_1 in L_n. Also, let M be the perfect matching in L_n which consists of all the edges joining these two paths.

There are two 3-edge-colourings of L_n for which M is a colour class: $\{A_0 \cup A_1, B_0 \cup B_1, M\}$ and $\{A_0 \cup B_1, B_0 \cup A_1, M\}$. We will call the latter colouring the *mixed colouring*. Furthermore, any other 3-edge-colouring $\{A, B, C\}$ of the cubic n-path $P_0 \cup M$ (where the edges in M are treated as half-edges) for which M is not a colour class extends uniquely to a 3-edge-colouring of L_n by adding to each colour class all the edges in the path P_1 which are copies of the edges of P_0 in that colour class. This reasoning establishes that $B_3(L(L_n))$ is isomorphic to $B_3(L(P))$ with one added triangle corresponding to the mixed colouring (where P is the cubic path on n vertices). The additional triangle shares the colour class M with the rest of the colouring complex.

We can easily prove that L_n is edge-colourful. A given edge $e_0 \in E(P_0)$ is coloured differently from any edge $e_1 \in E(P_1)$ in one of the colourings for which M is a colour class. Of course, in such a colouring, e_0 is also not coloured the same colour as any edge in M. We can also choose to colour e_0 differently from every other edge in P_0, and extend this 3-edge-colouring uniquely to L_n. By symmetry, we can also arrange for a given edge $e_1 \in E(P_1)$ to be coloured differently from any other given edge in L_n. Finally, if $a, b \in M$, then we can construct a 3-edge-colouring in which a and b are coloured differently by taking the edges incident with a in P_0 to be coloured with colours 1 and 2 in a 3-edge-colouring of P_0 in which only one pair of incident edges is coloured with colours 1 and 2. This 3-edge-colouring extends uniquely to L_n, and colours a differently from each other edge in M. Thus, L_n is edge-colourful.

Consequently, it suffices to show (by Lemma 2.4.2) that $B_3(L(L_n))$ has precisely $2n$ 3-colourings. Since the cubic n-path P is edge-reflexive (by Theorem 4.1.1) $B_3(L(P))$ has precisely n 3-colourings. Each such 3-colouring extends in two ways to the whole of $B_3(L(L_n))$, since we have two ways to colour the vertices of $B_3(L(L_n))$ (colour classes of $L(L_n)$) $A_0 \cup B_1$ and $B_0 \cup A_1$ which correspond to the mixed colouring. Thus, $B_3(L(L_n))$ has precisely $2n$ 3-colourings, as required. $\square$

The circular ladders are a more interesting case, as they have no half-edges. Unfortunately, the odd circular ladders are not edge-colourful. In any 3-edge-colouring of CL_{2n+1} ($n \geq 1$) each pair of the corresponding cycle edges in the Cartesian product is coloured identically. Thus, $B_3(CL_{2n+1})$ is isomorphic to $B_3(L(C_{2n+1})$, where C_{2n+1} is the cubic $(2n+1)$-cycle. However, even circular ladders are different.

Theorem 4.2.2. *For every even $n \geq 4$, the even circular ladder CL_n is edge-reflexive with respect to 3-edge-colourings.*

Proof. Let A_0 and B_0 be the perfect matchings of the first 2-regular n-cycle C_0' in CL_n, and let A_1 and B_1 be the corresponding perfect matchings of the second 2-regular n-cycle C_1' in CL_n. Also, let M be the perfect matching in CL_n which consists of all the edges joining these two cycles.

There are two 3-edge-colourings of CL_n for which M is a colour class: $\{A_0 \cup A_1, B_0 \cup B_1, M\}$ and $\{A_0 \cup B_1, B_0 \cup A_1, M\}$. We will call the latter colouring the *mixed colouring*. Furthermore, any other 3-edge-colouring $\{A, B, C\}$ of the cubic n-cycle $C_0' \cup M$ (where the edges in M are treated as half-edges) for which M is not a colour class extends uniquely to a 3-edge-colouring of CL_n by adding to each colour class all the edges in the cycle C_1' which are copies of the edges of C_0' in that colour class. This reasoning establishes that $B_3(L(CL_n))$ is isomorphic to $B_3(L(C_n))$ with one added triangle corresponding to the mixed colouring (where C_n is the cubic cycle on n vertices). The additional triangle shares the colour class M with the rest of the colouring complex.

We can easily prove that CL_n is edge-colourful. A given edge $e_0 \in E(C_0')$ is coloured differently from any edge $e_1 \in E(C_1')$ in one of the colourings for which M is a colour class. Of course, in such a colouring, e_0 is also not coloured the same colour as any edge in M. We can also choose to colour e_0 differently from every other edge in C_0', and extend this 3-edge-colouring uniquely to CL_n. By symmetry, we can also arrange for a given edge $e_1 \in E(C_1')$ to be coloured differently from any other given edge in CL_n. Finally, if $a, b \in M$, then we can construct a 3-edge-colouring in which a and b are coloured differently by taking the edges incident with a in C_0' to be coloured with colours 1 and 2 in a 3-edge-colouring of C_0' in which only one pair of incident edges is coloured with colours 1 and 2. This 3-edge-colouring extends uniquely to CL_n, and colours a differently from each other edge in M. Thus, CL_n is edge-colourful.

Consequently, it suffices to show (by Lemma 2.4.2) that $B_3(L(CL_n))$ has precisely $2n$ 3-colourings. Since the cubic n-cycle C_n is edge-reflexive (by Corollary 4.1.2) $B_3(L(C_n))$ has precisely n 3-colourings. Each such 3-colouring extends in two ways to the whole of $B_3(L(CL_n))$, since we have two ways to colour the vertices of $B_3(L(CL_n))$ (colour classes of $L(CL_n)$) $A_0 \cup B_1$ and $B_0 \cup A_1$ which correspond to the mixed colouring. Thus, $B_3(L(CL_n))$ has precisely $2n$ 3-colourings, as required. □

The Möbius ladders are a family of cubic graphs closely related to the circular ladders, so we might hope that some class of Möbius ladders would be edge-reflexive. We say that a graph M_n is a *Möbius ladder* if it is constructed from the circular ladder CL_n of order $2n$ (where the first 2-regular n-cycle C'_0 is on vertices $x_1, x_2, \ldots, x_n$ and the second 2-regular n-cycle C'_1 is on vertices $y_1, y_2, \ldots, y_n$) by deleting the edges $x_n x_1$ and $y_n y_1$ from CL_n and then replacing them with the edges $x_1 y_n$ and $x_n y_1$. Unfortunately, there are no edge-reflexive Möbius ladders.

Theorem 4.2.3. *Let $n \geq 4$. Then the Möbius ladder M_n is not edge-reflexive with respect to 3-edge-colourings.*

Proof. We divide this argument into two cases.

Case 1: n is even. In this case, all the vertices of M_n lie on a $4k$-cycle C'_{4k}, where $n = 2k$. Now, suppose that we are given a 3-edge-colouring c of M_n. Without loss of generality, we may say that $c(x_1x_2) = 1$ and $c(x_2x_3) = 2$. Now, observe that the edge y_1y_2 is incident with the edge x_2y_2, which satisfies $c(x_2y_2) = 3$. Thus, c colours y_1y_2 with either 1 or 2.

If $c(y_1y_2) = 2$, then $c(x_ix_{i+1}) \neq c(y_iy_{i+1})$ for each $i \in \{1, 2, \ldots, 2k\}$, so each edge x_iy_i must be coloured with colour 3. However, in that case, the colours 1 and 2 must alternate on C'_{4k}, from which it follows that $c(y_1y_2) = 1$, a contradiction. Thus, $c(y_1y_2) = 1$, so M_n is not edge-colourful.

Case 2: n is odd. In this case, all the vertices of M_n lie on a $(4k+2)$-cycle C'_{4k+2}, where $n = 2k + 1$. We can prove that M_n is edge-colourful. However, $B_3(L(M_n))$ has too many 3-colourings for M_n to be edge-reflexive. In order to illustrate these facts, we will first list the 3-edge-colourings of M_n.

Let M denote the perfect matching consisting of all edges of the form x_iy_i, A_1 denote $\{x_ix_{i+1} : i \text{ is odd}\} \cup \{x_{2k+1}y_1\}$, A_2 denote $\{x_ix_{i+1} : i \text{ is even}\}$, B_1 denote $\{y_iy_{i+1} : i \text{ is odd}\} \cup \{y_{2k+1}x_1\}$, and B_2 denote $\{y_iy_{i+1} : i \text{ is even}\}$. There is one 2-colouring of $A_1 \cup A_2$, and this colouring uniquely extends to the 3-edge-colouring of M_n given by $\{A_1 \cup B_2, A_2 \cup B_1, M\}$. Deleting any one of these three colour classes from M_n results in a Hamiltonian cycle, so this colouring is represented by an isolated triangle in $B_3(L(M_n))$. Otherwise, $A_1 \cup A_2$ is not 2-coloured. In this case, if x_1x_2 and $x_{2k+1}y_1$ are coloured differently, then the 3-colouring on $A_1 \cup A_2$ uniquely extends to a 3-edge-colouring of M_n in which edges of the form y_iy_{i+1} are coloured the same colour as their counterparts of the form x_ix_{i+1} (and $y_{2n+1}x_1$ is coloured the same colour as $x_{2k+1}y_1$). There are no valid 3-edge-colourings of M_n for which x_1x_2 and $x_{2k+1}y_1$ are coloured the same colour (unless $A_1 \cup A_2$ is 2-coloured, a case which we considered previously). Consequently, the 3-edge-colourings of $L(M_n)$ for which $A_1 \cup A_2$ are not 2-coloured are represented in $B_3(L(M_n))$ by a copy of $B_3(L(C_n))$ (where C_n is the odd, cubic cycle, corresponding in M_n to the edges of $A_1 \cup A_2 \cup M$, where we treat the edges of M as half-edges and require that x_1x_2 and $x_{2k+1}y_1$ must be coloured differently).

Now, the edge x_1x_2 is coloured differently from the edge y_1y_2 in the 3-edge-colouring of M_n given by $\{A_1 \cup B_2, A_2 \cup B_1, M\}$. Moreover, as C_n is edge-colourful by Corollary 4.1.2, there is a 3-edge-colouring of M_n in which $A_1 \cup A_2$ is not 2-coloured which colours x_1x_2 differently from any other given edge of M_n, except for the edge y_1y_2. Thus, by symmetry, for any pair of edges $e_1 \in A_1 \cup A_2 \cup B_1 \cup B_2$ and $e_2 \in E(M_n)$, there exists a 3-edge-colouring of M_n which colours e_1 and e_2 different colours. By a similar argument (applying Corollary 4.1.2 again) any pair of distinct edges $e_1, e_2 \in M$ are coloured differently in some 3-edge-colouring of M_n. Consequently, M_n is edge-colourful.

Now, by Corollary 4.1.2, $B_3(L(M_n))$ has $6n$ 3-colourings (since we can extend each 3-colouring of the copy of $B_3(L(C_n))$ in $B_3(L(M_n))$ to $B_3(L(M_n))$ in six ways). However, if M_n were edge-reflexive, then, according to Lemma 2.4.2, $B_3(L(M_n))$ would need to have exactly $2n$ 3-colourings. Thus, as $B_3(L(M_n))$ has too many 3-colourings, M_n is not edge-reflexive. □

4.3 Dual Graphs

We define the *inner dual* $T^\star$ of a planar, near-triangulation T as follows. Each triangular face s of T is represented by a vertex s in $T^\star$. Two vertices $s, t \in V(T^\star)$ are adjacent in $T^\star$ if their corresponding faces in T share an edge. Finally, if a triangular face s of T shares an edge with the outer face of T, then $s \in V(T^\star)$ is incident with a half-edge.

Additionally, given a graph G and vertex $v \notin V(G)$, we say that the *join* of G with the vertex v is the graph on the vertex set $V(G) \cup \{v\}$ which contains all the edges of G, as well as edges joining v to every other vertex in $V(G)$. Given a graph G, the *suspension* of G is the graph on the vertex set $V(G) \cup \{x, y\}$ (where $x, y \notin V(G)$) which contains all the edges of G, as well as edges joining x to every other vertex in $V(G)$ and edges joining y to every other vertex in $V(G)$. The *wheel graph* W_k is the graph formed from the cubic k-cycle C_k by making all the half-edges of C_k incident with a single new vertex.

Observation 4.3.1. *The inner dual $W_k^\star$ of the wheel graph W_k is isomorphic to the cubic k-cycle C_k.*

Observation 4.3.2. *Let SC_{2k} denote the suspension of the 2-regular $2k$-cycle C'_{2k}. Then the inner dual $SC_{2k}^\star$ of the triangulation of the plane graph SC_{2k} is isomorphic to the even circular ladder CL_{2k}.*

Surprisingly, the graphs $\hat{W}_k$ and $\hat{SC}_{2k}$ are both reflexive 4-colouring complexes.

Theorem 4.3.3. *The graph $\hat{W}_k$ ($k \geq 4$) is a reflexive 4-colouring complex.*

Proof. The graph $\hat{W}_k$ contains k 4-cliques. Thus, by Lemma 2.4.3, it suffices to show that $\hat{W}_k$ is colourful, and that $B_4(\hat{W}_k)$ has precisely k 4-colourings.

We can construct $\hat{W}_k$ as the join of $L(C_k)$ with a single central vertex w (where C_k is the cubic k-cycle). Thus, the 4-colourings of $\hat{W}_k$ are uniquely determined by the 3-colourings of $L(C_k)$. In each such 4-colouring, the central vertex of the wheel (call it w) is the sole vertex in one colour class of $\hat{W}_k$, and the other three colour classes are colour classes in a particular 3-colouring of $L(C_k)$. Thus, $B_4(\hat{W}_k)$ is isomorphic to the join of $B_3(L(C_k))$ with a single vertex $\{w\}$.

As $L(C_k)$ is colourful (by Corollary 4.1.2) it immediately follows that $\hat{W}_k$ is colourful. Moreover, by the same corollary, $B_4(\hat{W}_k)$ has precisely k 4-colourings. Thus, $\hat{W}_k$ is reflexive. □

Theorem 4.3.4. *Let SC_{2k} denote the suspension of the 2-regular $2k$-cycle C'_{2k} ($k \geq 2$). Then $\hat{SC}_{2k}$ is a reflexive 4-colouring complex.*

Proof. The graph $\hat{SC}_{2k}$ contains $4k$ 4-cliques. Thus, by Lemma 2.4.3, it suffices to show that $\hat{SC}_{2k}$ is colourful, and that $B_4(\hat{SC}_{2k})$ has precisely $4k$ 4-colourings.

The graph $\hat{SC}_{2k}$ can be constructed by taking the union of two copies of $\hat{W}_{2k}$, and then identifying the outer $2k$-cycles which bound the sole non-triangular face of each copy of $\hat{W}_{2k}$. Given a 4-colouring f of the first copy of $\hat{W}_{2k}$, the colouring f extends uniquely to all of $\hat{SC}_{2k}$, unless the bounding cycle of $\hat{W}_{2k}$ is 2-coloured. In this case, the colouring extends to all of $\hat{SC}_{2k}$ in two ways. We call the extension where the two central vertices (one in each copy of $\hat{W}_{2k}$) are coloured differently the mixed colouring. Thus, $B_4(\hat{SC}_{2k})$ is the union of an isomorphic copy of $B_4(\hat{W}_k)$ and a copy of K_4 representing the mixed-colouring, which have two vertices (colour classes of $\hat{SC}_{2k}$) in common.

As $\hat{W}_{2k}$ is colourful (by Theorem 4.3.3) it easily follows that $\hat{SC}_{2k}$ is colourful. Moreover, by the same theorem, $B_4(\hat{W}_{2k})$ has $2k$ 4-colourings, and, as each 4-colouring of the copy of $B_4(\hat{W}_{2k})$ in $B_4(\hat{SC}_{2k})$ extends in two ways to all of $B_4(\hat{SC}_{2k})$, $B_4(\hat{SC}_{2k})$ has precisely $4k$ 4-colourings. □

We consider the connection between these reflexive 4-colouring complexes and edge-reflexive inner duals rather suggestive, although we have not been able to generalize the relationship so far. On some level, one might expect there to exist some connection, stemming from the following classical result (originally formulated by Tait in [41]). But there are also differences between the colouring structures (and even the Kempe structures, as we note in Corollary 4.3.6) of the 4-colourings of near-triangulations of the plane and the 3-edge-colourings of cubic graphs, which make such a precise relationship surprising.

Theorem 4.3.5. *For any near-triangulation of the plane T, there is a bijection between the 4-colourings of T and the 3-edge-colourings of its inner dual $T^\star$.*

Proof. Consider a 4-colouring f of T with labels chosen from $\mathbb{Z}_2 \times \mathbb{Z}_2$. Now, we will attempt to construct a 3-edge-colouring g of $E(T^\star)$ with labels $(0,1)$, $(1,0)$ and $(1,1)$ as follows. Each full edge $st \in E(T^\star)$ crosses one edge $uv \in E(T)$. Thus, we may define $g(st) = f(u) + f(v)$. Similarly, a half-edge $e \in E(T^\star)$ crosses some full edge $uv \in E(T)$, so we may define $g(e) = f(u) + f(v)$. As f is a proper colouring, it immediately follows that this map is well-defined. Moreover, if e_1 and e_2 are distinct edges incident with the vertex s of $T^\star$, then $g(e_1) = f(u) + f(v)$ and either $g(e_2) = f(u) + f(w)$ or $g(e_2) = f(v) + f(w)$. In these equations, w is the third vertex in the triangle uvw of T. However, $f(u)+f(v) \neq f(u)+f(w)$ and $f(u) + f(v) \neq f(v) + f(w)$, so $g(e_1) \neq g(e_2)$. Consequently, g is a 3-edge colouring of $E(T^\star)$.

Conversely, given a 3-edge-colouring g of $E(T^\star)$ with labels $(0,1)$, $(1,0)$ and $(1,1)$, pick a vertex $u \in V(T)$, and define the map $f : V(T) \to \mathbb{Z}_2 \times \mathbb{Z}_2$ by the rule that $f(v)$ is the sum of the edges along a walk from u to v. Additionally, the map $g^\star : E(T) \to \mathbb{Z}_2 \times \mathbb{Z}_2 \setminus \{(0,0)\}$ is given by $g^\star(e) = g(e^\star)$, where $e^\star$ is the edge corresponding to e in the dual graph $T^\star$. The following claim establishes that the map f is well-defined.

