A Combinatorial Description of the Cup Product for Smooth Complete Toric Varieties

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Abstract

For any smooth variety X, there exists an associated vector space of *first-order deformations*. This vector space can be interpreted using sheaf cohomology; it is the first cohomology group $H^1(X, \mathcal{T}_X)$ where $\mathcal{T}_X$ is the tangent sheaf. One can ask when it is possible to "combine" two first-order deformations. The cup product takes elements of $H^1(X, \mathcal{T}_X) \times H^1(X, \mathcal{T}_X)$ and maps to the *obstruction space* $H^2(X, \mathcal{T}_X)$, and the vanishing of the cup product tells us precisely when this is possible. In this thesis we give a combinatorial description of the cup product (on the level of Čech cohomology) when X is a smooth, complete toric variety with an associated fan Σ. We also give an example of a smooth, complete toric 3-fold for which the cup product is nonvanishing.

Keywords: cup product, toric geometry, tangent sheaf, Čech cohomology, Euler sequence, deformation theory

Dedication

To my loving parents.

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Chapter 1

Introduction

1.1 Background and Motivation

Algebraic geometry is an expansive field of study in mathematics, which began as the study of solutions sets of systems of polynomial equations. Often we refer to these sets as *varieties*. The aim of toric geometry is to study those solution sets which also exhibit a combinatorial structure. By combinatorial we mean the solution sets correspond in some way to certain objects in discrete mathematics. Essentially, a variety is *toric* if it contains the algebraic torus as a dense open subset. There are many connections between toric geometry and polyhedra, combinatorics, and commutative algebra, which make this area particularly amenable to study. In this thesis we leverage these connections to give a combinatorial description of the *cup product*.

If X is a variety, then one may be interested in *deformations* of X. Roughly speaking, solution sets to polynomial systems can be deformed be varying the coefficients of the defining equations. The cup product provides information regarding when two first-order deformations can be combined. Specifically, we are interested in the vanishing of the cup product when X is a smooth, complete, toric variety. In this thesis we give conditions for when the cup product vanishes, and we provide an explicit example (in the form of a fan corresponding to a toric 3-fold) of when the cup product is nonzero, in Section 7.4.

For a smooth, complete toric variety X, isomorphism classes of first-order deformations are in correspondence with elements of the cohomology group $H^1(X, \mathcal{T}_X)$ [Ser07, §1.2.9]. This cohomology group has a concrete description as a vector space by [Ilt11], and we use this fact throughout. The cohomology group $H^2(X, \mathcal{T}_X)$ (called the *obstruction space*) describes the obstructions to lifting first-order deformations to higher order. This also has the structure of a vector space, which we describe in Chapter 5. The cup product can be thought of as a map of vector spaces, which combines two first-order deformations, and produces an element in the obstruction space:

$$\smile\colon H^1(X, \mathcal{T}_X) \times H^1(X, \mathcal{T}_X) \to H^2(X, \mathcal{T}_X).$$

The primary tool we use to describe these vector spaces is Čech cohomology. The Čech description gives us the ability to make explicit computations in these cohomology groups. The explicit nature of this representation allows us to describe a smooth, complete toric 3-fold for which the cup product does not vanish.

This thesis is structured as follows: in Chapter 2 we begin with a cursory introduction to the basics of algebraic geometry, introducing the relevant objects for this thesis — *varieties*, *divisors*, *sheaves*, and the *tangent sheaf*. This serves as background for Chapter 3, where we develop the language of toric geometry. This includes objects such as *cones* and *fans*, *torus invariant divisors*, and *sheaves associated to divisors*. Chapter 4 covers sheaf cohomology, where we define *Čech cohomology* and the *cup product*. In Chapter 5 we detail the isomorphism which allows us to view $H^1(X, \mathcal{T}_X)$ as a direct sum of simpler spaces (via *cohomology of boundary divisors*). Chapter 6 outlines the combinatorial description of both $H^1(X, \mathcal{T}_X)$ and $H^2(X, \mathcal{T}_X)$, and finally Chapter 7 contains the main results of the thesis.

1.2 Overview

The goal is to describe the cup product on X, when X is a smooth, complete, toric variety. As we mentioned, the cup product is a map taking two elements of $H^1(X, \mathcal{T}_X)$ and producing an element of $H^2(X, \mathcal{T}_X)$ which we denote by

$$\smile : H^1(X, \mathcal{T}_X) \times H^1(X, \mathcal{T}_X) \to H^2(X, \mathcal{T}_X).$$

The first step is to use the fact that there is a decomposition in cohomology of $H^p(X, \mathcal{T}_X)$, given by the character lattice of the torus. Let M be said character lattice, so that we have the grading

$$H^p(X, \mathcal{T}_X) = \bigoplus_{u \in M} H^p(X, \mathcal{T}_X)_u.$$

Given this grading of $H^p(X, \mathcal{T}_X)$, it is sufficient to compute the cup product

$$\smile : H^1(X, \mathcal{T}_X)_u \times H^1(X, \mathcal{T}_X)_{u'} \to H^2(X, \mathcal{T}_X)_{u+u'}$$

for all pairs $u, u' \in M$ instead. In fact we will see later that we need only consider pairs u, u' which satisfy conditions relating to the combinatorics of smooth, complete, toric varieties.

Next we use a result of [Ilt11] which gives a description of the cohomology groups

$$H^1(X, \mathcal{T}_X)_u \simeq V \qquad \text{and} \qquad H^1(X, \mathcal{T}_X)_{u'} \simeq V'$$

where V and V' are combinatorially described vector spaces relating to connected components of certain graphs. We extend this result via Theorem 6.9 to conclude that the cohomology group $H^2(X, \mathcal{T}_X)_{u+u'}$ is isomorphic to another combinatorially described vector space W. The goal of this thesis is to describe the cup product in terms of these vector spaces. That is, we characterize the cup product by describing the dashed arrow in the following diagram.

$$\begin{array}{ccc} V \times V' & \dashrightarrow & W \\ \downarrow \wr & & \downarrow \wr \\ H^1(X, \mathcal{T}_X)_u \times H^1(X, \mathcal{T}_X)_{u'} & \xrightarrow{\text{cup product}} & H^2(X, \mathcal{T}_X)_{u+u'} \end{array}$$

To describe a map of vector spaces in practice, one gives the map in terms of what it does to basis elements. There is no canonical basis for V or V', so instead we describe the map on elements of a spanning set. We give a set of spanning elements for V and V' in Section 6.2. Next, for an arbitrary pair of spanning elements, we trace through the isomorphism on the left and apply the cup product. This gives a Čech *representative* of the image in $H^2(X, \mathcal{T}_X)_{u+u'}$. The main difficulty is then to "lift" the representative for the element in $H^2(X, \mathcal{T}_X)_{u+u'}$ to a representative of the element in W. It turns out that this cup product image is actually a sum of two elements. However, by an argument in Chapter 7 we may consider the summands separately, which simplifies things somewhat. Finally, we lift the summands in Theorem 7.4 and Theorem 7.7. This gives us the combinatorial description that we were hoping for!

Chapter 2

Algebraic Geometry

In this chapter we cover a few of the concepts in algebraic geometry that are essential for this thesis. We start at the beginning with affine varieties, then work up to sheaves and divisors. For a gentle introduction to affine varieties (and the Nullstellensatz) see [CLO07] by Cox, Little, and O'Shea, particularly the first two chapters. For a more advanced treatment of the topics in this chapter, and for divisors, sheaves, etc., see the excellent [Har13].

2.1 Affine Varieties

We begin with the basic geometric object in algebraic geometry, the affine variety. Let k be an algebraically closed field of characteristic 0, and $k[\boldsymbol{x}] = k[x_1, \dots, x_n]$ be the ring of polynomials in n variables over k. The main objects of study in algebraic geometry are solutions to systems of the form

$$f_1(\boldsymbol{x}) = \cdots = f_k(\boldsymbol{x}) = 0,$$

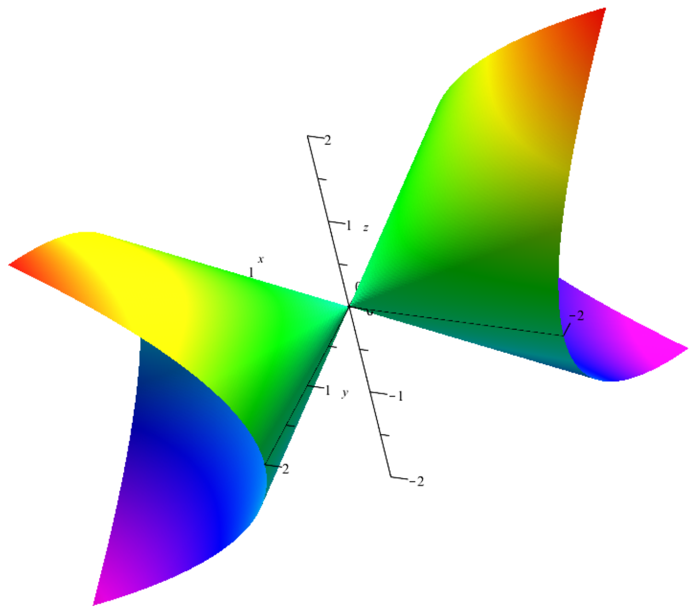
for a set of polynomials $\{f_i\} \subset k[\boldsymbol{x}]$. Observe that if a point $\boldsymbol{p} \in k^n$ satisfies $f_i(\boldsymbol{p}) = f_j(\boldsymbol{p}) = 0$ for some $f_i, f_j \in k[\boldsymbol{x}]$, then $f_i(\boldsymbol{p}) + f_j(\boldsymbol{p}) = 0$ as well. Similarly, $g(\boldsymbol{p})f_i(\boldsymbol{p}) = 0$ for any $g \in k[\boldsymbol{x}]$. So instead of considering solutions to systems of polynomials, we can instead look at solutions for the ideal $I \subset k[\boldsymbol{x}]$ generated by the polynomials of the aforementioned system. Since Hilbert's basis theorem implies that I is finitely generated, we need only look at the generators of I, which is a finite list. These solutions sets are what we call an affine variety.

Definition 2.1. Let $I \subset k[\boldsymbol{x}]$ be an ideal. Then

$$\boldsymbol{V}(I) = \{\boldsymbol{p} \in k^n \mid f(\boldsymbol{p}) = 0 \text{ for all } f \in I\}$$

is an **affine variety**.

Example 2.2. Consider the polynomial $xy - z^2 \in \mathbb{C}[x, y, z]$. Then $\boldsymbol{V}(xy - z^2)$ has real picture:



Instead of starting with a system of polynomials and obtaining an affine variety, we can go in the "other direction". Starting with an affine variety $V \subset k^n$, one can produce an ideal $I \subset k[\boldsymbol{x}]$. This is done by considering all polynomials which vanish on V:

$$\boldsymbol{I}(V) = \{f \in k[\boldsymbol{x}] \mid f(p) = 0 \text{ for all } p \in V\}.$$

This idea is used to define the most important algebraic object associated to V, it's coordinate ring.

Definition 2.3. Let $V \subset k^n$ be an affine variety. Then the **coordinate ring** of V is

$$k[V] = k[\boldsymbol{x}]/\boldsymbol{I}(V).$$

This association gives us the ability to "read off" facts about V by examining the coordinate ring $k[V]$ instead. For instance, V is irreducible (cannot be written as a union of non-empty varieties) if and only $k[V]$ is an integral domain, and two affine varieties are isomorphic if and only if their coordinate rings are isomorphic.

This is the classical way of approaching affine varieties. However the description we use for toric varieties requires an alternate definition. For this we will need to know what the spectrum of a ring is.

Definition 2.4. Let R be a commutative ring with unity. Then the **spectrum** of (and maximal spectrum) of R is

$$\operatorname{Spec} R = \{I \subset R \mid I \text{ a prime ideal}\}, \qquad \text{and} \qquad \operatorname{Specm} R = \{I \subset R \mid I \text{ a maximal ideal}\}.$$

Hilbert's Nullstellensatz [CLO07, §4] tells us that maximal ideals in $k[\boldsymbol{x}]$ are of the form

$$\mathfrak{m} = \langle x_1 - a_1, \ldots, x_n - a_n \rangle.$$

It follows that maximal ideals are in one-to-one correspondence with points in V (send ideals of the form $\langle x_i - a_i \rangle$ to the points $(a_1, \ldots, a_n)$), so one can write V as $V = \text{Specm}(k[V])$.

Example 2.5. Let $R = \mathbb{C}[x]$ be the ring of polynomials in one variable x. $\mathbb{C}$ is algebraically closed, so maximal ideals of $\mathbb{C}[\boldsymbol{x}]$ are of the form $\langle x - a \rangle$ for $a \in \mathbb{C}$. Each of these correspond uniquely to the point $a \in \mathbb{C}$, so that $\text{Specm}\, R$ is identified with the complex line. Similarly if $R = \mathbb{C}[x_1, \ldots, x_n]$, $\text{Specm}\, R$ can be identified with $\mathbb{C}^n$.

There are many important properties that an affine variety can possess, and one of the most important for this thesis is that of *normality*. Being normal is key in our discussion of toric varieties. A toric variety being normal is what allows us to use the "nice" combinatorial description via cones and fans. For an integral domain R with field of fractions K, we say that R is **normal** if R is integrally closed in K. That is, every element $k \in K$ that is a root of a monic polynomial $f \in R[x]$ is actually an element of R. We use this to define when an affine variety is normal.

Definition 2.6. An irreducible affine variety X is **normal** if its coordinate ring $k[X]$ is normal.

Finally, in our upcoming exposition on sheaves we require the notion of open and closed subsets of a variety $X \subset k^n$. That is, we need to describe the topology of an affine variety. When $k = \mathbb{C}$ there are two usual choices, the natural choice being the topology induced from the usual Euclidean topology on $\mathbb{C}^n$. The other is called the *Zariski* topology. In this topology, a subset $V \subset X$ is closed when V is a variety in k^n. The open sets are then the complements of closed sets. Now that we have the notion of topology on a variety, it is a good time to define when an affine variety is reducible or irreducible.

Definition 2.7. An affine variety X is **reducible** if there exist two closed proper subsets $V_1, V_2 \subsetneq X$ such that

$$X = V_1 \cup V_2.$$

If there is no such representation, then X is **irreducible**.

We will use the notion of *dimension* later on when introducing divisors on varieties. Since we have just talked about the topology of a variety, it is the natural time to define dimension. In fact, we will define dimension for any topological space.

Definition 2.8. Let X be a topological space. The **dimension** of X is the supremum of integers n such that there exists a chain of distinct closed irreducible subsets

$$X_0 \subset X_1 \subset \cdots \subset X_n \subset X.$$

Example 2.9. Let $X = \mathbb{C}$ with the Zariski topology. Then the only closed, irreducible subsets of X are individual points $z \in \mathbb{C}$ and the whole space itself. Thus the dimension of $\mathbb{C}$ is 1, as expected.

