

Community College Remedial Algebra: The Search for an Alternative

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Abstract

This dissertation is an attempt to add to the research on community college mathematics education, and to examine one way in which content delivery might be adapted from other levels to teach community college remedial mathematics courses. In order to address these issues, I adapted the idea of Design-Based Research and the Teaching Experiment to an entire class of students in order to examine whether Realistic Mathematics Education (RME) would be adaptable to the community college environment. I produced activities that would highlight the concepts students in the class were required to learn, but delivered them as a fantasy narrative using the principles of RME. I analyzed student submissions from the tasks embedded in these activities to determine whether RME had been a successful means by which to deliver the content and found not only that students had learned as much by this delivery method as by lecture, but that they had developed a sense of meaning from the mathematics in the process. These results suggest not only that methods designed for one population can effectively be used for others, but that community college students will be at least as successful under such a modified model. While teachers have always modified the work of others for their own purposes, the results of the research done for this dissertation support the idea that such modification is appropriate and effective; more specifically, it suggests that the time and effort required to modify methods for use in alternative environments is worth the sacrifice, and I would recommend that instructors at every level explore the myriad ways by which content can be delivered.

Keywords: community college; design-based research; mathematics education; Realistic Mathematics Education; task design, teaching experiment

For Mum
who gave me strength

For Lauren
who gave me courage

For Blanche
who gave me wings

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“When he took time to help the man up the mountain, lo, he scaled it himself.”

Tibetan Proverb

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List of Acronyms

A	Analysis
AMATYC	American Mathematical Association of Two-Year Colleges
AP	Advanced Placement
BOLD	Blood Oxygen Level Development
CAA	Concept-Audience-Approach
C-CORE	Change at the Core
DBR	Design-Based Research
DBTE	Design-Based Teaching Experiment
DOE	U.S. Department of Education
IOWO	Instituut voor de ontwikkeling van wiskundeonderwijs
OCTAE	Office of Career, Technical and Adult Education (U. S. Department of Education)
PBL	Problem-based learning
RME	Realistic Mathematics Education
SCL	Student-Centered Learning
SFU	Simon Fraser University
V	Visualization
VA	Visualization/Analytical (model)
WCC	Whatcom Community College
WWU	Western Washington University

Glossary

adjunct (faculty)	<p>A synonym for “part-time” (faculty) (q.v.). Adjunct faculty may teach a course load equivalent to or more than that of a full-time member of a faculty; thus the term “part-time” is misleading.</p> <p>Synonym: Contingent (faculty)</p>
community college	<p>Replaces the term “junior college” (q.v.) for public institutions.</p> <p>As described on page 40, a community college is an institution which:</p> <ul style="list-style-type: none">• is post-secondary and post-compulsory;• is situated in a binary higher education system (i.e. there is (at least) one higher education option)• targets non-traditional students• offers programs with distinguishable length of curriculum• has a lower status in the eyes of the public and among scholars, and receives less governmental financial assistance than do universities• promotes a curricular emphasis• combines curricular competence with social instruction• is “open-access” (any student who applies will be accepted for admission)
concept image	<p>“...all the cognitive structure in the individual's mind that is associated with a given concept.” (Tall & Vinner, 1981)</p>
contingent (faculty)	<p>A synonym for “adjunct (faculty)” (q.v.) but without what some consider to be a derogatory connotation.</p>
developmental	<p>A description of the level of mathematics undertaken in courses designed to prepare students for college-level mathematics; courses in developmental mathematics teach content mandated in primary or secondary school and without which students are unlikely to succeed in courses which rely on it as a foundation.</p> <p>Synonyms: Remedial; Pre-College</p>
in-service teacher	<p>A practicing teacher of students at any level</p>

instructor	In this work, a general term that applies to a teacher; an instructor may be a tenured university or community college professor, an adjunct or part-time teacher, or a primary or secondary classroom teacher, among others. Here the term is used in its most general sense.
junior college	A now somewhat archaic term (except for private colleges) used to describe institutions that offer courses from the first two years of post-secondary education. See also “community college”.
junior high (school)	A U.S. model for schools in the middle years; generally grades 7, 8 and 9, replaced by the middle school (q.v.) model in the 1970s.
middle school	A U.S. model for schools in the middle years; generally grades 6, 7 and 8, although some districts do not include grade 6; roughly equivalent with “junior school” or “lower secondary” school.
part-time (faculty)	See “adjunct (faculty)”
pre-college	See “developmental”
pre-service teacher	A student of education who has not yet begun teaching practice
primary school	Elementary school; generally grades 1 through 6
remedial	See “developmental”
secondary school	High school; generally grades 9 through 12
Wiskobas	“Wiskobas is an acronym for the Dutch translation of ‘mathematics in primary school’.” [Wiskunde in de basisschool] (Treffers, 1993, page 106)

Chapter 1.

Introduction

1.1. Background

I was not a “natural” mathematician. I know now, of course, that there is no such thing, but as a teenager, the intimidation techniques of one mathematics teacher in particular cultivated a resistance to mathematics and encouraged the idea that I was not capable of its study. I had enjoyed arithmetic in primary school, and I had done well enough in Algebra 1 in grade 9. I had really quite liked geometry in grade 10, and that was when they still taught “two-column proofs” in U.S. high schools. But in grade 11, my Algebra 2 teacher’s belief that girls had no business in mathematics (or science) was like a medal he liked to show off. His incitement that to get a better grade on an exam a student should “beg and grovel” for it was not only demeaning and terrifying, but did not address the issue. I didn’t want a better grade – well, I did, but not that way – I wanted to understand; I wanted to know *why* what I had written was wrong. Instead, all I got was a red “X” and an invitation to grovel. I did not see him after school, and I did not beg and grovel; I think I ended up with a “D” which, while technically passing, did not instill confidence in my ability to take on the “monster” that mathematics had become.

My early undergraduate experience did nothing to dissuade this perception; it could not, since I did not endeavor to undertake the study of any mathematics at that time. I had decided by age 15 that I would be a teacher of high school English, as that subject came easily to me. By the time I got to university, my relationship with mathematics was badly damaged, and I did not see the need to take mathematics courses until it was absolutely necessary. To make matters worse, I was far too young to undertake academic life when I finished high school. I was only 17, I had led a very sheltered life in a remote small town up to that point, and I had no sense of accountability to myself. I had squeaked by in high school by being just “smart enough” to pick up the essentials, but that does not work at university. Mother is not hovering over you asking if you turned in your assignment. While the freedom was liberating, it was also my undoing. I managed to finish two terms (under a “quarter” system this does

not even constitute one full academic year) before I went back home. I didn't really even know enough then to be embarrassed by my failure.

Growing up, though, higher education was never a case of "if"; it was always a given. Because of the culture in which I was raised, it was not long before I started to feel the pull of "going back to school", and in the U.S., that is possible for virtually anyone, thanks to the community college system. It is a large network made up of many campuses and many models under many districts. It is open and accessible and approachable. I mentioned my thoughts to a friend, and it happened that she was teaching a writing class and suggested I enroll.

It would take a few more years before I enrolled as a full-time student, but that writing class was the beginning of a new era for me. I got an A in the class and with it a shot of confidence that at least I could "do" college, even if I couldn't "do" math. I took a psychology class, and then a French class. Eventually, I knew I would need to tackle the "math monster", so I spoke to an advisor and took a placement test. I had done pretty well in Algebra 1, so I expected to take the one college-level class I would need for my degree and be done with it. I placed, however, into "Elementary Algebra", which meant that the college course I needed was not one or two, but three terms away. I was initially disappointed, but I did well in that first mathematics class. In fact, I did very well, and surprised myself not only with the realization that I was quite capable of mathematical thinking and learning, but with the awareness that I actually enjoyed it. All of the community college faculty were caring, passionate and encouraging – everything my high school teacher had not been.

After that first college success, I eventually took the college-level course I needed and enjoyed it as well. It was naturally much more challenging, but I found that the challenge was part of the enjoyment I felt – the challenge followed by the satisfaction of realizing that, more than simply being able to follow mechanical algorithms, I *understood*. I had to work at it, which I had not experienced before, but I understood. And then I took another (trigonometry) and the die was cast.

I was then at the end of my two-year tenure at the community college and getting ready to transfer to a university to complete my bachelor's degree. I decided that while I would not change my career path (teaching), I would change my focus from English to

mathematics, and I have loved the struggle ever since. I have also been aware from that time of the role that community colleges play in the academic pursuits of the less “traditional” student. Since joining the ranks of community college faculty, I have also become aware of the status of the community college in the world of post-secondary academic education, and the struggles these institutions endure of which neither public schools nor universities are necessarily even aware.

1.2. My Interests

My own background plays a large role in the motivation for the research described in this document. Over the course of several years I went from knowing I was not capable of learning mathematics to knowing just as well that I was; from disliking mathematics to enjoying it; from feeling like an incompetent student of mathematics to a confident teacher and one of its loudest advocates. In line with the change in my personal perspective was the then subconscious change in my self-belief. In hindsight I believe that I was learning to see mistakes as opportunities for growth rather than failures, and I try to pass that insight down to my students now.

Students seem to make common errors at various different levels in the mathematics sequence. Tracking those errors and the way that students react to their mistakes has always been an interest. Now that I am teaching at a community college, I have the opportunity to see not only where my own change of attitude fits within the spectrum of possible mind-sets, but also how changes made to the way material is presented can foster students’ positive outlooks.

As a matter of personal interest, I have done a considerable amount of reading and professional development around teaching community college students over the years, and I came to realize over time that a lesson can be broken down into three components: the concept to be taught, the audience to be addressed and the approach to be utilized in conveying the information. My experience as a teacher supports this structure, and although much of the reading I did also seemed to confirm it, I did not appreciate then the need to document research that supported my conviction. Later, while I was trying to describe the study executed for this dissertation, I realized that I had unconsciously been deconstructing teaching this way in order to talk about it, for years.

Once this unconscious thought became conscious, much of the other reading I had been doing for this research seemed to shine in a different light.

I knew I wanted to study the community college population; I wanted to find a way to investigate the problems students in remedial¹ courses have with algebra; and I wanted to find a way to present things in a different way – a way that would push students to want to engage with the material. These things seemed somewhat compartmentalized until I realized that my students were the audience, their courses dictated the concepts, and in my desire to find a way to present mathematics to them in a different way was a search for an approach. Having established this deconstruction of what I wanted to study, I was able to tackle each piece in a more refined way, but I was also, eventually, able to bring the three parts together in a way that I had not been able to see previously.

1.3. What this Research is About

This dissertation is meant in part to redress the paucity of research into community college mathematics education, and to contribute to the literature by exploring at least one way in which mathematics might be taught at the community college level other than by lecture. Students enrolled in remedial mathematics courses at community colleges are often already disillusioned by the education system at least with regards to their mathematics requirements, much as I was at one time; to require that they revisit the same material that they failed to grasp in high school using the same passive delivery modality is to invite skepticism and resignation rather than either enthusiasm or enjoyment. If we are to reach these students, we must find ways to meet them where they are academically. Finding different ways to teach the required concepts is one possible approach and one of the motives for this research.

The fact that research on community college mathematics education is lacking has not gone without notice. Vilma Mesa, a professor of mathematics education at the

¹ “The terms ‘remedial’, ‘developmental,’ and ‘pre-college’ are used in the literature to refer to courses that cover K-12 mandated content: arithmetic, algebra, and geometry. These courses do not carry college credit. The Conference Board of the Mathematical Sciences uses the term ‘pre-college.’ Most of the literature in higher education and in policy documents uses the term ‘remedial.’ The American Mathematical Association of Two-Year College uses the term ‘developmental.’” (Mesa, 2017; page 963, note 1); in the current document I will use these terms interchangeably.

University of Michigan, began writing about community colleges in about 2010 (Mesa, 2010; Mesa & Megginson, 2011; “Background”) and it has been a recurring theme in her contributions to the literature since that time. Other authors have also contributed, but mathematics education research on community colleges (and community college research on mathematics education) is still well behind that for either K-12 or universities. This dissertation is an opportunity to add to the effort: to bring to the fore the value of the community college environment and highlight the need to focus mathematics education research at these institutions. It is also a chance to add to the literature at least one approach by which community college students and practitioners may benefit from research at other levels. It is a description and an analysis of one modification in the environment of interest: a community college remedial mathematics course, and an invitation for others to try similar modifications.

1.4. The Goals of this Dissertation

This scholarship began as a lay-teacher’s experiment, developed in an attempt to make algebra more interesting to those required to take courses they had already failed, or in some cases, already “passed”. As an instructor, I wanted to create an environment in which students could feel free to explore, where they felt safe experimenting and even failing, but then felt comfortable, and even excited, about trying again. I had not read, then, about Realistic Mathematics Education, or about Design-Based Research or the Teaching Experiment. I was a teacher trying to make her content more relevant, or at least more fun.

When I first decided to try the teaching theory described in Chapter 5, I had recently read Jane McGonigal’s (2011) book *Reality Is Broken: Why Games Make Us Better and How They Can Change the World*, and I wanted to try to “gamify” the remedial algebra class I would be teaching the following term using some of the principles outlined in that book. I had little idea then that I would incorporate the results into a paper, let alone a doctoral dissertation. What mattered to me then was that I wanted to provide the students in that class with what they needed, but I wanted to give it to them in a way that was meaningful, and at least as effective as it would have been if delivered by lecture; in hindsight, I wanted well-designed tasks, and not having been satisfied with the tasks that I had previously seen or used, I determined to design my own.

I wanted to be able to say, at the end of that term, that these students had learned the algebra of linear functions and, specifically, that they could write the equation of a line, read and interpret the equation of a line, interpret the parts of the equation of a line, and graph a line from its equation. Nothing in this list stands out as unusual when compared with any other class I have taught at this level; what was to be different was the delivery, and hopefully as a result, the level of engagement.

In retrospect, I was asking the following questions:

- Can tasks be developed using a narrative or “game” structure in order to make them more interesting?
- Will students engage with such a structure as much as or more than with a conventional structure?
- Will students learn the requisite concepts when taught under this structure as well as or better than under a conventional structure?

To move from the enquiring teacher persona to that of researcher, these questions had to be formalized in order to investigate ways in which they might be answered. To that end, the goals of this dissertation are to **investigate how certain student-centered learning techniques can be carried out at community colleges and how these affect students’ learning**, and to **assess how teaching methods developed for use in primary or secondary school might be used in a community college remedial mathematics course**.

In order to accomplish these goals, I examine the history of the community college as an institution, the existing literature on the difficulties students have learning algebra and that on community college mathematics education in general, and describe and analyze an attempt at modifying a particular teaching method for use in the community college remedial mathematics classroom.

1.5. How This Dissertation is Structured

Since the domain of study for this research is the content that prepares students for college-level algebra, it is critical to investigate that content. In order to do so, I created a study targeting community college students taking a pre-college mathematics course, and used a blend of Design-Based Research (DBR) and the Teaching

Experiment (TE) into a model more appropriate for larger groups; in recognition of its origins I call this blend the Design-Based Teaching Experiment (DBTE).

With the exceptions of the first and last chapters, each chapter in this dissertation does one of two things: either it reviews and describes a particular component of the study done for the benefit of the research or it describes or analyzes the study itself. Chapter 2 establishes and defines the three components present in an episode of instruction: the concept to be taught, the audience to which it will be delivered, and the approach that will be used to convey it; for ease of reference, I refer to the combination of concept, audience and approach present in any lesson as the *Concept-Audience-Approach* (CAA) triad. With these three components identified, each can be considered individually and collectively in the chapters that follow.

Chapter 3 is a detailed evaluation of the existing literature on the *Concept* element of the CAA triad: an exploration into the many and various documented difficulties that students have in learning the algebra of linear functions. Many community college students have already had some exposure to these *Concepts*, but for various reasons they need to have them retaught or reinforced via pre-college (or “remedial”) courses before they can expect to be successful² in college-level, credit-bearing courses.

Chapter 4 explores the state of affairs in research into community college mathematics education, outlines the challenges of working with a community college population (the *Audience*), and narrows down the world of all possible students to a

² Success is defined by many college administrators as having passed a class with a sufficiently high grade to be prepared to advance to the next course in the sequence. At Whatcom Community College, this definition is interpreted to mean that a student must receive a grade of “C” (usually 73%) or higher. While a grade of “D” (usually 63%) or higher is frequently interpreted to be “passing”, it does not indicate that the student is sufficiently prepared to undertake the study of the material in the subsequent course.

According the Western Washington University transfer requirements web page, “Courses which are to apply to General University Requirements must be taken on an A through F grading scale, except for courses designated as S/U grading. They may not be taken with Pass/No Pass grading. Except for ENG 101, which requires a C- or better, the minimum passing grade for GUR courses is D- (Math courses must be passed with a grade of C- or higher if used as a prerequisite to another course)” (General Undergraduate Requirements (gur)).

Most universities in Washington state have similar policies; as many community college students have transfer to a university as their intended goal, university policies often inform the community college policies that reflect them.

subset who are mathematics students enrolled in remedial courses in the community college environment. On the surface, community college students are simply undergraduate post-secondary students in their first two years of study, but in fact many of them have already experienced some post-secondary education, and there are other critical differences between this population and that of first- and second-year university students. This chapter also details those differences, setting the stage for the need for further mathematics education research with this population.

In Chapter 5, I review Realistic Mathematics Education (RME), the theoretical perspective with which the study conducted for this dissertation was considered. I explain its origins for elementary education, how it came to be used for secondary education and in a university setting, and how I have adapted it for use with a community college population. This chapter is important not only because it provides the necessary framework for academic research, but because it provides the background for one of the key aims of this dissertation: to explore how alternative methods of delivery designed specifically for one environment might be used in another setting.

Chapter 6 outlines the composition of the teaching *Approach* developed for use in the study done for this dissertation. I review the fundamentals of task design, Design-Based Research and the Teaching Experiment. Task design is a function undertaken and developed by every teacher of mathematics, whether implicitly or explicitly. The *Approach* used here was developed for use in this particular context by blending and modifying Design-Based Research and the Teaching Experiment. The Teaching Experiment proved to be insufficient to address the fact that the study was to be done at a class-wide level rather than on an individual or small-group basis, and inadequate to address the fact that it was not only the students who were being evaluated but the method of delivery itself. For that reason, although the *Approach* was based on the Teaching Experiment, it also incorporates elements of Design-Based Research. These techniques, the modifications necessary for this study, and their place in the realm of task design are discussed.

Chapters 2 through 6 might have been presented in virtually any order; the content in each of these chapters is independent of the content in the others. Establishing the collective information contained in them together however was a necessary step in fashioning a foundation for Chapter 7. It is this chapter that brings

together the pieces in a way that allows for the examination of the questions at hand. In Chapter 2, I discuss how the “*Concept-Audience-Approach*” structure allows the process of “teaching” to be broken down into three constituent factors which can then be examined separately, while allowing for the fact that there are connections between them. Chapter 3 sets out the *Concepts* that will be examined in this dissertation, Chapter 4 describes the *Audience* who will undertake the study of those *Concepts*, Chapter 5 provides the theoretical framework that informed the study and Chapter 6 outlines the components used to fashion the *Approach* taken to deliver those *Concepts* to that *Audience*. The tasks designed to assess the *Concepts* in question are described in Chapter 7 as well as the means by which they were fashioned for this purpose and the method by which they were executed. The data presented in Chapter 7 is analyzed in Chapter 8 and the most prominent conclusions are drawn about the success of using the particular blend of RME, Design-Based Research and the Teaching Experiment.

Chapter 9 is a summary of the study performed for this dissertation, its conclusions, a reflection on the work undertaken to complete the study and its analysis, and a discussion about potential future questions that may arise from the results of this research.

Chapter 10 is the paper “On College Students and the Cartesian Connection”. It is a stand-alone study that I conducted as a PhD student as part of my work exploring mathematics education at the community college level. I have included here as it provides additional fine-grained analysis of a particular task.

Chapter 2.

Instructional Design

Much literature is devoted to instructional planning and instructional design. A number of authors describe what might easily be called the “structure” of teaching. Gorev and Kalimullin (2017) outline what they alternately call the “stages” or “phases” of instruction: motivation, substantial block, psychological relief, puzzles, intellectual warm-up and resume; Johnson (2000) refers to objective, input and activity as the three constituents of a lesson plan. Evenson, McIver, Ryan, Schwols, and Kendall (2013) devote an entire chapter to guidance for instructional planning where they describe the phases of creating an environment, developing understanding and applying knowledge.

While there are several perspectives in the literature on what constitutes “teaching”, these all seem to address the process of teaching, and none appears to consider the influence different *Audiences*, or the effect of teaching different *Concepts* might have on that process. This may be because for most teachers, the students they teach will always be high school students or always technical program registrants, and the topics they teach are mandated by their discipline and the curriculum they are required to convey. The *Audience* and the *Concept* are important facets of teaching however, and need to be considered – consciously or otherwise, just as much as does the teaching *Approach* that will be used to deliver the *Concept*. The *Audience* in the setting to be described in this document is community college students, a very different *Audience* from high school or university students, as will be discussed in more detail in Chapter 4. Anyone who has taught more than one type of student, more than one discipline, or even more than one level of the same discipline will appreciate the need for consideration of all of these elements. This chapter is a brief description of the three elements of *Concept*, *Audience* and *Approach*, their properties, and the ways in which they work together.

2.1. The Elements of Instructional Design

As with any instruction, a certain amount of preparation goes into the teaching of every session of a mathematics course. Even the most experienced teachers must take several factors into consideration, and while it may appear on the surface that a veteran

teacher can deliver a lesson without much forethought, several components must still be addressed if the lesson is to be successful: there is a *Concept* to be delivered, an *Audience* to receive it, and an *Approach* to be utilized. A teacher who has delivered a course countless times may not be aware of her preparation in the same way as a new teacher is, but the same elements of instruction are necessary in both cases.

These three components – the *Concept*, the *Audience* and the *Approach* – will be typeset in the balance of this document as they are seen here, capitalized and italicized, for the purpose of clarity. Furthermore, since the terms *Concept*, *Audience* and *Approach* will be used in a particular way in this thesis, I will define them for that purpose as follows:

During an academic term, the curriculum sets out broad themes that students are expected to learn. While it is hoped that, at the end of the term, a student will be able to demonstrate her understanding of an idea, it is impractical to attempt to teach only the “big ideas”. For the purpose of daily teaching, and so that students can learn these big ideas incrementally, themes are broken down into smaller blocks of content. The term ***Concept*** as used in this document refers to any of these smaller blocks of content. For example, the theme might be “graphing linear functions” but the *Concept* taught during one lesson might be “plotting ordered pairs” or “identifying the y -intercept” and during another session, “computing the slope”. It is important to note that the *Concept* is simply the content being taught; the term does not carry any implication of understanding or appropriate application.

The students learning a *Concept* during any given session make up the ***Audience***. While the *Concept* during that session might be “graphing ordered pairs”, the *Audience* might be high school Algebra 1 students, community college students or novice architects, for example. The same *Audience* will likely be exposed to many *Concepts*, and the same *Concepts* can be delivered to different *Audiences*. This relationship will be further discussed in section 2.5.

Since it is common while writing about research to discuss the research approach, it is important to distinguish that meaning of this word from the meaning used in this document. Thus, here the term ***Approach*** will be taken to mean the teaching technique used by a teacher when delivering a *Concept* to an *Audience*. Common

Approaches include lecture, guided discussion, the “flipped” classroom and activity-based learning. Many teachers use combinations of these, and it is possible to think of such a combination either as a single *Approach* or as separate *Approaches* being used to convey one *Concept*.

In each case, the use of one of the terms *Concept*, *Audience* or *Approach* recorded in this way should be assumed for this document to be considered as part of the CAA structure rather than as a stand-alone notion. In the places in this document where I have written “concept” rather than “*Concept*”, for example, I mean it in that location in the general, stand-alone sense of the word.

The three components of instruction can be summarized with a diagram:

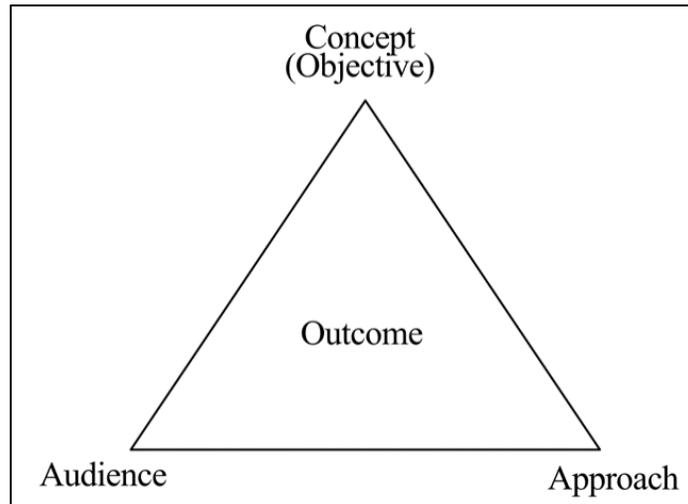


Figure 2.1. The Concept-Audience-Approach Triad

It is important to note that although two-way relationships are clearly observable, it is not a pair of vertices that defines a triangle but all three working together. Find a novel *Approach* to teaching a *Concept* from the algebra of linear equations but deliver it to an inappropriate *Audience* (for example) and it will fall flat if not fail outright. All three components must be considered together for instruction to be successful, at least from the perspective of delivery.

The basis of this dissertation is the exploration of a particular combination of the components of the CAA triad; thus it is worth beginning by regarding each in isolation and proceeding to a discussion of the interplay of each pair of these components. In this chapter, I investigate the particular CAA triad of “remedial” algebra of linear functions

(*Concept*), community college students (*Audience*) and Design-Based Teaching Experiment (*Approach*). Each of these components is developed in its own chapter in this dissertation: The *Concept* is examined in Chapter 3, the *Audience* in Chapter 4 and the *Approach* in Chapter 6. In the case of this dissertation, what is under investigation is the way in which the *Approach* is applied to a particular *Concept* with a particular *Audience*; therefore it will be useful to examine the relationships between the three components, which I do in section 2.5. I begin here by describing each of the components *Concept*, *Audience* and *Approach* in general, and proceed by examining the relationships between them.

2.2. The Concept

It seems fitting to begin this section with a description of one way in which concepts might be differentiated. Hewitt (1999) describes something as being “*arbitrary* if someone could only come to know it to be true by being informed of it by some external means”, while “things which are *necessary* can be worked out: it is only a matter of whether particular students have the awareness required to do so” (p. 3; *emphasis added*).

The decision to label coordinate pairs as (x, y) , for example, was arbitrary. To call the first position in this pair the “abscissa” and to associate it with a horizontal move and the second, the “ordinate”, with a vertical; to use x and y instead of, for example, e and n ; to use parentheses rather than brackets or “slashes” or some other grouping symbol; all of these choices were arbitrary. Any of them could be changed, and the mathematics being described by their use would not change. In order to know what the convention was, you would need to be told; it would not be possible to guess, or to extrapolate from the known.

On the other hand, 2^n describes a phenomenon that can be discovered (whether conventional notation is used to describe it or not). Students asked to fold a piece of paper multiple times and to record both the number of folds and the number of “layers” present with each folding, might work out the pattern that for n folds, there are 2^n layers. This pattern is necessary; it exists outside our desire to describe it. We might use any notation we like to explain it, but the underlying property cannot be changed.

The *Concept*, connected as it is to *Audience* and *Approach*, must be appropriate to both. A class of high school algebra students is the wrong *Audience* for a lecture on antiderivatives, and it would be ineffective to teach an arbitrary *Concept* (such as the use of the Greek letter Σ to indicate summation) using a “discovery” *Approach*, for example. The concepts that will be delivered during an academic term are unlikely to be much under the control of the teacher at the time of delivery. While a teacher may sit on a committee that decides which courses will deliver which content, at the moment she walks into class, the curriculum has been decided, and it is her job to ensure that students are exposed to the ideas the institution requires from the course in which those students are enrolled.

There are ways in which an individual instructor may exert some control, such as the order of delivery, the length of time spent on one *Concept*, the method and frequency of assessment, and so on, but deciding what content will be delivered precedes the delivery. Occasional curriculum changes require that instructors either create notes, tasks and activities from scratch, or at the very least reorder the content of a course they have taught before for delivery under the new curriculum requirements; this practice may assist in the delivery of that content in the way that makes the most sense to the instructor, but may also mean that when introduced to material “out of order” (according to the author or publisher of a textbook), students may not have been exposed to concepts developed in earlier chapters because the instructor desires to teach material found later in the textbook first. In this case the instructor must introduce any material in class upon which she plans for students to rely for that day’s lesson. Practices such as this require even seasoned instructors to spend time in preparation of the concepts they will deliver to their students.

It is possible for a *researcher* to select the *Concept* she will investigate. In this chapter, several concepts are examined from the curriculum of the algebra of linear functions. The specific *Concepts* discussed in this dissertation are outlined in section 3.5, and although *Concept* is one of the three critical components of the CAA triangle, these are predetermined for the purpose of this research in much the same way as curriculum concepts are fixed at the beginning of each academic term.

The *Concepts* in this study are some of the foundations of the algebra of linear functions: Cartesian coordinates, solutions to linear equations in two variables, slope, y -intercept, and slope–intercept form of a line.

2.3. The Audience

The population of students making up a class (the *Audience*) must be taken into consideration when deciding which *Approach(es)* to use, which examples will be given, and so on. Even within classes designed to deliver the same *Concepts*, delivery to different *Audiences* will require different *Approaches*. Most 15-year-old high school students in an introductory algebra class will not understand references to the pitch of a roof used when teaching about slope, for example, in the same way that a class of job-retraining carpenters at a technical college likely will. Equally, the *Audience* for a given instructor may always be high school students, but a class of grade 9 algebra students is a very different *Audience* from a class of Advanced Placement (AP) Calculus students.

As children get older, they generally progress through primary (elementary) and secondary school. There may be an argument for those in their last year of high school becoming adults in the eyes of the law toward the end of their tenure in that environment, but chronology aside, these students have, for the most part in North America, very little of what has come to be called “life experience”. By the same token, it can be argued that most undergraduate university students (again, in North America) are inexperienced. While we might call them “young adults” rather than children, and while they are at least somewhat more likely to have a bit of work experience and may even be living on their own, most still have little or no “life experience”. Unlike either high school or university undergraduates, community college students tend to have a considerable amount of “life experience”, although this population can be further classified by how much academic experience they possess.

Thus, while a teacher might find herself explaining the significance of the y -intercept, the population to which she is explaining it does matter. The study completed for this dissertation was conducted with an *Audience* of community college students enrolled in a remedial algebra course.

Students entering colleges and universities in the U.S. often have deficiencies in their knowledge of algebra that means they are unprepared for college-level courses both mentally and mathematically; a truth examined by Mesa (2017) and borne out by their performance on college entrance placement assessments (Van Campen, Sowers & Strother, 2013). Various terms in the vernacular describing curriculum designed to correct these deficiencies include “remedial”, “developmental” and “pre-college” algebra. Some institutions, and most community colleges (see section 4.2) offer a comprehensive sequence of such courses at different levels; others offer a single course that may or may not address all possible deficiencies; still others offer no such remediation.

Even though they are placed into courses designed to fill the gaps in their conceptual understanding, there are a number of places in the pre-college algebra curriculum where students can struggle to take up the required concepts. Algebra tends to be, for these students especially, a barrier to overcome rather than a subject to be learned, let alone enjoyed. Although about 90% of community colleges administer placement tests for mathematics and English, as many as 25% of students placed by one of these tests ends up in the wrong class (Ngo, Chi & Park, 2018). Many students arrive in pre-college classrooms because they were not successful in high school algebra, never took high school algebra, or took it sufficiently long ago that their memory of what it entails is skewed by time (see Karsenty, 2002). Their placement into developmental courses is already likely to be damaging both to their self-esteem and to their projected success (Mesa, 2012); if they are then inaccurately placed in courses lower than necessary (Ngo, Chi & Park, 2018), both belief in self-efficacy and projected success can only suffer further.

2.4. The Approach

The delivery method (the *Approach*) the teacher will use must be decided before students arrive in the classroom. Even if a teacher knows she is going to be teaching slope to high school algebra students, she will need to have made some notes for her lecture or devised the activities students will undertake; without these the instruction will not unfold efficiently; tasks designed “in the moment” rarely work as imagined to encourage discovery or uptake. Additionally, new teaching approaches are brought to our attention all the time, meaning that teachers are allowed (or required) to plan again with these new methods in mind.

As of 2014, the vast majority of teachers and lecturers still delivered concepts to undergraduate students by lecture: on average about 58% of the time (Wieman & Gilbert, 2014). While this means of delivering academic content does have still have recognizable value (Meyer & Hunt, 2017; Stearns, 2017), it has been shown that, used exclusively, lecture is dramatically inferior to approaches that require more active student participation for student engagement, uptake and retention (Mallin, 2017). Lecture has come to be known as a teacher-centred approach; its contrasting approach, the student-centred approach (also called Student Centred Learning, or SCL), while varied in its applications, is preferable because of the documented benefits to the learner (Laursen, Hassi, Kogan & Weston; 2014).

“Instruction” can be viewed narrowly or broadly. In the narrow view, it comprises only events that take place during a classroom session as interactions between students and teacher, such as lecture or classroom activity. Viewed broadly though, it includes a variety of teachers’ activities whose collective goal it is to support student learning. In the latter instance, teaching includes the preparation of activities and lecture notes, and endless hours spent simply thinking about how better to explain a difficult concept or designing a new and innovative activity. It also includes the peripheral tasks of answering questions, either during lecture or while circulating the classroom during an activity, and it also includes the myriad enterprises that follow on from classroom sessions, such as tutorials, writing and marking assessments and recording grades. In fact, the definition of teaching could include any action taken by the teacher that plays a role in conveying information to the student. It does not, therefore, include additional duties performed by an educator, such as committee and governance work, but does include virtually everything else a teacher does. I take this broad view of teaching for the balance of this dissertation.

The study presented here used the *Approach* called the Design-Based Teaching Experiment, which I discuss at length in Chapter 7.

2.5. Relationships between the *Concept*, the *Audience*, and the *Approach*

In section 2.1, I promised a more in-depth examination of the relationships between each pair of elements in the CAA structure. What follows is the promised examination for each pair in turn.

2.5.1. The *Audience-Concept Dyad*

The *Audience* has little or no influence on the *Concepts*. Whether the *Audience* is university or high school students, the *Concepts* presented will be those from the appropriate branch of mathematics for the course in which they are enrolled.

The *Concepts* may influence which subset of an *Audience* will enroll in a given class, as students will enroll in a course based on its requirement for their degree or program, or on their interest in the course content. The *Concepts* included in a given course do not affect whether the *Audience* will be high school, community college or technical college students however. The *Concepts* appropriate for a course in Elementary Statistics, for example, will be the same regardless of the students enrolled in the course.

We can say, therefore, that *Audience* and *Concept* are not influential of one another.

2.5.2. The *Concept-Approach Dyad*

It is in light of Hewitt's 1999 perspective (see section 2.2) that I address the *Concept-Approach* dyad. *Concepts* may be taught using one of several *Approaches*, and one *Approach* may be more suitable for teaching some *Concepts*. This argument is also at the heart of the lecture versus SCL debate (see section 2.4) in that while SCL has been shown to be highly effective, there is still evidence that lecture is a valuable delivery modality as well.

For example, *Concepts* that are arbitrary may be best taught by lecture, since they cannot be discovered organically; they are conventions rather than mathematical properties. *Concepts* that are necessary, on the other hand, may be better learned by

some form of SCL, since they are mathematical properties that exist in their own right and their discovery (rather than their dissemination) leads to deeper and more thorough understanding.

The suggestion that one way of learning a *Concept* might be more effective than another does not preclude a *Concept* being taught using any *Approach*. The choice of an *Approach* is unlikely to influence the *Concept* to be taught; it is more likely that a particular instructor may generally favor one *Approach* or another and at least attempt to apply that *Approach* to all, most, or at least some of her classes.

The *Concept* cannot therefore be influenced by the *Approach*, but the *Approach* may be influenced to some degree by the *Concept*.

2.5.3. The *Audience-Approach* Dyad

In the same way as the *Concept* does not influence the *Audience*, neither can the *Approach* affect who the *Audience* is; students enrolling in a course may or may not know the instructor or her teaching “style” before the first class session, but the individuals enrolled in a given class do not determine the *Audience*; the *Audience* is students at a particular type of institution. Whether it is the 8:00 am section of a Trigonometry class at Whatcom Community College or the 4:15 pm section of Pre-College Algebra at a different community college is irrelevant; the *Audience* is community college students.

The individuals in the *Audience* may not be known to the instructor either, before the first class session, but if the *Audience* is a class in high school, the students will be assumed to be properly, if minimally, prepared for the content of that course, and to have a certain amount of (or lack of) “life experience” or academic maturity; therefore even before she knows her students, an instructor knows her *Audience*, and that *Audience* is likely to play a role in the instructor’s decision to use a particular *Approach*, assuming that the instructor knows that there are multiple options.

Thus the *Approach* the teacher will take may be influenced by her knowledge of who the *Audience* is, and while the *Approach* does not influence the *makeup* of the *Audience*, it may *affect* a given *Audience* by being more or less engaging or effective than another *Approach*. One of the questions asked in this dissertation is whether the

Approach used in the study will influence the *Audience's* ability to absorb the *Concepts* as well or better than a lecture *Approach*; therefore it is important to observe that while the makeup of an *Audience* is out of the control of the teacher, the *Approach* used is not, and the teacher's knowledge of both *Audience* and *Approach* are important factors in the delivery of the *Concepts*.