Claim 1: Let $v_1, e_1, v_2, e_2, ..., v_n$ be a closed walk in T. Then $\sum_{i=1}^{n-1} g^\star(e_i) = (0,0)$.

Proof: Without loss of generality, we may assume that the closed walks under consideration have no repeated vertices, so assume that $v_1,, v_2, ..., v_n$ are distinct vertices. Certainly, our claim holds for walks of length two, so we may also assume that the edges $e_1, e_2, ..., e_{n-1}$ form a cycle of length at least three in T. We will use C to denote this cycle.

Now, consider the set of faces F which are bounded by the cycle C. Each such face is triangular, and $g^\star$ colours each of the face's edges with a different non-zero element of $\mathbb{Z}_2 \times \mathbb{Z}_2$. Thus, the sum of $g^\star$ on the closed walk bounding each such triangular face is zero. Now, sum $g^\star(e)$ over all three of the edges in every face $a \in F$. Notice that this sum equals $\sum_{i=1}^{n-1} g^\star(e_i)$, as each edge which is not contained in C appears twice in the sum. Additionally, observe that this sum is equal to zero, as the sum over each triangular face is zero. Hence, $\sum_{i=1}^{n-1} g^\star(e_i) = (0,0)$, as required.

As f is well-defined, all that remains is to show that f is a proper 4-colouring. So, let us consider adjacent vertices $v, w \in V(T)$. Constructing a closed walk containing u and the edge vw, we see that $f(v) + f(w) + g^\star(vw) = (0,0)$. However, $g^\star(vw) \neq (0,0)$, so $f(v) + f(w) \neq (0,0)$. Thus, $f(v) \neq f(w)$, as required. □

Corollary 4.3.6. *Suppose that f is a 4-colouring of a planar near-triangulation T, and let K be a Kempe chain of f. Then the edges $\{e_i^\star : e_i$ has exactly one end in $K\}$ form a union of Kempe chains in the dual colouring g of the inner dual graph $T^\star$.*

Proof. Firstly, we will compute how the values of the dual 3-edge-colouring g (as defined in the proof of Theorem 4.3.5) are changed by performing the Kempe change K on a 4-colouring f of T which has colours classes A, B, C and D. The value of g on an edge $e \in E(T^\star)$ which crosses the edge $uv \in E(T)$ will be unchanged if uv is an edge between two vertices in the Kempe chain K or if neither u nor v are in the Kempe chain K. By contrast, when $u \in K$ and $v \notin K$, the value of g on the edge st changes. If the Kempe chain K consists of vertices coloured with the colours A and B, while u is initially coloured with A and v is coloured with the colour C, then the value of $g(e)$ changes from $A+C$ to $B+C$. It then follows from the proof of Theorem 4.3.5 that we can transform one 3-edge-colouring of $T^\star$ into another 3-edge-colouring of $T^\star$ by changing the colours of the edges $\{e_i^\star : e_i$ has exactly one end in $K\}$ in the manner described above. It remains to show that this transformation fixes some colour class of $T^\star$. However, this follows immediately from the fact that $A + B = C + D$ in $\mathbb{Z}_2 \times \mathbb{Z}_2$, together with the observation that none of $A + C$, $A + D$, $B + C$ or $B + D$ equals $A + B$. □

At this time, the we do not fully understand the relationship between inner duals and reflexivity. However, we think that this is certainly a topic which merits further investigation.

4.4 Concluding Remarks

In this chapter, we have discussed a number of non-trivial, infinite families of reflexive graphs. Some of these examples, like the line graphs of cubic trees and cubic cycles (see Theorem 4.1.1 and Corollary 4.1.2) were previously known, while others are novel. In particular, we noted in Section 4.3 that the ladders and even circular ladders are edge-reflexive with respect to 3-colourings, while the near-triangulations of the plane $\hat{W}_k$ ($k \geq 4$) and $\hat{SC}_{2k}$ ($k \geq 2$) are reflexive 4-colouring complexes. We also discussed an interesting connection between reflexive 4-colouring complexes and edge-reflexive inner duals of near-triangulations of the plane in this section.

However, while it is of interest to list many non-trivial, infinite families of reflexive graphs, the most important section of this chapter (in terms of our work) is the proof of Theorem 4.1.1. Here, we drew the reader's attention to the notion of clustering, and explained how examining clusters within a colouring complex could be used to prove that graphs are reflexive. In Chapter 5, we will make improvements upon this method of clustering, in order to establish stronger results.

Chapter 5

Edge-Reflexive Outerplanar Graphs

Throughout this chapter, we will assume that cubic graphs satisfy the hypotheses of Theorem 1.3.1 below. We say that a graph G is *outerplanar* if there exists a planar drawing of G in which every vertex of G appears on the unbounded face of the drawing. In this chapter, we will focus our attention on classifying the edge-reflexive, cubic outerplanar graphs. Let us remark that every cubic outerplanar graph contains half-edges. The 3-edge-colourability of cubic outerplanar graphs was established by Fiorini in [17].

Theorem 1.3.1. *Let G be a connected, cubic, outerplanar graph. Then $X := L(G)$ is reflexive if and only if G is triangle-free.*

Proving Theorem 1.3.1 is an involved process. First, we must establish that G is never edge-reflexive, if G contains a triangle. We will also discuss the complexities which arise when G is not connected in Section 5.1. Subsequently, we prove that a 3-edge-colourable, connected cubic graph is edge-reflexive if all the cubic graphs obtained by cutting all the cut-edges of G are edge-reflexive. Note that Theorem 4.1.1 can be immediately derived as a corollary of this result. Finally, we address the 2-edge-connected case using a constructive proof.

In the process of establishing Theorem 1.3.1 we will also exhibit that (under certain conditions) subdividing an edge in a cubic graph preserves edge-reflexivity. This result not only implies Corollary 4.1.2 from the previous chapter, but also allows us to easily prove that most of the cubic theta graphs are edge-reflexive (see Section 5.4 for the definition of a cubic theta graph). However, it is not the case that subdividing edges always preserves edge-reflexivity. In particular, though $K_{3,3}$ is an edge-reflexive cubic graph, if we choose any edge $e \in E(K_{3,3})$ and subdivide that edge k times (where $k \geq 1$) then the resulting graph will not be edge-reflexive.

The main results of this chapter were submitted for publication in [29].

5.1 Triangles and Disconnected Graphs

Recall from Chapter 2 that (by Lemma 2.3.4) a cubic graph G containing a triangle is not edge-colourful. As a consequence, it is not edge-reflexive. This establishes the easier direction of the chapter's main theorem. Addressing graphs with more than one connected component is more difficult, though we can make progress using an old result due to Fisk. In order to state this result, we need to first define the categorical product of two graphs. Given a pair of graphs X and Y, their *categorical product* $X \times Y$ is the graph whose vertex set is $V(X) \times V(Y)$, and which satisfies $(x, y) \sim (x', y')$ if and only if $x \sim x'$ and $y \sim y'$, where $x, x' \in V(X)$ and $y, y' \in V(Y)$.

Theorem 5.1.1 (Fisk [20])**.** *Let X and Y be graphs on disjoint vertex sets in which all maximal cliques are triangles. Then the following graphs are isomorphic:*
(a) $B_3(X \cup Y) \cong B_3(X) \times B_3(Y)$;
(b) if X and Y are also connected, then $B_3(X \times Y) \cong B_3(X) \cup B_3(Y)$.

Proof. (a) Observe that f is a 3-colouring of $X \cup Y$ if and only if the restriction of f to X is a 3-colouring of X and the restriction of f to Y is a 3-colouring of Y. Thus, each vertex of $B_3(X \cup Y)$ is the disjoint union of a vertex of $B_3(X)$ with a vertex of $B_3(Y)$. Such unions are certainly in one-to-one correspondence with the elements of $V(B_3(X) \times B_3(Y))$, so it merely remains to show that two vertices $C, D \in V(B_3(X \cup Y))$ are adjacent if and only if $C \cap V(X) \sim D \cap V(X)$ in $B_3(X)$ and $C \cap V(Y) \sim D \cap V(Y)$ in $B_3(Y)$. However, this is self-evident, as C and D will be in a 3-colouring of $X \cup Y$ together if and only if their restrictions to each connected component of $X \cup Y$ are colour classes together in a 3-colouring of said connected component.

(b) The following claim is useful in establishing (b).

Claim 1: If f is a 3-colouring $f : K_3 \times K_3 \to K_3$ then f is a projection onto one of the factors of $K_3 \times K_3$ composed with an automorphism of K_3.

Proof: Let $v \in V(K_3)$, and suppose that there are distinct pairs (u, w) and (x, y) of $f^{-1}(v)$ such that $u \neq x$ and $w \neq y$. Then $(u, w) \sim (x, y)$, contradicting the fact that f is a 3-colouring. So, without loss of generality, we may assume that $u = x$ and $w \neq y$. It follows that all the first coordinates in $f^{-1}(v)$ are equal, as, otherwise, there would exist some other element (r, s) of $f^{-1}(v)$ such that $u \neq r$. However, this implies that $s = w$ and $s = y$, a contradiction.

Having established that all the first coordinates of vertices in $f^{-1}(v)$ are equal, consider the possibility that there exist vertices $p, q \in V(K_3)$ such that $f^{-1}(p)$ has all first coordinates equal, while $f^{-1}(q)$ has all second coordinates equal. In this case, there exist vertices p' and q' such that $f^{-1}(p) = p' \times K_3$ and $f^{-1}(q) = K_3 \times q'$. But then the vertex (p', q') belongs to both $f^{-1}(p)$ and $f^{-1}(q)$, which cannot be the case, since f is a 3-colouring. Thus, by symmetry, for every vertex $v \in V(K_3)$, there exists a vertex v' such that $f^{-1}(v) = v' \times K_3$. In other words, f is the projection onto the first factor of $K_3 \times K_3$, followed by the automorphism which sends v to v'. This establishes the claim.

Now, suppose that we are given a 3-colouring $f : X \times Y \to K_3$. Let R and S be triangles of X and Y, respectively. Then f restricted to $R \times S$ is a projection (without loss of generality) to the first factor. Now, suppose that R' is a triangle meeting R in some vertex v. As f restricted to $v \times S$ is a projection to the first factor, f is a projection onto the first factor of $R' \times S$. Continuing in this way, since X and Y are connected graphs in which all the maximal cliques are triangles, f is a projection to the first factor on all of $X \times Y$. Consequently, every 3-colouring of $X \times Y$ corresponds to either a 3-colouring of X or a 3-colouring of Y, so $B_3(X \times Y) \cong B_3(X) \cup B_3(Y)$. $\square$

Let us note that the connectivity assumption in part (b) of Theorem 5.1.1 cannot be omitted. In particular, consider the graph X consisting of the disjoint union of two copies of $L(K_{3,3})$. Equivalently, we can construct this graph as the product $(K_3 \cup K_3) \times K_3$. In this case, $B_3((K_3 \cup K_3) \times K_3)$ is isomorphic to four disjoint copies of $L(K_{3,3})$. By contrast, $B_3(K_3 \cup K_3) \cup B_3(K_3)$ is isomorphic to the disjoint union of K_3 and one copy of $L(K_{3,3})$.

Curiously, we also notice that $B_3(L(K_{3,3}) \cup L(K_{3,3})) \cong L(K_{3,3}) \cup L(K_{3,3}) \cup L(K_{3,3}) \cup L(K_{3,3})$. Fisk observed that this property (that $B_3(X) \cong X \cup X$) is shared by the line graph of the Coxeter graph, though few other examples are known [20].

Corollary 5.1.2. *Let X and Y be the line graphs of 3-edge-colourable cubic graphs G and H, respectively, where G and H each contain at least one half-edge. Then the disjoint union $X \cup Y$ is a reflexive graph with respect to 3-colourings if each of X and Y is reflexive.*

Proof. The graph $X \cup Y$ is reflexive if and only if $B_3(B_3(X \cup Y)) \cong X \cup Y$. By Theorem 5.1.1, this is equivalent to asking that $B_3(B_3(X) \times B_3(Y)) \cong X \cup Y$. Moreover, employing Observation 2.4.7, if X and Y are reflexive (and G and H each have at least one half-edge) then $B_3(X)$ and $B_3(Y)$ are connected graphs. Thus, $B_3(B_3(X) \times B_3(Y)) \cong B_3^2(X) \cup B_3^2(Y)$ by Theorem 5.1.1. Thus, $B_3^2(X \cup Y) \cong X \cup Y$, as required. $\square$

Unfortunately, the converse is not readily apparent. Worse, if either X or Y is not reflexive, then we cannot apply Observation 2.4.7, so we cannot conclude that $X \cup Y$ is not reflexive. For these reasons, we restrict ourselves to considering connected, cubic outerplanar graphs.

5.2 Cut-edges and Reflexivity

In this section, we will prove that a connected, cubic graph G is edge-reflexive whenever all graphs obtained from G by cutting all cut-edges are edge-reflexive. This result will allow us to restrict ourselves to merely studying 2-edge-connected, triangle-free, cubic outerplanar graphs.

Our approach to this proof is based on generalizing the proof of Theorem 4.1.1 from Chapter 4. In that proof, given a half-edge in an edge-reflexive graph G, making said half edge into a full edge joined to a new vertex (and adding half-edges in order to ensure the result was cubic) preserved edge-reflexivity. We then applied induction in order to prove that all cubic trees were reflexive. In Lemma 5.2.1, we show that, given a half-edge in an edge-reflexive graph G, making said half edge into a full edge joined to an edge-reflexive graph H also preserves edge-reflexivity. Then it follows by induction that we can reduce the study of G to the study of the graphs obtained from G by cutting all its cut-edges.

Let G be a cubic graph with a cut-edge e joining vertices a and b. By *cutting the edge e* we obtain two cubic graphs H and K which are obtained from $G-e$ by adding a half-edge to a and b, respectively. The added half-edge will be considered to be the same as the removed edge e, so that we can consider $E(H) \subseteq E(G)$, $E(K) \subseteq E(G)$, and $E(H) \cap E(K) = \{e\}$.

Lemma 5.2.1. *Let G be a cubic graph with a cut-edge e. Let H and K be cubic graphs obtained from G by cutting the edge e. If H and K are edge-reflexive, then G is edge-reflexive, too.*

Proof. We let $X = L(H)$, $Y = L(K)$, and $X' = L(G)$. Let w and x be the neighbours of e in X, and let y and z be the neighbours of e in Y. Before proceeding further, let us first observe that since X and Y are reflexive, they are also colourful. This implies that X' is also colourful. Thus, $\phi_{X'}$ is an injective homomorphism $X' \to B_3^2(X')$, and therefore it suffices to show that X' and $B_3^2(X')$ have the same number of triangles (by Lemma 2.4.2). Triangles in X' correspond to vertices in $V(G) = V(H) \cup V(K)$, while the triangles in $B_3^2(X')$ correspond to 3-colourings of $B_3(X')$.

Now, for any colour class $C \in V(B_3(X))$, let $\mathcal{F}_C$ be the set of colour classes of X' which coincide with C on the vertices of X. If $e \in C$, we also write $\mathcal{F}_C^e$ to denote the same set of colour classes. Observe that when $e \in C$ and $D \in \mathcal{F}_C$, we have $y, z \notin D$. On the other hand, if $e \notin C$ then exactly one of y and z is in D. In this case, we partition $\mathcal{F}_C$ into the subset $\mathcal{F}_C^y$ consisting of colour classes which contain y, and the subset $\mathcal{F}_C^z$ of colour classes containing z. We refer to the set $\mathcal{F}_C^e$, or the sets $\mathcal{F}_C^y$ and $\mathcal{F}_C^z$ as the *clusters* of $B_3(X')$ corresponding to C. Each cluster is an independent set in $B_3(X')$. We say that two clusters corresponding to $C, C' \in V(B_3(X))$ are *adjacent* if $CC' \in E(B_3(X))$.

<u>Claim 1:</u> For any triangle ACD in $B_3(X)$ with $e \in A$, the induced subgraph of $B_3(X')$ on $\mathcal{F}_A^e \cup \mathcal{F}_C^y \cup \mathcal{F}_D^z$ is isomorphic to $B_3(Y)$. The same is true for $\mathcal{F}_A^e \cup \mathcal{F}_C^z \cup \mathcal{F}_D^y$.

<u>Proof:</u> To prove the claim, consider the map $\psi : \mathcal{F}_A^e \cup \mathcal{F}_C^y \cup \mathcal{F}_D^z \to B_3(Y)$ which takes a colour class B of X' to the intersection of B with $V(Y)$. Since the restriction of B to $V(X)$ is identical to precisely one of A, C or D it is easy to see that ψ is injective. Further, we can form a colour class of X' from any colour class of Y by combining it with one of A, C and D, provided that they agree on containment of e. Hence ψ is surjective.

It remains to show that, for $B, B' \in \mathcal{F}_A^e \cup \mathcal{F}_C^y \cup \mathcal{F}_D^z$, B is adjacent to B' if and only if $\psi(B)$ is adjacent to $\psi(B')$. Suppose that $B \in \mathcal{F}_A^e$ and $B' \in \mathcal{F}_C^y$. If B and B' are adjacent in $B_3(X')$ then there is a colour class $F \in B_3(X')$ such that $BB'F$ is a triangle in $B_3(X')$. Note that $F \in \mathcal{F}_D^z$. The image of this triangle under ψ is also a triangle (as ψ maps 3-colourings of X' to 3-colourings of Y) and hence $\psi(B)$ is adjacent to $\psi(B')$. On the other hand, if $\psi(B)$ and $\psi(B')$ are adjacent in $B_3(Y)$, then there is a triangle $\psi(B)\psi(B')J \subseteq B_3(Y)$ since all edges in $B_3(Y)$ come from 3-colourings of Y. Now B, B' and $D \cup J$ form a colouring of X', and hence B and B' are adjacent in $B_3(X')$. The proofs for the cases when $B \in \mathcal{F}_A^e, B' \in \mathcal{F}_D^z$ and $B \in \mathcal{F}_C^y, B' \in \mathcal{F}_D^z$ are similar. This proves the claim.

Claim 2: Let ACD be a triangle in $B_3(X)$ with $e \in A$. Then the subgraph of $B_3(X')$ on colour classes in $\mathcal{F}_D^z \cup \mathcal{F}_C^y$ is connected and bipartite, and is isomorphic to $B_3(Y) - \phi_Y(e)$. Further, the graph B^* obtained from $B_3(Y)$ by deleting the edges of $B_3(Y) - \phi_Y(e)$ is connected and isomorphic to each of the bipartite graphs between $\mathcal{F}_A^e$ and $\mathcal{F}_C$, as well as between $\mathcal{F}_A^e$ and $\mathcal{F}_C^y \cup \mathcal{F}_D^z$.