Finally, we will need to know what a morphism of affine varieties is.

Definition 2.10. Let X, Y be two affine varieties. A map $f\colon X \to Y$ is a **morphism** when it has the form

$$f = (f_1, \ldots, f_m)$$

where the f_i lie in $k[X]$.

We now move on to sheaves on varieties.

2.2 Sheaves

A sheaf $\mathcal{F}$ on a variety X is a tool which organizes and tracks local data on X. It is a way of making precise the notion of translating between local data and global data; the idea being that if your local data agrees on overlapping open sets, then you can "glue" the data to get something global. By local, we mean for every open set $U \subset X$ there is an associated object $\mathcal{F}(U)$. Elements of $\mathcal{F}(U)$ are referred to as *sections* over U. These objects $\mathcal{F}(U)$ can be any object as long as they satisfy the sheaf axioms. Here we define what a sheaf of abelian groups is, but this definition can be extended to other objects in a natural way. We will also discuss sheaves of rings and modules later in the thesis.

Definition 2.11. A **sheaf of abelian groups** $\mathcal{F}$ on a topological space X is a collection of abelian groups $\mathcal{F}(U)$ and restriction maps $\mathrm{res}_{U,V}\colon \mathcal{F}(U) \to \mathcal{F}(V)$ for every inclusion of open sets $W \subset V \subset U \subset X$, such that the following are satisfied:

1. (**Restriction maps**) For every inclusion of open sets $W \subset V \subset U \subset X$, we have

   $$\mathrm{res}_{V,W} \circ \mathrm{res}_{U,V} = \mathrm{res}_{U,W},$$

   where $\mathrm{res}_{U,U} = \mathrm{id}_{\mathcal{F}(U)}$. If $V \subset U$ and the superset is clear via context, then we may also write $s|_V$ for $s \in \mathcal{F}(U)$.

2. (**Locality**) If $\{U_i\}_{i\in I}$ is an open cover of an open set $U \subset X$ and if $s, t \in \mathcal{F}(U)$ are sections over U which agree on every U_i then $s = t$. That is, if $\mathrm{res}_{U,U_i}(s) = \mathrm{res}_{U,U_i}(t)$ for all $i \in I$, then $s = t$.

3. (**Gluing**) If $\{U_i\}_{i\in I}$ is an open cover of an open set $U \subset X$, and if $s_i \in \mathcal{F}(U_i)$ satisfy

$$\mathrm{res}_{U_i, U_i \cap U_j}(s_i) = \mathrm{res}_{U_j, U_i \cap U_j}(s_j)$$

for all $i, j \in I$, then there is a section $s \in \mathcal{F}(U)$ such that $\mathrm{res}_{U,U_i}(s) = s_i$ for all i.

To illustrate this idea, we give the quintessential example of a sheaf.

Example 2.12. Let $X \subset \mathbb{C}^n$ be a variety endowed with the Zariski topology, and for any open set $U \subset X$ let

$$\mathcal{C}(U) = \{f \colon U \to \mathbb{C} \mid f \text{ is continuous}\}, \qquad \mathcal{C}(\emptyset) = 0.$$

This $\mathcal{C}$ is the sheaf of continuous functions on X. It is clear the restriction maps are given by restricting the domain of the continuous function. Both the locality and gluing condition follow from the fact that continuity is a locally defined property.

For the next example we need to know what localization is.

Definition 2.13. Let R be a commutative ring, and $f \in R$. Set $S = \{f^k \mid k \in \mathbb{N}\}$. Then the **localization** of R at f, denoted R_f, is the set

$$\left\{\frac{r}{s} \mid r \in R \text{ and } s \in S\right\},$$

modulo the equivalence relation

$$\frac{r}{s} \sim \frac{r'}{s'} \iff u(rs' - r's) = 0 \text{ for some } u \in S.$$

Remark 2.14. One extends this definition of localization at elements f to localization at ideals $\mathfrak{m}$ by replacing S in the definition with $R \setminus \mathfrak{m}$.

Example 2.15. Let $X = V(S)$ be an irreducible affine variety. The **structure sheaf** on X is defined on open subsets $U \subset X$ by

$$\mathcal{O}_X(U) = \bigcap_{x \in U} R_{\mathfrak{m}_x},$$

where $\mathfrak{m}_x \subset R$ is the maximal ideal corresponding to the point x, and $R_{\mathfrak{m}_x}$ is the localization of R at $\mathfrak{m}_x$. Here the intersection is viewed as taking place in the field of fractions of R (which is okay since we assume R is an integral domain). We can also think of elements of $\mathcal{O}_X(U)$ as rational functions defined on a neighbourhood containing the point x.

Remark 2.16. For a sheaf $\mathcal{F}$ on X and for a subset $U \subset X$, one may consider the restriction of the sheaf $\mathcal{F}$ to U. This is denoted by $\mathcal{F}|_U$, and is defined on opens $V \subset U$ by $\mathcal{F}|_U(V) = \mathcal{F}(V)$. See the end of [Har13, II.1] for details.

It is also useful to know what morphisms between sheaves are, as they are key to defining a dual sheaf. This is relevant as later on the tangent sheaf is defined in terms of a dual sheaf.

Definition 2.17. If $\mathcal{F}$ and $\mathcal{G}$ are sheaves on X, a **morphism of sheaves** $\varphi\colon \mathcal{F} \to \mathcal{G}$ consists of a collection of morphisms $\varphi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ for every open set U, such that for every inclusion of open sets $V \subset U$ the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi_U} & \mathcal{G}(U) \\ \Big\downarrow{\scriptstyle \mathrm{res}_{U,V}} & & \Big\downarrow{\scriptstyle \mathrm{res}_{U,V}} \\ \mathcal{F}(V) & \xrightarrow{\varphi_V} & \mathcal{G}(V) \end{array}$$

We move on, for now, to a more general class of varieties.

2.3 Abstract Varieties

While affine varieties are the basic objects in algebraic geometry, we are interested in more general objects. To this end we introduce the notion of an abstract variety. For more details on abstract varieties, see [CLS11, §3.0] and [Har13, II.4].

The abstractness means that the variety does not come equipped with an embedding into some ambient space. So in some sense this definition captures more of the intrinsic nature of varieties. Additionally, since we are interested in smooth, complete toric varieties arising from a fan, this definition is exactly what is needed to construct a variety from the combinatorial data in the fan.

The essential idea is to ”glue together“ some collection of affine varieties. So let I be an index set, and $\{V_i\}_{i\in I}$ some finite collection of affine varieties such that for all pairs $(i,j) \in I^2$ we have Zariski open sets $V_{ji} \subset V_i$ and isomorphisms $g_{ji}\colon V_{ji} \to V_{ij}$ which satisfy

1. $g_{ij} = g_{ji}^{-1}$
2. $g_{ji}(V_{ji} \cap V_{ki}) = V_{ij} \cap V_{kj}$ for all i, j, k, and $g_{ki} = g_{kj} \circ g_{ji}$ on $V_{ji} \cap V_{ki}$.

Then set Y to be the disjoint union of the $\{V_i\}$, and define the equivalence relation $\sim$ on Y by

$$x \sim y \iff x \in V_i,\ y \in V_j \text{ for some } i, j, \text{ with } g_{ji}(x) = y.$$

Let $X = Y/\sim$ be the quotient space defined by the equivalence relation $\sim$. Observe that X is covered by images of the affine varieties V_i, each with their own structure sheaf $\mathcal{O}_{V_i}$. The gluing isomorphisms ensure that the variety structure on intersections $V_i \cap V_j$ is preserved. This induces a sheaf $\mathcal{O}_X$ on X, which restricts to $\mathcal{O}_{V_i}$ on the image of V_i in X.

Definition 2.18. The quotient space $X = Y/\sim$, together with the sheaf $\mathcal{O}_X$, is an **abstract variety**.

We note here that an abstract variety is locally affine. To see this, for any $i \in I$ set $U_i = \{\bar{x} \in X \mid x \in V_i\}$. Then the map from $V_i \to U_i$ defined by $x \mapsto \bar{x}$ is a homeomorphism. Since V_i was assumed to be affine, so is U_i.

Example 2.19. Set $V_1 = \operatorname{Spec} \mathbb{C}[x, y]$, and $V_2 = \operatorname{Spec} \mathbb{C}[z, w]$. Then take $V_{21} = \operatorname{Spec} \mathbb{C}[x, y]_y$ and $V_{12} = \operatorname{Spec} \mathbb{C}[z, w]_w$. Transition maps g_{12} and g_{21} are induced by the k-algebra homomorphisms

$$\begin{aligned} g_{21}^* &\colon \mathbb{C}[z, w]_w \to \mathbb{C}[x, y]_y & \qquad g_{12}^*, &\colon \mathbb{C}[x, y]_y \to \mathbb{C}[z, w]_w, \\ z &\mapsto \frac{1}{y}, & x &\mapsto \frac{1}{w}, \\ w &\mapsto xy, & y &\mapsto zw. \end{aligned}$$

This variety may be recognized as the blowup of $\mathbb{C}^2$ at the origin.

2.4 Divisors

For more about divisors on a variety, see [CLS11, §4.0] and [Har13, II.6]. Let X be a variety. If the goal is to understand X completely, we might want to understand all subvarieties of X as well. It turns out that studying codimension-1 subvarieties of X reveals much about the geometry of X itself. We call the irreducible codimension-1 subvarieties *prime divisors*, which leads to an invariant of X, called the divisor class group. In our case we care about divisors since, in the toric case, we can use them to give a nice decomposition in cohomology of the tangent sheaf.

Definition 2.20. Let $D \subset X$ be an irreducible codimension-1 subvariety. We call D a **prime divisor** on X.

Example 2.21. If X is a smooth irreducible plane curve, prime divisors consist of the points of X.

Definition 2.22. The free abelian group generated by prime divisors of X is denoted by $\operatorname{Div}(X)$, and elements of $\operatorname{Div}(X)$ are called **Weil divisors**.

Remark 2.23. This means that a Weil divisor D on X is a formal sum

$$D = \sum_i a_i \cdot D_i \qquad a_i \in \mathbb{Z},$$

where only finitely many a_i are nonzero and D_i are prime divisors on X.

Remark 2.24. In fact there are two kinds of divisors that are usually considered, Weil divisors and Cartier divisors. Since we are considering the smooth case, there is an isomorphism between the group of Weil divisors and the group of Cartier divisors. Hence for the sake of length we only mention Weil divisors here. The idea behind Cartier divisors is that locally, they are given by rational functions. For details see [CLS11, §4.0]

2.5 Tangent Sheaf

We first introduce the tangent sheaf $\mathcal{T}_X$ as the dual of the cotangent sheaf Ω^1_X. The notion of dual sheaf requires a few more definitions. Many of the important sheaves in this thesis are of a specific type.

Definition 2.25. Let X be a variety with structure sheaf $\mathcal{O}_X$, and let $\mathcal{F}$ be a sheaf on X. Then $\mathcal{F}$ is a **sheaf of $\mathcal{O}_X$-modules** when for $U \subset X$ the object $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module. Further, when $V \subset U$ the restriction maps $\mathcal{F}(U) \to \mathcal{F}(V)$ are compatible with the restriction maps $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$.

Definition 2.26. Let $\mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules. A **morphism of sheaves of $\mathcal{O}_X$-modules** is a morhpism of the underlying sheaves of abelien groups $\varphi \colon \mathcal{F} \to \mathcal{G}$ such that for all $U \subset X$, the map $\varphi_U \colon \mathcal{F}(U) \to \mathcal{G}(U)$ is a homomorphism of $\mathcal{O}_X(U)$-modules. We denote by $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is the set of all such morphisms.

Definition 2.27. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules on a variety X. Define the **sheaf of local morphisms** $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ on open sets $U \subset X$ by

$$\mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(U) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}|_U, \mathcal{G}|_U).$$

So sections over U consist of morphisms of sheaves restricted to U. At first glace it is not immediately obvious that this satisfies the sheaf axioms, but one can show that the locality and gluing conditions hold. As a side-note, this sheaf is also known as "sheaf hom".

Next, we need to know how to take the dual of a sheaf of $\mathcal{O}_X$-modules.

Definition 2.28. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules on an affine variety X. Then the **dual sheaf** of $\mathcal{F}$, denoted by $\mathcal{F}^\vee$, is defined by

$$\mathcal{F}^\vee = \mathscr{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X).$$

The cotangent sheaf is defined using the module of *Kähler differentials*.

Definition 2.29. The **module of Kähler differentials**, denoted by $\Omega_{\mathcal{O}_X(U)/\mathbb{C}}$, is the $\mathcal{O}_X(U)$-module generated by the symbols $\mathrm{d}f$ for $f \in \mathcal{O}_X(U)$, where the Leibniz rule is satisfied:

1. $\mathrm{d}(cf + g) = c\mathrm{d}f + \mathrm{d}g$ for all $c \in \mathbb{C}$ and $f, g \in \mathcal{O}_X(U)$.

2. $\mathrm{d}(fg) = f\mathrm{d}g + g\mathrm{d}f$ for all $f, g \in \mathcal{O}_X(U)$.

We can create a sheaf using the modules by specifying what the sections over affine open sets are.

Definition 2.30. The **cotangent sheaf** Ω^1_X on a variety X is a sheaf of $\mathcal{O}_X$-modules, defined on affine open sets $U \subset X$ by

$$\Omega^1_X(U) = \Omega_{\mathcal{O}_X(U)/\mathbb{C}}.$$

Definition 2.31. The **tangent sheaf** $\mathcal{T}_X$ on a variety X is the dual sheaf

$$\mathcal{T}_X = (\Omega^1_X)^\vee = \mathscr{H}om_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X).$$

Duality preserves the coherent property, so $\mathcal{T}_X$ is a coherent sheaf.

This definition is rather unwieldy, so in fact we use an alternate definition in terms of derivations.

Definition 2.32. Let R be a $\mathbb{C}$-algebra and M an R-module. A $\mathbb{C}$-**derivation** is a $\mathbb{C}$-linear map $\delta\colon R \to M$ which satisfies

$$\delta(fg) = \delta(f)g + f\delta(g)$$

for all $f, g \in R$.

Now we know what a derivation is, we can make the following remark.

Remark 2.33. Let R be a $\mathbb{C}$-algebra and M an R-module. The module $\Omega_{R/\mathbb{C}}$ of Kähler differentials, together with the $\mathbb{C}$-derivation $d\colon R \to \Omega_{R/\mathbb{C}}$ given by $f \mapsto df$, satisfies the following universal property. For any R-module M, and for any $\mathbb{C}$-derivation $d'\colon R \to M$ there is a unique R-module homomorphism f making the following diagram commute:

$$\begin{array}{ccc} \Omega_{R/\mathbb{C}} & \overset{f}{\dashrightarrow} & M \\ {\scriptstyle d}\uparrow & \nearrow_{d'=f\circ d} & \\ B & & \end{array}$$

In fact this can be taken to be the defining property of $\Omega_{R/\mathbb{C}}$. See e.g. [Har13, §II.8].

