Optimal Investment and Consumption Strategy for a Retiree Under Stochastic Force of Mortality

by

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Abstract

With an increase in the self-driven retirement plans during past few decades, more and more retirees are managing their retirement portfolio on their own. Therefore, they need to know the optimal amount of consumption they can afford each year, and the optimal proportion of wealth they should invest in the financial market. In this project, we study the optimization strategy proposed by Delong and Chen (2016). Their model determines the optimal consumption and investment strategy for a retiree facing (1) a minimum lifetime consumption, (2) a stochastic force of mortality following a geometric Brownian motion process, (3) an annuity income, and (4) non-exponential discounting of future income. We use a modified version of the Cox, Ingersoll, and Ross (1985) model to capture the stochastic mortality intensity of the retiree and, subsequently, determine a new optimal consumption and investment strategy using their framework. We use an expansion method to solve the classic Hamilton-Jacobi-Bellman equation by perturbing the non-exponential discounting parameter using partial differential equations.

Keywords: Retirement planning, optimal consumption, optimal investment, non-exponential discounting, mortality intensity model, deterministic income.

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Chapter 1

Introduction

With an increase in the number of individuals managing their retirement portfolio, more attention has been drawn towards determining the optimal amount of wealth an individual should invest and consume during their retirement. The most common objectives of an individual are maintaining a certain standard of living in retirement without running out of funds before dying, and simultaneously maximizing utility of consumption.

Recent evidence suggests that the expected lifetime of an individual has significantly increased during the past decades. Therefore, it becomes necessary for retirees (or their wealth managers) to manage their portfolio of wealth efficiently to meet their retirement objectives. The goal of this study is to determine the optimal proportion of wealth a retiree should invest in the risky asset and the optimal amount she should consume during her retirement to meet her desired objectives stated above.

There are numerous factors that individuals should consider to successfully plan for their retirement. Some of the most crucial factors are:

– Inflation: The amount of money required to maintain a certain standard of living today will not be enough to maintain the same standard of living in the future. For instance, annual inflation of as low as 2% might lead to nearly double the consumption costs for a retiree over the next 30 years.

– Longevity: Living longer than expected can result in less consumption during the later years of retirement and might lead to financial ruin.

– Lifestyle: Individuals might be wishing to maintain the same lifestyle in their retirement as they enjoyed during their working years (e.g., housing, consumption). However, the costs involved in following the same lifestyle might change significantly for a retiree. For instance, an individual might want to settle down in a new country after retirement where the cost of living might be significantly different from their current cost of living. A person might move to a smaller or bigger house for their remaining lifetime.

– Bequest motive: The decision to leave a bequest will affect the withdrawal strategy of the retiree during their remaining lifetime. Hence, it might lead to a lower amount available for consumption during their retirement.

– Healthcare: An individual who tends to be sick more often will have to consider higher health care costs during their retirement phase, and hence will consume less.

In the literature, numerous studies have incorporated the factors that affect an individual's optimal consumption and investment strategy. The fundamental model of individual savings and consumption was originally proposed by Fisher (1930), which was later extended by Yaari (1964) to include bequests in determining the optimal consumption behaviour of an individual having a deterministic lifetime. Later on, Yaari (1965) introduces stochastic future lifetimes in the optimization problem to reflect that the individuals should adjust their optimal consumption according to their expected survival rates. This was done by maximizing the utility function, originally proposed by Fisher (1930). The investments were being made in the risk-free assets only. The seminal work of Merton (1969, 1971) introduces risky assets that follow a geometric Brownian motion (gBm) process in the asset portfolio and establish closed-form solutions for the optimal consumption and investment strategy of an individual. Moreover, he combines multiple risky assets into a mutual fund (using the so-called mutual fund theorem) in the optimization problem, and generalizes its solution for a wide class of utility functions. His results serve as the basis for many extensions. Some of these relevant to our study are:

– Longevity: An individual's lifetime can be affected by a change in her health conditions and, therefore, it becomes necessary to adjust her optimal consumption and investment strategy accordingly. Chen and Milevsky (2003) discuss the use of longevity insurance to hedge against longevity risk and determine the optimal investment strategy for the retiree. Later, Milevsky and Huang (2011) examine the impact of longevity risk aversion on the optimal consumption strategy of an individual having deterministic force of mortality and investing in risk-free assets only. Their study proposes that an individual's optimal withdrawal rate should be changed dynamically with a change in their longevity risk aversion, i.e., the risk aversion parameter in the utility function. This was later extended to the stochastic force of mortality by Huang et al. (2012), where the individual adjusts her optimization strategy as quickly as the information becomes available via her health status.

– Non-exponential discounting: Using an exponential function to discount future income (or utility) has been common in the literature when it comes to solving an optimization problem. This is because an individual's preferences remain similar throughout time and the dynamic programming principle can be applied to determine the closed-form solutions directly. However, many studies suggest that individuals have different risk-aversions over time, and they discount their near future income at a much higher rate than the exponential rate, and later year income at a much lower rate; see for example Loewenstein and Prelec (1992), Luttmer and Mariotti (2003), Ekeland and Lazrak (2006), Ekeland and Pirvu (2008), and Ekeland et al. (2012). Considering the same idea, Morris and Postlewaite (1997) discuss in detail the concept of non-exponential discounting in the intertemporal consumption problem. The use of non-exponential discounting leads to time-inconsistent optimization problems which deter the use of the classic Bellman equation. Björk and Murgoci (2014) suggest the application of game theory to solve these time-inconsistent problems by determining the subgame perfect equilibrium points. This is done by introducing a system of linear Hamilton-Jacobi-Bellman (HJB) equations and, then, determining the equilibrium strategy and equilibrium value function. Later, using this, Dong and Sircar (2014) solve the Merton (1971) optimization problem for non-exponential discounting function. Since the non-local HJB equations cannot be solved explicitly, they propose an expansion method by undertaking a small perturbation of the hyperbolic discount function to determine the first-order approximation of the optimal consumption and investment strategy.

Delong and Chen (2016) combine these streams of literature and determine the optimal consumption and investment strategy for the retiree facing longevity risk aversion and non-exponential discounting. Moreover, the retiree is assumed to maintain a certain standard of living and, therefore, she spends a minimum fixed amount on her consumption every year, which she finances with a life annuity. The authors extend the method suggested by Huang et al. (2012) to incorporate longevity risk aversion, deterministic future income, and a general utility function. Furthermore, they use a gBm process to model the stochastic force of mortality and discount the future income using a non-exponential discounting function as proposed by Dong and Sircar (2014).

In this project report, we replicate the results derived using the Delong and Chen (2016) methodology and perform a sensitivity analysis on the overall optimal consumption level, investment, and wealth strategies. Additionally, we introduce a modified Cox, Ingersoll, and Ross (1985; CIR hereafter) model to capture the stochastic force of mortality of the retiree, and subsequently determine the optimal consumption and investment strategy in Delong and Chen (2016).

The rest of this report is organized as follows. Chapter 2 discusses some key concepts which will be used throughout this study. Chapter 3 discusses the Merton framework, which serves as the base in Delong and Chen (2016), and the current study. Chapter 4 explains the optimization using the methodology suggested by Delong and Chen (2016). Chapter 5 introduces a modified CIR mortality intensity model in Delong and Chen (2016) framework and compares the numerical results to those of Chapter 4.

Chapter 2

Background

We begin this report by introducing a few concepts from economics and actuarial science as these serve as the basis for most existing research in retirement. This project uses all of these concepts, and it is essential to understand them.

2.1 Utility Function

Utility can be defined as usefulness received from consuming. Mathematically speaking, the utility function measures the welfare or satisfaction of a consumer as a function of real goods such as food, clothing, and shelter. In the optimal consumption and investment context, it can be defined as the amount of future satisfaction to be received by an individual who wishes to maximize her utility of consumption by making different investment choices. Each individual has different risk preferences which drive their decision to invest and consume, which in turn lead to distinct utilities. These preferences will affect the optimal consumption and investment decision of each individual, and it is, therefore, necessary to have a separate utility function for each individual in the optimization problem. The constant absolute risk aversion (CARA) and constant relative risk aversion (CRRA) utility functions are two classic functions, based on Morgenstern and von Neumann (1953). Merton (1971) uses a hyperbolic absolute risk aversion (HARA) utility model, that helps him solve his specific optimization problems in a closed form. Like the utility function used by von Neumann and Morgenstern, which is expressed as a desire to maximize final payouts while mitigating risk, the HARA utility function is of the form

$$V(C) = \frac{(1-\gamma)}{\gamma}\left(\frac{\beta C}{1-\gamma} + \eta\right)^{\gamma}, \tag{2.1}$$

where C is the amount of consumption (economic variable of interest), and γ, β, and η are constants subject to following restrictions:

$$\gamma \neq 1, \quad \beta > 0, \quad \left(\frac{\beta C}{1-\gamma} + \eta\right) > 0, \quad \text{and} \quad \eta = 1 \quad \text{if} \quad \gamma \to -\infty.$$

Absolute risk aversion is defined as the investor's reaction to uncertainty for a dollar change in their wealth, or an absolute change in the economic variable of interest. For the HARA utility function, the absolute risk aversion is given by

$$A(C) = -\frac{\frac{\partial^2 V(C)}{\partial C^2}}{\frac{\partial V(C)}{\partial C}} = \frac{1}{\left(\frac{C}{1-\gamma} + \frac{\eta}{\beta}\right)} > 0. \tag{2.2}$$

Absolute risk aversion can be increasing, decreasing, or constant depending upon the individual's risk preferences. For the HARA family, the first derivative of absolute risk aversion function in Equation (2.2) given by

$$A'(C) \equiv \frac{\partial A(C)}{\partial C} = \frac{-1}{(1-\gamma)\left(\frac{C}{1-\gamma} + \frac{\eta}{\beta}\right)^2},$$

Consequently, utility functions with increasing, decreasing and constant absolute risk aversion can be constructed under the following conditions:

- Increasing Absolute Risk Aversion: $A'(C) > 0$, if $1 < \gamma < \infty$.
- Decreasing Absolute Risk Aversion: $A'(C) < 0$, if $-\infty < \gamma < 1$.
- Constant Absolute Risk Aversion, (or CARA): $A'(C) = 0$, if $\gamma \to +\infty$. Under CARA, the utility function can be expressed as $U(C) = \frac{-e^{-\eta C}}{\eta}$.

Relative risk aversion is defined as the investor's reaction to uncertainty for a proportional change in their wealth, or a relative change in the economic variable of interest. The relative risk aversion is given by

$$R(C) = A(C)C.$$

Similar to absolute risk aversion, relative risk aversion can be increasing, decreasing, and constant. For the HARA family, the first derivative of relative risk aversion can be determined by the first derivative of the relative risk aversion function described below.

$$R'(C) \equiv \frac{\partial R(C)}{\partial C} = \frac{\frac{\eta}{\beta}}{\left(\frac{C}{1-\gamma} + \frac{\eta}{\beta}\right)^2}.$$

Therefore, HARA utility functions with increasing, decreasing and constant relative risk aversion can be constructed under the following conditions:

- Increasing Relative Risk Aversion: $R'(C) > 0$, if $\eta > 0$ and $\gamma \neq 1$.
- Decreasing Relative Risk Aversion: $R'(C) < 0$, if $\eta < 0$ and $-\infty < \gamma < 1$.

– Constant Relative Risk Aversion, (or CRRA): $R'(C) = 0$, if $\eta = 0$. Under CRRA, the utility function can be expressed as $U(C) = \frac{C^{1-\gamma}}{1-\gamma}$ for $\gamma \neq 1$, or $U(C) = \log(C)$ for $\gamma = 1$.

The most commonly used utility functions in the optimization literature are the constant absolute risk aversion (CARA), and constant relative risk aversion (CRRA). These utility functions have the advantage of allowing us to solve complex optimization problems in closed form for a large collection of problems, without making any additional assumptions.

2.2 Non-Exponential Discounting of Future Income

The time value of money refers to the concept that the money available now is worth more than the same amount of money in the future, due to factors such as inflation and potential earning capacity. For example, 10 dollars today is different than 10 dollars five years from now. An individual's decision to invest and consume today is dependent upon their future income (pension or investment income), and, therefore, it becomes necessary to evaluate the present value of their future income accurately. The classic approach to evaluate the present value of future income is to discount back using an exponential function, which has a fixed discounting rate (i.e., constant over time) multiplied by the number of years of discounting. This means that time duration does not have any effect on the discount rate while discounting future income. However, this does not seem to reflect the true behaviour of an individual's subjective discounting. Individuals tend to have a higher discount rate for immediate income than for future income. Their valuation rate tends to decrease rapidly for short period delays, and less rapidly for longer period delays (e.g., Loewenstein and Prelec, 1992; Luttmer and Mariotti, 2003; Ekeland and Lazrak, 2006). A typical example points out that while people prefer two apples in 21 days to one apple in 20 days, they also prefer one apple now to two apples tomorrow. Such behaviour cannot be described by exponential discounting. Non-exponential discounting (such as hyperbolic discounting) tends to do a better job. A typical non-exponential discount function is of the form

$$\phi(t) = e^{-\rho t} \frac{1}{(1+\delta t)^{\epsilon}}, \quad 0 \leq t \leq T, \tag{2.3}$$

where ρ, δ and ϵ are the input factors used to determine the behaviour of an individual. As ρ goes up, the individual increases her discount rate for the entire time horizon. When δ goes up, the individual's discount rate increases for shorter horizons and decreases for longer ones. The parameter ϵ determines how close the discount rate is to a constant rate. In Figure 2.1, $\epsilon = 0$ is the exponential discounting function, and as the variable ϵ increases, the discount function becomes less convex, thus representing the true behaviour of the individual's subjective discounting. We use the non-exponential discount factor of Equation (2.3) as employed in Delong and Chen (2016).

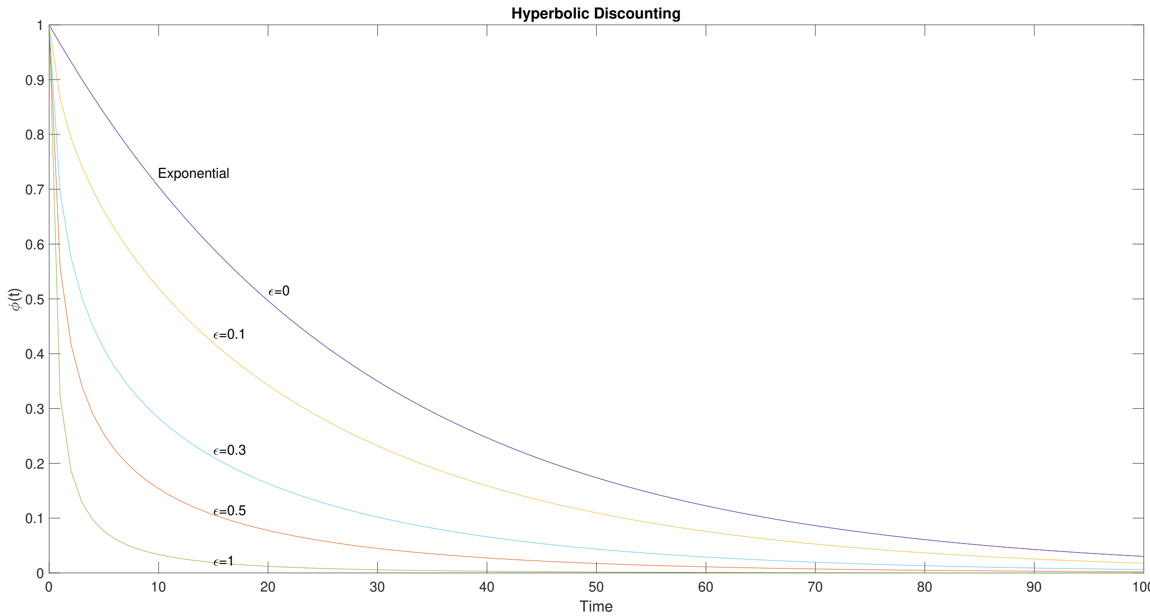

Figure 2.1: **Exponential and Hyperbolic Discount Functions.**
This figure shows the effect of a change in the hyperbolic discounting parameter ϵ on the hyperbolic discount function of Equation (2.3). We assume $\delta = 2$, $\rho = 0.035$. The parameter $\epsilon = 0$ represents exponential discounting.

2.3 Bequest

A bequest is the amount of money an individual plans to leave behind when she dies. This can be a fixed amount, a proportional amount of wealth, or in the form of fixed assets. In an optimization context, an individual determines the discounted value of the bequest amount through her utility function. Thereafter, this amount is accounted for in the optimization model to determine the optimal consumption and investment strategy.

2.4 Mortality Effects on Optimal Investment and Consumption

An individual's decision to invest and consume their wealth depends highly upon their health condition. A rational individual may tend to save more and consume less if she is expecting to live longer. On the contrary, an individual might consume at a higher pace if she expects to die soon. For instance, a person diagnosed with cancer is likely to have a much higher consumption of wealth than a healthy individual.

In an optimization framework, the effect of mortality can be modelled using a discrete-time approach, i.e., adjusting the optimal investment and consumption strategy every t years, according to the expected remaining lifetime (see Milevsky and Huang, 2011). However, a stochastic mortality model can account for the randomness in future mortality intensity between these t years continuously, and immediately adjust the optimal investment and consumption strategy (see Luciano and Vigna, 2008; Norberg, 2010; Huang et al., 2012). In this study, we generalize the work of Delong and Chen (2016) and use a mean-reverting modified Cox, Ingersoll, and Ross (1985) model to represent the stochastic force of mortality of an individual and determine the resulting optimal consumption and investment strategy. This is discussed in detail in Chapter 5.