The makeup of the *Audience* is therefore not influenced by the *Approach*, but the choice of *Approach* may be influenced by the teacher's knowledge of the makeup of the *Audience*.

2.6. Chapter Summary

Of all the components in the CAA triad, the *Approach* is the only one that can be affected by the makeup of the others. This may be the reason *Audience* and *Concept* are not generally considered when discussing instructional design; *Audience* and *Concept* are not affected by the *Approach*; but the *Approach* can be influenced by either the *Audience* or the *Concept*, or both, which is why it is important to consider all three components. This perspective is also useful in the current context, since it explains the nature of the *Audience* and the *Concepts* as "fixed" which allowed for the study of the impact of the *Approach*. The *Audience* was students enrolled at a (particular) community college; the course being investigated dictated the *Concepts* to be taught, and therefore to be analyzed; the only element that might vary was the *Approach*, and it was this component that was under investigation.

In this document, in the role of research practitioner, I attempted the *Approach* of the Design-Based Teaching Experiment (using Realistic Mathematics Education) with an *Audience* of community college students and *Concepts* found in the curriculum of remedial algebra. In the next chapter I discuss the *Concepts* at length, in Chapter 4 I discuss the *Audience*, and in Chapter 5 and Chapter 6 the *Approach*; in Chapter 7 I discuss the outcomes of the use of the Design-Based Teaching Experiment on the particular CAA configuration found in the study for this dissertation.

Chapter 3.

Linear Functions: Making the Connection

This chapter is a brief review of some of the available literature regarding how students learn algebra and common difficulties students face in their attempts to learn various aspects of the subject. The quantity of literature addressing “difficulties” learning algebra is staggering and, as such, what is reviewed here is necessarily only a sampling. The content of this chapter is a discussion of the issues that arise for algebra students learning about linear equations. The focus is on equations in two variables, but there is also some discussion about equations in one variable.

In 1996, Judit Moschkovich wrote:

Linear functions is a complex domain where the development of connected pieces of conceptual knowledge is essential for competence.[...] Conceptual understanding in this domain involves more than using procedures to manipulate equations or graphs lines; it involves understanding the connections between the two representations (algebraic and graphical), knowing which objects are relevant in each representation, and knowing which objects are dependent and independent. (p. 242)

This statement not only beautifully and concisely summarizes the goal of learning about linear functions; it also outlines many of the ways and combinations of ways in which students may fail to be “competent”. It is not enough to understand slope and intercept, nor to understand the roles of x and y . What counts as competence in this domain is the *connections* of those pieces to one another.

Coming to grips with the connections is one of the biggest hurdles students face. Once they have had at least some exposure, many students can compute a slope, but cannot explain what it says about the relationship between x and y . Many can plot the y -intercept on a graph from the equation, but are unable to verbalize the conditions under which it occurs or what its significance is. Students can often graph a line given an equation in slope–intercept form, but they may not be able to tell you what they did to accomplish this task, or in what way the results are meaningful. The mechanics may be there, but the *connections* often are not. In Moschkovich’s words then, these students do not have “conceptual understanding in this domain”; they are not competent.

3.1. A Note on Vernacular

U.S. remedial college algebra curriculum typically progresses from the general idea of “variable” to that of algebraic expressions containing variables and their simplification (combining like terms, distribution to remove parentheses, etc.) Once the ways by which expressions may be manipulated have been established, the curriculum introduces the idea of “equation” and students learn to solve simple linear equations (those for which any non-constant term has a maximum degree of 1) containing only one variable, and at first only one instance of the variable. A second instance of the same variable, and instances of the variable on both sides of the equation round out this objective. Occasionally the idea of graphing the solution to an equation is included in which the value is marked on a real number line. While it is possible for a linear equation in one variable to have no solutions (e.g. $1 = 0$) or an infinite number of solutions (e.g. $x = x$), and while these cases are usually addressed in the curriculum, the implication is that *most* linear equations in one variable have a unique numerical solution. This implication, whilst not necessarily true, allows students to focus on learning how to solve equations for which there is a solution while not neglecting the other possibilities completely.

Upon establishing methods by which linear equations in one variable may be solved, the curriculum proceeds to linear equations in two variables, in which the idea of a relationship between two distinct variables can be explored. While the solution to a linear equation in one variable “usually” has a single, unique numerical solution, a solution to a linear equation in two variables is a pair of numbers for which the *relationship* between them is described by the equation. For example, in the equation $y = 2x$, y is always twice x ; in $x - y = 1$, the value of x is always one more than the value of y . Because it is the relationship that makes the statement of equation true, there is an infinite number of pairs of numbers that will satisfy the equation.

The idea of function is conveyed in the remedial algebra curriculum during the course that follows on from the one described in this dissertation. The students in the study conducted for this research were not exposed to the idea of function, of domain, codomain or range, or of mapping (except for the use of ordered pairs as used to graph an equation in two variables). The literature on learning algebra, however, seems strongly inclined to the use of the term “function”, and as such many references cited

here utilize that terminology. This conflict in the language between the curriculum and the literature might easily be confusing, but I take the stance that the idea of “linear equation in two variables” is a precursor to that of “function”. I do not claim that the terms are equivalent, but throughout the remainder of this report, the two terms may appear to be used somewhat interchangeably. Where the term “function” is used, it is because it relates to something that appears in the literature; where I use “linear equation in two variables” I refer to an aspect of the curriculum. In a few places it might be possible to use either expression, in which case I have used the term that seemed to me at the time to be the more appropriate.

3.2. How Students Learn the Algebra of Linear Functions

How we learn anything is an enormous question. A great deal of research has gone into the science of cognition and learning; learning mathematics in general and learning algebra in particular are often topics of discourse in this realm (e.g. Anderson, Qin, Sohn, Stenger, & Carter, 2003; Blessing & Anderson, 1996; Hohensee, 2014; Qin et al., 2004). In this section, I summarize the results of some of this research on learning algebra in an attempt to expose the key factors required to accomplish the goal of becoming algebraically “competent”.

A few studies have shed some interesting light on the physical processes of learning (or at least practicing) algebra. Blessing and Anderson (1996) examined the brain activity of adult experts learning “artificial algebra” in an attempt to understand how people learn algebra. The results were a close fit to the *Power Law of Learning* (or the *Power Law of Practice*; broadly the idea that the time required to master a skill follows a log-log model) (Newell and Rosenbloom, 1981, as cited in Lacroix & Cousineau, 2007),.

Anderson, Qin, Shon, Stenger and Carter (2003) produced a quantitative analysis of the Blood Oxygen Level Development (BOLD) response recorded as subjects solved relatively simple algebraic equations while undergoing brain imaging. In 2004, Qin, Carter, Silk, Stenger, Fissell, Goode & Anderson repeated this experiment with young learners of algebra. The astounding result was that “the active areas in children’s algebra equation learning are similar to areas active in adults” (p. 5690). This realization is remarkable in that it demonstrates that learning may take place in the same way, using the same regions of the brain, regardless of the age of the learner.

Although the results of these neurological studies are interesting, they leave us no better off in terms of knowing *how* someone learns algebra. The studies summarized above all address a question of physiology, but cognition is defined as “the mental action or process of acquiring knowledge and understanding through thought, experience, and the senses” (“cognition”). To answer the question at hand, it is necessary to attempt to determine what “mental action” is being taken en route to “acquiring knowledge and understanding” of algebra.

Chiu, Kessel, Moschkovich and Muñoz-Nuñez (2001) discovered that, during the process of learning algebra, a student may learn to see a pattern in a different way than she previously did, but instead of replacing an earlier concept image with a new one, the new idea is instead included in the concept image alongside or as a modification of the old idea.

Huntley, Marcus, Kahan and Miller (2007) conducted a study “designed to gain insight into how high-school students taking a third-year³ mathematics course approach a variety of algebra problems” and to “gain insight into students’ thinking” (p. 118). The conclusions at which they arrived are that students rely heavily on symbol manipulation. In fact, manipulation was the first strategy utilized by all of their subjects, and the only strategy applied by most subjects. After being asked a scripted, probing question about whether the problem could be solved a different way, subjects (collectively) suggested graphing, substitution, “guess and check”, performing the steps in a different order, “cross-multiplication” and “FOIL” as alternatives to their previous attempts. The authors’ comment about subject responses before and after the probe was that although many did not think to try a different strategy before probing, after probing, “it was as if the floodgates opened and out poured several possible solution strategies” (p. 126). The Huntley, Marcus, Kahan and Miller (2007) study answers some interesting questions about the process students follow when attempting to solve linear equations, but it does not teach us how they learn the algebra of linear functions. A footnote in this article states “We acknowledge the fact that neither behaviorists nor constructivists believe that we can know exactly what students are thinking” (p. 118).

³ Equivalent to grade 11 in Canada and the U.S.

In the same way as identifying all the components of a machine that are working is much more daunting a task than identifying any that are not working, examining how, what or in what way students do not learn may be informative in discovering how they do. To that end, I now turn to the topic of considering the places where the learning of algebra has been observed to be problematic.

3.3. Common Difficulties

If the previous section attempts to speak to the means by which students acquire algebraic concepts, the current one addresses the ways in which they do not do so. There is substantial documentation describing various difficulties that students have with learning algebra, much of which will be referenced in the balance of this chapter, some of which also attempts to provide solutions to the problem under discussion. Difficulties described in the literature are many and diverse, but they can, for the most part, be broadly sorted into a few categories, including difficulties involving graphing linear equations, those in which notation is problematic or in which manipulation of expressions is inappropriate, and those which involve making a connection between algebraic and graphical representations. Having identified several categories of difficulties, I address some of the more commonly identified problems, suggest some opportunities for further research and, where appropriate, discuss potential solutions to the stated problems. Since algebra is generally taught by first introducing the idea of expressions and equations containing only one variable, before a second variable is introduced, I advance here in the same manner, beginning with the algebra of linear equations in one variable and proceed to that for linear equations in two variables holding a (linear) relationship with one another.

3.4. The Algebra of Linear Equations

In the U.S., elementary algebra is commonly taught in steps of increasing complexity. Students are first exposed to the idea of variable and shown how to manipulate expressions containing one variable, then given equations in one variable and shown how to solve them. Eventually, linear equations in two variables are introduced. The sequence as described here is vastly oversimplified, but it serves as a

minimal example of the steps taken in the curriculum by most U.S. institutions. I address here some of the difficulties researched in the topic of one-variable linear equations.

3.4.1. Notation and manipulation

Research points to various difficulties occurring in the computation of the components of linear functions. Many of these difficulties result from problems with the notation used, while others may come about as a result of the student's inability to manipulate an algebraic equation. These difficulties are issues in their own right, and I address them in that light.

3.4.1.1. Embodiment

Nogueira de Lima and Tall (2008) investigated an embodiment perception idea that students see the constituent terms of an equation such as $3x + 1 = 7$ as "things" that can physically be moved around the page. The authors explain:

This shifting of symbols is seen as an application of human embodiment, picking up things and moving them around. It works fine for simplifying expressions such as $4a + 3b + 2a$ where the term $2a$ can be shifted next to the $4a$ and combined to give $6a + 3b$, but fails for equations where shifting terms to the other side requires additional actions. ... The concern is that students attempt to perform similar actions within an equation, aware that they must also perform "a kind of additional 'magic' to get the correct solution. (p. 4)

However, it doesn't take much variation to cause difficulties for students (and teachers) handling expressions.... For instance, faced with $3 + 2x$, a student not knowing what x is may recognize the part $3 + 2$ and add these together. (p. 7)

I would add that, if students do not see $2x$ as 2 times x to begin with, it is very possible that the addition of the numbers in $3 + 2x$, which is much more familiar, appears to be the only action they can take. It is not so much that they are attempting to combine 'unlike' terms, as that they are unaware that they are working with terms in the first place, or that, not recognizing $2x$ as representing multiplication, it does not occur to them that there should be an application of the correct order of operations. Additionally, students with little depth of knowledge frequently assume when presented with algebraic structures that they are required to do something. If nothing presents itself as an obvious

task that they have been shown explicitly, they will invent a task rather than accept that perhaps there is nothing that can be done to such an expression.

3.4.1.2. The equal sign

Knuth, Alibali, McNeil, Weinberg and Stephens (2005) made the observation that

[T]here is abundant literature that suggests students do not view the equal sign as a symbol of equivalence (i.e., a symbol that denotes a relationship between two quantities), but rather as an announcement of the result or answer of an arithmetic operation (e.g. Falkner, Levi, & Carpenter, 1999; Kieran, 1981; McNeil & Alibali, in press;⁴ Rittle-Johnson & Alibali, 1999).

And later,

A relational view of the equal sign is essential to understanding that the transformations performed in the process of solving an equation preserve the equation (i.e., the transformed equations are equivalent) — an idea that many students find difficult, and that is not an explicit focus of typical instruction. (p. 69)

Kieran (1981) appears to have agreed: “Even college students taking calculus, when asked to find the derivative of a function, frequently seem to be using the equal sign merely as a link between steps” (p. 324).

The problem in Pirie and Martin’s (1997) view is that students frequently do not see an equation as having sides; they often perceive the “=” symbol not as a statement of equality, but as an operator; a signal to do something to the things on one side of the symbol in order to determine their collective value, which should then be placed (or already be present) on the other side of the symbol.

Pirie and Martin (1997) explained their view on this issue with the equals sign as follows:

Learning related to linear equations is probably the first time that the images that students have for the meaning of the equals sign, are challenged. Until this point, the fundamental image of ‘=’ as ‘indicating the result of an operation’ has been sufficient to deal with all the symbolic expressions they have encountered (Matz, 1982; Mevarech and Yitschak, 1983; Booth 1984). Failure to acknowledge this potential problem, and deal with it at this point, can lead to pupils *ostensibly* demonstrating an ability to

⁴ This article was published in 2005.

solve linear equations, but masking a deeper lack of understanding. (p. 159; *emphasis original*)

3.4.1.3. The balance model

The “balance model” (Vlassis, 2002) is a common analogy to balancing a scale frequently used to explain to students that what is done to one side of an equation must also be done to the other. The analogy works as long as what is being “done” to both sides is subtraction of a positive quantity or division of an integer multiple. Once the equation to be solved has a subtraction operator in it, however, the analogy no longer makes sense (Pirie and Martin, 1997; Vlassis, 2002).

While learning to solve linear equations in one variable, there is a critical elevation in demands placed on cognition with the presence of the variable on both sides of the equation. Whereas it was possible to solve equations in which the variable appeared only on one side of the equal sign simply by using the operation inverse to the one present in the equation, solving equations in which the variable appears on both sides of the equal sign cannot be approached in this primitive manner. It is this intellectual crossroads that Filloy & Rojano (1989) called the “didactic cut”.

Of the students in her 2002 study, Vlassis wrote, “Once they had arrived at the equations, the students were able to begin finding the solution; in most cases this was by means of trial and error ... [and] the students felt that their methods for solving problems where [*sic*] cumbersome and tedious” (p. 346). It is not clear when, if ever, the students in Vlassis’ study determined a more efficient way to solve such problems. Vlassis called one of the categories into which her subjects’ work fell “formalisation”, claiming that, in this category, “All of the students assimilated the principle demonstrated by the scales, i.e. performing the same operation on both sides” (p. 347; *emphasis original*). The same group, however, made errors revealing that their equal treatment of both sides of an equation was based on misconceptions about what the two sides represented.

Pirie and Martin (1997) claim that the perception that there is an exact place in the curriculum (i.e. Filloy & Rojano’s (1989) “didactic cut”) where students begin to encounter this difficulty may, in fact, be inaccurate (p. 161). In the description of a case study being examined by Pirie and Martin, the teacher used the idea not of a balance, but of a fence to represent equality. The instructor, Mr. Alwyn, explained to the students

that, “The equals sign [...] is actually a fence that says something like ‘this side must have exactly the same number as this side’.” He wrote the following statement on the chalk board:

$$\square + \square + 18 = \square + 53$$

and reminded students of the special role that the “=” plays as a fence dividing things that must be equal. He then added one more restriction: the number must be the same in all of the boxes.

In this way, the students in Mr. Alwyn’s class, having seen and talked about the symbol “=” from the start as one that describes a relationship rather than one that dictates an operation, apparently never experienced the difficulty with the symbol that many students do. Instead, they battled with the particular problems Alwyn gave them to practice with until the general strategy began to reveal itself. Pirie and Martin concluded, “By the end of the second lesson, none of [the students] (in Mr. Alwyn’s class) was still solving the equations by random trial and error” (p. 171), and “their techniques immediately alert them to a possible problem, a need to fold back and establish, a little more precisely, what they were doing” (p. 173).

This statement is particularly encouraging when it is realized that Alwyn’s class in the study described was comprised of “less able” students. Interestingly, however, the Pirie and Martin study was published in 1997 and, over twenty years on, it is still remarkably common to hear teachers using a “balance” model approach to teaching solution of linear equations.

3.4.2. Use and interpretation of variables

Interpretation of variables is another theme explored by several authors. In an echo of Nogueira de Lima and Tall’s (2008) “embodiment”, Knuth et al. (2005) wrote that “students considered literal symbols as objects[...] Few students considered them as specific unknowns[...], and fewer still as generalized numbers[...] or variables” (p. 69).

The topic of Hewitt’s (2012) paper is difficulties with formal algebraic notation, and issues around this subject such as “interpret[ing] someone else’s notation” and “order of operations within an expression” (p. 141) were analyzed comprehensively.

Hewitt (2012) also looked at formal algebraic notation as “an example of a social convention” (p. 142) and explored the ways in which this convention is arbitrary and ways in which it is necessary (Hewitt, 1999, 2012). In the 2012 article, Hewitt’s focus was not just on the problem, but on one potential solution of his own creation. Grid Algebra (Hewitt, 2007) is a software program in which students begin exploring what movement across a multiplication grid does to a particular number, and then (eventually) apply the patterns they uncover to explore algebraic notation. Upon discovering the “path” a number can take, students are given relatively complex numerical expressions and tasked with recreating the path taken to arrive at the resulting notation. While the notation is all numerical at first, the goal is eventually to be able to re-create a “path” with a variable. Since the idea of inverse operations is built into the tasks from the beginning, the result was that:

[...] students were able to gain a success of around 70% with questions on solving linear equations after only three lessons from a starting point of never having been introduced to letters [standing in for values] nor having met formal notation.[...] They also did not experience, or quickly overcame some of the common difficulties...[with] the equals sign, seeing expressions as processes and not so much as objects, learning to accept formal notation and reading order of operations within that notation. (pp. 156-157)

While Hewitt’s (2012) approach was very different from Mr. Alwyn’s as described in Pirie and Martin (1997), they both addressed the problem students often have with algebraic notation in a similar manner by speaking to the need to achieve solutions not just by inverting operations, but by reversing the order in which the operations were performed. They also had in common a result that the notation does not ever seem to be a hurdle for students who are introduced to the idea of manipulations on particular numbers first and given time to explore the ways in which operations on numbers affect them. The students in both studies grasped algebraic manipulation quickly in comparison to those in more conventional classrooms. This commonality suggests that perhaps it is not the notation that is the problem in the first place, nor the introduction of a variable on both sides of an equation, but the way in which students are conventionally taught what the notation means.

3.5. The Algebra of Linear Functions

Once the requisite amount of time has been spent in this arena, linear equations are introduced in which a second variable is present, and the various things that can be done with these equations is explored: graphing, solving for one variable in terms of the other, solving systems of linear equations, etc. I continue now with a discussion of the difficulties faced when dealing with some of the various features of linear equations in two variables.

3.5.1. Graphing

However mechanically simple, the ability to graph a relation by producing and plotting ordered pairs depends on fundamental concepts such as understanding the notation of Cartesian coordinates, knowing how to represent and interpret an ordered pair, understanding that lines are infinite collections of points, and comprehending slope, among others. The concepts outlined in this section are those analyzed in section 8.2.1.

3.5.1.1. *Graphing: Ordered Pairs*

Little coverage has been afforded in the literature to students' struggle with Cartesian coordinates; perhaps it is an easy concept to remediate or perhaps it has simply been overlooked. It has certainly been the experience of this author that some students confuse the roles of the coordinate values, essentially reversing the effect each value plays in the position of a point on the plane; I have also seen students use the notation of Cartesian coordinates to indicate slope, the x and the y being interpreted as "rise" and "run" (or "run" and "rise"), respectively, an idea similar to that observed by Barr (1980; see subsection 3.5.1.4). If one observer can easily identify two such errors in a domain, it cannot be the case that the topic is unworthy of investigation; however, I can find no research regarding students' difficulties with the elementary concept of Cartesian coordinates.

I unearthed one paper that describes a manipulative approach to learning coordinates in various three-dimensional spaces (Koss, 2011), one that addresses difficulty learning curl (Jung and Lee, 2012), numerous papers and articles addressing specialized coordinates in advanced mathematics courses and references to the use of Cartesian coordinates in not-strictly-mathematical treatments of problems in specialized

fields, such as medicine and mechanics. There seems to be little or no research, however, on the causes of student difficulties with Cartesian coordinates on the plane. As critical as this idea is to the connection between algebraic and graphical representations of linear equations, it seems clear that there is a need for research into this topic to take place.

3.5.1.2. Graphing: Lines as collections of points

One of the critical connections an algebra student must make about linear equations in two variables is that there is a relationship between the coordinates of a point and the inclusion of that point on the graph of a line. The terminology mathematicians use to express this idea is that an equation containing two variables succinctly and explicitly expresses a *relation* between the two variables. The graph of a linear equation demonstrates this relationship.

Considerable attention is given to this idea in Chapter 10 of this dissertation, where I discuss the idea, identified by Moschkovich, Schoenfeld & Arcavi (1993), that “A point is on the graph of the line L if and only if its coordinates satisfy the equation of L ” (p.73), an idea the authors define as “the Cartesian Connection”. Bell and Janvier (1981) stated that according to a 1977 study by Kerslake, “the infinity of possible points on a line was appreciated by only 10-20% [of students in Kerslake’s study]” (p. 36).

Most of the literature I found regarding student uptake of linear functions addresses a particular aspect of linear equations in two variables (e.g. the slope, the equals sign, the manipulation of terms). Knuth’s (2000b) investigation is the only research I have found to date that takes on the challenge of uncovering whether students see that an equation is a way to represent a relationship. This is another area, then, that is ripe for examination.

3.5.1.3. Graphing: Slope

Various studies have investigated students’ issues perceiving slope as a ratio (e.g. Barr, 1980; Simon and Blume, 1994); mistaking the value of slope for height (e.g. Leinhardt, Zaslavsky & Stein, 1990); recognizing slope as a rate of change (e.g. Bell and Janvier, 1981; Orton, 1984) and connecting either the “steepness” or the numerical ratio to a physical model (e.g. Simon and Blume, 1994) to name a few.

Barr (1980) examined post-secondary technical students' retention of various "facts" about slope. In one question, 83% of students in his study correctly selected an image representing a negative slope, meaning that 17% of participants cannot even identify from an image when a slope is considered positive or negative, to say nothing of computing its value or understanding its implications. In a subsequent question, Barr's students were asked what the value of the slope was in an equation in slope–intercept form ($y = mx + b$). Although the equation is in a familiar form, only 62% of students could identify the slope. Given an equation in the form $y = b + mx$, a format frequently used for linear regression models, only 56% were able to identify the slope correctly.

Given its importance, it is perhaps not surprising that slope appears to be the most commonly addressed deficiency in the literature, far outweighing research into others. What follows is a further breakdown of this issue.

3.5.1.3.1. Slope as a ratio.

Barr (1980) highlighted one reason that students sometimes do not see the slope of a line as a ratio, by asking for example, "is 3 a ratio?". (p. 6) He summarized that what is needed is "a 'feeling' for number. A fluency, not only with the fact that $3 = \frac{3}{1}$, or $\frac{3}{4} = \frac{6}{8} = 0.75 = 75\%$, but with the many different notions of numeracy" (p. 6). This statement hedges the argument that slope is the issue in favor of one calling for more fundamental abilities. Simon and Blume (1994) have explained that what is missing in students' uptake of slope as a ratio is specifically their ability "to recognize the ratio as the measure of the attribute in question" (p. 184). In light of Barr (1980), this need for a "'feeling' for number" may be, at least for some students, overlooking those more fundamental deficiencies.

Hohensee (2014) found that, without guidance, students can actually "regress" in their ability to think about linear equations as expressions of proportional relations when learning about quadratic equations, a phenomenon known as "(unproductive) backward transfer" (pp. 136, 161).

3.5.1.3.2. Slope as a rate of change.

Orton (1984) gave this example of a question about rate of change: "Water is flowing into a tank at a constant rate, such that for each unit increase in the time the

depth of water increases by 2 units” (p. 24). He found that even if students appear to have understood a statement such as this, they are often unable to answer questions such as “What is the rate of increase in the depth when $x = 2\frac{1}{2}$?” Several students answered with the y -value from the graph where $x = 2\frac{1}{2}$ rather than the rate. Particularly significant about this result is that the study’s participants were not high school or pre-college algebra students, but calculus students. Orton’s (1984) study also points in the direction of the tendency students have to perceive the dependent variable always as indicating height.

Stump (1999) analyzed the results of a purpose-made survey and found that pre-service teachers (students of education who have not yet begun teaching practice) and in-service teachers could easily interpret slope as a rate of growth, but the same teachers had trouble with questions that asked about slope as a speed and as a rate of increase.

Bell and Janvier (1981) summarized Kerslake’s 1977 report by commenting that, “the relationship between straight lines and their equations was understood by 5-30% [of British secondary students] (depending on age)” (p. 36). In their own study, Bell and Janvier (1981) “showed fully the confusion among the *rate* of increase (gradient), *amount* of increase (interval) and greatest value” (p. 38; *emphasis original*).

3.5.1.3.3. Connecting the algebraic and the graphical representations.

Part of Judit Moschkovich’s (1996) statement in the introduction to this chapter bears repeating here: “Conceptual understanding in this domain [linear functions] involves more than using procedures to manipulate equations or graph lines; it involves understanding the connections between the two representations” (p. 242). In other words, success in learning the algebra of linear functions requires understanding the connections between the variables and between the representations, not just an ability to manipulate them. Yet, to the point that they begin to learn about graphing linear equations, manipulation is all most students have done, and this itself causes problems. Connecting the algebra to the slope is a key skill, but it is part of a larger problem: connecting the algebraic representation of a line with the graph of the line; seeing the equation and the graph as two versions of the same thing. I will address this larger issue in section 3.6.

3.5.1.3.4. Connecting slope to a physical model.

Stump (1999) explained the importance of connecting the idea of slope to a physical model:

[A]lthough slope has its reference in geometry, slope is a rate of change and therefore it also has meaning in formulae, tables, physical situations, and verbal descriptions. Furthermore, slope is closely related to the concept of derivative.... It is a deep mathematical idea that threads its way throughout the curriculum.... Meaningful learning of mathematics includes forming relationships between conceptual and procedural knowledge (Hiebert & Lefevre, 1986, pp. 125, 127).

Leinhardt, Zaslavsky & Stein (1990) made the observation that height is a commonly used (dependent) variable in the early algebra curriculum; this makes good sense, even if only from the perspective that it is not a big leap for students to record a taller height farther up the axis of the dependent variable. They went on to say, however, that when comparing two lines on the same graph and asked which shows the greater speed, students will select the line that has the higher values for its dependent variable, and the authors assessed that this is an error for the reason that students are mistaking the function's value for the slope.

In another example by these authors, students were asked to view a graph of a wide jar being filled with water, then sketch another graph showing a narrow jar being filled. The authors commented that "The students took account of the fact that the water would be higher in the narrower jar but failed to note that the rate (the steepness or slope of the graph) would be greater also" (p. 39).

A subtly different conclusion arose in McDermott, Rosenquist and VanZee (1986). From a question in which the positions of two objects are shown over time on the same graph, these authors stated:

Most incorrect answers seem to be due to failure to realize that the information about the velocity cannot be extracted from the height.[...] [M]any students focus on [the] difference in height, rather than on the difference in slope, to determine which object has the greater speed. (p. 504)

3.5.1.4. Graphing: The y -intercept

Davis (2007) wrote about the importance of the y -intercept in the following way:

The y -intercept holds the potential to promote the Cartesian connection because of its presence within the slope–intercept form of an equation. In other words, the y -intercept leads to the straightforward identification of a point on an oblique line since it is the y -coordinate when the x -coordinate is zero. Knuth (2000[b]) researched students who had mathematical backgrounds ranging from first-year algebra to calculus and found that students did not apply the Cartesian connection to translate a graphical representation to an algebraic one. Instead, students preferred to move in the opposite direction, from algebraic to graphical representations.... (p. 388)

The translation of which Davis speaks is really a comment on making connections, and an important theme that recurs, first in Knuth, as discussed here, and again in the discussion about connecting graphs and equations near the end of this chapter.

Knuth (2000a) illuminated one of the problems students have with the y -intercept: “Students are routinely given tasks that require translations in the equation-to-graph direction.... (Leinhardt, Zaslavsky and Stein, 1990) ... [They] may have difficulty on tasks in which they must proceed in the graph-to-equation direction” (p. 53). Davis’ (2007) answer to this problem was to suggest not only a delay in introducing the formal terminology, but that in the interim, students be allowed, and in fact encouraged, to create their own terminology. His reasoning for the use of this tactic was in part that terminology created for a particular purpose will have more meaning to the individuals using the concept they are naming. Davis supported the idea of “natural language”, recalling its promotion by Lakatos in 1976 and Lampert in 1991. “At the same time”, however, Davis reminded us that “students’ use of natural language may result in misunderstandings through a process of semantic contamination (Pimm, 1987). This can happen when the more common definition for a term contrasts with its use in the mathematics classroom” (p. 389). That Davis gives us a potential solution to the problem of the y -intercept is encouraging, but it is interesting to note that students may be hesitant to use their own terminology unless it has been sanctioned, in some way, by the teacher (Davis, 2007, p. 401).

Hattikudur et al. (2012) began their exploration of students’ learning about slope and y -intercept by arguing that the “real world” offers more intuition with regards to slope as a rate, but:

students in one study found it difficult to plot a y -intercept when graphing the linear relationship between the number of scoops of ice cream in an ice

cream cone and the amount of money it costs (Davis, 2007). Students were reluctant to plot a point where the ice cream cone would be purchased without ice cream (the y -intercept), believing that no one would ever purchase the cone alone in the real world. (p. 231)

Hattikudur et al. concluded that the y -intercept is actually more difficult for students to grasp as a concept than is slope. They reported errors such as plotting a point on the y -axis but at the wrong value position, plotting the initial value associated with an “input” value of 1 rather than 0 (and some of these students then connected this point to the origin), and changing the slope of a line rather than moving it from the origin.⁵ In general, Hattikudur et al. explained, there is “a tendency for students to begin a line at the origin” (p. 235) and “students are not as successful at graphing y -intercept as they are at graphing slope” (p. 239).

In the article discussed in section 3.5.1.3, Barr (1980) examined student responses to questions which gave the equations of lines in various forms and found in one question that if the equation was given in slope–intercept form ($y = mx + b$), 81% of students could correctly identify the y -intercept, but only 45% of students could accurately indicate that they were aware that the sign of the gradient did not affect the sign of the y -intercept. For another item, 20% of respondents selected an answer that suggested that they did not consider that a line could pass through the origin (i.e. that the constant term could be zero). In a third item, 26% of respondents in Barr’s study gave an incorrect response that suggested that they may have mistaken the gradient for the x -intercept.

3.6. Writing the Equation of a Line

In 3.5.1.3.3, I examined the connection between the slope as a ratio or a rate and the use of the numerical value in creating a graphical representation; here, as promised, I look at a different connection: that of moving between the algebraic representation of linear equation and the graph of such an equation. This connection is the idea conveyed by Davis (2007) and Knuth (2000a) discussed briefly above, as a “translation” between representations.

⁵ The tasks performed by participants were not described in this paper, so results are difficult to interpret.

I have already summarized one of Knuth's (2000a) findings as he set out "to discuss a possible reason for the inadequate and often absent connections that students made between [algebraic and graphical representations of functions]" (p. 48). Another of his conclusions was that "After students have been exposed to this connection, teachers generally assume that little or no review is needed (Schoenfeld, Smith and Arcavi, 1993)" (p. 52). According to Knuth (2000a), then, the issue is with the instruction:

The nature of the instruction that students receive, in both the representations that are emphasized and the kinds of translation tasks that are presented, may significantly contribute to the difficulties that many students have in connecting equations and graphs....

[M]any students seemed to perceive that the graph was unnecessary or that it served only to support their algebraic solution methods rather than be a means to a solution in and of itself.... Many students failed to realize – after converting the equations into slope–intercept form and calculating the slope of each line by selecting points from each line – that the points used in calculating each slope were themselves solutions to the equations. (pp. 52-53)

The ability to see that the graph of a line and its algebraic equation are two different ways to represent the same information is missing for these students. Regardless of where the fault lies, if students are not seeing the connections between the "pieces of conceptual knowledge" present in the algebra of linear functions, then they are not "competent" (Moschkovich, 1996) in the discipline; furthermore, their lack of competence in one domain of learning can both propagate into their next algebraic experience and influence previous learning, a phenomenon nicely summarized by Hohensee's (2014) "(unproductive) backward transfer" (p. 161).

3.7. Chapter Summary

Students continue to struggle to digest the concepts of algebra, and individual teachers continue to explore ways to address those struggles. Research cited here ranges from 1935 to the present and, macroscopically at least, researchers are still investigating the same problems now as they were eighty years ago.

A particular difficulty has attracted either a great deal of research attention or very little. The areas that require additional attention seem to be around fundamental concepts. Perhaps it is because so little attention has been paid to the issues students

have with these fundamental concepts that they struggle with more intermediate ideas such as the real meaning of slope, or the existence of a relationship between variables.

The commonalities in Pirie and Martin (1997) and Hewitt (2012) suggest that fundamental issues may be at stake. It would be impractical to re-teach or remediate all of the essential underlying ideas during the course of an academic (college) term if for no other reason that there is already insufficient time for the existing content to be conveyed. Although it is beyond the scope of this dissertation, it may prove beneficial to look for alternatives to the current structure of college curriculum.

Regardless of the reason, the minor attention in the current literature provides rich veins that may yet be exploited. The field of mathematics education has, for eighty years (or more), investigated various student difficulties learning the algebra of linear equations and functions, but not much has changed in the way learning is perceived or the way the subject is taught. There is, then, still considerable work to be done.

Chapter 4.

Community College Mathematics Education

In this chapter, I describe the problem encountered when trying to define “community college” and summarize my choice of a working definition adequate to the undertaking in this thesis. I outline the history of the community college as an institution, explore the existing literature on community college mathematics education, and describe the community college student population and some of ways in which this population is different from those in other tertiary educational establishments.

4.1. A Note on the Definition of “Community College”

I have undertaken the work associated with this dissertation from within the traditions of Simon Fraser University in British Columbia, but my own experience of the “community college” is biased by my familiarity both as a student and as faculty at several community colleges in the U.S. The community college is a global phenomenon, but it is an entity that defies a single definition. Institutions called “community college” the world over are, by their nature, created and maintained to serve the communities in which they are situated, which means that they are inherently different from one another. These enterprises serve many and varied purposes for the students who attend them, but because they can be very different, a single definition is not an easy thing to secure.

For the purpose of this dissertation, however, it will be important to have a consistent description of what a community college is. Rosalind Raby (2009) describes the problem thus: “Although [community colleges] play a cohesive role in national education forums, a basic understanding of their construct remains elusive due to the fact that these institutions are defined by local needs” (p. 3). Her solution is to find the areas of overlap between institutions that serve the educational needs of the communities in which they are found and use these to set the minimum qualifications an institution must meet in order to be considered a community college in a global sense. To that end, for the purpose the work that follows, I take a “community college” to be defined as by Raby (2009) as an institution which:

1. is post-secondary and post-compulsory;
2. is situated in a binary higher education system (i.e. there is (at least) one higher education option other than the one in question);

3. targets non-traditional students;
4. offers programs with varying lengths of curriculum;
5. has a lower status in the eyes of the public and among scholars, and receives less governmental financial assistance than do universities;
6. promotes a curricular emphasis;
7. combines curricular competence with social instruction;
8. is “open-access” (any student who applies will be accepted for admission).

Note that this list of descriptors can be applied to an educational entity anywhere in the world, although my own experience, and the experience upon which I have based the work of this dissertation, is that of U.S. community colleges (although even these can be very different from one another). I began work on this dissertation with a very clear idea in my mind of what a community college was. I attended three community colleges in 1986 and from 1989 to 1992, and I have taught mathematics at three different community colleges from 1997 to 1999 and between 2011 and the present. In hindsight, this exposure is very limited. Not only is the sample on which I have drawn my many unconscious assumptions very small, but all of these colleges were either in California, Washington state or Texas. All of them were based on very similar community needs, and my perception of at least four of them was that of a novice (either a student or an un-invested, first-year adjunct instructor), which meant that I had a very narrow view of how they were run or what their missions were.

I would be disappointed if any work that I produced were to have the effect of restricting the definition of a community college to Raby's list provided above; community colleges are valuable in part because they are all so different. There are two distinct community colleges in the town where the research for this dissertation was conducted, and both are thriving, valuable commodities in the community; they happen to serve very different needs, and while there is sometimes a sense of competition between them for students, these two colleges collaborate to ensure that as many as possible of the educational needs of the community are met. To require a community college to submit to a hard-and-fast definition would be to dampen the essence of the very thing I am attempting to uplift by its study.