We now let $k = \mathbb{C}$. Denote the set of all $\mathbb{C}$-derivations $R \to M$ by $\mathrm{Der}_{\mathbb{C}}(R, M)$. To define a sheaf on X, we specify what the sections of an open affine subset look like. If $U \simeq \operatorname{Spec} R$ is an affine open subset of a variety X, then set $\mathcal{T}_X(U) = \mathrm{Der}_{\mathbb{C}}(R, R)$. Later on when working with the tangent sheaf, this is the way we are thinking of sections over open sets.

2.5.1 Lie Bracket

If we are going to use the derivations interpretation of the tangent sheaf, then one needs to be able to multiply derivations in order to have a ring structure. In this section we describe how that multiplication works.

Multiplication of two derivations f, g is given by the Lie bracket $[\ ,\ ]$, defined by $[f, g] = f \circ g - g \circ f$. This definition preserves the property of being a derivation. Namely, for some h we have

$$\begin{aligned}
d(h \circ [f, g]) &= d(h \circ f \circ g - h \circ g \circ f), \\
&= d(h \circ f \circ g) - d(h \circ g \circ f), \\
&= d(h) \circ f \circ g + h \circ d(f \circ g), \\
&= d(h) \circ f \circ g + h \circ (d(f) \circ g + f \circ d(h)) - d(h) \circ g \circ f - h(d(g) \circ f + g \circ d(f)), \\
&= d(h) \circ [f, g] + h \circ d([f, g]).
\end{aligned}$$

Since the Leibniz rule is satisfied, this is indeed a derivation.

2.6 Coherent and Quasicoherent Sheaves

When we wish to describe sheaf cohomology, the main tool we use is Čech cohomology. When the sheaves are particularly nice, Čech cohomology groups are isomorphic to the cohomology groups given by the general definition [Har13, §III.4.5]. In this section we make precise what we mean by "nice".

Definition 2.34. Let R be a commutative ring, and M an R-module. The **localization** of M at $f \in R$ is denoted M_f. This is an R_f-module satisfying the same equivalence relation as for localizing rings, together with the module structure for M.

We now introduce a new sheaf, which we denote by $\tilde{M}$. Since it is enough to specify a sheaf on a basis of open sets (see Lemma 2.1 in [Per07, III.2] for example), we define $\tilde{M}$ on distinguished open subsets (which form a basis for the topology on our variety).

Definition 2.35. Let R be a commutative ring and M an R-module. The **sheaf associated to** M on $\operatorname{Spec} R$ is denoted by $\tilde{M}$, and defined on distinguished open sets $U_f = \operatorname{Spec} R_f$ by

$$\tilde{M}(U_f) = M_f.$$

Definition 2.36. A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules on a variety X is **quasicoherent** if there is an affine open cover $\{U_i = \operatorname{Spec} R_i\}$ of X, such that for each i there is an R_i-module M_i with $\mathcal{F}|_{U_i} \simeq \tilde{M}_i$.

A sheaf is coherent if, in addition, each M_i is finitely generated. Every relevant sheaf in this thesis is quasicoherent.

Theorem 2.37. *The cotangent sheaf Ω^1_X is a coherent sheaf.*

Proof. Follows from [Har13, §II.8.15]. □

For every quasicoherent sheaf $\mathcal{F}$ it is possible to find a "good cover", for which each piece has vanishing higher cohomology. See e.g. Chapter 4. That is to say an open cover $\{U_i\}_i$ of X is "good" when any finite intersection $U = \cap_j U_j$ satisfies $H^p(U, \mathcal{F}|_U) = 0$ for all $p > 0$. See, for example, 4.11 in [Har13, III.4]. This implies that Čech cohomology is sufficient to compute the cohomology groups we are interested in.

Chapter 3

Toric Geometry

In this chapter we will give an overview of toric varieties, and the reasons why they are an appealing object of study. For more details, see the excellent book [CLS11]. We will begin by providing a definition.

Definition 3.1. An n-dimensional variety X is **toric** if the algebraic torus $(\mathbb{C}^*)^n \subset X$ is dense in X, and the natural action of the torus on itself extends to all of X.

Example 3.2. Let $X \subset \mathbb{C}^4$ be the variety defined by the vanishing of $xy - zw$. X is 3-dimensional, and has the torus

$$(\mathbb{C}^*)^3 \simeq \{(x_1, x_2, x_3, x_1 x_2 x_3^{-1}) \mid x_i \in \mathbb{C}^*\}$$

as a dense open subset.

Restricting our attention to the class of toric varieties yields several benefits. Compared to a general abstract variety, a toric variety is easier to understand, compute examples of, and provides a fertile testing ground in algebraic geometry. For these reasons, toric varieties are a well studied class of objects. In this thesis we restrict ourselves to toric varieties which are *normal*, as these are precisely the ones which allow for a combinatorial description via a fan. This description is what allows to compute concrete examples of abstract notions, in our case this is the cup product map.

While the dense open subset characterization of toric varieties is important, there is another description which allows us to work explicitly. Normal toric varieties admit a description via objects in a rational vector space, whose dimension is that of the algebraic torus embedded in X. These objects are called cones, which can be arranged in a collection called a fan. Cones correspond to affine toric varieties, while fans correspond to more general toric varieties.

As we will see, information about the variety X is encoded in the combinatorics of the fan. This encoding allows us to make explicit computations, helpful when providing an example of the nonvanishing of the cup product. We will begin with affine toric varieties.

3.1 Affine Toric Varieties

We will assume the ground field is $\mathbb{C}$, although most of what follows would be fine if we only assumed a ground field which is algebraically closed. We have seen already that a *variety* V is the set of common zeroes of a collection of polynomials $\{f_1, \ldots, f_s\} \subset \mathbb{C}[x_1, \ldots, x_r]$. In what follows we give a way to construct affine toric varieties via combinatorics, rather than the embedded torus idea. We follow closely the notation found in [CLS11]. That means, among other things, that we use the following definition for lattice.

Definition 3.3. A **lattice** is a free abelian group of finite rank.

This means that a lattice N of rank n is isomorphic to $\mathbb{Z}^n$, by sending the n generators of N to the standard basis vectors of $\mathbb{Z}^n$. Thus for the rest of this thesis, N is a lattice of rank n, with a corresponding dual lattice M. For us, M is the character lattice of the torus.

Definition 3.4. A **character** of a torus T is a map $\chi\colon T \to \mathbb{C}^*$.

Example 3.5. Let $T = (\mathbb{C}^*)^n$ and $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$. Then $\chi^m\colon T \to \mathbb{C}^*$ defined by

$$(t_1, \ldots, t_n) \mapsto t_1^{m_1} \cdots t_n^{m_n}$$

is a character. In fact, *all* characters arise in this manner, so $M = \mathrm{Hom}(T, \mathbb{C}^*) \simeq \mathbb{Z}^n$.

The dual to the character lattice is the lattice of one-parameter subgroups.

Definition 3.6. Let $T = (\mathbb{C}^*)^n$. A **one-parameter subgroup** of T is a morphism $\lambda\colon \mathbb{C}^* \to T$ that is a group homomorphism.

Example 3.7. For $u = (u_1, \ldots, u_n) \in \mathbb{Z}^n$ there is a one-parameter subgroup $\lambda^u\colon \mathbb{C}^* \to (\mathbb{C}^*)^n$ given by

$$t \mapsto (t^{u_1}, \ldots, t^{u_n}).$$

In fact, all one-parameter subgroups arise in this manner, so $N = \mathrm{Hom}(\mathbb{C}^*, T) \simeq \mathbb{Z}$.

We let $\langle \cdot, \cdot \rangle\colon N \times M \to \mathbb{Q}$ be the natural bilinear pairing. The associated $\mathbb{Q}$-vector spaces are denoted $N_{\mathbb{Q}} = N \otimes \mathbb{Q}$ and $M_{\mathbb{Q}} = M \otimes \mathbb{Q}$. With this in mind, we describe how to construct an affine toric variety from combinatorial data in the lattice N. The basic "building blocks" are cones.

Definition 3.8. A **convex polyhedral cone** $\sigma \subset N_{\mathbb{Q}}$ is the set of all positive combinations of a finite set of lattice points $V = \{v_1, \ldots, v_s\} \in N$. Explicitly,

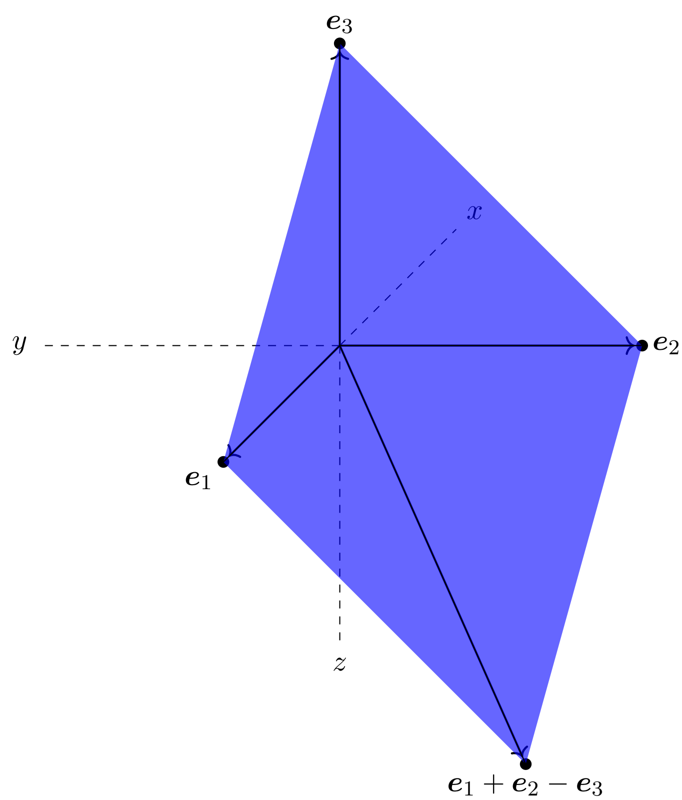
$$\sigma = \mathrm{Cone}(V) = \{\lambda_1 v_1 + \ldots + \lambda_s v_s \mid \lambda_i \geq 0\} \subset N_{\mathbb{Q}}.$$

We say that σ is *generated* by $\{v_1, \ldots, v_s\}$. From now on when we say "cone", we mean a convex polyhedral cone.

Example 3.9. Let $\boldsymbol{e}_i$ be the standard basis vectors for $\mathbb{Q}^3$ and set

$$V = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_1 + \boldsymbol{e}_2 - \boldsymbol{e}_3\}.$$

Then $\mathrm{Cone}(V) \subset N_{\mathbb{Q}}$ looks like



Given a cone $\sigma \subset N_{\mathbb{Q}}$, there is a naturally occurring cone in the dual space, called the dual cone.

Definition 3.10. For a cone σ, the **dual cone** $\sigma^\vee \subset M_{\mathbb{Q}}$ consists of all elements of the dual vector space $M_{\mathbb{Q}}$ which pair non-negatively with every element of σ. Explicitly,

$$\sigma^\vee = \{m \in M_{\mathbb{Q}} \mid \langle x, m \rangle \geq 0 \text{ for all } x \in \sigma\}.$$

On can show that (see [CLS11, §1.2.4] for instance) $(\sigma^\vee)^\vee = \sigma$, and that $\sigma^\vee$ is a cone in $M_{\mathbb{Q}}$. Now, fix $m \in M_{\mathbb{Q}}$ and consider the following two sets:

$$H_m^+ = \{x \in N_{\mathbb{Q}} \mid \langle x, m \rangle \geq 0\} \qquad \text{and} \qquad H_m = \{x \in N_{\mathbb{Q}} \mid \langle x, m \rangle = 0\}$$

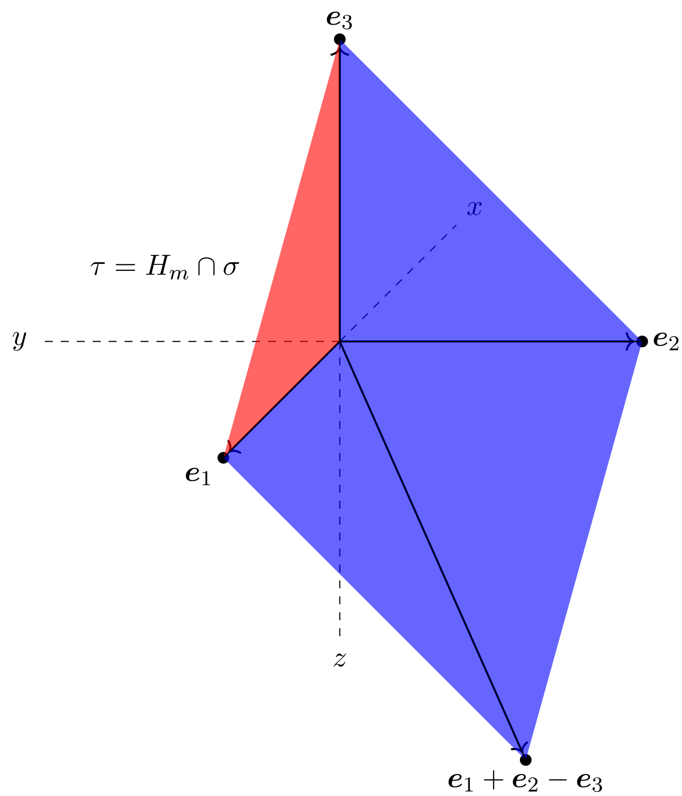
Definition 3.11. We call H_m a **supporting hyperplane** of a cone σ if $\sigma \subset H_m^+$.

Example 3.12. Continuing our previous example, set $m = (0, 1, 0) \in M_{\mathbb{Q}}$. Then H_m is a supporting hyperplane for $\mathrm{Cone}(V)$.

One of the important properties of a cone is the notion of a *face*, defined using supporting hyperplanes.

Definition 3.13. We say that τ is a **face** of a cone $\sigma \subset N_{\mathbb{Q}}$ if $\tau = H_m \cap \sigma$ for some $m \in \sigma^{\vee}$.

Example 3.14. In our continuing example, a face of σ can be obtained by setting $m = (0,1,0) \in M_{\mathbb{Q}}$. For this choice of m, the supporting hyperplane lies on the xz-axis. Thus by intersecting with σ we get the indicated red region τ, which is a face of σ.



We are now ready to construct the affine variety U_σ associated to a cone $\sigma \subset N_{\mathbb{Q}}$. For technical reasons, one constructs U_σ via the corresponding dual cone under the pairing $\langle \cdot, \cdot \rangle$.