2.5 The Black and Scholes Market

In Black and Scholes (1973), an asset price follows a gBm process given by

$$dX(t) = \alpha X(t)dt + \sigma X(t)dZ(t), \tag{2.4}$$

where $X(t)$ is the price of an asset, α is the drift parameter, σ is the volatility parameter and $Z(t)$ is the time-t value of the standard Brownian motion $\{Z(t)\}_{t\geq 0}$. The geometric Brownian motion is a continuous-time stochastic process which has a constant drift and constant volatility parameter with continuous asset paths. It is a Markov process, which means that future asset prices given the present prices are independent of past prices.

If an asset price follows a gBm process, then the asset return between time s and t, i.e., $\log X(t) - \log X(s)$, is normally distributed. This can be expressed as

$$\log X(t) - \log X(s) \sim \mathcal{N}\left(\left(\alpha - \frac{\sigma^2}{2}\right)(t-s), \sigma^2(t-s)\right), \quad t > s. \tag{2.5}$$

In Equation (2.5), the mean and variance of asset returns remain constant over time, and therefore, the asset returns are stationary.

2.6 Hamilton-Jacobi-Bellman Equation

The Hamilton-Jacobi-Bellman (HJB) equation is one of the tools used to solve an optimal control problem in continuous time. It is a non-linear partial differential equation whose solution is the value function. When the optimization problem is stochastic, the HJB equation becomes a second-order partial differential equation. To solve optimization problems under non-exponential discounting, one can approximate this second-order partial differential equation. We use this method to determine the optimal consumption and investment strategy for a retiree, who discounts her future income using hyperbolic discounting, i.e., see Chapter 4.

Chapter 3

Optimal Consumption and Portfolio Rules: Merton's Model

3.1 Introduction

Before 1969, a number of portfolio selection models has been introduced to determine the optimal portfolio choice of an investor over a one-period horizon, e.g., Markowitz (1952). Building on this rich literature, Samuelson (1969) combines the problem of optimal portfolio selection and consumption in a multi-period discrete-time model.

Merton (1969) extended Samuelson (1969), using several key economic and financial concepts to solve the optimal consumption and investment strategy for a rational individual who maximizes her utility in a continuous-time model, where the only source of income is the return obtained from investments. Specifically, stochastic dynamic programming and the HJB equation are used to derive a generalized optimal consumption and an investment strategy for an individual under CARA and CRRA utility functions.

The Merton (1969) model was further extended by Merton (1971) who uses a HARA utility function instead of the CARA and CRRA classes. Indeed, the HARA utility family can accommodate increasing, decreasing or constant risk aversions (see Section 2.1 for more details). Merton (1971) also considers portfolio separation (i.e., the so-called mutual fund theorem), deducing that if asset prices follow a gBm processes, then the assets can be combined into one risky asset without any loss of generality. Finally, his model allows for deterministic non-capital income, and the optimal consumption and investment strategy are derived under these new assumptions. The framework used in his paper applies to a wide class of economic models that deal with decision-making under uncertainty.

Closely related to this framework, Delong and Chen (2016) find that the optimal consumption and investment strategy are dependent upon the stochastic mortality of the retiree in continuous time. They assumed that the retiree has bequest goals and non-capital income, which she discounts using non-exponential discounting. This paper is discussed in detail in the next chapter.

For the rest of this chapter, we consider the work of Merton (1969) and Merton (1971) as these will allow us to understand Delong and Chen's contributions better.

3.2 Framework

Merton (1969) assumes that the only source of income is the asset returns, and that the explicit solution to the optimization strategy can be determined when an individual's risk preferences follow a CARA or CRRA utility function, and the asset portfolio contains one risky and one risk-free asset only. The risky asset price is assumed to follow a gBm process. First, we introduce the budget equation for the wealth process under multiple risky assets. This case will be reduced to the one risky asset case afterwards. A sequel from the same author (i.e., Merton, 1971), investigates the case of multiple risky assets further.

3.2.1 Budget Equations

The continuous-time budget equation, describing the wealth process $W(t)$ of an individual at time t can be represented as the following stochastic differential equation[1]

$$dW(t) = \left[\sum_{i=1}^{m} w_i(t)\alpha_i W(t) - C(t)\right] dt + \sum_{i=1}^{m} w_i(t)\sigma_i W(t) dZ_i(t), \tag{3.1}$$

where $Z_i(t)$ is a standard Brownian motion process for asset i at time t, potentially correlated with the other Brownian motions. Variable $C(t)$ is consumption per unit of time at time t, $w_i(t)$ is the proportion of wealth invested in asset i at time t, such that $\sum_{i=1}^{m} w_i(t) = 1$, and T is the terminal time. Parameter α_i is the constant expected rate of return for asset i, and σ_i^2 is the variance per unit of time for the process $Z_i(t)$.

3.3 Optimal Rules for the Two-Asset Scenario

In Merton (1969), the optimal solution to the multiple asset scenario is determined, but a closed-form solution cannot be obtained. However, if it is assumed that the asset portfolio contains one risky asset and one risk-free asset, a closed-form solution can be obtained and the optimization solutions are easier to interpret. Therefore, for simplicity, it is assumed that an individual invests a proportion $w(t)$ of her wealth at time t in the risky asset, and a proportion $1 - w(t)$ of her wealth in the risk-free asset. The time-t return for the risky asset follows a gBm process given in Equation (2.4), whereas the return on the risk-free asset is a constant r. Therefore, the continuous-time budget of Equation (3.1) can be written under

[1]See Ito (1957) for a detailed discussion on stochastic differential equations.

the two-asset scenario as

$$dW(t) = [(w(t)(\alpha - r) + r)W(t) - C(t)]\, dt + w(t)\sigma W(t) dZ(t). \tag{3.2}$$

Now, the problem of choosing the optimal portfolio $w(t)$ and the optimal consumption $C(t)$ can be formulated as

$$\max_{\{C(t),w(t)\}} \quad \mathbb{E}\left[\int_0^T e^{-\rho t} U(C(t))dt + B(W(T),T)\right], \tag{3.3}$$

subject to the wealth dynamics of Equation (3.2), where $C(t) \geq 0$ and $W(t) > 0$. Function $U(C)$ is the utility of consumption C, which is assumed to be concave in wealth. Function $B(W(T),T)$, on the other hand, represents the bequest valuation function at time T for a wealth level of W. The bequest valuation function is of the form

$$B(W(T),T) = e^{-\rho T} \frac{b(T)\Big(W(T)\Big)^{\gamma}}{\gamma}$$

where $b(T)$ is the weight assigned to the bequest by an individual. To determine the value of $b(T)$, see Equation (3.7).

Since the optimal portfolio and consumption strategy during the initial period affect the optimal strategy for later periods, it becomes essential to account for future optimization decisions now. Therefore, Equation (3.3) needs to be re-expressed in the form of a HJB equation to account for future consumption and investment decisions in making optimal investment and consumption decisions today. This breaks the optimization problem of Equation (3.3) into a sequence of simpler sub-problems which can be solved using stochastic dynamic programming.

Let us define

$$I(W(t),t) \equiv \max_{\{C(s),w(s)\}} \quad \mathbb{E}_t\left[\int_t^T e^{-\rho s} U(C(s))ds + B(W(T),T)\right], \tag{3.4}$$

with the following terminal condition:

$$I(W(T),T) = B(W(T),T).$$

The optimal investment and consumption rule can be determined by solving the following differential equation

$$\begin{aligned} 0 = \max_{\{C(t),w(t)\}} \Bigg[& e^{-\rho t} U(C(t)) + \frac{\partial I_t}{\partial t} + \frac{\partial I_t}{\partial W} \left((w(t)(\alpha - r) + r)W(t) - C(t)\right) \\ & + \frac{1}{2}\frac{\partial^2 I_t}{\partial W^2} \sigma^2 w^2(t) W^2(t)\Bigg], \end{aligned} \tag{3.5}$$

where $I_t \equiv I(W(t), t)$ and $t \in [0, T]$.

For a detailed derivation of the optimality equation, see Dreyfus (1960), and for detailed steps to apply these concepts to Equation (3.4), see Merton (1969).
The optimal solutions can be determined by solving Equation (3.5). Yet, an assumption needs to be made about the utility function. Merton (1969) assumes that these are in the CRRA (i.e., iso-elastic utility) or CARA (i.e., exponential utility) families.

3.3.1 Constant Relative Risk Aversion

Using the CRRA utility function, Equation (3.5) can be expressed as follow

$$0 = \left(\frac{(1-\gamma)}{\gamma} \frac{\partial I_t}{\partial W} \right)^{\gamma/(\gamma-1)} e^{-\rho t/(1-\gamma)} + \frac{\partial I_t}{\partial t} + \frac{\partial I_t}{\partial W} rW - \frac{(\alpha - r)^2}{2\sigma^2} \frac{(\frac{\partial I_t}{\partial W})^2}{(\frac{\partial^2 I_t}{\partial W^2})}. \tag{3.6}$$

Subsequently, the optimal consumption and investment strategy can be found as[2]

$$C^*(t) = \left(e^{\rho t} \frac{\partial I_t}{\partial W} \right)^{1/(\gamma-1)},$$

and

$$w^*(t) = \frac{-(\alpha - r) \frac{\partial I_t}{\partial W}}{\sigma^2 W \frac{\partial^2 I_t}{\partial W^2}}.$$

To determine the function I_t in Equation (3.6), a trial solution $\bar{I}(W(t), t)$ is introduced

$$\bar{I}(W(t), t) = \frac{b(t)}{\gamma} e^{-\rho t} (W(t))^\gamma,$$

where $b(t)$ is a function of time t. In order for $\bar{I}(W(t), t)$ to be the solution of Equation (3.6), $b(t)$ must satisfy the following ordinary differential equation,

$$\frac{\partial b(t)}{\partial t} = yb(t) - (1-\gamma)(b(t))^{\frac{-\gamma}{1-\gamma}}, \tag{3.7}$$

with the following terminal condition $b(T) = \xi^{1-\gamma}$, and $y \equiv \rho - \gamma((\alpha - r)^2 / 2\sigma^2(1-\gamma) + r)$.

The resulting rules for optimal consumption and portfolio selection are

$$C^*(t) = (b(t))^{\frac{1}{\gamma-1}} W(t),$$

and

$$w^*(t) = \frac{(\alpha - r)}{\sigma^2(1-\gamma)}.$$

[2]For a detailed derivation of these equations, see page 250 of Merton (1969).

Therefore, the solution of Equation (3.7) is

$$b(t) = \left((1 + (\psi\xi - 1)e^{\psi(t-T)})/\psi\right)^{1-\gamma},$$

where $\psi = y/(1-\gamma)$.

If $\bar{I}(W(t), t)$ is a real-valued function that is concave (i.e., $\frac{\partial^2 \bar{I}_t}{\partial W^2} < 0$), and if the solution is feasible (i.e., $C^*(t) \geq 0$), then $\bar{I}(W(t), t)$ is the solution for Equation (3.6). These conditions are satisfied under the CRRA utility since

$$[1 + (\psi\xi - 1)e^{\psi(t-T)}]/\psi > 0, \quad 0 \leq t \leq T,$$

for all values of ψ when $T < \infty$. Therefore, the optimal consumption and investment strategy can be represented as

$$C^*(t) = \begin{cases} \left(\psi/(1 + (\psi\xi - 1)e^{\psi(t-T)}\right) W(t) & \text{if } \psi \neq 0 \\ \frac{1}{(T-t+\xi)} W(t) & \text{if } \psi = 0 \end{cases}$$

and

$$w^*(t) = \frac{(\alpha - r)}{\sigma^2(1-\gamma)},$$

which is independent of $W(t)$ and t.

3.3.2 Constant Absolute Risk Aversion

Another class of utility function under which optimal solutions to Equation (3.5) can be determined is CARA. Using a similar methodology as that used in the previous subsection, the optimal consumption and investment strategy can be deduced as

$$C^*(t) = rW(t) + \left[\frac{\rho - r + (\alpha - r)^2/2\sigma^2}{\eta r}\right],$$

and

$$w^*(t) = \frac{(\alpha - r)}{\eta r \sigma^2 W(t)}.$$

As one can notice, the optimal investment strategy at time t depends upon the wealth level at time t under the CARA utility, whereas the optimal investment strategy is independent of the level of wealth under CRRA utility. This means that under the CARA utility, when an individual becomes wealthier and wealthier (as $W(t) \to \infty$) her proportion of investment in the risky asset keeps on declining since she can achieve the same utility of consumption by investing a smaller proportion in the risky asset.

3.4 Generalization of Merton's Results

The optimal investment and consumption strategy is determined under a particular set of assumptions that restricts the use of the model. Therefore, Merton (1971) extends the optimization model to allow for:

- Explicit solution for multiple risky assets in the asset portfolio;
- A general utility function;
- Non-capital income.

3.4.1 Optimization Framework for Multiple Risky Assets

If an asset portfolio has m risky assets and one risk-free asset such that the total number of assets in the portfolio is $m+1$, then the optimization problem can be defined as

$$J(W(t), \boldsymbol{X}(t)) = \max_{\{C(t), \boldsymbol{w}(t)\}} \quad \mathbb{E}_t \left[\int_t^T U(C(s))ds + B(W(T)) \right], \tag{3.8}$$

where $\boldsymbol{w}(t)$ is the vector of proportion of wealth invested in each risky asset, such that $\sum_{i=1}^m w_i(t) = 1$. Function $U(C(t)$ is the utility of consumption C at time t and $B(W(T))$ is the bequest valuation function of wealth at time t, both of which are assumed to be concave. The process $\boldsymbol{X}(t)$ is the vector of $m+1$ asset prices at time t. The horizon for which the optimization problem is solved is fixed at the outset to be T years.

Equation (3.8) is simplified using the HJB equation and the techniques of stochastic dynamic programming. Therefore, the optimal strategy can be determined by solving the following,[3]

$$\begin{aligned} 0 = U[G,T] + J_t + J_W[rW - G] + \sum_{i=1}^m J_i \alpha_i X_i(t) + \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^m J_{ij} \sigma_{ij} X_i(t) X_j(t) \\ - \frac{J_W}{J_{WW}} \sum_{j=1}^m J_{jw} X_j(t)(\alpha_j - r) + \frac{J_W^2}{2J_{WW}} \sum_{i=1}^m \sum_{i=1}^m v_{ij}(\alpha_i - r)(\alpha_j - r) \\ - \frac{1}{2J_{WW}} \sum_{i=1}^m \sum_{j=1}^m J_{iw} J_{jw} \sigma_{ij} X_i(t) X_j(t), \end{aligned} \tag{3.9}$$

where σ_{ij} represents the i^{th} row and j^{th} column in $m \times m$ variance-covariance matrix, v_{ij} represents the i^{th} row and j^{th} column of the inverse of the variance-covariance matrix, and r is the risk-free rate of return. Expressions J_W and J_k are the partial derivative of function $J(W(t), \boldsymbol{X}(t))$ with respect to wealth $W(t)$ and the k^{th} asset price $X_k(t)$, respectively. The function G is the inverse function of U_C, which represents the first derivative of the

[3]See page 381 of Merton (1971) for a detailed derivation of the Equation (3.9).

discounted utility $U(C(t))$ with respect to consumption C at time t. In Equation (3.9), $X_i(t)$ is assumed to be an Ito process with α_i and σ_i (see Equation (5) in Merton, 1971). Parameter σ_i is the standard deviation of the i^{th} asset, i.e., $\sqrt{\sigma_{ii}}$.

Using the boundary condition $J(W(T), \boldsymbol{X}(T)) = B(W(T))$ of Equation (3.8), the optimal proportion of investment in risky asset can be calculated using

$$w_i^*(t) = -\frac{J_W}{J_{WW}W(t)} \sum_{j=1}^{m} v_{ij}(\alpha_j - r) - \frac{J_{iW}X_i(t)}{J_{WW}W(t)}, \quad i = 1, ..., m,$$

where $w_i^*(t)$ is the proportion of wealth invested in i^{th} risky asset at time t.

3.4.2 Combining Multiple Risky Assets in One Asset

Individuals prefer to invest in more than one risky asset at a particular time t to reduce their idiosyncratic risk. Therefore, Merton (1971) introduces the concept of the separation theorem, originally proposed by Cass and Stiglitz (1970), which combines all the risky assets as in a mutual fund. The theorem entails that if the price of each risky asset follows a gBm process, there exists a unique pair of mutual funds which is constructed from linear combinations of these risky assets.

This unique pair of mutual funds is created using the mean-variance framework proposed by Markowitz (1952). Under the mean-variance framework, each mutual fund is created such that it has minimum variance for a certain expected return. Therefore, all these mutual funds lie on the efficient frontier of the expected return-variance space such that an individual will be indifferent among choosing a linear combination of any two mutual funds lying on the efficient frontier. A key advantage of using this theorem is that it reduces the transactional cost for an investor who purchases each security individually, and it helps to solve the complex optimization problems by reducing the number of assets in the portfolio. However, to apply this theorem in the Merton (1971) framework, the asset price needs to be independent of an individual's preferences (i.e., utility function) and wealth distribution. To satisfy these conditions, Merton (1971) assumes that the expected rate of return and the variance of each individual risky security are constants, i.e., α_i and σ_{ij} in Equation (3.9) are constant. This, in turn, leads to the asset price $X_i(t)$ in Equation (3.9) to behave as a gBm.

Now, the pair of unique mutual funds consist of investments being made in several risky assets. Therefore, Merton (1971) defines δ_i to be the proportion of first mutual fund's value held in the the i^{th} risky asset, and Υ_i to be the proportion of the second mutual fund's value held in the same i^{th} risky asset. These can be determined as[4]

$$\begin{aligned}\delta_i &= \frac{h}{\nu}\sum_{j=1}^{m} v_{ij}(\alpha_j - r),\\ \Upsilon_i &= \frac{(h-1)}{\nu}\sum_{j=1}^{m} v_{ij}(\alpha_j - r),\end{aligned} \tag{3.10}$$

where h and ν are arbitrary constants, and $\nu \neq 0$.