4.2. A Brief History of the Community College in the United States

U.S. Community colleges today serve very different purposes from those their originators intended. The idea of the community college was inspired in the late 19th century by a desire to keep the scholarly research activities of the universities of the day “pure” and to provide a place for those viewed as not suited to academia still to have a place to acquire a post-secondary education. Initially, students could attend “junior college” as an extension of their time in high school. The courses offered at junior colleges during this time were only those traditionally offered during the first two years at university; this structure was meant to provide some students with the academic background required for research (i.e. better preparation for and transfer to a university), and others with a trajectory that would make them more attractive as workers or employees (Mesa, 2017).

The two major benefits of this new structure at the time to the universities were that they would no longer need to offer elementary mathematics courses, thereby enabling them to cater to students at more advanced levels, and that students attending universities would, by default, be seeking research-oriented degrees (Mesa, 2017). “[T]he junior college was also a way to keep universities selective and specifically focused on research and advanced disciplinary training” (Beach, 2011, p. 73).

Thus, the first two functions served by community colleges were “transfer to a four-year institution and general education” (Mesa, 2017; p. 950). During the course of the 20th century, the focus of post-secondary education changed from politico-social preparation to vocational training (Beach, 2011), thereby creating a third function: that of vocational or technical education (Mesa, 2017). Over time, community colleges also came to serve the additional functions of community leisure (now often called “continuing education”), remediation and worker retraining (Mesa, 2017).

The U.S. Department of Education (DOE) lists “community colleges” under the Office of Career, Technical and Adult Education (OCTAE) along with two other divisions: “adult education and literacy” and “career and technical education” (Office of Career, Technical, and Adult Education). That the U.S. Department of Education groups these three types of institutions together administratively is telling. This grouping suggests that

in the eyes of the DOE, the purpose meant to be served by community colleges is in some way equivalent to those served by literacy and technical programs. This statement is not meant to diminish the important work done by those other systems, but it does highlight two important issues: that the DOE views the work of community colleges as being targeted toward, or at least inclusive of, those needing to achieve basic literacy or technical (i.e. non-academic) skills; and that community colleges need to be distinguished from universities by their functions.

This distinction is no surprise, since it is the basis upon which community colleges were conceived; but that community colleges are not specifically distinguished from literacy and technical education environments means they are viewed with the same lens as those structures. Community colleges frequently offer literacy classes and most have technical programs, but today these programs are in addition to pre-baccalaureate academic courses. In fact, then, community colleges belong in both camps, (academic and literacy/technical) but are treated as though they are in neither.

4.3. Community College Research in Mathematics Education

Much effort has been focused on researching mathematics education but little of that deals with community college students or faculty. There has also been considerable research into the community college as an entity since 1974 when the Center for the Study of Community Colleges was established, and there are multiple journals whose focus is community college issues⁶. In spite of this wide selection of research repositories however, the research does not, by and large, address community college *mathematics* education.

The *Two-Year College Mathematics Journal* was published between 1970 and 1983, and The American Mathematical Association of Two-Year Colleges (AMATYC) began publishing *The AMATYC Review* in 2002 and retired it in favor of *The MathAMATYC Educator* in 2008. Many of the contributors to the *MathAMATYC Educator* are community college research-practitioners, but there is still a great deal of capacity for

⁶ Some examples include *Community College Journal*, *Community College Journal of Research and Practice*, *Community College Review*, *Community College Leadership*, *Journal of Applied Research in the Community College* and *New Directions for Community Colleges*.

more research and considerable opportunity for community college practitioner research. There are no other such publications.

Informal Internet searches quickly reveal about a dozen journals whose focus is mathematics education, but none of these addresses community college populations as a rule. Furthermore, the *MathAMATYC Educator* does not appear on most of these lists, possibly because while it does offer articles based on empirical studies, it does not necessarily do so regularly, according to Johanna Debrecht, the journal's current editor, favoring instead to produce an occasional special issue (Debrecht, 2019; personal correspondence). *MathAMATYC Educator* assistant editor George Alexander explained that its predecessor, *The AMATYC Review*, did print empirical studies during its publication, but "had a more general overall calling, and included more for abstract mathematical results as well as two year mathematics" (Alexander, 2019; personal correspondence).

In 2017, Vilma Mesa conducted an exhaustive analysis of academic journal articles addressing community college issues between 1970 and 2014, dividing this period between that time before the inception of the *Community College Journal of Research & Practice* in 1975 and after, since this was the first such journal to target both community college researchers and practitioners. In the 35 years between 1970 and 2004, Mesa identified 98 articles as having abstracts containing relevant keywords ("mathematics," "junior college," "two-year colleges," or "adults"), an average rate of 2.9 articles per year. Mesa retained for inclusion in the study only 40 of this 98, since these were the only ones that studied two-year college students (1.2 articles per year); only 25 of the 40 retained also addressed mathematics education (0.26 articles per year). Mesa's table of results for this time span shows a decidedly increasing trend during the period between 1970 and 2004 both in the number of journal articles addressing community college populations and in those dealing with mathematics education. Between 2000 and 2004 for example, the last four years of the "early" period, there were 36 articles (9 articles per year) of which 23 were retained (5.8 articles per year), and 18 articles (4.5 articles per year) that addressed mathematics education.

In the years between 2005 and 2014, Mesa found that "the majority of the scholarship [...] continues to be conducted in the field of higher education (rather than

mathematics education)” (p. 954), but that research is well differentiated. Topics found in these later submissions included:

...low success rates in developmental mathematics courses (Attewell, Lavin, Domina, & Levey, 2006; Bahr, 2008, 2010; Bailey, 2009; Bos, Melguizo, Prather, & Kosiewicz, 2011; Calcagno & Long, 2008; Melguizo et al., 2008); specific aspects of instruction: faculty, students, and content inside the classroom (Leckrone, 2014; Mesa, 2010c, 2011; Mesa, Celis, & Lande, 2014; Mesa & Lande, 2014; Sitomer, 2014); and curriculum reform (Van Campen, Sowers, & Strother, 2013). (page 954)

Mesa found an additional 98 articles in the later nine-year span, a rate of 10.9 articles per year, a great improvement on 2.9 articles per year. Furthermore, of the later 98 articles published, a full 81 addressed community colleges (9 articles per year), and 50 of these (5.6 articles per year) addressed mathematics education. Not only were there a great deal more publications in the near-decade between 2005 and 2014, but a significant proportion more of these dealt with mathematics education; Mesa was able to categorize her final list into several topics: students, curriculum, faculty and instruction.

Lack of professional attention from practitioners in particular should not be interpreted as a lack of interest. University faculty are expected to publish original research in their disciplines, while their counterparts at community colleges are not generally expected to do so. Scholarship in mathematics education is likewise primarily the domain of university researcher in part for the same reason as is mathematics, and in part because community colleges do not generally employ education faculty. Williams (2014) lists several additional “barriers to engagement in scholarship” (p. 35) of community college faculty. Among these are time and financial restrictions, as well as the “difficulty of faculty members...to see scholarship as being an integral part of the overall mission [of their institutions]” (p. 35), and a sense of inferiority when comparing themselves to their university counterparts. “Faculties at community colleges execute the mission [of their institutions] by focusing primarily on instruction. ...[R]esearch endeavors are voluntary...” (Cohen & Brawer, 2007, as cited in Williams 2014, p. 32). The fact that community colleges are not perceived as research institutions is a natural extension of their historical development, but it may play a role in the poor reputation that the institution of community college suffers publicly.

4.4. The Community College Population

One of the reasons for their reputation of inferiority may be the nature of their populations. Roughly half of all undergraduate students start their tertiary academic career at community college (Mesa, Wladis & Watkins, 2014; Mesa, 2017). An institution that serves such a large proportion of the population of students, that has placed on it such high demands and must attempt to meet those demands with restricted resources deserves to be acknowledged and certainly needs to be studied.

U. S. Community colleges serve about half of all first- and second-year undergraduates, and almost all college-bound students not initially accepted by universities, thanks to open enrollment policies; most community colleges in the United States practice such policies (Scherer & Anson, 2014). As such, community colleges provide a structure of tertiary education for a higher proportion of adults and veterans than do universities, and they make college enrollment possible for many who would not otherwise be able to attend. Mesa (2017) puts the role of the community college in perspective succinctly with the statement “No other institution of higher education has such a complex set of demands and such diversity of functions” (p. 950). Community colleges do not (for the most part) provide the third- and fourth-year experience that universities do, and they serve a different population than do universities; thus it is important to recognize the institution of community college in its own right, rather than assuming that the undergraduate experience is the same at community college as at university.

[Community colleges] have high levels of attrition, low rates of degree completion and transfer, and a disproportionate investment in remedial education relative to other higher education institutions. Such problems are specially heightened in mathematics. Some argue that community colleges’ multiple functions and diverse goals, together with the vicissitudes of federal, state, and local funding are breeding grounds for such ‘failure’. (Grubb, 1999; Jacobs, 2011; Labaree, 1997; all as cited in Mesa, 2017, p. 950)

However, these institutions were designed to take in students that the universities did not want. It is not surprising that completion rates are low when the population is considered in comparison to its university counterparts. In 2010, about half of all undergraduates taking a mathematics course were doing so at a community college, but these students were also more likely to work at least part-time and to be

older than their university counterparts. A higher proportion were students of color, and more women attended part-time than did at universities (Mesa, 2017).

Since community colleges were designed to attract these very populations, it should not be a surprise that students enrolling at a community college are more likely to be under-prepared to be successful in a college career. They have either been out of the school environment for some time (they are older), they are not academically ready for college (they have been rejected by the university system), or their attention is divided (they work or are raising children in addition to taking classes). The focus has been on the “failure” rate at community colleges when it should instead be on the success of getting people into higher education who would not otherwise have had the opportunity or the financial, social or academic foundation. That is not to say that the system is perfect, nor that it should simply accept its attrition and completion rates without attempting to improve them, but community colleges are not the “failures” or the subordinate institutions that their detractors claim.

4.5. Community College Mathematics Education

Fortunately, the perception around community colleges as providing sub-standard education has begun to change, in large part due to the work of Vilma Mesa and colleagues since about 2010. Education researchers have been looking to community college students and instructors more in recent years as a subject worthwhile of study (e.g. Mesa, 2017; Burn, White & Mesa, 2017; Gomez et al., 2015; Lande & Mesa, 2016), although mathematics education is still an underrepresented topic in this setting.

Students entering community colleges in the U.S. are often unprepared for college-level mathematics courses (between 50% and 67%, according to various surveys cited in Mesa, 2017); these students are required to enrol in mathematics courses that cover content typically taught in middle and high school. Various terms in the vernacular describing these courses include “remedial”, “developmental” and “pre-college” algebra. These terms are more or less interchangeable, although some institutions prefer specific nomenclature (Mesa, 2017). The research conducted for the current dissertation is situated in one such course.

4.5.1. Community College Faculty

As mentioned in section 4.3, the focus of community college faculty tends to be on teaching, not on the scholarship of teaching or on research in their disciplines. The data for Williams' 2014 dissertation⁷ included only full-time community college faculty for a variety of reasons (Williams, 2014; p. 15), in spite of the fact that part-time faculty make up about two thirds of the teaching faculty at community colleges (Williams, 2014; Mesa, 2017). Between 1995 and 2010, the number of university students enrolled in mathematics courses dropped by 4%, but enrollment in the same courses at community colleges remained a robust 57% during the same two years. 44% of community college mathematics courses were taught by part-time faculty, most of them pre-college courses; at the same time, 57% of students enrolled in a mathematics course at a community college were taking pre-college courses; only 11% of university mathematics students were enrolled in such courses. "That is, by 2010, two-year colleges enrolled almost four times as many remedial mathematics students as other institutions of higher education" (Mesa, 2017; p. 951).

Part-time faculty (also called "adjunct" or "contingent" faculty at various institutions) must be equally qualified to teach community college courses as their full-time counterparts, but there are limitations on their functions within the departments for which they teach, such as not being permitted to teach more than 75% (or some other fraction) of an academic year or being asked or expected to teach more than 100% of a "full-time" teaching load; unavailability of courses during particular terms; lower pay rates; and an undeserved reputation as under-qualified, for example. "There was also concern that part-time faculty would dilute the quality of community college education because the so-called alienated freeway flyers taught about 30 to 40 percent of all full-time equivalent contact hours" (Beach, 2011).

It is imperative to recall that the community college system, now often decried as somehow inferior, was created to be the way it is. During the 1970s and 80s, the turn to more part-time instructors was meant to relieve the financial struggle encountered by community colleges that were being left out of the funding opportunities afforded to universities (Beach, 2011). Part-time faculty who are qualified to teach college

⁷ *The Scholarship of Teaching at Community Colleges*

mathematics courses often take multiple part-time positions in order to make up for the lower pay and inadequate teaching load. The fiscal savings to the colleges for which they teach are undeniable, but the consequences are not surprising. Faculty teaching part-time at two or more colleges have less time to prepare for classes or meet with students. This lack of availability lends itself to the perception that the teacher does not care about her students or the classes she is teaching, and to the perception she is delivering an inferior academic experience.

4.5.2. Professional Development

In addition to a growing interest in the community college student, there is a burgeoning trend in the professional development of tertiary faculty in educational strategies (e.g. Mesa, Celis & Lande, 2014; Edwards, Sandoval, & McNamara, 2015). Although they ultimately convey the same content, secondary school mathematics teachers and tertiary faculty teaching pre-college mathematics undergo vastly different preparation for teaching. Pre-service high school teachers in the U.S. are required to complete a certification program that typically includes coursework in teaching techniques, student development, and culture and equity training, among other topics; a visit to any university education department web site will verify that these students are required to take several pedagogy and psychology courses in addition to courses in their discipline if they hope to be put forward for teaching certification. Secondary pre-service teachers must also undergo several phases of practicum and internship before becoming certified to teach without supervision (“Secondary Education, PreProf”).

In stark contrast, college and university instructors are not required to obtain a licence to teach. As a result, while they may be experts in their respective disciplines, they are not necessarily exposed to the principles of how people learn, alternative methods for teaching, classroom and community cultures and traditions, the needs of students as individual learners or the dynamics of a class as a body of learners. A tertiary educator is nearly always required to have a higher degree in her discipline, but her knowledge of teaching as a discipline in its own right is frequently limited to the methods to which she was exposed as a student herself.

The teaching and learning community in general has become more and more aware in recent years that the inclusion of student-centered learning as a teaching

strategy is more effective than lecture alone (e.g. Aditomo, Goodyear, Bliuc & Ellis, 2013), and tertiary institutions are beginning to take up this mantle alongside their secondary counterparts, for example by providing professional development programs that address possible alternatives to lecture (e.g. Wright, 2011).

One such program is “Change at the Core” (C-Core), developed at Western Washington University (WWU) in Bellingham, Washington. C-Core began in 2014 when secondary teacher education researchers at WWU, studying the results of student-centered training of pre-service secondary teachers, identified a need for tertiary education faculty training.

A Collaborative Model for Undergraduate STEM Education Reform (C-Core) is an NSF-funded institutional transformation project. Three institutions, Western Washington University, Whatcom Community College, and Skagit Valley College, are working together to transform courses in Biology, Chemistry, Computer Science, Engineering, Environmental Science, Geology, Mathematics, and Physics from teacher-to student-centered learning environments. Currently, over 100 STEM faculty are participating in professional development workshops, implementing and observing evidence-based teaching and learning practices, and collaborating on curriculum changes and course alignment across the three institutions. (“Science, Mathematics, and Technology Education”)

In the course of five years (2014-2019) and three separate but overlapping cohorts of participants, the aims of the C-Core program were largely achieved, and changes are being seen in the hiring, tenure and promotion processes that reflect the changes the participating bodies have undergone (Geary, 2019). Critically, the program was extended multiple times allowing not only for continuity among teaching colleagues in different cohorts, but also for new and ongoing interest and participation.

According to Shannon Warren, a spokesperson for the project, “cohort 3 really pushed some departments and divisions to a tipping point so now *more* instructors understand and value student-centered teaching and learning than [do] not” (personal correspondence; emphasis original).

4.5.3. Mathematics Courses Provided at the Community College Level

Mathematics courses offered at community colleges include many of the courses offered to university students in their first two years of undergraduate study. Courses

that teach algebra, trigonometry and calculus are common in both types of institutions, and although many community colleges also have courses in linear algebra, differential equations and sequences and series in their catalogues, these courses are frequently not filled, or only offered once every one or two years due to historically insufficient enrollment. Courses beyond this level are not seen at the community college; any student needing or wanting to take more, or higher levels of mathematics would likely be a mathematics, physics or engineering major. All of these students would be destined to transfer to a university and thus required to take at least half the credits required for graduation from that institution at that institution. Taking additional courses at the community college before transferring does not benefit the student, and in fact can become a financial liability.

Courses below college-level algebra (frequently called College Algebra or Precalculus 1) are offered at some universities, but those institutions are not in the business of remediation. Recall that the history of community colleges includes a need to keep university seats available for “real” scholars; offering remediation simply doesn’t fall in line with that mission. Recall also that community colleges early took on the challenge of providing remediation as one of their six functions. With community colleges providing robust remediation, there is even less need for universities to address this need.

While there does exist an overlap of basic undergraduate mathematics provided at both types of institutions, community colleges are the only place for students to obtain thorough mathematics remediation in preparation for college-level courses. Universities see remediation as the job of the community college; to an extent this perspective can be explained by the historical role of the community college in alleviating universities from the need to provide such remediation.

4.6. Community College Resources

Colleges founded in the early days of the California Community College system enjoyed guaranteed state and federal funding. Between 1907 and 1921, this funding came from the state legislature; in 1917, funds from state and county coffers were earmarked for community colleges; in 1921, the Junior College Act passed, establishing

a “permanent Junior College Fund to further support these fledgling institutions” (Beach, 2011, p. 72)

Beach (2011) tells us that the Junior College Act dedicated \$2,000 to each district and \$1,600 to each college annually, and an additional \$90 for each unit of average daily attendance. He does not state for how long these amounts continued to be “permanent”, nor whether they have been increased in line with the cost of living. Additional funds were available to California junior colleges from “junior college district taxes, high school taxes, and fees for nonresident students” (Beach, 2011; p. 72). “Economic development policies” provided additional funding in the 1970s and 80s (Beach, 2011; p. 76). In the late 1940s though, California state legislators became concerned about growing competition between “adult education programs, junior colleges, teachers [sic] colleges, state colleges, and the University of California” (Beach, 2011, p. 78), and given the general sentiment among the committee reporting on the then-current state of the community college in California, the die was cast for universities to receive priority over community colleges whenever there was a funding decision to be made.

“Legislative and professional oversight over the junior college had always been geared toward the priorities of the university, and this arrangement would later be consecrated in...the California model, which was later emulated around the country” (Beach, 2011; p. 77). Its place in the minds of university administrators, even early on, as the dumping ground for those not suited to university level work meant that community colleges were destined to be the underdog in any battle over funding or other resources. The system was designed to be this way, and community colleges have been struggling to maintain their many functions with limited resources ever since. The process by which resources are allocated by bodies in competition for those resources has been replicated across the U.S., embedded as it is in the California model.

4.7. Chapter Summary

That community colleges are an important part of the post-secondary education system is undeniable; that they are under-resourced is of little debate. What has been less obvious is the need to study the community college: its student body, its faculty, its strengths and challenges. There is considerably more literature published now than was

previously the case, but there is more work to do, especially if the community college is to throw off the mantle of inferiority that it has worn since its inception. Those taking up the scholarship of community college mathematics education can certainly do so by standing on the foundation laid by those researching mathematics education at both K-12 and university institutions. The only element that varies between these traditions is the *Audience*, which is more diverse and more widely experienced than its K-12 or university counterparts. As such the community college population is possibly even more able, rather than less, to adapt to the challenges with which it is presented.

Community colleges may have begun from a desire to keep out undesirable students from getting in the way of “real” academic work at universities, and they may still suffer a greater proportion of failures than do universities, but they serve multiple, vital functions in the modern educational system, providing a second chance for some, a first chance for others, and a means by which to change directions to others still. As Mesa (2017) has observed, no other academic tradition has had so much expected of it and been given so few resources with which to carry out those expectations. Research into the community college population may even highlight some of the ways in which these institutions benefit the communities in which they are situated, which might, in turn, provide funding agencies the impetus and the motivation needed to support community colleges in order that they can better do the things expected of them.

Many of the students who attend these institutions intend to continue their education beyond the first two years, transferring to a university to complete a baccalaureate degree. These students begin their post-secondary education at community colleges for a variety of reasons, but this goal in particular means that community colleges must prepare these students by providing first- and second-year courses and instruction that parallel or exceed those found at universities. Because they only provide first- and second-year courses, community colleges are positioned to meet this demand exceptionally well, even under-resourced.

Since community colleges provide services to these and other students that universities do not, and since the universities that later accept these students in transfer benefit alongside the students themselves, not to mention greater society, it seems only reasonable to attempt to find some ways to approach teaching these students in a manner that will increase success and transfer rates.

Chapter 5.

Realistic Mathematics Education

Realistic Mathematics Education (RME) is a now well-established theoretical perspective that relies on the idea that rather than teach an abstract concept in a manner that emphasizes mechanical manipulation, the concept should be embedded in a “realistic” problem so as to make its solution, and the emergent mathematics, meaningful (Cobb, Zhao & Visnovska, 2008). RME presents students with real problems to be solved, and the problems motivate the mathematics, rather than the other way around.

This chapter examines the theoretical perspective known as Realistic Mathematics Education. I provide a brief history of the theory and discuss its potential for teaching algebra; specifically, I explore the use of RME for teaching pre-college algebra as a way of addressing the difficult history and troublesome relationship many community college students have with mathematics. I also discuss the potential benefits and drawbacks to using RME in this setting and explain how RME can be used to teach specific algebraic *Concepts*. I hope to convince the reader not only that RME is a robust and viable method with which to teach, but that it has a place in the contemporary college setting in addition to the primary and secondary settings for which it was first conceived.

5.1. Realistic Mathematics Education: A Brief History

Realistic Mathematics Education (RME), began as the “Wiskobas”⁸ project when, in 1968, Edu Wijdeveld, Fred Goffree and Adrian Treffers began collaborating to reform mathematics education in the Netherlands. Hans Freudenthal joined the Wiskobas team in 1971 (van den Heuvel-Panhuizen & Drijvers, 2014), and “put Wiskobas on the track of realistic mathematics education” (Treffers, 1993, p. 89) and away from the “New Math” movement that was growing in the Netherlands. In 1973, the project was extended to the secondary level with the name “Wiskivon” (van den Heuvel-Panhuizen & Drijvers, 2014).

⁸ Wiskobas is an acronym for the Dutch translation of ‘mathematics in primary school.’ [Wiskunde in de basisschool] (Treffers, 1993, p. 106)

When Freudenthal joined the Wiskobas team, he did so as its first director, an appointment which coincided with the founding of the Institute for the Development of Mathematics Education (IOWO)⁹ and Wiskobas' membership in that organization (van den Heuvel-Panhuizen & Drijvers, 2014). RME, which stems from Freudenthal's "didactical phenomenology", is the idea that rather than teach an abstract concept in a manner that emphasizes mechanical manipulation, the concept should be embedded in a "realistic" problem so as to make its solution, and the emergent mathematics, meaningful (Cobb, Zhao & Visnovska, 2008). Freudenthal was an avid proponent of "mathematics as a human activity" (van den Heuvel-Panhuizen, 2000, p. 3); he advocated that "mathematics education should take its point of departure primarily in *mathematics as an activity*, and not in mathematics as a ready-made-system" (Gravemeijer & Doorman, 1999, p. 116; emphasis original), and he believed that "mathematics must be connected to reality, stay close to children's experience, and be relevant to society, in order to be of human value" (van den Heuvel-Panhuizen, 2000, p. 3); he stressed guided reinvention (Cobb, Zhao & Visnovska, 2008; van den Heuvel-Panhuizen, 2000); and he promoted the delivery of concepts via "rich contexts" (Treffers, 1993; van den Heuvel-Panhuizen, 2014, p. 523). According to Cobb, Zhao & Visnovska (2008), RME can be summarized by three precepts: learning should be "experientially real, ... justifiable in terms of the end points, ... [and] support the process of vertical mathematization" (p. 106). The last of these is "the notion that the 'lower' activity offers a necessary basis of experience for the 'higher' activity" (Treffers, 1987; p. 62), which satisfies the *level* principle (q.v., below) of RME.

5.2. RME's Features

Adrien Treffers played a key role in developing RME, not least in developing the core tenets of the theory. These were originally given by Treffers (1987) as "phenomenological exploration by means of contexts, bridging by vertical instruments, selfreliance (sic), pupils' own constructions and productions, interactivity, and intertwining of learning strands" (pp. 247-250). van den Heuvel-Panhuizen (2000)

⁹ IOWO (Instituut voor de ontwikkeling van wiskundeonderwijs) is "the oldest predecessor of the Freudenthal Institute" (van den Heuvel-Panhuizen, 2000, p. 3); the latter is the contemporary study group at Utrecht University in the Netherlands.

adapted and extended these five tenets to six principles, and it is from these that RME can be identified in a contemporary classroom setting:

the **activity** principle “means that in RME students are treated as active participants in the learning process [and] emphasizes that mathematics is best learned by doing mathematics” (Treffers, 1993; pp. 104-105);

the **reality** principle “expresses the importance that is attached to the goal of mathematics education including students’ ability to apply mathematics in solving ‘real-life’ problems.... [and] means that mathematics education should start from problem situations that are meaningful to students, which offers them opportunities to attach meaning to the mathematical constructs they develop while solving problems” (van den Heuvel-Panhuizen & Drijvers, 2014; p. 523);

the **level** principle “underlines that learning mathematics means students pass various levels of understanding: from informal context-related solutions, through creating various levels of shortcuts and schematizations, to acquiring insight into how concepts and strategies are related” (van den Heuvel-Panhuizen & Drijvers, 2014; p. 523. See also Treffers, 1987; pp. 242-244);

the **inter-twinement** principle “means mathematical content domains such as number, geometry, measurement, and data handling are not considered as isolated curriculum chapters but as heavily integrated” (van den Heuvel-Panhuizen & Drijvers, 2014; p. 523);

the **interaction** (or **interactivity**) principle, “signifies that learning mathematics is not only an individual activity but also a social activity [which is the reason] RME favors whole-class discussions and group work which offer students opportunities to share their strategies and inventions with others” (van den Heuvel-Panhuizen & Drijvers, 2014; p. 523);

the **guidance** principle refers to “[o]ne of Freudenthal’s key principles for mathematics education...”, that students should have “a ‘guided’ opportunity to ‘re-invent’ mathematics” (van den Heuvel-Panhuizen, 2000; p. 9).

RME can be viewed as a formal, well-structured and well-documented framework for instruction of mathematics in primary and secondary schools. It is a multi-faceted, contemporary theory that has been used in Indonesia (see for example Sitorus and Masrayati, 2016; Zakaria and Syamaun, 2017) and the UK (see for example Dickinson and Hough, 2012; Hough and Gough, 2007), and is known in the US as “Mathematics in Context” (Dickinson and Hough, 2012, p. 4). As a contemporary theory, it has been the basis for other emergent frameworks such as design research (q.v. in section 6.2; see also for example Cobb, Jackson and Dunlap, 2016; Gravemeijer, 1994; Cobb, Confrey,

diSessa, Lehrer and Schauble, 2003). While no single theory will suit all instruction, Realistic Mathematics Education is recognized by many in the mathematics education community as a viable, stable tool for teaching mathematics at the primary and secondary levels as well as in the university setting, and provides a strong foundation on which to build or with which to blend other theories. Ongoing work and the continued existence of the Freudenthal Institute at Utrecht University is evidence that RME has an established foothold among accepted frameworks for research in mathematics education. An example of an RME lesson is provided in section 5.4.3.

5.3. The Use of RME to Teach Algebra at Various Levels

While RME was developed to teach primary school mathematics (Treffers, 1987), since one of its distinguishing features is the integration of mathematical domains (Treffers, 1987) there is at least some support for the idea that RME can be used to teach algebra. Webb, van der Kooij and Geist (2011) explored its use in introducing logarithms; Beaugris (2013) adapted it to teach abstract algebra and Drijvers, Goddijn and Kindt (2011) mentioned it, almost in passing, in a book chapter. Beyond these examples however, there is little research around the use of RME in algebra as an isolated domain.

Contexts for Learning Mathematics is an example of a curriculum using RME that contains a unit on algebra. “*Contexts for Learning Mathematics* is a series of 24 units on the topics of number, operation, and algebra, K-6, developed by teacher educators, mathematicians, classroom teachers, and researchers from Mathematics in the City and the Freudenthal Institute” (Heinemann). By the publisher’s own admission however, the extent to which algebra is included is limited to the “early” algebra that might be taught in grades 1-6.

5.4. RME for Algebra in the Contemporary Pre-College Setting

In this section, I examine RME as a potential method for teaching in the specific setting of the community college remedial algebra classroom. It can be argued that if RME can be used successfully at primary, secondary and higher tertiary levels, then it should certainly be exploitable at the lower tertiary level, and specifically in the remedial

algebra classroom, where secondary mathematics concepts are being learned (or revised) by tertiary students. In particular, since community college students frequently have a great deal more life experience than students learning algebra for the first time, the aspect of RME that relies on addressing situations informally and experientially before learning them formally can readily be utilized here.

5.4.1. The Problem

As I discuss in subsection 5.5.2, Freeman et al. (2014) produced convincing results that undergraduate STEM students who are actively engaged in learning are more likely to retain the concepts they learn than those who learn the same content in lecture-format classes. If students learn better and retain more under active learning models, it can be viewed as detrimental to continue delivering content under passive models. Yet the vast majority of mathematics courses are still taught lecture-style: in 1990, this was 94% of two-year college mathematics classes (Albers, Loftsgaarden, Rung & Watkins, 1992); by 2010, the rate was down for Elementary algebra at courses offered at two-year colleges, but only to 76%¹⁰ (Blair, Kirkman & Maxwell, 2013). The continuing practice of delivering mathematics by the lecture format may be particularly harmful in classes where pre-college content is taught, as the students enrolled in such classes often have a historically troublesome relationship with the subject. It is likely that this difficult relationship plays a role in their underperformance before they arrive at the community college and place into pre-college courses; it is the same relationship that causes them to develop a dislike, distrust or even an anxiety about the subject. To deliver the same content with which these students already have a challenging relationship in the same way that it was delivered to them the first time can only strengthen the feelings they have about the content and their competence.

There are many SCL models, one of which is RME, but whether RME is appropriate for use at the college level is under-researched. This lack of examination indicates a clear need to investigate its application in that environment. As with the use of RME to teach algebra, there is a notable scarcity of research on the use of RME at the tertiary level. One exception is Gravemeijer and Doorman's (1999) "Context

¹⁰ A new report from the Conference Board of Mathematical Sciences was conducted in 2015, but that survey did not ask instructors about their instructional methods.

Problems in Realistic Mathematics Education: A Calculus Course as an Example". This thoughtful, thorough and well-balanced paper describes RME historically as well as examining its place in the calculus classroom. The conclusion drawn by the authors is that RME has a rightful place in the teaching of mathematics at any level. A few other exceptions exist, such as Beaugris' (2013) study with students of abstract algebra, Kwon's (2002) paper regarding its use teaching ordinary differential equations and Webb, van der Kooij and Geist's (2011) logarithms study.

A handful of papers can be put forth as examples that discuss how RME may be used to teach pre-service teachers. These students are undeniably undergraduates and the results from this collection are useful to an exploration of how RME can be used with adults. They do not, however, enlighten the search for information about undergraduate *mathematics* students' response and uptake when curriculum is delivered with a "guided re-invention" and "context-rich" approach. Here again, then, a convincing argument cannot be made for the efficacy of RME in the specific setting of pre-college algebra at the community college, but a strong case can be made for the need for such research to take place.

5.4.2. One Possible Solution

Freeman et al. (2014) define "active learning" thus: "Active learning engages students in the process of learning through activities and/or discussion in class, as opposed to passively listening to an expert. It emphasizes higher-order thinking and often involves group work. (pp. 8413-8414). Many specific teaching approaches can be considered "active learning" approaches, "...including vaguely defined 'cooperative group activities in class,' in-class worksheets, clickers, problem-based learning (PBL), and studio classrooms, with intensities ranging from 10% to 100% of class time" (Freeman et al., 2014, p. 8414). Since the task design (q.v. in section 6.1) feature of the Design-Based Research (q.v. in section 6.2) *Approach* taken for the study discussed in this document used "cooperative group activities in class" and "problem-based learning" in the form of RME, Freeman et al.'s (2014) definition can be applied to the setting in this study.

Freeman et al.'s (2014) definition very closely mimics RME's *activity* and *interaction* principles, which (as a reminder) state that "in RME, students are treated as

active participants in the learning process [and] emphasizes that mathematics is best learned by doing mathematics” (Treffers, 1993, pp. 104-105), and “learning mathematics is not only an individual activity but also a social activity [which is the reason] RME favors whole-class discussions and group work” (van den Heuvel-Panhuizen & Drijvers, 2014, p. 523).

A great deal of research exists that describes various methods for inviting students to participate actively in their learning (see for example Bennett, 2014; Lenz, 2015). The method employed by RME is to give students a genuine problem to solve. Instead of teaching mechanical algorithms followed by superficial applications, as is the wont of contemporary pre-college algebra textbooks, RME utilizes its six principles to present a complex problem at the outset, and students must think critically, laterally and collaboratively in order to solve it (Hough & Gough, 2007).

Having established that “active” learning can be more effective than passive for student engagement and concept retention, I now describe one particular application of RME for delivering content in a way that provides students an opportunity to engage.

5.4.3. An Example

An RME problem outlined in Treffers (1993) describes a pair of 1991 newspaper articles that reported artist Andrew Scott’s reproduction of Van Gogh’s “Vase with Sunflowers” in a farm field in Scotland. Specifically, the two articles reported different measurements for the area covered by the reproduction, and the students (in-service teachers in the case of Treffers’ report) were asked which of the articles was more accurate and why. They were also asked how the error in one of the articles might have occurred, which required them to think not just about how to find an area, but how the task of finding an area is accomplished and why the process for doing so works. Lastly, they were asked why, when the newspapers obtained the measurements in square yards, they reported the areas in square meters and square kilometers, respectively.

The Scott problem is a real problem in that the scenario (reproducing Van Gogh’s image on a large scale in an alternative medium) really occurred. More importantly however, it was realistic: imaginable, probable, possible in the students’ minds – and presented a genuine situation about which students needed to think critically. Whether or

not they were interested in horticulture or in Van Gogh, students had to apply the concepts of measurement, unit conversion and levels of accuracy, and to perform their own computations in order to determine not only which was the correct answer, but *why* the wrong answer was wrong. Issues of rounding, appropriate operation and plausibility all arose in the ensuing discussion.

In light of the difficulty many pre-college tertiary students have accepting mathematics delivered conventionally and given the results from research on active learning and the motivation¹¹ expected by realistic problems and guided re-invention, RME seems an easy candidate for teaching in this specific setting. Students are offered the opportunity to explore rather than memorize, to develop strategies rather than simply absorb (or not absorb) disjointed facts and formulas, and to investigate their own ways of understanding a situation before being asked to approach it with unfamiliar tools. In short, the use of RME to teach algebra to disenfranchised students seems an obvious choice.

5.5. Potential Benefits and Possible Drawbacks of RME for Algebra in a Pre-College Setting

In this section, I examine both some of the potential benefits and possible drawbacks to using RME to teach algebra, and specifically, to using it in the college classroom, rather than the primary or secondary classroom for which it was envisioned.

5.5.1. Possible Drawbacks

As previously mentioned, the vast majority of writing around RME is in relation to its use in primary and secondary school. At these as well as the tertiary level, there may be complications to using any “alternative” teaching or learning model. Preparation for class sessions using RME is an extracted affair, as finding or creating scenarios, stories or situations that fit the curriculum takes a great deal of time and very careful thought; weaving them into the instruction in a way that results in the desired concepts being learned takes even more. Creativity is a requirement but cannot be summoned at will. Even if a great idea strikes, taking it from concept to design to implementation takes long

¹¹ This conclusion is not unanimous among mathematics education researchers; see Zakaria and Syamaun (2017, p. 38) for a counterexample.

hours, days and weeks of careful exploration, trial and revision. A story, newspaper clipping, game or exploration designed to be used over the course of two or three class sessions can take many weeks to prepare conceptually, and even more time and resources to prepare the materials required for its implementation.

While any method of teaching requires at least some effort on the part of the teacher, RME may require more than many other approaches. Instructors working in isolation will struggle to find ways to deliver an entire course through the use of RME. With adequate support and collaboration, instructors will find it easier to deliver content using RME than on their own, but it will likely still be a challenge, given the amount of forethought that must go into the preparation of each concept.

At the tertiary level specifically, apart from the numerous and very real possible administrative obstacles, assessment may be complicated. In the truest model of RME, students work collaboratively during class sessions, but tertiary academic records are still very heavily weighted on the ability of the individual, and particularly the ability of faculty to be able to produce documented evidence of that academic ability.