For a cone $\sigma \in N_{\mathbb{Q}}$, consider the intersection $\sigma^{\vee} \cap M$ of the dual cone with the lattice M. This gives an affine semigroup S_σ (which is finitely generated by Gordan's Lemma, see [CLS11, §1.2.17]). From this semigroup, one constructs the semigroup algebra

$$\mathbb{C}[S_\sigma] = \left\{ \sum_{m \in S_\sigma} c_m \chi^m \mid c_m \in \mathbb{C}, \text{ and all but finitely many } c_m \text{ are } 0 \right\},$$

with multiplication induced by $\chi^m \cdot \chi^{m'} = \chi^{m+m'}$. Now we set $U_\sigma = \operatorname{Spec} \mathbb{C}[S_\sigma]$. This is an affine toric variety by [CLS11, §1.1.14].

We can use these affine toric varieties to construct more general toric varieties, which arise by "gluing together" affine toric varieties U_σ. In fact, one can simply glue the cones σ together. This creates an object called a fan, usually denoted by Σ.

3.2 Toric Varieties

In this section we describe the combinatorial aspect of toric varieties, which allows us to give nice descriptions of several geometric properties. We begin by defining what a fan is.

Definition 3.15. A **fan** Σ in $N_{\mathbb{Q}}$ is a finite collection of cones satisfying two properties:

(a) Σ *is closed under taking faces.* For all $\sigma \in \Sigma$, every face of σ lies in Σ.

(b) *Compatibility condition.* For all $\sigma, \tau \in \Sigma$, the intersection $\sigma \cap \tau$ is a face of both σ and τ.

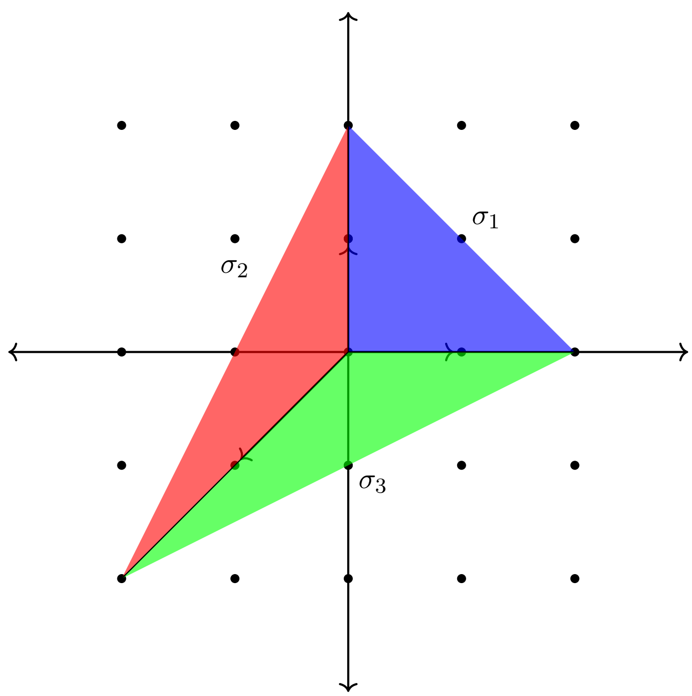
A fan Σ lives in the vector space $N_{\mathbb{Q}}$, and the **support** of Σ is

$$\operatorname{Supp}(\Sigma) = \bigcup_{\sigma \in \Sigma} \sigma \subset N_{\mathbb{Q}}.$$

Example 3.16. Consider the lattice points $\boldsymbol{v}_1 = (1, 0)$, $\boldsymbol{v}_2 = (0, 1)$, and $\boldsymbol{v}_3 = (-1, -1)$, and let ρ_i be the ray generated by $\boldsymbol{v}_i$. From the $\boldsymbol{v}_i$ we can specify the cones

$$\sigma_1 = \operatorname{Cone}(\boldsymbol{v}_1, \boldsymbol{v}_2), \qquad \sigma_2 = \operatorname{Cone}(\boldsymbol{v}_2, \boldsymbol{v}_3), \qquad \sigma_3 = \operatorname{Cone}(\boldsymbol{v}_1, \boldsymbol{v}_3).$$

The set $\Sigma = \{\sigma_1, \sigma_2, \sigma_3, \rho_1, \rho_2, \rho_3, (0, 0)\}$ is a fan in $N_{\mathbb{Q}}$. The fan Σ looks like:



In fact this is the fan for the projective plane $\mathbb{P}^2$ which has support $\operatorname{Supp}(\Sigma) = \mathbb{Q}^2$.

It is often convenient to refer to cones of Σ of a specific dimension, especially the one-dimensional cones.

Definition 3.17. The set of k-dimensional cones of a fan Σ is denoted by $\Sigma(k)$. The elements of $\Sigma(1)$ are called **rays** of Σ.

For any fan $\Sigma \in N_{\mathbb{Q}}$ one can construct a toric variety X by first constructing U_σ for each $\sigma \in \Sigma$, then by gluing the U_σ together along intersections (the details of which can be found in [CLS11, §3.1]). This works due to property (b) in the definition of a fan. If σ and τ are cones of a fan Σ, then $\sigma \cap \tau$ is again a cone, and has associated affine variety $U_{\sigma\cap\tau}$. There are natural inclusions $U_{\sigma\cap\tau} \subset U_\sigma$ and $U_{\sigma\cap\tau} \subset U_\tau$. Hence we can glue U_σ and U_τ along the common open subset $U_{\sigma\cap\tau}$. The point of this is to say that the set $\{U_\sigma \mid \sigma \in \Sigma\}$ forms an affine open cover of X. Since an inclusion of cones corresponds to an inclusion of open sets, we can restrict our attention to *maximal* cones of Σ. Now we turn our attention to the notion of *completeness*.

Definition 3.18. A fan Σ in $N_{\mathbb{Q}}$ is **complete** when

$$\operatorname{Supp}(\Sigma) = N_{\mathbb{Q}}.$$

Example 3.19. The fan Σ for $\mathbb{P}^2$ is complete, since $\operatorname{Supp}(\Sigma)$ is all of $N_{\mathbb{Q}}$.

By [CLS11, §3.4.6] a toric variety X is complete precisely when the fan Σ is complete (note this is yet another example of properties of toric varieties being encoded in the information of the fan). In brief, completeness of a variety is the algebraic analogue of compactness as a topological space. More precisely, a variety X is complete when for every variety Z, the projection map $X \times Z \to Z$ is closed in the Zariski topology.

Throughout this thesis we assume X (and therefore Σ) is complete, so it suffices to consider n-dimensional cones for our open cover. We denote this affine open cover by

$$\mathcal{U} = \{U_\sigma \mid \sigma \in \Sigma(n)\}.$$

It is also useful to be able to talk about those elements of the lattice N which generate the rays of Σ.

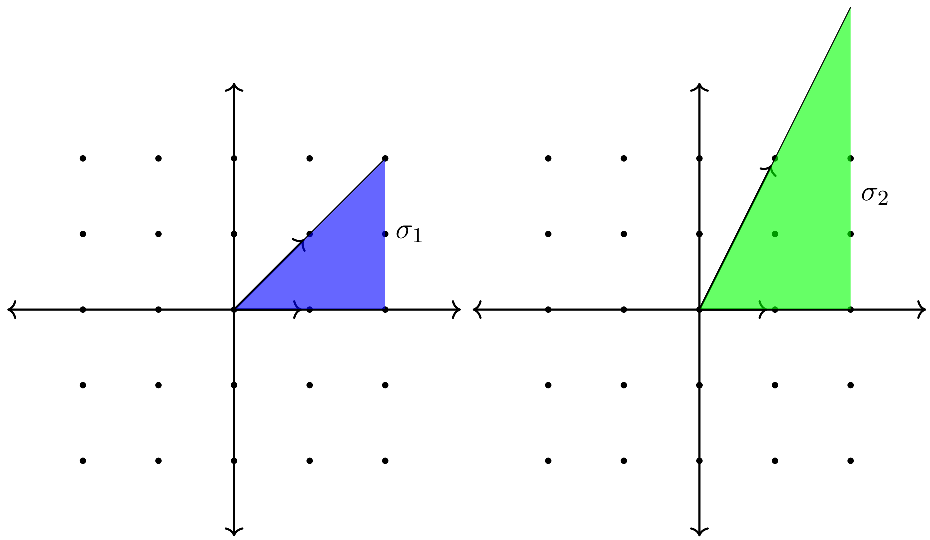
Definition 3.20. For each ray $\rho \in \Sigma(1)$ there is an associated semigroup $\rho \cap N$. Since ρ is a ray, $\rho \cap N$ has a unique generator, which we denote by $\boldsymbol{v}_\rho$. This is called the **ray generator** for ρ.

Observe that for any cone σ, the set of ray generators $\{\boldsymbol{v}_\rho \mid \rho \in \sigma(1)\}$ is a set of generators for σ as a cone. That is, we can specify cones just by giving their ray generators.

One of most appealing reasons to study toric geometry is the fact that important properties of the variety X can be "read off" of the data of the fan Σ. We have seen already that completeness is one such property, and now we will see that smoothness is another. As with most properties of a fan Σ, smoothness is defined in terms of the cones of Σ. That is, Σ is smooth when each cone of Σ is smooth. But when are cones smooth?

Definition 3.21. A cone $\sigma \subset N_{\mathbb{Q}}$ generated by $\{\boldsymbol{v}_\rho \mid \rho \in \sigma(1)\}$ is **smooth** when the set $\{v_\rho \mid \rho \in \sigma(1)\}$ is a subset of some $\mathbb{Z}$-basis of N.

Example 3.22. For example, consider the the cones σ_1 and σ_2 in $N = \mathbb{Q}^2$.



The cone σ_1 is generated by $(1,0)$ and $(1,1)$. These form a $\mathbb{Z}$-basis for our lattice, so σ_1 is smooth by definition. However, the ray generators for σ_2 are $(1,0)$ and $(1,2)$. These cannot form part of a basis since, for example, the lattice point $(1,1)$ cannot be expressed as a $\mathbb{Z}$-linear combination of $(1,0)$ and $(1,2)$.

Having the definition of smoothness for cones now allows us to define smoothness for fans in the natural way.

Definition 3.23. A fan Σ is smooth when every cone $\sigma \in \Sigma$ is smooth.

There is another nice theorem relating the smoothness of the fan to the smoothness of the associated toric variety.

Theorem 3.24 ([CLS11])**.** *Let X be a toric variety with associated fan Σ. Then X is smooth as a variety if and only if Σ is a smooth fan.*

3.3 Torus Invariant Divisors

One of the nice properties of smooth toric varieties is the decomposition in cohomology of the tangent sheaf, in terms of certain sheaves soon to be described. These sheaves arise from torus invariant divisors. Recall that a divisor D on a variety X is a finite $\mathbb{Z}$-combination of codimension-1 subvarieties (prime divisors). We briefly explain how the torus invariant prime divisors are found.

Let X be a toric variety of dimension n with associated fan Σ. First, recall that for a cone $\sigma \in \Sigma$ we have an associated affine open U_σ. In fact, the points in U_σ are in one-to-one correspondence with semigroup homomorphisms $\sigma^\vee \cap M = S_\sigma \to \mathbb{C}$ [CLS11, §1.3.1]. Let $\gamma_\sigma \in U_\sigma$ be the point corresponding to the semigroup homomorphism $S_\sigma \to \mathbb{C}$ defined by

$$m \mapsto \begin{cases} 1 & \text{if } m \in \sigma^\perp \cap M, \\ 0 & \text{otherwise.} \end{cases}$$

Then the torus orbit associated to the cone σ is given by

$$O(\sigma) = T_N \cdot \gamma_\sigma,$$

where T_N denotes the torus. Then [CLS11, §3.2.6] tells us that if the dimension of σ is k, then the dimension of $O(\sigma)$ is $n - k$.

When $k = 1$, taking the closure gives an irreducible codimension one subvariety, which is exactly the definition of a prime divisor. Thus, for any ray $\rho \in \Sigma(1)$ we have a torus-invariant prime divisor, denoted D_ρ. In fact any torus invariant divisor D on X can represented as a sum

$$D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho, \qquad a_\rho \in \mathbb{Z}.$$

See [CLS11, §4.1] for details.

3.4 Sheaves Associated to Divisors

Later on in Theorem 5.2 we see that for toric varieties, there is a decomposition in cohomology for the tangent sheaf $\mathcal{T}_X$ in terms of the sheaves $\mathcal{O}_X(D_\rho)$. These sheaves $\mathcal{O}_X(D_\rho)$ are associated to the torus invariant boundary divisor D_ρ. In this section we briefly cover the association between the divisor and the sheaf. For the details of this association see [CLS11, §4]. These sheaves are what allow us to decompose the cohomology for the tangent sheaf $\mathcal{T}_X$ as a direct sum. Recall that a variety X has an associated sheaf $\mathcal{O}_X$ called the *structure sheaf*. Over an open subset $U \subset X$, the set $\mathcal{O}_X(U)$ consists of those rational functions of X which are defined everywhere on U. The sheaf of a divisor is closely related to $\mathcal{O}_X$.

First we will describe *principal divisors*. These are divisors which arise from elements of $\mathbb{C}(X)^*$. The idea is to generalize the notion of order of vanishing at a point.

Example 3.25. Consider the set of rational functions in one variable x, denoted by $\mathbb{C}(x)$. Elements $f \in \mathbb{C}(x)$ are of the form $f(x) = \frac{h(x)}{g(x)}$. A natural question is, how does f behave at points $\alpha \in \mathbb{C}$? Since $\mathbb{C}[x]$ is a unique factorization domain, it is possible to factor out all copies of $(x - \alpha)$ from both $h(x)$ and $g(x)$. This means that f can be written uniquely as

$$f(x) = \frac{h(x)}{g(x)} = (x - \alpha)^n \frac{h_0(x)}{g_0(x)},$$

for some $h_0(x)$ and $g_0(x)$. Then the value of n is the order of vanishing of f at the point α.

In the example, the variety we are considering was just $\mathbb{C}$, and the prime divisors of $\mathbb{C}$ are just the points α. Now, for a variety X and prime divisor D, we generalize by constructing a function (called a *valuation*) $\nu_D \colon \mathbb{C}(X)^* \to \mathbb{Z}$ for any prime divisor D. Then the order of vanishing of f along D is the integer $\nu_D(f)$. We can use this valuation to construct a Weil divisor for every $f \in \mathbb{C}(X)^*$ by setting $\mathrm{div}(f) = \sum_D \nu_D(f)D$. Now we're ready to describe what the sections of $\mathcal{O}_X(D)$ look like over an open set U.

Definition 3.26. The **sheaf of a Weil divisor** D on a toric variety X is defined on open subsets $U \subset X$ by

$$\mathcal{O}_X(D)(U) = \{f \in \mathbb{C}(X)^* \mid \mathrm{div}(f) + D|_U \geq 0\} \cup \{0\}.$$

The sum $\mathrm{div}(f) + D$ is done componentwise, and the condition $\mathrm{div}(f) + D \geq 0$ means that all coefficients are greater or equal to 0.