Therefore, the proportion of investment in the risk-free asset ($m+1^{\text{th}}$ asset) for each of the above mutual fund will be,

$$\begin{aligned}\delta_{m+1} &= 1 - \sum_{i=1}^{m}\delta_i,\\ \Upsilon_{m+1} &= 1 - \sum_{i=1}^{m}\Upsilon_i.\end{aligned}$$

If there is an economy in which all asset prices behave as gBms, then an individual can invest in two financial intermediaries (i.e., two mutual funds), which hold all the risky securities. The proportions δ_i and Υ_i are independent of individual preferences, wealth distribution and age distribution.

The optimal combination of risky assets can be determined by comparing the above equations with the classic Tobin-Markowitz framework, where we choose one of the funds to be the risk-free asset (i.e., set $\eta = 1$) and the other fund to hold only risky assets. Therefore, the proportion δ_i are determined by finding the locus of points of composite returns which minimizes the variance for a given mean in the mean-standard deviation space. Thereafter, the δ_i are determined at the point at which the locus is tangent to a line drawn from the origin, representing the risk-free rate r.[5]

Consequently, the composite asset price consisting of m risky assets follows the process

$$dX(t) = \alpha X(t)dt + \sigma X(t)dZ, \tag{3.11}$$

[4]See page 384-388 of Merton (1971) for a detailed derivation of the Equation (3.10).

[5]See Page 388 of Merton (1971) for more details.

where

$$\alpha = \frac{\sum_{k=1}^{m}\sum_{j=1}^{m} v_{kj}(\alpha_j - r)\alpha_k}{\sum_{i=1}^{m}\sum_{j=1}^{m} v_{ij}(\alpha_j - r)},$$

$$\sigma^2 = \sum_{k=1}^{m}\sum_{j=1}^{m} \delta_k \delta_j \sigma_{kj},$$

$$dZ(t) = \sum_{k=1}^{m} \delta_k \sigma_k dZ_k(t)/\sigma,$$

where σ_i is the standard deviation of the i^{th} asset, i.e., $\sqrt{\sigma_{ii}}$.

Using the above price dynamics, the optimization problem is reduced to one mutual fund and one risk-free asset, where the mutual fund represents the combined behaviour of the m risky assets. One of the key advantages of using the separation theorem is that whenever asset prices are assumed to behave as gBms, the optimization problem can be solved by assuming one mutual fund following a gBm process, and one risk-free asset only, without any loss of generality. This further helps in determining the closed-form solution while using specific utility functions.

3.4.3 General Utility Function

As discussed in Section 2.1, HARA is a general utility function which can represent both CARA and CRRA utility with different levels of risk-aversion. Using a HARA utility, it is possible to solve the optimization problem in closed form. This closed-form solution reflects a range of consumer preferences by changing a few parameters in the HARA utility function. Using the HARA utility, the discounted utility $U(C,t)$ can be written as $e^{-\rho t}V(C)$, where[6]

$$V(C) = \frac{(1-\gamma)}{\gamma}\left(\frac{\beta C}{1-\gamma} + \eta\right)^{\gamma},$$

subject to constraints:

$$\gamma \neq 1; \quad \beta > 0; \quad \left(\frac{\beta C}{1-\gamma} + \eta\right) > 0; \quad \eta = 1 \text{ if } \gamma = -\infty.$$

Now, if we assume that the mutual fund follows the dynamics of Equation (3.11), then Equation (3.9) can be simplified to determine the optimal consumption and investment strategy as

[6]Under quadratic utility, the members of HARA family with $\gamma > 1$ are only defined for restricted range of consumption, namely $0 < C < (\gamma - 1)\eta/\beta$.

$$C^*(t) = \frac{\left(\rho - \gamma\Big(r + \frac{(\alpha-r)^2}{2(1-\gamma)\sigma^2}\Big)\right)\left[W(t) + \frac{(1-\gamma)\eta}{\beta r}(1-e^{r(t-T)})\right]}{(1-\gamma)\left(1 - \exp\left[\frac{\rho-\gamma\left(r+\frac{(\alpha-r)^2}{2(1-\gamma)\sigma^2}\right)}{1-\gamma}(t-T)\right]\right)} - \frac{(1-\gamma)\eta}{\beta}, \tag{3.12}$$

and,

$$w^*(t)W(t) = \frac{(\alpha-r)}{(1-\gamma)\sigma^2}W(t) + \frac{\eta(\alpha-r)}{\beta r\sigma^2}(1-e^{r(t-T)}). \tag{3.13}$$

where $w^*(t)$ is the proportion of wealth invested in the mutual fund (i.e., pool of risky assets).

The main feature of Equations (3.12) and (3.13) is that the consumption and investment functions are linear in wealth, i.e., if wealth increases (decreases), so does the consumption, whereas the proportion of optimal investment in risky assets behaves in the other direction, i.e., with an increase (decrease) in wealth the optimal investment in risky asset decreases (increases).

3.4.4 Introduction of Non-Capital Income

An individual might have other sources of income, such as a pension income. Therefore, incorporating exogenous income sources in the model will result in determining a more adequate consumption pattern and investment strategy. In Merton (1971), these exogenous income sources are assumed to be deterministic (i.e., $Y(t) \equiv Y$), for the remaining lifetime of an individual since a stochastic income can lead to complicated results and non-explicit solutions. The stream of future income is discounted using the risk-free rate of return to determine its value today. Thereafter, this value is added to the current level of wealth and the optimal consumption level and investment strategy are determined. These can be represented as

$$C^*(t) = \frac{\left(\rho - \gamma\Big(r + \frac{(\alpha-r)^2}{2(1-\gamma)\sigma^2}\Big)\right)\left[W(t) + \frac{Y(1-e^{r(t-T)})}{r} + \frac{(1-\gamma)\eta}{\beta r}(1-e^{r(t-T)})\right]}{(1-\gamma)\left(1 - \exp\left[\frac{\rho-\gamma\left(r+\frac{(\alpha-r)^2}{2(1-\gamma)\sigma^2}\right)}{1-\gamma}(t-T)\right]\right)} - \frac{(1-\gamma)\eta}{\beta},$$

and,

$$w^*(t)W(t) = \frac{(\alpha-r)}{(1-\gamma)\sigma^2}\left(W(t) + \frac{Y(1-e^{r(t-T)})}{r}\right) + \frac{\eta(\alpha-r)}{\beta r\sigma^2}(1-e^{r(t-T)}).$$

Chapter 4

Optimization Strategy Under Stochastic Force of Mortality

Building upon the optimization framework introduced by Merton (1969) and Merton (1971), Delong and Chen (2016) solve for the optimal consumption and investment problem for a retiree. The latter adjusts her optimal consumption and investment strategy continuously based on the changes in her stochastic force of mortality. In Delong and Chen (2016), a retiree is assumed to possess a certain amount of wealth $W(0) > 0$ when she retires, out of which she invests a certain proportion in a lifetime-income annuity. The remaining wealth is invested in a risky and risk-free asset. The model assumes that there are no other sources of income apart from (1) the asset returns, (2) the annuity income. The retiree has some bequest goals. Moreover, a retiree wants to maintain a certain standard of living when she retires. Consequently, she plans to spend a certain minimum amount of money on her consumption every year during her retirement, which she can adjust upwards if the realized returns on her investments exceed her expectation. Furthermore, the retiree is assumed to check on her health condition and is fully aware of her expected future lifetime. Therefore, she adjusts her investment and consumption strategy according to her stochastic mortality intensity. For example, if the retiree is diagnosed with a particular type of disease, say cancer, and she expects to die soon, then her consumption will rise rapidly as she expects to live for less years and wants to maximize her utility of consumption. On the other hand, if she is cured from a chronic disease, then she will expect to live longer and will have to cut down on her consumption, to afford for the additional years in her expected future lifetime. The retiree's risk preferences are assumed to follow a CRRA utility and she discounts her future income using a non-exponential discounting function (i.e., hyperbolic discounting as explained in Section 2.2). This provides a more realistic value of her future income while determining the optimal investment and consumption strategy, but also leads to complicated optimal solutions which are, therefore, solved using first-order differential HJB equations (see Section 2.6). Finally, the results obtained using the model match the underlying intuition that the retiree increases (decreases) her consumption for an increase (decrease) in her mortality intensity.

4.1 Model

4.1.1 Financial Market

The financial market comprises risky and a risk-free asset, where the rate of return on the risk-free asset is assumed to be r, and the risky asset is assumed to follow a gBm process with drift parameter α, and volatility parameter σ.

4.1.2 Stochastic Mortality

The stochastic mortality intensity $\lambda(t)$ of a retiree is assumed to evolve randomly over time. This means that the retiree's health can worsen or improve unexpectedly during her lifetime. This is modelled using the following gBm process:

$$d\lambda(t) = \mu_\lambda \lambda(t) dt + \sigma_\lambda \lambda(t) dZ_\lambda(t), \tag{4.1}$$

where μ_λ and σ_λ are the drift and volatility parameter of the mortality intensity process, respectively.

The expected future lifetime of the retiree is determined using a conditional survival probability such that

$$\mathbb{P}(\tau \geq t | \mathcal{F}_t^\lambda) = e^{-\int_0^t \lambda(u) du}, \quad 0 \leq t \leq T,$$

where $\mathcal{F}_t^\lambda$ is the filtration which contains the information with respect to the mortality intensity up to time t, τ is the time at which the retiree dies, and T is the maximum lifetime of a retiree assumed at the outset. The process $\lambda(t)$ is assumed to be bounded from above and below, i.e., $\lambda_{\min} \leq \lambda(t) \leq \lambda_{\max}$ for any $0 \leq t \leq T$.[1]

Since the optimal investment and consumption strategy depend upon the stochastic mortality intensity, we introduce a second-order differential operator for the mortality process as

$$\mathcal{L}_\lambda v(t, \lambda) = \frac{\partial}{\partial \lambda} \left(v(t, \lambda) \mu_\lambda \lambda(t) \right) + \frac{1}{2} \frac{\partial^2}{\partial \lambda^2} \left(v(t, \lambda) \sigma_\lambda^2 \lambda(t)^2 \right), \tag{4.2}$$

for some function $v(t, \lambda)$ that is twice-continuously differentiable with respect to λ.

We will use the second-order differential operator of Equation (4.2) in the next section to determine the first-order equilibrium consumption and investment strategy.

[1] The gBm process is not a bounded process, but to determine the mortality intensity process numerically, Delong and Chen (2016) set a sufficiently high value for the upper bound of the mortality process, in an ad-hoc fashion.

4.1.3 Annuity Income

The retiree at the time of retirement (i.e., $t = 0$) is assumed to have an initial level of wealth of $\hat{x} > 0$. We assume that a retiree invest a certain proportion of their initial wealth $\pi\hat{x}$ in a lifetime-income annuity with an annual annuity income of $a(\pi\hat{x})$. This annuity income is determined by an annuity provider after incorporating the retiree's survival probabilities. Therefore, the annuity income $a(\pi\hat{x})$ for a given amount of investment in annuity $\pi\hat{x}$ can be calculated as

$$a(\pi\hat{x}) \int_0^T e^{-rs}\hat{p}(s)ds = \pi\hat{x},$$

where $\hat{p}(s), 0 \leq s \leq T$ denotes the survival probabilities of a retiree aged 65, determined by the insurer. Therefore, the following condition holds:

$$a(\pi\hat{x})\frac{1 - e^{-rT}}{r} > \pi\hat{x} \tag{4.3}$$

because of the presence of survival credits in annuity investment, the discounted value of lifetime-income annuity must be greater than the amount invested in it. In this study, the annuity income $a(\pi\hat{x})$ is determined as the minimum value at which Equation (4.3) is satisfied. The choice of using the risk-free rate of return r as the discount rate is the most reasonable one, as the discount rate cannot be less than r in the economy.

4.1.4 Consumption Function

One of the major goals of the retiree is to maintain a certain standard of living in her retirement. Therefore, she wants to have a minimum consumption of c^* every year during her remaining lifetime. It is interpreted as a habit level of consumption for a retiree, which is generally represented as a function of past consumption amounts, see, for example, Munk (2008). In this report, the minimum consumption amount is assumed to satisfy

$$(1 - \pi)\hat{x} \geq \int_0^T (c^* - a(\pi\hat{x}))e^{-rt}dt,$$

or

$$c^* \leq \frac{(1 - \pi)\hat{x}}{\frac{1-e^{-rT}}{r}} + a(\pi\hat{x}). \tag{4.4}$$

The above condition states that the minimum consumption c^* cannot exceed the combined value of the annuity income and the risk-free income from remaining wealth at time 0, i.e., $(1 - \pi)\hat{x}$, at retirement. In other words, the minimum consumption amount is determined assuming that the retiree will earn at most the risk-free rate of return on all of her investments along with the annuity income during her remaining lifetime.

One of the variables among π and c^* is assumed known in the model at the outset, and the other one is determined using Equation (4.4). If the inequality (4.4) is not satisfied, the retiree will not be able to meet the minimum consumption amounts and, therefore, the optimization problem cannot be solved.

After determining the minimum consumption amount, the retiree wishes to maximize her consumption. Consequently, she invests her remaining wealth in the financial market and increases her consumption according to the excess returns realized from risky assets. Therefore, the consumption function c at time t can be defined as

$$c(t) = c^* + u(t), \quad 0 \leq t \leq T, \tag{4.5}$$

where $u(t)$ is defined as the excess of the optimal realized consumption over minimum consumption at time t.

4.2 Optimization Problem

The wealth process of the retiree an instant after t (i.e., $t + \Delta t$, where Δt is very small) depends upon her optimal investment and consumption decision at time t. The retiree invests a proportion $w(t)$ of her wealth in risky assets and a proportion $1 - w(t)$ in the risk-free asset with rate of return r. Then, she receives a certain annuity income $a(\pi\hat{x})$ at time t, and spends an amount of $c^* + u(t)$ on her consumption at time t. Therefore, the change in the wealth of a retiree, for a very short time period at time t, can be described as

$$\begin{aligned} dW^{w,u}(t) &= w(t)W^{w,u}(t)(\alpha dt + \sigma dZ(t)) + (1 - w(t))W^{w,u}(t)rdt \\ &\quad - (c^* + u(t))dt + a(\pi\hat{x})dt, \quad 0 \leq t \leq T, \end{aligned}$$

where $W^{w,u}(t)$ is the controlled wealth of the retiree with respect to their optimal consumption u, and the optimal investment strategy w at time t. The initial wealth $W^{w,u}(0) = (1 - \pi)\hat{x}$.

The retiree wants to maximize her lifetime consumption of wealth and have some bequest goals. Therefore, the optimization problem for the retiree at retirement date $t = 0$ that has a CRRA utility function with a coefficient of risk aversion $1 - \gamma$ can be described as

$$\sup_{\{w(t),u(t)\}} \mathbb{E}\left[\int_0^T \phi(t)(u(t))^\gamma dt + q\phi(T)(W^{w,u}(T))^\gamma\right], \tag{4.6}$$

where q is the weight assigned by the retiree to her bequest. The function $\phi(t)$ is the hyperbolic discount function used by a retiree to value her future income (see Section 2.2).

First, we define the value function for the optimization problem as

$$v^{w,u}(t, W(t), \lambda(t)) = \mathbb{E}\left[\int_t^T \phi(s - t)(u(s))^\gamma ds + q\phi(T - t)(W^{w,u}(T))^\gamma\right], \tag{4.7}$$

for $(t, W, \lambda) \in [0, T] \times \mathbb{R} \times \mathcal{K}$, where $\mathcal{K}$ is the range of mortality intensity λ, i.e., $\mathcal{K}=[\lambda_{\min}, \lambda_{\max}]$. Moreover, the optimal value function is

$$v(t, W(t), \lambda(t)) = v^{w^*,u^*}(t, W(t), \lambda(t)) = \sup_{\{w(t),u(t)\}} v^{w,u}(t, W(t), \lambda(t)), \tag{4.8}$$

for $(t, W, \lambda) \in [0, T] \times \mathbb{R} \times \mathcal{K}$.

The value function of Equation (4.7) can be solved using stochastic dynamic programming if the exponential discounting is used to value the future income. However, Delong and Chen (2016) use non-exponential discounting which results in obtaining time-inconsistent optimization strategies, thus preventing the use of stochastic dynamic programming techniques. This is because the optimization strategy derived at time t_1 for the period $[t_1, T]$ is different from the strategy derived at time t_2 for the period $[t_2, T]$ due to a change in an individual's risk preferences and their discount factor.