Some students push back when they learn that the course in which they are enrolled is to be delivered using unconventional methods. While I do not have data to support any claims I might make about such students, I have observed that they are often the ones who prefer to hide in the back of the classroom and do not want to interact with the instructor or other students. These students can frequently be heard to say, “I do better with lecture”, or “I don’t like working in groups”. They are quite accustomed to flying under the radar, and they are often (but not always) students who will pass a class, but not excel. A good deal more might be said about these students’ attitude about learning in general and about their more flexible counterparts (see for example Dweck, 2007; Boaler, 2016) but it is outside the scope of this dissertation to do so here.

I would not pretend to claim that the modality I have described here guarantees that all students will learn better and remember longer, nor that there are no students who do learn better by passively listening to a lecture, but the vast majority of students who end up in pre-college algebra at a community college have already been let down and turned off by the traditional academic model. For these students, delivering the

same content the same way and expecting different results seems pointless and harmful. If they have any chance of progressing, something needs to change in order for these students to learn algebra in *this* course when they couldn't learn it before; since it cannot be the curriculum, or the time of day or the number of weeks in the term, it will have to be the means by which the curriculum is delivered. Multiple reasonable alternatives exist, and others are always being investigated, but Realistic Mathematics Education is one which has been tried and found to be a strong influence for positive change at many academic levels, and therefore reasonable to try it with a community college *Audience*.

5.5.2. Potential Benefits

Even if each of these possible roadblocks can be overcome, the question remains: Is RME a good idea in this setting? One of my aims in this chapter has been to investigate theoretically whether it is appropriate in any setting other than for which it was designed, and specifically whether it is *particularly* appropriate in the community college remedial algebra classroom because of the path so many students have taken to arrive there. In spite of the above drawbacks, RME is a strong candidate for helping students to learn algebra when they have failed to do so in conventional environments. Since RME falls neatly into the realm of "active learning", it can provide students with a purpose and an opportunity for exploring and learning mathematics. Having developed a set of skills in order to solve a problem embedded in a scenario that they view as at least possible, if Freeman et al. are correct they are more likely to retain their new knowledge, and for longer than under conventional methods. Many other benefits have already been discussed about the use of RME in other settings, but they are likely to apply in the tertiary classroom as well. Active learning gets and keeps students engaged, and engaged students learn more deeply and remember longer (Freeman et al., 2014). The use of plausible scenarios means that students are more likely to see the point in doing the necessary mathematics; even if they don't care about the scenario, they can at least see the reasons that such concepts as they are being asked to learn are useful. Since the application comes before the mathematics, there is a stronger motivation for developing strategies, and eventually mathematics, necessary to solve the given problem (van den Heuvel-Panhuizen & Drijvers, 2014).

No single teaching framework will be without drawbacks, and none will singlehandedly be the exclusive answer to whatever the particular problem is that researchers and educators want to address, but RME may be a practicable means by which to approach the current condition in which pre-college tertiary students often find themselves.

5.6. The Meaning of “Realistic”

It is important to clarify the meaning of “realistic” in the sense in which it was originally meant in RME. In English, and in particular in the setting of conventional mathematics education, the word “realistic” will likely be interpreted to mean “real”, “actual” or “factual”. In RME however, “realistic” is more accurately translated “imaginable”.

The Dutch translation of ‘to imagine’ is ‘zich REALISEren’. It is this emphasis on making something real in your mind that gave RME its name. For the problems presented to the students, this means that the context can be one from the real world, but this is not always necessary. The fantasy world of fairy tales and even the formal world of mathematics can provide suitable context for a problem, as long as they are real in the student’s mind. (van den Heuvel-Panhuizen, 2000, p. 4; *emphasis original*)

This perspective opens up myriad opportunities for scenarios in which to embed mathematics. By the reality principle, “mathematics education should start from problem situations that are meaningful to students” (van den Heuvel-Panhuizen & Drijvers, 2014, p. 523). While contemporary textbooks use applications problems to demonstrate some of the ways in which learned concepts can be applied, they make the assumption that the concepts have been learned well enough to be applied in the first place.

5.7. Chapter Summary

Realistic Mathematics Education is now more than 50 years old, but there is still a considerable amount of research being undertaken, albeit little of it at the college level. It has been shown under many conditions to be a feasible perspective with which to teach and learn mathematics. I have examined RME from the perspectives of its history and its contemporary application and found that it can be as useful for teaching algebra as for arithmetic, and for university as for primary students. While, as with any theoretical

framework, RME has possible drawbacks, it also has the potential to provide great benefits. With sufficient resources, the community college mathematics classroom can be a vital place of exploration and imagination, and while there may be many ways in which to achieve this ideal, Realistic Mathematics Education has a position high on the list of candidates. Reinforcing the existing research with examinations of how RME actually works in the pre-college classroom is a vital step in extending its place in mathematics education research.

While RME provides the theoretical perspective which informed the approach to instruction for this dissertation, there is still a need to outline the particular way in which it was applied in that study. In order to test the applicability of RME in the community college mathematics classroom, I needed to develop an *Approach* for working with community college students. That *Approach* is the Design-Based Teaching Experiment, which I describe in the following chapter.

Chapter 6.

Toward the Study: Design-Based Research and the Teaching Experiment

The degree of success of an instructional session will depend at least to some extent on the level of preparation that went into it. In the classroom, an activity will succeed or fail to a greater or lesser extent, and often, that will be the end of it. For the purposes of research however, the outcome of the activity, or of any individual component or pair of components in the CAA structure, may be viewed through one or more theoretical lenses. The current research relies on Freudenthal's Realistic Mathematics Education (RME) (Treffers, 1993). This chapter specifically describes the implementation of the *Approach* of the Design-Based Teaching Experiment (DBTE) which is an amalgamation of Design-Based Research and the Teaching Experiment.

6.1. Task Design

Task design is something that all teachers do, whether formally, rigorously and explicitly, or informally, casually and implicitly or even unconsciously. A task in mathematics education can be simple or complex. Tasks can be designed in the moment, or they can be created over years and incorporate multiple iterations. They can be used once and discarded or they can be developed, documented and analyzed. In practice, it is likely that for most teachers of mathematics, their experience of Task design is a combination of the above that falls somewhere between the extremes.

Antonetti & Stice's (2018) comment, "A well-designed task allows learners to make meaning even if the curriculum is based upon a known set of accepted facts, dates, and ideas" (p.5), very nicely captures the aim of task design and begins to suggest how it might be used in the domain of mathematics education. Jones & Pepin define task design as "mathematical tasks (including tasks in the form of digital resources and tools) that are developed and designed in, or for, mathematics teaching, or in, or for, mathematics teacher education" (2016, p. 107). Superficially, this definition seems obvious, but it is interesting to note that it describes a phenomenon specifically aimed at teaching mathematics as opposed other subjects or disciplines. This specificity suggests that the way a mathematical task is designed is crucial to the way a student

relates to mathematical content. In a 2015 article titled “Frameworks and Principles for Task Design”, Kieran, Doorman, and Ohtani comment that:

...the work of Hans Freudenthal at IOWO, of Alan Bell at Nottingham, and their colleagues during the 1970s reflected the beginnings of the new community of mathematics education researchers’ efforts to grapple with the interaction between curriculum materials and the quality of mathematical teaching and learning – a dimension on which curriculum development efforts over the previous several decades had yielded little information. (p. 23)

It is not surprising that research into task design as a way to approach improvement of teaching practice and Realistic Mathematics Education as a teaching theory were being developed at the same time and involved many of the same research practitioners. RME was a response to the impact at the time of the “New Math” and an attempt to give students an opportunity to “make meaning” (Antonetti & Stice, 2018).

A task for use in the mathematics classroom may be conventional or unconventional. It can easily be argued that a teacher’s goal should always be to create well-designed tasks, but the nature of the task can vary greatly while still providing learners to “make meaning”. By way of example, if the purpose of a task is for students to appreciate the meaning of the slope of a linear function as a ratio, a conventional task might be to have them graph distance over time and then explore the idea of “miles per hour” by interpolating the speed of a car given a range of times. Alternatively, this task might be designed simply to ask students to “find the average speed of the car between one hour and three hours after it began its journey”. Little thought would need to go into answering this question, and students might even get the right answer, but it is very uncertain whether they would appreciate the ratio aspect. A different approach might be to ask students to start by creating the graph themselves from a limited amount of data, and using it to find the speed of the car between 1 and 3 hours, 4 and 10 hours, 0 and 2 hours, and 5 and 6 hours. By creating the graph, students will be much more likely to assimilate the idea of the ratio of distance to time; by finding multiple responses, they will see that the ratio remains constant even though both the time and the distance do not.

An unconventional task that conveys the same content might ask students to determine the direction of travel to find a treasure, find their way out of a mythical forest or research how steep a roof must be to discourage an unsafe buildup of snow. Adult students might even appreciate the conventional approach from a purely pragmatic point

of view, but by overlaying a narrative, this task becomes memorable and potentially even enjoyable. Importantly for the RME aspect of the study described, it also invites students to find meaning in the mathematics they are learning.

6.2. Design-Based Research

Research into task design is a form of Design-Based Research. Wang and Hannafin (2005) define Design-Based Research (DBR) as “a systematic but flexible methodology aimed to improve educational practices through iterative analysis, design, development, and implementation, based on collaboration among researchers and practitioners in real-world settings, and leading to contextually-sensitive design principles and theories” (p. 6). Notable in this definition are the ideas of flexibility, improving practices and analysis. In the context of the study done for this dissertation, Design-Based Research forms a part of the *Approach*.

Design-Based Research has many alternative names and is used in several domains, but van den Akker, Gravemeijer, McKenney and Nieveen (2006) kindly provide the following definition that “encompasses most variations of educational design research:

...a series of approaches, with the intent of producing new theories, artifacts, and practices that account for and potentially impact learning and teaching in naturalistic settings.’ (Barab and Squire, 2004; as cited in van den Akker, Gravemeijer, McKenney, and Nieveen, 2006, p. 4)

Van den Akker, Gravemeijer, McKenney, and Nieveen (2006) describe Design-Based Research as being interventionist, iterative, process-oriented, utility-oriented and theory-oriented. Wang and Hannafin (2005) note that “many characteristics are not unique to design-based research, but rather the nature of their use varies and the approaches are often extended in design-based research” (p. 7). Note, however, that some of the same characteristics appear here as do in task design; namely flexibility, intervention and contextualization. That these also occur in the RME framework makes Realistic Mathematics Education a good fit as a device with which to observe a Design-Based study.

According to Bakker & van Erde (2015), “Educational design-based research (DBR) can be characterized as research in which the design of educational materials

(e.g., computer tools, learning activities, or a professional development program) is a crucial part of the research. That is, the design of learning environments is interwoven with the testing or developing of theory” (p. 430). In the case of the study described here, the design of the educational materials and learning activities was indeed a crucial part of the research, and that design was interleaved with the testing of RME in the community college setting.

Bakker & van Erde explain that:

Design-based research is claimed to have the potential to bridge the gap between educational practice and theory, because it aims both at developing theories about domain-specific learning and the means that are designed to support that learning. DBR thus produces both useful products (e.g., educational materials) and accompanying scientific insights into how these products can be used in education.[...] A key characteristic of DBR is that educational ideas for student or teacher learning are formulated in the design but can be adjusted during the empirical testing of these ideas, for example if a design idea does not quite work as anticipated. In most other interventionist research approaches design and testing are cleanly separated. (McKenney and Reeves, 2012, van den Akker et al., 2006, as cited in Bakker & van Erde, 2015, p. 430)

The bridging described by Bakker & van Erde was precisely the reason the study described in this chapter was undertaken, and the potential for insight was one of the anticipated outcomes of its use. In order to conduct the study, student materials and teacher guides needed to be produced and it was hoped that these would provide a useful alternative to lecture while still conveying the information to which the curriculum requires students be exposed. In addition to attempting both to teach and research at the same time, adaptation is one of the key components of DBR; it shares both of these with the approach known as the Teaching Experiment although they manifest in those models in different ways.

Cobb, Jackson & Dunlap (2016) explain, “the development of design research in mathematics education [...] builds on two prior lines of research: the constructivist teaching experiment and Realistic Mathematics Education” (p. 7). I have already discussed Realistic Mathematics Education in detail; I now turn to a discussion of the Teaching Experiment.

6.3. The Teaching Experiment

The Teaching Experiment was developed by Steffe in 1983 (Cobb, Jackson & Dunlap, 2016; Cobb & Steffe, 1983; Steffe and D'Ambrosio 1996, as cited in Komorek & Duit, 2004), as a “conceptual tool that researchers use in the organization of their activities. It is primarily an exploratory tool, derived from Piaget's clinical interview and aimed at exploring students' mathematics.” (Steffe and Thompson, 2000, p. 273) The Teaching Experiment is, therefore, a way to utilize the advantages of the clinical interview over multiple sessions. The teacher-researcher generates a series of tasks to be performed during a clinical interview and records the interviewees' responses, as in a clinical interview, but also their learning. A single interview may produce results that would be familiar in form to anyone who has ever conducted such an interview, but since the structure of the Teaching Experiment involves multiple sessions, details recorded from one session can be carried over to others (Steffe & Thompson, 2000). While the teacher-researcher performs her interview in the role of researcher, she has also created the sessions with the goal that the students will actually learn the concepts being examined during the sessions in addition to simply recording their responses, correct or not (Steffe & Thompson, 2000); hence the Teaching Experiment is also an interventionist approach. Furthermore, the cycle of interviewing, assessing, modifying and repeating is a feature of the Teaching Experiment not present in the typical clinical interview.

Several components should be present in a Teaching Experiment: “a teaching *agent*, one or more *students*, a *witness* of the teaching episodes, and a method of *recording* what transpires during the episode” (Steffe & Thompson, 2000; emphasis added). The witness is not commonly the teacher-researcher, but in the case of the sessions reported here the author served the role of both teaching agent (*the agent*) and witness (*the witness*). Instead of one-on-one or small-group interviews as is usual, the experiment in this the study performed for this dissertation involved teaching an entire class of students (*the students*). The record of students' responses (*the record*) consists of submitted in-class group work and responses to a questionnaire which were analyzed.

Engelhardt, Corpuz, Ozimek and Rebello (2004) summarized the Teaching Experiment as “incorporat[ing] three components: modeling, teaching episodes, and individual or group interviews” (p. 1). The small groups and desire for learning to take

place suggest that the Teaching Experiment is much like a tutorial session, but the Teaching Experiment has the additional layer that the “sessions” (interviews) are recorded for analysis, and that the process is repeated for the benefit of both the student and the researcher. The authors used the Teaching Experiment in a physics class and compared it with clinical interviews as a means of determining student understanding. One of the most remarkable conclusions drawn from their comparison was that both methods provide a glimpse into student comprehension, but at different points in the learning process:

Clinical interviews provided details of how students *currently* understand a particular physics concept. They reveal areas where students are confused, but cannot always reveal how best to create a change in students’ thinking as this would violate the rule that one should not teach during an interview. Through a teaching experiment one can discover which technique will produce a change and can follow that change. (p. 4; *emphasis original*)

That the Teaching Experiment was the basis of the *Approach* used for the current study is irrefutable, but there are sufficient differences from that approach that it cannot be claimed that it is simply a “modified” version of the Teaching Experiment. Both the larger group size and the written submissions in place of interview data contributed to an ability to analyze the data more broadly than would have been possible in a true Teaching Experiment, encompassing documented group responses and multiple students at one time as it did. Furthermore, iterations were not performed on activity materials before the next session with the same students, but before the next class group to which the course was delivered¹².

Finally, it was not only the student responses being analysed, but the *Approach* being utilized in the form of the activity materials, as is the goal of Design-Based Research. For these reasons, the remainder of this document will refer to the *Approach* used here as a Design-Based Teaching Experiment (DBTE), which honors and utilizes the intersection of the Teaching Experiment and Design-Based Research traditions from which it was devised.

¹² The iterations undertaken to the materials and their content is not discussed in this document.

6.4. The Design-Based Teaching Experiment

The Teaching Experiment and the Design-Based Teaching Experiment are similar in that they both comprise a task or series of tasks which are both delivered and analyzed by the teacher-researcher who records the subjects' responses as well as their learning. The structures of both involve multiple sessions, which means that details recorded from one session can be carried over to others. While the teacher-researcher performs her delivery of content in the role of researcher, she has also created the sessions with the goal that the student will actually learn the desired concepts during the sessions. The cycle of delivery, assessment, modification and repetition is also common to both. There is, as required by the Teaching Experiment, a teaching agent, one or more students, a witness of the teaching episodes (even if that is the teacher-researcher), and a method of recording what transpires during the episode; iterations of the initial experiment are performed, and the purpose of the sessions is twofold: to record and analyze student responses to prompts, but also to teach the students the concept(s) being studied.

Here, though, the similarities end.

In the Design-Based Teaching Experiment, the "subject" is an entire class of students instead of individuals or (very) small groups of students; the data consist of written records only, and no interviews are conducted; iterations are performed, but they cannot be accomplished with the same group of students. The approach described here and utilized in the study conducted for this dissertation will require further analysis in order to determine its strength, usefulness and validity.

As previously mentioned, I call the *Approach* used for the current research the "Design-Based Teaching Experiment". The components of the Teaching Experiment are modified here so as to provide a list of features for the DBTE: a teaching *agent*, a class of *students*, a *witness* of the teaching episodes, and a method of *recording* what transpires during the episode.

Composed as it is of Design-Based Research and the Teaching Experiment, but modified as was necessary in this instance, the Design-Based Teaching Experiment has the several key qualities, listed below. The annotations in parentheses in each item description indicate from which device it was taken. The DBTE is:

- interventionist (DBR and TE) – one of the goals of the sessions is to teach rather than simply record;
- iterative (DBR and TE) – each application is modified for use in another session, albeit with different students;
- process-oriented (DBR and TE) – the purpose is to develop deep understanding rather than algorithmic memorization;
- utility-oriented (DBR and TE) – the purpose is to provide both the students and the teacher/researcher with practical tools;
- theory-oriented (DBR) – one of the aims is to test the application of the theory; in this case RME.

6.5. Chapter Summary

The Design-Based Teaching Experiment is a model developed from the already-established methods Design-Based Research and the Teaching Experiment, modified to meet the needs of the current study. Both DBR and the Teaching Experiment provided the desired features of teaching-while-researching and design and development of materials, but both contained elements that made them impractical for exclusive use in the current study. The Teaching Experiment provided the opportunity for teaching-while-researching but was originally designed for repeated use with the same small groups. From Design-Based Research “design, development, and implementation” (Wang and Hannafin, 2005, p. 6) were sought-after elements, but “collaboration among researchers and practitioners” (Wang and Hannafin, 2005, p. 6) was not a desirable feature during this study. In addition, it is hoped that the results of this dissertation will lead to “contextually-sensitive design principles” (Wang and Hannafin, 2005, p. 6), but it was not originally an intent of the design of the study.

Mathematics Education is well-known as a field of research that heavily utilizes, extends, adapts and combines earlier theory, and this study has been no exception. The Design-Based Teaching Experiment is an amalgamation of constructs, each of which had some properties I wanted to incorporate into the study, but none of which would work in that setting on its own.

Chapter 7.

The Study: Design-Based Teaching Experiment

Having described the *Concepts* and the *Audience* of interest in the study undertaken for this dissertation and having disclosed the means by which the *Approach* was fashioned, this chapter describes the *Approach* in further detail.

7.1. Research Questions

The informal goals I stated in section 1.4 are restated here for easy reference; they were:

- to investigate how certain student-centered learning techniques can be carried out at community colleges,
- to assess how teaching methods developed for use in primary or secondary school might be used in community college remedial mathematics course, and
- to assess the materials being developed for this investigation for their usefulness to others.

These questions can now be rephrased in a tenor more in accordance with their role in a research manuscript:

1. How do community college remedial algebra students engage with the algebra of linear equations in two variables using the DBTE?
2. How can methods used in primary and secondary classrooms be implemented in community college classrooms to teach the algebra of linear equations in two variables?

It is these questions that formulate the basis of the research performed for this dissertation. In order to answer these questions, I describe in this chapter the way in which the Design-Based Teaching Experiment was. I also describe the specific course in which the Design-Based Teaching Experiment was utilized, the class of students who participated in the study and the particular activities that were designed for implementation so that student responses to these might be evaluated. The analysis undertaken in order to answer these questions constitutes the substance of Chapter 8 in this document.

7.2. The Study

The study conducted for this dissertation was developed to explore whether learning remedial algebra could be made more engaging while ensuring that students still learned the concepts required of the curriculum. The success of the study would indicate in part that the materials produced for its execution had been successful in their role of providing the required content under the method being evaluated. The materials used were produced with the features of Realistic Mathematics in mind, and for this reason, examining the *Approach* would also mean examining the effectiveness of the materials.

7.2.1. Motivation

I have always sought new, better ways to teach. The conventional lecture model does not work for everyone, but from experience, neither does any other single approach. The inspiration to try the technique described here came from a book and a conversation. A colleague had recommended Jane McGonigal's (2011) book *Reality Is Broken: Why Games Make Us Better and How They Can Change the World* as it had inspired her to rethink teaching elementary statistics using activities to supplement each of the concepts. About a third of the way through reading McGonigal's book, I had an epiphany. What if I could teach a whole course using one storyline? I decided to use the characteristics of games McGonigal describes that keep players coming back for more to keep students engaged. I hoped that if I could embed the mathematics into a storyline and then add layers of "leveling up" and "epic wins" (McGonigal, 2011), I could shift the focus from the drudgery of learning mathematics to the excitement of playing (while learning mathematics).

The first concept I would be teaching in the remedial algebra class in which I wanted to try this experiment was that of Cartesian coordinates, followed by concepts related to graphing lines. I decided on a map as the basis for the first few concepts as they were easy to visualize implementing in that setting and I created the first few activities, which are described below in detail. After a conversation with my very creative sister, I had not just a few activities, but most of the unit. There was a system of roles for each member of each group of four and in which students (players) could "level up"; we created a fictional monetary system that students could use to earn points or buy levels

or activity hints; and we had an overarching theme for the story that would run through the entire unit.

The narrative layer used in the study provides the motivation for the students to undertake the challenges presented to students. Many if not most of the game-like features described here are completely unnecessary for learning mathematics, but I include them because without them the picture of the study and any reason for analysis of student work under the *Approach* utilized is incomplete.

7.2.2. The Activities and Tasks

In section 7.3.1 I describe each of the activities and their tasks in detail. In what follows here, I describe the factors considered in creating them. For the purpose of clarity, I will use the term “activity” to mean a series of tasks, usually around one course learning objective, collected onto one set of instructions and a corresponding worksheet; I will use “task” to mean the individual items students were expected to perform in order to complete an activity; thus there are several activities represented in the study, each of which contains multiple tasks.

Since the purpose of the study was to investigate how students engaged with algebra under an RME delivery, I introduced a narrative layer that would be complex enough to accept various problems to be solved, and activities involving tasks that would invite solutions to those problems. The concepts that were required by the curriculum to be addressed were ordered pairs, lines as collections of points, the y -intercept, the slope, and the relationship between the equation of a line and the graph of that equation. An overview of the activities and embedded tasks is provided here, but these will be revisited in much greater detail in section 7.3.1.

To begin, students working in small groups were given a map of a fictional continent and asked to label the cities and towns on the map. The conventional way in which Cartesian coordinates are written was described in the scenario via a narrative apparatus, and students were expected to use Cartesian coordinates to complete the task. In order to express the idea that a line is an infinite collection of points, students were asked to find cities and towns that fell on a line between two locations.

One of the tasks students performed during the study was writing of the equation of a line, a concept whose requirement was explained as its being the flight path of a messenger drone. The y -intercept was introduced by explaining that a drone's flight path between two locations passes through an important political feature, and the slope was interpreted as a direction of flight.

The activities created for the study conducted for this dissertation are complex and multi-faceted. They incorporate the use of a realistic (in the RME sense) scenario to introduce and scaffold concepts from the algebra of linear equations in two variables, they rely on students' previous experiences to make sense of the situational information, and they require the use of creative problem-solving in order to complete them. In order to accomplish all of these things, tasks were designed using methods from multiple approaches.

Characteristics were borrowed from Design-Based Research and the approach known as the Teaching Experiment to inform the research approach and to formulate a means by which the method of researching-while-teaching could be implemented while studying an entire class of students. Ultimately, task design, Design-Based Research and the Teaching Experiment all played a role in the design of the study done for this dissertation. Each of these devices is described in more detail below, and their combined features used to define the approach used to carry out the study. I now explain the role of each of the devices task design, Design-Based Research and the Teaching Experiment in light of the study.

7.3. Description of the Setting

In at least one Washington state community college, the remedial algebra curriculum includes a unit that exposes students to the fundamental principles of linear equations in two variables. Students are expected to acquire the concepts of Cartesian coordinates, ordered pairs as solutions to a linear function, slope, x - and y -intercepts, linear graphs and solutions to systems of linear equations. This course is conventionally lecture-based, which requires no active learning; students can be physically present without absorbing the information they need to pass a conventional exam successfully.

7.3.1. Activities

In subsection 7.2.2, I described the particular way in which I use the terms “activity” and “task”. “Activity” refers to a single set of instructions and its related worksheet; each activity was expected to take student teams about one class session to complete. “Task” refers to a single item whose completion is required within an activity from a set of instructions that contains several such items. An activity frequently aligns with one or two learning objectives, and each task is designed to make use of the principles of RME to convey the required *Concept(s)*. In the descriptions that follow, each task within an activity is described and it is labeled with the RME principles with which it is imbued. For the sake of brevity in the following descriptions, those principles are re-stated here with an abbreviated definition of each:

- the *activity* principle – students are active participants in their learning
- the *reality* principle – mathematics should start from meaningful problems
- the *level* principle – students pass through levels of formality and understanding
- the *inter-twinement* principle – content domains are integrated
- the *interaction* principle – learning mathematics is a social endeavor and an opportunity to share strategies
- the *guidance* principle – students are guided to ‘re-invent’ mathematics

The full definitions can be refreshed by referring to section 5.2.

7.3.1.1. Introductory Activity

At the start of the unit that covers linear equations in two variables and their graphs, an activity is given to student groups in which a story is relayed (the *reality* principle) about a valuable missing artifact requiring students’ expertise to help locate. Teams (the *interaction* principle) are given maps of a country with fifteen cities, towns and “taers” (towers) labeled on it; the map has a grid overlaid on it (no axes are suggested), and some but not all of the cities fall neatly at intersections of grid lines. The land mass is not identifiable (i.e. it is a map of an imaginary continent), so any activities requiring measurements to be taken, for example, cannot be researched on the Internet.

Several activities are then delivered around the same distal goal of finding the missing item.

An introductory activity is provided in order to familiarize students with the model for the way in which the unit was to progress, and the narrative. It has the added benefit that it lets them accomplish an elementary task within their new groups, affording a sense of accomplishment and increasing confidence. Since the only mathematics required to undertake the introductory activity is that with which students are expected already to be confident, this activity also serves as a segue from the world of “conventional” mathematics to that of RME. The introductory activity in this study can be seen in Figure 7.1 and Figure 7.2; the map used is shown in Figure 7.3; the student groups responded by completing the worksheet shown in Figure 7.4.

Mission: The Call to Taer Shantara

Task: Planning the Journey

You received the call two weeks ago: The Gem of Shantara has gone missing. It is priceless, of course, but more importantly, it is imbued with magical properties, which in the wrong hands, could be disastrous.

The Gem of Shantara, like its counterparts, is the power behind the seat at Shantara. The other castles, Aerie, Zephyr Keep, Taer Elladorn, Griffonclaw, Taer Valior, Stonefeather Spire, Taer Zanathar and The Starhaunt, each has its own powerful Gem. At least they did. It is said that it was the mysterious disappearance of the Gems of the Starhaunt and Taers Elladorn and Zanathar that caused them to fall, although there is no concrete evidence to support this theory.

It took all your powers of persuasion to convince the magistrate of Pylas Maradal that you must go to the aid of your countrymen. They are a strong and proud people, and they would not have asked lightly for your help. It is likely that your position in the Order of Masters and your family's connection to Taer Shantara is the only reason that your sister-in-law Marshall Jaq'qal wrote to you in the first place.

The magistrate has given you leave to journey to Taer Shantara with your House to help the Shantarians search for clues about the disappearance of the Gem. A few other Houses will be going as well, but you will travel with your own people. You must decide how to travel, and how much to take.

Your first task is to plan your journey to Taer Shantara. Determine the following and record your decisions and the details, along with any supporting calculations in your mission log. Remember that your mission log is an official document; it should be complete, showing all necessary calculations, but it should also be neat and legible. The magistrate will pay for you to keep and turn over your mission log, but you may earn a bonus if it is clear, concise, legible and easy to follow.

1. Using the map of Valenar, determine how far it is from Pylas Maradal to Taer Shantara, in miles. You will likely need to estimate to a certain degree, but be as accurate as you can.
2. Decide whether you will go on foot, and carry all your own gear (slow, but economical); on horseback (fast, but expensive - you must pay to feed and care for your horses on the journey); or go on foot with one shared pack mule (a compromise of the two other extremes). You may need to decide on this issue in conjunction with the next two. Assume that in the absence of animals, you can walk at a rate of 3 miles per hour, and that you will travel for 10 hours per day.

Figure 7.1. Introductory activity (part 1)

3. Determine how long it will take you to get from Pylas Maradal to Taer Shantara. Remember that you will need to stop overnight to sleep, and also to stop along the way to rest, eat, and replenish supplies. You may meet other people, and interacting with them will take time as well. Your chosen method of travel will also influence this calculation.

4. Because of your position, you already have the essential gear that you will need for the journey. This includes:

- sturdy shoes
- all necessary clothing and headgear
- hygiene items
- canteen and eating utensils (plate, cup, fork, etc.)
- day pack
- sleeping roll
- small knife
- compass

In addition, each House already has

- First Aid Kit
- Essential tools (saw, axe, mallet, rope, small shovel, etc.)
- Fire starters

You will need to transport the above items on your journey.

The costs of various other critical items are given below:

- 1 horse^α (with tack and equipment): Ψ^β1,500
- 1 pack mule^γ (with tack and equipment): Ψ500
- 1-person tent: Ψ60
- 5-person tent: Ψ200
- Package: Enough food to feed a House on the road for 5 days^ε: Ψ2,000
- Package: Food for a horse or mule for 3 days: Ψ50

Determine the cost of the journey based on the above, and on the decisions you made in the previous two entries.

^α A Valenarian horse can travel at about 6 miles per hour, on average, but can only travel for 6 hours per day

^β The currency of Valenar, pronounced "sigh", or sometimes "puh-SIGH".

^γ A pack mule can travel 4 miles per hour for 8 hours per day. You'll be able to keep up with it, because you are carrying much less yourselves.

^ε You will be able to replenish your food supplies in Jal Paeridor, but you will still need to pay for them.

Figure 7.2. Introductory activity (part 2)

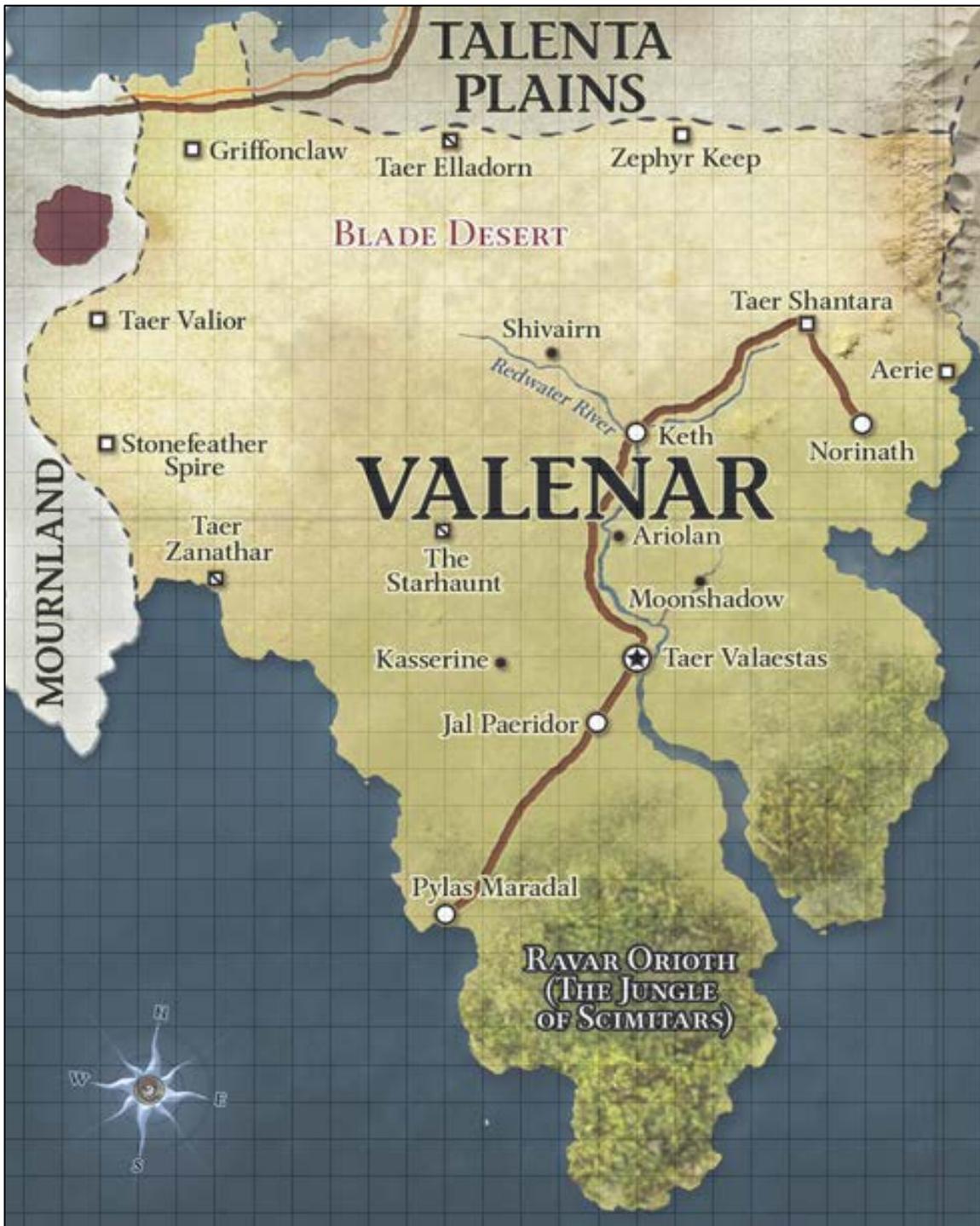


Figure 7.3. Map used in the study

This image is modified from one on the "Endor" article on the Star Wars wiki at FANDOM and is licensed under the Creative Commons Attribution-Share Alike License.

Mission Log for: _____

House of: _____

Mission: The Call to Taer Shantara

Task: Planning the Journey

Jaq'qal has begged our assistance to help Taer Shantara recover the Gem of Shantara. Magistrate Pyrfal has agreed, and we leave in three days' time.

The distance to Taer Shantara from Pylas Maradal is _____ hahrs. It would be so much easier to calculate this distance if the blessed Shantarians would convert to using miles, as we do, but they do resist progress!

We will travel _____, and will therefore cover _____ hahrs (_____ miles) a day; we should be at the gates of Taer Shantara in _____.

The cost of the journey is expected to be ψ _____.

Comments:

Our supporting calculations can be found on the following page.

Figure 7.4. Introductory activity worksheet

7.3.1.2. Graphing: Ordered Pairs

Conventionally taught, I generally introduced Cartesian coordinates as a location x units to the right of and y units above a set point and recorded in the form (x, y) . Both the set point (the origin) and the decision to measure to the right first and up second are mathematically arbitrary (see Hewitt, 1999). Without context, students sometimes struggle to remember which direction the first coordinate represents and which the second.

In the RME version of the *Concept* delivery described here, the “history” of the land is given in which the capital city is identified and described as the “origin” of commerce. Through the narrative students are told about an ancient cartographer who designed a system of location markers that measure each city, town or tower as being some distance east and some distance north from the capital (the *reality* principle). If a city is west of the capital, the city’s “east” value is negative¹³; if a city is south of the capital, its “north” value is negative. With this information in hand, teams are tasked with drawing in the country’s “longitude line” and its “latitude line” (the axes) and labeling all the cities with their coordinates (the *activity*, *reality*, *level*, *inter-twinement* and *interaction* principles; see section 5.2 for details).

While the decision to measure north and east from the origin is still arbitrary, the origin at least has a reason for its name. Because it is the location of a city, and in fact the capital, it is relatively easy for students to accept this choice of location for the center of the activity. The transparency of the arbitrary nature of this location may also help later promote the idea that Cartesian axes may be located wherever is convenient. An auxiliary story is told about how the mapmaker who “invented” the system (Xenon Yarvancia) initially used an abbreviation of (e, n) , (short for (east, north)) for his location coordinates; being vain, however, he decided instead to use his initials, (x, y) . This narrative is unabashedly superficial, much more so than in other parts of the scenario, but students accept it as a part of their instructor’s fantastical storyline, and a plausible requirement for continuing. It may help that the system we actually use for geolocation (latitude and longitude) is similar to the notation students use in this activity. Given current satellite technology, many adult students bring with them the idea of latitude and

¹³ Students in this course are assumed to have knowledge of and experience with positive and negative real numbers.

longitude (the *level* and *reality* principles), and whether or not they know how they are commonly written, they can identify with the scenario with which they are presented. The Cartesian coordinate system is a reasonable proxy for latitude and longitude, right down to the non-integer and signed values for some of the cities on the map (although the coordinates are given in the order (N, W) in the latter case; a different arbitrary convention).