Proof: $B_3(Y) - \phi_Y(e)$ is connected by Lemma 2.4.5. Furthermore, every edge of $B_3(Y) - \phi_Y(e)$ is contained in a triangle in $B_3(Y)$ using no other edge of $B_3(Y) - \phi_Y(e)$. Therefore, B^* is also connected since the edges of such triangles can be used in B^* to replace any of the removed the edges. Now, suppose that ACD is a triangle in $B_3(X)$, where $e \in A$. Then the bipartite graph between $\mathcal{F}_D^z$ and $\mathcal{F}_C^y$ is isomorphic to $B_3(Y) - \phi_Y(e)$, while the bipartite graphs between $\mathcal{F}_A^e$ and $\mathcal{F}_C$ and between $\mathcal{F}_A^e$ and $\mathcal{F}_C^y \cup \mathcal{F}_D^z$ are each isomorphic to B^*, in the latter case, by Claim 1. Hence both of these are also connected, establishing the claim.

As in the proof of Theorem 4.1.1, we say that a colouring is *constant* on a set S of vertices if S is contained in a colour class of the colouring. If a colouring χ of $B_3(X')$ is constant on each of $\mathcal{F}_C^y$ and $\mathcal{F}_C^z$, we say that χ is *near-constant* on $\mathcal{F}_C$.

Claim 3: Let ACD be a triangle in $B_3(X)$, where $e \in A$, and let χ be a colouring of $B_3(X')$ that is constant on one of $\mathcal{F}_C$ or $\mathcal{F}_D$. Then χ is constant on each of $\mathcal{F}_A^e$, $\mathcal{F}_C$ and $\mathcal{F}_D$.

Proof: If χ is constant on $\mathcal{F}_C$, then $\mathcal{F}_A^e$ and $\mathcal{F}_D$ form a connected bipartite graph (by Claim 2) on which χ uses only two colours. Therefore, χ is constant on each of $\mathcal{F}_A^e$ and $\mathcal{F}_D$. The case where χ is constant on $\mathcal{F}_D$ is similar. This proves the claim.

Claim 4: If χ is constant on $\mathcal{F}_A^e$, then it is near-constant on $\mathcal{F}_C$ and $\mathcal{F}_D$. If χ is near-constant on one of $\mathcal{F}_C$ and $\mathcal{F}_D$, then it is also near-constant on the other, together using only two colours on $\mathcal{F}_C \cup \mathcal{F}_D$, and is constant on $\mathcal{F}_A^e$, using the third colour.

Proof: To prove the claim, we first observe that if χ is constant on $\mathcal{F}_A^e$, then $\mathcal{F}_C^y$ and $\mathcal{F}_D^z$ form a connected bipartite graph on which χ uses only two colours. Hence, χ is constant on each of $\mathcal{F}_C^y$ and $\mathcal{F}_D^z$ (and, similarly, $\mathcal{F}_C^z$ and $\mathcal{F}_D^y$). It follows immediately that χ is near-constant on $\mathcal{F}_C$ and $\mathcal{F}_D$.

Suppose now that χ is near-constant on $\mathcal{F}_C$. By Claim 3, we may assume that χ is not constant on $\mathcal{F}_C$, so it uses different colours on $\mathcal{F}_C^y$ and $\mathcal{F}_C^z$. Then there is only one colour left for the whole of $\mathcal{F}_A^e$ (noting from Claim 2 that every vertex of $\mathcal{F}_A^e$ is adjacent to some vertex in each of $\mathcal{F}_C^y$ and $\mathcal{F}_C^z$). So χ is constant on $\mathcal{F}_A^e$, and hence near-constant on $\mathcal{F}_D$. Again, the case where χ is near-constant on $\mathcal{F}_D$ is similar. This completes the proof of the claim.

Our next goal is to count all of the 3-colourings of $B_3(X')$. All such colourings are partitioned into three classes.

Class 1: Colourings that are constant on $\mathcal{F}_C$ for some $C \in V(B_3(X)) \setminus \phi_X(e)$. By Lemma 2.4.5, $B_3(X) - \phi_X(e)$ is connected. This fact, repeatedly combined with Claim 3, implies that such a colouring χ is constant on $\mathcal{F}_A^e$ for every $A \in \phi_X(e)$. Note that every colouring of $B_3(X)$ gives rise to a colouring of $B_3(X')$ in the obvious way: If $\mathcal{D} \in V(B_3^2(X))$ is a colour class in a colouring of $B_3(X)$, we let $\mathcal{D}' = \bigcup_{D \in \mathcal{D}} \mathcal{F}_D \in V(B_3^2(X'))$. This correspondence yields a bijection between colourings of $B_3(X)$ and the colourings of $B_3(X')$ that are of Class 1. In particular, the number of colourings of Class 1 is equal to the number of triangles of X, since X is reflexive. Of course, this is equal to the number of vertices of H.

Class 2: Colourings that are constant on $\mathcal{F}_A^e$ for some $A \in \phi_X(e)$, but not on $\mathcal{F}_C$ for any $C \in V(B_3(X)) \setminus \phi_X(e)$. In this case, Claim 4 implies that any such colouring is near-constant on $\mathcal{F}_C$ for each $C \in V(B_3(X)) \setminus \phi_X(e)$. Since $B_3(X) - \phi_X(e)$ is connected, the same two colours are used for every $\mathcal{F}_C$ and the colour of each $\mathcal{F}_C^y$ and $\mathcal{F}_C^z$ are determined by the choice of any one of them. We can therefore construct exactly one colouring of $B_3(X')$ in this way.

Class 3: Colourings that are non-constant on each $\mathcal{F}_A^e$, $A \in \phi_X(e)$. In this case, we will show that any such colouring is completely determined by its restriction to an arbitrarily chosen triangle of clusters $\mathcal{F}_A^e, \mathcal{F}_C^y, \mathcal{F}_D^z$. The vertices of these clusters can be coloured according to any colouring of $B_3(Y)$, except for the colouring in which the clusters are themselves colour classes. Therefore, the number of colourings covered by this case is one less than the number of triangles in Y.

Suppose now that χ is a colouring of $B_3(X')$ of class 3, and consider its restriction to $\mathcal{F}_A^e \cup \mathcal{F}_C^y \cup \mathcal{F}_D^z$. Our aim is to prove that this restriction determines χ. To see this, we employ an inductive argument. First observe, by Lemma 2.4.4, that χ is determined on $\mathcal{F}_D^y$ and $\mathcal{F}_C^z$, since these clusters form a connected bipartite graph. Moreover, by Claim 4, neither $\mathcal{F}_C$ nor $\mathcal{F}_D$ may be near-constant. Similarly, χ is determined on any clusters adjacent to $\mathcal{F}_A^e$. Again, by Lemma 2.4.4, noting that χ is not near-constant on $\mathcal{F}_C$, χ is also determined on $\mathcal{F}_B^e$ for any $B \in \phi_X(e)$ which is adjacent to C, since the bipartite graph between $\mathcal{F}_B^e$ and $\mathcal{F}_F$ is connected, where F is the third vertex of the triangle in $B_3(X)$ containing B and C. Similarly to the argument for $\mathcal{F}_A^e$, χ is also determined on $\mathcal{F}_F$, and $\mathcal{F}_F$ cannot be near-constant. Repeating this argument, since $B_3(X)$ is connected, we conclude that we have uniquely determined the whole of the colouring χ.

By the preceding cases, in total, the number of colourings of $B_3(X')$, and therefore the number of triangles in $B_3^2(X')$, is equal to the sum of the number of triangles of X and Y. Since X' also has this many triangles, we conclude by Lemma 2.4.2 that the injection $\phi_{X'} : X' \to B_3^2(X')$ is in fact an isomorphism. □

As a direct corollary of Lemma 5.2.1 we obtain Theorem 4.1.1. The lemma also enables us to restrict ourselves to 2-edge-connected cubic graphs.

Corollary 5.2.2. *Suppose that all graphs obtained from a connected, cubic graph G by cutting all cut-edges of G are edge-reflexive. Then G is edge-reflexive.*

5.3 2-Edge-Connected, Triangle-Free, Outerplanar Graphs

Corollary 5.2.2 shows that in order to prove Theorem 1.3.1, it suffices to prove the following.

Lemma 5.3.1. *Every 2-edge-connected, cubic, triangle-free outerplanar graph is edge-reflexive.*

We can show that any 2-edge-connected, triangle-free, cubic, outerplanar graph G can be constructed from a cubic 4-cycle by repeatedly applying the following two operations:

1. *Adding a 4-cycle*: Given an edge $e = v_1v_2$ in a cubic graph H, incident with half-edges e_1 and e_2 (respectively), as well as the full edges f_1 and f_2, add two new vertices v_3, v_4 and form a 4-cycle $v_1v_2v_3v_4$, where e_1 joins v_1 and v_4, e_2 joins v_2 and v_3, and e_3 joins v_3 and v_4. Finally, add half-edges e_{v_3} and e_{v_4} incident with v_3 and v_4, respectively.

2. *Subdividing an edge*: Given an edge $e = v_1v_2$ in a cubic graph H, where v_1 and v_2 are incident with half-edges e_1 and e_2 (respectively), as well as with full edges f_1 and f_2, subdivide the edge e into two edges e' and e'' by inserting a new vertex v. Then add a half-edge g incident with v, in order to form a new cubic graph.

These two operations with the corresponding notation that will be used when speaking about them are depicted in Figure 5.1.

The following lemma verifies our claim that any 2-edge-connected, triangle-free, cubic, outerplanar graph G can be constructed from a cubic 4-cycle by repeatedly applying these operations.

Lemma 5.3.2. *A 2-edge-connected, triangle-free, cubic, outerplanar graph G can be constructed from a cubic 4-cycle by repeatedly applying operations 1 and 2.*

Proof. We proceed by induction on $n = |V(G)|$, where $n \geq 4$. The base case is the cubic 4-cycle. Now, suppose that any 2-edge-connected, triangle-free, cubic, outerplanar graph H which satisfies $|V(H)| < n$ can be constructed from a cubic 4-cycle by repeatedly applying

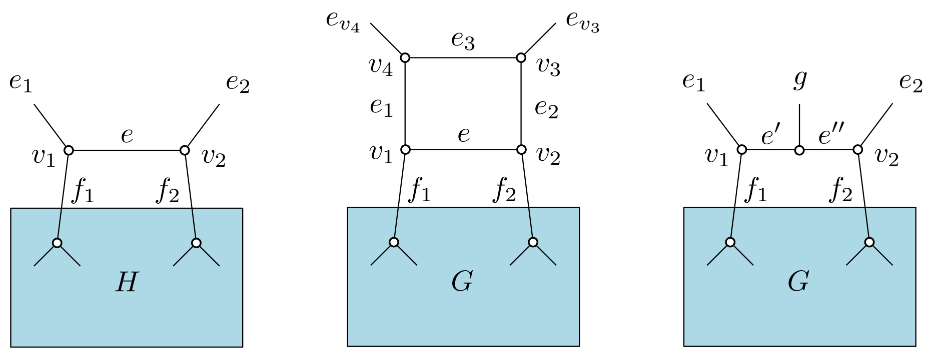


Figure 5.1: Adding a 4-cycle and subdividing an edge

operations 1 and 2. Let G be a 2-edge-connected, triangle-free, cubic, outerplanar graph such that $|V(G)| = n$.

Fix a planar drawing of G in which every vertex of G appears on the unbounded face of the drawing. We will call the edges on the unbounded face of this drawing the outer edges of G, and the other edges of G will be called the inner edges of G. For every inner edge $uv \in E(G)$, the endpoints of uv are a 2-separator.

Now, choose an inner edge uv so that the smallest component of $G - \{u, v\}$ has as few vertices as possible. As this smallest component is minimal, it cannot contain an inner edge, so this smallest component is just a path P. The path P has at least three edges, since G does not contain any triangles, multi-edges or loops. If G has exactly three edges uw, wx and xv, then we can construct a 2-edge-connected, triangle-free, cubic, outerplanar graph H' from $G - wx$ by adding a half-edge to each of u and v. By our induction hypothesis, H' can be constructed from a cubic 4-cycle by repeatedly applying operations 1 and 2, and then G can be constructed from H' by adding a 4-cycle.

If G has at least four edges uu_1, u_1u_2, ..., u_kv, where $k \geq 3$, then we can contract the edge $u_{k-1}u_k$ in order to obtain a graph H'' which can be constructed from a cubic 4-cycle by repeatedly applying operations 1 and 2. Then, by subdividing the edge $u_{k-2}u_{k-1}$, we can construct G from H''. Therefore, by induction, G can be constructed from a cubic 4-cycle by repeatedly applying operations 1 and 2 for all $|V(G)| = n \in \mathbb{N}$.

□

As a consequence of this lemma, in order to prove Lemma 5.3.1, it suffices to show that operations 1 and 2 preserve edge-reflexivity.

Lemma 5.3.3. *Operation 1 preserves edge-reflexivity.*

Proof. Let G be the graph obtained from H by adding a 4-cycle, let $X = L(G)$, $Y = L(H)$, and take all the vertex and edge labels of H, G, X and Y to be as we defined in Operation 1. Furthermore, assume that Y is reflexive. We have to show that the homomorphism $\phi_X : X \to B_3^2(X)$ is an isomorphism. As before, we will establish that X is colourful and then count the number of colourings of $B_3(X)$.

Claim 1: The graph X is colourful.

Proof: We are given that Y is colourful. Since every 3-colouring of Y extends to X, it immediately follows that, for any pair of vertices $u, v \in V(X) \setminus \{e_3, e_{v_3}, e_{v_4}\}$, there exists a colouring of X in which u and v are coloured differently. Moreover, as there is a colouring of Y which colours f_1 and e_2 differently, there is a colouring of X in which e_1 and e_2 have the same colour. Now, by performing an $\{e_3, e_{v_3}, e_{v_4}\}$ Kempe change, if needed, we can ensure that for any $u \in V(X) \setminus \{e_3, e_{v_3}, e_{v_4}\}$ and $v \in \{e_3, e_{v_3}, e_{v_4}\}$, there exists a colouring in which u and v are coloured differently. Finally, as there exists a colouring in Y which colours e_1 and e_2 differently, we can extend this colouring to X in order to obtain a colouring in which e_{v_3} and e_{v_4} are coloured differently. This completes the claim.

Now, we partition the vertices of $B_3(X)$ into the seven clusters shown in Figure 5.2, where (as we noted in chapters 2 and 4) we denote by $\mathcal{C}_{x_1 x_2 \ldots x_t}$ the set of colour classes of X which contain the vertices $x_1, x_2, \ldots, x_t \in \{e, e_1, e_3, e_2, f_1, f_2\}$. In other words,

$$\mathcal{C}_{x_1 x_2 \ldots x_t} = \bigcap_{i=1}^{t} \phi_X(x_i).$$

In Figure 5.2 we also have the clusters $\mathcal{C}_{\widehat{e_3} f_1 f_2}$ and $\mathcal{C}_{\widehat{e_3} e}$, where $\widehat{e_3}$ indicates that the colour classes in this cluster *do not* contain e_3. Note that $\mathcal{C}_{e_3 f_1 f_2} \cup \mathcal{C}_{\widehat{e_3} f_1 f_2}$ is a partition of $\mathcal{C}_{f_1 f_2}$.

The 3-colourings of X fall into three types as indicated by triangles in Figure 5.2. The subgraph of $B_3(X)$ consisting of all triangles of type $i \in \{1, 2, 3\}$ will be denoted by $\mathcal{T}_i$. Note that $\mathcal{T}_i$ contains only those vertices from the corresponding three clusters that appear as colour classes in $\mathcal{T}_i$. Thus, $\mathcal{T}_i$ is obtained from the induced subgraph on the three clusters by removing the isolated vertices (which must participate in colourings of the neighboring $\mathcal{T}_j$, but not in $\mathcal{T}_i$). We also denote the subgraph of $\mathcal{T}_i$ induced by the union of the two adjacent clusters $C_{x_1 \ldots x_t}$ and $C_{w_1 \ldots w_s}$ by $C_{x_1 \ldots x_t} C_{w_1 \ldots w_s}$.

Claim 2: $\mathcal{T}_2 \cup \mathcal{T}_3$ is isomorphic to $B_3(Y)$. Under this isomorphism, $\mathcal{C}_{ee_3}$ is mapped onto $\phi_Y(e)$.

Proof: Consider the map $\psi : \mathcal{T}_2 \cup \mathcal{T}_3 \to B_3(Y)$ which takes a colour class C of X to its intersection with $V(Y)$. As every colouring c of Y extends uniquely to a colouring c' of X with $c'(e) = c'(e_3)$, ψ is a bijection. Moreover, this unique extension also establishes that ψ is a graph homomorphism, since it follows that ABC is a colouring of X which lies in $\mathcal{T}_2 \cup \mathcal{T}_3$ if and only if $\psi(A)\psi(B)\psi(C)$ is a colouring of Y. Thus, our claim is established.

Since Y is reflexive and e has degree 4 in Y, Lemma 2.4.5 shows that $B_3(Y) - \phi_Y(e)$ has precisely two components. Hence, it follows from Claim 2 that $\mathcal{C}_{e_1 f_2} \mathcal{C}_{e_2 f_1}$ and $\mathcal{C}_{e_1 e_2} \mathcal{C}_{\widehat{e_3} f_1 f_2}$ are bipartite and connected subgraphs of $\mathcal{T}_2 \cup \mathcal{T}_3$.

Claim 3: $\mathcal{T}_1$ is isomorphic to $\mathcal{T}_2$ and $\mathcal{C}_{\widehat{e_3} e} \mathcal{C}_{e_3 f_1 f_2}$ is isomorphic to $\mathcal{C}_{ee_3} \mathcal{C}_{\widehat{e_3} f_1 f_2}$.