Ekeland and Lazrak (2006), Ekeland and Pirvu (2008), and Björk and Murgoci (2014) propose a game-theoretic approach to solve these time-inconsistent optimization strategies. The rationale is the following: each player at time t can freely choose a strategy to control their wealth at that time and pass the wealth to the next player at time $t + \Delta t$, where Δt is a small positive number. The next player has different risk preferences and will choose another investment and consumption strategy. So the idea is that the player at time t cannot force the players ahead to choose a particular strategy and, therefore, will be better off by considering other player's optimization strategies in advance. If the player at time t knows that all players ahead will choose an equilibrium strategy (w^*, u^*), then it is optimal for this player to choose that strategy as well. Thus, using Björk and Murgoci (2014), and Ekeland et al. (2012), the optimal value function of Equation (4.8) should satisfy the following non-local HJB equation:

$$\begin{aligned} &\frac{d}{dt} v(t, W(t), \lambda(t)) + \sup_{w(t),u(t)} \Big[u(t)^\gamma + \mathcal{L}_W^{w(t),u(t)} v(t, W(t), \lambda(t))\Big] \\ &+ \mathcal{L}_\lambda v(t, W(t), \lambda(t)) - \lambda(t) v(t, W(t), \lambda(t)) \\ &= -\mathbb{E}\Big[\int_t^T \phi'(s-t)(u^*(s))^\gamma ds + q\phi'(T-t)(W^{w^*,u^*}(T))^\gamma | W(t) = W, \lambda(t) = \lambda\Big], \end{aligned} \tag{4.9}$$

with the following terminal condition:

$$v(T, W(T), \lambda(T)) = q(W(T))^\gamma,$$

for $(t, W(t), \lambda(t)) \in [0, T) \times \mathbb{R} \times \mathcal{K}$, where

$$(w^*, u^*) = \arg \sup_{w(t),u(t)} \Big(u(t)^\gamma + \mathcal{L}_W^{w(t),u(t)} v(t, W(t), \lambda(t))\Big). \tag{4.10}$$

Function $\mathcal{L}_\lambda v(t, W(t), \lambda(t))$ is the second-order differential operator of the mortality intensity process defined in Equation (4.2). The second-order differential operator for the wealth process, $\mathcal{L}_W v(t, W(t), \lambda(t))$ is given as

$$
\begin{aligned}
\mathcal{L}_W v(t, W) = & \frac{\partial}{\partial W} v(t, W)\Big(w(t)W(t)(\alpha - r) + W(t)r + a(\pi\hat{x}) - c^* - u(t)\Big) \\
& + \frac{1}{2}\frac{\partial^2}{\partial W^2} v(t, W) w(t)^2 W(t)^2 \sigma^2.
\end{aligned} \tag{4.11}
$$

Moreover, the function $\phi(t)$ is the first-order approximation of hyperbolic discounting function

$$
\phi(t) = e^{-\rho t}\frac{1}{(1+\delta t)^\epsilon}, \quad 0 \le t \le T,
$$

such that

$$
\phi(t) \approx e^{-\rho t}\Big(1 - \log(1+\delta t)\epsilon\Big) + \mathcal{O}(\epsilon^2), \tag{4.12}
$$

or

$$
\phi(t) \approx e^{-\rho t}\Big(1 + V(t)\epsilon\Big) + \mathcal{O}(\epsilon^2),
$$

where $V(t) = -\log(1+\delta t)$.

4.2.1 Solving the Optimization Problem

Solving the HJB equation of Equation (4.9) is quite complicated and a direct solution under the non-exponential discounting factor is not possible. Dong and Sircar (2014) suggest the expansion method, which allows determining the approximation of the first-order equilibrium value function as well as the first-order equilibrium consumption and investment strategy. The method provides a better approximation when the hyperbolic discounting factor is closer to the exponential discounting factor, or when ϵ tends to 0 in Equation (4.12). Therefore, the first-order approximation of the equilibrium consumption $\tilde{u}^*(t, W, \lambda)$ and equilibrium investment $\tilde{w}^*(t, W, \lambda)$ strategies are as follows:[2]

$$
\tilde{w}^*(t, W, \lambda) = \frac{\alpha - r}{\sigma^2(1-\gamma)}\frac{W + g(t)}{W}, \tag{4.13}
$$

for $(t, W, \lambda) \in [0, T] \times \mathbb{R} \times \mathcal{K}$, and

$$
\tilde{u}^*(t, W, \lambda) = (W + g(t))(f(t,\lambda))^{\frac{1}{\gamma-1}}\left(1 - \frac{F(t,\lambda)}{f(t,\lambda)(1-\gamma)}\epsilon\right), \tag{4.14}
$$

for $(t, W, \lambda) \in [0, T] \times \mathbb{R} \times \mathcal{K}$.

The functions $f(t, \lambda)$, $F(t, \lambda)$ and $g(t)$ are obtained from solving the following partial dif-

[2]See Delong and Chen (2016) for the detailed steps to obtain these equilibrium strategies.

ferential equations (PDEs):

$$
\begin{aligned}
&\frac{\partial}{\partial t} f(t,\lambda) + \mathcal{L}_\lambda f(t,\lambda) - \left(\lambda + \rho - r\gamma - \frac{1}{2}\frac{\gamma}{1-\gamma}\frac{(\alpha - r)^2}{\sigma^2}\right) f(t,\lambda) \\
&+ (1-\gamma)(f(t,\lambda))^{\frac{\gamma}{\gamma-1}} = 0,
\end{aligned} \tag{4.15}
$$

with the terminal condition $f(T,\lambda) = q$, and $(t,\lambda) \in [0,T) \times \mathcal{K}$. Specifically,

$$
\begin{aligned}
g(t) &= \int_t^T (a(\pi\hat{x}) - c^*)e^{-r(s-t)} ds \\
&= (a(\pi\hat{x}) - c^*)\frac{1 - e^{-r(T-t)}}{r}, \quad 0 \le t \le T.
\end{aligned} \tag{4.16}
$$

Then, the function $F(t,\lambda)$ is the solution of the following PDE:

$$
\begin{aligned}
&\frac{\partial}{\partial t} F(t,\lambda) + \mathcal{L}_\lambda F(t,\lambda) - \left(\lambda + \rho - r\gamma - \frac{1}{2}\frac{\gamma}{1-\gamma}\frac{(\alpha - r)^2}{\sigma^2} + \gamma(f(t,\lambda))^{\frac{1}{\gamma-1}}\right) F(t,\lambda) \\
&+ Q(t,\lambda) = 0,
\end{aligned} \tag{4.17}
$$

with the terminal condition $F(T,\lambda) = 0$, and $(t,\lambda) \in [0,T) \times \mathcal{K}$, where

$$
Q(t,\lambda) = \int_t^T \frac{\partial}{\partial t} V(s-t) P^s(t,\lambda) ds + q^{\frac{1}{1-\gamma}} \frac{\partial}{\partial t} V(T-t) P^T(t,\lambda), \tag{4.18}
$$

and the function $P^s(t,\lambda)$ is the unique solution to PDE

$$
\frac{\partial}{\partial t} P^s(t,\lambda) + \mathcal{L}_\lambda P^s(t,\lambda) - \left(\lambda + \rho - r\gamma - \frac{1}{2}\frac{\gamma}{1-\gamma}\frac{(\alpha - r)^2}{\sigma^2} + \gamma(f(t,\lambda))^{\frac{1}{\gamma-1}}\right) P^s(t,\lambda) = 0, \tag{4.19}
$$

with the terminal condition $P^s(s,\lambda) = (f(s,\lambda))^{\frac{\gamma}{\gamma-1}}$, and $(t,\lambda) \in [0,s) \times \mathcal{K}$.

4.2.2 Methods Used to Solve the Optimization Problem

The mortality parameters of Equation (4.1) are determined using the calibration method, e.g., Huang et al. (2012) and Shen and Wei (2016). First, we consider the 10, 20 and 30 years survival probabilities for a male aged 65 using the 2014 Polish life table. Then, we fit a deterministic Gompertz mortality distribution to these survival probabilities to determine the initial mortality intensity (i.e., $\lambda(0) = 0.0215$) and the expected future lifetime of the retiree (15.86 years). Thereafter, we choose the value of σ_λ to be 0.15, as suggested by Huang et al. (2012), and consequently, we determine the equivalent value of the drift parameter, $\mu_\lambda = 0.096$ by equating the stochastic and deterministic mortality to be equal at time 0.

We solve the PDEs of Equations (4.15), (4.17), (4.18) and (4.19) using a combination of explicit and implicit finite difference methods. The methods allow us to determine the value of the function at each time step, starting from time T, recursively for each λ. However, to solve the PDEs, we need to know the upper bound of the mortality intensity process $\lambda_{\max}$. Therefore, we set $\lambda_{\max}$ to be high enough and assume the mortality intensity to be bounded (i.e., $\lambda_{\max} = 2$). For instance, in the case of solving $f(t, \lambda)$, we follow the following steps:

1. Create an $m \times n$ matrix with m being the number of mortality intensity steps from $\lambda_{\min}$ to $\lambda_{\max}$, and n is the number of time steps from time 0 to T, with the difference between each time being Δt.

2. Determine the optimal size of time step Δt, and the optimal mortality intensity step size $\Delta\lambda$, so that efficient results can be derived. Setting Δt and $\Delta\lambda$ too high might results in non-efficient results. On the other hand, setting Δt and $\Delta\lambda$ too small will lead to excessive computing time. We set the number of mortality intensity steps (NAS) to be 200, which in turn leads to a $\Delta\lambda$ of 0.00995, and determine the value of Δt using the following condition

$$\Delta t < \frac{\sigma_\lambda^2}{\text{NAS}^2}, \tag{4.20}$$

   Therefore, we set

$$\Delta t = 0.99 \frac{\sigma_\lambda^2}{\text{NAS}^2},$$

   The condition of Equation (4.20) in necessary to maintain stability in the grid, and to execute explicit and implicit finite methods efficiently.

3. After determining the right dimensions, we determine the value of the function for each λ at the time step $T - \Delta t$ using the terminal value of the function at time T, i.e., $f(T, \lambda)$.

4. We follow the above step recursively to determine the value of the function at each time step for each λ until time 0.

The other PDEs are solved in a similar fashion.

4.3 Results

We derive the numerical results using the data presented in Table 4.1. Since the mortality process is not a bounded process, we set $\lambda_{\max}$ to be sufficiently high at 2. For computational ease, we choose $\lambda_{\min}$ to be equal to 0.01. We assume that a retiree invests 50% of her initial wealth in a lifetime annuity product, receiving a lifetime annuity income determined using Equation (4.3). Consequently, the minimum consumption amount is determined as the maximum amount of c^* satisfying the Equation (4.4).

The probability that the retiree aged 65 survives for 35 years is estimated to be 1.7×10^{-6}, using the 2014 Polish life table. We, therefore, choose the maximum lifetime of a retiree T

Table 4.1: **Input Parameters.**

Parameter	Value	Parameter	Value
α	0.05	μ_λ	0.096
σ	0.25	σ_λ	0.15
r	0.02	δ	2
γ	0.1	q	10
T	35	$\hat{x}$	100
ρ	0.035	π	0.5
$\lambda_{\min}$	0.01	$\lambda_{\max}$	2
ϵ	0.05		

to be 35 years. According to Delong and Chen (2016), the optimal choice for a retiree is to invest all her wealth in a lifetime annuity income and, thereafter, determine the optimal consumption and investment strategy. However, we assume that most of the retirees are willing to invest some portion of their wealth in the financial market to increase their future consumption. Therefore, we assume that the retiree invests 50% of her wealth in the financial market and 50% in lifetime-income annuity. Since we are using the 2014 Polish mortality table, we use the average risk-free rate of return in Poland during the last 10 years, i.e., $r = 0.02$. This is consistent with most of the developed economies around the world.

4.3.1 Optimal Consumption Strategy Under Fixed Wealth

First, we determine solely the impact of stochastic mortality intensity on the equilibrium consumption strategy of Equation (4.14). Therefore, we assume that the wealth amount remains fixed at 1 for each time t, i.e., $W + g(t) = 1$ in Equation (4.14). By doing this, we focus only on the longevity and mortality risk of the retiree.

In Figure 4.1, as one can notice, the amount of consumption increases with an increase in the mortality intensity of the retiree. This is consistent with our intuition that a retiree will consume more of her wealth if she expects to die soon. However, this relationship is observed only for shorter times because the retiree is not concerned about her bequest goals during the initial years as the probability of surviving until the end is negligible. As the retiree approaches her maximal lifetime, T, the stochastic mortality intensity does not have a huge impact on her equilibrium consumption strategy.

Next, we determine the longevity risk of the retiree by varying her stochastic mortality volatility (i.e., σ_λ). We analyze the impact of increases in σ_λ from 0.15 to 0.30 for two specific initial mortality intensities, i.e., $\lambda(0) = 0.03$ and $\lambda(0) = 0.126$ in Figure 4.2. With an increase in the mortality intensity volatility, the retiree encounters higher uncertainty concerning her expected future lifetime. Therefore, she consumes less of her wealth to have enough funds to sustain herself if she lives more than her initial expected lifetime. The impact of increasing parameter σ_λ is not significant when we consider mortality intensity to be as low as 0.03. However, when mortality intensity is 0.126, the retiree reduces her consumption by about 10% every year for her remaining lifetime.

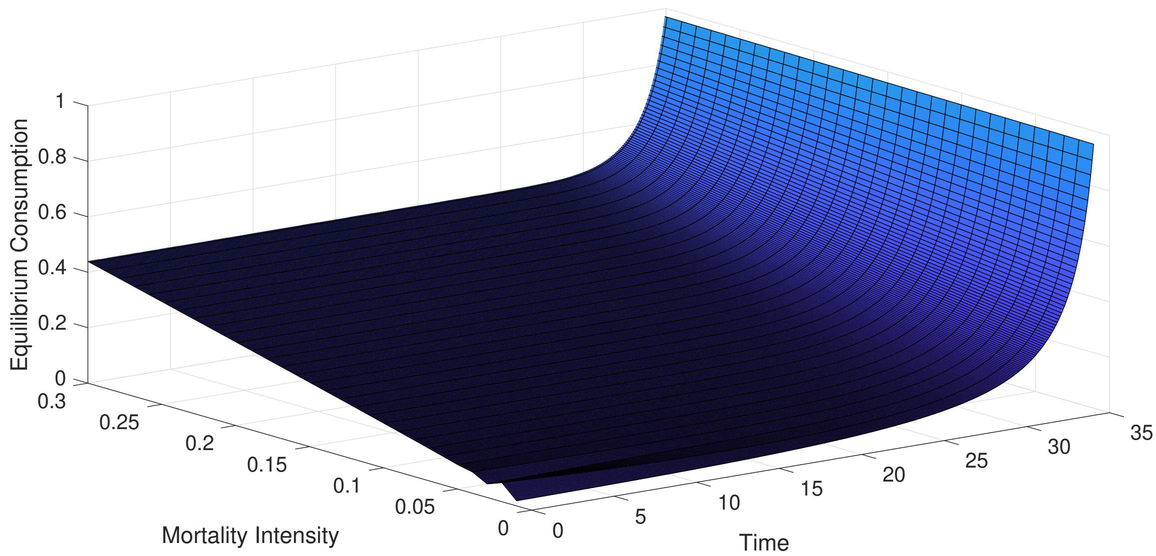

Figure 4.1: **Optimal Consumption Strategy Under the Geometric Brownian Motion Mortality Intensity Model With a Fixed Amount of Wealth 1.**
This figure shows the equilibrium consumption amount above the minimum consumption amount for a varying mortality intensity following the gBm process of Equation (4.1). Here, we assume that the amount of wealth including annuity-income at time t, i.e., $W + g(t)$ in Equation (4.14) is equal to 1 at each time t.

At last, we determine the impact of the hyperbolic discounting parameter ϵ in the non-exponential discounting function of Equation (2.3) on the optimal consumption strategy of Equation (4.14). We compare the optimal consumption strategy when $\epsilon = 0$ (i.e., exponential distribution), $\epsilon = 0.05$, and $\epsilon = 0.1$ for an initial mortality intensity of $\lambda(0) = 0.03$ and $\lambda(0) = 0.126$ in Figure 4.3. As ϵ increases, the retiree discounts back her future income at a higher rate and wants to consume more now since she is not willing to postpone her consumption. This increases her consumption during each period of time. As we can see in Figure 4.3, the retiree spends more on her consumption as ϵ increases. Similar to the previous case, the impact is much higher when $\lambda(0) = 0.126$ than when $\lambda(0) = 0.03$.

4.3.2 Optimal Consumption, Investment and Wealth Process

In the previous subsection, we focused on determining the optimal consumption strategy with a fixed level of wealth at each time t. In this subsection, we determine how optimal consumption, investment, and wealth processes evolve without any conditions on the wealth process. We performed 10,000 simulations on the wealth process to determine the optimal consumption, investment and wealth processes. Thereafter, we perform a sensitivity analysis to determine the impact of a change in the main assumptions of this study on the overall consumption, investment and wealth of the retiree.