The instructions for the Cartesian coordinates activity are given in Figure 7.5 and Figure 7.6, and the student worksheet is shown as Figure 7.7.

Task: Mapping the Terrain

You will need to know something of map-reading before undertaking your journey.

Valenarian cartographers are highly skilled, and their services frequently sought by intellectuals all over the world. They do much, much more than make and read maps. Over the years, they have developed a reputation as innovators and their council is sought by kings and scholars alike.

Many centuries ago, the young mapmaker Renée Décartes developed a shorthand system for describing the location of any place on a map in relation to the capital city. This system replaces the name of a town or village with a locator address that Décartes called "coordinates". So useful was this new system that Décartes' coordinates and the system in which they function were named after him; they have been called the Cartesian Coordinate System for as long as anyone can remember.

To use the system, locate the capital city on any map (usually represented by a star, inscribed in a circle), and draw a vertical line through it. This line is called the "longitude (or longitudinal) line" of the capital. Now draw a line horizontally through the capital. This line is called the "latitude (or latitudinal) line" of the capital. We call the place where these two lines meet (0,0); that is the capital is 0 units east of itself (longitude 0), and 0 units north of itself (latitude 0). Then any place on the map can be described by how far east and north it is of the capital (always giving the east value first, and then the north).

For example, the city of Keth can be described by giving the location coordinates (0,6), and the town of Moonshadow is at the location (1.75,2) (approximately).

What say you of towns and villages that are not east, but west, you ask? Ah, but west is simply the opposite of east, and so the negation symbol is used. The same is true of south, since it is the opposite of north. Thus the location of the village of Kasserine is (-3.5,-0.2) (approximately).

Accuracy is more important in some applications than in others, but for now, it will suffice to be able to give relatively approximate locations. Make list of the locations of all the villages, towns and cities on the Valenarian map and include it as a reference page in your Mission Log.

Figure 7.5. Cartesian coordinates activity (part 1)

A little trivia: After D cartes created the system now named for him, a virtually forgotten scribe called Xenon Yarvancia determined that it would be easier to refer to the (east, north) coordinates by some abbreviation. He is thought to have considered using (e,n) as a way of abbreviating a coordinate pair, but being very vain, he decided instead to name them after himself. Thus, a coordinate pair whose coordinate *values* are unknown are often referred to as an (x,y) pair.

With this new standard in place for coordinates, it quickly became customary to refer to the latitudinal line of the capital as the x -axis, (because the Valenarian world "revolves" around the capital) and the longitudinal line as the y -axis.

Figure 7.6. Cartesian coordinates activity (part 2)

Mission Log for: _____

House of: _____

Reference Page:

the Cartesian locations of Valenarian villages, towns and cities:

Name	Classification	Location
Aerie	tower	
Ariolan	village	
Griffonclaw	tower	
Jal Paeridor	city	
Kasserine	village	(-3.5,-0.2)
Keth	city	(0,6)
Moonshadow	village	(1.75,2)
Norinath	city	
Pylas Maradal	city	
Shivairn	village	
The Starhaunt	tower (ruin)	
Stonefeather Spire	tower	
Taer Elladorn	tower (ruin)	
Taer Shantara	tower	
Taer Valaestas	capital	
Taer Valior	tower	
Taer Zanathar	tower (ruin)	
Zephyr Keep	tower	

Figure 7.7. Cartesian coordinates worksheet

7.3.1.3. Graphing: Lines as Collections of Points

My traditional delivery of the concept of an (x, y) solution to a linear equation in two variables included an explanation that substitution of the values from a given (x, y) pair into the equation must make the algebraic statement true. To determine solutions, candidate numerical pairs are “plugged in” to the equation to be tested. If the result is a true statement, the pair is a solution; otherwise it is not a solution. This property of linear equations and their solutions is what Moschkovich, Schoenfeld & Arcavi (1993) refer to as “the Cartesian Connection” (p.73). This concept is one for which students seem to fall cleanly on one side of understanding or the other: either they do or they do not recognize that for a point to be on a line the coordinates of its ordered pair must make the equation of that line a true statement. The Cartesian Connection appears to serve, therefore, as a kind of dividing line between students who can only follow mechanical algorithms and implement elementary procedures and those who are able to extend their use of these basic abilities and utilize the properties of algebraic structures to assess more complex questions involving linear equations in two variables. This idea is discussed in detail in Chapter 10 of this dissertation.

Students in the class in this example are assumed to know how to solve linear equations in one variable, and to know what it means for a single value to be a *solution* to such an equation. One particularly evident problem that students have with the extension of the concept of linear equations in one variable into that of linear equations, which have two variables, is that whereas there was only one solution to the former, there are infinite solutions to the latter. Students have come to think of a statement such as “ $x = -3$ ” as *the* solution, and when they discover that $(4, -2)$ is a solution to the equation $3x + 5y = 2$, they often believe they have found the *unique* solution and stop checking other points.

By contrast, under the RME delivery used in this class, the storyline begun in the first activity continues here with the requirement that teams plan to send a message from one map location to another by means of a flying drone (the *reality* principle; see section 5.2 for details). The drone, they are told, can only fly in a straight line, and must be programmed to fly in the correct direction and deliver its message (land) at the correct position. While it is too simple to accept a complex flight program, the drone can transmit its message along its route if it is programmed with the required locations (coordinates).

The problem the teams are given is that they must notify a distant city of their intentions to travel there, but they must also alert friendly forces at locations along the drone's flight path. As they program the drone with the various location coordinates, they are using the idea that one linear equation has many solutions (the *reality*, *activity* and *level* principles; see section 5.2 for details). This approach helps reinforce the idea that a line is a *collection* of points; a concept from planar geometry with which many students in this course have only a tentative acquaintance (the *level*, *inter-twinement* and *guidance* principles; see section 5.2 for details).

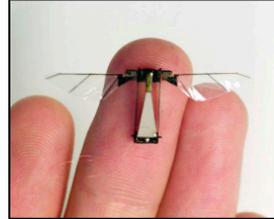
The activity and worksheet associated with this concept are provided in Figure 7.8 and Figure 7.9

Mission: The Call to Taer Shantara

Task: Sending a Message

You are ready to send a message to Jaq'qal, letting her know that you are coming to her aid. The fastest way to get word to Taer Shantara is by drone.

Drones are incredibly small flying machines, and although remarkably fast, they are very primitive, and can only carry messages in a straight line. Valenarians have an expression: "As the drone flies" means the distance between two places not by road, but in a straight line, by air.



A Valenarian Messenger Drone

The longer a drone must stay in the air, the bigger the battery must be, but this means it will cost more to send.

1. Determine the distance^β from Pylas Maradal to Taer Shantara so that you can prepare your drone at minimal cost.

There are people in Taer Valaestas who might help by providing food and shelter on your arrival there, if they knew you were coming; if you could get a message to them. They will need at least a day's notice of your arrival however.

You could send a second drone to alert them of your plans, but this would be costly.

2. Find a way to use the drone you are already sending to Taer Shantara to drop a message for Taer Valaestas^γ. Be sure to give the coordinates of the drop location.
3. Would it be possible to drop another message for Keth using the same drone?
4. Draw the drone's path on your map of Valenar and label the Cartesian locations of the start and end of the drone's journey as well as any drop locations.

Document your work and save it to submit with your Mission Log.

^β A hint is available.

^γ A hint is available.

Figure 7.8. Points as solutions to lines activity

Mission Log for: _____

House of: _____

Task: Sending a Message

We are ready to begin our arduous journey. We must notify Jaq'qal of our departure, so we are preparing to send a messenger drone. The distance the drone will need to cross is _____ miles "as the drone flies".

Supporting calculations:

We will program the messenger drone drop a message at _____ as well, to let the good people of Taer Valaestas know we are coming. Many of us have friends and relations in that most glorious city, and we would all be glad of a soft bed and a hot meal after such a journey.

Notes, maps, diagrams and other supporting work:

Figure 7.9. Points as solutions of lines worksheet

7.3.1.4. Graphing: Slope

In this study, slope was introduced within the parameters of the now-established storyline. Initially referred to in the narrative as the “directional command”, the fundamental idea of the relationships between the changes in vertical and horizontal directions was explained, but in words, rather than by a pre-assembled formula. The description of the drone as too primitive to accept anything but a single value as its directional command was meant to encourage students to see that while there are two components to the direction, they work together as a single number which is the ratio of the two component values.

The concept of slope is one that many students struggle with, and this difficulty is extremely well documented (e.g. Barr, 1980; Bell and Janvier, 1981; Leinhardt, Zaslavsky & Stein, 1990; Orton, 1984; Simon and Blume, 1994). The hope in introducing slope via a realistic scenario was that students would see the slope as a relationship first and a numerical value second, but also that the connection between these two expressions of that value would become apparent.

7.3.1.5. Graphing: y -Intercept

The y -axis was explained within the narrative as a “local convention”: the line running north/south through any location is the “longitudinal” line of that location. Since the capital city is at the origin, the line that runs north and south there is the longitudinal line of that city. The y -intercept is the location where the drone crosses the y -axis. Algebraically this point serves as something of an anchor, being a vertical (or horizontal) shift of the line with the same slope through the origin. The narrative was intended to explain the importance of being more than zero units north or south of the origin while being exactly zero units east or west of the origin (the y -intercept).

7.3.1.6. Writing the Equation of a Line

My conventional lecture of this *Concept* included a demonstration of the means by which the formula for slope (given two points (x_1, y_1) and (x_2, y_2) , the slope m of the line containing these two points is $m = \frac{y_2 - y_1}{x_2 - x_1}$), can be algebraically manipulated into the point-slope form of the equation of a line ($y - y_1 = m(x - x_1)$); this equation is, in turn, manipulated into the slope-intercept form ($y = mx + b$), and the latter can then be evaluated for various values of x . One of the selected values is always, at some point

during the conventional lecture, zero, and with this, students are shown that in slope–intercept form, the value of “ b ” is the same as the y -intercept of the graph of the function. Mathematicians and instructors of mathematics know the many reasons the slope–intercept form of the equation of a line is so powerful, but students do not necessarily appreciate the subtle power of the form, primarily because they have not absorbed the fundamental mechanics of the situation, let alone observed the reasons for manipulating an equation. In many students’ eyes at this point in the proceedings, zero is just one of the numbers the professor used in her examples; that it has any particular special meaning is lost on many students.

In the RME version, the story continues with a description of the means by which the drones introduced in the previous activity are programmed (the *reality* principle). The program is written so that the programmer needs only to input two values.¹⁴ With additional narrative guidance, teams then program their drones to deliver a message (the *activity, reality, level, interaction* and *guidance* principles).

Once again, the need to solve a problem preceded the mathematics. Students were guided so that they would finish the unit with the necessary algebraic skills to succeed in subsequent courses, but they developed the parameters of the equation of a line in slope–intercept form for themselves (with their teams) from a realistic, albeit imaginary, need to do so in order to solve a problem.

The tasks for graphing linear equations included individual instructions for finding the y -intercept and the slope. Each of these concepts is described in terms of the narrative below and the activity and worksheets used for this section are provided at the end of these descriptions as Figure 7.10, Figure 7.11 and Figure 7.12.

7.3.1.6.1. Slope–intercept Form of Equation

The slope–intercept form of the equation of a line is a linear combination of slope and y -intercept. The connection is often lost to the fact that “ b ” is the value produced for y by the equation under the special condition that $x = 0$. In the RME version, students

¹⁴ The two values are a direction of flight (the slope, when the land mass is viewed from above, as on a map), and the location (the y -intercept) at which the drone will cross the longitude line (the y -axis), which runs vertically through the capital city; the students were not given this information, but asked implicitly to determine what the two values must be.

were required to find the particular combination of the two values they need in order to create this equation. As a reminder, well-prepared students had read through an introduction to the concept in the textbook, so in addition to the terminology used in the narrative, they had seen the nomenclature used in the conventional setting. This preparation meant that they already knew the form in which the equation they were asked for should be, and they were determining the combination they needed based on work they had done to find the slope and the y -intercept.

7.3.1.6.2. Point-Slope Form of Equation

The point-slope form of the equation of a line is a modification of the average rate of change (slope) formula. This idea was not explicitly covered in the class, but it was addressed briefly as a “trivia” note in the narrative.

Task: Programming the Drone

Your drone is very primitive and can only take a very simple program to get it to its destination. The program is standardized, so that drones can be programmed for any route, no matter where they are actually located; every drone's program is entered in exactly the same way.

The program uses the Cartesian coordinate system, and is based on two parameters:

- A. Its "directional command" (how far north and east ^{δ} of the starting point its destination is) and
- B. Where the drone crosses the longitude line of the capital city.

The first of these is calculated by comparing the Cartesian coordinates of the two locations. The difference between the two y coordinates tells the drone how far north to fly, and the difference between the two x coordinates tells it how far east to fly. But it can only take a single number as a directional command. Refer to your officer's manual (page 269) if you need to refresh your memory about how to calculate this number for programming your drone's directional command.

The second parameter is a number associated with the fact that the capital city is at longitude 0. We could have used the longitudinal line of any city, or in fact even an uninhabited location in the desert as our reference, but using the capital has two advantages: One is that it pays homage to our benevolent regent, and the other is the convenience of the value of 0 in performing calculations!

The drone's program uses these two parameters together in a single string. The drone interprets the information to determine both its direction and its specific path. The string is a kind of calculation in itself: the drone checks its longitude every picosecond, and it uses the string to determine what its latitude should be at that instant. This is why the program needs a reference point (the point on the line longitude 0) and a directional command.

^{δ} North and east is used for programming drones instead of "east and north", as in a Cartesian location. It is not critical to the program that you understand why this is, but your House may earn a bonus if it can present a compelling argument about why the "roles" of east and north are reversed as compared to plotting Cartesian locations.

Figure 7.10. Equation of a line activity (part 1)

To program your drone, follow these instructions:

1. Draw the straight line(s) of your drone's route on your map.
2. Determine your drone's directional command(s)^ε.
3. Determine the coordinates of your drone's longitudinal crossing point(s).
4. Write the program(s) for your drone's flight to Taer Shantara^η.
5. Determine the minimal instruction you need to give the drone so that it will drop its message for Taer Valaestas^θ.
6. Determine the minimal instruction you need to give the drone so that it will stop exactly at Taer Shantara (and not just keep flying until its battery dies).

Make notes of your work in your Mission Log.

A little trivia: When drone technology was first being developed, two different schools of thought predominated regarding the standards by which the machines should be programmed. One (the one we use today) was to use the parameters of longitude and direction; the other used longitude and latitude. Occasionally, an "old school" programmer will give a technician the drone's program using the other archaic parameters, and it is considered an indication of skill if the technician can convert the program into the one needed for the drone quickly and easily. For most technicians, it is a matter of pride that they can receive an alternative program and simply shrug and convert the program with a few back-of-the-envelope calculations.

^ε A hint is available.

^η A hint is available

^θ A hint is available.

Figure 7.11. Equation of a line activity (part 2)

Task: Programming the Drone

Our drone must be programmed to fly from here, Pylas Maradal, to Taer Shantara, to deliver the message to Jaq'qal that we are on our way.

- The directional command is _____
Supporting calculations: _____

- The longitudinal crossing point is _____
Supporting calculations: _____

- The program is therefore _____.
Supporting calculations: _____

- The input instruction to the drone to drop a message for Taer Valaestas is **[drop: ____]** and the instruction to stop is **[stop: ____]**.
Supporting calculations: _____

Figure 7.12. Equation of a line worksheet

7.3.1.7. Graphing Systems of Linear Equations

In courses taught under the lecture model, I almost always introduced systems of linear equations and their solutions via the “graphing” method. Within the lecture model, this approach nicely set the stage for what it means to be a solution to a system; students can literally see that the intersection of two lines on the same graph is a point, but also that it is the only point that satisfies both. This realization can be problematic for some students, since they find themselves now returning to a scenario in which, once again, there is only one solution, but its form is the same as the scenario in which there were an infinite number of solutions. That the special cases of “infinite solutions” (*coincident* lines) and “no solution” (an *inconsistent* system) are frequently taught at the same time can further deepen the confusion.

Having successfully graphed a single line in the RME course, teams receive the continuation of the narrative: enemy drones have been intercepted and their flight paths decoded, but their purpose is unclear. Teams are given the flight path of an intercepted drone and left to discover that different teams have different linear equations (the *interaction* principle). Once teams decide to collaborate, they learn that the two (or more) flight paths they collectively have to work with cross paths, and that the intersection of the paths is the location of the missing artifact (the *activity, reality, level* and *guidance* principles).

The final activity for this unit and its associated worksheets are given as Figure 7.13 and Figure 7.14.

Task: Decoding the Enemy

Your house has made good progress along the route to Taer Shantara. You have been en route for nearly a month, but you arrived at Jal Paeridor several days earlier than you anticipated and your friends and family there made sure you were well fed. You all bathed and slept well for two nights, restocking your food and supplies in anticipation of continuing on your journey. Your next stop was to be Taer Valaestas, only about a week away.

But while you were in Jal Paeridor, you received word that a spy at Taer Elladorn working for your house has learned of some drone activity that she thinks you should be aware of. It seems that messages dropped by numerous drones have been intercepted as they passed over Taer Elladorn. The messages the drones were carrying all make reference to "The Gem".

Hackers have been able to use the drones' emission signals to determine that the majority of the drones (if not all of them) were programmed with the flight path $y = -\frac{17}{11}x - \frac{93}{22}$.

The drone's "stop" instructions were heavily encrypted, so there is no way to tell where the message was destined for, nor even in which direction it was travelling.

This information could be crucial to the location of one or even all of the missing Gems.

Use the drones' programs to determine its destination.^{α β}

α A hint is available

β A hint is available

Figure 7.13. Systems of linear equations activity

Task: Decoding the Enemy

One of our spies at The Starhaunt has information about messenger drones being sent back and forth along the path $y = -\frac{17}{11}x - \frac{93}{22}$. If we could determine the drones' "stop" instruction, we could send a drone ourselves with instructions that would reveal the criminal party in The Starhaunt, not to mention at the drones' destination.

The messages the drones were carrying are all questions and answers or instructions about what to do with "The Gem". Someone at The Starhaunt knows where one of the missing Gems is! Is it the Gem of Shantara, or one of the other missing Gems?

Clearly the "drop" instruction for these drones is [drop: -5], but what is the "stop" instruction? We must learn this if we are to progress in our search for the Gem of Shantara, even if it means it is not the Gem being referenced in the intercepted messages!

We have unraveled the mystery! The "stop" instruction for all of the intercepted drones is [**stop:** ____], so we now know that the drones' destination is place called _____, and whose Cartesian coordinates are (____, ____).

The question now, of course is whether to continue to Taer Shantara as planned, or divert our course to this new destination in order to search for the missing Gems...

supporting calculations and notes:

Figure 7.14. Systems of linear equations worksheet

7.3.2. Individual Questionnaire

At the end of the series of activities on linear functions, students answered a written questionnaire to determine the extent of their uptake of the concepts in that unit. Another reason for considering individual students' responses was to determine whether they had successfully learned the concepts under the Design-Based Teaching Experiment delivery modality. While the activities were team endeavours, the questionnaire was a measurement of individual learning.

The questionnaire to which students responded is shown in Figure 7.15. The analysis of the students' responses to these questions are presented in Chapter 8, as well as analysis of the groups' responses to the team activities.

Name: _____

Unit 1 Questionnaire

1. On the graph paper you were given, plot the points $(0,0)$ and $(-5,3.5)$ and sketch the line that goes through these two points. Use a ruler for accuracy.
2. Does your line also go through the point $(-13,9.1)$. Support your answer algebraically, not just by “eyeballing it”.
3. The equation of the line in the above questions is $y = -\frac{7}{10}x$. Find the point on this line whose x -coordinate is -11 .
4. Write the equation of the line that goes through the points $(-5,3.5)$ and $(4.5,9)$.

Figure 7.15. The Individual Questionnaire

7.4. Method

In order to respond to the research questions, activities were produced that required that students use the obligatory concepts. Each activity corresponded with one or more learning objectives, and RME was invoked to write each activity as a problem embedded in a narrative. Students worked in teams to address the challenges in the activities.

The student responses were collected and analyzed in order to determine whether the RME device had been an effective overlay to the course, what students had learned about linear functions under this scheme and what the benefits were of the DBTE for future research. Data were collected and analysed in two phases: team activities and a written questionnaire.

The team activities were conducted for two reasons. The first was that RME is based in part on the idea that learning mathematics is a collaborative (as well as an individual) activity (the *interaction* principle; see section 5.2 for details). The use of team-based activities invokes the *interaction* principle directly, and the other principles can then be embedded into these activities. As one of the questions asked by this research is whether an approach designed for one setting can be used effectively in another, the approach in question must be implemented fully for any answer to that question to have meaning.

One of the objectives of the DBTE was that students actually learn the *Concepts* being discussed during an experimental session. Thus, the second reason for utilizing team-based activities was to deliver content effectively (i.e. to teach), even if the responses to that instruction was later to be scrutinized (i.e. for research).

As the activities were carried out in teams, most of the time the members of a particular team would produce answers identical or nearly identical to those of their teammates. Occasionally however an individual student would be absent on the day an activity was undertaken by his teammates and would need to “make up” the required activity. When compared to the answers provided by his teammates, the response submitted by the absent student on these “make-up” activities frequently provided insight into the thinking of an individual student and offered interesting substance for analysis. These anomalies could not be planned, so they were exploited when they did arise.

I noted whether responses were consistently correct or incorrect across a group of four, and in what ways. I also observed whether all four members of a group had supplied their answers in the same format. I made note of interesting assumptions, errors and approaches and considered these in light of the existing literature and under the application of RME.

Since RME was being investigated for its effectiveness at the community college level, an individually completed written questionnaire was distributed at the end of the unit in order to examine what students had learned about linear equations under the RME scheme. Having been exposed to the *Concepts* in a team setting it was hoped that each student would acquire the necessary outcomes; to assess to what extent this had happened, the written questionnaire was produced. Each question on the written questionnaire was analysed individually across all student responses. Here again, I followed interesting or unusual assumptions, errors and approaches. In the case of the questionnaire, I was asking to what extent the concept had been learned under the DBTE, so in effect, both the students and the DBTE model were under assessment. The questionnaire was designed to examine student uptake of outcomes (Question 1), whether they had made the Cartesian connection (Questions 2 and 3) and whether they were able to demonstrate a sequence of skills and the analytical leap necessary to produce the equation of a line given two points on that line (Question 4). The degree of accuracy in responses to the last of these questions would be an indication of the extent to which students had learned the algebra of linear equations in two variables as required by the course outcomes, thus allowing for an informal means by which to compare their success in the course to that of students in a more conventionally delivered course.

7.5. Implementation of DBTE

Using a DBTE *Approach*, I taught all of these concepts by presenting a scenario at the start of the unit in which a distal goal is described. Students then worked in teams of four to accomplish tasks and reach proximal goals, each of which was designed to align with a course learning objective. Each task was embedded within an activity in the master scenario, and while many of the chapters of the story are fantastical, each is sufficiently believable to be considered realistic by the RME definition of that word.

In addition, the particular narrative composed for use in the study designed for this dissertation was intended to bring a sense of play to the experience. Gaming is a familiar enterprise to many community college students, either because they engage in various games themselves or because their children do, and certainly there is a cultural trend to gaming as recreation. Creating a fantastic setting in which to investigate concepts is to invite students to play, and therefore to engage in a way that they have not likely done before, at least in a classroom setting, and was intended to provide an environment for exploration that is less threatening than a conventional lecture-based class.

Students were assigned into teams of four that as much as possible included students displaying varying algebraic ability, and they remained in these teams throughout the academic term. Class meetings were daily 50-minute sessions, and students were expected to prepare for class in advance by reading and attempting a minimal number of basic exercises, and then to spend most of the class time engaged in the activities that were given to them on any particular day. Retnowati, Ayres, and Sweller (2018) remind us that not every approach is right for every student at all times; specifically, they warn that students with gaps in prior knowledge usually benefit from learning in groups, but students with adequate prior knowledge do not, because the information they are purportedly learning is already in place and therefore redundant. Loes and Pascarella (2017) reinforce this caveat with their own that it is mostly White students and those most underprepared for college who benefit from collaborative learning in the first year of undergraduate studies.

Three observations are important here. One is that it is not practical to teach half the class using one method and the other half by a different method. If students do not have knowledge gaps in one area, they are likely to have them in others, especially in the case of a remedial course, in which case they are likely to benefit from the collaborative environment in at least some sessions if not in all of them. Furthermore, even if one attempted such a “split” method, it would apply to different groupings of students for each concept. While it could certainly be argued that collaborative learning might be more effective if it is interleaved with independent learning, it is certainly not worth exchanging the benefits it provides for some redundancy.

The second critical observation is that counsel against collaborative learning denies the learner the additional social benefits it carries. Collaborative learning helps the mathematically *stronger* members of the group learn the concept more deeply. Explaining an idea to another person requires the interpreting student to formulate the idea into a cohesive package in her own mind in order that it can be delivered. “[R]epresentations constructed during collaborative problem solving frequently need to bridge different viewpoints and thus need to be more abstract than is required for each viewpoint alone” (Schwartz, 1995, as cited in Ploetzner, Dillenbourg, Praier & Traum, 1999, p. 8). There are also non-instructional benefits, such as increased awareness and tolerance of issues of diversity (Loes, Culver and Trolan, 2018). Recalling that community colleges are charged with social as well as academic education, and that this study is an investigation of just that, *Audience* puts the social aspect in perspective for this document. While it is outside the scope of this dissertation to discuss levels of abstraction and issues of diversity, the welcome, free, additional potential benefits of any method should not be dismissed out of hand.

The third is that the class in which this study was undertaken was populated by exactly the type of students Retnowati, Ayres, and Sweller (2018) suggest will benefit the most: those with gaps in prior knowledge. Had they adequate prior knowledge, they would not have found themselves in a remedial course in the first place. As the primary benefits of collaborative learning are extremely applicable, its use in the study presented here is justified and, moreover, necessary as a component of RME. As the instructor, during class sessions I moved between groups answering logistical questions and clarifying instructions; as researcher, I retained for analysis the work produced by the student groups in these sessions.

7.6. Additional Information

Two major defining features can be identified to describe the delivery of the course for this study: a requirement that students “pre-read” the content for each class session, and tasks that students performed which utilized that content during a class session. Throughout the term, students were expected to read the section in their textbook on which the activities of the next class session would be based, and to attempt a few simple exercises presented alongside the text, in the margins of the pages they were reading. The reason for asking them to pre-read was that each student who did so

would be able to contribute to their group's conversation at a fundamental level by virtue of their recent exposure to the concepts, while retaining the feature of RME that individuals bring different experiential strengths to a group setting.

Many if not most students claimed to meet the pre-reading expectation, at least in part, at least most of the time. If a student prepared as she were asked, she arrived in class having exposed herself to the underlying terminology and mathematics required for the tasks that would follow and she performed three to five exercises for which models were readily available. Even if she had not understood everything she had read, pre-reading allowed exposure to concepts that would be discussed and used the following day, and the student who pre-read would have acquired some social currency by being at least familiar with those ideas. A student who did not so prepare might struggle with the mathematics of the tasks, but the team was expected to work together and any shortfall of comprehensions addressed in that setting first. Students who could not get help from their team mates often took it upon themselves to "catch up" by skimming the textbook before engaging with their group. The pre-reading and activities features cannot be separated in this research, as they relied on one another both in the classroom and in the analysis of the data that resulted from the study.

7.7. Participants and Data

The study was implemented with a class of 24 community college students¹⁵, 23 of whom completed the unit on linear equations in two variables. Students represented reasonably diverse populations in terms of sex, age and ethnicity and were assigned to groups of four in order to take advantage of the benefits of collaborative learning.

In this remedial course, students are assumed to have little or no knowledge of the formal algebraic or graphical representation of a line. In fact, as is common, some students started the term with previous exposure to the idea of linear structures. As the curriculum calls for a rigorous treatment of the subject, some redundancy is almost certain for at least some students. A student with prior knowledge in the remedial mathematics setting is at a social and emotional advantage, however. He has a

¹⁵ One student completed the in-class activities, and his efforts are included in the analysis of that assessment; however, this student did not complete the end-of-unit questionnaire, having withdrawn from the course.

commodity that the others need, which puts him in a position of authority. In a setting where many students are present because of their demonstrated lack of skill, this advantage can be an enormous enhancement to confidence, which is an asset in itself. The pre-reading expectation described in section 7.6 was intended to provide all students with at least some of this social currency, thereby leveling the playing field to some degree.

The data analyzed in the study was the activity worksheets produced by students during group interaction and the written questionnaire distributed at the end of the unit. The worksheets were templates that students completed individually, but in collaboration with the other members of their group, while the questionnaire was completed individually with no input from others.

7.8. Chapter Summary

The class in which this study was undertaken was unremarkable for the population served by the institution, and the domain of study was one required en route to those students' satisfaction of degree requirements. The model used to teach the class was highly unconventional however, and even more so for the setting in which it was implemented. The *Approach*, therefore, was implemented in order to explore whether the adaptations applied would be effective in the setting described here, and whether its use might be of benefit to other research-practitioners. Under investigation were the use of RME in a new setting, and the DBTE as a means to teach-while-researching students' learning around linear functions. In Chapter 8, I provide an analysis of the results of the responses to the activities and questionnaire described in the present chapter.

Chapter 8.

Results and Analysis

In this chapter, I report the results of the tasks undertaken by students in the RME classroom under a Design-Based Teaching Experiment *Approach* and analyze those results. I describe the ways in which the tasks were successful in conveying concepts conventionally delivered by lecture, and the ways in which they fell short, in which case I also discuss the modifications made to those tasks. I highlight additional ways that the data can be analyzed and provide insight into the line of demarcation between a student's desire to "do" a problem and her desire to understand a mathematical concept. I finish with a discussion of the results particular to this study and repercussions for the future.

8.1. Overview

In this chapter, I analyze student responses to questions about the concepts found in a unit on linear functions as a means by which to investigate the comprehension of those *Concepts* as delivered to an *Audience* of community college mathematics students in a remedial algebra course. In particular, when analyzing student submissions, I examined both the responses submitted by teams of students to activities presented in the RME style and individual student responses to a questionnaire. This method of examination allowed me to compare the known areas of difficulty students have learning algebra with what they actually did learn under the RME delivery. As I outlined in Chapter 3, student difficulties learning algebra can be classified into major categories including difficulties involving graphing linear equations, those in which notation is problematic or in which manipulation of expressions is inappropriate, and those which involve making a connection between algebraic and graphical representations. For the purpose of this dissertation, I focused on the specific theme of graphing linear equations in two variables (functions) and looked at the component *Concepts* under that theme, including ordered pairs, lines as collections of points, slope (as a ratio), the y -intercept, writing the equation of a line and solving systems of linear equations by graphing. In examining individual student responses to "conventional" questions after a unit delivered in an "unconventional" manner, I was looking for similarities and differences in the quality of responses between the students in the study

and students in classes I had previously taught more conventionally. I reasoned that if study student responses to these questions were similar in quality to those given by students taught conventionally, that this similarity could be taken as evidence that the approach under investigation had worked: that community college remedial algebra students could learn as much under the RME model as under lecture.

Importantly, this analysis accepts Mesa's (2017) challenge to take up what can be learned from K-12 research in mathematics education and extend it into the community college setting. Specifically, this report addresses the question of what happens when a teaching strategy developed for use in the K-12 environment is introduced in the remedial tertiary setting, and what adaptations, if any, must be made in order to attempt such an implementation. Although the students at the center of the study are a very different population from that for which RME was conceived, any approach that allows students to bring prior knowledge to the table will almost certainly empower those students. Recognizing that it was alienating and demoralizing to ask adult literacy and numeracy students to use textbooks written for primary students, textbooks were eventually developed that addressed the same concepts but which used language appropriate to the *Audience*. In the same way, RME can be used to teach adults, but the manner in which the activities are presented must be appropriate if they are to be effective.

To obtain the concepts required by the curriculum for the course in this study, students were expected to read from their textbooks before coming to class and to attempt a small number of basic exercises that followed the models and examples in the reading. On arriving for class, students worked in teams to complete tasks that used the *Concepts* about which they had read. If a student was not present, he could still submit the required worksheet, but he would either need to work out the solution on his own or ask his team for help. The expectation was that students would collaborate to answer the tasks presented within the activities, for which reason it is not surprising that within any team, the responses were often similar if not identical.

8.1.1. A Note on Vernacular

Students frequently refer to "a line (between two points)" when what they are describing is actually a *segment* of "a line between two points". The activities in this

study used the “two points” in this description as the starting and ending points, respectively, of a journey. As such, while students were strictly working with a line segment, their use of the term “line” was not corrected either in speech or in writing. The ability to determine an algebraic expression for the line *containing* two points requires the collection and application of multiple skills and, as their teacher, I was more concerned with investigating students’ ability to use these skills than of their use of the correct related terminology; as a researcher I was more interested in the use of the *Approach* to convey the ideas of coordinates, slope and equation of a line than I was with whether students learned the absolutely correct nomenclature. As students mature academically and mathematically, such a correction is not difficult to make, and I was not concerned about the impact it would have on these students’ progress. In light of this disclosure, I have also frequently used the term “line” in the descriptions of student responses when it would have been more correct to use the term “line segment” in keeping with the spirit of the *Approach*. One wonders what Pimm (1987) might have to say about the word “between” as it is used here.

Additionally, recall that the terms “function” and “linear equation in two variables” are used somewhat interchangeably in the discussion of the study, as explained in section 3.1.

8.2. Team Activities on Linear Functions

What follows is an analysis of the multi-stage activities in which teams in the study were expected to learn about ordered pairs, points as solutions to linear equations in two variables and the equation of a line from two points on that line; in keeping with the narrative, the instructions were given to student groups in that spirit. The tasks undertaken by groups required them to estimate and record coordinate locations, find the “directional command” (the slope) and the “longitudinal crossing point” (the y -intercept) of the path of a “drone” traveling between two locations (along a line), and then to put these last two values into their correct positions in a drone’s flight “program” (the slope–intercept form of the equation of a line). Each subheading below aligns with one of the subheadings under section 3.5. For each task, I review the requirements of the related activity and the information students were given, and briefly describe the means by which analysis was performed, as this process was different for most of the tasks. I also describe the results of the analysis and comment on the implications.

8.2.1. Graphing

The responses analyzed in this section align with the activities and tasks described in section 3.5.1, and they reference the map, instructions and worksheets found in Figure 7.1 through Figure 7.14. Students worked in groups of three or four, but each student submitted their own work.

8.2.1.1. Graphing: Ordered Pairs (see section 7.3.1.2)

In this activity, students were asked to identify the Cartesian coordinates for locations on the map (see Figure 7.3). Having been given a story that explained the convention by which such coordinates are written (see Figure 7.5), students completed a table in which the names of the locations were listed with the coordinates of some of those locations. Since they were applying a convention they had just learned to an artificial situation, it is reasonable to expect that their estimations of the locations might not be terribly accurate. In fact, the most common question students had was what to do about all the locations that were not on “exact” (i.e. integer) grid lines. Once they were satisfied that they were expected to estimate and that they would not be penalized for not knowing with absolute certainty whether a value was 4.75 or 4.8, they were all quite happy to complete the task estimating all of the coordinate values.

In analyzing student responses to this task, I first compared the coordinate values across each group of four students and found some minor evidence that a few students had simply copied the values from another student. This behavior was permitted, but potentially indicates a lack of engagement. In one case, the level of accuracy of one student’s responses was different from the other members of his group; given that he was absent on the day the activity was done in class, this indicates that he performed the task himself, without the input of the other members of his group, but also that he did not simply copy the work of the others. That his results were still reasonably accurate, although different from those of the rest of his group, suggests that he understood well the objective of the task.

I also compared the responses between groups and found that while different groups approximated the decimal values of locations differently, all six groups found acceptable approximations of all coordinate values. While the individual questionnaires might return different results, this activity appears to have created an adequate

environment for students to take up the idea of Cartesian coordinates. The task included fifteen locations, which I expected to be sufficient practice using the convention of Cartesian coordinates.

8.2.1.2. Graphing: Lines as collections of points (see section 7.3.1.3)

While any location on the plane might be represented as an ordered pair, a specific subset of all possible ordered pairs reveals a relationship between the two coordinates such that all points in that subset fall on a line. The task designed to convey this idea asked students to “Find a way ... to drop a message for Taer Valaestas” for which they would need to know the location (the coordinates) of that location.

This task was designed to provide a foundation for the idea that a line is made up of many points, and that multiple points can be found on any one line. The messenger drone was introduced and students were asked to determine the coordinates of a location part way along the drone’s flight path. In analyzing the responses to this task, I looked for ways in which students approached the problem in unexpected ways.

The second task in this activity was designed to help students see that an ordered pair between two endpoints was also a point on their “line”. Furthermore, students were then asked, “Would it be possible to drop another message for Keth using the same drone?” The coordinates for Keth cannot reasonably be said to satisfy the equation of the line between the endpoints, even approximately. This result was meant to demonstrate that not every point in the plane will satisfy the requirement of being on a line. All of these instructions were given in the spirit of the narrative in order to make the activity have some meaning, however fanciful.

An important question in light of the current research was the location of the point on the “path” associated with the city of “Taer Valaestas”. The drone was described as being required to fly between the locations of Pylas Maradal and Taer Shantara. This flight path passes fairly closely over Taer Valaestas, which, being the capital city, has map coordinates (0,0). Even before giving this activity to students, this location seemed too trivial and I determined to find a more “interesting” position for future instances; however, only seven students gave the anticipated response of (0,0), and all of these submissions came with unexpected complications.