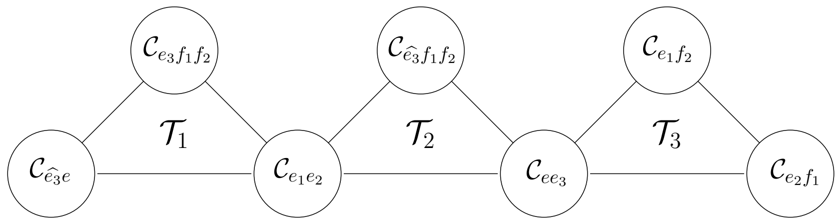


Figure 5.2: Partitioning $B_3(X)$ into clusters

Proof: Note that the colourings of X forming $\mathcal{T}_1$ and $\mathcal{T}_2$ have e_1 and e_2 coloured the same. Consider the map $\psi : \mathcal{T}_2 \to \mathcal{T}_1$ which takes a colour class $C \in V(\mathcal{T}_2)$ to the colour class of X in $\mathcal{T}_1$ which results from performing a Kempe change on $\{e_3, e_{v_4}, e_{v_3}\}$. Observe that this map is well-defined since e_{v_3} and e_{v_4} are coloured the same and hence the colours of e_1 and e_2 remain the same. As ψ and ψ^{-1} are both invertible (as Kempe changes are reversible), ψ is a bijection. Similarly, ψ is a graph homomorphism, since it follows from the existence and reversibility of the Kempe change discussed that ABC is a triangle of $\mathcal{T}_2$ if and only if $\psi(A)\psi(B)\psi(C)$ is a triangle of $\mathcal{T}_1$. These observations confirm our claim.

Claim 4: $\mathcal{C}_{\widehat{e_3}f_1f_2}\mathcal{C}_{ee_3}$ is isomorphic to $\mathcal{C}_{\widehat{e_3}f_1f_2}\mathcal{C}_{e_1e_2}$.

Proof: Consider the map $\psi : \mathcal{T}_2 \to \mathcal{T}_2$ which takes a colour class $C \in V(\mathcal{T}_2)$ to the colour class of X in $\mathcal{T}_2$ which results from performing a Kempe change on $\{e, e_1, e_2, e_3\}$. Observe that this map is well-defined, as such a Kempe change always exists and leaves e_1 and e_2 the same colour, which remains different from the shared colour of e and e_3. Also, f_1 and f_2 remain the same colour, which is different from the colour of e_3. As Kempe changes are reversible, ψ is a bijection. Similarly, ψ is a graph homomorphism, since it follows from the existence and reversibility of the Kempe change discussed that ABC is a colouring of $\mathcal{T}_2$ if and only if $\psi(A)\psi(B)\psi(C)$ is a colouring of $\mathcal{T}_2$.

Claim 4 implies that $\mathcal{C}_{ee_3}\mathcal{C}_{\widehat{e_3}f_1f_2}$ is bipartite and connected. By Claim 3 it follows immediately that $\mathcal{C}_{\widehat{e_3}e}\mathcal{C}_{e_3f_1f_2}$ is bipartite and connected.

Now, let χ be a colouring of $B_3(X)$. Suppose first that χ is constant on $\mathcal{C}_{ee_3} \cap \mathcal{T}_2$. We claim that χ is constant on the whole cluster $\mathcal{C}_{ee_3}$. For a contradiction, suppose that this claim is false. As χ is constant on $\mathcal{C}_{ee_3} \cap \mathcal{T}_2$, we know that χ is constant on all three clusters of $\mathcal{T}_2$ (since $\mathcal{C}_{e_1e_2}\mathcal{C}_{\widehat{e_3}f_1f_2}$ is bipartite and connected). Now, as Y is reflexive and $\mathcal{T}_2 \cup \mathcal{T}_3$ is isomorphic to $B_3(Y)$ (by Claim 2), the colour class of $\mathcal{T}_2 \cup \mathcal{T}_3 \cong B_3(Y)$ containing $\mathcal{C}_{ee_3} \cap \mathcal{T}_2$ must be of the form $\phi_Y(v)$, for some vertex $v \in V(Y)$. If $v = e$ or $v = e_3$, we are done, so we may assume otherwise. As $\mathcal{C}_{\widehat{e_3}f_1f_2}$ and $\mathcal{C}_{e_1e_2} \cap \mathcal{T}_2$ are both nonempty (since Y is colourful) the colour class containing $\mathcal{C}_{ee_3} \cap \mathcal{T}_2$ in $B_3(Y)$ cannot be any of $\phi_Y(f_1)$, $\phi_Y(f_2)$, $\phi_Y(e_1)$, $\phi_Y(e_2)$, $\phi_Y(e_{v_4})$ or $\phi_Y(e_{v_3})$. However, for any vertex $v \in V(Y) \setminus \{e, e_1, e_3, e_2, f_1, f_2, e_{v_4}, e_{v_3}\}$, by Claim 4, if $\phi_Y(v) \cap \mathcal{C}_{ee_3} \cap \mathcal{T}_2 \neq \emptyset$, then $\phi_Y(v) \cap \mathcal{C}_{e_1e_2} \cap \mathcal{T}_2 \neq \emptyset$. But this contradicts the fact that χ is constant on $\mathcal{C}_{ee_3} \cap \mathcal{T}_2$, completing our claim.

Consequently, if χ is constant on $\mathcal{C}_{ee_3} \cap \mathcal{T}_2$, then χ is constant on the whole cluster $\mathcal{C}_{ee_3}$, and hence, on all the clusters of $B_3(X)$. There are four such colourings of $B_3(X)$, compared to two such colourings of $B_3(Y)$.

Let us now consider the case when χ is not constant on $\mathcal{C}_{ee_3} \cap \mathcal{T}_2$. In this case, by Lemma 2.4.4 and Claim 4, χ is determined and non-constant on $\mathcal{C}_{e_1e_2} \cap \mathcal{T}_2$ and determined on $\mathcal{C}_{\widehat{e_3}f_1f_2}$. By Claim 3 and Lemma 2.4.4, it then follows that χ is completely determined on $\mathcal{T}_1$. Thus, the colouring χ of $B_3(X)$ is completely determined by its restriction to $\mathcal{T}_2 \cup \mathcal{T}_3$, which is isomorphic to $B_3(Y)$. Each such colouring is determined uniquely by a colouring of $B_3(Y)$ which is not constant on some cluster of $B_3(X)$. Since two colourings of $B_3(Y)$ correspond to case 1, we get two fewer colourings in case 2, so we obtain two less colourings than the number of colourings of $B_3(Y)$.

Thus, in total, $B_3(X)$ has two more colourings than $B_3(Y)$ had, which is precisely the number of additional triangles in X. Consequently, $X \cong B_3^2(X)$, as required. □

Lemma 5.3.4. *Operation 2 preserves edge-reflexivity.*

Proof. Again, let $X = L(G)$, $Y = L(H)$, and we assume that Y is reflexive. As in our previous arguments, we will demonstrate that the homomorphism $\phi_X : X \to B_3^2(X)$ is an isomorphism by proving that X is colourful and then counting the number of colourings of $B_3(X)$. We also find it useful to define graphs $H_{e'}$ and $H_{e''}$ as shown in Figure 5.3, as well as their line graphs $X_{e'}$ and $X_{e''}$, respectively. Note that $H_{e'}$ and $H_{e''}$ are both isomorphic to H, but have some of their edges labelled differently. Through this labeling we can view $E(H_{e'})$ and $E(H_{e''})$ as subsets of $E(G)$.

Figure 5.3 also shows the correspondence of 3-edge-colourings of $H_{e'}$ and $H_{e''}$ with certain 3-edge-colourings of G. Observe that $H_{e'}$ ($H_{e''}$) has precisely the 3-edge colourings of G in which the colour of e' is equal to the colour of e_2 (the colour of e'' is equal to the colour of e_1) and $H_{e'} \cong H \cong H_{e''}$.

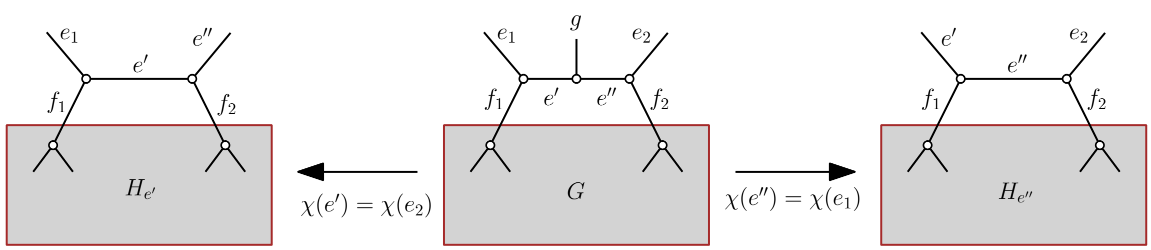


Figure 5.3: Two graphs isomorphic to H whose 3-edge-colourings correspond to certain subsets of the 3-edge-colourings of G

<u>Claim 1:</u> G is edge-colourful.

<u>Proof:</u> Let $u, v \in V(X)$. We examine four cases.

Case 1: If $u, v \notin \{e', e'', e_1, e_2, g\}$, then a 3-colouring $c_{e'}$ of $X_{e'}$ with $c_{e'}(u) \neq c_{e'}(v)$ can be extended to a colouring of X in which $c(u) \neq c(v)$. As $X_{e'}$ is colourful, this establishes case 1.

Case 2: $u, v \in \{e', e'', e_1, e_2, g\}$: As $X_{e'}$ is colourful, there exists a 3-edge colouring of G with $c(f_1) \neq c(f_2)$ and $c(e') = c(e_2)$. Since $c(e')$ is different from $c(f_1)$ and $c(f_2)$, we have that $c(e'') \neq c(e_1)$, $c(e_1) \neq c(e_2)$ and $c(e_2) \neq c(g)$. By the same argument on $X_{e''}$, we also establish that there is a colouring distinguishing e', e_2 and e_1, g.

Case 3: If $u \in \{e', e'', e_1, e_2\}$ and $v \notin \{e', e'', e_1, e_2, g\}$, then we may always arrange a colouring of X in which $c(f_1) = c(f_2)$, as $X_{e'}$ is colourful. In the event that $c(u) = c(v)$, we can then perform a Kempe change on the vertices of the set $\{e', e'', e_1, e_2\}$ in order to arrange for $c(u) \neq c(v)$. This establishes case 3.

Case 4: $u = g$ and $v \notin \{e', e'', e_1, e_2, g\}$: In case 2, we showed that there is a 3-edge colouring of G with $c(e') = c(f_2)$. So, for each vertex $v \notin \{g, e_2\}$ we can arrange for $c(g) \neq c(v)$ by performing a Kempe change on the path g, e'', e_2 (if needed). This completes the claim.

Now, as X is colourful, it suffices to establish (by Lemma 2.4.2) that $B_3(X)$ has precisely one more 3-colouring than $B_3(Y)$. In order to prove this fact, we will again partition $B_3(X)$ into *clusters* of the form $\mathcal{C}_{ab} = \phi_X(a) \cap \phi_X(b)$, where $a, b \in V(X)$. We will consider the partition into clusters as depicted in Figure 5.4.

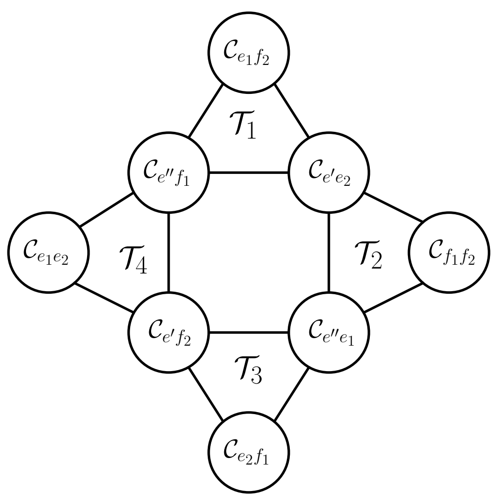


Figure 5.4: Partitioning $B_3(X)$ into clusters

As in the proof of Lemma 5.3.3, let $\mathcal{T}_i$ $(1 \leq i \leq 4)$ be the subgraph of $B_3(X)$ on all colour sets participating in 3-colourings of X whose edges are between the three clusters of $\mathcal{T}_i$, as shown in Figure 5.4. We will refer to the subgraph consisting of all the edges in $B_3(X)$ joining the clusters C_{uv} and $\mathcal{C}_{xy}$ as the edge $\mathcal{C}_{uv}\mathcal{C}_{xy}$ of our cluster partition. When $\mathcal{C}_{uv}\mathcal{C}_{xy}$ is nonempty, it is contained in precisely one of the subgraphs $\mathcal{T}_i$, $i \in \{1, 2, 3, 4\}$.

Claim 2: The following subgraphs of $B_3(X)$ are isomorphic: $\mathcal{T}_1 \cup \mathcal{T}_2 \cong B_3(X_{e'}) \cong B_3(Y) \cong B_3(X_{e''}) \cong \mathcal{T}_2 \cup \mathcal{T}_3$.

Proof: That $B_3(X_{e'}) \cong B_3(Y) \cong B_3(X_{e''})$ follows immediately from their definitions. So, it suffices to prove that the maps $\psi_1 : \mathcal{T}_1 \cup \mathcal{T}_2 \to B_3(X_{e'})$ and $\psi_2 : \mathcal{T}_2 \cup \mathcal{T}_3 \to B_3(X_{e''})$, each of which takes a colour class C of X to its restriction as indicated in Figure 5.3, are isomorphisms. As these two arguments are identical, we will only establish the claim for the map ψ_1.

As every colouring c of $X_{e'}$ extends uniquely to a colouring of X and $\mathcal{T}_1 \cup \mathcal{T}_2$ includes the whole $\mathcal{C}_{e'e_2}$, ψ_1 is a bijection. Moreover, this unique extension also establishes that ψ_1 is a graph homomorphism, since it follows that a colouring ABC of X is in $\mathcal{T}_1 \cup \mathcal{T}_2$ if and only if $\psi_1(A)\psi_1(B)\psi_1(C)$ is a colouring of $X_{e'}$. Thus, Claim 2 is resolved.

Claim 3: The edges of the cluster partition in Figure 5.4 representing $\mathcal{C}_{e_1f_2}\mathcal{C}_{e''f_1}$, $\mathcal{C}_{f_1f_2}\mathcal{C}_{e''e_1}$, $\mathcal{C}_{e'e_2}\mathcal{C}_{f_1f_2}$ and $\mathcal{C}_{e'f_2}\mathcal{C}_{e_2f_1}$ all represent connected, bipartite subgraphs of $B_3(X)$. Moreover, $\mathcal{C}_{f_1f_2}\mathcal{C}_{e''e_1} \cong \mathcal{C}_{e'e_2}\mathcal{C}_{f_1f_2}$.

Proof: Recall that, by Claim 2, $\mathcal{T}_1 \cup \mathcal{T}_2 \cong B_3(X_{e'}) \cong B_3(Y) \cong B_3(X_{e''}) \cong \mathcal{T}_2 \cup \mathcal{T}_3$. Notice that the graph $X_{e'}$ is reflexive and e' has degree 4 in $X_{e'}$. Through the isomorphism $\mathcal{T}_1 \cup \mathcal{T}_2 \cong B_3(X_{e'})$, $\phi_{X_{e'}}(e')$ corresponds to the cluster $\mathcal{C}_{e'e_2}$ in $\mathcal{T}_1 \cup \mathcal{T}_2$. Therefore, $\mathcal{C}_{e_1f_2}\mathcal{C}_{e''f_1}$ and $\mathcal{C}_{f_1f_2}\mathcal{C}_{e''e_1}$ are bipartite and connected by Lemma 2.4.5. Similarly, as $X_{e''}$ is reflexive and e'' has degree 4 in $X_{e''}$, $\mathcal{C}_{e'e_2}\mathcal{C}_{f_1f_2}$ and $\mathcal{C}_{e'f_2}\mathcal{C}_{e_2f_1}$ are bipartite and connected.

Now, consider the map $\psi : \mathcal{T}_2 \to \mathcal{T}_2$, which takes a colour class $C \in V(\mathcal{T}_2)$ to the colour class C' which results from performing a Kempe change on $\{e_1, e', e'', e_2\}$. Observe that this map is well-defined, as such a Kempe change always exists and leaves e_1 and e'' the same colour, which remains different from the shared colour of e' and e_2. Moreover, as Kempe changes are reversible, ψ is a bijection, and ABC is a triangle in $\mathcal{T}_2$ if and only if $\psi(A)\psi(B)\psi(C)$ is a triangle in $\mathcal{T}_2$. Consequently, ψ is a graph isomorphism. It remains to note that ψ maps $\mathcal{C}_{f_1f_2}\mathcal{C}_{e''e_1}$ onto $\mathcal{C}_{e'e_2}\mathcal{C}_{f_1f_2}$. This establishes Claim 3.

We will now discuss the structure of 3-colourings of $B_3(X)$. If we consider our cluster partition as an 8-vertex graph, each 3-colouring of that graph determines a 3-colouring of $B_3(X)$ in which each cluster is monochromatic (the colouring is *constant on the cluster*). There are other 3-colourings of $B_3(X)$. To understand them, we first show that each such colouring is determined by its restriction to certain subgraphs of $B_3(X)$.

Claim 4: For every 3-coloring of $B_3(X)$, its restriction to $\mathcal{T}_1 \cup \mathcal{T}_3$ determines the colouring on $\mathcal{T}_4$.

Proof: Observe that deleting either e' and f_2 or e'' and f_1 separates X into two components, one of which only contains vertices in $\{e_1, e_2, e', e'', g\}$. Consequently, we can obtain from any colouring of X represented by a triangle in $\mathcal{T}_4$ a triangle in $\mathcal{T}_1$ through a Kempe change on $\{e_1, e', g\}$ and a triangle in $\mathcal{T}_3$ through a Kempe change on $\{e_2, e'', g\}$. This shows that

$\mathcal{C}_{e''f_1} = \mathcal{T}_1 \cap \mathcal{T}_3$ and $\mathcal{C}_{e'f_2} = \mathcal{T}_3 \cap \mathcal{T}_4$. Thus, the colouring of $\mathcal{C}_{e''f_1}$ is determined by the colouring of $\mathcal{C}_{e''f_1} \cap \mathcal{T}_1$, and the colouring of $\mathcal{C}_{e'f_2}$ is completely determined by the colouring of $\mathcal{C}_{e'f_2} \cap \mathcal{T}_3$. Now, every vertex in $\mathcal{C}_{e_1e_2}$ is in a triangle in $\mathcal{T}_4$ and hence also its colour is determined.

<u>Claim 5:</u> If a colouring χ of X is constant on $\mathcal{C}_{e''e_1} \cap \mathcal{T}_2$, then χ is constant on the whole cluster $\mathcal{C}_{e''e_1}$. If χ is constant on $\mathcal{C}_{e'e_2} \cap \mathcal{T}_2$, then it is constant on $\mathcal{C}_{e'e_2}$.