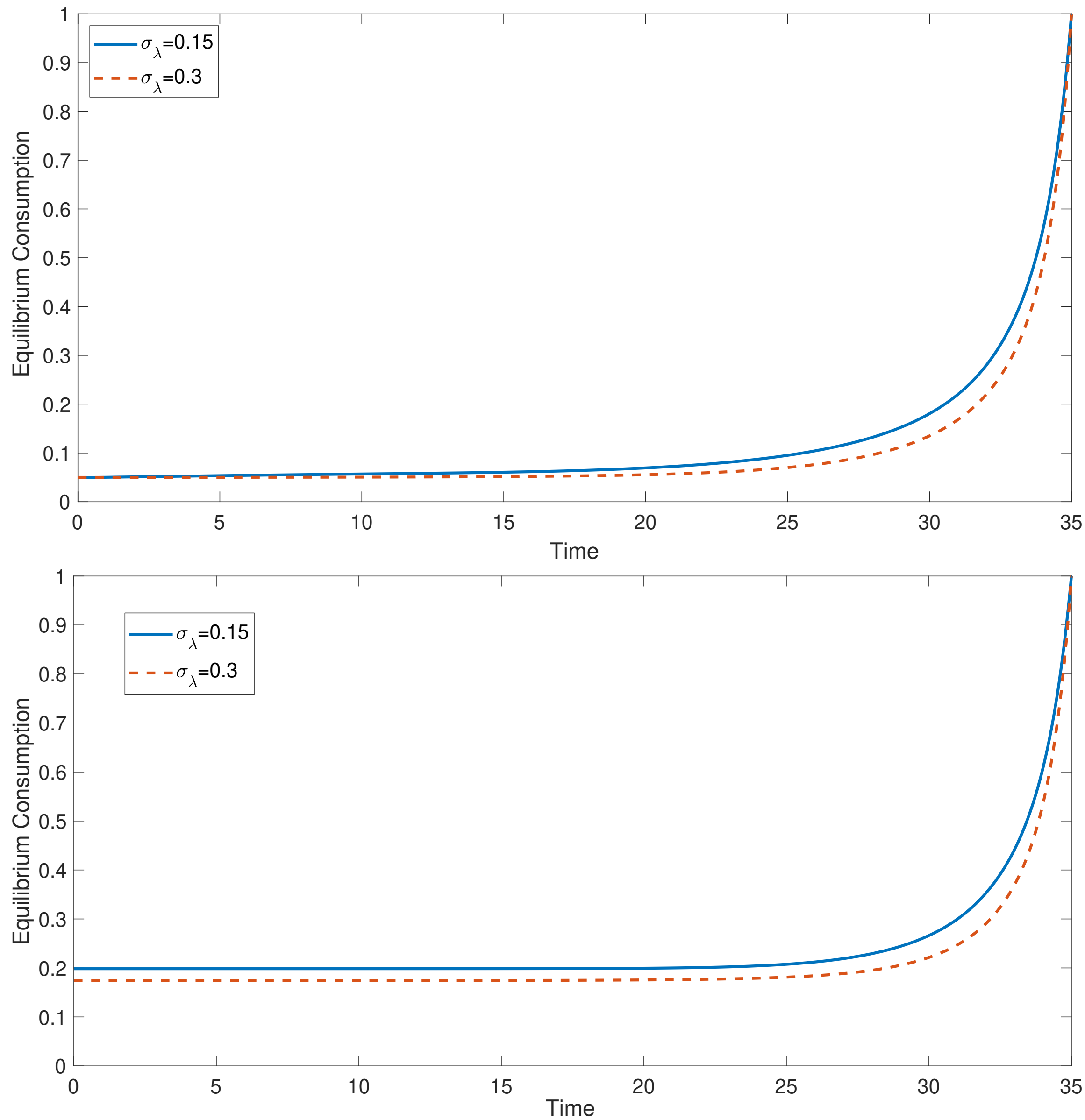

Figure 4.2: **Optimal Consumption Strategy For Varying σ_λ Under the Geometric Brownian Motion Mortality Intensity Model.**
These figures show the equilibrium consumption amount for a selected mortality intensities of $\lambda(0) = 0.03$ (top panel) and $\lambda(0) = 0.126$ (bottom panel) with two different mortality intensity volatility parameters, i.e., $\sigma_\lambda = 0.15$ (in blue) or $\sigma_\lambda = 0.3$ (in red) for a fixed amount of wealth (i.e., 1) at each time t. The mortality intensity process is assumed to follow the gBm model of Equation (4.1).

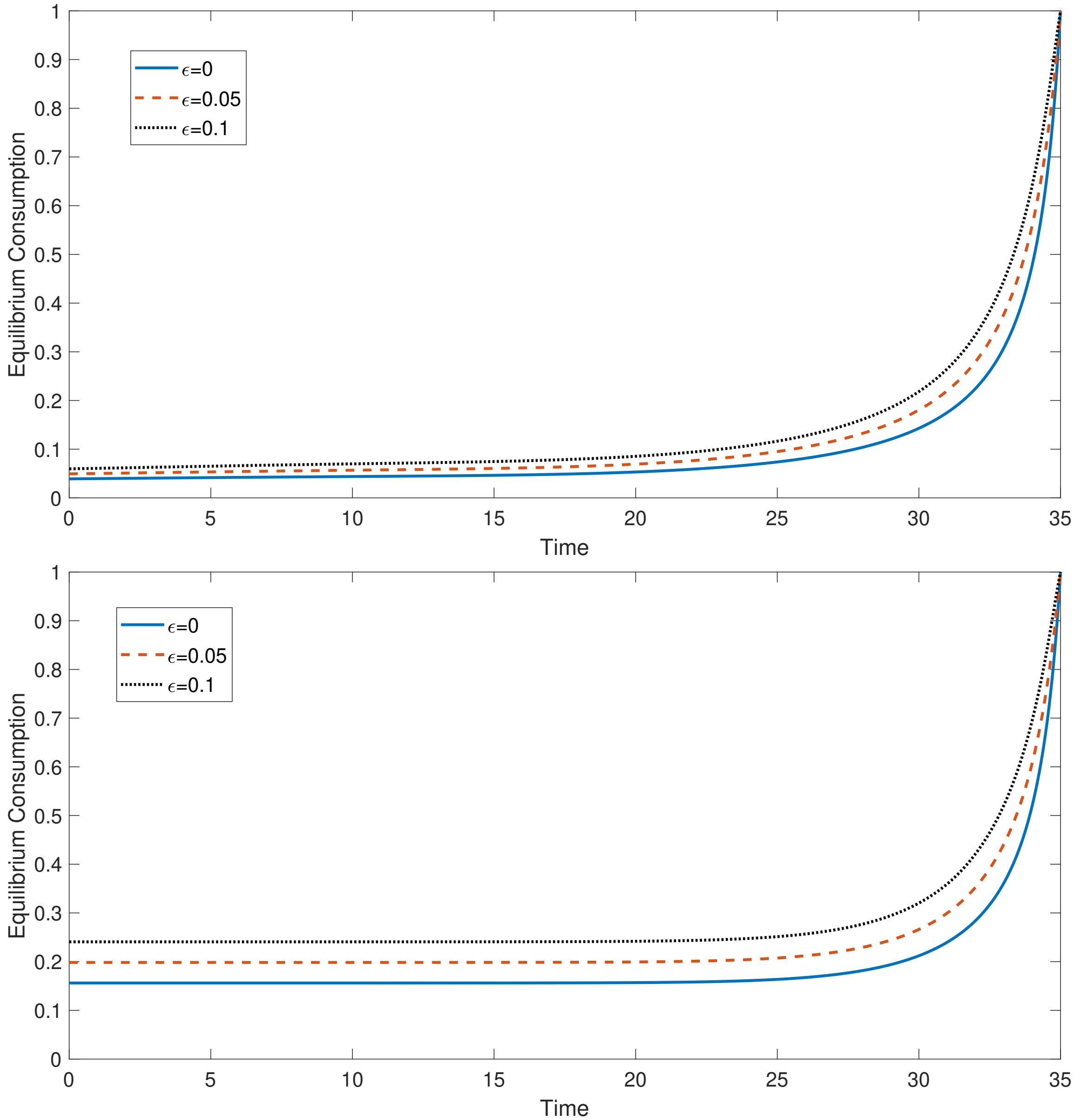

Figure 4.3: **Optimal Consumption Strategy For Varying ϵ Under the Geometric Brownian Motion Mortality Intensity Model.**

These figures show the equilibrium consumption amount for a selected mortality intensities of $\lambda(0) = 0.03$ (top panel) and $\lambda(0) = 0.126$ (bottom panel) for three different values of hyperbolic discounting parameter ϵ. The solid line (in blue) shows the case when $\epsilon = 0$, the dashed line (in red) shows the case when $\epsilon = 0.05$, and the dotted line (in black) shows the case when $\epsilon = 0.1$. The mortality intensity process is assumed to follow the gBm model of Equation (4.1).

First, we increase the asset return volatility parameter σ from 0.25 to 0.4 in Figure 4.4. Accordingly, the uncertainty with respect to the asset return increases. Therefore, the retiree reduces her proportion of investment in the risky asset by almost 50% of the initial proportion of investment. As a result, the retiree's optimal consumption declines by 5% to 10% since the amount of consumption above the minimum consumption level is realized from the risky asset return only. On the other hand, the wealth process remains somewhat insensitive to a change in the market volatility because the retiree consumes less if she invests less in the risky asset. The effect on the wealth process is less than 1% with an increase in the market volatility from 0.25 to 0.4.

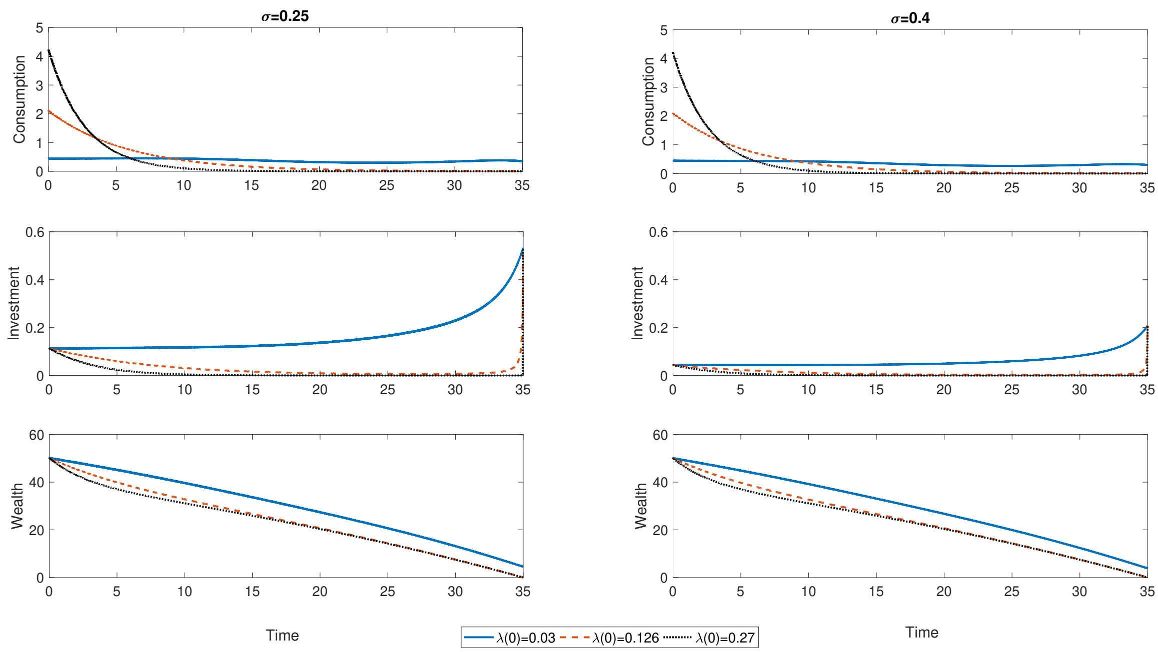

Figure 4.4: **Optimal Consumption, Investment, and Wealth of the Retiree Under the Geometric Brownian Motion Mortality When the Market Volatility Increases from 0.25 to 0.4.**

This figure shows the impact of an increase in asset return volatility σ from 0.25 (left panels) to 0.4 (right panels) on the retiree's equilibrium consumption, investment and wealth. The solid line (in blue) represents the fixed mortality intensity $\lambda(0) = 0.03$, the dashed line (in red) is the case when $\lambda(0) = 0.126$, and the dotted line (in black) is the case when $\lambda(0) = 0.27$. The mortality intensity process is assumed to follow the gBm model of Equation (4.1).

Next, in Figure 4.5 we analyze the behaviour of optimal consumption, investment, and wealth of the retiree with a change in the risk-aversion parameter γ. With an increase in the γ parameter from 0.1 to 0.2, the retiree's coefficient of relative risk aversion decreases from 0.9 to 0.8. Consequently, the retiree becomes less risk-averse towards her consumption and bequest goals, and therefore, consumes more during the initial periods. This, in turn, leads to a higher proportion of wealth being allocated towards the risky asset. As one can see in Figure 4.5 and Tables 4.2 and 4.3, the equilibrium consumption amount increases by about 10% during the initial period (i.e., first 15 years), and by about 1% to 2% in later years. This is because the retiree is not concerned about her bequest goal during the initial periods. Correspondingly, the retiree increases her investment in the risky asset by 1% to 5% on average to increase her future consumption. Similar to the previous case, the wealth process remains insensitive due to the offsetting effects of excess consumption and excess returns.

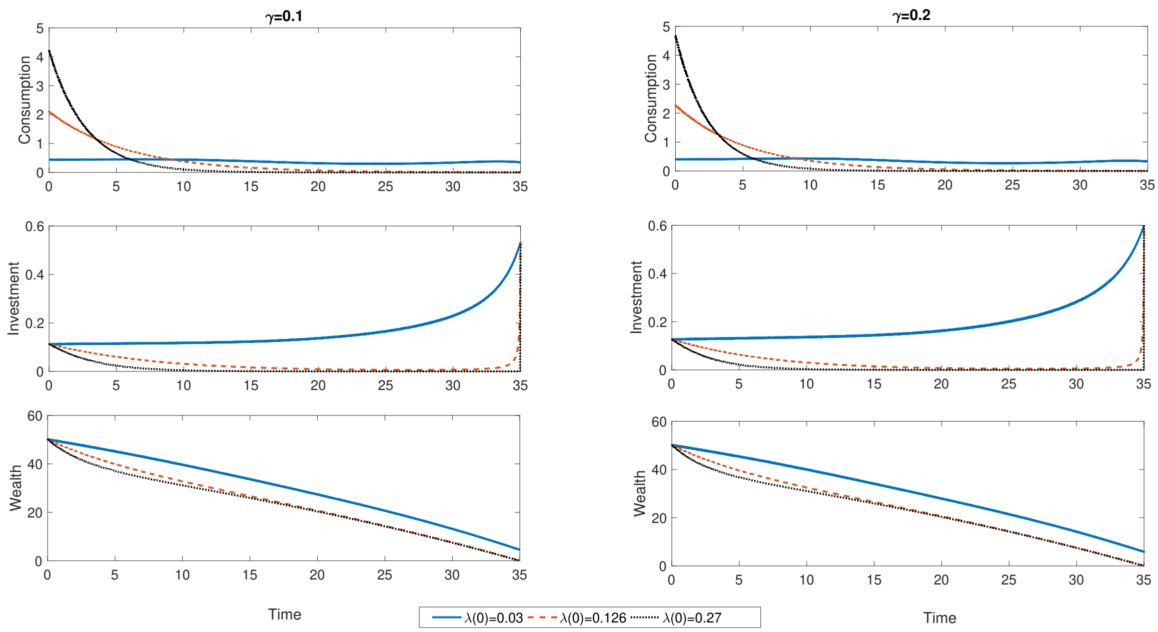

Figure 4.5: **Optimal Consumption, Investment, and Wealth of the Retiree Under the Geometric Brownian Motion Mortality When the Risk-Aversion Increases from 0.1 to 0.2.**
This figure shows the impact of an increase in the risk-aversion parameter γ from 0.1 (left panels) to 0.2 (right panels) on the retiree's equilibrium consumption, investment and wealth. The solidline (in blue) represents the fixed mortality intensity $\lambda(0) = 0.03$, the dashed line (in red) is for the case when $\lambda(0) = 0.126$, and the dotted line (in black) is for the case when $\lambda(0) = 0.27$. The mortality intensity process is assumed to follow the gBm model of Equation (4.1).

Finally, we investigate the impact of an assumption change on the optimal consumption and investment strategy of the retiree in Tables 4.2 and 4.3. The base cases uses the assumptions set of Table 4.1.

For the case when retiree's initial wealth $\hat{x}$ is increased from 100 to 200, the optimal amount of consumption also doubles, and the proportion of investment in the risky asset remains the same due to the fact that the retiree has a CRRA utility function, which states that the investment decision is independent of the amount of wealth. On the other hand, when the weight assigned to the bequest q increases from 10 to 20, the retiree invests more in the risky asset to achieve the bequest goal. This increase in her investment strategy is more apparent after 10 years when the bequest motive kicks in. The optimal consumption amount remains similar to the case when $q = 10$, reflecting the fact that the risky asset's excess returns are used toward meeting the excess of bequest goal. An increase in the risk free rate r from 0.02 to 0.04, motivates the retiree to allocate a higher proportion of her wealth towards risk-free assets. This results in lowering down her proportion of investment in the risky asset to almost one-third of her base case investment proportion. However, the amount of consumption remains similar to the base case during most of the time. This can be explained by a lower absolute average return for the risky asset, due to lower proportion of wealth invested, and at the same time an increase in the risk-free asset return, which indeed offsets each other, to some extent.

If the retiree plans to cut down her investment in a lifetime-income annuity (i.e., parameter π) from 50% to 25% of her initial wealth, she have to find a way to secure her minimum consumption (i.e., c^*) for her remaining expected lifetime. Therefore, she adjusts her optimal investment strategy by investing a lower proportion of her wealth in the risky asset, and more towards the risk-free asset. This reduces her optimal consumption strategy by around 1% to 5%.

Table 4.2: **Optimal Consumption Strategy When the Stochastic Mortality Intensity Follows a Geometric Brownian Motion.**

Panel A: $\boldsymbol{\lambda(0) = 0.03}$.				
Time	0	10	20	30
Base Case	0.44	0.44	0.32	0.34
Modified Cases				
$\pi = 0.25$	0.43	0.43	0.31	0.33
$\sigma = 0.4$	0.44	0.42	0.29	0.30
$\alpha = 0.08$	0.42	0.56	0.50	0.69
$\hat{x} = 200$	0.88	0.88	0.64	0.68
$q = 20$	0.37	0.41	0.26	0.26
$r = 0.04$	0.43	0.50	0.42	0.52
$\gamma = 0.2$	0.39	0.43	0.30	0.31
$\rho = 0.02$	0.33	0.40	0.32	0.40
$\delta = 1$	0.41	0.42	0.31	0.34
$\delta = 3$	0.47	0.46	0.33	0.33
$\epsilon = 0$	0.36	0.38	0.30	0.36
$\epsilon = 0.1$	0.52	0.49	0.33	0.30
$\sigma_\lambda = 0.3$	0.52	0.41	0.28	0.26
Panel B: $\boldsymbol{\lambda(0) = 0.126}$.				
Time	0	10	20	30
Base Case	2.08	0.38	0.07	0.01
Modified Cases				
$\pi = 0.25$	2.03	0.37	0.07	0.01
$\sigma = 0.4$	2.09	0.36	0.06	0.01
$\alpha = 0.08$	2.06	0.49	0.11	0.02
$\hat{x} = 200$	4.17	0.75	0.13	0.02
$q = 20$	2.08	0.38	0.07	0.01
$r = 0.04$	2.07	0.43	0.09	0.02
$\gamma = 0.2$	2.24	0.35	0.05	0.01
$\rho = 0.02$	1.94	0.41	0.08	0.02
$\delta = 1$	1.89	0.41	0.09	0.02
$\delta = 3$	2.27	0.35	0.05	0.01
$\epsilon = 0$	1.64	0.45	0.12	0.03
$\epsilon = 0.1$	2.53	0.30	0.04	0.00
$\sigma_\lambda = 0.3$	1.83	0.42	0.10	0.02
Panel C: $\boldsymbol{\lambda(0) = 0.27}$.				
Time	0	10	20	30
Base Case	4.18	0.10	0.00	0.00
Modified Cases				
$\pi = 0.25$	4.08	0.10	0.00	0.00
$\sigma = 0.4$	4.19	0.10	0.00	0.00
$\alpha = 0.08$	4.15	0.14	0.00	0.00
$\hat{x} = 200$	8.36	0.21	0.01	0.00
$q = 20$	4.18	0.10	0.00	0.00
$r = 0.04$	4.17	0.12	0.00	0.00
$\gamma = 0.2$	4.63	0.07	0.00	0.00
$\rho = 0.02$	4.01	0.12	0.00	0.00
$\delta = 1$	3.80	0.13	0.00	0.00
$\delta = 3$	4.56	0.08	0.00	0.00
$\epsilon = 0$	3.35	0.18	0.01	0.00
$\epsilon = 0.1$	5.01	0.06	0.00	0.00
$\sigma_\lambda = 0.3$	3.69	0.14	0.01	0.00

This table represents the sensitivity of the optimal consumption strategy when the input variables are changed. Panel A shows the sensitivity for a fixed initial mortality intensity of 0.03, whereas Panel B and Panel C showcase sensitivity for mortality intensity of 0.126 and 0.27, respectively. The mortality intensity process is assumed to follow a gBm process.