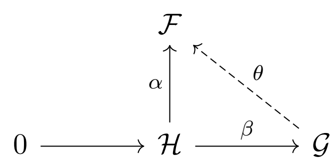
We note here that $\mathcal{O}_X(D)$ is a coherent sheaf of $\mathcal{O}_X$ modules [CLS11, §4.0.27]. In particular, this allows us to use Čech cohomology to represent $H^1(X, \mathcal{O}_X(D_\rho))_u$. We now move on to the central tool used in this thesis, Čech cohomology.

Chapter 4

Sheaf Cohomology

For an expanded treatment of sheaf cohomology, see [Har13, III.1,2] or [CLS11, §9.0] for a short primer. We have defined earlier in Section 2.2 what a sheaf is. Now we define some notions essential for defining sheaf cohomology.

Definition 4.1. A sheaf $\mathcal{F}$ is **injective** when for any sheaf homomorphisms $\alpha\colon \mathcal{H} \to \mathcal{F}$ and $\beta\colon \mathcal{H} \to \mathcal{G}$ there exists a sheaf homomorphism $\theta\colon \mathcal{G} \to \mathcal{F}$ making the following diagram commute:



Every sheaf $\mathcal{F}$ of abelian groups has an injective resolution (see e.g. [Har13, III.1])

$$0 \to \mathcal{F} \to \mathcal{F}^0 \xrightarrow{d^0} \mathcal{F}^1 \xrightarrow{d^1} \cdots$$

which is denoted by $F^{\bullet}$. Considering the global sections of this resolution gives a complex of abelian groups

$$\Gamma(X, \mathcal{F}^{\bullet})\colon \Gamma(X, \mathcal{F}^0) \xrightarrow{d^0} \Gamma(X, \mathcal{F}^1) \xrightarrow{d^1} \Gamma(X, \mathcal{F}^2) \xrightarrow{d^2} \cdots$$

Here, the term *complex* means that the maps d^i satisfy $d^i \circ d^{i-1} = 0$.

Definition 4.2. Let $\mathcal{F}$ be a sheaf of abelian groups, with $0 \to \mathcal{F} \to \mathcal{F}^{\bullet}$ an injective resolution. Then the p**-th cohomology group** of $\mathcal{F}$ with respect to a topological space X is

$$H^p(X, \mathcal{F}) = \ker d^p / \operatorname{im} d^{p-1}.$$

Note that the cohomology groups might seem to depend on the choice of injective resolution — in fact it can be shown that they are independent of choice of resolution. Different choices of resolution give isomorphic cohomology groups, see e.g. [Har13, III.1].

Remark 4.3. We often need to compute the cohomology of a sheaf $\mathcal{F}$ of rings or $\mathcal{O}_X$-modules. In these cases we consider $\mathcal{F}$ as a sheaf of abelian groups instead, and compute cohomology by applying Definition 4.2.

While this definition is powerful and general, it is often impractical to calculate cohomology groups this way. Fortunately, there is a tool which can compute cohomology groups in many nice cases (including the cases we are interested in).

4.1 Čech Cohomology

We now describe the main tool used in this thesis, Čech cohomology, named after the mathematician Eduard Čech (1892–1960). See [Har13, §III.4], [CLS11, §9.0], or [Bos12, §7.6] for more details. Let $\mathcal{F}$ be a sheaf of abelian groups on a variety X. The idea is to cover X with open sets $\{U_i\}_{i\in I}$ and compute sections of $\mathcal{F}$ over all the U_i, as well as all possible intersections. So suppose that X has an open cover $\mathcal{U} = \{U_i\}_{i\in I}$, and set

$$U_{i_0,\dots,i_p} = U_{i_0} \cap \ldots \cap U_{i_p}.$$

Then for a sheaf $\mathcal{F}$ on X we have the following definition.

Definition 4.4. The group of **Čech p-cochains** on $\mathcal{U}$ with values in $\mathcal{F}$ is

$$\check{C}^p(\mathcal{U},\mathcal{F}) = \prod_{i_0,\dots,i_p} \mathcal{F}(U_{i_0,\dots,i_p}).$$

Note that the product runs over all possible choices for indices. That is, there are $(p+1)!$ multiplicands in the product. Now to turn these Čech groups into a complex we need a differential, or coboundary, map.

Definition 4.5. The **p-th Čech differential** $d^p\colon \check{C}^p(\mathcal{U},\mathcal{F}) \to \check{C}^{p+1}(\mathcal{U},\mathcal{F})$ is defined by

$$(d^p\alpha)_{i_0,\dots,i_{p+1}} = \sum_{k=0}^{p+1}(-1)^k \alpha_{i_0,\dots,\hat{i}_k,\dots,i_{p+1}}|_{U_{i_0,\dots,i_{p+1}}}.$$

Here, $\hat{i}_k$ means omit the index i_k.

One can verify that $d^{p+1}\circ d^p = 0$, so that

$$\check{C}^\bullet(\mathcal{U},\mathcal{F})\colon 0 \to \check{C}^0(\mathcal{U},\mathcal{F}) \xrightarrow{d^0} \check{C}^1(\mathcal{U},\mathcal{F}) \xrightarrow{d^1} \check{C}^2(\mathcal{U},\mathcal{F}) \xrightarrow{d^2} \cdots$$

is a *complex*, called the Čech complex. Once we have $\check{C}^\bullet(\mathcal{U},\mathcal{F})$, we can then take the cohomology of this complex.

Definition 4.6. For a sheaf $\mathcal{F}$ and cover $\mathcal{U}$, the **p-th Čech cohomology group** is

$$\check{H}^p(\mathcal{U},\mathcal{F}) = H^p(\check{C}^\bullet(\mathcal{U},\mathcal{F})) = \ker d^p / \operatorname{im} d^{p-1}.$$

Observe that $\check{H}^0(\mathcal{U},\mathcal{F}) = \ker d^0$. If $(a_i)_{i\in I}$ is an element of $\check{C}^0(\mathcal{U},\mathcal{F})$ then $d^0(a_i)_{ij} = 0$ implies $a_i = a_j$ on the intersection $U_i \cap U_j$. Since the global sections $\mathcal{F}(X)$ are exactly those elements which satisfy this rule, it follows that $\check{H}^0(\mathcal{U},\mathcal{F}) = \mathcal{F}(X) = H^0(X,\mathcal{F})$.

This equality doesn't necessarily hold for $p > 0$, but when $\mathcal{F}$ and $\mathcal{U}$ are sufficiently nice we do get isomorphic cohomology groups, as we see in the following theorem.

Theorem 4.7. *Let $\mathcal{U}$ be an affine open cover of a variety X, and let $\mathcal{F}$ be a quasicoherent sheaf on X. Then $\check{H}^p(\mathcal{U},\mathcal{F}) \simeq H^p(X,\mathcal{F})$.*

Proof. See the proof in [Har13, §III.4.5]. □

Since we are always taking the natural open affine cover $\{U_\sigma\}_{\sigma\in\Sigma}$, and the sheaves $\mathcal{T}_X$ and $\mathcal{O}_X(D_\rho)$ are coherent (and hence quasicoherent), this theorem tells us we may freely use Čech cohomology to describe the cohomology groups we are interested in.

While we are mainly interested in the sheaf cohomology of a smooth complete toric variety, our combinatorial description of $H^1(X,\mathcal{T}_X)$ and $H^2(X,\mathcal{T}_X)$ relies on *singular cohomology groups* $H^p(Z,\mathbb{C})$ where Z is a topological space. In our situation the topological space Z is locally contractible, which allows us to use sheaf cohomology instead. For each open set $U \subset Z$, let $\mathcal{F}(U) = \mathbb{C}$. This defines a presheaf $\mathcal{F}$ on Z, whose sheafification is the *constant sheaf.* By [Bre12, §III.1], the sheaf cohomology of this constant sheaf is the same as the singular cohomology. Note that the constant sheaf is not coherent, so 4.7 doesn't apply. However we can still use Čech cohomology to compute these groups, since we will be using a cover where all intersections are locally contractible. See e.g. [God58, §II.5.2].

4.2 Alternating and Ordered Čech Cohomology

While the definition of singular Čech p-cochains is natural (in the sense that there is no choice involved with the indices), it is unwieldy to work with, as the number of summands in the group $\check{C}^p(\mathcal{U},\mathcal{F})$ grows factorially in p. This, along with other reasons, make it attractive to look for other definitions for Čech cochains and complexes, which preserve the cohomology. To this end, we have the following definition.

Definition 4.8. The group of **alternating Čech p-cochains** on $\mathcal{U}$ with values in $\mathcal{F}$ is

$$\check{C}^p_{\text{alt}}(\mathcal{U},\mathcal{F}) \hookrightarrow \check{C}^p(\mathcal{U},\mathcal{F}),$$

and consists of those p-cochains α which satisfy

$$\alpha_{i_0,\ldots,i_p} = \begin{cases} 0 & \text{if } i_0,\ldots,i_p \text{ are not pairwise distinct,} \\ \operatorname{sgn}(\pi)\cdot \alpha_{i_{\pi(0)},\ldots,i_{\pi(p)}} & \text{for any } \pi \in S_{p+1}. \end{cases} \tag{4.1}$$

Here S_{p+1} is the group of permutations on $p+1$ symbols.

Using the same definition for the differential on Čech p-cochains, we obtain the complex $\check{C}^\bullet_{\text{alt}}(\mathcal{U},\mathcal{F})$ of alternating Čech p-cochains. Looking at the cohomology groups, we obtain

$$\check{H}^p_{\text{alt}}(\mathcal{U},\mathcal{F}) = H^p(\check{C}^\bullet_{\text{alt}}(\mathcal{U},\mathcal{F})).$$

If an ordering $<$ on the indices I is fixed, then we can define another important Čech complex.

Definition 4.9. The group of **ordered Čech p-cochains** on $\mathcal{U}$ with values in $\mathcal{F}$ is

$$\check{C}^p_{\text{ord}}(\mathcal{U},\mathcal{F}) = \bigoplus_{i_0<\ldots<i_p} \mathcal{F}(U_{i_0,\ldots,i_p}).$$

Again we obtain the cohomology group

$$\check{H}^p_{\text{ord}}(\mathcal{U},\mathcal{F}) = H^p(\check{C}^\bullet_{\text{ord}}(\mathcal{U},\mathcal{F})).$$

This definition is useful when computing explicit examples of cohomology groups.

One can check that $\check{H}^p_{\text{alt}}(\mathcal{U},\mathcal{F}) \simeq \check{H}^p_{\text{ord}}(\mathcal{U},\mathcal{F})$ by $c\colon \check{C}^\bullet_{\text{ord}}(\mathcal{U},\mathcal{F}) \to \check{C}^\bullet_{\text{alt}}(\mathcal{U},\mathcal{F})$, where

$$c(s)_{i_0,\ldots,i_p} = \begin{cases} 0 & \text{if } i_j = i_k \text{ for some } i \neq k, \\ \operatorname{sgn}(\sigma) s_{i_{\sigma(0)},\ldots,i_{\sigma(p)}} & \text{if } i_{\sigma(0)} < \cdots < i_{\sigma(p)}. \end{cases}$$

Less straightforward is the fact that $\check{H}^p_{\text{alt}}(\mathcal{U},\mathcal{F})$ is isomorphic to $\check{H}^p(\mathcal{U},\mathcal{F})$.

Theorem 4.10. *Let $\mathcal{U}$ be an affine open cover of a variety X. Then for all $p \geq 0$, the inclusion $\iota\colon \check{C}^\bullet_{\text{alt}}(\mathcal{U},\mathcal{F}) \hookrightarrow \check{C}^\bullet(\mathcal{U},\mathcal{F})$ induces isomorphisms of cohomology groups*

$$\check{H}^p_{\text{alt}}(\mathcal{U},\mathcal{F}) \simeq \check{H}^p(\mathcal{U},\mathcal{F}).$$

Proof. See the full proof in [Bos12, §7.6]. The idea is to show that ι is a homotopy equivalence by producing a complex homomorphism $q\colon \check{C}^\bullet(\mathcal{U},\mathcal{F}) \to \check{C}^\bullet_{\text{alt}}(\mathcal{U},\mathcal{F})$ such that $q \circ \iota$ is the identity on $\check{C}^\bullet_{\text{alt}}(\mathcal{U},\mathcal{F})$ and $i \circ q$ is homotopic to the identity on $\check{C}^\bullet(\mathcal{U},\mathcal{F})$. This implies that the cohomology groups of $\check{C}^\bullet(\mathcal{U},\mathcal{F})$ and $\check{C}^\bullet_{\text{alt}}(\mathcal{U},\mathcal{F})$ are isomorphic, and that the induced map on cohomology from q is an isomorphism.

The standard choice for q used by Bosch is described by $\tilde{\gamma}_p \colon \check{C}^p(\mathcal{U}, \mathcal{F}) \to \check{C}^p_{\text{alt}}(\mathcal{U}, \mathcal{F})$ where

$$\tilde{\gamma}_p(g)_{i_0,\dots,i_p} = \begin{cases} 0 & \text{if } i_0, \dots, i_p \text{ are not pairwise distinct,} \\ \operatorname{sgn}(\pi) g_{i_{\pi(0)},\dots,i_{\pi(p)}} & \text{otherwise.} \end{cases}$$

Here π is the permutation in S_{p+1} satisfying $i_{\pi(0)} < \ldots < i_{\pi(p)}$ (once an ordering has been fixed on the indices i_j). We alter the choice of q by using the following morphism instead:

$$\tilde{\alpha}_p(g)_{i_0,\dots,i_p} = \begin{cases} 0 & \text{if } i_0, \dots, i_p \text{ are not pairwise distinct,} \\ \frac{1}{(p+1)!} \sum_{\pi \in S_{p+1}} \operatorname{sgn}(\pi) g_{i_{\pi(0)},\dots,i_{\pi(p)}} & \text{otherwise.} \end{cases} \tag{4.2}$$

This gives a morphism of complexes $q' \colon \check{C}^\bullet(\mathcal{U}, \mathcal{F}) \to \check{C}^\bullet(\mathcal{U}, \mathcal{F})$. To see that q' is a morphism of complexes, we have to check that for each p the following diagram commutes:

$$\begin{array}{ccc} \check{C}^p(\mathcal{U}, \mathcal{F}) & \xrightarrow{d^p} & \check{C}^{p+1}(\mathcal{U}, \mathcal{F}) \\ \downarrow{\scriptstyle \tilde{\alpha}_p} & & \downarrow{\scriptstyle \tilde{\alpha}_{p+1}} \\ \check{C}^p_{\text{alt}}(\mathcal{U}, \mathcal{F}) & \xrightarrow{d^p} & \check{C}^{p+1}_{\text{alt}}(\mathcal{U}, \mathcal{F}) \end{array}$$

That is, we need to show that $\tilde{\alpha}_{p+1} \circ d^p = d^p \circ \tilde{\alpha}_p$. Let $g \in \check{C}^p(\mathcal{U}, \mathcal{F})$ be contained in one summand. Without loss of generality, say $g \in \mathcal{F}(U_{j_0} \cap \cdots \cap U_{j_p})$. We may assume that there are no repeated indices (else both $\tilde{\alpha}_{p+1} \circ d^p(g) = d^p \circ \tilde{\alpha}_p(g) = 0$). Now let $i_0, \dots, i_p$ be a set of nonrepeating indices. For each $0 \le \ell \le p+1$ there is a unique permutation $\sigma \in S_{p+2}$ such that $\sigma(\ell) = \ell$ and

$$(j_0, \dots, j_p) = (i_{\sigma(0)}, \dots, \widehat{i_k}, \dots, i_{\sigma(p+1)}). \tag{4.3}$$

Now computing the image for g we see that on the one hand we have

$$\begin{aligned} d^p\left(\tilde{\alpha}_p(g)\right)_{i_0,\dots,i_{p+1}} &= \sum_{k=0}^{p+1} (-1)^k \tilde{\alpha}_p(g)_{i_0,\dots,\widehat{i_k},\dots,i_{p+1}}, \\ &= \sum_{k=0}^{p+1} (-1)^k \frac{1}{(p+1)!} \sum_{\pi \in S_{p+1}} \operatorname{sgn}(\pi) g_{i_{\pi(0)},\dots,\widehat{i_k},\dots,i_{\pi(p+1)}}, \\ &= \frac{(-1)^\ell}{(p+1)!} \operatorname{sgn}(\sigma) g_{j_0,\dots,j_p}. \end{aligned}$$

The last equality follows from (4.3), and the fact that g was assumed to lie in one summand.