<u>Proof:</u> For a contradiction, suppose that this claim is false. If χ is constant on $\mathcal{C}_{e''e_1} \cap \mathcal{T}_2$, then (by Claim 3) it is constant on each of $\mathcal{C}_{e'e_2} \cap \mathcal{T}_2$ and $\mathcal{C}_{f_1f_2}$. Now, as $\mathcal{T}_2 \cup \mathcal{T}_3 \cong B_3(X_{e''})$ is an isomorphic copy of the colouring complex of a reflexive graph, the colour class containing $\mathcal{C}_{e''e_1} \cap \mathcal{T}_2$ in $B_3(X_{e''})$ must be of the form $\phi_{X_{e''}}(v)$, for some vertex $v \in V(X_{e''})$. If $v = e''$, we are done, so we may assume otherwise. As $\mathcal{C}_{f_1f_2}$ and $\mathcal{C}_{e'e_2} \cap \mathcal{T}_2$ are both nonempty (since $X_{e''}$ is colourful) the colour class containing $\mathcal{C}_{e''e_1} \cap \mathcal{T}_2$ in $B_3(X_{e''})$ cannot be any of $\phi_{X_{e''}}(f_1)$, $\phi_{X_{e''}}(f_2)$, $\phi_{X_{e''}}(e'')$, $\phi_{X_{e''}}(e_2)$ or $\phi_{X_{e''}}(e')$. However, when the vertex $v \in V(X_{e''}) \setminus \{e_2, e', e'', f_1, f_2\}$, then $\phi_{X_{e''}}(v) \cap \mathcal{C}_{e'e_2} \cap \mathcal{T}_2 \neq \emptyset$ by Claim 3. But this contradicts the fact that χ is constant on $\mathcal{C}_{e''e_1} \cap \mathcal{T}_2$, completing our claim. The proof of the second statement is the same.

<u>Claim 6:</u> The subgraphs of $B_3(X)$ corresponding to $\mathcal{T}_4$ and $\mathcal{T}_3$ are isomorphic, and $\mathcal{C}_{e''f_1}\mathcal{C}_{e'f_2}$ is bipartite and connected.

<u>Proof:</u> As in the proof of Claim 4, we consider the map $\psi : \mathcal{T}_4 \to \mathcal{T}_3$ induced on $B_3(X)$ by performing a Kempe change in X on $\{g, e'', e_2\}$. This map is a bijection between $\mathcal{T}_4$ and $\mathcal{T}_3$. Moreover, ABC is a triangle in $\mathcal{T}_4$ if and only if $\psi(A)\psi(B)\psi(C)$ is a triangle in $\mathcal{T}_3$, so this is indeed a graph isomorphism.

<u>Claim 7:</u> If a 3-colouring χ of $B_3(X)$ is non-constant on $\mathcal{C}_{e''e_1} \cap \mathcal{T}_2$, then the restriction of χ to $(\mathcal{C}_{e''e_1} \cap \mathcal{T}_2) \cup (\mathcal{C}_{e'f_2} \cap \mathcal{T}_3)$ determines χ on the whole of $\mathcal{T}_3$.

<u>Proof:</u> Suppose that $A \in (\mathcal{C}_{e''e_1} \cap \mathcal{T}_3) \setminus (\mathcal{C}_{e''e_1} \cap \mathcal{T}_2)$. As $\mathcal{T}_2 \cup \mathcal{T}_3 \cong B_3(X_{e''})$ and $B_3(X_{e''}) \setminus \phi_{X_{e''}}(e')$ is bipartite and connected (by Lemma 2.4.5) there exists a path in $\mathcal{C}_{e''e_1}\mathcal{C}_{e_2f_1}$ from A to some vertex $D \in \mathcal{C}_{e''e_1} \cap \mathcal{T}_2$. Since $D \in \mathcal{C}_{e''e_1} \cap \mathcal{T}_2$, its colour is determined. Since the colouring on $\mathcal{C}_{e'f_2} \cap \mathcal{T}_3$ is determined, we conclude that χ is determined on the whole path from D to A. Thus, the colour of A is determined. Now, as A was chosen arbitrarily, it follows that the colouring is determined on $\mathcal{C}_{e''e_1} \cap \mathcal{T}_3$, from which it follows that χ is determined on all of $\mathcal{T}_3$. This proves the claim.

<u>Claim 8:</u> If a 3-colouring χ of $B_3(X)$ is non-constant on $\mathcal{C}_{e''e_1} \cap \mathcal{T}_2$, then the restriction of χ to $(\mathcal{C}_{e''e_1} \cap \mathcal{T}_2) \cup (\mathcal{C}_{e''f_1} \cap \mathcal{T}_4)$ determines χ on the whole of $\mathcal{T}_4$.

<u>Proof:</u> Firstly, let $A \in \mathcal{C}_{e'f_2}$ be a vertex of $B_3(X)$ in a connected component K of $\mathcal{C}_{e'f_2}\mathcal{C}_{e''e_1}$. Since e_2 is a half-edge in $X_{e''}$ and $X_{e''}$ is reflexive, $B_3(X_{e''}) \setminus \phi_{X_{e''}}(e_2)$ is connected (by Lemma 2.4.5). Thus, there exists a path from A to some vertex $D \in \mathcal{C}_{e''e_1} \cap \mathcal{T}_2$, which is also

in the connected component K. Now, the colour class containing D in $X_{e'}$ is of the form $\phi_{X_{e'}}(x)$, for some $x \in V(X_{e'})$ (because $\mathcal{T}_1 \cup \mathcal{T}_2$ is isomorphic to $B_3(X_{e'})$). The vertex D is also adjacent to some vertex $D' \in \mathcal{C}_{e'f_2}$, which is in turn adjacent (by the aforementioned isomorphism between $\mathcal{T}_3$ and $\mathcal{T}_4$) to a vertex $D'' \in \mathcal{C}_{e''f_1}$ whose colour is determined.

If D and D'' are coloured with the same colour, then they both belong to $\phi_{X_{e'}}(x)$. Therefore, $x \in D$ and $x \in D''$. However, by Claim 6, unless $x \in \{g, e'', e_2\}$, $x \in D$ implies that $x \notin D''$. Meanwhile, if $x \in \{e'', e_2\}$, then it easily follows that χ is constant on $\mathcal{C}_{e''e_1} \cap \mathcal{T}_2$ (contradicting our original assumption). Finally, x cannot be equal to g. So, in each case, we have a contradiction. Thus, D and D'' must be coloured differently.

Since D and D'' are coloured differently, the colour of D' is determined. Consequently, each connected component K of $\mathcal{C}_{e'f_2}\mathcal{C}_{e''e_1}$ must contain some vertex $D' \in \mathcal{C}_{e'f_2}$ whose colour is determined.

Now, we make an argument similar to that in the proof of Claim 7. Suppose that $A \in \mathcal{C}_{e'f_2}$. By using the isomorphism between $\mathcal{T}_3$ and $\mathcal{T}_4$, we see that, in the connected component K' of $\mathcal{C}_{e'f_2}\mathcal{C}_{e_1e_2}$ containing A, there is a path from A to some vertex $D' \in \mathcal{C}_{e'f_2}$ whose colour is determined. Since the colour of D' and the colours of vertices in $\mathcal{C}_{e''f_1} \cap \mathcal{T}_4$ are determined, the neighbor of D' on this path has its colour determined. By iterating this argument, we conclude that all the vertices on this path have their colour determined. Now, as A was chosen arbitrarily, it follows that the colouring is determined on $\mathcal{C}_{e'f_2}$, from which it follows that the colouring on all of $\mathcal{T}_4$ is determined. This proves the claim.

Now, consider an arbitrary 3-colouring χ of $B_3(X)$ and its restriction χ' on $\mathcal{T}_1 \cup \mathcal{T}_2 \cong B_3(X_{e'})$. We consider two cases.

Firstly, if χ is constant on $\mathcal{C}_{e''e_1} \cap \mathcal{T}_2$, then by Claim 3, χ is constant on each cluster of $\mathcal{T}_2$. Thus, applying Claim 5 and then Claim 3 again, we observe that χ is constant on all the clusters of $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$. It then follows from Claim 4 that χ is constant on every cluster of $B_3(X)$. There are three such colourings, compared to two colourings χ' of $\mathcal{T}_1 \cup \mathcal{T}_2$ which have this form.

Secondly, if χ is not constant on $\mathcal{C}_{e''e_1} \cap \mathcal{T}_2$, we need to show that χ is uniquely determined on all of $B_3(X)$ by its restriction χ' to $\mathcal{T}_1 \cup \mathcal{T}_2$. So, we apply Claim 8. The 3-colouring χ of $B_3(X)$ is determined and non-constant on $\mathcal{C}_{e''e_1} \cap \mathcal{T}_2$, and determined on $\mathcal{C}_{e''f_1}$ by the 3-colouring χ' of $\mathcal{T}_1 \cup \mathcal{T}_2$. Thus, the colouring is determined on $\mathcal{T}_4$. Consequently, the 3-colouring is determined and non-constant on $\mathcal{C}_{e''e_1} \cap \mathcal{T}_2$, and determined on $\mathcal{C}_{e'f_2}$, so by Claim 7, it is determined on $\mathcal{T}_3$.

Hence, all 3-colourings of $\mathcal{T}_1 \cup \mathcal{T}_2 \cong B_3(Y)$ uniquely extend to $B_3(X)$ (except in the case when $\mathcal{C}_{e''f_1}$ and $\mathcal{C}_{e''e_1}$ are coloured identically, in which case the colouring extends in two ways to $B_3(X)$). This shows that $B_3(X)$ has precisely one more 3-colouring than $B_3(Y)$, as required. $\square$

Proof of Lemma 5.3.1 and of Theorem 1.3.1. By Lemma 2.3.4, G cannot be edge-reflexive if it contains a triangle. So let us assume that G is triangle-free. By applying Lemma 5.3.3 and Lemma 5.3.4 repeatedly, we can construct any 2-edge-connected, triangle-free, cubic, outerplanar graph from a cubic 4-cycle (which is edge-reflexive). This establishes Lemma 5.3.1. Then, taking this result together with Corollary 5.2.2, Theorem 1.3.1 follows immediately. □

5.4 Subdivisions and Reflexive Theta Graphs

In Lemma 5.3.4 we showed that, under certain circumstances, the graph G, which we obtained from an edge-reflexive cubic graph H by subdividing an edge e of H, is edge-reflexive. However, there exist edge-reflexive graphs H where, regardless of how many times we subdivide one of its edges, the result will never be edge-reflexive. One such example is the non-planar graph $K_{3,3}$. We illustrate that $K_{3,3}$ is edge-reflexive in Figure 5.5.

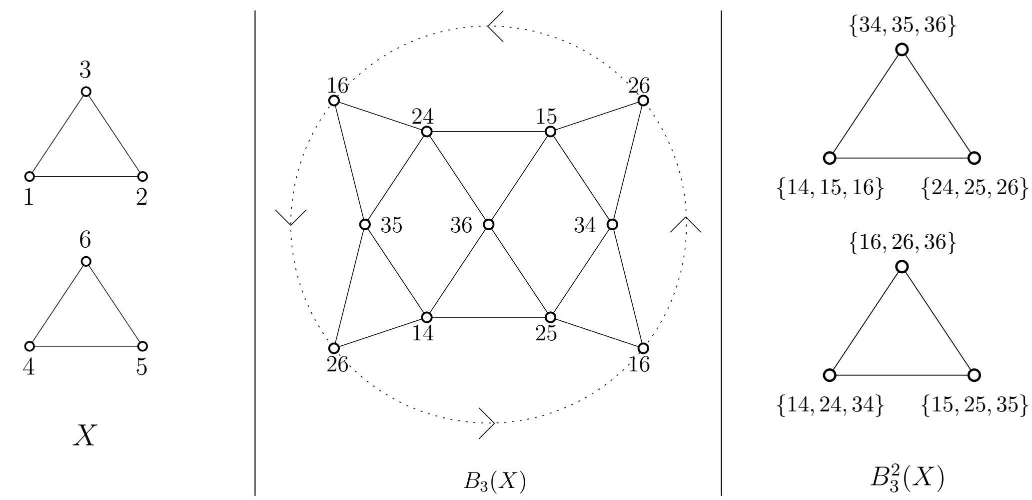


Figure 5.5: The disjoint union of two triangles X, its 3-colouring complex $B_3(X)$ (drawn in the projective plane) which is isomorphic to $L(K_{3,3})$, and the graph $B_3^2(X)$. This shows that $K_{3,3}$ is edge-reflexive.

In fact, there is a more general family.

Theorem 5.4.1. *Suppose that G is an edge-reflexive cubic graph without half-edges. Then no graph H which results from subdividing a single edge of G k times ($k \geq 1$) is edge-reflexive.*

In the proof we will employ the well-known Parity Lemma.

Lemma 5.4.2 (Parity Lemma)**.** *Suppose that a cubic graph G is edge-coloured. Let n_1, n_2 and n_3 be the number of half-edges of G in each of the three colour classes. Then n_1, n_2 and n_3 are congruent modulo 2.*

Proof. The number of half-edges in a colour class is equal to the number n of vertices in G, minus twice the number of full edges in the same colour class. Thus, $n \equiv n_1 \equiv n_2 \equiv n_3 \pmod 2$. □

Proof of Theorem 5.4.1. Let the graph H be obtained from G by subdividing $e = uv$ $k \geq 1$ times. The Parity Lemma applied to the cubic graph H' obtained from $H - e$ by adding two half-edges shows that H is not 3-edge-colourable when $k = 1$ and that it is not edge-colourful if $k \geq 2$, since in every 3-edge-colouring of H', the half-edges are coloured the same colour. □

Theorem 5.4.1 shows that subdividing a single edge in $K_{3,3}$ yields a graph that is not edge-reflexive. Of course, this is still not the full story, as some subdivisions of $K_{3,3}$ are edge-reflexive. For example, using a computer, it can be shown that if we subdivide each edge of $K_{3,3}$ once, the resulting graph is edge-reflexive. At this time, we do not fully understand the relation between subdividing edges and edge-reflexivity. However, we can still use Lemma 5.3.4 to help identify additional infinite families of edge-reflexive graphs. For example, this lemma is instrumental in proving our next result.

We construct the *cubic theta graph* $T_{k,l,m}$ $(k, l, m \geq 1)$ as follows. Begin with three paths of lengths k, l and m, respectively. Label their vertices $u_0, u_1, \ldots, u_k$, $v_0, v_1, \ldots, v_l$ and $w_0, w_1, \ldots, w_m$. Then identify the vertices u_0, v_0 and w_0, as well as the vertices u_k, v_l and w_m. Finally, add half-edges to make the graph cubic. Observe that $T_{k,l,m} \cong T_{l,k,m} \cong T_{k,m,l}$, and hence we may assume that $k \leq l \leq m$.

A number of small theta graphs are not edge-reflexive. In particular, $T_{1,1,m}$ is not edge-reflexive for any $m \geq 1$ by the same argument as we used to prove Theorem 5.4.1. The graph $T_{1,2,m}$ is not edge-reflexive for any $m \geq 1$, since it contains a triangle. Additionally, using a computer, we found that $T_{2,2,2}$, $T_{2,2,3}$, $T_{2,2,4}$, $T_{2,3,3}$, $T_{2,3,4}$ and $T_{3,3,3}$ are not edge-reflexive. However, all other theta graphs are edge-reflexive.

Theorem 5.4.3. *The cubic theta graphs $T_{1,1,m}$, $T_{1,2,m}$ $(m \geq 1)$ $T_{2,2,2}$, $T_{2,2,3}$, $T_{2,2,4}$, $T_{2,3,3}$, $T_{2,3,4}$ and $T_{3,3,3}$ are not edge-reflexive. All other cubic theta graphs are edge-reflexive.*

Proof. As mentioned above, the graphs listed in the statement of the theorem are not edge-reflexive. To show that all other cubic theta graphs are edge-reflexive, we have verified by using a computer that $T_{1,3,3}$, $T_{2,2,5}$, $T_{2,3,5}$, $T_{2,4,4}$ and $T_{3,3,4}$ are edge-reflexive. Since any other cubic theta graph can be obtained from one of these by subdividing edges, Lemma 5.3.4 implies that they are all edge-reflexive. □

5.5 Concluding Remarks

At this point, we have not only established that the triangle-free, connected, cubic, outer-planar graphs are edge-reflexive, but have gone considerably further. That the cubic theta graphs are edge-reflexive shows us that internal vertices are not an insuperable obstacle to edge-reflexivity for planar graphs. Moreover, the line graph of $K_{3,3}$ is a non-planar reflexive graph. We also proved in Theorem 4.2.2 that the even circular ladders (a class of graphs without any half-edges) are edge-reflexive, while the cubic cycles C_n, where $n \geq 4$ provide us with examples of edge-reflexive graphs with n half-edges. Of course, the cubic vertex is an edge-reflexive graph with three half-edges, and, by the Parity Lemma (Lemma 5.4.2) no edge-reflexive cubic graph can have precisely 1 or 2 half-edges.

Perhaps even more importantly, Corollary 5.2.2 establishes that we may reduce the study of a connected, edge-reflexive graph G to examining all the graphs obtained from G by cutting all the cut-edges of G; and Lemma 5.3.4 establishes conditions under which subdividing an edge of G preserves the edge-reflexivity of G. Together, these results offer us a deeper understanding of just what can be said about the edge-reflexive cubic graphs, while pointing us in the direction of future problems.

Are there circumstances under which subdividing every edge of an edge-reflexive graph G preserves edge-reflexivity? Can we generalize our results on cubic theta graphs to graphs where we identify the ends of four or more paths? This area remains wide-open, and, despite making considerable advances, we have only scratched its surface.

Chapter 6

Very Colourful Graphs

In this chapter, we explore some differences between the 4-colouring complex of a near-triangulation of the plane T and the 3-colouring complex of the line graph X of its inner dual (a cubic graph) G. In Section 4.3, we observed that certain families of reflexive near-triangulations of the plane correspond to families of edge-reflexive, 3-edge-colourable cubic graphs through the inner dual operation. Moreover, as Theorem 4.3.5 illustrates, the inner dual operation provides a natural correspondence between the 3-edge-colourings of cubic graphs and the 4-colourings of planar near-triangulations. However, studying the 4-colouring complexes of triangulations of the plane through the lens of the line graphs of cubic graphs has some serious limitations.