One such response is shown in Figure 8.1, and a different way of documenting the same approach is given in Figure 8.2. While creative, this submission is a good example of a task that has been designed without consideration for that creativity. The point of this task was to get students to see the need to travel between two points in order to establish that a third point could be on the same line. These two teams' creativity meant that the importance of the relationship between two points was lost.

We will program the messenger drone drop a message at 0,0 as well, to let the good people of Taer Valaestas know we are coming. Many of us have friends and relations in that most glorious city, and we would all be glad of a soft bed and a hot meal after such a journey.

Notes, maps, diagrams and other supporting work:

Drone drop #1: Pylas Maradal \rightarrow Taer Valaestas
 $(-5, -7)$ $(0, 0)$

$$5^2 + 7^2 = 8.6^2$$
$$8.6 \cdot 200 = 1720$$

Drone drop #2: Taer Valaestas \rightarrow Taer Shantara
 $(0, 0)$ $(4.5, 9)$

$$4.5^2 + 9^2 = 10^2$$
$$10 \cdot 200 = 2000$$

Figure 8.1. Assumptions that complicate – example 1

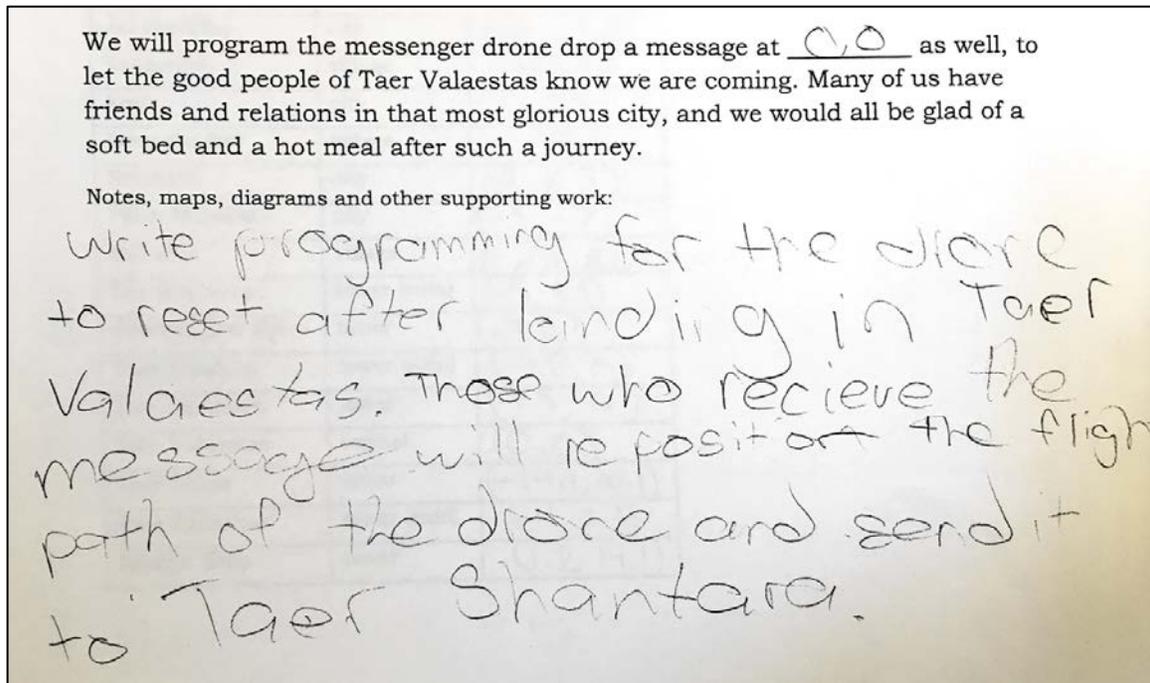


Figure 8.2. Assumptions that complicate – example 2

If a line is drawn between Pylas Maradal and Taer Shantara, it passes near $(0,0)$ and directly through such points as $(-1,0)$ and $(-0.5,0.8)$ among others significant to the map, the x - and y -intercepts and one which might be described as being on a road on the map. Six students gave one of these two points. Although the instructions were to give the “drop” coordinates for the city Taer Valaestas, several students instead gave the coordinates of one of these other locations.

The team whose work is shown above in Figure 8.2 on the other hand, adjusted their plan in a different way. Instead of joining segments of two lines as the teams from Figure 8.1 and Figure 8.2, or “missing the mark” of $(0,0)$ as several other teams did, they moved the path so that it would travel over $(0,0)$ and created the role of a “carrier” who would convey the message to its final destination. This team’s approach is indicative of their close attention to detail in light of the challenge created by the inaccuracies inherent in the problem they were given.

In hindsight, the responses $(-1,0)$ and $(-0.5,0.8)$ may be more correct than at first glance. In light of the “Assumptions that complicate” examples they are certainly more in line with the purpose of the task. The line drawn between the two endpoints does not actually pass directly through the point $(0,0)$. The task was designed with the

assumption that all the students would feel the line between the two points in question passed sufficiently closely to $(0, 0)$ as to be a “good enough” answer to the problem. Students who answered this question with either $(-1, 0)$ and $(-0.5, 0.8)$ actually gave the coordinates (by observation at least) of points that are on the line, in spite of the fact that they are not particularly close to the requested “drop” location. These students, then, appear to have made the Cartesian Connection, at least at an elementary level, while those who “correctly” answered with $(0, 0)$ may not have, giving instead a response requiring no analytical thought and no accuracy, and in some cases missing the point of the task.

Beyond the above responses, five students (one group and another individual student) gave the coordinates of $(-1, -1.9)$, which is the location of a different city than was requested, Jal Paeridor. The coordinates of this city lie on a line between Pylas Maradal and another location; it appears that the students providing this response used the same tactic as those in the “Assumptions that complicate” examples with an additional modification.

One group (four students) gave the response $(1, -0.25)$. It is not clear why this point was provided. If the coordinate values are reversed, the resulting point $(-0.25, 1)$ is somewhat close to the point $(-0.5, 0.8)$ which is a point on the desired line, but as it is not very close, this speculation is not instructive.

Having completed the previous activity in which they were expected to identify the coordinate values of points in the plane, students were able to identify points if they knew the location of the city for which they were being asked. For 13 out of 22 students, the coordinates they recorded were either correct at face value, or better, points that fell on the line they had been asked to consider. That so many of these students were able to reconcile the importance of the location of a point on a line is encouraging.

This activity serves as an example of one that “fell short”. It required modifications to be made to the information provided to students so that it would be clearer that they were required to send their message directly to its final destination. In addition, the map would be redrawn so that the location of the intermediate location was much more obviously on the line and so that it was not a trivial position.

8.2.1.3. Graphing: Slope (see section 7.3.1.4)

The starting and ending locations of the drone's (correct) flight path have coordinates $(-5, -7)$ and $(4.5, 9)$ respectively. Students were told that the "directional command" is a comparison of the distance north and the distance east the drone must fly to get to its destination and asked to calculate the "directional command" for the flight path of a drone flying between these two locations. In analyzing student responses to this task I considered both the form in which the answer was given and the means by which the answer was achieved.

All students but one (who did not submit a worksheet for this activity at all) interpreted the meaning of "directional command" in such a way that its calculation was the same as that for "slope", and most students used the coordinates of the points they had recorded in the Cartesian Coordinates activity and the slope formula to find the desired value of $\frac{32}{19} \approx 1.6842$. 11 of 23 students gave a value of $\frac{16}{9.5}$, which is the value obtained by using the slope formula, but without simplification. Four others (all members of the same group) gave the fraction $\frac{15.9}{9.5}$, having been more exacting in establishing their coordinates in the previous activity (again, without simplifying), and three more gave $\frac{32}{19}$, having simplified the ratio. In all, 18 students gave their answers as fractions. These responses were all regarded as correct given that students were measuring with very course tools, and the goal was to capture the idea of slope rather than accuracy of measurement.

Of particular interest is the large number of students who were able not only to produce a fraction for this value, but who also left that fraction in a form which makes it easy to see that it was obtained by taking the ratio of the vertical change and the horizontal change between two points. A representative of this response is shown in Figure 8.3.

• The directional command is $x = \frac{16}{9.5}$;

Supporting calculations:

$$(-5, -7) \quad (4.5, 9)$$

$$\frac{9 - (-7)}{4.5 - (-5)} = \frac{16}{9.5}$$

Figure 8.3. The slope as a ratio

That so many students left their results in the form of a fraction, and in particular a fraction containing a decimal, speaks to the possibility that they were seeing the number they produced as a ratio, even if such a “mixed” format might not be regarded as strictly simplified. Students at this level frequently avoid using fractions, and frequently make errors when subtracting negative integers. 18 of 23 students accomplished both of these operations with confidence.

Not all students completed the above task correctly. In one team, three members correctly used the slope formula and gave their answers as $\frac{16}{9.5}$ (although one of these gave his answer as $x = \frac{16}{9.5}$), but their fourth member gave the entire equation of a line in slope–intercept form for his answer; there is evidence of the use of the slope formula on his paper, and although he appears to have used the formula to find it, one of his points does not fit the situation. This student’s response is shown in Figure 8.4 and comments follow the figure.

Our drone must be programmed to fly from here, Pylas Maradal, to Tacr Shantara, to deliver the message to Jaq'qal that we are on our way.

• The directional command is $y = \frac{5}{3}x + 1.5$; $\frac{16}{9.5}$

Supporting calculations:

$$(x_1, y_1) \quad (x_2, y_2)$$

$$\frac{y_2 - y_1}{x_2 - x_1}$$

$$(+1, +3) \quad (-5, -7) \quad \frac{3 - (-7)}{1 - (-5)} = \frac{10}{6} = \frac{5}{3}$$

$$y = \frac{5}{3}x + 1.5$$

slope = $\frac{5}{3}$

Figure 8.4. Making the problem fit the solution

This student’s work is remarkable for several reasons, the most prominent of which is that he did not answer the question posed; when asked for the “directional command”, he responded with the entire equation. He has the slope answer indicated in a box, but he also boxed the entire equation in a second location, and he has written the

equation in the space given for the answer. Another curious observation is his record of the ratio $\frac{16}{9.5}$, which is the correct, if unsimplified, ratio of the change in the vertical direction to the horizontal between the two locations (i.e. the correct slope). He does not indicate how he arrived at this ratio, nor does he use it in the remainder of the work.

One of the points this student used in computing the slope was $(-5, -7)$ (in spite of the corresponding point on his own Cartesian Coordinates worksheet being recorded as $(-5, -6.8)$); the other appears originally to have been $(-1, -3)$ and changed to $(+1, +3)$. On first glance neither the reason for his choice of coordinates nor his reason for changing the signs is apparent; there is no “town” at either $(-1, -3)$ or $(+1, +3)$. On further investigation however, the slope of the line between the two endpoints is sufficiently close to $\frac{5}{3}$ to give the illusion, on inspection alone, that this is its value; in fact the value of the slope when algebraically determined is different from this student’s slope only by about 0.0175. Given a “starting point” of $(-5, -7)$, this slope would indeed mean that the point $(+1,+3)$ was on *his* line. It appears that he has determined a slope that is “close enough” by visual inspection, and then selected a point that satisfies the resulting line. Having established the necessary parameters, he then went about the process of determining the equation of the line between those two points. Although he performed these mechanical procedures correctly, he did so with incorrect values. He did not carry over the coordinates of the first point from his own earlier record, and he “found” the second point by manipulating the situation to match the level of understanding of which he was able.

This student was absent on the day this activity was done in class. The corresponding task on his individual questionnaire (see section 8.4 below) was correctly computed, but there the points were given. This student has demonstrated that he can find the slope of the line between two points, but he may not be able to determine which points are appropriate to use in a given situation.

Another team did not parse the information in the narrative correctly and ended up with a slope of 1.2. This error is based on their having selected the wrong destination for their calculations. Although this group too would benefit from more attention to detail, the remainder of their work was correct, including the use of the slope formula to determine the slope.

Errors in computation or interpretation will always result in incorrect responses, but in this activity, with the exception of one student's apparently random choice of location and four students' use of an incorrect location, every student completed this task using the appropriate tools and arrived at an appropriate answer. While it is impossible to say which particular part of the system of DBTE is responsible for these results, the combination appears to hold considerable merit.

8.2.1.4. Graphing; The y -Intercept (see subsection 7.3.1.5)

The y -intercept of a line is a critical concept for two reasons: it is the graphical representation of the case where the input value is zero, which is important in myriad applications, and it is a way of “anchoring” the graph of a line to the plane when creating a sketch of the line. In the activity in which students were asked to find the y -intercept of the line they were using, they were given general instruction on finding the parts that make up a drone's program and asked to determine the values for those parts for their particular situation. The parts in question were the slope and the y -intercept, although in keeping with the narrative, these were called the “directional command” and the drone's “longitudinal crossing point” respectively. In the narrative, a description is given of both the significance and the benefit of evaluating an equation for $x = 0$ followed by the instructions “Determine the coordinates of your drone's longitudinal crossing point”. For a student able to connect the $x = 0$ comment with the instruction to find the y -intercept, the task is an easy one, requiring only that the position be determined of the line as it crosses the y -axis. This does not occur at a “nice” location however, and most students determined the value by observation and approximation.

Determined algebraically, the y -intercept in this task can be found to have a value of $\frac{27}{19}$, or approximately 1.4211. Only one team of four students arrived at (approximately) this value, and they did so by using the slope–intercept form of a line ($y = mx + b$), replacing m with 1.68 (a good decimal approximation of the correct slope ratio) and x and y with -5 and -7 respectively; solving the resulting equation for b returned the correct y -intercept (based on their approximation of the slope). It is not possible to know whether these students understood the y -intercept any better than students in the other groups, but they were at least successful in following a mechanical procedure.

Figure 5.8 shows a representative sample of this group's response to the y -intercept task. The absence of the ordered pair format was ignored in the analysis.

• The longitudinal crossing point is 1.4;

Supporting calculations:

$$y = mx + b$$

$$-7 = 1.68(-5) + b$$

$$-7 = -8.4 + b \rightarrow 1.4 = b$$

$$+8.4 \quad +8.4$$

Figure 8.5. Determining the y -intercept

In general, during guided instruction following on from self-exposure and group discussion and negotiation, students found the slope easily. On the other hand, the y -intercept seemed to present problems for these students. The group that determined the slope using an incorrect endpoint used a correct procedure to do so; two members of this group used a correct algebraic procedure to find the y -intercept as well. Although their y -intercept was incorrect, it can at least be said that these two students were able to make appropriate use of the algebraic tools they had. These students' response is represented in Figure 8.6.

The longitudinal crossing point is (0, -1);

Supporting calculations:

$$6.2 = (1.2)b + b$$

$$6.2 = 7.2 + b$$

$$6.2 - 7.2 = b$$

$$b = -1$$

Figure 8.6. Finding the y -intercept with the incorrect slope

The other two members of this group provided the points $(-4, 0)$ and $(0, -4)$. One of these students explained that she had observed where the line drawn between the two points this group used crossed the y -axis, but such a line does not cross at either of the two locations given by these two students. It is not clear how they determined these coordinates.

In another group, two members found the correct y -intercept using the slope-intercept form of a line and the values they had for x , y and m to find b , but the other two

members of this group, although they arrived at the same value, did so by observation of the graph (or by simply copying the results produced by the others).

In all, 19 students were able to find an appropriate, if not correct value for the y -intercept, but of these, only 7 did so using an algebraic approach. The meaning of the y -intercept was clearly apparent to most of the students who found it by observation, and it might be argued that they were not told explicitly that they must use algebra to find the answer. Even among these students, not all of them felt the need to find the y -intercept with any accuracy. While students were easily able to find an accurate slope of the line between two given points, they were less able to find an accurate y -intercept. The need for this value to be accurate does not appear to be important, at least to these students, without explanation, and so it falls to the task designer to imbue it with some feature that will make its accuracy more attractive.

8.2.1.5. Writing the Equation of a Line (see section 7.3.1.6)

Where each of the previous tasks in this activity addressed one distinct *Concept*, the overall activity provided a scaffolding from which students were expected to combine the *Concepts* from the earlier tasks into the ability to write the equation of a line given two points on the line in the last task. It was expected that students would put the “directional command” (slope) and “longitudinal crossing point” (y -intercept) from the previous two tasks into place in the equation of a line. All 23 students correctly (or at least adequately) determined the slope and y -intercept, but only 16 of these were able to use directly the values they had established to construct the equation of the line in the form requested of them.

The remaining seven students used some form of formula which indicated their need to re-compute a value for slope as though they had not just done so two tasks before, in two cases arriving at different values than they had previously. One of these papers is a good representative of the apparent need to recomputing the slope. The paper in question is shown in Figure 8.7 and comments follow the figure.

Our drone must be programmed to fly from here, Pylas Marada, to the
Shantara, to deliver the message to Jaq'qal that we are on our way.

- The directional command is $\frac{16}{9.5}$;

Supporting calculations:

$$\text{slope} = \frac{\text{vert change}}{\text{Horizontal chan}} = \frac{\text{rise}}{\text{run}} = \frac{16}{9.5}$$

$$\frac{y_2 - y_1}{x_2 - x_1}$$

- The longitudinal crossing point is $(0, 1.5)$; moon shadow

Supporting calculations:

$$y = \frac{16}{9.5}x +$$

- The program is therefore $y = \frac{16}{9.5}x + 1.5$.

Supporting calculations:

$$\text{slope} = m = \frac{\text{Rise}}{\text{Run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{9 - (-7)}{4.5 - (-5)} = \frac{16}{9.5}$$

Figure 8.7. Recomputing the slope

Given her justification in the first task that “slope=vert change/Horizontal chan...=rise/run”, the student whose work is shown here appears to have determined the slope by visual inspection in the first task. Her work on the third task includes a similar justification: that “slope = m = rise/run”, but in the second case this annotation is extended into the slope formula. What makes this submission fascinating is the student not only computed the slope twice moments apart (once for the specific instruction in the first task and once to write the equation of the line), but she accomplished this task by two different methods. This student’s approach may seem redundant and unnecessary, but in fact it could be used as a model by which to bridge the gap between the intuitive and algebraic definitions of slope. Furthermore, if multiple points on a line were known, the slope could be determined both by inspection and algebraically multiple times. When this work always returned the same value, it might be easier for students to make the connection between observable traits and their algebraic determination, but also to begin to see the slope as a ratio, and a ratio as a fixed value. While this task does not quite qualify as having “fallen short”, this student’s response brings to light an opportunity to reinforce the exercise and possibly extend it.

Re-computation is common during exam conditions, but during the in-class activity it was an unexpected result. Students were working in a casual, relatively relaxed environment, and working collaboratively with no restrictions on what resources they might use to help them answer the questions they were asked. The fact that a student can correctly find a slope given two points, but then needs to re-compute the same slope when asked to write the equation of a line suggests either that she has not made the connection between the slope of a line and the role that it plays within the equation of a line or between the slope she perceives as the one for “direction” and the one for “real algebra”. At the very least she has not noticed or not attempted to bring the result of one task into the performance of the next. This is perhaps not a surprising result from a course at this level. Since many of these students are seeing the material either for the first time or for the first time in a long time, it is understandable that perhaps they do not understand deeply the relationship between the equation of a line and the parts that make up that equation. It may also be a symptom of a system in which students are often shown an algorithm and then required to perform that algorithm on multiple disconnected exercises as is the wont of most pre-college algebra textbooks.

Many students who tend to work by mechanical manipulation without generalizing will usually put the “ m ” value in the correct place in the “template” $y = mx + b$; students who re-compute the slope are lacking understanding of the purpose for having already found the desired value. It may be that these students are so used to finding the equation of a line by purely mechanical means that on being asked to do so, their only way forward is to work through the steps “find the slope”, “plug in x and y ”, “solve for b ”, “write out the equation”. That they have already found “ m ” eludes them because it does not fit into the process that they have memorized for this task.

8.3. Summary of Team Activities

The team tasks were designed to introduce and reinforce the content required of the curriculum by making use of the activity, reality, level, inter-twinement, interaction and guidance principles of RME. While designed for use at the primary school level, their implementation with a community college *Audience* was not exceptionally demanding beyond the time and creative energy that would have been required to deliver them to any *Audience*. The community college population is highly adaptive and receptive to

challenge, which makes this population a good candidate for research into any new technique.

Working in groups, students were highly successful describing locations using ordered pairs. The task that was set for them required that they utilize the convention of Cartesian coordinates, but it presented them with locations having imprecise coordinates, a substantial extension from most textbook applications. Applying the concept in question in the narrative setting required a considerable amount of flexibility and creativity. Most teams easily rose to the challenge.

When answering the task regarding the location of a particular city (a point) on a flight path (on a line), most students answered this somewhat ambiguous task correctly, but several students gave reasonably accurate estimates of other locations. This type of response does not speak to an inability to determine whether a given point is on a line but instead seems to indicate that these students either did not carefully read or did not understand the instructions. While changes could be made to the instructions that might clarify them, leaving them as they are might provide a useful lesson in checking one's assumptions. Many students in remedial algebra classes tend to skim over the details or neglect them altogether. Since one of the tenets of community college is to provide a combination of curricular competence and social instruction, instructions such as those described here are well placed to afford that opportunity. It cannot go without comment that for this task and at least one other, the teacher also learned from the students about some of the ways in which they tend to think "around" a problem, and about some additional elements to consider in designing tasks.

All students correctly analyzed and computed the slope, insofar as its meaning was identified. Although some teams did not correctly interpret the instructions given for finding the slope, all students but one correctly interpreted the directional command as "slope". 19 out of 23 students also exhibited proficiency in finding the y -intercept, although some did so by observation or by using a correct algebraic procedure but incorrect values for x or y . Finally, the vast majority of students (23 of 24) correctly used the slope and y -intercept to create the equation of the line that went through the two given points, even if some of them had to recalculate the slope in order to do so. In comparison to lecture-based courses I have taught in the past, the collaborative and narrative-based approach to delivering the *Concepts* related to linear functions and their

graphs was considerably more successful in producing student work that correctly used the information they were given to accomplish the required tasks. Because of the nature of the collaborative group, it is of course impossible to say whether each individual student successfully grasped the concepts from the activities, or whether some of them simply copied out the work of their teammates. This behavior was anticipated, and in fact encouraged. It is not unusual for understanding to follow mechanics (Skemp, 1987) and the collaborative environment was intended to ensure that every student would be able to turn in work with some confidence. To determine whether students had achieved individual understanding, they were asked to complete a written questionnaire.

8.4. Individual Responses to the Written Questionnaire

At the end of the series of activities on linear functions, students responded individually to a written questionnaire to determine their uptake of the concepts in that unit. Since the *Approach* was under investigation, the individual responses were used to determine whether students had successfully learned the concepts under the Design-Based Teaching Experiment. While the activities were team endeavours, the questionnaires were independent efforts, and as such they were analyzed individually. The questionnaire appears in Figure 7.15 in section 7.3.2.

8.4.1. Analysis of Question 1:

Question 1: [P]lot the points $(0, 0)$ and $(-5, 3.5)$ and sketch the line that goes through these two points. Use a ruler for accuracy.

All respondents to this question were able, working independently, to plot a Cartesian point one of whose coordinates was a negative integer and the other of which was a non-integer value. Plotting a Cartesian point is trivial for the expert, but for students in a class in pre-college algebra, this task holds several traps, such as the potential for reversing the roles of the x and y values and plotting positive values when negative values are required. That 100% of students successfully answered this question is noteworthy. One of these students did not follow the instructions completely and his graph does not contain the line that passes through the two points, but he was at least able to plot the points. All of the other students completed the question by correctly sketching the line between the two points.

A representation of the correct response to Question 1 is shown in Figure 8.8 as a link to, and an aid in the discussion of the analysis of Question 2. The point $(-13, 9.1)$ from Question 2 has been added.

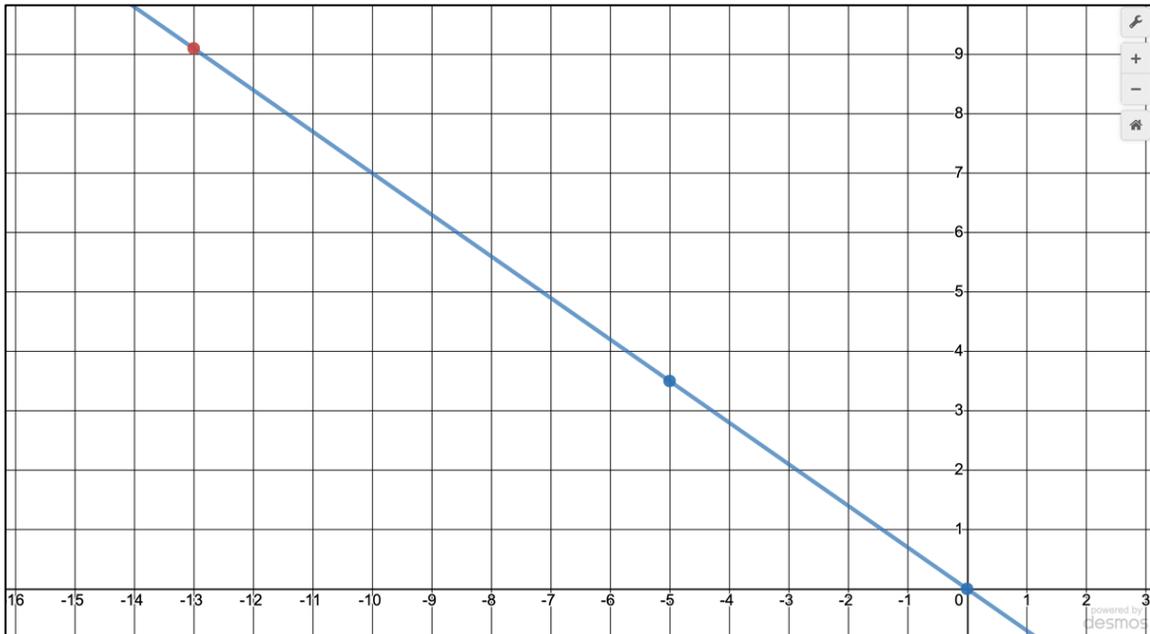


Figure 8.8. The two required points and the line through them. the point $(-13, 9.1)$ has been added.

8.4.2. Analysis of Question 2:

Where Question 1 asks students to plot the points the points $(0, 0)$ and $(-5, 3.5)$ and sketch the line through them, Question 2 asks about the potential for another point to be on the same line:

Question 2: Does this line also go through the point $(-13, 9.1)$?

This question provides a new ordered pair and students are asked to determine whether this point is on the line they drew in the previous question. Note that the “new” point appears to be on the line on observation. It was expected that students who were unable to approach the question analytically would answer by estimating that the line they had drawn passed sufficiently closely to $(-13, 9.1)$ to claim that it passed through that point, while students who had some analytical skills would write the equation of the line in the previous question and test the x value of -13 to determine whether y should

be 9.1 or some other value. In fact, the point $(-13, 9.1)$ does lie on this line, but it is not possible to know this with certainty without some algebraic analysis.

Determining whether a point such as $(-13, 9.1)$ is on a line is a nontrivial task for many students in a course such as this, as the point in question contains both a negative coordinate value and a non-integer coordinate value. Not all students answered this question correctly, but many more did than in my prior experience.

Instructions for this questionnaire item included a requirement to support the answer with algebra; if a student did not provide any algebraic reasoning for their response, it was assumed that they did not use any, and that they relied on observation alone to determine the point's location in relation to the line. 19 students approached this question with the use of a formula (i.e. analytically), even if some of these did not see that approach through or successfully answer the question.

Students took one of several approaches to answering this question: most students either used the equation of the line, having observed that it had been provided for a subsequent question, or they determined the equation of the line and tested the point $(-13, 9.1)$ in some way; a few used observation alone or some other approach.

Six students noticed that the subsequent question gave the equation of the line through the points with they had been provided, and they used this equation to answer this task. These students either tested whether replacing the x with the value -13 would produce a y value of 9.1 (or in one case whether $y = 9.1$ would result in x being -13) or whether replacing both x and y with -13 and 9.1 respectively would produce a statement of equivalence. One of the students taking this approach was not quite able to reconcile the process with his response. Figure 8.9 shows this student's attempt to use the equation given in the subsequent question.

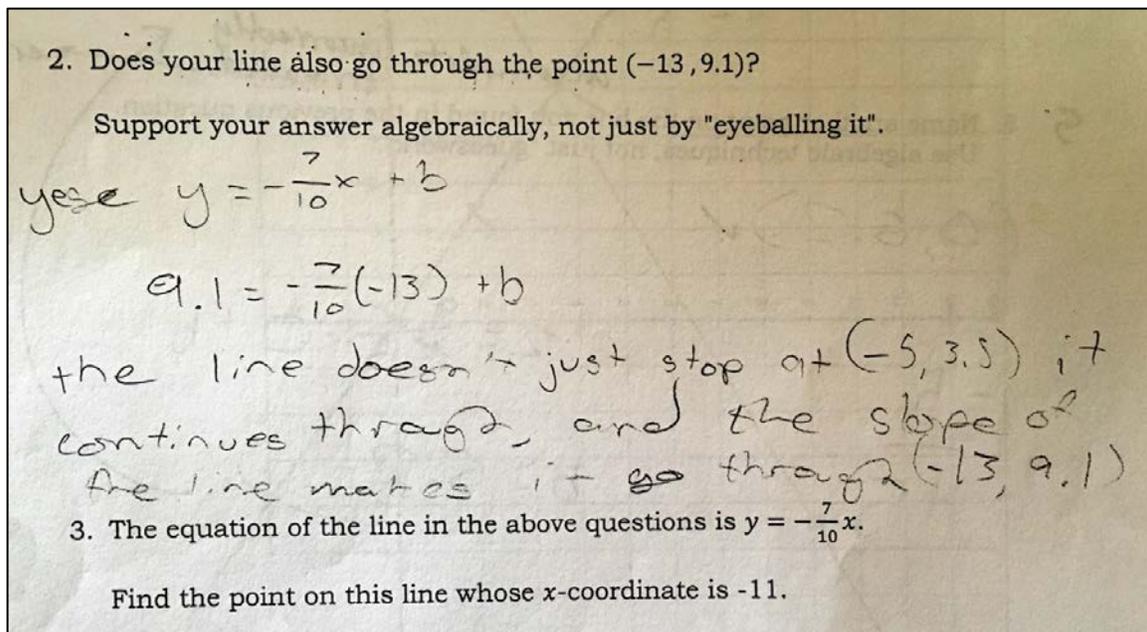


Figure 8.9. Noticing the equation in the next question

The apparent need to place a “ b ” in the equation (or perhaps the absence of a b value in the given equation) resulted in a dissonance this student was unable to rectify; instead, he gave essentially a verbal description of what he thought the line should do, and how it would go about doing so. There is no evidence that this student actually verified that the y value would be 9.1 for an x value of -13 . In his defense, this student did at least recognize that lines do not “end” in the way they did in the narrative delivery of the same concept. All of the other students taking this approach completed the task satisfactorily.

A second approach involved computing the slope of the line through the two points given and applying it to an x - y pair to create an equation. 13 students approached the question in this manner, all of whom correctly determined the slope, correctly determined the equation and, with two exceptions, correctly determined the corresponding value for y . As is common, it is the exceptions that are interesting as they might give a glimpse of the difficulty students have parsing information. One of the exceptions here is a student who managed to create the equation of the line, but then replaced the “ b ” value in her template with 9.1. This student’s submission is shown in Figure 8.10, and additional comments follow the figure.

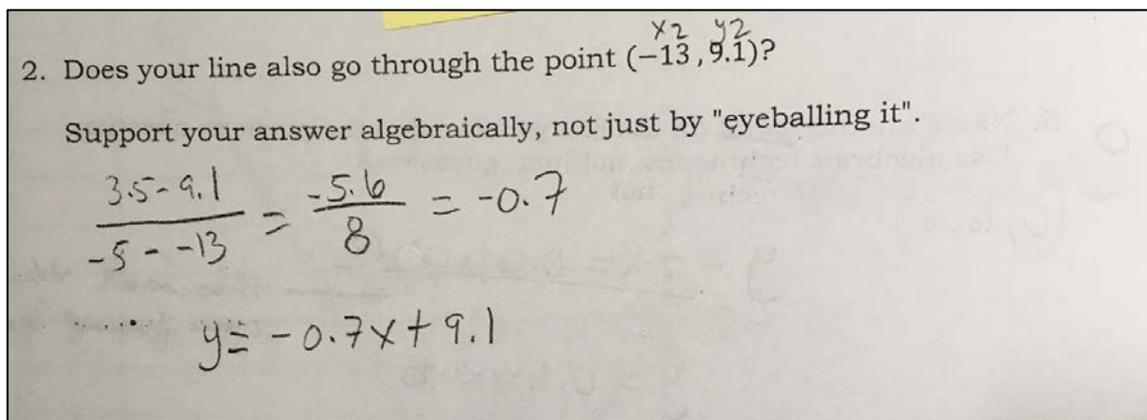


Figure 8.10. Reinventing the y -intercept

Since it did not result from a calculation, she appears to have assumed that the value of " b " she sought must be the "missing" element in the equation of the line. This assumption is an indication of the student's need to use all the numbers she has been given to work with, even though she clearly does not appreciate their real meaning (see Carpenter, Lindquist, Matthews, & Silver, 1983)

The other exception in this approach is a student who worked through the entire problem very thoroughly, finding the slope of the line between the given points using the slope formula and applying that slope and a point to a slope-intercept template to find a y -intercept, then using the now known slope and y -intercept to complete the equation of the line before substituting the value of -13 for x to confirm the y -value of 9.1 . One error made by this student was the values he substituted for x and y in computing the slope; he selected the x -coordinate from one point but the y -coordinate from the other.

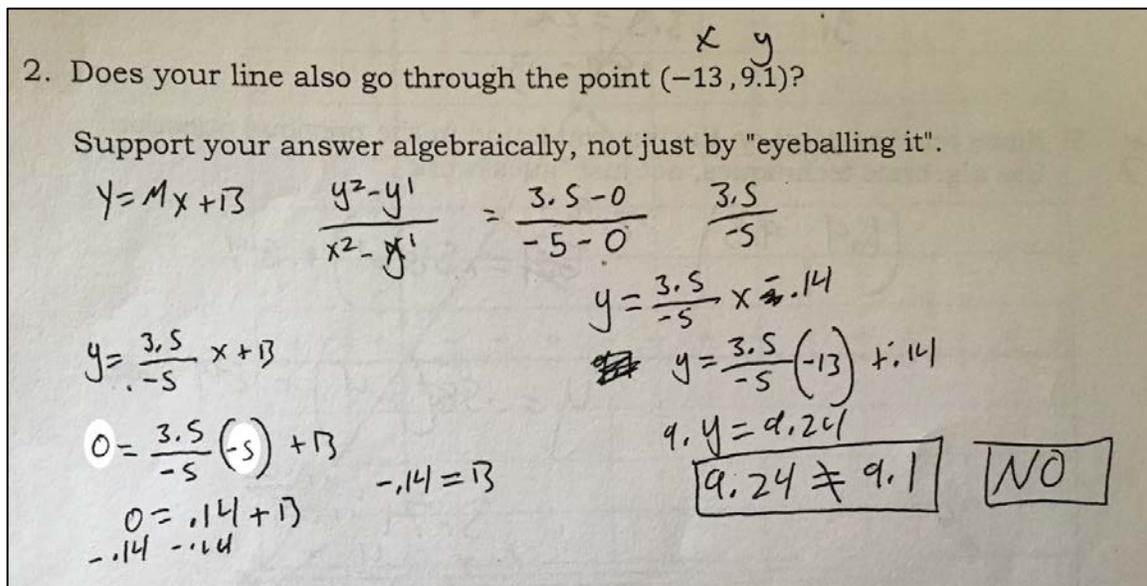


Figure 8.11. Coordinate values from different points

It appears that this student either needs the security of an algorithm to proceed or at least anticipates that an algorithmic approach is required of him, as he used a dummy variable “ b ” in his equation in spite of the fact that the y -intercept had been given. This student did not notice that to be the case or did not make the connection that if the point $(0,0)$ is on a line, it is the y -intercept for that line. Again then, while observation alone is often insufficient to answer a question that requires precision, failure to use observation at all is risky.

The process this student followed was otherwise sound, and the conclusion to which he came was correct based on the results of his error. In spite of having arrived at the wrong answer, he demonstrated a sound understanding of the process of writing the equation of a line and the ability to use that equation in an appropriate, informative way. Figure 8.11 shows this student’s work. The “mixed up” coordinates have been highlighted in the image.

17 out of the 23 questionnaire respondents provided well-formed responses to this question. Two additional “approaches” were observed in students’ answers to this question. One was observation (or assumed to be observation, as no supporting work was provided); the other was to write the equation of the line, but not to use it to answer the question. One student answering the question “Does your line go through the point $(-13,9.1)$?” answered “No. it goes through $(-10,7)$ and $(-15,10.5)$ ”. This response is

intriguing because this student was able to find two points that lie on the line presumably with some kind of reasoned approach (no supporting work was provided), but she was unable to reconcile the fact that the requested point also met the criteria she used for the others. This student has made the Cartesian connection discussed at length in Chapter 10 as she has determined not one but two points that lie on the line; but she has made what might perhaps be called an “incomplete” connection in that she has not been convinced that her two points might not exclude the one in question.