Table 4.3: **Optimal Investment in the Risky Asset When the Stochastic Mortality Intensity Follows a Geometric Brownian Motion.**

Panel A: $\lambda(0) = 0.03$.

Time	0	10	20	30
Base Case	11.20%	11.80%	13.69%	22.90%
Modified Cases				
$\pi = 0.25$	7.29%	7.72%	9.06%	16.59%
$\sigma = 0.4$	4.38%	4.42%	4.96%	8.28%
$\alpha = 0.08$	22.40%	28.41%	37.61%	63.93%
$\hat{x} = 200$	11.20%	11.70%	13.69%	22.99%
$q = 20$	11.20%	12.29%	15.34%	27.51%
$r = 0.04$	3.73%	4.12%	5.03%	8.47%
$\gamma = 0.2$	12.60%	13.49%	16.32%	28.14%
$\rho = 0.02$	11.20%	12.71%	15.61%	26.09%
$\delta = 1$	11.20%	11.99%	14.55%	24.57%
$\delta = 3$	11.20%	11.43%	12.98%	21.36%
$\epsilon = 0$	11.20%	12.53%	15.97%	27.29%
$\epsilon = 0.1$	11.20%	10.90%	11.72%	18.76%
$\sigma_\lambda = 0.3$	11.20%	11.44%	13.55%	24.38%

Panel B: $\lambda(0) = 0.126$.

Time	0	10	20	30
Base Case	11.20%	3.13%	0.91%	0.69%
Modified Cases				
$\pi = 0.25$	7.29%	1.91%	0.54%	0.42%
$\sigma = 0.4$	4.38%	1.16%	0.32%	0.24%
$\alpha = 0.08$	22.40%	8.00%	2.95%	2.88%
$\hat{x} = 200$	11.20%	3.10%	0.91%	0.70%
$q = 20$	11.20%	3.10%	0.93%	0.98%
$r = 0.04$	3.73%	1.10%	0.35%	0.28%
$\gamma = 0.2$	12.60%	3.00%	0.77%	0.57%
$\rho = 0.02$	11.20%	3.56%	1.20%	1.04%
$\delta = 1$	11.20%	3.66%	1.31%	1.14%
$\delta = 3$	11.20%	2.63%	0.65%	0.43%
$\epsilon = 0$	11.20%	4.57%	2.08%	2.21%
$\epsilon = 0.1$	11.20%	2.08%	0.40%	0.21%
$\sigma_\lambda = 0.3$	11.20%	3.90%	1.45%	1.29%

Panel C: $\lambda(0) = 0.27$.

Time	0	10	20	30
Base Case	11.20%	0.45%	0.02%	0.00%
Modified Cases				
$\pi = 0.25$	7.29%	0.27%	0.01%	0.00%
$\sigma = 0.4$	4.38%	0.16%	0.01%	0.00%
$\alpha = 0.08$	22.40%	1.17%	0.06%	0.01%
$\hat{x} = 200$	11.20%	0.44%	0.02%	0.00%
$q = 20$	11.20%	0.44%	0.02%	0.00%
$r = 0.04$	3.73%	0.16%	0.01%	0.00%
$\gamma = 0.2$	12.60%	0.32%	0.01%	0.00%
$\rho = 0.02$	11.20%	0.52%	0.02%	0.00%
$\delta = 1$	11.20%	0.63%	0.03%	0.00%
$\delta = 3$	11.20%	0.31%	0.01%	0.00%
$\epsilon = 0$	11.20%	0.96%	0.08%	0.02%
$\epsilon = 0.1$	11.20%	0.20%	0.00%	0.00%
$\sigma_\lambda = 0.3$	11.20%	0.70%	0.04%	0.01%

This table represents the sensitivity of the optimal investment strategy in risky assets when the input variables are changed. Panel A shows the sensitivity for a fixed initial mortality intensity of 0.03, whereas Panel B and Panel C showcase sensitivity for mortality intensity of 0.126 and 0.27, respectively. The mortality intensity process is assumed to follow a gBm process.

Chapter 5

Using a Different Process to Model Mortality

In general, a retiree's mortality increases with her age after reaching the age of 65. In other words, the probability that a retiree dies increases with her age. Therefore, there should be an upward trend in the mortality intensity process of a retiree. Using a gBm model, as discussed in the previous chapter, results in modelling these mortality intensities with large amount of uncertainty. One of the goals of this study is to predict the behaviour of a retiree when her health condition changes, and the gBm might not be the best process for that purpose. For instance, under the gBm mortality intensity, a retiree recovering from a chronic disease will continue to have a higher mortality intensity than the average until her new health information has been updated in the mortality intensity model. In reality, the retiree's mortality intensity should revert back to the average mortality intensity, and therefore, to model this, we will now use a mean-reverting modified CIR model. The dynamics of the modified CIR model can be represented as

$$d\lambda(t) = k\left(\theta_\lambda(t) - \lambda(t)\right)dt + \sqrt{\lambda(t)}\sigma_\lambda dZ_\lambda(t), \tag{5.1}$$

where

$$\theta_\lambda(t) = \lambda_0 + \mu_\lambda t, \quad k \geq 0, \quad T \geq t \geq 0, \quad \lambda(t) \geq 0.$$

Moreover, variable k is the speed at which $\lambda(t)$ reverts back to its time-dependent mean, $\theta_\lambda(t)$. The deterministic function $\theta_\lambda(t)$ is the mean of the mortality intensity process which captures the increasing trend of the mortality intensity process with time. Parameters μ_λ and σ_λ are fixed constants.

The second-order differential operator for the modified CIR mortality process can be represented as

$$\mathcal{L}_\lambda v(t,\lambda) = \frac{\partial}{\partial\lambda}\Bigg(v(t,\lambda)k\left(\theta_\lambda(t) - \lambda(t)\right)\Bigg) + \frac{1}{2}\frac{\partial^2}{\partial\lambda^2}\left(v(t,\lambda)\sigma_\lambda^2\lambda(t)\right), \tag{5.2}$$

for some function $v(t, \lambda)$ that is twice-continuously differentiable with respect to λ. For more details, see Appendix A.

We determine the value of parameter k by assuming that the value of $\lambda(0)$ is 0.0215, μ_λ is 0.096, and σ_λ is 0.15, and thereafter calibrating the model in Equation (5.1) to match the gBm dynamics of Equation (4.1). We perform 100,000 Monte Carlo simulations on both mortality intensity processes and determine the 2.5^{th} and 97.5^{th} percentiles for both mortality intensity process from time 0 to 35. Moreover, we find that when $k = 0.001$, both 2.5^{th} and 97.5^{th} percentiles under gBm and modified CIR mortality intensity process behaves similarly. We use Euler's discretization method to simulate from Equation (5.1). In this discretization method, $\lambda(t)$ can be a negative number, which will lead to non-real value of the $\lambda(t)$ process. Therefore, we assume that whenever $\lambda(t) \leq 0$, then $\lambda(t) = 0.0001$. In other words, $\lambda(t) = \max(\lambda(t), 0.0001)$ in our simulations.

5.1 Optimal Consumption Strategy Under Fixed Wealth

Similar to the numerical results obtained in the previous chapter, we change the mortality intensity model to Equation (5.1), and obtain the optimal consumption strategy for a retiree when her wealth amount $W + g(t)$, is fixed to 1. In Figure 5.1, we find that the new optimal consumption strategy have a similar pattern when compared to Delong and Chen's framework (see Figure 4.1). The slope of the optimal consumption strategy, i.e., the change in the equilibrium consumption strategy with respect to the change in mortality intensity, is lower under the modified CIR mortality process than the gBm mortality process. This is explained by the mean-reverting feature of the modified CIR mortality intensity process which leads to less randomness as compared to the gBm process used to generate Figure 4.1.

Next, we determine the impact of longevity risk, i.e., increase in the parameter σ_λ from 0.15 to 0.3. In Figure 5.2, we determine the impact of longevity risk on the optimal consumption strategy when $\lambda(0) = 0.03$ and $\lambda(0) = 0.126$, respectively. The optimal consumption amount remains at least 0.2 when mortality intensity is as low as 0.03. In contrast with Figure 4.2, the impact of change in σ_λ is much higher here. Similarly, when $\lambda(0) = 0.126$, the change in the optimal consumption is larger in the modified CIR model as compared to the gBm model. This is because the mean reversion speed k of the modified CIR model is very low: it takes a long time to reach the stationary mean of the process. In the meantime, if the volatility parameter σ_λ increases, it results in higher uncertainty of the expected future lifetime of the retiree than the gBm process. Hence, the retiree decreases her consumption at a higher rate in the modified CIR mortality than the gBm process.

Finally, we investigate the impact of a change in hyperbolic discounting parameter ϵ on the optimal consumption strategy. Comparing Figure 5.3 to Figure 4.3, we find that the retiree increases her consumption with an increase in parameter ϵ under both mortality

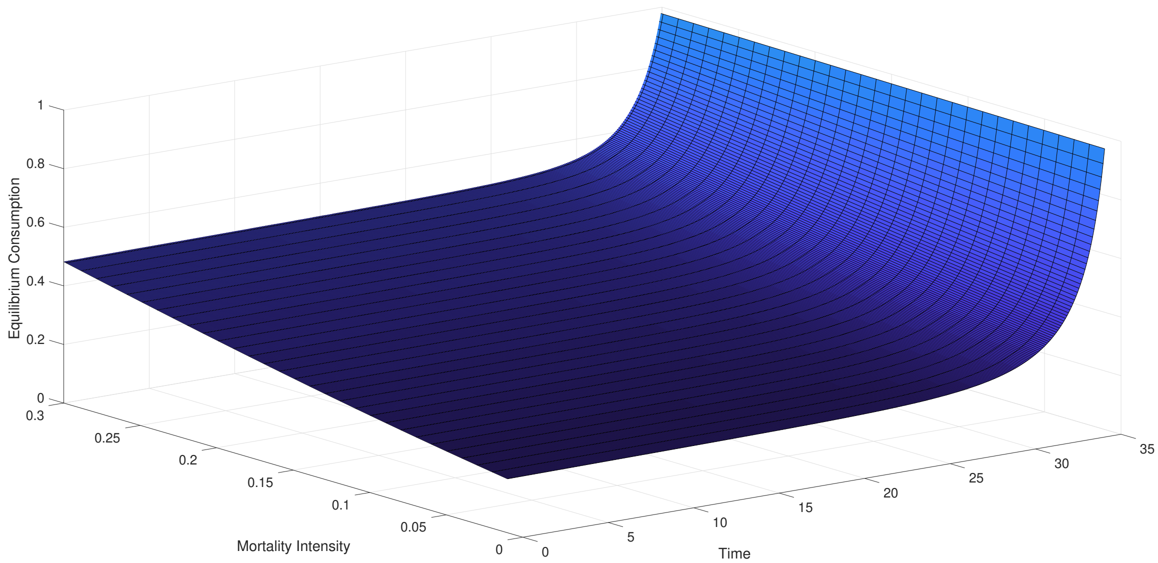

Figure 5.1: **Optimal Consumption Strategy Under the Modified Cox-Ingersoll-Ross Mortality Model With a Fixed Amount of Wealth 1.**
This figure shows the equilibrium consumption amount above the minimum consumption amount for a varying mortality intensity following the modified CIR process of Equation (5.1). Here, we assume that the amount of wealth including annuity-income at time t, i.e., $W + g(t)$ is equal to 1.

intensity models. Furthermore, the impact of such a change on optimal consumption is higher under the modified CIR mortality model when initial mortality intensity $\lambda(0)$ is 0.03, but remains similar when $\lambda(0)$ is 0.126. This is because of the presence of $\sqrt{\lambda(t)}$ factor in the CIR volatility component of Equation (5.1). When $\lambda(t)$ is as low as 0.03, the CIR overall volatility decreases rapidly as compared to the gBm mortality model. This further results in higher certainty to the retiree with respect to her future expected lifetime and, therefore, the retiree has a higher consumption under CIR mortality intensity.

5.2 Optimal Consumption, Investment and Wealth Under Modified CIR Mortality Intensity

In this section, we determine the overall consumption, investment and wealth of the retiree without any conditions on her wealth. As shown in Figure 5.4, the consumption amounts are higher during the initial years since the retiree does not care much about her bequest goals during these years. The consumption amounts become stable after a few years when the retiree starts to consume less to save more for her future bequest.

We determine the impact of change in the asset returns' volatility on the overall optimal consumption, investment and wealth processes. When σ increases from 0.25 to 0.4, the retiree prefers to invest a lesser proportion of her wealth in the risky asset and, therefore, reduces her optimal consumption. Comparing Figure 5.4 to 4.4, one can notice that

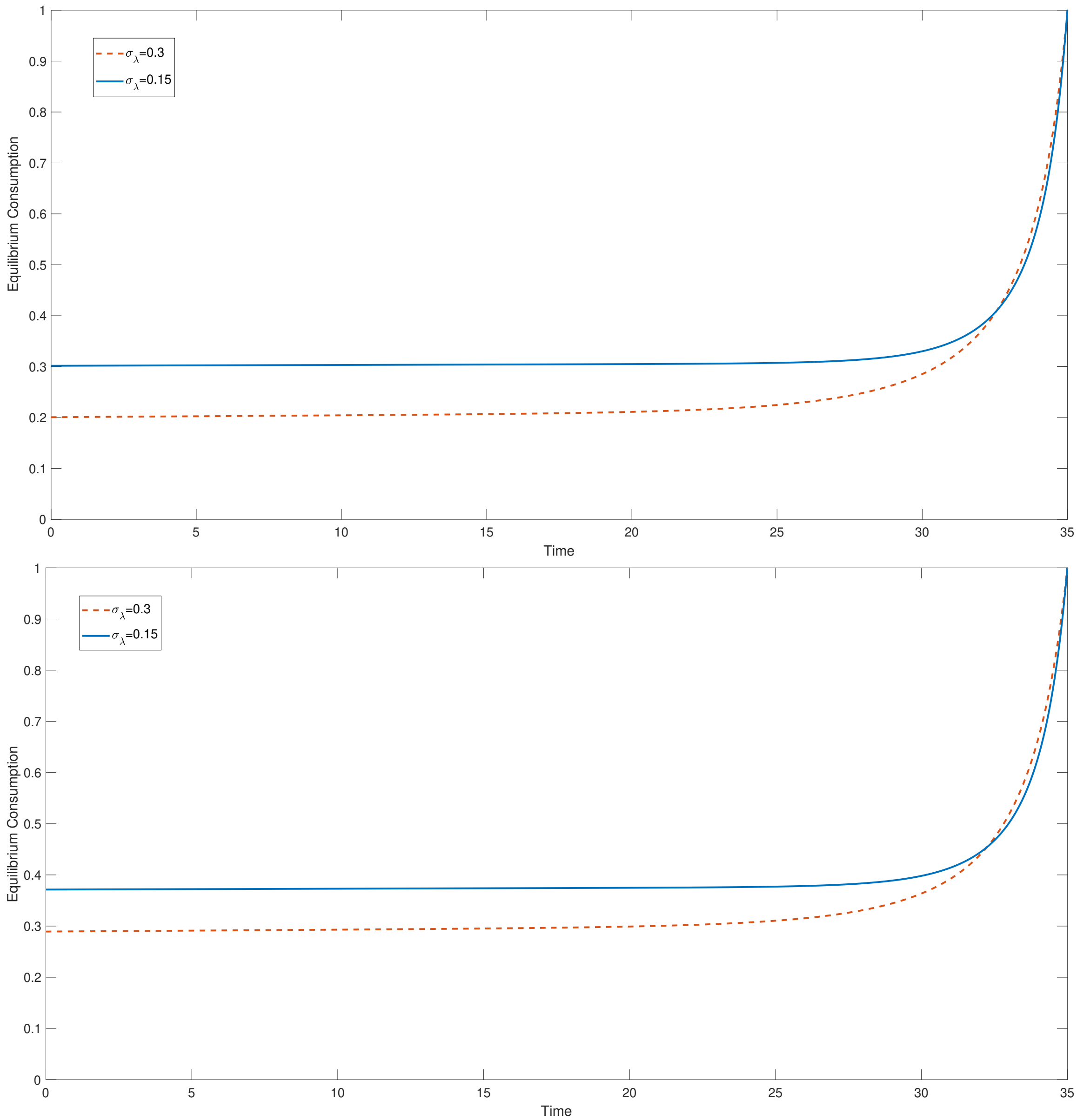

Figure 5.2: **Optimal Consumption Strategy For Varying σ_λ Under the Modified Cox-Ingersoll-Ross Mortality Model.**

These figures show the equilibrium consumption amount for a selected mortality intensities of $\lambda(0) = 0.03$ (top panel) and $\lambda(0) = 0.126$ (bottom panel) with two different mortality intensity volatility parameters, i.e., $\sigma_\lambda = 0.15$ (in blue) or $\sigma_\lambda = 0.3$ (in red) for a fixed amount of wealth (i.e., 1) at each time t. The mortality intensity process is assumed to follow the modified CIR mortality intensity model of Equation (5.1).