Firstly, a pair of 4-colourings may be adjacent in $B_4(T)$, despite there existing no sequence of Kempe changes transforming one of these 4-colourings into the other. Recall from Chapter 2 that we say two colourings are adjacent if they share at least one colour class. We have already seen a good example of this phenomenon in Figure 3.6. In Figure 3.6, the 4-colourings $\{\{1,7,9\},\{2,4,10,12\},\{3,5,11\},\{6,8\}\}$ and $\{\{1,8,9\},\{2,4,10,12\},\{3,6,11\},\{5,7\}\}$ share the colour class $\{2,4,10,12\}$, so they are adjacent, but there is no Kempe change transforming one of these colourings into the other colouring. Another good example is the colouring complex of the Icosahedron, which is drawn in Figure 3.4. Its 4-colouring complex is connected, but no pair of its ten 4-colourings are Kempe equivalent. By contrast, connectedness in $B_3(X)$ (where X is the line graph of a cubic graph) is entirely determined by the Kempe structure of X.

Further, we find it illuminating to examine the structure of very colourful graphs. We say that a graph is *very colourful* (for k-colourings) if, for any pair of non-adjacent vertices $u, v \in V(G)$, there exists a k-colouring f of G such that $f(u) = f(v)$. In Section 6.1, we examine very colourful graphs for 3-colourings, while very colourful graphs for 4-colourings are studied in Section 6.2. This property is useful in studying the 3-colouring complex of the line graph of a cubic graph. We will establish that very colourful graphs are colourful (for the line graphs of cubic graphs with no half-edges) while the converse is false. We also prove that, if X is the line graph of a cubic graph G, then the 3-colouring complex $B_3(X)$ of X must be very colourful.

As checking that a graph is very colourful involves no more computation than checking that a graph is colourful, while there are fewer very colourful graphs then colourful graphs, this property is frequently useful in shortening lengthy computations. For a comparison of the frequency of very colourful graphs to the frequency of colourful graphs, see Tables 6.1 and 6.2. There are also more theoretical applications, such as the short proof we will offer that a 3-colouring complex which is also the line graph of a cubic graph (in which each vertex is incident with at most one half-edge) cannot have 1, 2 or 3 half-edges.

Table 6.1: The frequency of colourful and very colourful graphs among the line graphs of 2-connected, triangle-free, cubic graphs G with no half edges

$\|V(G)\|$	**Very Colourful**	**Colourful**	**Number of Graphs**
8	1	1	2
10	1	2	6
12	3	5	22
14	7	22	109
16	24	178	788
18	105	2649	7772
20	884	48045	97292

Table 6.2: The frequency of colourful and very colourful graphs among the line graphs of 2-connected, triangle-free, subcubic graphs without half-edges G

$\|V(G)\|$	**Very Colourful**	**Colourful**	**Number of Graphs**
8	10	16	23
9	17	30	48
10	56	107	148
11	135	259	399
12	487	1023	1339
13	1538	3111	4395
14	6006	13134	16183

Unfortunately, in the context of triangulations of the plane, some of the results which make the notion of a very colourful graph useful for 3-colouring complexes break down. As a consequence, the notion of a very colourful graph has far less utility when studying the 4-colouring complex of a planar near-triangulation. This difference concretely illustrates the additional complexities which arise in moving from studying 3-colouring complexes to examining 4-colouring complexes.

6.1 Cubic Graphs and their Edge-Colourings

In this section, when we refer to a graph $X = L(G)$, we mean the line graph of a cubic graph G (which may contain half-edges). The name 'very colourful' was motivated by the observation that the very colourful line graphs of cubic graphs are strictly contained within the set of all colourful line graphs of cubic graphs (with one exception).

Theorem 6.1.1. *Let G be a cubic graph with no half-edges. Moreover, suppose that G does not contain the complete graph on four vertices as a subgraph. If $X = L(G)$ is very colourful with respect to 3-colourings, then X is colourful with respect to 3-colourings.*

Proof. Fix an arbitrary edge $a \in E(G)$. We will show that we can always arrange for a to be coloured differently from any other non-incident edge $b \in E(G)$ (obviously, a is coloured differently from any other incident edge b in any 3-edge-colouring, so we don't consider these edges). Notice that b is incident with four edges, none of which are a. If a and some edge incident with b (call it c) are not incident, then there is a 3-edge-colouring f of G which colours a and c the same colour (since $L(G)$ is very colourful). Thus, $f(a) \neq f(b)$. As G is cubic and has no multiedges, we can only have all four of the edges incident with b also incident with a if G is isomorphic to the complete graph on four vertices. □

This result naturally extends to cubic graphs with half-edges whenever G is triangle-free.

Theorem 6.1.2. *Let G be a 3-edge-colourable, triangle-free, cubic graph. If $X = L(G)$ is very colourful, then X is colourful.*

Proof. Again, we fix $a \in E(G)$ and consider a non-incident edge $b \in E(G)$. If a and some neighbour of b (call it c) are not incident, then there is a 3-edge-colouring f of G such that $f(a) = f(c)$ (since $L(G)$ is very colourful). Thus, $f(a) \neq f(b)$, as required.

Otherwise, all of the neighbours of b must be incident with a. In the event that b is a full edge, it immediately follows that each endpoint of b can have at most degree two (otherwise, G would contain a triangle). If both endpoints have degree two, then a must be a full edge, as the two edges incident with b cannot be incident (otherwise, G would contain a triangle). In this case, as $L(G)$ is very colourful, there exists a 3-edge-colouring of G which colours the two edges incident with b the same colour. Otherwise, the single edge incident with b clearly only has one colour, or b is incident with no other edges. Consequently, in each of these three cases, we can arrange for edges a and b to be coloured differently by performing a Kempe change (if needed) on $\{b\}$ which fixes the colour of all the edges of G which are incident with b (there are 0, 1 or 2 such edges).

When b is a half-edge, we see again that the endpoint of b must have degree at most two (otherwise, a triangle or multiedge would be formed). As before, we can then change the colour of b so that it differs from the colour of a through a Kempe change, if needed. □

Our claim that the set of very colourful line graphs of cubic graphs (without half-edges) lies strictly within the set of colourful line graphs of cubic graphs (with one exception) can now be established by exhibiting a cubic graph without any half-edges whose line graph is colourful, but not very colourful. Such a graph is drawn in Figure 6.1, and this example also shows that the containment in Theorem 6.1.2 is strict. The edges labelled 1 and 2 must be coloured differently in any 3-edge-colouring. This (and the claim that the graph is edge-colourful) can be checked by listing each of the four unlabelled 3-edge-colourings of the graph.

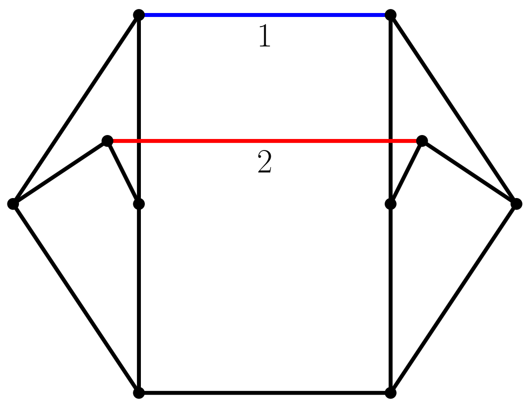


Figure 6.1: A cubic graph whose line graph is colourful, but not very colourful

We also note here that if a subcubic graph contains a triangle, then its line graph may be very colourful without being colourful. A small example is drawn in Figure 6.2. The green edges of this graph (which are labelled with 1 and 2) must be coloured the same colour in any 3-edge-colouring. However, such graphs lie outside of our main area of interest.

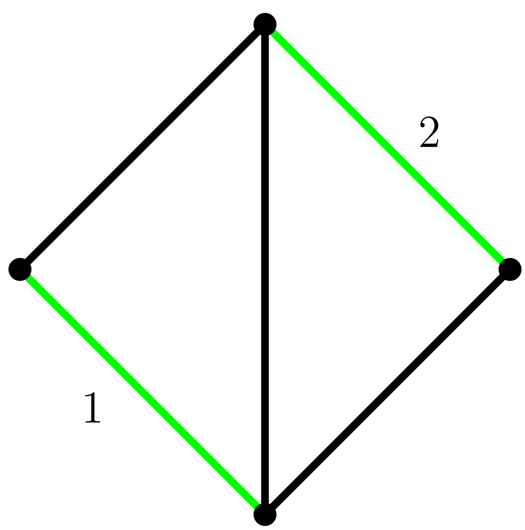


Figure 6.2: A subcubic graph whose line graph is very colourful, but not colourful

For our purpose, the most interesting class of very colourful graphs is certainly the set of 3-colouring complexes outlined in our next result.

Theorem 6.1.3. *Let X be a 3-colourable graph, where each edge of X is in at least one triangle, and X has no isolated vertices. Then its 3-colouring complex $B_3(X)$ is very colourful.*

Proof. We will prove the contrapositive. Suppose that there exists a pair of colour classes C and D of some graph X (in which every edge is contained in some triangle and without isolated vertices) such that C and D are coloured differently in all 3-colourings of $B_3(X)$ of the form $\{\phi_X(a), \phi_X(b), \phi_X(c)\}$, where a, b and c are vertices forming some triangle T of X. By the same argument as we used in the proof of Lemma 2.1.2, $\{\phi_X(a), \phi_X(b), \phi_X(c)\}$ is a 3-colouring of $B_3(X)$ for each triangle abc of X. Thus, for each triangle T of X, exactly one vertex in T must be in C and exactly one vertex in T must be in D. Hence, $V(X)\setminus(C\cup D)$ is an independent set (since each edge of X is contained in a triangle) from which it follows that $\{C, D, (C \cup D)^c\}$ is a 3-colouring of X. Consequently, C and D are adjacent in $B_3(X)$. $\square$

Fisk proved in [20] that a k-colouring complex $B_k(X)$ (where each each edge of X is in at least one k-clique, and X has no isolated vertices) is colourful, so perhaps Theorem 6.1.3 should not be a surprise.

Theorem 6.1.4 (Fisk [20])**.** *Let X be a k-colourable graph, where each each edge of X is in at least one k-clique, and X has no isolated vertices. Then the k-colouring complex $B_k(X)$ is a colourful graph.*

Proof. Let C and D be distinct vertices of $B_k(X)$ (that is to say, distinct colour classes of X). As each vertex $v \in V(X)$ is in some k-clique, C and D must differ on some vertex v in a k-clique of X. As this is a k-clique, each of its vertices must be coloured a different colour, so there exists some vertex $u_C \in C \setminus D$ and some vertex $u_D \in D \setminus C$. Now, consider a k-clique of X which contains u_C. Call this clique $q_1 q_2 \ldots q_k$. Then $\{\phi_X(q_1), \phi_X(q_2), \ldots, \phi_X(q_k)\}$ is a k-colouring of $B_k(X)$ which colours the vertices C and D differently. $\square$

One consequence of Theorem 6.1.4 is that, if $Y = B_k(X)$ (where every edge of X is in at least one k-clique, and X has no isolated vertices) then, by Observation 2.3.1, $B_k^2(Y)$ contains an isomorphic copy of Y as a subgraph. Theorem 6.1.3 allows us to derive an interesting strengthening of this result for 3-colouring complexes.

Corollary 6.1.5. *Let $Y = B_3(X)$ be the 3-colouring complex of some graph X, where every edge of X is in at least one triangle, and X has no isolated vertices. Then $B_3^2(Y)$ contains an isomorphic copy Y' of Y as an induced subgraph.*

Proof. It suffices to establish that, for any two non-adjacent vertices x and y of Y, $\phi_Y(x) \cap \phi_Y(y) \neq \emptyset$, as it then follows that $\{\phi_Y(x), \phi_Y(y), A\}$ cannot be a 3-colouring of $B_3(Y)$ for any choice of A. In that case, since Y is colourful, $B_3^2(Y)$ contains an isomorphic copy Y' of Y as a subgraph with vertices $V(Y') = \{\phi_Y(x) : x \in V(Y)\}$ and edges $\{\phi_Y(x)\phi_Y(y) : xy \in E(Y)\}$. Moreover, as $\phi_Y(x)$ is not adjacent to $\phi_Y(y)$ for any two non-adjacent vertices x and y of Y, $\{\phi_Y(x)\phi_Y(y) : xy \in E(Y)\}$ must be a complete listing of the edges of $B_3^2(Y)[V(Y')]$.

It remains to prove our initial claim. So, consider two non-adjacent vertices x and y of Y. By Theorem 6.1.3, there exists some colouring c of Y such that $c(x) = c(y)$. Hence, $\phi_Y(x) \cap \phi_Y(y) \neq \emptyset$, as required. $\square$

In light of Theorem 6.1.3 and Theorem 6.1.4, we can also characterize the 3-colouring complexes $Y = B_3(X)$ (where every edge of X is in at least one triangle, and X has no isolated vertices) as follows: if Y is such a 3-colouring complex, then, for any pair of vertices $u, v \in V(Y)$, we may identify u and v or join u and v with an edge without increasing the chromatic number. The following theorem is another straightforward consequence.

Theorem 6.1.6. *Let G be a cubic graph in which each vertex is incident with at most one half-edge. Moreover, suppose that G has either 1, 2 or 3 half-edges. Then $Y = L(G)$ is not the 3-colouring complex $B_3(X)$ of any graph X which has no isolated vertices, and in which edge of X is in at least one triangle.*

Proof. For a contradiction, suppose that the line graph $Y = L(G)$ of some cubic graph G with either 1, 2 or 3 half-edges is a 3-colouring complex $Y = B_3(X)$ for an X as described in Theorem 6.1.6. By Lemma 2.1.2, G must be 3-edge-colourable. If G has only one half-edge, then we saw in chapter 5 that G is not 3-edge colourable by the Parity Lemma (Lemma 5.4.2), a contradiction.

If G has two half-edges u and v, then, again by the Parity Lemma, u and v must be coloured the same colour in any 3-edge-colouring of G. Thus, G is not edge-colourful, so $L(G)$ is not the 3-colouring complex $B_3(X)$ of any graph X without isolated vertices, and in which every edge of X is in at least one triangle, by Theorem 6.1.4. This is another contradiction.

Finally, if G has three non-incident half-edges u, v and w, then, by the Parity Lemma, u, v and w must each be coloured a different colour in any 3-edge-colouring of G. Consequently, $L(G)$ is not very colourful. By Theorem 6.1.3, it follows that $L(G)$ is not the 3-colouring complex $B_3(X)$ of any graph X without isolated vertices, and in which every edge of X is in at least one triangle, so we obtain a contradiction in this last case as well. $\square$

6.2 Near-Triangulations of the Plane

We can obtain natural analogues of our results regarding very colourful line graphs of cubic graphs and 3-colouring complexes in terms of near-triangulations of the plane and their 4-colouring complexes. In particular, we still find that the very colourful triangulations of the plane must be colourful under fairly weak assumptions.

Theorem 6.2.1. *Let T be a near-triangulation of the plane. Moreover, suppose that for every pair $u, v \in V(T)$, $N(u) \nsubseteq N(v)$. If T is very colourful, then T is colourful.*

Proof. Fix an arbitrary vertex $v \in V(T)$. We will show that we can always arrange for v to be coloured differently from any other non-adjacent vertex $u \in V(T)$. Observe that $N(u) \setminus N(v) \neq \emptyset$, so we can choose a vertex $w \in N(u) \setminus N(v)$. As T is very colourful, there exists a 4-colouring f of T which colours w and v identically, unless $w \in N(v)$. However, $w \notin N(v)$, so f colours u and v differently. $\square$

However, while we found that the 3-colouring complex of the line graph of a cubic graph must be very colourful in Theorem 6.1.3, it is not the case that the 4-colouring complex of a triangulation of the plane must be a very colourful graph. The smallest example of a triangulation of the plane T such that $B_4(T)$ is not very colourful is given below.

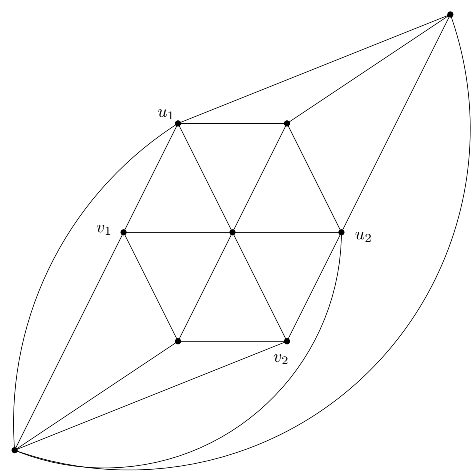


Figure 6.3: A triangulation of the plane T such that $B_4(T)$ is not very colourful

By listing all the 4-colourings of $B_4(T)$, it can be verified that $\{v_1, v_2\}$ and $\{u_1, u_2\}$ are a pair of vertices of $B_4(T)$ (colour classes of T) which must be coloured differently in every 4-colouring of $B_4(T)$. However, they are not adjacent in $B_4(T)$, since $T[V(T) \setminus (\{v_1, v_2\} \cup \{u_1, u_2\})]$ is a 5-cycle. Thus, $B_4(T)$ is not very colourful.

As a consequence, the computational utility of this 'very colourful' notion is significantly reduced when studying 4-colouring complexes. It is also not clear that there exist useful 4-colouring complex analogues of Corollary 6.1.5 or the characterization of 3-colouring complexes which we used to prove Theorem 6.1.6.

6.3 Concluding Remarks

Evidently, while a useful notion in studying 3-colouring complexes, very colourful graphs have considerably less utility when we examine 4-colouring complexes, even when we restrict our attention to near-triangulations of the plane. In both of these cases, very colourful graphs must automatically be colourful under fairly weak assumptions. However, the fact that 3-colouring complexes are very colourful under weak assumptions, which has both computational and theoretical applications, does not extend to the 4-colouring complexes. This challenge illustrates one of the many complexities in moving from the realm of 3-colouring complexes, where adjacency is simply a matter of Kempe equivalence, to the more difficult study of 4-colouring complexes.

Chapter 7

Open Problems and Future Research

7.1 Extensions of Tutte's Question

There are a number of interesting generalizations of Tutte's question (see Chapter 3 for the details of Tutte's original problem) which remain unaddressed in this thesis. Do there exist 4-connected triangulations of the plane whose 4-colouring complexes have arbitrarily many components of each parity? Do 5-connected triangulations of the plane exist whose 4-colouring complexes have arbitrarily many components? Of course, there is also the specific conjecture we discussed in Chapter 3 itself.