Clearly these students still have a few issues when processing algebraic tasks. Students who lack the necessary motivation to approach a problem in an analytical manner tend to rely on their observational ability. As in life, observation alone is sometimes adequate, but just as often, it lacks the accuracy or precision needed for a particular application. Algebra is a difficult hurdle for many students to begin with, and they can be resistant to applying much effort to a subject they feel is not worth their labor. Although the majority of students who undertook this task performed very well, those few who did not confirm for us that there is still room for improvement in the way in which mathematics is delivered.

8.4.3. Analysis of Question 3

This question was designed to examine how students approached the task of utilizing the coordinates of a point with the equation of a line, and for that reason, the equation was provided.

Question 3: The equation of the line in the above questions is $y = -\frac{7}{10}x$.
Find the point on this line whose x -coordinate is -11 .

This question examines whether the student understands that the equation of the line is a description of the relationship between the two variables. It can also be utilized to determine which students approach such a task with a purely visual justification and which will use an analytical approach. The answer to this question, easily discovered by replacing x with -11 in the given equation, is -7.7 . It is a simple matter to determine this value, which is likely the reason that several students did not show any supporting calculations.

Getting students to demonstrate the processes they undertake is a longstanding battle. Teachers want students to show their work so that the teacher can see what logic the student is using to arrive at an answer, correct or otherwise. Students often resist writing down anything more than absolutely necessary. There are several working theories as to why this is, such as a desire to appear not to need to work in detail, but to be “smart” enough to do all one’s work in one’s head. It is outside the scope of this dissertation to speculate on this phenomenon, but it is an area with a great deal of potential for further research.

16 students (70%) returned the correct answer, and all of these did so by replacing x with -11 in the given formula and calculating the corresponding y value. The remaining 7 failed to provide any evidence of any computation that might have been performed to achieve their answers, and all 7 returned incorrect answers. Of these, three gave either 7.6 or 7.8 as their answer; this value is sufficiently close to the correct value to speculate with at least some confidence that these students approximated their answers by observation. Two additional students gave 8.75 as the requested y value and one gave 3.5; with no supporting computation, it is impossible to know how these students came to believe that these were appropriate values.

The responses to this task add to the observation made with the previous question, that students who do not have the wherewithal to answer a question with use of analytical reasoning rely heavily on their observation of a situation.

8.4.4. Analysis of Question 4

While each of Questions 1 through 3 on the written questionnaire was designed to investigate student responses to distinct outcomes, Question 4 was designed to examine how students would approach a task that required them to use a combination of skills.

Question 4: Write the equation of the line that goes through the points $(-5, 3.5)$ and $(4.5, 9)$.

This task gave students no clues within its text as to how to approach the problem, although having just completed the previous three questions likely primed many students to a potential tactic. Its successful completion requires the inclusive

knowledge that the two given points are on one line, and that the equation of that line demonstrates a relationship between the two coordinates in a general sense. The correct response is $y = \frac{11}{19}x + \frac{243}{38}$, the form of which was familiar to students, but the components of which required an unusual level of accuracy for a class at this level.

Many students were comfortable writing the slope in fractional form, which is not surprising given their work on the question addressed in section 8.2.1.3; most, however, did not write the y -intercept in that form. Slope is traditionally taught as being computed as a ratio of two numbers (i.e. “rise over run”), and its form as a ratio is therefore the first form in which it appears when analyzing information about a line between two points. When determined this way, it does not require any manipulation in order to be put into ratio form. Once computations need to be done, however, even if the fractional form of the slope is used in performing those computations, it is highly likely that the student performing the computation will do so on a calculator. As most calculators produce their answers first in decimal form, this is the form in which the value is first seen and frequently, therefore, recorded by the student. Unless there is explicit instruction and an easy means by which the intercept value can be converted to a fraction, students are not likely to do so. For that reason, most of the responses to this question took the form $y = \frac{11}{19}x + 6.395$.

The most common initially recorded slope was $\frac{5.5}{9.5}$. As with the slope work done during the activity described in section 8.2.1.3, students found the ratio of the changes in the vertical and horizontal directions and recorded the result without simplification. Two students recorded the reciprocal of the difference quotient; two performed at least one subtraction incorrectly, another correctly determined the value $\frac{5.5}{9.5}$, but then abandoned that approach in favor of one that returned a slope of $-\frac{3}{5}$, and one did not know how to proceed. Of the students who found the slope to be $\frac{5.5}{9.5}$ or $\frac{9.5}{5.5}$, only four students left the fraction in this form. Nine students converted their slope to a decimal, and three converted it to a “nice” fraction, a simple undertaking on many programmable calculators. One of the 23 students completing the questionnaire did not supply any response to this question and one recorded an equation whose relevance to the problem at hand is unclear.

Having found the slope, whether correctly or not, all students, except for the one who did not answer the question, recognized that an acceptable form for the equation of a line was $y = mx + b$. Not all students arrived at the correct equation, but every one of them wrote their answer in the above form. One of these did all the work required to write the requested equation, and although she did not put the results of her work into an equation, she did write the general form of the equation on the page. This student seems to have been able to keep in mind all of the various skills required in writing an equation from two points, and she had recalled the form for the equation of a line. What she lacks is either the knowledge of the roles played by the various parts she found or an ability to see the importance of representing the relationship in the condensed form of an equation (or perhaps both). This student's work is shown in Figure 8.12. Her lack of organization speaks to her ability to do the various computations, but possibly not to realize that the information she has found has a place in a larger whole. Alternatively, she may have felt that she had done so much work that there was no reason to write her results out again in a different format. This last possibility would speak to a need to teach effective written communications of mathematical processes, but that topic is beyond the scope of this research.

4. Write the equation of the line that goes through the points $(-5, 3.5)$ and $(4.5, 9)$.

$$3.5 = -0.7 \cdot -5$$

$$3.5 = 3.5$$

$$9 = 4.5 \cdot .57$$

$$9 - 3.5$$

$$4.5 - -5$$

$$y = 3.5 - .57 \cdot -5$$

$$y = mx + b$$

$$3.5 = -5 \cdot .57$$

$$3.5 = -2.85 + b$$

$$+2.85 \quad +2.85$$

$$6.35 = b$$

$$3.5 - 9$$

$$-5 - 4.5 \quad -5.5$$

$$\underline{-9.5}$$

$$1 = 4.5 \cdot .57 + b$$

$$9 = 2.56 + b$$

$$-2.56 \quad -2.56$$

$$\frac{5.5}{4.5} \cdot 57$$

Figure 8.12. All the work, but no answer

Incorrect final responses took one of several forms: four students were unable to compute the value for the y -intercept or used inappropriate methods to do so; one student reversed the roles of x and y in the computation of the value for the y -intercept, one mixed the x value from one of the points with the y -coordinate from the other, and one student did all the correct work but did not write the results she found in the form of an equation (see Figure 8.12 above).

Eleven final responses were correct outright, and an additional three were correct in light of one of the errors listed above; more than half of the students in this remedial algebra class were able to demonstrate the skills necessary to complete a comprehensive task about linear functions using proficiencies accumulated over the course of about two weeks. While this statistic is not earth shattering and could certainly stand to be improved, it demonstrates that at the very least, students taught algebra using a DBTE based on RME may be able to grasp the basic concepts required by the curriculum and to assimilate them into a cumulative skill set as are students in a conventionally taught (i.e. lecture) course covering the same *Concepts*.

8.4.5. Summary of Individual Questionnaire

It would be an interesting exercise to compare the results of the individual problems on this questionnaire with the responses to the tasks in the related activities. A casual observation can easily be made between the students who made errors or incorrect assumptions in the activity phase and their poorer performance on the questionnaire. It is beyond the scope of this dissertation to include an additional robust cross-analysis, but it provides the incentive for such an analysis, and another opportunity for deeper research into student motivation and comprehension.

The high number of analytical approaches to this questionnaire is in stark contrast to the ratio of such responses I have experienced in lecture courses in general. It is impossible to know whether the students in this class using an analytic approach to problems are doing so because they are more engaged or whether it is because they are required to pre-read the text; certainly the combination of pre-reading and the DBTE *Approach* appears to be very effective. Students were, by and large, able to learn about the algebra of linear functions under this approach as they would have done under a lecture model.

8.5. Chapter Summary

The “prior knowledge” of an *Audience* of secondary students comes from limited “life experience” and of a tertiary *Audience* from greater experience. Within any class of either, a group of three or four individuals working together will find that each individual often has something different and valuable to contribute, and that the group is stronger because of its differences. It is this aspect of RME that was being exploited when students were required to complete activities in groups.

The idea in this study was to investigate the use of the DBTE in delivering content normally conveyed by lecture, with the hope of providing additional interest and greater engagement for remedial algebra students to motivate deeper learning. The results of the study are promising. Of interest was whether the level of uptake of *Concepts* by a community college *Audience* with an *Approach* that utilized a Design-Based Teaching Experiment would be equal to that of a classroom without the DBTE *Approach*. Analysis of the data indicates that students were able to integrate the required content to an acceptable degree and, anecdotally at least, to a degree in some instances much greater than under a traditional model.

The size of this study was small and, on the surface, it seems that all that has been accomplished is that I have verified what is already known about student-centered learning: that it is an effective approach for teaching mathematics, but the results actually say much more than that. They tell us that community college students in remedial algebra classes can learn the required course content under an RME model, which means that there is support for the use of the narrative and/or “realistic” scenario overlay in teaching mathematics, and for the continuing use of Student-Centered Learning in general; they tell us that the methods used in primary and secondary classrooms can also be used to teach at tertiary institutions, and specifically at community colleges; and they support the idea that testing not just student responses, but the *Approach* being used to deliver course content materials is a worthwhile exercise with valuable results. This chapter borrows from K-12 the idea that student-centered learning is a more effective model than a teacher-centered approach, but it extends the idea by acknowledging the additional contributions that can be made by students with greater maturity and acumen.

Chapter 9.

Discussion and Conclusion

9.1. Summary of this Work

In this dissertation, I have summarized the available literature on the difficulties students have in learning the algebra of linear functions and I have represented in detail the community college student population. I have outlined the perspective with which I view the design of instruction and the component parts which are critical to its success. I have reviewed the theoretical perspective known as Realistic Mathematics Education and I have described the way in which I combined the methods known as Design-Based Research and the Teaching Experiment in order to assume the study conducted for this dissertation. I have described a new method, which I named the Design-Based Teaching Experiment, and I used this method to explore how RME might be used in a community college setting. I investigated whether community college students can learn algebra in under conditions originally designed for use in primary school, by analyzing student responses to both team-based and individual tasks to establish the effectiveness of the method I was testing. The aim at the outset was to try to engage students. Things did not go *precisely* as I expected, but much of the infrastructure worked well, and students were extremely engaged.

Through the eyes of a teacher, I can say that the implementation of the narrative and the related activities was an unmitigated success. Students enjoyed the quest and the puzzle. Even if the narrative was thin in places, the added layer made coming to class interesting. I have even had students express their enjoyment of the class several terms on.

As a researcher my declaration of success is much more restrained, but I make it nonetheless. I set out to try a technique in the community college classroom that had been designed for primary school, and I was able to demonstrate not only that students learned the requisite concepts under this model, but also that the model itself was an important artifact, worth study in its own right. I wanted to create teaching materials for myself and others that would make learning algebra more interesting, but also that could

be revised and reused. Documenting the responses of the students to the activities in this study proved to be instructive in my roles as both a teacher and a researcher.

9.2. Research Questions

The questions I set out to address with this work were these:

1. How do community college remedial algebra students engage with the algebra of linear equations in two variables using the DBTE?
2. How can methods used in primary and secondary classrooms be implemented in community college classrooms to teach the algebra of linear equations in two variables?

The tools used to make these determinations needed not to be forged, but to be modified and combined from existing tools. The Teaching Experiment was designed for use with small groups and individuals, but it provided a sound foundation from which to create the necessary structure to do the current research, and Design-Based Research was employed to provide the structure necessary for analyzing whole-class data. The adaptations required to RME were minor, mostly in the realm of the language used to express the narrative. The specific “realistic” scenario could have been anything and the one that I chose would likely have worked with grade four students as well as it did with college students, as long as the language and mathematical content was appropriate for the learners. It is appropriate to add that while the modifications required to RME were minimal, the amount of time and energy required to implement an RME activity was great, but no greater than it would have been in its originally intended environment. While it is possible for students to learn the algebra of linear functions under any model if they have sufficient support with which to absorb the information, it is equally possible, perhaps with better uptake, with an RME perspective; what is more, students are more engaged and as a result the increase in their content knowledge is greater as was borne out in their responses to the written questionnaire.

The DBTE *Approach* allowed for the opportunity to register student responses for analysis not only of their learning, but also of the effectiveness of the *Approach*. The structure of the DBTE allowed for analysis of students’ responses on a class-wide scale without losing the ability to teach the content. This structure provided a framework from which to analyze and document the results of the use of RME in a new setting by looking at student comprehension on a particular task. Student responses and errors were

recorded and analyzed in much the same way as they might have been in a clinical interview or error analysis. Their responses were used not only to assess their current state of knowledge, however, but to provide an opportunity for instructor feedback and for improvements to the materials themselves.

9.3. Contributions & Limitations

One of the limitations of the use of the DBTE, as already mentioned, is that its application is time-consuming. As an approach to teaching, it requires considerable creativity and an immense commitment of time. As a research tool, it requires iteration and re-application. While it is possible to report some findings after one application, the DBTE may well be better suited to iterative or longitudinal studies. Since one of the purposes of the DBTE is the development and refinement of teaching products, an extended period of study will likely only improve those products, and alongside them, the method used for their implementation.

While much of what is included here is based on the existing literature, this dissertation attempts to strengthen the argument that the community college population is significantly different from either the K-12 or the university population, and as such deserves its own research community. Community college success and retention rates, accessibility, teacher-student ratios and research culture are all very different from those in the university environment, where university, primary and secondary students, teachers and teaching are recognized as important topics of research. University researchers do not, by and large, study community college teaching and learning; there are valuable exceptions, but there is much room for growth. Community colleges seldom have a research culture of their own and while the atmosphere is changing, there is currently little support for this change, meaning that many community college researchers must find their own way and their own resources. I hope that this document adds strength to the argument that community college students, faculty and teaching methods deserve study and that the research-practitioners at these institutions may be in the best position to conduct the necessary research. If I am successful in making that argument, I sincerely hope that I can add to the case for a modicum of both professional support, and some kind of recognition for the time it takes to perform research if it is done in such a way as to be useful.

This dissertation also aims to contribute to the literature supporting Student-Centered Learning in general by strengthening the argument that SCL techniques, including RME, have the potential to be successful methods to increase learning, but also by adding to the literature that demonstrates that techniques designed for use in one setting can be used – with or without adaptation, depending on the particular circumstances – in other settings quite successfully. While RME was initially designed for use in the primary classroom, it has been successfully applied in the secondary classroom, and extended in this study for use at the tertiary level. I have demonstrated here that it can make not just an adequate transfer to the community college but an excellent one.

Student success rates are an important talking point among community college administrators. The more tools instructors have for increasing those rates, the higher the rates are likely to be. Perhaps more importantly for this dissertation, I have shown that Vilma Mesa’s 2017 challenge – that “Researchers need to be informed by findings from K-12 and university settings, but they need to be conscious that there will be more than likely be a need to reinterpret and redefine constructs to fit the community college context” (p. 962) – can be taken up, and that the special population that is community college remedial algebra students can be successful.

9.4. Personal Growth

There is likely little that needs to be said about the growth that must inevitably come from undertaking the creation of a document such as a doctoral dissertation. The work is demanding, time-consuming, rigorous and thorough; it is also highly rewarding and immensely satisfying to dig so deeply into one idea and to emerge from the process having made even some small difference.

The research and writing part of the process comes at the end of a coursework sequence that in and of itself has made me a better teacher. When I began the PhD program in Mathematics Education at Simon Fraser University, I felt I was already a good teacher – curious and willing to experiment in order to find better ways to teach – but even before the end of my first term as a PhD student, I found myself questioning much more deeply the way in which I processed information and how that impacted the way I passed that information on to others; I became aware at a conscious level of

things that teachers do – myself included – that can affect the way students gain knowledge. I learned to observe not just my students for the things that worked and did not work for them, but myself for the things I did that helped and hindered others' learning.

Doctoral work is not only about becoming a better practitioner however; it is about becoming a researcher, a tradition I had never before undertaken. I received my master's degree in Mathematics in 2002 on the back of a "Project-Based" capstone. The only "research" I undertook then was to learn a little about the original author of the theorem that I would prove before my committee by way of demonstrating that I could perform graduate-level mathematics – not graduate-level research. I have no regrets from that time; my M.S. in mathematics was rewarding in itself and provided me with everything I needed at the time, namely the credentials I needed to teach undergraduate mathematics at the community college level. Having begun a PhD program in Education, however, I quickly learned that my research skills would need to be developed. The program faculty made this process almost seamless, and although I do not consider myself a "seasoned" researcher, I do now have the tools with which to continue researching the ideas that are important to me and that will contribute to the community college research-practitioners' tradition, as young as it is.

For me, this work comes in the middle of an already satisfying career, but it has provided me with a number of valuable tools and lessons. It has reminded me what it is like to work full-time while "going to school", the work-school balancing act being something that many of my own students must navigate. Experiencing that challenge again myself has been a humbling reminder of the demands on one's time and resources. It has reminded me of the power of persistence, a "lesson" I try to convey to my students. It has reinvigorated my love of learning and sparked an embryonic interest in research. It has humbled me to be able to work with some dedicated and passionate faculty. It has empowered me to be told often and in various ways and by many people that what I was doing was "impressive" or "important" or "valuable". It has added to my self-confidence, which I know will also benefit others, as I am aware that teachers are role models for their students, whether they choose to be or not.

In the process of writing this document I have learned a great deal about myself, about my professional community and about my interests, strengths and abilities. I have

also learned about research. Perhaps the biggest lesson I have learned in that domain is that it is never complete. Although a time will inevitably come when I have dotted the last “i” and crossed the last “t” in this disquisition, it would be foolish to think this product could not have been improved. As with anything worth doing, there is always room for improvement. If my work on the study for this discourse is to have been for a purpose, it will have been to shed a bit of light on one small, dark corner of mathematics education, and hopefully to provide a new approach to teaching and researching in that domain. As the materials that I used in the study I conducted might always be improved from one iteration to the next, so might this document, given more time with which to accomplish that task.

Likewise, the method I have developed from my position on the legendary shoulders of giants will also, always, be ready for improvement. I hope in the next year or two to revisit the tasks again that I delivered for the study reported here, but I also hope to revisit the DBTE, explore its strengths and investigate which of the features of Design-Based Research and the Teaching Experiment I left out might be reincorporated. I would also like to examine the method itself more closely and see what more it has to offer to the mathematics education research community.

What might come next for me is still unknown, although “research-practitioner” is a more and more attractive title to me every day. I will certainly continue to teach community college mathematics. I am very fortunate to have found not only my professional niche, but the ideal college in which to practice my discipline. The culmination of a PhD is, in one way, the conclusion of a life-long dream, but in another, it is just the beginning.

9.5. Implications and Recommendations

Although only the first iteration of the application of DBTE is reported here, I have now executed the particular use of the DBTE using RME as applied in the study reported here multiple times; each time I have made modifications that make the *Concepts* more valuable as tools to the students, and the materials more valuable to the instructor as a way to gauge student comprehension. In addition to finding not only that the *Approach* described here more actively engages students than does lecture, it also appears to hold up when the same students are presented with questions about linear functions in a

more conventional manner. The study revealed multiple areas for improvement to, and suggestions for iterations in the tasks. Many of these improvements were incorporated into later sessions (future classes), but given the nature of the delivery approach, it was not possible to perform iterations with the same group of students as would have been expected in a traditional Teaching Experiment. That said, the documentation of strengths and weakness of, provide a structure for both teaching and research that does not generally exist in the teacher's vernacular. It might be described as a kind of one-teacher Lesson Study (Stigler and Hiebert, 2009) in which the teacher examines her *Approach* to teaching a *Concept* to a particular *Audience* and fine-tunes it for the next class session or the next time the course is taught. Like the Lesson Study, Design-Based Research and the Teaching Experiment both include a collaborative aspect that was not utilized in this application of the Design-Based Teaching Experiment. Perhaps the DBTE is ripe for an iteration of its own.

Student-centered learning (SCL) is not a new construct; it has been around since at least 1951 (Rachman, 1987), although it can be argued that it became a “modern” phenomenon in the early 1970s (e.g. Foster, 1970; Clasen & Bowman, 1974). That SCL is more effective than lecture alone has become difficult to dispute, given the ongoing and increasing research into its benefits (e.g. Wilson, Sztajn, Edgington & Myers, 2015; Dondlinger, McLeod & Vasinda, 2016; Osmanoglu & Dincer, 2018). Although the community college specifically, and tertiary education in general, is now enjoying some attention from Mathematics Education researchers, there is a great deal of room for more, and for specifics, such as whether RME, designed for use in elementary classrooms, can be effective in other environments.

The technique described in this dissertation, designed for mathematics education might be just as useful in a Physics or English classroom, or perhaps it might be of some use in a secondary or university setting. Because I am fortunate enough to teach at a community college where the faculty across and between disciplines view one another as allies and collaborating colleagues, and where unusually, faculty research-practitioners are not rare and isolated creatures, there are multiple opportunities for the DBTE to be trialed in other disciplines, and perhaps, as mentioned above, with the collaborative component restored. I developed the DBTE for mathematics research-practitioners however, and it is my fervent hope that it is of some use to that community.

Having concluded the study, it is tempting to claim that the research is also finished, but the very nature of the method selected for this study is that it is iterative. It is not enough to claim that a chapter has ended and close the book. In order for this work to have been meaningful it must be continued. I expect to continue much of that work, but I sincerely hope that others find this contribution useful and can add to its continued growth.

9.6. Concluding Statement

This dissertation demonstrates that RME is a valid perspective through which to view the remedial community college classroom, and that flexible teaching methods are valuable to both teachers and researchers. It also highlights the known issue of preparation and modification: given the already overwhelming demands on their time, many post-secondary educators may be understandably unwilling to put in the considerable effort required to create, deliver, analyze and modify tasks and activities such as those described in this report even once, let alone multiple times. Although this is not the place for a discussion about workload issues, that question is one that will likely need to be addressed if academic professional development is to be sufficiently robust as to be meaningful.

If students are to benefit, it is critical that more college and university instructors become informed of the benefits of SCL. Certainly as more instructors use such methods, their students who themselves become educators are more likely to employ alternatives to lecture, but the field seems to be resistant to this type of challenge to the status quo. It is hoped that this dissertation helps to convince the reader that SCL is an approach that benefits students, but also that it need not mean reinventing the wheel.

While the leap had already been made supporting the use of RME in various academic settings, very little had been reported about its use in the community college setting specifically, and the current research contributes to that body of knowledge by addressing RME in the community college environment explicitly and primarily. This is not to claim that the modality described here guarantees that all students will learn better and remember longer, nor that there are no students who learn better by listening passively to a lecture, but many students who end up in pre-college algebra at community college have already been let down and turned off by the traditional

academic model. For these students, delivering the same content the same way and expecting different results seems at the very least unproductive. If students have any chance of progressing, something needs to change in order for them to learn algebra in *this* course when they could not learn it before; since it cannot be the curriculum, or the time of day or the number of weeks in the term that changes, it will have to be the means by which the curriculum is delivered. Multiple reasonable alternatives exist, and others are always being investigated, but Design-Based Research and the Teaching Experiment, with their focus on teaching, iteration and improvement, together with the “realistic” perspective offered by Realistic Mathematics Education is a combination which, it can now be said, has been tried and found to be a strong candidate for positive change in the community college remedial algebra setting.

It is often said that the only people who read a dissertation are the aspirant and her committee. I know that statement is not completely true as I have cited one such document in the creation of this one and referenced several others. I am not sufficiently naïve to believe that this document will ever be seen as ground-breaking, but I do sincerely hope that it is of use to others, and not only to me. If a student researching Student Centered Learning, Realistic Mathematics Education or Community Colleges runs across this document, I hope it is useful and informative, and that it may be used to further improve some aspect of community college or mathematics education, or maybe both. More importantly still, I hope it can be used to improve upon the technique I have developed, because nothing worth doing is ever finished.

Chapter 10.

On College Students, Linear Equations and the Cartesian Connection

This chapter was originally compiled as a paper early in my research into community college remedial algebra students. I was investigating a particular phenomenon that I had noticed in the classroom and wanted to explore more deeply. The analysis of the data collected for this paper revealed a surprising split between students who approach a problem using analytical tools and those who rely exclusively on what they can observe.

The reason for including this study as part of this dissertation is that it provides a perspective for a deeper level of analysis than was possible with the study conducted specifically for this dissertation. The paper reports in detail the method by which I analyzed student responses to the question on a quiz from which I first noticed the phenomenon and describes it as a type of code-switching. Students with the ability to use an analytical approach have, at their disposal, tools by which to solve a problem as well as an informal way of verifying their results; students who do not have such an ability must rely solely on observation, which may result in an inaccurate conclusion. This realization has colored my thinking about learning mathematics ever since, and as such supplies an underlying perspective to this dissertation.

The content of this chapter is a paper co-authored with Rina Zazkis and submitted to a refereed journal. As a stand-alone paper, it contains some of the content referred to elsewhere in this dissertation, which means that there is some redundancy with the previous chapters. It was written approximately a year before I began writing this dissertation.

10.1. Introduction: Background and Brief Overview of Related Literature

It is critical that students understand connections between algebraic and graphical representations of functions. Certainly, if they do not understand the component parts, there is nothing for them to connect, but it is the connecting of concepts that signals competence in the discipline of algebra (Moschkovich, 1996). In

this study we focus on student work on a Task designed to examine one of these connections: in particular, the connection between the coordinates of a point and its being located on a specific line. This is what Moschkovich, Schoenfeld & Arcavi (1993) refer to as “the Cartesian Connection”: “A point is on the graph of the line L if and only if its coordinates satisfy the equation of L ” (p.73).

Moon, Brenner, Jacob, & Okamoto (2013) expanded the notion of Cartesian Connection from focusing on a line to any general Cartesian representations of symbolic equations: “A point is on the graph of the mathematical relation $R(x, y) = 0$ if and only if its coordinates satisfy $R(x, y) = 0$.” While the Moschkovich et al. (1993) focus on linear functions is sufficient for our study, we follow Moon et al (2013) in restricting the use of the term “representation” to the standard graphical and symbolic representations used in Algebra.

10.2. On Equations and Graphs

There is no shortage of research indicating that students experience difficulties learning the various components of the algebra of linear functions (e.g., Leinhardt, Zaslavsky, & Stein, 1990), and that they often do not adequately comprehend the meanings of the individual components (e.g., Hattikudur, S., Prather, R., Asquith, P., Alibali, M., Knuth, E., & Nathan, 2012). Difficulties described in the literature are many and diverse, but they can, for the most part, be broadly sorted into a few categories, including difficulties involving graphing linear equations (e.g., Davis, 2007; Hattikudur et al., 2012), those in which notation is problematic or in which manipulation of expressions is inappropriate (e.g., Nogueira de Lima and Tall, 2008; Knuth, Alibali, McNeil, Weinberg and Stephens, 2005), and those which involve making a connection between algebraic and graphical representations (e.g., Davis, 2007; Knuth, 2000a, 2000b).

In particular, students have limited understanding of connections between equations and graphs (e.g., Knuth, 2000b). In exploring students’ connections between graphical and algebraic representation of functions, Knuth (2000a) noted students’ overwhelming reliance on algebraic methods, even in tasks where the visual or graphical approach appeared more appropriate. Exploring the reasons for inadequate and often absent connections, Knuth noted students’ use of and overreliance on familiar routine approaches in cases where a graphical approach led to an immediate solution.

Moschkovich (1996) stated that “Conceptual understanding in this domain [linear functions] involves more than using procedures to manipulate equations or graph lines; it involves understanding the connections between the two representations” (p. 242). In other words, success in learning the algebra of linear functions requires understanding the connections between the variables and between the representations, not just an ability to manipulate them.

While Knuth (2000a, 2000b) and Moschkovich (1996) focused on secondary school students, Moon et al. (2013) investigated the (extended) Cartesian Connection among prospective secondary school teachers and reported participants’ difficulties with the Cartesian Connection in the context of conic curves. These researchers highlighted the importance of the Cartesian Connection as a “big idea” and noted that “these seemingly easy ideas appear in more complicated forms when learners attempt to make connections among representations in complex mathematical contexts” (ibid., p. 222). We contribute to the investigation of the Cartesian Connection by focusing on college students.

10.3. On Community College Mathematics Education

There is an increasing interest in community college mathematics education, based on particular characteristics of students (e.g., maturity, family and job status, returning to education) (Mesa, 2017). In particular, many students entering community colleges in the U.S. are often unprepared for college-level mathematics courses (between 50% and 67%, according to various surveys cited in Mesa, 2017) and are required to enroll in courses that cover content typically taught in middle and high school mathematics. Various terms in the vernacular describing the curriculum in these courses include “remedial”, “developmental” and “pre-college” algebra. Our research is situated in one such course.

Mesa (2017) noted that researchers working in the community college setting “need to be informed by findings for K-12 and university settings, but they need to be conscious that there will more than likely be a need to reinterpret and redefine constructs to fit the community college context” (p. 962). In her thorough review of research conducted in community colleges Mesa noted several features (p. 962):

1. it assumes a deficit view on the various objects of investigation: students, instructors, and curriculum;
2. it devotes only superficial attention to mathematics and specific aspects of learning the content;
3. it uses scholarship from K-12 and university settings as the guide for assessing work in the community college context;
4. it does not take up questions that matter to practitioners; and
5. it lacks theoretical support

Our research contributes to addressing deficiencies identified in 2, 4 and 5. Specifically, we focus on a particular mathematics topic, that of linear functions and their graphs; we address it by using and refining a theoretical lens and draw conclusions relevant for the practice of teaching a remedial mathematics curriculum.

10.4. Research Questions

Our goal was to examine whether students attended to the Cartesian Connection when responding to a carefully designed task. In particular, we were interested in the way in which they would use given information (two given points and the illustration of the graph of the line containing those points) with the idea that another point is located on the same line. The following research questions guided our analysis: What approaches do students use (in the Task) to determine the coordinates of a point on a line? In particular, do they apply a visual/graphical or an analytical/algebraic approach in order to determine the solution? What characterizes each approach?

10.5. Theoretical Perspective

In analysing how students think during tasks in linear algebra, Sierpinska (2000) made the distinction between what she calls “modes of thinking [...] synthetic-geometric, analytic-arithmetic and analytic-structural” (p. 233). Synthetic-geometric was described as “practical”, while analytic-arithmetic and analytic-structural were identified as different modes of “theoretical” thinking.

Sierpinska attributed her distinction between “practical” and “theoretical” thinking to the work of Vygotsky; specifically, she has adapted Vygotsky’s “distinction between

spontaneous and scientific concepts” (p. 211) to mathematics education. She outlined thoroughly the transition of the model from Vygotsky’s general application to learning and development, to its use in mathematics education and to its application in a particular college-level course.

Relating the identified modes of thinking to the study of lines and equations
Sierpinska noted:

[I]n the synthetic mode, a straight line is seen as a pre-given object of a certain shape. One can speak of the properties of the straight line, but these properties will only describe the line, they will not define it. In the analytic mode, the straight line is defined as a certain [sic] specific relationship between the coordinates of points. (Sierpinska, 2000, p. 233).

In another vein, Moschkovich (1996) claimed that:

Linear functions is a complex domain where the development of connected pieces of conceptual knowledge is essential for competence.... Conceptual understanding in this domain involves more than using procedures to manipulate equations or graph lines; it involves understanding the *connections* between the two representations. (Moschkovich, 1996, p. 242; emphasis added)

The Cartesian Connection is the particular connection between the two representations that is of our interest in this study.

Note that while both Sierpinska and Moschkovich discuss relationships or connections as related to functions/equations and graphs/lines, Sierpinska attends to modes of thinking (synthetic and analytic) while Moschkovich attends to representations (graphical and algebraic). When students engage in a task, they consider an external representation and apply a mode of thinking. However, what is visible for a researcher in students’ solutions is the approaches that were applied. As such, when a student attends mainly to the graph and derives the information from observing the graph, we consider the approach as “visual” or “graphical”, likely resulting from the synthetic mode of thinking. When a student involves in the solution algebraic/symbolic manipulation, we consider the approach as “analytical” or “algebraic”, which may or may not be connected to Sierpinska’s “analytic” mode of thinking.

10.6. Methods

10.6.1. Participants and setting

The study was carried out with 65 community college students from 2 different classes of 34 and 31 students respectively, of whom 32 were female, 32 male and 1 other, representing reasonably diverse populations in terms of age and ethnicity. At the time of data collection, the students were enrolled in an introductory algebra course, part of which was devoted to exploration of linear equations and their related graphs. In this remedial course, students are assumed to have little or no knowledge of the formal algebraic or graphical structure of a line. Some students will have entered the course with some previous exposure to the idea of linear structures; however, the curriculum calls for a rigorous treatment of the subject, and as such, during the unit on linear functions, students learned (or re-learned) the concepts of Cartesian coordinates, slope, intercept, points on a line, graphical interpretation of linear equations, the writing and interpretation of linear equations in the forms $y = mx + b$ (slope–intercept form) and $y - y_1 = m(x - x_1)$ (point-slope form), and solutions to systems of linear equations by graphing. Both sections of the course from which students' work was considered were taught using a Student-Centered Learning approach known as the Teaching Experiment under the theoretical lens of Realistic Mathematics Education. In this setting, the problems to be solved motivate the mathematics to be learned, and students are interviewed in order to discover their prior knowledge but also their assimilation of the content of the course.

The observation that led to the investigation outlined in this chapter was that some students appeared to be using a given graph as the only necessary source of information about points on a line. If the line did not pass precisely through the “crosshairs” of a grid, these students appeared to be assuming that any perceived difference between proposed and actual coordinates was coincidental and therefore insignificant, and they did not consider algebraic means available to them by which they might confirm the information derived from the graph. This classroom observation served as a rationale for the design of the Task analysed in this report.

10.6.2. The Task

The Task, presented in Figure 10.1, was designed to assess students' understanding of the "Cartesian Connection", that is, what it means for a point to be on a line. Students were provided with the graph of a line; two points were marked on the graph and their coordinates were indicated in the text as $(0, -2)$ and $(13, 3)$. Given the x -value of a third point ($x = 8$), they were asked to find the corresponding y -value, that is, to find the value of y in the ordered pair $(8, y)$.

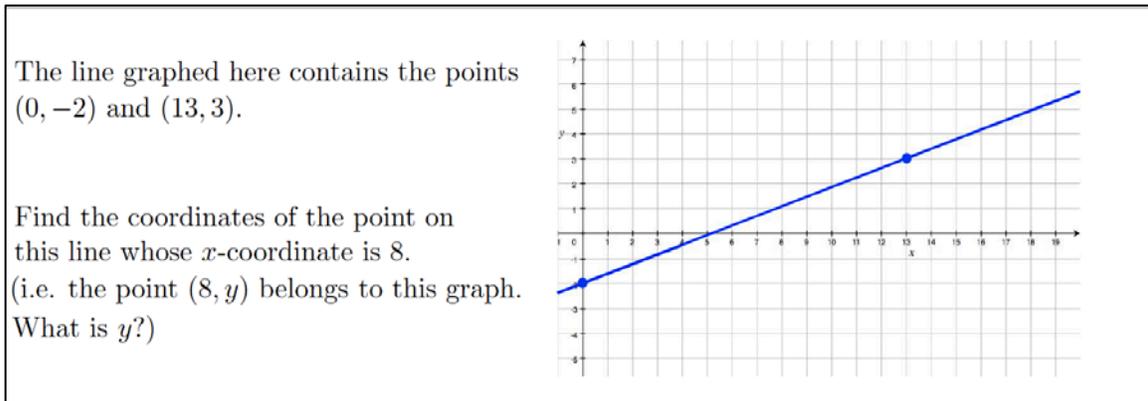


Figure 10.1. The Task

The slope of $\frac{5}{13}$ for the given line was purposely selected based on the ratios of Fibonacci numbers. That is, the ratios of consecutive Fibonacci numbers approach a limit; hence they appear to be close to each other even in the beginning of the Fibonacci sequence. As a result, when explored visually, the slope of $\frac{5}{13}$ looks similar to $\frac{2}{5}$ or $\frac{3}{8}$ and the y -coordinate corresponding to $x = 5$ or $x = 8$ appears very close to an integer value. As such, the information detected visually from the graph, while ignoring the provided coordinates of the two given points on the line, might be misleading.

The Task design was inspired by "area puzzles", such as the one shown in Figure 10.2.

In the first round of analysis we distinguished between responses of students who relied solely on observation of the graph (22 responses, visual approach) and responses that included some relevant algebra (41 responses, analytical/algebraic approach); however, simply partitioning responses into these two groups said little about the differences between or within the two groups. In an effort to examine the differences between the ways in which students used the information they had been given, the responses in were scrutinized more closely and the steps taken by students were noted and catalogued. Specifically, we assessed whether each student had found the slope of the line between the two given points, whether they had used that slope to build an equation for the line, and whether they had used their equation to compute the y -coordinate for the given x -coordinate. Our focus was on the chosen approach, rather than on the correctness of the approach; if a student determined an incorrect slope but then used that slope in the creation of an equation, that response was credited with being analytical.