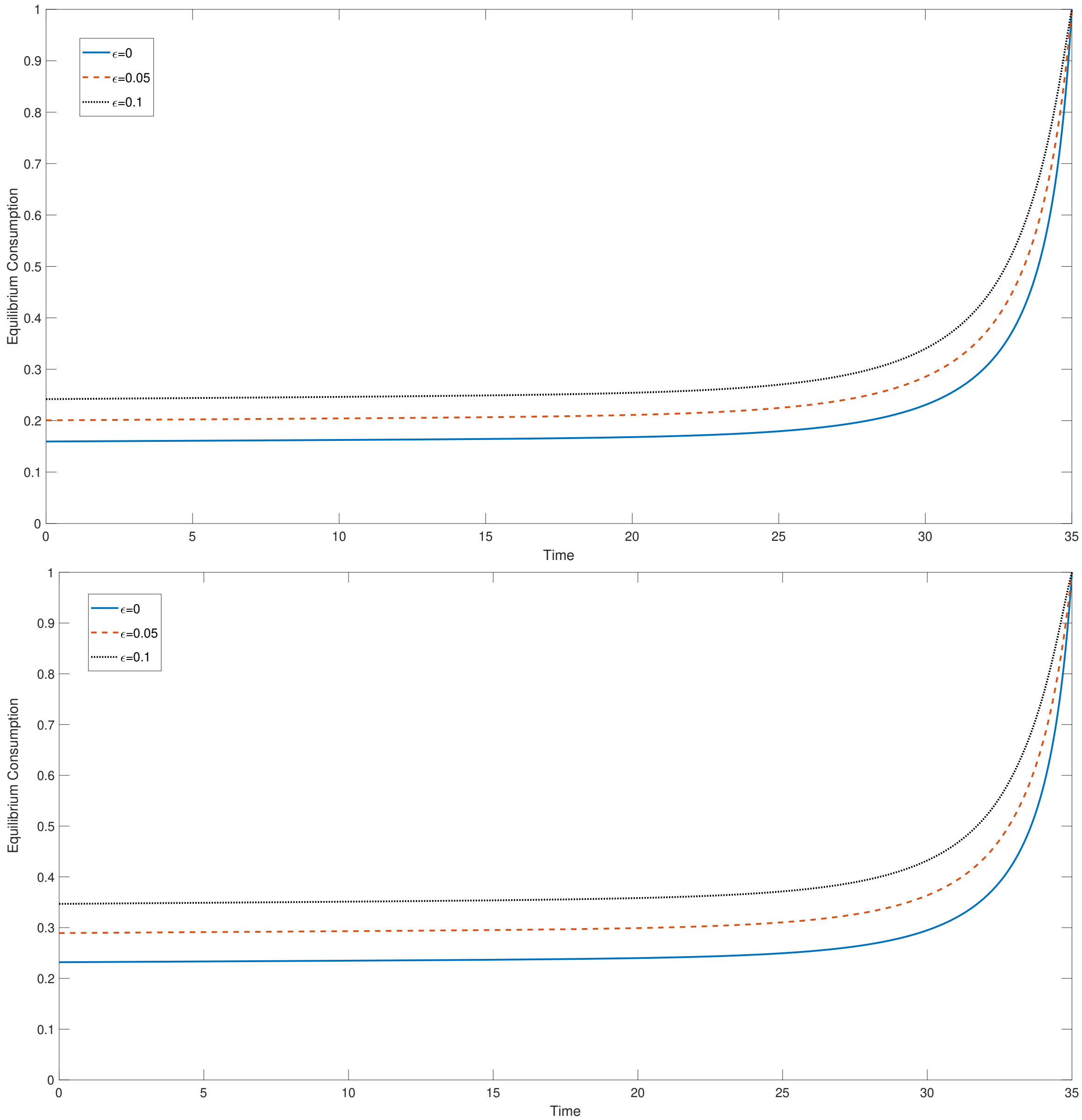

Figure 5.3: **Optimal Consumption Strategy For Varying ϵ Under the Modified Cox-Ingersoll-Ross Mortality Model.**

These figures show the equilibrium consumption amount for a selected mortality intensities of $\lambda(0) = 0.03$ (top panel) and $\lambda(0) = 0.126$ (bottom panel) for three different values of hyperbolic discounting parameter ϵ. The solid line (in blue) shows the case when $\epsilon = 0$, the dashed line (in red) shows the case when $\epsilon = 0.05$, and the dotted line (in black) shows the case when $\epsilon = 0.1$. The mortality intensity process is assumed to follow the modified CIR mortality intensity model of Equation (5.1).

the impact of change in the asset volatility on the optimal consumption and investment strategy of the retiree is similar under both mortality intensity models. This is because both mortality intensity models are calibrated such that they behave similarly, and both models are independent of market volatility parameter σ. Therefore, a similar impact (i.e., the percentage change) is noticed in the optimal consumption and investment strategy of the retiree under both mortality intensity models for a change in market volatility.

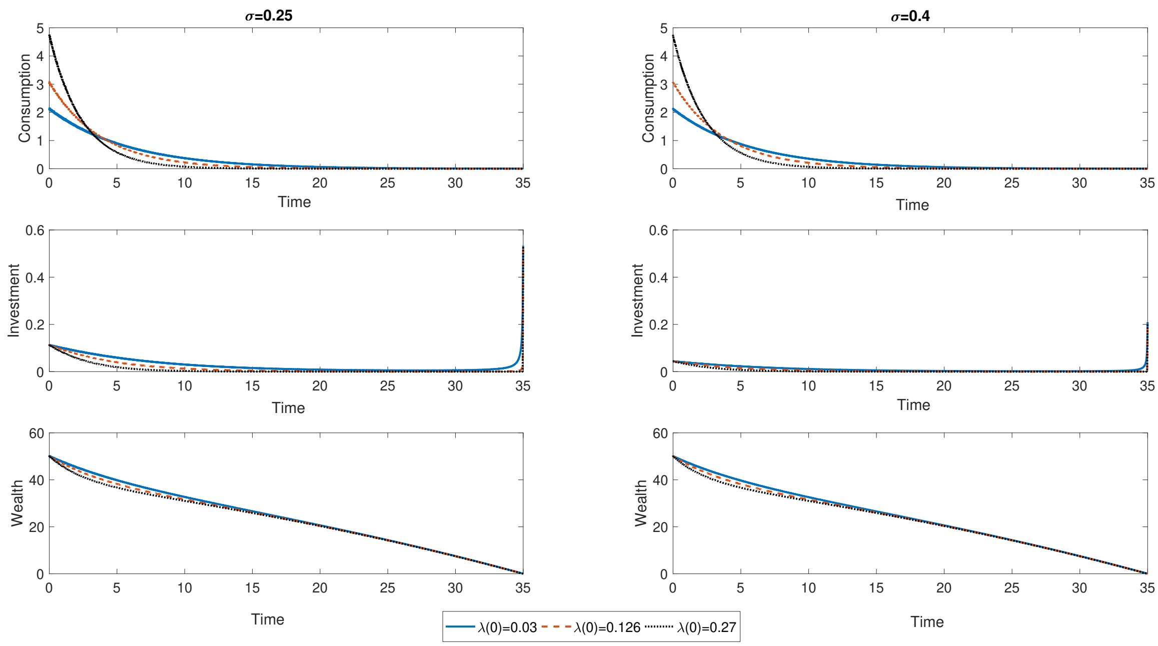

Figure 5.4: **Optimal Consumption, Investment, and Wealth of the Retiree Under the Modified Cox-Ingersoll-Ross Mortality When the Market Volatility Increases from 0.25 to 0.4.**
This figure shows the impact of an increase in asset return volatility σ from 0.25 (left panels) to 0.4 (right panels) on the retiree's equilibrium consumption, investment and wealth. The solid line (in blue) represents the fixed mortality intensity $\lambda(0) = 0.03$, the dashed line (in red) is the case when $\lambda(0) = 0.126$, and the dotted line (in black) is the case when $\lambda(0) = 0.27$. The mortality intensity process is assumed to follow the modified CIR model of Equation (5.1).

Similarly, a change in the risk-aversion parameter of the retiree from 0.1 to 0.2, impacts on the overall consumption, investment and wealth of the retiree. This increase in γ results in a reduction of coefficient of relative risk-aversion from 0.9 to 0.8. Therefore, the retiree becomes less risk-averse and prefers to take more risk by investing more in the risky asset. More specifically, she increases her investment in risky assets by 1% to 4% during the first 10 years, but becomes more cautious and reduces her investment in the risky assets when her bequest goal kicks in. Similarly, her consumption increases during the first 10 years and then declines over her remaining lifetime. Comparing Figure 5.5 to 4.5, we find that the impact of a change in parameter γ on the optimal consumption and investment strategy remains similar under both mortality models.

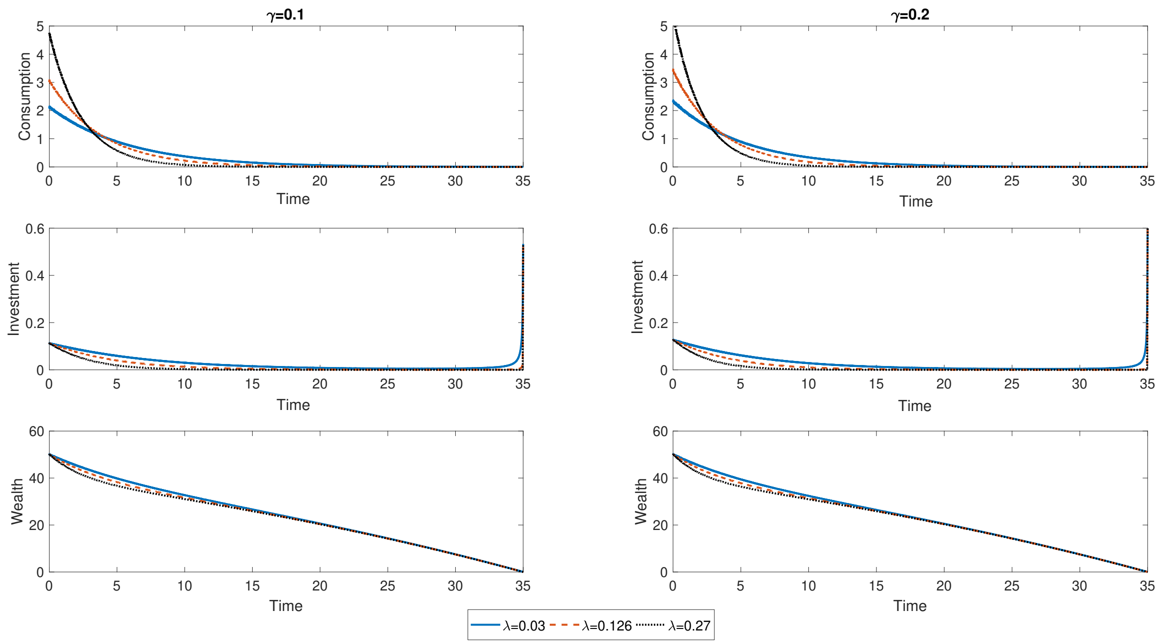

Figure 5.5: **Optimal Consumption, Investment, and Wealth of the Retiree Under the Modified Cox-Ingersoll-Ross Mortality When the Risk-Aversion Increases from 0.1 to 0.2.**
This figure shows the impact of an increase in the risk-aversion parameter γ from 0.1 (left panels) to 0.2 (right panels) on the retiree's equilibrium consumption, investment and wealth. The solidline (in blue) represents the fixed mortality intensity $\lambda(0) = 0.03$, the dashed line (in red) is for the case when $\lambda(0) = 0.126$, and the dotted line (in black) is for the case when $\lambda(0) = 0.27$. The mortality intensity process is assumed to follow the modified CIR model of Equation (5.1).

At last, we perform sensitivity analysis on the optimal consumption and investment strategy of the retiree with a change in an input parameter described in Table 4.1. These results are shown in Tables 5.1 and 5.2. We find that, broadly speaking, the behaviour of the retiree with a change in an assumption remains the same under both mortality intensity models. However, the severity of these assumptions varies for both these models. For instance, a change in financial market assumptions (i.e., α, σ, and r) results in similar percentage changes in optimal consumption and investment strategy for both mortality models. Moreover, variables such as the weight assigned to the bequest q, initial wealth $\hat{x}$, and the proportion of wealth invested in annuity π are also equally sensitive for both mortality models. On the other hand, the optimal consumption and investment strategy are highly sensitive under the modified CIR model than the gBm model for a change in the discounting parameters (i.e., ρ, δ, and ϵ) and the mortality intensity parameters (i.e., μ_λ and σ_λ).

Table 5.1: **Optimal Consumption Strategy When the Stochastic Mortality Intensity Follows the Modified Cox-Ingersoll-Ross Model**

Panel A: $\lambda(0) = 0.03$.

Time	0	10	20	30
Base Case	2.10	0.37	0.06	0.00
Modified Cases				
$\pi = 0.25$	2.05	0.36	0.06	0.00
$\sigma = 0.4$	2.11	0.35	0.05	0.00
$\alpha = 0.08$	2.07	0.47	0.10	0.02
$\hat{x} = 200$	4.21	0.74	0.12	0.01
$q = 20$	2.11	0.37	0.06	0.01
$r = 0.04$	2.09	0.43	0.08	0.02
$\gamma = 0.2$	2.30	0.34	0.05	0.01
$\rho = 0.02$	1.92	0.41	0.08	0.01
$\delta = 1$	2.11	0.37	0.06	0.01
$\delta = 3$	1.67	0.45	0.11	0.03
$\epsilon = 0$	2.54	0.30	0.03	0.00
$\epsilon = 0.1$	1.94	0.41	0.08	0.01
$\sigma_\lambda = 0.3$	3.17	0.20	0.01	0.00

Panel B: $\lambda(0) = 0.126$.

Time	0	10	20	30
Base Case	3.04	0.22	0.02	0.00
Modified Cases				
$\pi = 0.25$	2.97	0.22	0.02	0.00
$\sigma = 0.4$	3.04	0.21	0.01	0.00
$\alpha = 0.08$	3.01	0.28	0.03	0.00
$\hat{x} = 200$	6.08	0.44	0.03	0.00
$q = 20$	3.04	0.22	0.02	0.00
$r = 0.04$	3.02	0.25	0.02	0.00
$\gamma = 0.2$	3.39	0.18	0.01	0.00
$\rho = 0.02$	2.78	0.26	0.02	0.00
$\delta = 1$	3.04	0.22	0.02	0.00
$\delta = 3$	2.43	0.31	0.04	0.00
$\epsilon = 0$	3.64	0.15	0.01	0.00
$\epsilon = 0.1$	2.85	0.25	0.02	0.00
$\sigma_\lambda = 0.3$	3.90	0.13	0.00	0.00

Panel C: $\lambda(0) = 0.27$.

Time	0	10	20	30
Base Case	4.69	0.07	0.00	0.00
Modified Cases				
$\pi = 0.25$	4.58	0.07	0.00	0.00
$\sigma = 0.4$	4.70	0.07	0.00	0.00
$\alpha = 0.08$	4.66	0.09	0.00	0.00
$\hat{x} = 200$	9.39	0.14	0.00	0.00
$q = 20$	4.69	0.07	0.00	0.00
$r = 0.04$	4.68	0.08	0.00	0.00
$\gamma = 0.2$	5.32	0.04	0.00	0.00
$\rho = 0.02$	4.29	0.09	0.00	0.00
$\delta = 1$	4.69	0.07	0.00	0.00
$\delta = 3$	3.80	0.13	0.00	0.00
$\epsilon = 0$	5.59	0.04	0.00	0.00
$\epsilon = 0.1$	4.48	0.08	0.00	0.00
$\sigma_\lambda = 0.3$	5.15	0.05	0.00	0.00

This table represents the sensitivity of the optimal consumption strategy when the input variables are changed. Panel A shows the sensitivity for a fixed initial mortality intensity of 0.03, whereas Panel B and Panel C showcase sensitivity for mortality intensity of 0.126 and 0.27, respectively. The mortality intensity process is assumed to follow the modified CIR process.

Table 5.2: **Optimal Investment in the Risky Asset When the Stochastic Mortality Intensity Follows the Modified Cox-Ingersoll-Ross Model.**

Panel A: $\lambda(0) = 0.03$.

Time	0	10	20	30
Base Case	11.20%	2.98%	0.84%	0.59%
Modified Cases				
$\pi = 0.25$	7.29%	1.83%	0.50%	0.35%
$\sigma = 0.4$	4.38%	1.12%	0.29%	0.20%
$\alpha = 0.08$	22.40%	7.60%	2.74%	2.38%
$\hat{x} = 200$	11.20%	2.97%	0.83%	0.58%
$q = 20$	11.20%	2.98%	0.90%	0.91%
$r = 0.04$	3.73%	1.07%	0.32%	0.24%
$\gamma = 0.2$	12.60%	2.83%	0.65%	0.45%
$\rho = 0.02$	11.20%	3.54%	1.17%	0.94%
$\delta = 1$	11.20%	3.00%	0.83%	0.58%
$\delta = 3$	11.20%	4.39%	1.87%	1.80%
$\epsilon = 0$	11.20%	4.39%	1.87%	1.80%
$\epsilon = 0.1$	11.20%	4.39%	1.87%	1.80%
$\sigma_\lambda = 0.3$	11.20%	4.39%	1.87%	1.80%

Panel B: $\lambda(0) = 0.126$.