On the other hand we have

$$\begin{aligned}\tilde{\alpha}_{p+1}\left(d^p(g)\right)_{i_0,\dots,i_{p+1}} &= \frac{1}{(p+2)!}\sum_{\pi\in S_{p+2}} \operatorname{sgn}(\pi) d^p(g)_{i_{\pi(0)},\dots,i_{\pi(p+1)}},\\ &= \frac{1}{(p+2)!}\sum_{\pi\in S_{p+2}} \operatorname{sgn}(\pi)\sum_{k=0}^{p+1}(-1)^k g_{i_{\pi(0)},\dots,\widehat{i_{\pi(k)}},\dots,i_{\pi(p+1)}}.\end{aligned}$$

Observe that for each $0 \le k \le p+1$, there is exactly one permutation $\pi \in S_{p+2}$ such that $\pi(k) = \ell$ and

$$(j_0, \dots, j_p) = (i_{\pi(0)}, \dots, \widehat{i_\ell}, \dots, i_{\pi(p+1)}).$$

In fact the sign of this $\pi \in S_{p+2}$ is the same as the sign of σ as in (4.3). Thus the factor of $(p+2)$ cancels in the denominator and we are left with

$$\tilde{\alpha}_{p+1}\left(d^p(g)\right)_{i_0,\dots,i_{p+1}} = \frac{(-1)^\ell}{(p+1)!\operatorname{sgn}(\sigma) g_{j_0,\dots,j_p}},$$

as desired. This plus linearity proves that the maps $\tilde{\alpha}_p$ yield a morphism of complexes.

Now, one checks that q' maps elements of $\check{C}^p(\mathcal{U},\mathcal{F})$ to alternating cochains in $\check{C}^p_{\text{alt}}(\mathcal{U},\mathcal{F})$. Then it is clear that q' is identity when restricted to $\check{C}^p_{\text{alt}}(\mathcal{U},\mathcal{F})$. The only thing left to check is that q' is homotopic to the identity. Since $\alpha_0 - \text{id}$ is the zero map (now α_0 is a map on the indices as in [Bos12, §7.6.1]), one can construct homotopies h_n by following the proof in Bosch. □

We will use this q' when translating the cup product (which is typically defined in terms of the Čech complex $\check{C}^\bullet(\mathcal{U},\mathcal{F})$) to the alternating Čech complex.

4.3 Cup Product

Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of a variety X, and $\mathcal{F}$ a sheaf of rings on X. Then one can form the abelian group

$$\check{C}^*(\mathcal{U},\mathcal{F}) = \bigoplus_{p\ge 0}\check{C}^p(\mathcal{U},\mathcal{F}).$$

Definition 4.11. The **cup product**[1] on $\check{C}^*(\mathcal{U},\mathcal{F})$ is defined via the linear extension of maps

$$\check{C}^p(\mathcal{U},\mathcal{F}) \times \check{C}^q(\mathcal{U},\mathcal{F}) \to \check{C}^{p+q}(\mathcal{U},\mathcal{F}), \qquad (\alpha,\beta)\mapsto \alpha \smile \beta,$$

[1]See the exercises of [Bos12, §7.6] for example.

where

$$(\alpha \smile \beta)_{i_0,\dots,i_{p+q}} = \alpha_{i_0,\dots,i_p}|_{U_{i_0,\dots,i_{p+q}}} \cdot \beta_{i_p,\dots,i_{p+q}}|_{U_{i_0,\dots,i_{p+q}}}.$$

This cup product induces a ring structure on $\check{H}^*(\mathcal{U},\mathcal{F}) = \bigoplus_{p\geq 0} \check{H}^p(\mathcal{U},\mathcal{F})$. Since $\mathcal{T}_X$ is a sheaf of rings, the cup product induces a map

$$\smile\colon H^1(X,\mathcal{T}_X) \times H^1(X,\mathcal{T}_X) \to H^2(X,\mathcal{T}_X).$$

While the cup product is defined on the Čech group $\check{C}^*(\mathcal{U},\mathcal{F})$, it is actually more convenient to have a description on the level of the alternating Čech complex instead. To do this, take the map defined after Theorem 4.10 and compose with the cup product map. This yields a new cup product map

$$\smile^*\colon \check{C}^p_{\text{alt}}(\mathcal{U},\mathcal{F}) \times \check{C}^q_{\text{alt}}(\mathcal{U},\mathcal{F}) \to \check{C}^{p+q}_{\text{alt}}(\mathcal{U},\mathcal{F})$$

where

$$(\alpha \smile^* \beta)_{i_0,\dots,i_{p+q}} = \frac{1}{(p+q+1)!} \sum_{\pi\in S_{p+q+1}} (\alpha \smile \beta)_{\pi(i_0),\dots,\pi(i_{p+q})}.$$

This is the cup product map we are going to be using, and we are interested in the case when $p = q = 1$. We emphasize that here, multiplication is given by the Lie bracket $[\ ,\ ]$, but may write this more simply by juxtaposition later.

Theorem 4.12. *Let X be a smooth, complete toric variety with associated fan Σ. The cup product map*

$$H^1(X,\mathcal{T}_X) \times H^1(X,\mathcal{T}_X) \to H^2(X,\mathcal{T}_X)$$

on the level of alternating Čech cocycles is given by $(a_{\sigma\tau}) \times (b_{\gamma\alpha}) \mapsto (c_{\sigma\tau\gamma})$, where

$$c_{\sigma\tau\gamma} = \frac{1}{6}\Big([a_{\sigma\tau}, b_{\tau\gamma}] + [a_{\gamma\sigma}, b_{\sigma\tau}] + [a_{\tau\gamma}, b_{\gamma\sigma}] - [a_{\tau\sigma}, b_{\sigma\gamma}] - [a_{\sigma\gamma}, b_{\gamma\tau}] - [a_{\gamma\tau}, b_{\tau\sigma}]\Big),$$

and $(a_{\sigma\tau}) \in \check{H}^1(\mathcal{U},\mathcal{T}_X)$ is indexed by the cones $\sigma, \tau \in \Sigma$, and $(b_{\gamma\alpha}) \in \check{H}^1(\mathcal{U},\mathcal{T}_X)$ is indexed by the cones $\gamma, \alpha \in \Sigma$.

Proof. Let $\mathcal{U} = \{U_\sigma \mid \sigma \in \Sigma(n)\}$ be our usual affine open cover of X, and let $(a_{\sigma\tau})$ and $(b_{\gamma\alpha})$ be Čech cocycles as above. Then under the cup product map we obtain the cocycle

$$(a_{\sigma\tau} b_{\tau\gamma}) \in \check{H}^2(\mathcal{U},\mathcal{T}_X),$$

where the σ, τ, γ are indexing cones for $\check{C}^2(\mathcal{U}, \mathcal{T}_X)$. Applying the isomorphism from (4.2) gives the cocycle above.

□

Chapter 5

Dualized Euler Sequence

One of the key steps in giving a combinatorial interpretation of the cup product map is the decomposition in cohomology of the tangent sheaf into a direct sum of sheaves associated to boundary divisors. For a smooth, complete, toric variety X with associated fan Σ, this section covers the isomorphisms

$$H^p(X, \mathcal{T}_X) \simeq \bigoplus_{\rho \in \Sigma(1)} H^p(X, \mathcal{O}_X(D_\rho))$$

for $p > 0$. We then describe the isomorphism in cohomology on the Čech level. For more details see [CLS11, §8.1].

5.1 Euler Sequence

In section 2.2, we defined the cotangent sheaf Ω^1_X on X, as well as the tangent sheaf $\mathcal{T}_X$. We have the following theorem, which is key in obtaining the isomorphisms we desire.

Theorem 5.1. *[Jac94] Let X be a smooth, complete toric variety associated to the fan Σ. Then there is an exact sequence*

$$0 \to \Omega^1_X \to \bigoplus_{\rho \in \Sigma(1)} \mathcal{O}_X(-D_\rho) \to \operatorname{Pic}(X) \otimes_{\mathbb{Z}} \mathcal{O}_X \to 0.$$

One may see this as the toric generalization of the *Euler sequence* of $\mathbb{P}^n$

$$0 \to \Omega^1_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(-1)^{n+1} \to \mathcal{O}_{\mathbb{P}^n} \to 0,$$

found in [Har13]. The fan for $\mathbb{P}^n$ is determined by the ray generators $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$, where $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ are the standard basis vectors for $\mathbb{Q}^n$ and $\boldsymbol{e}_0 = -\sum_{i=1}^n \boldsymbol{e}_i$. So there are $n+1$ boundary divisors, which accounts for the power of $n+1$. Since the class group for $\mathbb{P}^n$ is one dimensional, these divisors are linearly equivalent. For convenience we may just write $\mathcal{O}_{\mathbb{P}^n}(-1)$.

We now turn our attention to the cohomology of the dualized Euler sequence in the case when X is a smooth, complete toric variety.

5.2 Cohomology of the Tangent Sheaf

This decomposition in cohomology for the tangent sheaf is due to Jaczewski in 1994.

Theorem 5.2. *[Jac94] Let X be a smooth, complete toric variety arising from a fan Σ, with boundary divisors $\{D_\rho \mid \rho \in \Sigma(1)\}$. Then for $p \geq 1$ we have*

$$\bigoplus_{\rho\in\Sigma(1)} H^p(X, \mathcal{O}_X(D_\rho)) \simeq H^p(X, \mathcal{T}_X)$$

as M-graded groups.

Proof. Dualizing the sequence found in Theorem 5.1 gives the sequence

$$0 \to (\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathcal{O}_X)^\vee \to \bigoplus_{\rho\in\Sigma(1)} \mathcal{O}_X(D_\rho) \to \mathcal{T}_X \to 0.$$

Note that $\mathrm{Pic}(X)$ is a finite abelian group, so that $(\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathcal{O}_X)^\vee$ is a direct sum of copies of $\mathcal{O}_X$. Setting $\mathcal{F} = (\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathcal{O}_X)^\vee$ we obtain a long exact sequence in cohomology

$$\begin{aligned} 0 \to H^0(X, \mathcal{F}) \to \bigoplus_{\rho\in\Sigma(1)} H^0(X, \mathcal{O}_X(D_\rho)) \to H^0(X, \mathcal{T}_X) \to \\ H^1(X, \mathcal{F}) \to \bigoplus_{\rho\in\Sigma(1)} H^1(X, \mathcal{O}_X(D_\rho)) \to H^1(X, \mathcal{T}_X) \to \dots \end{aligned}$$

Using the fact that cohomology commutes with direct sums [Har13, III.2.9.1] and that higher cohomology ($p > 0$) of $\mathcal{O}_X$ vanishes [Ful93, §3.5] gives the result. □

In order to explicitly describe the cup product map, we need a concrete description of the isomorphism between $\bigoplus_{\rho\in\Sigma(1)} H^1(X, \mathcal{O}(D_\rho))_u$ and $H^1(X, \mathcal{T}_X)_u$. That description comes in the next theorem. Before stating the theorem, we first need a lemma encoding an important fact about divisor sheaves.

Lemma 5.3. *The sheaf $\mathcal{O}_X(-D_\rho)$ is isomorphic to $\mathcal{O}_X(D_\rho)^\vee = \mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X(D_\rho), \mathcal{O}_X)$ via the isomorphism $\chi^m \mapsto [\chi^w \mapsto \chi^{m+w}]$.*

Proof. First note that for Cartier divisors D and E the sheaf $\mathcal{O}_X(D) \otimes \mathcal{O}_X(E)$ is isomorphic to $\mathcal{O}_X(D + E)$. For an open set U, define a sheaf homomorphism by sending simple tensors $f \otimes g$ to fg. In fact this is an isomorphism when $\mathcal{O}_X(D)(U)$ is trivial. It is well known that such an isomorphism induces an isomorphism

$$\mathcal{O}_X(E) \simeq \mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X(D), \mathcal{O}_X(D + E)).$$

Now set $E = -D$ so that $\mathcal{O}_X(D) \otimes \mathcal{O}_X(-D) \simeq \mathcal{O}_X$. It follows that

$$\mathcal{O}_X(-D) = \mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X(D_\rho), \mathcal{O}_X) = \mathcal{O}_X(D)^\vee.$$

□

We now have a brief aside to discuss a particular kind of derivation, important in the next theorem. In the description of the cup product, we are required to multiply certain derivations of the form $\partial(\rho, u)(\chi^m) = \langle \boldsymbol{v}_\rho, m\rangle \chi^{u+m}$. More precisely, we have the following definition.

Definition 5.4. Fix a ray $\rho \in \Sigma(1)$ and a weight $u \in M$. Then let $\partial(\rho, u)$ be a derivation on characters be defined by

$$\chi^m \mapsto \langle \boldsymbol{v}_\rho, u\rangle \chi^{u+m}.$$

We now record the product of two of these derivations in the following lemma.