10.8. The Visual Approach

In total, 22 of the 63 students responded using only a visual approach and based their answers on observing the provided graph. For these students, no visible attempts to manipulate the given information was present on their submissions.

The graph, if inspected casually, appears to include the point with coordinates $(8,1)$, although the actual coordinates of the point in question are $(8, \frac{14}{13})$. Based on this appearance, 15 students indicated this point as their response to the Task with no further explanation (although one student wrote “refer to the graph” as part of his response). Figure 10.3 exemplifies such responses.

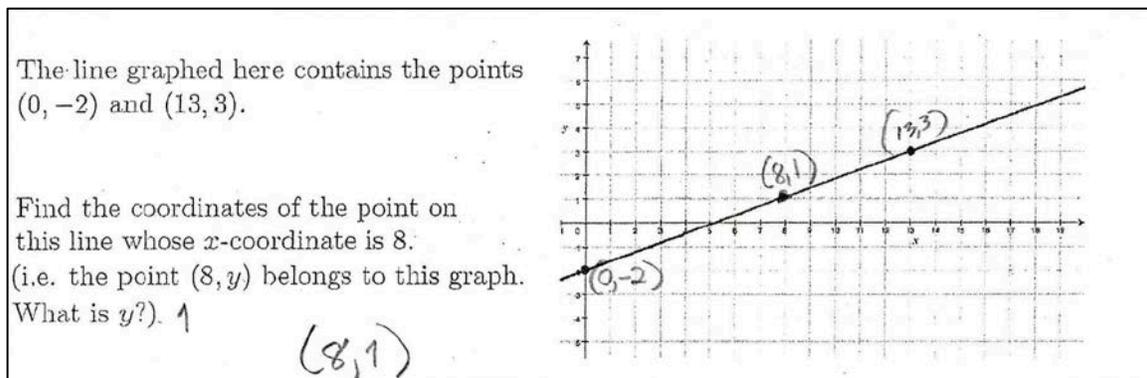


Figure 10.3. Robin’s response to the Task

Note that Robin¹⁶ also added the point $(8, 1)$ to the graph.

The remaining 7 of the 22 “observational” responses noted the y -coordinate as being not exactly 1, but “close to 1”; these 7 students, however, provided little or no additional explanation. The following responses exemplify a “close to 1” determination:

- “ $(8, 1.5)$ not exactly at 1 a little over one”
- “ y would have to be around 1.2 because the line doesn’t go through the exact point “
- “1 or 1.2ish”

These students recognized that $(8, 1)$ was inaccurate, but they did not employ an algebraic approach to determine the exact value.

10.9. The Analytical/Algebraic approach

The 41 responses in which students used, or attempted to use, algebraic means to answer any part of the question were also examined further. Note that we labeled as “analytical/algebraic” responses that included some reference to concepts related to linear functions, such as “slope” or “intercept”, rather than being free from visual examination of the graph. This means that students conducted or attempted some analysis of the Task, regardless of the ways in which they provided the final answer.

The identifying feature in these 41 responses was that the students determined the slope of the line; however, of the 41 who determined the slope, only 36 proceeded to

¹⁶ All student names are pseudonyms

determine the equation of the line (that is, 5 did not use the slope they found in an equation), and only 29 of the 36 responses utilized the Cartesian Connection (that is, 7 did not use their analytical results to determine the desired y -coordinate). In what follows we attend to the 29 who provided a slope and an equation, and then used these to answer the question; then we provide further detail for the remaining 12 (the above mentioned 5 and 7) responses.

10.9.1. On the use of equation and the Cartesian Connection.

To reiterate, only 29 of the 41 students, having determined the slope of the line, proceeded with writing down the equation and then using this equation in finding the y -coordinate. Figure 10.4 exemplifies this approach.

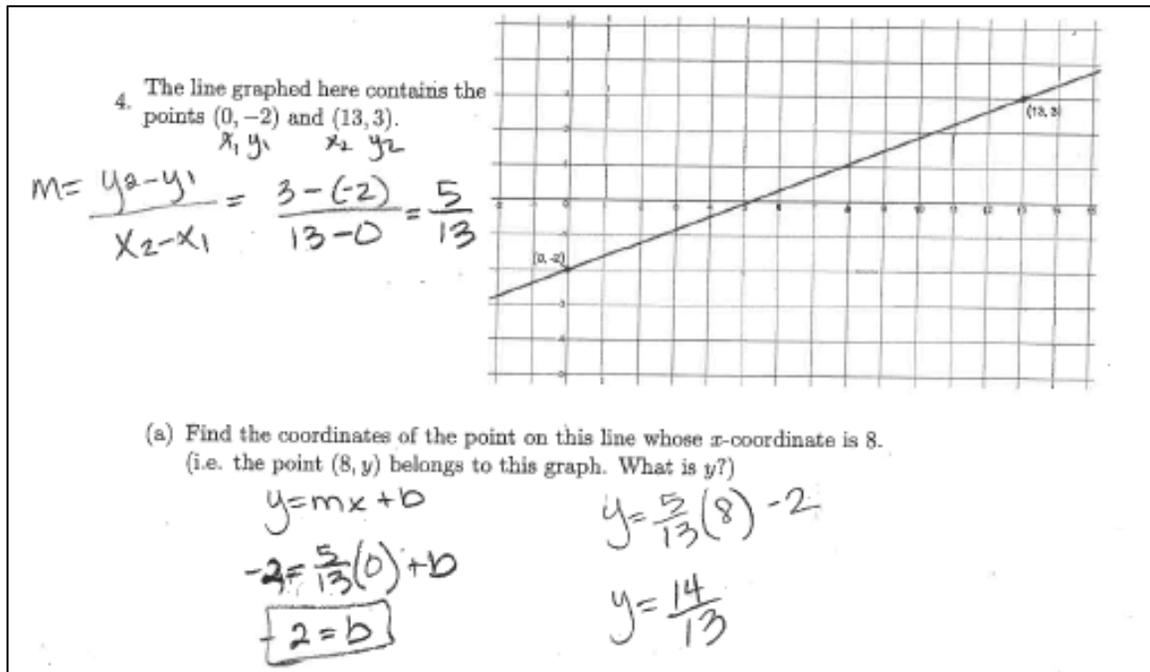


Figure 10.4. Applying the Cartesian Connection

Determining the coordinates of the point on the line from the equation of the line clearly points to the understanding of the Cartesian Connection, even though this term was not used in the instruction. However, determining the equation appeared insufficient to utilize the Cartesian Connection, as we discuss below.

As mentioned above, 36 students determined the equation of the line, but only 29 continued to use it in the solution. The 7 students who wrote an equation but did not

ultimately use it did not make the Cartesian Connection; they were able to perform mechanical algebraic computations but not to incorporate what they found towards the solution of the Task. As such, they ultimately approached the Task by visual inspection of the graph. Angela's and Jake's responses exemplify this situation in Figure 10.5 and Figure 10.6, respectively.

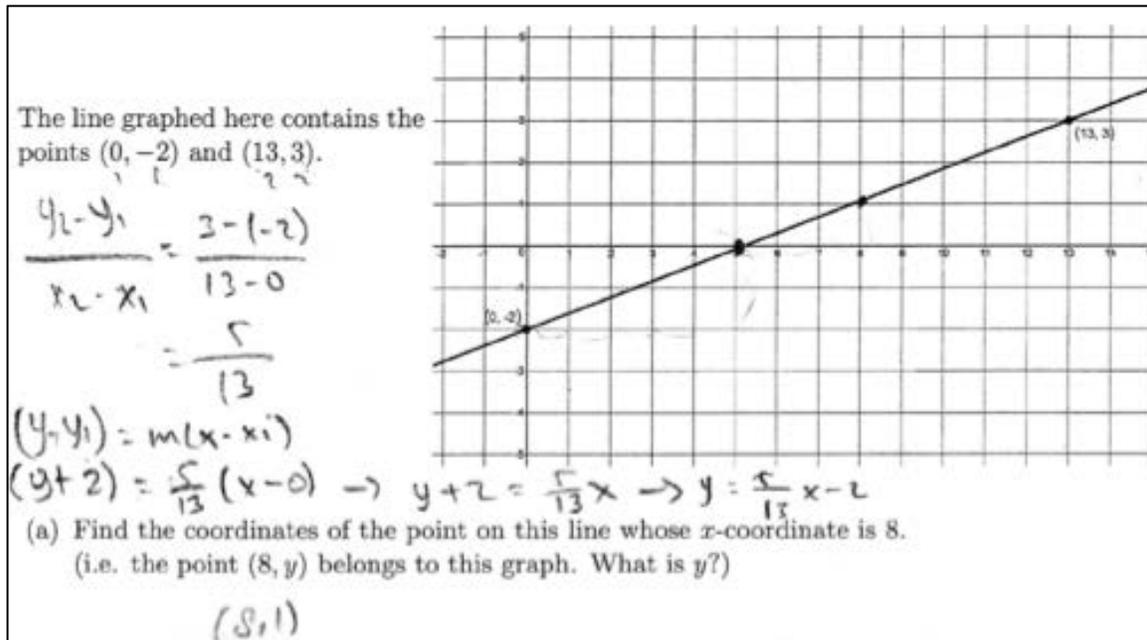


Figure 10.5. Angela's solution

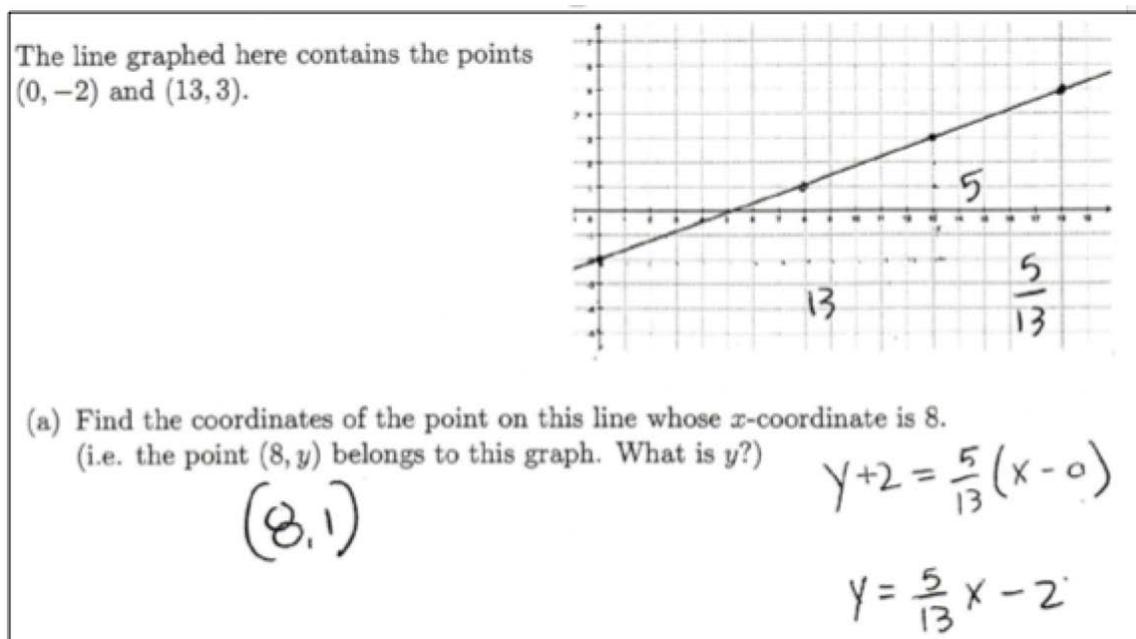


Figure 10.6. Jake's solution

Note the particular detail that Angela provided: she recorded the formula for the slope, substituted the given coordinates into the equation of the line and used this form to determine the value of “ b ” (the y -intercept), and then recorded the equation in the form $y = mx + b$ form. Nevertheless, $(8,1)$ was submitted as the solution. In comparison, while Jake’s solution provides the same equation and the same answer, it does not rely on a formula to find the slope of the line. Jake’s solution indicates visible traces that the slope was determined by counting the differences in the coordinates of the points on the provided graph, and importantly, by bypassing a perceived point with x -coordinate 8. We elaborate on the use of these two methods of finding the slope in the next section.

10.9.2. On determining the slope

Students’ ability to determine the slope was not necessarily translated into their determination of the equation of the line. As mentioned, of the 41 students who determined the slope of the line, 5 did not use it in an equation. Having determined the slope, they resorted to determining the sought coordinate from observation of the graph. These students seemed to be aware that more than observation was required of them, but they were missing critical skills that might otherwise allow them to complete the Task differently.

We initially thought that the method a student used to find the slope of a line – using a formula or counting the differences in coordinates from the graph – might be a good indicator of their inclination towards either a visual or analytical approach. However this assumption was abandoned as on one hand 5 students demonstrated their use of the Cartesian Connection having determined the slope by counting, and on the other hand, of the 5 students who did not use the slope they found to develop an equation for the line, 3 found the slope by using the formula.

The students’ approach to finding the slope can, however, still provide some insight. Of the 41 students who determined a slope for a line, 33 gave clear evidence of use of the slope formula, 5 left visible traces of “counting” on their papers and 3 others indicated the correct slope, but left no indication as to how they found it; it was assumed that these last 3 had found the slope by counting without leaving a trace. Of note is that among the 29 students who relied on the Cartesian Connection in their solutions, 24

determined the slope by formula and 5 by counting; that is, the algebraic approach for finding the slope was not a predictor of how the rest of the solution was developed.

The caveat is that determining the slope by counting and then proceeding to use it in an equation, while procedurally correct, resulted in some complications. For example, two students determined the slope to be $\frac{2}{5}$. This is the “trap” caused by the use of Fibonacci numbers in the Task and counting the slope between the given point $(0, -2)$ and an incorrectly visually determined point $(5,0)$.

10.9.3. Reconsidering the Cartesian Connection

Moschkovich et al (1993) formed the Cartesian Connection as an “if and only if” statement: “A point is on the graph of the line L if and only if its coordinates satisfy the equation of L ”. In a previous section we claimed that students who found the coordinates of the point by using the equation of the line had implemented the Cartesian Connection. We refine this claim now and suggest that these students relied on the “if” part of the statement, and that we have no evidence of their use of the “only if” part; that is to say, a complete implementation of the Cartesian Connection is evidenced when, having calculated the sought coordinates the solution is checked against the graph to determine if it “makes sense”: if the found point belongs to the graph by a reasonable estimation.

For example, Chris reversed roles of the x - and y -coordinates when replacing them with their values in the equation he wrote, which resulted in his final answer “ $y = \frac{50}{13}$ ”. It is clear that he did not proceed to check this answer against the graph. As such, he implemented only one direction of the Cartesian Connection. Lauren used the (perceived) x -intercept $(5,0)$ as y -intercept, then otherwise correctly finished the Task based on this error but had no confidence in her final answer. Having written an equation of the line (using 5 for the y -intercept), she correctly evaluated this equation for $x = 8$ and found the y -value to be $\frac{105}{13}$. However, she wrote “im stuck” (sic) in the answer space (see Figure 10.7).

VISUAL: responses relied only on inspection of the graph	22	replied with "1"		17		34 No evidence of Cartesian Connection
		replied with "close to 1"		5		
ALGEBRAIC/ANALYTIC: responses included "algebraic" parts; determined slope	41	did not proceed to determine equation, relied on visual inspection		5		
		determined equation	36	did not use equation	7	
				used equation	29	29 Evidence of Cartesian Connection
TOTAL	63				63	

Table 10.1. Evidence of Cartesian Connection (Summary)

In summary, analysis revealed a clear distinction between students who determined and used an equation of the line passing through the two given points and those who did not.

- 29 students wrote and used an equation in responding to the Task (see example in Figure 10.4). These responses demonstrated evidence of relying on the Cartesian Connection, at least in one direction;
- 7 responses included an equation but did not use it in the solution and these students ultimately relied on observation (see examples in Figure 10.5 and Figure 10.6);
- 5 responses included the slope of the line, but did not proceed towards the equation;
- 22 responses relied solely on inspection of the graph (see example in Figure 10.3).

10.11. Results and Analysis: Interviews

Individual interviews with 3 students – Janice, Edna and Gina – were conducted at the end of the course. At the interview they were presented with the Task and asked to talk about how they would approach the problem. These students were selected to represent different responses to the Task written prior to the interview. In particular, Janice utilized the analytical/algebraic approach, while Edna and Gina appeared to derive information visually by considering the graph. None of the interviewees provided

any indication during the interview that they recognized the Task from their prior work. In what follows we summarise each interview and then provide an analysis of the three cases.

10.11.1. Interview with Janice

In Janice's work on the written Task she found the slope of the line algebraically using the two given points, then composed the equation of the line and substituted 8 for x in this equation to find $y = \frac{14}{13}$. While her general approach during the interview mirrored her written work, it provided a further insight into her understanding of the Cartesian Connection.

As the first step in approaching the Task in interview, Janice confirmed that the given points $(0, -2)$ and $(13, 3)$ were indeed on the line. She clearly coordinated the visual information provided by the graph and the Cartesian representation of points as ordered pairs.

Janice: First I'm going to look at the points that it says it contains and see if they're actually there.

Interviewer: OK

Janice: So, that's zero, negative two and then thirteen, three. So those are on the line.

She then proceeded to find the slope.

Janice: So first I'm going to find the slope and I could do it by counting over there [points at graph] ... But it that's not accurate sometimes, so we're going to do y -two minus y -one over x -two minus x -one, and so that's going to be five over thirteen. There we go. I must have counted that wrong. [...] OK so, [counts on graph] one, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve thirteen, so that's the slope, five over thirteen, so then we're going to... We have a, we have a ... the y -intercept is going to be negative two, so y equals five thirteen x minus two.

Janice starts to find the slope by counting, but then she appears not to trust the counting strategy, as "that's not accurate sometimes". Instead, she uses the formula to find the slope. It is interesting to note that while Janice voiced the formula – y -two minus y -one over x -two minus x -one – she did not write down the formula, but substituted the

relevant numbers as she spoke, and computed $\frac{5}{13}$. Noting the discord between her initial count and the information obtained by using the formula, she counted again and confirmed the numerical result. She then confirmed the y -intercept, first by pointing to the ordered pair $(0, -2)$ and then by checking the graph, and wrote down the equation of the line, $y = \frac{5}{13}x - 2$. The interviewer prompted Janice to explain her thinking.

Interviewer: Ok, so before you go on, I'm just curious about something maybe if you could sort of vocalize what you were thinking.

Janice: Sure.

Interviewer: Why did you go to find the slope, when I'm asking you for the coordinates of a point?

Janice: Well, because I wanted to get the... the my brain, the way my brain works is I want to get the, the um, equation so I can test the points and fill it in for that spot.

Janice indicates the need to find the equation in order to “test the points and fill it in for that spot”. Referring to “that spot” Janice pointed to y in $(8, y)$. We note in Janice’s description here a clear indication of her awareness and use of the Cartesian Connection. We discuss this further in the next section. She then completed the Task, substituting 8 for x in her equation and computing $y = \frac{14}{13}$ and explained further:

Janice: That's why I started to get the equation. Because after I have the equation then I can, if I have a number, I can plug it in here and figure out what the other one should be.

Looking for the strength of belief (Ginsburg, 1981) the interviewer presented Janice with the visual approach:

Interviewer: What often happens is a student will say “Hmmm, there's 8, hmmm, maybe one”. And they'll just observe that one looks about right. So they guess one. Well, fourteen thirteenths is actually very close to one.

Janice: But it's not one. If you're buying something that's a pound, instead of being fourteen thirteenths, then you just got ripped off.

Janice objected to the imprecise potential solution suggested by the interviewer and provided a realistic reference where the need for an exact measure is desirable.

10.11.2. Interview with Edna

Edna indicated on her written work that the y coordinate was 1. In the beginning of the interview Edna focused on calculating the slope, frequently referring to her vague memory. It appeared that she initially focused on the first line of the Task (“the line graphed here contains the points $(0, -2)$ and $(13, 3)$ ”) and this triggered for her the need for a calculation.

Edna: I'm trying to remember how we did it. [...] Oh, wait, I might remember how to do this. It's where you put like 3 minus 13? over... No, wait. I'm going to kind of mark which one's x ... and then which one's y .

Interviewer: OK

Edna: It'll be easier to see when it's laid out like that. 2 y , ok. Um, x ok, then, but, negative two minus three, zero minus 13. That would be 1 over, and then, 13 minus... [long pause]

Interviewer: So, what is... the piece that you're doing right now: what is it that you're... what is that telling you?

Edna: Let's see, that's telling me, what like, so y equals $m x$ plus b .

Interviewer: OK.

Edna: And that's, if I remember correctly, I might have just figured out what the $m x$ is.

Interviewer: OK. What does it mean? What does it tell you?

Edna: So then, could, like, go like, up 1 and then like, well, no that would be rise over run. Uh, yeah, so then I would go up 1 and then over 13.

Note that Edna correctly interpreted the slope as “rise over run”. At this point the interviewer decided to ignore Edna’s incorrect calculation of the slope (which resulted in $\frac{1}{13}$ as Edna mistakenly calculated “negative 2 minus 3” as 1) and directed her attention explicitly to finding the y -coordinate in $(8, y)$.

Interviewer: OK. But I'm still asking, what are the coordinates of the point whose x -coordinate is 8?

Edna: x -coordinate is 8... It'd be 8-comma-1.

Interviewer: So, does that help you figure out what it is you're really looking for? Does that answer the question? Is that all you need to do?

Edna: 'What is y '?... Um, well I think it does because it basically just shows what y is right there. I mean we've got the 8 for the x , which is right down there. And then over is the one. And it lines up perfectly on the graph.

Interviewer: OK. So that might be enough.

Edna: Yeah.

Note that Edna's initial approach was to find the slope, but that she did not seem to need it when prompted for the coordinate.

While Edna appeared convinced in her solution, the interviewer decided to pursue.

Interviewer: I'm gonna just push you just a little bit further [...] Are you completely convinced that that 8-comma-1 is accurate?

Edna: Mmm... Like On the graph? I mean, no graphs are never perfect... So...

Interviewer: Well this one was drawn by a computer so it should be pretty, pretty close but even so...

Edna: I mean, it looks pretty close. It looks I mean slightly off, by a little bit.

Following up on Edna's acknowledgement of "slightly off, by a little bit" the interviewer asked her to consider the given points.

Interviewer: So, the [slope] work that you started doing down here might be a way of taking it just a step further and seeing if we can get more accurate. So, let me ask you one more question.

Edna: Mm hm.

Interviewer: The two points that I did give you – the $(0, -2)$ and $(13, 3)$ – is either one of those special in any way? Or are they just two random points?

Edna: Yeah, the negative two is important cause it's the b .

Having recognized “the b ” (the y -intercept) and having previously calculated the slope, Edna wrote down the equation of the line. However, she was not confident how the equation was to be used.

Interviewer: OK. And then if that's the equation of the line, then how could you find out whether 1 was the correct y coordinate here for the x coordinate 8?

Edna: You could plug it into the x .

Interviewer: OK

Edna: And solve for, well, that's 8, we're trying to find y ... So, we might not be able to plug it in cause it's not the same...

Interviewer: Not the same as what?

Edna: The... well, it's a y not an x . I feel like they shouldn't be able to work, like, you shouldn't be able to plug them together because of that.

[...]

Interviewer: Can you find the y using that x , that's the question.

Edna: I feel like, for this you have to plug it in somewhere. But I don't remember where and I feel like it has to do something with the mx like you plug it in where the x is, so like you, uh...

Interviewer: Okay. Do you want to try that and see what happens?

[...]

Edna: OK, so, I... start, okay, let's go over it, so like, I mean it goes all the way over to thirteen... it doesn't go up one.

Even after acknowledging the need to “plug it into the x ” Edna needed guided prompting in order to substitute the value of x into the equation and find a corresponding y , having confused the “ y ” in the desired ordered pair $(8, y)$ as the place where she should plug in the 8.

However, in saying “go over it” and identifying “it has to do something with the mx ” Edna focused on the graph, counting the difference in x and y coordinates of the given points. This led her to discover the discord between her count or “rise” and previously calculated value of 1, as “it doesn't go up one”.

Edna: I feel like I switched those somehow.

Interviewer: You actually wrote it down just fine. So... you subtracted your y 's. You did negative two minus three. [...]

Edna: Oh, would that be negative five?

Then Edna was guided to recognize that the slope she identified as $\frac{-5}{-13}$, was actually equal to $\frac{5}{13}$, and following some struggle with fractions, she completed the Task correctly.

10.11.3. Interview with Gina

Unlike Edna, Gina immediately responded with (8,1) as the sought coordinates.

Gina: y is 1.

Interviewer: Does it look like it's exactly one?

Gina: No, it looks like it's a little bit over one.

Interviewer: But if it's not exactly one then how else do you think you could find out what it was exactly.

Gina acknowledged that y was “a little bit over one” and was asked to think how to find the exact value. Then, with some help from the interviewer, she calculated the slope as $\frac{5}{13}$, but was not sure whether this is “rise-over-run” or “run-over-rise”.

Interviewer: Can you use that [pointing to $\frac{5}{13}$] to decide whether slope is rise over run or run over rise?

Gina: It's rise over run.

Interviewer: How can you tell?

Gina: One two three four five... and then run thirteen.

Interviewer: Good.

Gina: So, yeah. Mm hm. So, slope is rise over run.

Interviewer: Can you see that in this formula at all? Can you see how the y s would relate to the rise [laughs] not just because they rhyme - and the x 's relate to the run?

Gina: Yeah cause the y always goes up.

Interviewer: Yeah.

Gina: And the x s always go either left or right.

Interviewer: So, this formula's actually a hint that it's rise over run, isn't it? Yeah.

Gina: Oh. I see.

Having clarified the slope, the interviewer attempted to direct Gina back to the Task. The excerpt below demonstrates that Gina considers creating a table as a possible approach. Then to fill in the table she needs "some type of equation", but she does not provide an indication of what to do with the equation, even if it were given to her.

Interviewer: I'd like to look at the original question again. We said, or you said, you thought the point was about one, but it looked like it wasn't exactly one. Once you decided it wasn't exactly one, and you wanted to find out what it exactly was, you went and found the slope. How's that going to help? Is there something else you need to do?

Gina: Mm, that's a good question. [long pause] Well, I can also go by creating a table x and y .

Interviewer: OK.

Gina: And if x is eight... Maybe this is not the right... 'cause I was going to do like when x is 8, what is y ?

Interviewer: Mm hm. That's a good approach, but what do you need to be able to do that?

Gina: Well, if I put...[pause]

Interviewer: You said 'if x is eight what would y be?' What would you do with that eight to find out?

Gina: Then I will do x equals eight plus y ? I have to create some type of equation.

Interviewer: Aha. What would you do with an equation if you had it?

Gina: I have to find that y value.

Interviewer: Right. So what do you need? What kinds of parts are there in the equation for a line?

Gina: Like a slope.

Interviewer: Mm hm.

Gina: In the equation.

Interviewer: If you had an equation, if I had given you an equation what do you think it might look like.

Gina: Maybe y equals eight, I mean x ... No, wait, y isn't equal to that. I was going to say plus 8 but not, that does not go in there. [laughs]. It has to say eight equals... something.

In what followed Gina was reminded how to find the equation of the line, after which, with some additional prompting, she substituted the value of x and found the corresponding value of y .

10.12. Cross-Case Analysis of the Interviews

Technically, the analytical approach to the Task consists of 3 steps: (1) calculate the slope, (2) determine the equation of the line, and (3) substitute 8 for the x -value in the equation to find the corresponding y -value. While Janice completed the three steps without any prompting, Gina needed guidance to proceed beyond step 1 and Edna needed guidance to proceed beyond step 2. Gina's initial responses at the interview are similar to answers of 5 participants who, in responding to the Task in the written questionnaire, calculated the slope, but determined the y -coordinate from inspecting the graph. Similarly, Edna's initial interview responses are similar to answers of the 7 participants who, in their written responses to the Task, provided the equation of the line, but did not use it to determine the answer (see Angela's work in Figure 10.5 and Jake's in Figure 10.6). That is, the calculation of the slope and of the line's equation may have had some ceremonial effect of following a procedure, recalling and imitating learned approaches, but did not influence the solution. Advantageously, the interview setting provided both Gina and Edna with sufficient guidance to complete the Task correctly, revealing in the process their struggle and incomplete links between visual information and information accessed analytically.

Janice was the only interviewee who showed explicit evidence of recognising the Cartesian Connection. For both Gina and Edna this connection was not evident. Even having the equation of the line at hand, Gina considered constructing a table as an option for finding the y -coordinate of the point and was not sure how the equation should be determined and used. Edna had less difficulty constructing the equation of the line but was unable on her own to determine how to use it.

However, all three interviewees identified an additional facet of the Cartesian Connection: that related to the slope. That is, all related the formula of the slope and the expression “rise-over-run” to counting of the units between the x -coordinates and the y -coordinates of the given points using the graph. Janice noted “I could do it by counting over there... But it that's not accurate sometimes”, so she used the learned formula. Edna initially miscalculated the slope, but considering the graphical interpretation helped her correct her mistake. Gina was initially unsure whether the slope was determined by “rise-over-run” or “run-over-rise”, but counting the distances in terms of units on the graph and considering the place of y 's in the numerator helped her decision. This additional facet of the Cartesian Connection was instrumental towards completing the solution.

10.13. Discussion and Conclusion

Our analysis clearly distinguished between two groups of students: those who relied on a visual/graphical approach and those who implemented analytical/algebraic methods, either completely or at least in part. However, a more appropriate distinction is not between “visualizers” and “analysers”, but between those who have made and implemented connections between various representations and those who have not.

To elaborate, our analysis of the written responses distinguished among those who relied on the Cartesian Connection, as elaborated in the strategy of finding the slope and the intercept of a line, their places in an equation and the use of that equation to find the sought coordinates of a point on the line ($n = 29$), those who were able to perform some algebraic manipulation, but who had not made the connection to how these manipulations could serve the solution of the particular Task and relied on observing the graph ($n = 12$) and those who relied solely information determined visually from the graph ($n = 22$).

These results were confirmed in the interview setting, where Janice elaborated on the approaches that relied on the Cartesian Connection, while Gina and Edna implemented initially the visual approach, even when they demonstrated familiarity with some analytical tools.

Furthermore, the interviews provided strong evidence of the importance of the interplay between analytical and visual approaches. For example, Janice confirmed the information regarding the slope obtained algebraically by considering the graph; Edna was able to consider and correct her incorrect calculation of the slope by considering the graph; the graphical information obtained from counting the units helped Gina connect the known formula for the slope to the “rise-over-run” idea.

Our analysis provided a clear indication that alongside teaching the mechanics of algebra a greater effort should be made in helping students make the essential connections between visual and analytical representations, specifically the Cartesian Connection. While this general recommendation is not novel, our analysis elaborated on a helpful additional facet of the Cartesian Connection, that of connecting visual and analytical representations of the slope.

The symbiotic relationship between visualization (V) and analysis (A) was described by Zazkis, Dubinsky & Dauterman (1996) as a VA model (see Figure 10.8).

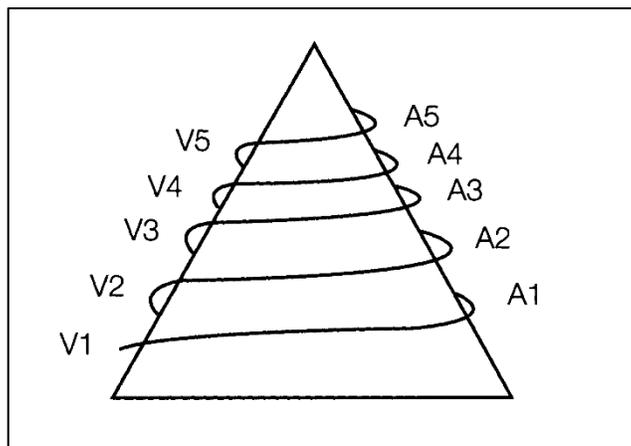


Figure 10.8. Visualization/Analysis model (from Zazkis, Dubinsky & Dauterman (1996))

Moving between the V and A sides on the triangular image metaphorically represents the idea that visualization helps in achieving a higher step in analysis, that the higher one gets in the process (on the triangle), the closer the visualization and the analysis become, and that a rather steep move to a higher analysis can be achieved via a less steep move via visualization.

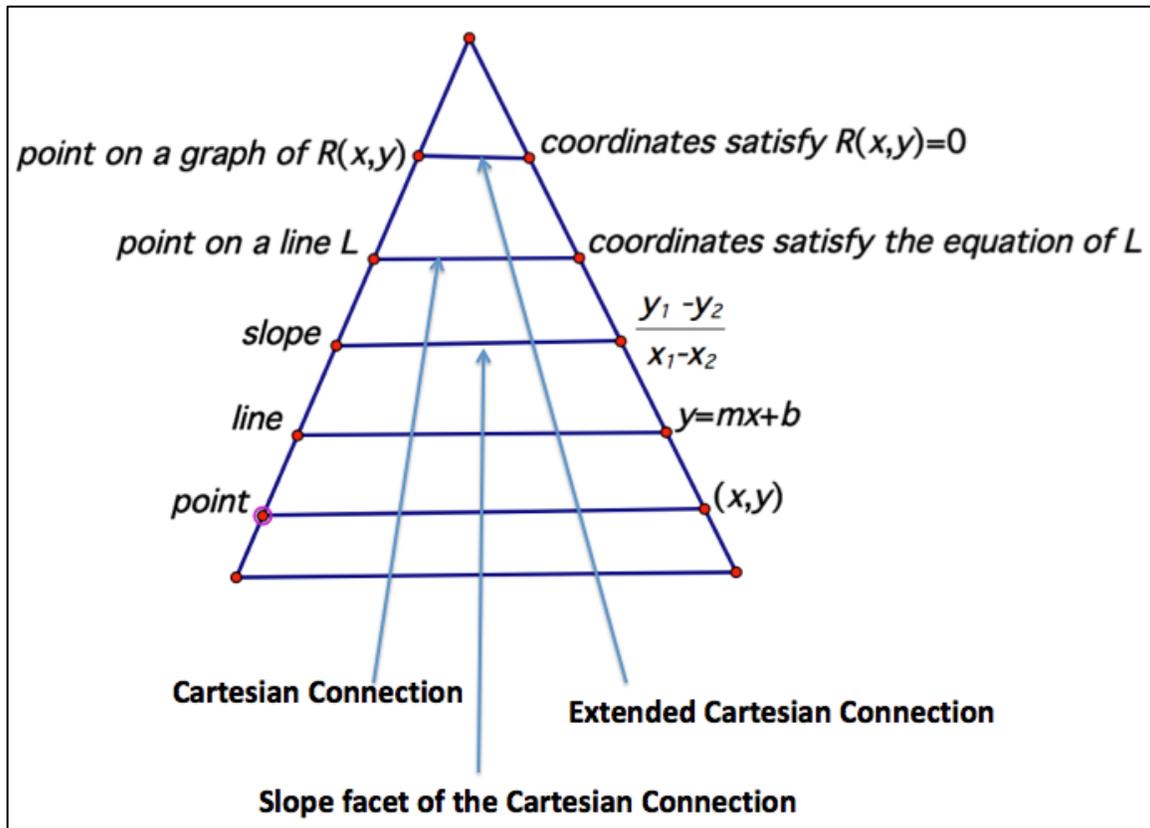


Figure 10.9. Facets of the Cartesian Connection

Within the general description of the VA model, our analysis provided an additional elaboration. While Moon et al. (2013) generalized the Cartesian Connection from a line, as formalized by Moschkovich et al (1993), to a general curve on the Cartesian plane, we elaborated to an additional facet related to the Cartesian Connection: that of relating the slope to the rise-over-run equation. As is shown in Figure 10.9, at the “lower” or initial stage of learning, students connect a graphical representation of points on the coordinate system to the analytical representation of ordered pairs of numbers. Then they connect the graph of a line to the linear equation. In particular, the slope of the line, as determined visually as “rise-over-run” is connected analytically to the equation determining the slope. The next step is the celebrated Cartesian Connection, which is further generalized from a line to any curve on the Cartesian plane.

Our results appear in discord with the results of Knuth (2000a, 2000b); participants in his study relied on algebraic manipulations to get results that were readily available from exploring the graph. Knuth’s students avoided visual approaches when

those were absolutely appropriate in the given tasks. This apparent contrast in results demonstrates that no one approach is generally preferable to the other in all cases. It is the flexible use of representations, connections among them and ability to move among graphical/visual and analytical/algebraic interpretations that contributes to proficiency and competence in a subject. The need for this flexibility should guide the instruction. Furthermore, the apparent conflict between our results and those of Knuth (2000a, 2000b) also suggest that additional research would be beneficial into how students determine when a particular approach is appropriate.

Mathematics education has long been searching for better ways to convey mathematics, to understand what happens when learning takes place, and to help students make connections. Making connections is, as Moschkovich (1996) indicated, vital to competence. Explicit attention to mathematical connections should be an integral part of teaching mathematics at all levels. However, it is of crucial importance in teaching remedial mathematics courses in the community college setting, as students enrolled in remedial mathematics courses may have not experienced instruction that capitalises on mathematical connections. We agree with Arcavi (2003) that “learning to understand and be competent in the handling of multiple representations can be a long-winded, context dependent, nonlinear and even tortuous process for students” (p.235). However, we believe that engaging in this “tortuous process” is essential for students’ success.

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