Time	0	10	20	30
Base Case	11.20%	1.27%	0.14%	0.04%
Modified Cases				
$\pi = 0.25$	7.29%	0.77%	0.08%	0.03%
$\sigma = 0.4$	4.38%	0.47%	0.05%	0.01%
$\alpha = 0.08$	22.40%	3.29%	0.48%	0.18%
$\hat{x} = 200$	11.20%	1.27%	0.14%	0.04%
$q = 20$	11.20%	1.27%	0.15%	0.07%
$r = 0.04$	3.73%	0.46%	0.06%	0.02%
$\gamma = 0.2$	12.60%	1.04%	0.08%	0.02%
$\rho = 0.02$	11.20%	1.63%	0.23%	0.09%
$\delta = 1$	11.20%	1.28%	0.14%	0.04%
$\delta = 3$	11.20%	2.22%	0.45%	0.22%
$\epsilon = 0$	11.20%	2.22%	0.45%	0.22%
$\epsilon = 0.1$	11.20%	2.22%	0.45%	0.22%
$\sigma_\lambda = 0.3$	11.20%	2.22%	0.45%	0.22%

Panel C: $\lambda(0) = 0.27$.

Time	0	10	20	30
Base Case	11.20%	0.27%	0.01%	0.00%
Modified Cases				
$\pi = 0.25$	7.29%	0.16%	0.00%	0.00%
$\sigma = 0.4$	4.38%	0.10%	0.00%	0.00%
$\alpha = 0.08$	22.40%	0.70%	0.02%	0.00%
$\hat{x} = 200$	11.20%	0.27%	0.01%	0.00%
$q = 20$	11.20%	0.27%	0.01%	0.00%
$r = 0.04$	3.73%	0.10%	0.00%	0.00%
$\gamma = 0.2$	12.60%	0.17%	0.00%	0.00%
$\rho = 0.02$	11.20%	0.40%	0.01%	0.00%
$\delta = 1$	11.20%	0.27%	0.01%	0.00%
$\delta = 3$	11.20%	0.62%	0.03%	0.00%
$\epsilon = 0$	11.20%	0.62%	0.03%	0.00%
$\epsilon = 0.1$	11.20%	0.62%	0.03%	0.00%
$\sigma_\lambda = 0.3$	11.20%	0.62%	0.03%	0.00%

This table represents the sensitivity of the optimal investment strategy in risky assets when the input variables are changed. Panel A shows the sensitivity for a fixed initial mortality intensity of 0.03, whereas Panel B and Panel C showcase sensitivity for mortality intensity of 0.126 and 0.27, respectively. The mortality intensity process is assumed to follow the modified CIR process.

5.3 Impact of a Change in Mean Reversion Speed

The mean reversion speed k of the modified CIR model describes the rate at which the mortality intensity process reverts to its expected mean $\theta_\lambda(t)$. A change in the value of parameter k directly affects the drift of Equation (5.1). As the value of parameter k rises, the retiree becomes more certain about her future expected lifetime and, therefore, consumes more. Figure 5.6 shows that as value of parameter k increases, the retiree spends more on her consumption. Similarly, she reduces her proportion of investment in the risky asset as she is more certain about her future lifetime, and can increase her consumption without investing more in the risky asset, as shown in Figure 5.6. The wealth process remains the same after 10 years when the consumption and investment strategy becomes similar as well because the bequest goal is of more importance after the first 10 years.

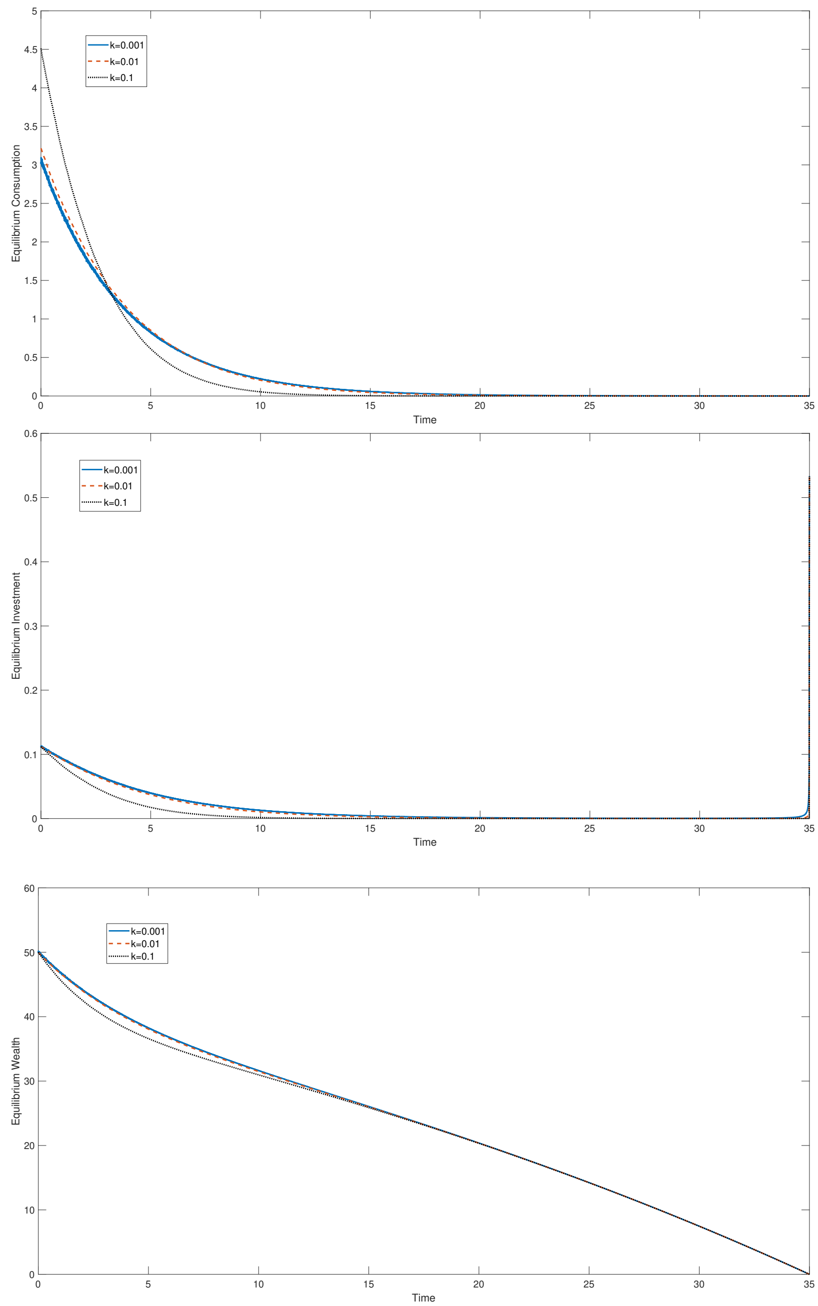

Figure 5.6: **Optimal Consumption, Investment, and Wealth of the Retiree Under the Modified Cox-Ingersoll-Ross Mortality with Varying Mean Reversion Speed.**
This figure shows the impact of an increase in the mean reversion speed k for a selected mortality intensity of $\lambda(0) = 0.126$ on the retiree's equilibrium consumption (top panel), investment (middle panel), and wealth (bottom panel). The solid line (in blue) represents the mean reversion speed $k = 0.001$, the dashed line (in red) is the case when $k = 0.01$, and the dotted line (in black) is the case when $k = 0.1$. The mortality intensity process is assumed to follow the modified CIR model of Equation (5.1).

Chapter 6

Conclusion

With an increase in the number of self-driven retirement plans, more retirees are now concerned with determining the optimal amount of wealth they should consume and invest such that they can fulfill their retirement objectives. Some of their major retirement goals are: maintaining a minimum standard of living, having enough funds before they die, leaving a bequest, and maximizing their utility of consumption by dynamically changing their consumption according to their health status.

In this project, we found the optimal consumption and investment strategy for the retiree who wishes to maximize her utility of consumption. First, we replicated the results of the Delong and Chen (2016) framework that extends the Merton (1971) model by incorporating for (1) a minimum lifetime consumption, (2) a stochastic force of mortality that follows a gBm process, (3) an annuity income, and (4) a non-exponential discounting of future income. Also, we performed a sensitivity analysis on the optimal consumption and investment strategy, and found that the optimization strategy is most sensitive to a change in the mortality intensity parameter σ_λ and non-exponential discounting parameters (i.e., ϵ and δ).

Second, we believe that there exists a mean-reverting trend in the retiree's mortality intensity. Therefore, we introduced a modified CIR mortality intensity model with a variable drift in Delong and Chen's (2016) framework to determine a new optimal consumption and investment strategy for the retiree. Broadly speaking, we found that the optimization strategy under the modified CIR mortality intensity model to be more sensitive than the gBm model for a change in the non-exponential discounting parameters (i.e., ϵ and δ), and a change in the mortality intensity volatility parameter (i.e., σ_λ). However, the optimization strategy remains equally sensitive for a change in the asset return and volatility parameters (i.e., α, σ and r) under both mortality intensity models.

The current model assumes that a retiree purchases a lifetime-income annuity at the exact date of her retirement. For future work, it will be interesting to see if Milevsky and Robinson (2000) theory determining the optimal age to purchase a life-time income annuity applicable to the current study. A delay in retiree's decision to buy a lifetime-income annuity may result in an increase in the amount of annuity income received from an annuity provider. This is because of the pooling of mortality risk by an annuity provider which provides a higher yield to the retirees who live longer than their expected lifetime. This yield rate is higher than what it could be achieved through retirees investments made outside this pool. One way Delong and Chen's model may account for the delay in purchasing the lifetime-income annuity is by investing the discounted amount of annuity investment in a risk-free asset. Other strategies to account for this delay in purchase of a lifetime-income annuity remains open for discussion.

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Appendix A

More Details on the Partial Differential Equations for the Modified CIR Mortality Intensity

The solutions to the PDEs of Equations (4.15), (4.17), (4.18) and (4.19) cannot be derived explicitly. Therefore, we use the explicit and implicit finite difference methods (FDM) to solve these PDEs.

First, we simply the second-order differential operator for the modified CIR mortality intensity process given in Equation (5.2):

$$\mathcal{L}_\lambda v(t,\lambda) = \frac{\partial}{\partial \lambda}\Bigg(v(t,\lambda) k \left(\theta_\lambda(t) - \lambda(t)\right) \Bigg) + \frac{1}{2}\frac{\partial^2}{\partial \lambda^2}\Big(v(t,\lambda)\sigma_\lambda^2 \lambda(t) \Big). \tag{A.1}$$

The first term on the right side of Equation (A.1) can be simplified as

$$\frac{\partial}{\partial \lambda}\Bigg(v(t,\lambda) k \left(\theta_\lambda(t) - \lambda(t)\right) \Bigg) = v(t,\lambda)(-k) + k\left(\theta_\lambda(t) - \lambda(t)\right)\frac{\partial}{\partial \lambda} v(t,\lambda),$$

and the second term can be expanded as

$$\begin{aligned}
\frac{1}{2}\frac{\partial^2}{\partial \lambda^2}\Big(v(t,\lambda)\sigma_\lambda^2 \lambda(t) \Big) &= \frac{1}{2}\frac{\partial}{\partial \lambda}\frac{\partial}{\partial \lambda}\Big(v(t,\lambda)\sigma_\lambda^2 \lambda(t) \Big) \\
&= \frac{1}{2}\frac{\partial}{\partial \lambda}\Bigg(v(t,\lambda)\sigma_\lambda^2 + (\sigma_\lambda^2 \lambda(t))\frac{\partial}{\partial \lambda}(v(t,\lambda) \Bigg) \\
&= \frac{1}{2}\Bigg(\sigma_\lambda^2 \frac{\partial}{\partial \lambda} v(t,\lambda) + \sigma_\lambda^2 \lambda(t)\frac{\partial^2}{\partial \lambda^2} v(t,\lambda) + \sigma_\lambda^2 \frac{\partial}{\partial \lambda} v(t,\lambda) \Bigg) \\
&= \frac{1}{2}\sigma_\lambda^2 \Bigg(2\frac{\partial}{\partial \lambda} v(t,\lambda) + \lambda(t)\frac{\partial^2}{\partial \lambda^2} v(t,\lambda) \Bigg).
\end{aligned}$$

Therefore, the second-order differential operator of Equation (A.1) can be written as

$$\mathcal{L}_\lambda v(t,\lambda) = v(t,\lambda)(-k) + \frac{\partial}{\partial \lambda} v(t,\lambda)\Bigg(\sigma_\lambda^2 + k\left(\theta_\lambda(t) - \lambda(t)\right)\Bigg) + \frac{\partial^2}{\partial \lambda^2} v(t,\lambda)\Bigg(\frac{1}{2}\sigma_\lambda^2 \lambda(t)\Bigg). \quad (A.2)$$

A.1 Solution to Equation (4.15) under the Modified CIR Mortality Intensity

In this section, we approximate the value of first and second-order derivatives in Equation (4.15), and use numerical methods afterwards. Let us assume that $C = \lambda + \rho - r\gamma - \frac{1}{2}\frac{\gamma}{1-\gamma}\frac{(\alpha-r)^2}{\sigma^2}$. Using this, the PDE of Equation (4.15) can be written as

$$\frac{\partial}{\partial t} f(t,\lambda) + \mathcal{L}_\lambda f(t,\lambda) - C f(t,\lambda) + (1-\gamma)(f(t,\lambda))^{\frac{\gamma}{\gamma-1}} = 0. \quad (A.3)$$

We use the simplified expression of $\mathcal{L}_\lambda v(t,\lambda)$ from Equation (A.2) in Equation (A.3). Therefore, we obtain

$$\begin{aligned} &\frac{\partial}{\partial t} f(t,\lambda) + \frac{\partial}{\partial \lambda} f(t,\lambda)\Bigg(\sigma_\lambda^2 + k\left(\theta_\lambda(t) - \lambda(t)\right)\Bigg) + \frac{\partial^2}{\partial \lambda^2} f(t,\lambda)\Bigg(\frac{1}{2}\sigma_\lambda^2 \lambda(t)\Bigg) \\ &+ f(t,\lambda)(-k - C) + (1-\gamma)(f(t,\lambda))^{\frac{\gamma}{\gamma-1}} = 0. \end{aligned} \quad (A.4)$$

We then use numerical methods to solve for the derivatives $\frac{\partial}{\partial t} f(t,\lambda)$, $\frac{\partial}{\partial \lambda} f(t,\lambda)$, and $\frac{\partial^2}{\partial \lambda^2} f(t,\lambda)$ such that

$$\frac{\partial}{\partial t} f(t,\lambda) = \frac{f(t,\lambda) - f(t-\Delta t,\lambda)}{\Delta t},$$

$$\frac{\partial}{\partial \lambda} f(t,\lambda) = \frac{f(t,\lambda) - f(t,\lambda-\Delta\lambda)}{\Delta\lambda},$$

and

$$\frac{\partial^2}{\partial \lambda^2} f(t,\lambda) = \frac{f(t,\lambda+\Delta\lambda) - 2f(t,\lambda) + f(t,\lambda-\Delta\lambda)}{(\Delta\lambda)^2}.$$

We input the approximated value of the derivatives in Equation (A.4) and obtain

$$\begin{aligned} \frac{f(t-\Delta t,\lambda)}{\Delta t} = \frac{f(t,\lambda)}{\Delta t} &+ \frac{f(t,\lambda) - f(t,\lambda-\Delta\lambda)}{\Delta\lambda}\Bigg(\sigma_\lambda^2 + k\left(\theta_\lambda(t) - \lambda(t)\right)\Bigg) \\ &+ \frac{f(t,\lambda+\Delta\lambda) - 2f(t,\lambda) + f(t,\lambda-\Delta\lambda)}{(\Delta\lambda)^2}\Bigg(\frac{1}{2}\sigma_\lambda^2 \lambda(t)\Bigg) \\ &+ f(t,\lambda)(-k - C) + (1-\gamma)(f(t,\lambda))^{\frac{\gamma}{\gamma-1}} \end{aligned}$$

This can further be simplified as

$$\frac{f(t-\Delta t,\lambda)}{\Delta t} = c(t) f(t,\lambda) + b(t) f(t,\lambda-\Delta\lambda) + a(t) f(t,\lambda+\Delta\lambda) + (1-\gamma)(f(t,\lambda))^{\frac{\gamma}{\gamma-1}}, \quad (A.5)$$

where

$$a(t) = \frac{1}{2}\frac{\sigma_\lambda^2 \lambda(t)}{(\Delta\lambda)^2},$$

$$b(t) = \left(\frac{-\sigma_\lambda^2 - k\left(\theta_\lambda(t) - \lambda(t)\right)}{\Delta\lambda}\right) + \frac{1}{2}\frac{\sigma_\lambda^2 \lambda(t)}{(\Delta\lambda)^2},$$

$$c(t) = \frac{1}{\Delta t} + \left(\frac{\sigma_\lambda^2 + k\left(\theta_\lambda(t) - \lambda(t)\right)}{\Delta\lambda}\right) - \frac{\sigma_\lambda^2 \lambda(t)}{(\Delta\lambda)^2} - k - C.$$

Using the terminal condition $f(T, \lambda) = q$, we can determine the value of $f(T - \Delta t, \lambda)$ for each value of λ using Equation (A.5). This process is repeated to determine the value of $f(t, \lambda)$ for each time step and mortality process value using the FDM.

A similar methodology is used to solve the PDEs of Equations (4.17), (4.18), and (4.19).