Lemma 5.5. *Fix rays $\rho, \rho' \in \Sigma(1)$ and weights $u, u' \in M$. Then for the derivations $\partial(\rho, u)$ and $\partial(\rho', u')$ the Lie bracket is given by*

$$[\partial(\rho, u), \partial(\rho', u')] = \langle \boldsymbol{v}_\rho, u'\rangle \partial(\rho', u + u') - \langle \boldsymbol{v}_{\rho'}, u\rangle \partial(\rho, u + u').$$

Proof. The derivation $[\partial(\rho, u), \partial(\rho', u')]$ can be specified by where it maps characters χ^m. Straightforward computations reveal that

$$\begin{aligned}[\partial(\rho, u), \partial(\rho', u')](\chi^m) &= \partial(\rho, u) \circ \partial(\rho', u')(\chi^m) - \partial(\rho', u')\partial(\rho, u)(\chi^m),\\ &= (\langle \boldsymbol{v}_{\rho'}, m\rangle \langle \boldsymbol{v}_\rho, u'\rangle + \langle \boldsymbol{v}_{\rho'}, m\rangle \langle \boldsymbol{v}_\rho, m\rangle)\chi^{u+u'+m}\\ &\quad - (\langle \boldsymbol{v}_\rho, m\rangle \langle \boldsymbol{v}_{\rho'}, u\rangle + \langle \boldsymbol{v}_\rho, m\rangle \langle \boldsymbol{v}_{\rho'}, m\rangle)\chi^{u+u'+m}.\end{aligned}$$

Cancelling terms in this expression gives the result. □

We now describe how to map cocycles via the isomorphism given by the Euler sequence.

Theorem 5.6. *Let $\bigoplus_{\rho \in \Sigma(1)} H^1(X, \mathcal{O}_X(D_\rho))_u$ have Čech representation $\bigoplus_{\rho \in \Sigma(1)} \check{H}^1(\mathcal{U}, \mathcal{O}_X(D_\rho))_u$, (as a subquotient of $\bigoplus_{\rho \in \Sigma(1)} \oplus_{\sigma,\tau \in \Sigma} H^0(U_{\sigma\tau}, \mathcal{O}_X(D_\rho))_u$) and let $H^1(X, \mathcal{T}_X)_u$ have the Čech representation $\check{H}^1(\mathcal{U}, \mathcal{T}_X)_u$ (as a subquotient of $\bigoplus_{\sigma,\tau \in \Sigma} H^0(U_{\sigma\tau}, \mathcal{T}_X)_u$). Then a Čech representative $(f^{(\rho)}_{\sigma\tau})\chi^u \in \bigoplus_{\rho \in \Sigma(1)} \check{H}^1(\mathcal{U}, \mathcal{O}_X(D_\rho))_u$ is mapped to*

$$\sum_{\rho \in \Sigma(1)} (f^{(\rho)}_{\sigma\tau})\partial(\rho, u) \in \check{H}^1(\mathcal{U}, \mathcal{T}_X)_u.$$

Proof. This comes from dualizing the sequence

$$0 \to \Omega^1_X \to \bigoplus_{\rho\in\Sigma(1)} \mathcal{O}_X(-D_\rho) \to \operatorname{Pic}(X) \otimes_{\mathbb{Z}} \mathcal{O}_X \to 0.$$

found in Theorem 5.1. The required isomorphism is the map

$$\varphi\colon \bigoplus_{\rho\in\Sigma(1)} H^1(X, \mathcal{O}(D_\rho)) \to H^1(X, \mathcal{T}_X).$$

To describe this, we need to know what the original map

$$\Phi\colon \Omega^1_X \to \bigoplus_{\rho\in\Sigma(1)} \mathcal{O}(-D_\rho)$$

was, then dualize. From the proof of [CLS11, §8.1.6] we have the following commutative square, excised from a larger diagram:

$$\begin{array}{ccc} \Omega^1_X & \xrightarrow{\ \alpha\ } & M \otimes \mathcal{O}_X \\ \Big\downarrow{\scriptstyle \Phi} & & \Big\downarrow{\scriptstyle \beta} \\ \bigoplus_{\rho\in\Sigma(1)} \mathcal{O}_X(-D_\rho) & \xrightarrow{\ \gamma\ } & \bigoplus_{\rho\in\Sigma(1)} \mathcal{O}_X \end{array}$$

The map $\alpha\colon \Omega^1_X \to M \otimes \mathcal{O}_X$ is natural, it is given by

$$\alpha(d\chi^m) = m \otimes \chi^m,$$

where χ^m is a character of the torus. Similarly, $\gamma\colon\ \oplus_{\rho\in\Sigma(1)} \mathcal{O}_X(-D_\rho) \to \oplus_{\rho\in\Sigma(1)}\mathcal{O}_X$ is also natural: the sections of $\mathcal{O}_X(-D_\rho)(U)$ are those elements of $\mathcal{O}_X(U)$ which vanish on $\operatorname{Supp} D_\rho$ by [CLS11, §4.0.28], so γ is just the inclusion.

The map $\beta\colon M \otimes \mathcal{O}_X \to \oplus_\rho \mathcal{O}_X$ is more interesting. It is defined by (see [CLS11, §4.1.3])

$$\beta(m \otimes \chi^w) = \sum_{\rho\in\Sigma(1)} \langle \boldsymbol{v}_\rho, m\rangle \chi^w.$$

Since the square commutes, it must be the case that $\gamma \circ \Phi = \beta \circ \alpha$ (or if we view γ as the inclusion map, $\Phi = \beta \circ \alpha$). So if we restrict to the summand indexed by the ray ρ in the image, we have

$$\Phi(d\chi^m) = \langle v_\rho, m\rangle \chi^m.$$

After dualizing, we obtain the map

$$\Phi^*\colon \operatorname{Hom}(\mathcal{O}_X(-D_\rho), \mathcal{O}_X) \to \operatorname{Hom}(\Omega_X, \mathcal{O}_X),$$

which is given by precomposition. That is, Φ^* is defined by

$$\Phi^*(f) = f \circ \Phi.$$

Now we use the fact that $\mathcal{O}_X(D_\rho)$ and $\mathscr{H}\!om_{\mathcal{O}_X}(\mathcal{O}_X(-D_\rho), \mathcal{O}_X)$ are isomorphic via the isomorphism

$$\chi^m \mapsto [\chi^w \mapsto \chi^{m+w}].$$

Putting this all together we conclude that for the ray ρ, the map $\mathcal{O}_X \to \mathcal{T}_X$ is given by

$$\chi^m \mapsto [\chi^w \mapsto \chi^{w+m}] \mapsto \underbrace{[\chi^w \mapsto \langle \boldsymbol{v}_\rho, w\rangle \chi^{w+m}]}_{\partial(\rho,m)}.$$

So our map φ that we were interested in,

$$\varphi\colon \bigoplus_{\rho\in\Sigma(1)} H^1(X, \mathcal{O}_X(-D_\rho)) \to H^1(X, \mathcal{T}_X),$$

is given by $\varphi(\chi^m) = \partial(\rho, m)$ when restricted to the summand corresponding to the ray ρ. Each of these $\partial(\rho, m)$ are actually derivations living in $H^1(X, \mathcal{T}_X)_m$, since $H^1(X, \mathcal{T}_X)$ carries an M-grading. Therefore for each $m \in M$, the overall map will be given by the sum over $\rho \in \Sigma(1)$ of the $\partial(\rho, m)$. $\square$

Chapter 6

Cohomology of Boundary Divisors

6.1 Combinatorial Description

We rely heavily on the fact that the dual lattice M gives a grading of the Čech complex, which allows us to decompose cohomology groups as a direct sum over elements in M. We can use this grading, as well as a combinatorial description of each graded piece to calculate cohomology.

In this thesis, we are always taking the open cover $\mathcal{U} = \{U_\sigma \mid \sigma \in \Sigma(n)\}$, i.e. the collection of affine opens corresponding to n-dimensional cones. Since we assume X is complete (and thus $\mathrm{Supp}(\Sigma) = N_{\mathbb{Q}}$), this collection covers X. Recall that for a torus-invariant divisor $D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$ on X, the alternating Čech complex $\check{C}^p_{\mathrm{alt}}(\mathcal{U}, \mathcal{O}_X)$ is the subgroup of

$$\check{C}^p(\mathcal{U}, \mathcal{O}_X(D)) = \bigoplus_{\sigma_0, \ldots, \sigma_p \in \Sigma(n)} H^0(U_{\sigma_0} \cap \cdots \cap U_{\sigma_p}, \mathcal{O}_X(D))$$

whose elements satisfy condition (4.1). Now, by [CLS11, §4.3.3] we may write

$$H^0(U_{\sigma_0} \cap \cdots \cap U_{\sigma_p}, \mathcal{O}_X(D)) = \bigoplus_{u \in M} H^0(U_{\sigma_0} \cap \cdots \cap U_{\sigma_p}, \mathcal{O}_X(D))_u$$

where

$$H^0(U_{\sigma_0} \cap \cdots \cap U_{\sigma_p}, \mathcal{O}_X(D))_u = \begin{cases} \mathbb{C} \cdot \chi^u & \langle \boldsymbol{v}_\rho, u \rangle \geq -a_\rho \text{ for all } \rho \in \sigma_0 \cap \cdots \cap \sigma_p(1) \\ 0 & \text{otherwise.} \end{cases}$$

This is a grading of the Čech complex, and induces a natural decomposition in cohomology:

$$H^p(X, \mathcal{O}_X(D)) = \bigoplus_{u \in M} H^p(X, \mathcal{O}_X(D))_u.$$

Theorem 6.1. *[CLS11] The Čech complex $\check{C}^p_{alt}$ is graded by the dual lattice M. That is for a torus invariant divisor $D = \sum_{\rho\in\Sigma(1)} a_\rho D_\rho$ we have*

$$H^p(X, \mathcal{O}_X(D)) = \bigoplus_{m\in M} H^p(X, \mathcal{O}_X(D))_m$$

where

$$H^0(U_\sigma, \mathcal{O}_X(D))_m = \begin{cases} \mathbb{C}\cdot\chi^m & \langle \boldsymbol{v}_\rho, m\rangle \geq -a_\rho \text{ for all } \rho\in\sigma(1),\\ 0 & \text{otherwise.}\end{cases}$$

Proof. See [CLS11, §9.1]. □

We now define the combinatorial object which encodes information about $H^1(X, \mathcal{O}_X(D_\rho))_u$.

Definition 6.2. Fix a ray $\rho \in \Sigma(1)$, and a weight $u \in M$. Then define the graph $\Gamma_\rho(u)$ as follows. The vertices of $\Gamma_\rho(u)$ are ray generators $\boldsymbol{v}_\tau$, such that $\tau \neq \rho$ and $\langle \boldsymbol{v}_\tau, u\rangle < 0$. The edges of $\Gamma_\rho(u)$ are pairs of ray generators that lie in a common cone of Σ.

Theorem 6.3. *[Ilt11] For a smooth, complete toric variety X, we have $H^1(X, \mathcal{O}_X(D_\rho))_u = 0$ if $\langle \boldsymbol{v}_\rho, u\rangle \neq -1$. Otherwise*

$$\dim H^1(X, \mathcal{O}_X(D_\rho))_u = \max\{0, \dim H^0(\Gamma_\rho(u), \mathbb{C}) - 1\}.$$

We now describe how to calculate $H^p(X, \mathcal{O}(D_\rho))_u$ for $p > 0$, using a different combinatorial object $\mathfrak{S}^p_{D_\rho,u}$. When $p = 1$, we have the following equality involving the so called *reduced cohomology* (see Definition 6.8)

$$\dim H^0(\Gamma_\rho(u), \mathbb{C}) - 1 = \dim \tilde{H}^0(\mathfrak{S}^1_{D_\rho,u}, \mathbb{C}),$$

so this result extends [Ilt11]. Before stating our theorem we need a few definitions.

Definition 6.4. Let $D = \sum_{\rho\in\Sigma(1)} a_\rho D_\rho$ be a torus invariant divisor and let $u \in M$. Then define the set $V_{D,u} \subset N_\mathbb{Q}$ as

$$V_{D,u} := \bigcup_{\sigma\in\Sigma} \operatorname{Conv}(\boldsymbol{v}_\tau \mid \tau\in\sigma(1) \text{ and } \langle \boldsymbol{v}_\tau, u\rangle < -a_\rho).$$

We also need to define a similar but subtly different object. This require the notion of a *support function*.

Definition 6.5. Let Σ be a fan in $N_\mathbb{Q}$ and $D = \sum_{\rho\in\Sigma(1)} a_\rho D_\rho$ a torus invariant Cartier divisor. Define the **support function** $\varphi_D\colon \operatorname{Supp}(\Sigma) \to \mathbb{Q}$ on ray generators by

$$\varphi_D(\boldsymbol{v}_\rho) = -a_\rho,$$

then extending linearly on each cone.

This means that φ_D is a piecewise linear function on $\operatorname{Supp}(\Sigma)$. By this we mean φ_D is linear on each cone $\sigma \in \Sigma$.

Definition 6.6. With the same setup as the previously, we define $V_{D,u}^{\mathrm{supp}} \subset \operatorname{Supp}(\Sigma)$ as

$$V_{D,u}^{\mathrm{supp}} := \{v \in \operatorname{Supp}(\Sigma) \mid \langle v, u\rangle < \varphi_D(v)\}.$$

We are assuming that Σ is smooth, hence $\sigma \in \Sigma$ is smooth by definition. Thus the ray generators are part of a $\mathbb{Z}$-basis of N. It follows that φ_D is unique.

Definition 6.7. we denote by $\mathfrak{S}_{D,u}^p$ the ***p*-skeleton** of $V_{D,u}$. That is, $\mathfrak{S}_{D,u}^p$ is the union of polyhedra of $V_{D,u}$ whose dimension is less than or equal to p.

In order to streamline notation, and avoid the special case when $p = 0$, we use the notion of *reduced cohomology*. Essentially we are just wrapping the special case into the definition.

Definition 6.8. Let Z be a topological space. The **reduced cohomology** for Z with coefficients in $\mathbb{C}$ is

$$\tilde{H}^p(Z, \mathbb{C}) = \begin{cases} H^p(Z, \mathbb{C}) & p \geq 1, \\ \tilde{H}^0(Z, \mathbb{C}) & p = 0. \end{cases}$$

Here, $\tilde{H}^0(Z, \mathbb{C})$ is defined as the cokernel of the map $f^*\colon \mathbb{C} = H^0(\{\mathrm{pt}\}, \mathbb{C}) \to H^0(Z, \mathbb{C})$, which is induced by the continuous map of topological spaces $f\colon Z \to \{\mathrm{pt}\}$.

After some careful thought, one realizes that this means $\tilde{H}^0(Z, \mathbb{C}) = 0$ if and only if Z is path-connected. We now have the terminology to state the theorem.