Conjecture 3.4.1. *Suppose that T is a triangulation of the plane, and that its 4-colouring complex $B_4(T)$ has at least two components. Then $B_4(T)$ has a component of even parity and a component of odd parity.*

We have checked this conjecture up to 13 vertices using SageMath. For the reader who wishes to confirm this for themselves, a copy of our code is available in Appendix A.

Finally, it may be of interest to consider Tutte's problem on surfaces of higher genus, such as the torus. Are examples still relatively rare in this context, or are colouring complexes with many components far more common?

7.2 Reflexive Graphs

Just as there remain many interesting problems relating to the number of connected components of 4-colouring complexes, providing a characterization of the edge-reflexive cubic graphs or the reflexive triangulations of the plane remains wide open. While we made some progress in this direction in chapters 4, 5 and 6, even more questions remain unaddressed. In this section, we first summarize these questions and some suggestive computational evidence. Then we briefly discuss two other closely related problems Fisk raised in [20].

In Section 5.4, we discussed how subdividing an edge in a cubic graph effected its edge-reflexivity. Under certain circumstances, edge-reflexivity was preserved, but we also noted that if we subdivide a single edge of $K_{3,3}$, the result is not an edge-reflexive graph. However, the result of subdividing each edge of $K_{3,3}$ once is edge-reflexive (this can be checked with a computer). Indeed, while the result of subdividing each edge of K_4 once is not an edge-reflexive graph, the results of performing this operation on each of the cubic graphs (with no half-edges) on six vertices is edge-reflexive, as are the results of performing this operation on each of the cubic graphs without half-edges on eight vertices. While not enough evidence for a conjecture, this observation does suggest the following question.

Question 7.2.1. *Suppose that G is a connected cubic graph with no half-edges and of order at least 6. Let H be the graph which results from subdividing each edge of G once. Do there exist any such graphs H which are not edge-reflexive?*

Another class of potentially edge-reflexive cubic graphs we consider particularly interesting are the fusenes (also known as *hexagonal graphs*). We say that G is a *fusene* if G is a 2-connected plane graph, in which every interior face is a hexagon, all vertices of G have degree three (after adding half-edges) and only vertices on the boundary of the outer face are permitted to be incident with half-edges.

Question 7.2.2. *Do there exist any fusenes that are not edge-reflexive?*

We are not aware of any examples, and have confirmed that none exist with nine or fewer hexagonal faces. Code for this lengthy computation is included in Appendix A. Given an edge-reflexive fusene graph, we can often obtain an infinite family of edge-reflexive fusenes by using the operation of adding a 4-cycle followed by two subdivisions. However, not all fusenes are obtained in this way.

Finally, we provide a few interesting examples of edge-reflexive graphs which do not fit into the classes we have described in the preceding paragraphs in hopes that they might provide inspiration for additional research.

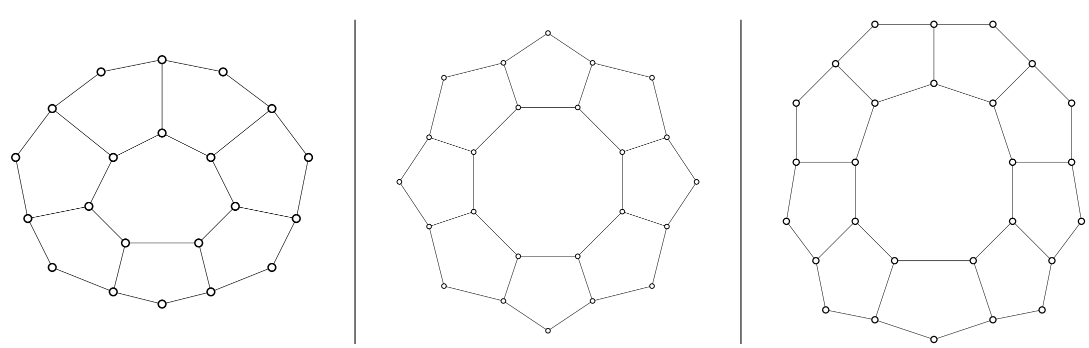


Figure 7.1: A variation on circular ladders, where the 4-cycles in the ladders are replaced with 5-cycles. These three cubic graphs have reflexive line graphs.

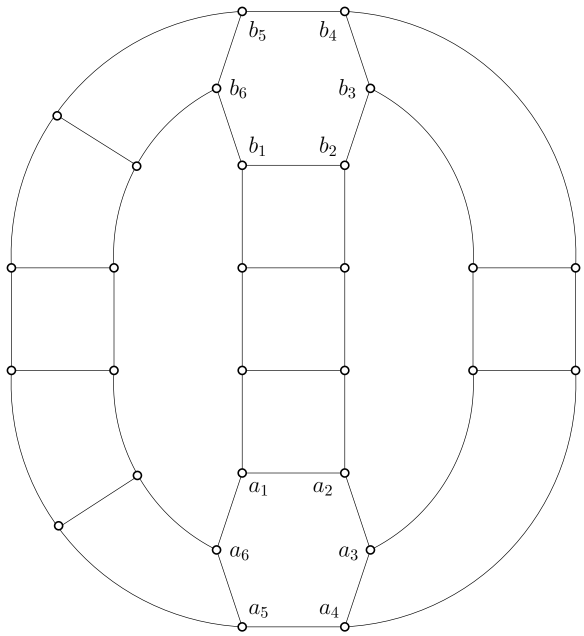


Figure 7.2: A cubic graph constructed by attaching three ladders to a pair of 6-cycles. Many graphs G of this form satisfy $L(G) \cong B_3^2(L(G))$.

In fact, we suspect that the graph in Figure 7.2 is one of the smallest examples in an infinite family of edge-reflexive cubic graphs with no half-edges. We will call a graph a *theta ladder* $TL(l, m, n)$ if it is constructed as follows. Begin with two 6-cycles $a_1a_2a_3a_4a_5a_6$ and $b_1b_2b_3b_4b_5b_6$. Now, join the edge a_1a_2 to the edge b_1b_2 with a ladder of length l. Similarly, the edges a_3a_4 and b_3b_4 are connected by a ladder of length m, while a_5a_6 and b_5b_6 are connected by a ladder of length n. The example given in Figure 7.2 is the theta ladder $TL(3, 3, 5)$.

Using a computer, we have confirmed that $TL(1, 1, 1)$, $TL(1, 3, 3)$, $TL(1, 3, 5)$, $TL(1, 3, 7)$, $TL(1, 3, 9)$, $TL(1, 5, 5)$, $TL(1, 5, 7)$, $TL(3, 3, 3)$, $TL(3, 3, 5)$, $TL(3, 3, 7)$ and $TL(3, 5, 5)$ are edge-reflexive. Those $TL(l, m, n)$ for which at least one parameter is even and $l+m+n \leq 13$ were found to not be edge-reflexive, and the same holds for $TL(1, 1, 3)$ (the only odd-odd-odd exception). Based on this evidence, we ask the following question.

Question 7.2.3. (a) *Do there exist any theta ladder graphs $TL(l, m, n)$, where $l, m, n \geq 3$ are all odd, that are not edge-reflexive?*

(b) *Do there exist any theta ladder graphs $TL(l, m, n)$, where l is even, that are edge-reflexive?*

Before proceeding onto other subjects, we note two variations on the notion of reflexivity which were introduced by Fisk. Firstly, we will say that a graph X is *self-dual* if $B_k(X) \cong X$. Examples of graphs with this property appear rare, although the triangle and the line graph of the cubic cycle C_5 are examples of 3-colouring complexes with this property. Additionally, Fisk noted that some graphs X had colouring complexes isomorphic to the disjoint union of X with itself (ie. $B_k(X) \cong X \cup X$). We noted that $L(K_{3,3}) \cup L(K_{3,3})$ has this property in Section 5.1. Another example (originally discovered by Fisk) is the line graph of the Coxeter graph. Progress on either of these problems would be interesting.

7.3 Very Colourful Graphs and Too Colourful Graphs

In Chapter 6, we saw that the notion of a very colourful graph appears to be an interesting strengthening of the idea of a colourful graph, particularly in the context of 3-edge-colouring cubic graphs. In particular, we know that 3-colouring complexes satisfying fairly weak conditions have this property (by Theorem 6.1.3) so examining the very colourful line graphs of cubic graphs more closely could lead us towards an interesting partial characterization of the 3-colouring complexes. Perhaps unsurprisingly, it appears that the 2-connected, 3-edge-colourable, subcubic graphs of large girth may have very colourful line graphs. One notable example of such a graph is the famous Tutte-Coxeter graph (the smallest cubic graph without half-edges of girth 8), which we can show has a very colourful line graph using a computer.

Table 7.1: 2-connected, subcubic graphs without half-edges and with girth > 6 which have very colourful line graphs

$\lvert V(G)\rvert$	**Number of all graphs**	**Number of v.c. graphs**
10	2	2
11	4	4
12	8	8
13	16	16
14	39	39
15	102	102
16	300	300

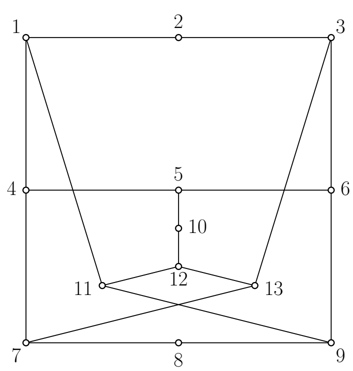


Figure 7.3: A 2-connected, subcubic graph of girth six which does not have a very colourful line graph

Conjecture 7.3.1. *Suppose that G is a 2-connected, 3-edge-colourable, subcubic graph (without half-edges) of girth at least seven. Then $L(G)$ is very colourful.*

Another approach to strengthening the notion of a colourful graph is to call a graph X *too colourful* if, for any trio of distinct, non-adjacent vertices $u, v, w \in V(X)$, there exists a 3-colouring such that $c(u) \neq c(v)$, $c(u) \neq c(w)$ and $c(v) \neq c(w)$. We assume here that $u, v, w \in V(X)$ are non-adjacent vertices, as, otherwise, it immediately follows that $X = L(G)$ will not have this property for any non-trivial line graph of a 3-edge-colourable cubic graph. In particular, if G contains any full edge $e = uv$, u is incident with edges e_1 and e_2, and v is incident with edges, e_3 and e_4, then no 3-edge-colouring is possible which satisfies $c(e_1) \neq c(e_3)$ and $c(e_2) \neq c(e_3)$.

Cubic graphs without half-edges on 14, 16 and 18 vertices exist whose line graphs are too colourful. Of course, there are also simpler examples, if we relax the condition that our graph has no half-edges. For instance, it is easy to show that the line graphs of subcubic trees have this property.

Observation 7.3.1. *Let G be a cubic tree with at least three edges which are pairwise non-incident. Then $X = L(G)$ is too colourful.*

Proof. Select three pairwise non-incident edges u, v and w from G. As G is a tree, there exists a unique path between each pair of these edges. Consider the longest of these paths. Without loss of generality, assume that this path begins at u, reaches v and then ends at w. Then we can colour u with colour 1, alternate between colours 1 and 2 until reaching v, colour v with 3, alternate between colours 3 and 1 until reaching w, and then colour w with 2. 3-colouring the remaining edges is trivial. □

Furthermore, suppose that G is a 3-edge-colourable, connected, triangle-free cubic graph with no half-edges. Under these conditions, if $X = L(G)$ is too colourful, then X is also very colourful.

Observation 7.3.2. *Let G be a 3-edge-colourable, connected, cubic graph with girth at least 4. If $X = L(G)$ is too colourful, then X is very colourful.*

Proof. Fix an arbitrary edge $a \in E(G)$. We want to show that we can always arrange for a to be coloured the same colour as a given, non-incident edge b. As G is cubic, b is incident with precisely four edges. Two of these edges (e_1 and e_2) are incident with the endpoint v of b and the other two (f_1 and f_2) are incident with the endpoint u of b.

As G is triangle-free, at most one edge incident with each of u and v is incident with a. Thus, there exist edges e_i and f_j ($i, j \in \{1, 2\}$) which are incident with b and not incident with a. Moreover, e_i and f_j are not incident, since G is triangle-free. Consequently, as G is too colourful (while a, e_i and f_j are pairwise not incident) there must be some 3-edge-colouring of G in which a, e_i and f_j are each coloured different colours. It immediately follows that a and b are coloured the same colour in this colouring. □

Unfortunately, at least among the line graphs of 3-edge-colourable cubic graphs without half-edges, too colourful graphs are rare. We can easily show that a 3-edge-colourable cubic graph G with no half-edges must be cyclically 4-edge-connected, in order for $X = L(G)$ to be too colourful. A graph G is said to be *cyclically k-edge-connected* if at least k edges must be removed from G in order to disconnect G into two components, each containing a cycle. Such a collection of edges is called a *cyclic k-edge cut*.

In order to prove this result, we need a version of the Parity Lemma for edge cuts. Let $U \subseteq V(G)$ and let $\delta(U)$ be the set of edges with precisely one endvertex in U (notice that half-edges incident with vertices of U are in $\delta(U)$). The set $\delta(U)$ is then called an *edge cut* if U is a proper nonempty subset of V.

Lemma 7.3.3. *Let G be a cubic graph which has been 3-edge-coloured. Let $U \subseteq V(G)$, and let $\delta(U)$ be the set of edges with precisely one endvertex in U. Let n_1, n_2 and n_3 be the number of edges of $\delta(U)$ which are contained in each of the three colour classes. Then n_1, n_2 and n_3 are congruent modulo 2.*

Proof. Let C_i be the colour class containing n_i edges from $\delta(U)$. Then the number of edges in $C_i \cap \delta(U)$ is equal to the number of vertices in U, minus twice the number of edges of $E(G) \cap C_i$ which have both endvertices in U. Consequently, $|U| \equiv n_1 \equiv n_2 \equiv n_3 \pmod 2$. □

Lemma 7.3.4. *Let G be a 3-edge-colourable, connected, cubic graph with no half edges. Furthermore, suppose that $X = L(G)$ is too colourful. Then G is cyclically 4-edge-connected.*

Proof. For a contradiction, suppose that G contains a cyclic k-edge cut. If $k = 1$, then the existence of a 1-edge cut contradicts Lemma 7.3.3. If $k = 2$, then the two edges in the cut K must be coloured the same colour by Lemma 7.3.3. As $L(G)$ is too colourful, it follows that every edge of G must be incident with one of those two edges. However, as G must also be cubic, this does not leave enough freedom to construct any cycles in $G - K$. In particular, it is not possible to construct a cubic cycle subject to the condition that every edge in the cycle must be incident with k chosen vertices for $k \leq 2$.

If $k = 3$, then the three edges in the cut K must each be coloured a different colour by Lemma 7.3.3. Thus, $L(G)$ is not very colourful, unless all three edges of K are pairwise incident. However, the three edges in K cannot be pairwise incident, since G is cubic, and each component of $G - K$ must contain a cycle. So, by Observation 7.3.2, it follows that G must contain a triangle abc. Let e_a be the third edge incident with the vertex a. Then the edges bc and e_a must be coloured the same colour, so, provided there exists some edge $e \in E(G)$ which is not incident with either bc or e_a, $L(G)$ is not too colourful, a contradiction. As G is cubic, a graph for which all edges $e \in E(G)$ must be incident with either bc or e_a can have at most eight edges, but this is too few edges for a cyclic 3-edge cut to exist. □

If a graph G satisfying the hypotheses of Lemma 7.3.4 has a cyclic 4-edge cut K, then either all edges of K must be in the same colour class, or two edges of K must be in one colour class and two edges of K in another colour class by Lemma 7.3.3. In either case, if K contains a matching of cardinality 3, it follows that $L(G)$ is not too colourful, a contradiction. However, if some edges of K are incident, then it might be possible to construct a cyclic 4-edge cut.

Based upon an extensive computation ($|V(G)| \leq 20$) it appears that the only 3-edge-colourable, connected, cubic graphs without half-edges, whose line graphs are too colourful, may be the cyclically 5-edge-connected, bicubic graphs. We feel that this problem merits investigation.

Question 7.3.5. *Suppose that G is a 3-edge-colourable, connected, cubic graph without half edges. Furthermore, assume that $X = L(G)$ is too colourful. Does it follow that G must be a cyclically 5-edge-connected, bipartite graph?*

Bibliography

[1] Kenneth I Appel and Wolfgang Haken. *Every planar map is four colorable*, volume 98. American Mathematical Soc., 1989.

[2] Arash Asadi, Zdeněk Dvořák, Luke Postle, and Robin Thomas. Sub-exponentially many 3-colorings of triangle-free planar graphs. *Journal of Combinatorial Theory, Series B*, 103(6):706 – 712, 2013. URL: `http://www.sciencedirect.com/science/article/pii/S0095895613000634`, `doi:https://doi.org/10.1016/j.jctb.2013.09.001`.

[3] Yves Aubry, Jean-Christophe Godin, and Olivier Togni. Free choosability of outerplanar graphs. *Graphs and Combinatorics*, 32(3):851–859, 2016.

[4] Julie Beier, Janet Fierson, Ruth Haas, Heather M Russell, and Kara Shavo. Classifying coloring graphs. *Discrete Mathematics*, 339(8):2100–2112, 2016.

[5] S.M. Belcastro and R. Haas. Counting edge-Kempe-equivalence classes for 3-edge-colored cubic graphs. *Discrete Mathematics*, 325:77–84, 2014.

[6] S.M. Belcastro and R. Haas. Triangle-free uniquely 3-edge colorable cubic graphs. *ArXiv e-prints*, August 2015. `arXiv:1508.06934`.

[7] S.M. Belcastro and R. Haas. Edge-kempe-equivalence graphs of class-1 regular graphs. *Australasian Journal of Combinatorics*, 69:197–214, 01 2017.

[8] Norman Biggs. Pictures. In D.J.A. Welsh and D.R. Woodall, editors, *Combinatorics: proceedings of the Conference on Combinatorial Mathematics held at the Mathematical Institute, Oxford.*, pages 1–17. Institute of Mathematics and its Applications, Southend-on-Sea, 1972.

[9] Marthe Bonamy, Nicolas Bousquet, Carl Feghali, and Matthew Johnson. On a conjecture of Mohar concerning Kempe equivalence of regular graphs. *Journal of Combinatorial Theory, Series B*, 135:179–199, 2019.

[10] J.A. Bondy and U.S.R. Murty. *Graph Theory*. Springer Publishing Company, Incorporated, 1st edition, 2008.

[11] Luis Cereceda, Jan Van Den Heuvel, and Matthew Johnson. Connectedness of the graph of vertex-colourings. *Discrete Mathematics*, 308(5-6):913–919, 2008.

[12] V. Chvátal and J. Sichler. Chromatic automorphisms of graphs. *Journal of Combinatorial Theory, Series B*, 14(3):209–215, 1973. URL: `http://www.sciencedirect.com/science/article/pii/009589567390004X`, `doi:https://doi.org/10.1016/0095-8956(73)90004-X`.

[13] Harold S.M. Coxeter. Self-dual configurations and regular graphs. *Bulletin of the American Mathematical Society*, 56(5):413–455, 1950.

[14] Harold S.M. Coxeter. The four-color map problem, 1840-1890. *The Mathematics Teacher*, 52(4):283–289, 1959.

[15] Blanche Descartes. Network-colourings. *The Mathematical Gazette*, 32(299):67–69, 1948. URL: `http://www.jstor.org/stable/3610702`.

[16] Carl Feghali, Matthew Johnson, and Daniël Paulusma. Kempe equivalence of colourings of cubic graphs. *Electronic notes in discrete mathematics*, 49:243–249, 2015.

[17] Stanley Fiorini. On the chromatic index of outerplanar graphs. *Journal of Combinatorial Theory, Series B*, 18(1):35–38, 1975. URL: `http://www.sciencedirect.com/science/article/pii/009589567590060X`, `doi:https://doi.org/10.1016/0095-8956(75)90060-X`.

[18] Steve Fisk. Geometric coloring theory. *Advances in Mathematics*, 24(3):298–340, 1977.

[19] Steve Fisk. Variations on coloring, surfaces and higher-dimensional manifolds. *Advances in Mathematics*, 25(3):226–266, 1977. URL: `http://www.sciencedirect.com/science/article/pii/0001870877900755`, `doi:https://doi.org/10.1016/0001-8708(77)90075-5`.

[20] Steve Fisk. Cobordism and functoriality of colorings. *Advances in Mathematics*, 27:177–211, 1980.

[21] Steve Fisk. *Coloring Theories*. American Mathematical Society, Providence, R.I., 1989.

[22] Steve Fisk, Daniel Abbw-Jackson, and Dan Kleitman. Helly-type theorems about sets. *Discrete Mathematics*, 32(1):19–25, 1980.

[23] Andrew J Goodall, SD NOBLE, M Noy, et al. The tutte polynomial characterizes simple outerplanar graphs. *Combinatorics, Probability and Computing*, 20(4):609–616, 2011.

[24] Harald Gropp. VI.7 configurations. In Charles J. Colburn and Jeffrey H. Dinitz, editors, *Handbook of Combinatorial Designs, Discrete Mathematics and its Applications (Second ed.)*, pages 353–355. Chapman & Hall/CRC, Boca Raton, 2007.

[25] Jonathan L Gross and Jay Yellen. *Graph theory and its applications*. CRC press, 2005.

[26] P.J. Heawood. Map-colour theorem. *Quarterly Journal of Mathematics*, 24:332–339, 1890.

[27] Wilfried Imrich and Sandi Klavzar. *Product graphs: structure and recognition*. Wiley, 2000.

[28] Alfred B. Kempe. On the geographical problem of the four colours. *American Journal of Mathematics*, 2(3):193–200, 1879.

[29] Fiachra Knox, Bojan Mohar, and Nathan Singer. Reflexive coloring complexes for 3-edge-colorings of cubic graphs. *arXiv preprint arXiv:2004.06788*, 2020.

[30] Jessica McDonald, Bojan Mohar, and Diego Scheide. Kempe equivalence of edge-colorings in subcubic and subquartic graphs. *Journal of Graph theory*, 70(2):226–239, 2012.

[31] Bojan Mohar. Akempic triangulations with 4 odd vertices. *Discrete Mathematics*, 54:23–29, 1985.

[32] Bojan Mohar. *Kempe Equivalence of Colorings*, pages 287–297. Birkhäuser Basel, Basel, 2007. `doi:10.1007/978-3-7643-7400-6_22`.

[33] Bojan Mohar and Jesús Salas. A new Kempe invariant and the (non)-ergodicity of the wang–swendsen–koteckỳ algorithm. *Journal of Physics A: Mathematical and Theoretical*, 42(22):225204, 2009.

[34] Bojan Mohar and Nathan Singer. The last temptation of William T. Tutte. *European Journal of Combinatorics*, 2020. `doi:10.1016/j.ejc.2020.103221`.

[35] Bojan Mohar and Carsten Thomassen. *Graphs on surfaces*, volume 16. Johns Hopkins University Press Baltimore, 2001.

[36] Naomi Nishimura. Introduction to reconfiguration. *Algorithms*, 11(4):52, 2018.

[37] Neil Robertson, Paul Seymour, and Robin Thomas. Tutte's edge-colouring conjecture. *Journal of Combinatorial Theory, Series B*, 70(1):166–183, 1997.

[38] Thomas L. Saaty. Thirteen colorful variations on guthrie's four-color conjecture. *American Mathematical Monthly*, 79(1):2–43, 1972.

[39] Horst Sachs. Perfect matchings in hexagonal systems. *Combinatorica*, 4:89–99, 03 1984.

[40] Michael Stiebitz, Diego Scheide, Bjarne Toft, and Lene M. Favrholdt. *Graph edge coloring: Vizing's theorem and Goldberg's conjecture*, volume 75. John Wiley & Sons, 2012.

[41] Peter Tait. 10. remarks on the previous communication. *Proceedings of the Royal Society of Edinburgh*, 10:729–729, 1880. `doi:10.1017/S0370164600044643`.

[42] Carsten Thomassen. The number of k-colorings of a graph on a fixed surface. *Discrete Mathematics*, 306(23):3145–3153, 2006.

[43] Carsten Thomassen. Many 3-colorings of triangle-free planar graphs. *Journal of Combinatorial Theory, Series B*, 97(3):334 – 349, 2007. URL: `http://www.sciencedirect.com/science/article/pii/S0095895606000864`, `doi:https://doi.org/10.1016/j.jctb.2006.06.005`.

[44] William T. Tutte. A non-hamiltonian graph. *Canadian Mathematical Bulletin*, 3(1):1–5, 1960.

[45] William T. Tutte. On the algebraic theory of graph colorings. *Journal of Combinatorial Theory*, 1(1):15–50, 1966.

[46] William T. Tutte. Even and odd 4-colorings. In *Proof Techniques in Graph Theory*, pages 161–169. Academic Press, New York, 1969.

[47] William T. Tutte. Some comments on the four-colour problem. Available at: http://www.sfu.ca/~mohar/Notes/Tutte_CampinasBrazil_Notes.pdf, 1999. Speaker's notes at the "Workshop in Combinatorics and Discrete Structures — In honour of Prof. W. T. Tutte, UNICAMP, Campinas, SP, Brazil, August 2–4, 1999". URL: http://www.sfu.ca/~mohar/Notes/Tutte_CampinasBrazil_Notes.pdf.

[48] Hassler Whitney and William T. Tutte. Kempe chains and the four colour problem. In *Hassler Whitney Collected Papers*, pages 185–225. Springer, 1992.

Appendix A

Code

In this section, we provide the computer code used in a number of the computations discussed in this dissertation. These programs all use SageMath, a free, open-source mathematics software system.

As we performed an extremely large number of similar computations in the course of our research, this appendix will not provide the reader with a complete listing of all the programs we have written. Instead, we aim to present enough examples that the reader can fill in the missing details with a relatively modest effort.

In Section 3.4, we made the following conjecture based upon "a large number of computations." This conjecture was reiterated when we discussed directions for future research in Chapter 7. There we noted that we had checked all triangulations of the plane on thirteen or fewer vertices.

Conjecture 3.4.1. *Suppose that T is a triangulation of the plane, and that its 4-colouring complex $B_4(T)$ has at least two components. Then $B_4(T)$ has a component of even parity and a component of odd parity.*

The program written below checks all triangulations of the plane on thirteen vertices.

```
from sage.graphs.graph_coloring import all_graph_colorings
from sage.graphs.connectivity import connected_components_subgraphs

def B(G):
# A function which takes as input a graph G and returns its colouring
# complex B(G).
    parts = []
# We create an empty list to store all the simplices of B(G).
    for C in all_graph_colorings(G,4):
# We iterate over all the 4-colourings of G.
        parts.append([(Set(C[k])) for k in C])
# For each 4-colouring, we append to our list of 3-simplices the set
# of its 4 colour classes.
```

```
    X = SimplicialComplex(parts)
# We construct a simplicial complex with the sets in parts as our facets.
    X2 = X.graph()
# We take the 1-skeleton of X in order to obtain a graph.
    return X2

r = 0
m = 0
n = 0
l = 0
# We set a number of variables to 0.  We will use m to count the
# number of triangulations with B(T) connected, n for the number of
# triangulations where B(T) has two components and l for the number
# of triangulations where B(T) has at least 3 components.  We use r to
# count the number of counterexamples where B(T) has two components.

for T in graphs.triangulations(13, minimum_degree=3, minimum_connectivity=3):
# For each triangulation T on 13 vertices, we compute its 4-colouring
# complex B(T).
    B1 = B(T)
    if B1.is_connected():
        m += 1
# If B(T) is connected, increment m.
    else:
        if B1.connected_components_number() > 2:
            l += 1
            L = connected_components_subgraphs(B1)
            L2 = []
            for j in L:
                v = j.vertices()[0]
                degSum = 0
                for i in v:
                    degSum = degSum + T.degree(i)
                Par = 2 * v.cardinality() - degSum + T.order() - 3
                L2.append(Par)
            print(L2)
# If B(T) has at least three components, increment m, compute the parity
# of each component using Theorem 1.1.1, and then construct a list of
# these parities.
        else:
            n += 1
            L = connected_components_subgraphs(B1)
            L2 = []
            for j in L:
                v = j.vertices()[0]
                degSum = 0
                for i in v:
```

```
                    degSum = degSum + T.degree(i)
                Par = 2 * v.cardinality() - degSum + T.order() - 3
                L2.append(Par)
            s = sum(L2)
            if is_even(s):
                r+=1
                print(L2)
# If B(T) has two components, use Theorem 1.1.1 to check for a
# counterexample.  If there is a counterexample, increment r and
# list the parities of its components.

print "Done"
print(r)
print(l+n)
print(n+m+l)
# When the program has finished running, we print Done, the number of
# counterexamples T for which B(T) has two components, the number of
#T for which B(T) has at least two components and the total number of
#3-connected, planar triangulations T on thirteen vertices.
```

After a lengthy computation, this program outputs nine three element lists and ten four element lists. Each element is a pair of integers with different parities. Then it outputs 0 (the number of counterexamples found where $B_4(T)$ has two components) 8796 (the number of triangulations checked for which $B_4(T)$ has at least two components) and 49566 (the total number of 3-connected triangulations of the plane on thirteen vertices).

In order to determine which fusenes are edge-reflexive (as we discussed in Section 7.2) we need to do a bit more work.

```
from sage.graphs.graph_coloring import all_graph_colorings

def B(G):
# A function which takes as input a graph G and returns its colouring
# complex B(G).  This is the same function as in the previous
# example, except that we compute 3-colourings, rather
# than 4-colourings.
    parts = []
    for C in all_graph_colorings(G,3):
        parts.append([(Set(C[k])) for k in C])
    X = SimplicialComplex(parts)
    X2 = X.graph()
    X3 = X2.canonical_label()
    return X3

def Hat(G):
# A function which takes as input a graph and returns a
# graph in which all edges of the original graph now appear in a
```

```
# triangle.
    k = Set(G.vertices()).cardinality()
# k is the cardinality of the vertex set of G.
    for e in G.edges(labels=False):
# We iterate over each edge e of G.
        A = Set(G.neighbors(e[0])).intersection(Set(G.neighbors(e[1])))
# Set A to be the common neighbours of the ends of the edge e.
        if not A:
# If there are no such common neighbours, proceed.
            G.add_vertex(name=k)
            G.add_edge(e[0],k)
            G.add_edge(k,e[1])
# Add a new vertex with a label which is unused in the canonical
# labelling, and then add two edges to make the new vertex a
# common neighbour of the ends of e.
            k = k+1
# Now, increment k, so that we don't repeat labels.
    return G

n = 0
m = 0
# We set n and m equal to 0.  The variable n will count
# the number of fusenes with a reflexive line graph, while m
# will count the number of fusenes which do not have a
# reflexive line graph.

for g in graphs.fusenes(9):
    H = g.line_graph()
    H2 = Hat(H)
    G = H2.canonical_label()
# For each fusene g with 9 hexagonal faces, we take its line
# graph (using the hat operation to guarantee that every
# edge will be in a triangle).  The graph G is the resulting graph,
# given a canonical labelling.
    B1 = B(G)
    B2 = B(B1)
# Now, we apply B twice, in order to construct $B^{2}(G)$.
    A = G.clique_complex()
    A2 = A.facets()
    F1 = B2.clique_complex()
    F2 = F1.facets()
    if A2.cardinality() == F2.cardinality():
        n += 1
    else:
        m += 1
# Now, we use the injectivity of the map \phi_{G}.  As
# this map is injective, G is isomorphic to B^{2}(G)
```

```
# if and only if these two graphs have the same
# number of triangles.

print("Done")
print(m)
print(n+m)
# When the program has finished running, we print Done, the
# number of fusenes which do not have reflexive line graphs,
# and then the total number of fusenes checked.
```

In the end, this program outputs 0 (the number of fusenes checked whose line graphs are not reflexive) and 7036 (the total number of fusenes checked). This particular computation checked all fusenes with nine hexagonal faces.

Finally, the following computation determines the frequency of colourful and very colourful graphs among the line graphs of the 2-connected, cubic graphs G with sixteen vertices and no half-edges. Similar computations yield the data we displayed in tables 6.1, 6.2 and 7.1.

```
from sage.graphs.graph_coloring import all_graph_colorings

def Hat(G):
# A function which takes as input a graph and returns a
# graph in which all edges of the original graph now appear in a
# triangle.  This is the same function as in the previous example.
    k = Set(G.vertices()).cardinality()
    for e in G.edges(labels=False):
        A = Set(G.neighbors(e[0])).intersection(Set(G.neighbors(e[1])))
        if not A:
            G.add_vertex(name=k)
            G.add_edge(e[0],k)
            G.add_edge(k,e[1])
            k = k+1
    return G

def getColourings(G):
# This function makes a list of all the 3-colourings of G, formatting that
# list so the colourings can be easily processed.
    colourings = []
# We create an empty list to store all the 3-colourings of G.
    for C in all_graph_colorings(G,3):
        colourings.append([(Set(C[k])) for k in C])
# For each 3-colouring, we append it to our list of 3-colourings.  The
# colouring is stored as a set of colour classes.
    return colourings

def Colu(colourings,u):
# This function takes the list of 3-colourings outputted by the previous
```

```
# function, as well as a vertex u of G.  It returns the list of colour
# classes of G which contain u.
    classes1 = set()
# We create an empty set to fill with colour classes
    for D in colourings:
        for A in D:
            if u in A:
                classes1.add(A)
    classes2 = list(classes1)
# For each colour class A in each 3-colouring D of G, if u is in A, we add
# A to our set of colour classes.  We then reprocess the resulting set as
# a list.
    return classes2

def colourful(G):
# This function takes a graph G as input and returns either the string
# colourful, if G is colourful, or False, if G is not colourful.
    vertices = G.vertices()
    n = len(vertices)
    x = 0
    K = getColourings(G)
    for u in vertices:
        U = Colu(K,u)
# For each vertex u of G, we form a list U of all the colour classes
# containing u.
        for v in vertices:
            if (u != v):
                m = 0
                for i in range(len(U)):
                    if v not in U[i]:
                        m += 1
# For each vertex v of G, which is not equal to u, and colour class
# U[i] of G, if v is not in U[i], increment m
                if m > 0:
                    x += 1
# If we found any colour class U[i] which contained u, but not v,
# then m > 0, so increment x.

    if x == ((len(vertices)) ** 2) - n:
        return "colourful"
    else:
        return False
# If every pair of distinct vertices have some colouring in which
# they are coloured differently, then we return the string
# colourful.  Otherwise, we return False.

def veryColourful(G):
```

```
# This function takes a graph G as input and returns either the string
# veryColourful, if G is very colourful, or False, if G is not very colourful.
    vertices = G.vertices()
    n = len(vertices)
    x = 0
    K = getColourings(G)
    for u in vertices:
        U = Colu(K,u)
# For each vertex u of G, we form a list U of all the colour classes
# containing u.
        for v in vertices:
            if (u != v) and (v not in G.neighbors(u)):
                m1 = 0
                for i in range(len(U)):
                    if v in U[i]:
                        m1 += 1
# For each vertex v of G, which is not in the closed neighbourhood
# of u, and colour class U[i] of G, if v is in U[i], increment m1.
                if m1 == 0:
                    x += 1
# If no colour class U[i] contains v, then u and v are never coloured the
# same colour and m1 = 0, so we increment x.

    if x == 0:
        return "veryColourful"
    else:
        return False
# If x was ever incremented, then G is not very colourful and we return
# False.  If x never incremented, then x is very colourful and we return
# the string veryColourful.

k = 0
m = 0
n = 0
# We set three variables equal to 0.  The variable k will count
# the number of line graphs of 2-connected, cubic graphs which
# are colourful, n the number which are very colourful and
# m the number which are not very colourful.

for H in graphs.nauty_geng("16 -d3 -D3 -t -C -l"):
# Notice here that we consider only triangle-free graphs H, since
# H cannot contain a triangle if its line graph is colourful.
    H2 = H.line_graph(labels = False)
    G = Hat(H2)
    G2 = G.canonical_label()
# We take a canonical labelling of the hat graph of the line graph
# of H.  This graph is G2.
```

```
    if colourful(G2) == "colourful":
        k += 1
    if veryColourful(G2) == "veryColourful":
        n += 1
    else:
        m += 1
# Finally, for each graph G2, we check whether or not it is colourful
# and whether or not it is very colourful.
print "Done"
print(k)
print(n)
print(n+m)
# Upon completion, this program prints the string Done, the number
# colourful graphs found, the number of very colourful graphs found
# and then the total number of graphs examined.
```

This program outputs 178, 24 and then 788. So, we conclude that there are 178 line graphs of 2-connected, cubic graphs (without half-edges) G on sixteen vertices which are colourful, as well as 24 which are very colourful. Both of these quantities are considerably smaller than the 788 2-connected, triangle-free, cubic graphs (without half-edges) G which we counted.