Theorem 6.9. *Let X be a smooth complete toric variety, and $D = D_\rho$ a torus invariant boundary divisor of X. Then for $p > 0$ we have*

$$\dim H^p(X, \mathcal{O}_X(D))_u = \begin{cases} 0 & \text{if } \langle \boldsymbol{v}_\rho, u\rangle \neq -1, \\ \dim \tilde{H}^{p-1}\left(\mathfrak{S}_{D,u}^p, \mathbb{C}\right) & \text{otherwise.} \end{cases}$$

Proof. Suppose that $\langle \boldsymbol{v}_\rho, u\rangle \neq -1$, and $p > 1$. Since X is smooth, the divisor D is Cartier. Thus by [CLS11, §9.1.3] we have the following isomorphisms in cohomology:

$$H^p(X, \mathcal{O}_X(D))_u \simeq \tilde{H}^{p-1}(V_{D,u}^{\mathrm{supp}}, \mathbb{C}) \simeq \tilde{H}^{p-1}(V_{D,u}, \mathbb{C}).$$

The condition $\langle \boldsymbol{v}_\rho, u\rangle \neq -1$ simplifies the definition of $V_{D,u}$ to

$$V_{D,u} = \bigcup_{\sigma\in\Sigma} \operatorname{Conv}(\boldsymbol{v}_\tau \mid \tau \in \sigma(1) \text{ and } \langle \boldsymbol{v}_\tau, u\rangle < 0).$$

But observe, this is equivalent to taking the trivial divisor $D = 0$. It then follows that

$$H^p(X, \mathcal{O}(D))_u \simeq \tilde{H}^{p-1}(V_{0,u}^{\text{supp}}, \mathbb{C}).$$

Now observe that if $D = 0$, the support function φ_D is the zero function. Therefore

$$V_{0,u}^{\text{supp}} = \{v \in \text{Supp}(\Sigma) \mid \langle v, u \rangle < 0\},$$

which is clearly contractible. Since the i-th cohomology of a contractible space is 0 for $i > 1$, it follows that $H^p(X, \mathcal{O}_X(D))_u = 0$. If instead we had $p = 1$, then

$$H^1(X, \mathcal{O}_X(D))_u \simeq \tilde{H}^0(V_{D,u}^{\text{supp}}, \mathbb{C}) = 0,$$

since $V_{D,m}^{\text{supp}}$ is path connected.

Suppose instead that $\langle \boldsymbol{v}_\rho, u \rangle = -1$ and $p > 1$. Again we have isomorphisms

$$H^p(X, \mathcal{O}_X(D))_u \simeq \tilde{H}^{p-1}(V_{D,u}, \mathbb{C}).$$

If $p > 1$, the reduced cohomology $\tilde{H}^{p-1}(V_{D,u}, \mathbb{C})$ is just $H^{p-1}(V_{D,u}, \mathbb{C})$. Since $V_{D,u}$ is a union of polyhedra, and our Čech covering is sufficiently fine, we may use *simplicial cohomology* (see [Hat02, §2.27] for instance). Computing the i-th cohomology group of $V_{D,u}$ is the same as computing the i-cohomology group of the $(i+1)$-skeleton of $V_{D,u}$, hence the result follows in this case as well.

Finally if $p = 1$, then we need to consider $\tilde{H}^0(V_{D,u}, \mathbb{C})$, the dimension of which is one less than the number of connected components of $V_{D,u}$. By replacing each conjugand

$$\text{Conv}(\boldsymbol{v}_\tau \mid \tau \in \sigma(1) \text{ and } \langle \boldsymbol{v}_\tau, u \rangle < 0)$$

of $V_{D,u}$ with its 1-skeleton we do not change the connectivity of the set. This gives us exactly $\mathfrak{S}^1_{D,u}$, so we are done. □

6.2 Čech Description of Spanning Elements

In the process of computing the cup product, we require an explicit description of the map

$$\psi \colon \bigoplus_{\langle \boldsymbol{v}_\rho, u \rangle = -1} \tilde{H}^0(\Gamma_\rho(u), \mathbb{C}) \to \bigoplus_{\rho \in \Sigma(1)} H^1(X, \mathcal{O}_X(D_\rho))_u, \tag{6.1}$$

with respect to the Čech representation. Since $\Gamma_\rho(u)$ is a graph, we can label its components as $\mathcal{C} = \{c_1, \ldots, c_s\}$. The zeroth cohomology for this graph counts the connected components, so

$$H^0(\Gamma_\rho(u),\mathbb{C}) \simeq \bigoplus_{c\in\mathcal{C}} \mathbb{C} = \mathbb{C}^s.$$

The zeroth reduced cohomology is a quotient of the above, so a spanning set of $\tilde{H}^0(\Gamma_\rho(u),\mathbb{C})$ can be obtained from a spanning set of $H^0(\Gamma_\rho(u),\mathbb{C})$ in the natural way.

An obvious choice for a spanning set of $H^0(\Gamma_\rho(u),\mathbb{C})$ is given by elements

$$e^c_{\rho,u} = \begin{cases} 1 & \text{on the component } c, \\ 0 & \text{otherwise.} \end{cases}$$

Viewing these in $\tilde{H}^0(\Gamma_\rho(u),\mathbb{C})$ instead yields the spanning set we desire. Similarly, the graph $\Gamma_{\rho'}(u')$ has components $\mathcal{D} = \{d_1,\ldots,d_r\}$. So elements $e^d_{\rho',u'}$ (defined similarly as above) span $\tilde{H}^0(\Gamma_{\rho'}(u'),\mathbb{C})$.

All the isomorphisms in cohomology are being represented on the level of Čech cocycles, so we have to interpret these spanning sets as Čech cocycles as well. The Čech cohomology groups are a direct sum which ranges over cones in Σ. To specify an element, one must give a value in $\mathbb{C}$ for each cone in Σ. Thus, for any $\sigma \in \Sigma$ let

$$e^c_{\rho,u}(\sigma) = \begin{cases} 1 & \text{if } \sigma\cap c \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases} \qquad \text{and} \qquad e^d_{\rho',u'}(\sigma) = \begin{cases} 1 & \text{if } \sigma\cap d \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases} \tag{6.2}$$

Since $\{e^c_{\rho,u}\}_{c\in\mathcal{C}}$ and $\{e^d_{\rho',u'}\}_{d\in\mathcal{D}}$ are spanning sets, it follows that the Čech representations form a spanning set for their respective Čech cohomology groups. Now we have this representation, we can apply the isomorphism $\tilde{H}^0(\Gamma_\rho(u),\mathbb{C}) \to H^1(X,\mathcal{O}_X(D_\rho))_u$.

Lemma 6.10. *Let $e^c_{\rho,u}(\sigma)$ be the Čech representation of the element $e^c_{\rho,u}$ in the spanning set for $\tilde{H}^0(\Gamma_\rho(u),\mathbb{C})$. Then under the isomorphism*

$$\tilde{H}^0(\Gamma_\rho(u),\mathbb{C}) \to H^1(X,\mathcal{O}_X(D_\rho))_u,$$

$e^c_{\rho,u}(\sigma)$ is mapped to the Čech representative

$$e^c_{\rho,u}(\sigma) - e^c_{\rho,u}(\tau) \in \check{H}^1_{\text{alt}}(\mathcal{U},\mathcal{O}_X(D_\rho))_u,$$

where $\check{C}^1_{\text{alt}}(\mathcal{U},\mathcal{O}_X(D_\rho))_u$ is indexed by the cones σ,τ.

Proof. The following argument is adapted from the proof of Theorem 9.1.3 in [CLS11]. For each $\sigma \in \Sigma$, consider the exact sequence

$$0 \to H^0(U_\sigma,\mathcal{O}_X(D_\rho))_u \to \mathbb{C} \to H^0(\Gamma_\rho(u)\cap\sigma,\mathbb{C}) \to 0.$$

By the definition of $\Gamma_\rho(u)$, it is clear that $\Gamma_\rho(u) \cap \sigma$ is connected when it is nonempty. Therefore, $H^0(\Gamma_\rho(u) \cap \sigma, \mathbb{C}) = \mathbb{C}$ whenever $\Gamma_\rho(u) \cap \sigma \neq \emptyset$. Note that if $\Gamma_\rho(u) \cap \sigma \neq \emptyset$, there is a ray $\rho' \neq \rho \in \sigma(1)$ such that $\langle \boldsymbol{v}_{\rho'}, u \rangle < 0$. Since

$$H^0(U_\sigma, \mathcal{O}_X(D_\rho))_u = \mathbb{C}$$

whenever $\langle \boldsymbol{v}_{\rho'}, u \rangle \geq 0$ for all $\rho' \in \sigma(1)$, this means we have the sequence

$$0 \to 0 \to \mathbb{C} \to \mathbb{C} \to 0,$$

which is exact. However if $\Gamma_\rho(u) \cap \sigma = \emptyset$, then for all $\rho' \neq \rho \in \sigma(1)$ we have $\langle \boldsymbol{v}_{\rho'}, u \rangle \geq 0$. But since Σ is complete, it follows that $\langle \boldsymbol{v}_\rho, u \rangle \geq 0$, so that we have the sequence

$$0 \to \mathbb{C} \to \mathbb{C} \to 0 \to 0,$$

which is also exact.

Now for $\gamma = (\sigma_0, \ldots, \sigma_p) \in \Sigma^{p+1}$ let $U_{\sigma_\gamma} = U_{\sigma_0} \cap \cdots \cap U_{\sigma_p}$, and $\sigma_\gamma = \sigma_0 \cap \cdots \cap \sigma_p$. Since taking the direct sum of exact sequences preserves exactness, we can construct the exact sequence

$$0 \to \bigoplus_{\gamma \in \Sigma^{p+1}} H^0(U_{\sigma_\gamma}, \mathcal{O}_X(D_\rho))_u \to \bigoplus_{\gamma \in \Sigma^{p+1}} \mathbb{C} \to \bigoplus_{\gamma \in \Sigma^{p+1}} H^0(\Gamma_\rho(u) \cap \sigma_\gamma, \mathbb{C}) \to 0.$$

by taking the direct sum over all $\gamma \in \Sigma^{p+1}$. Recognizing that $\bigoplus_{\gamma \in \Sigma^{p+1}} H^0(U_{\sigma_\gamma}, \mathcal{O}_X(D_\rho))_u$ is the group of p-th Čech cochains, we write

$$0 \to \check{C}^p(\mathcal{U}, \mathcal{O}_X(D_\rho))_u \to \bigoplus_{\gamma \in \Sigma^{p+1}} \mathbb{C} \to \bigoplus_{\gamma \in \Sigma^{p+1}} H^0(\Gamma_\rho(u) \cap \sigma_\gamma, \mathbb{C}) \to 0.$$

where $\mathcal{U}$ is the canonical open affine cover of X given by $\{U_\sigma \mid \sigma \in \Sigma\}$.

Since $\check{C}(\mathcal{U}, \mathcal{O}_X(D_\rho))_u$ is a cochain complex, it comes equipped with a differential. The other terms of the sequence can be endowed with Čech-like differentials d_B^p and d_D^p, in order to obtain complexes $B^\bullet$ and $D^\bullet$ fitting into the exact sequence of complexes

$$0 \to \check{C}^\bullet(\mathcal{U}, \mathcal{O}_X(D_\rho))_u \to B^\bullet \to D^\bullet \to 0.$$

This gives rise to the induced long exact sequence in cohomology

$$0 \to H^0(\check{C}^\bullet(\mathcal{U}, \mathcal{O}_X(D_\rho))_u) \to H^0(B^\bullet) \to H^0(D^\bullet) \to H^1(\check{C}^\bullet(\mathcal{U}, \mathcal{O}_X(D_\rho))_u) \to H^1(B^\bullet) \to \cdots \tag{6.3}$$

as in [Har13, §III.1]. In this case our map ψ in (6.1) is the *connecting homomorphism* from $H^0(B^\bullet)$ through to $H^1(\check{C}^\bullet(\mathcal{U}, \mathcal{O}_X(D_\rho))_u)$. Using the theory of Kozsul complexes [CLS11,

§9.1.3], one can show that the complex $B^\bullet$ has the following simple cohomology:

$$H^p(B^\bullet) = \begin{cases} \mathbb{C} & \text{if } p = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus by focusing our attention to the beginning of (6.3) we obtain the exact sequence

$$0 \to H^0(\check{C}^\bullet(\mathcal{U}, \mathcal{O}_X(D_\rho))_u) \to H^0(B^\bullet) \to \mathbb{C} \to H^1(\check{C}^\bullet(\mathcal{U}, \mathcal{O}_X(D_\rho))_u) \to 0.$$

Let d^p denote the differential for the complex $\check{C}^\bullet(\mathcal{U}, \mathcal{O}_X(D_\rho))_u$. The map we want is found by considering the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \check{C}^0(\mathcal{U}, \mathcal{O}_X(D_\rho))_u & \longrightarrow & B^0 & \longrightarrow & D^0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d^0} & & \downarrow{\scriptstyle d_B^0} & & \downarrow{\scriptstyle d_D^0} & & \\
0 & \longrightarrow & \check{C}^1(\mathcal{U}, \mathcal{O}_X(D_\rho))_u & \longrightarrow & B^1 & \longrightarrow & D^1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d^1} & & \downarrow{\scriptstyle d_B^1} & & \downarrow{\scriptstyle d_D^1} & & \\
0 & \longrightarrow & \check{C}^2(\mathcal{U}, \mathcal{O}_X(D_\rho))_u & \longrightarrow & B^2 & \longrightarrow & D^2 & \longrightarrow & 0
\end{array}$$

The connecting homomorphism is a map $\ker d_D^0 \to \ker d^1$ which vanishes on $\operatorname{im} d^0$. To obtain this map, pick $f \in \ker d_D^0$, and lift to $\bar{f} \in B^0$ (which exists since the top row is exact). Now let $y = d_B^0(\bar{f})$ be the image of f under d_B^0. Then since the middle row is exact, this lifts to some $\bar{y} \in \check{C}^1(\mathcal{U}, \mathcal{O}_X(D_\rho))_u$. The only map that isn't the identity map is d_B^0, which is the Čech differential. Let (f_σ) be the Čech representation of f so that

$$(f_\sigma) \in \bigoplus_{\sigma \in \Sigma} H^0(\Gamma_\rho(u) \cap \sigma, \mathbb{C}).$$

Then the Čech representation of the image of f is

$$(f_\tau - f_\sigma) \in \check{C}^1(\mathcal{U}, \mathcal{O}_X(D_\rho))_u = \bigoplus_{\sigma, \tau \in \Sigma} H^0(U_{\sigma\tau}, \mathcal{O}_X(D_\rho))_u.$$

Since $\check{C}^\bullet(\mathcal{U}, \mathcal{O}_X(D_\rho))_u$ is a complex, and $(f_\sigma - f_\tau)$ is an element of $\operatorname{im} d_B^0$ it follows that $(f_\sigma - f_\tau)$ is an element of $\ker d^1$. A standard diagram chase shows that this image is unique modulo $\operatorname{im} d^0$. $\square$

Chapter 7

Main Theorems

This chapter covers the main results of this thesis: Theorem 7.4 which details the first lift, and Theorem 7.7 which completes the second lift to the combinatorial description. We now recall what the overall goal is. Our goal is to describe the map in the top level of Figure 7.1, indicated by the dashed arrow. To this end, we have taken Čech representations of

$$
\begin{array}{ccc}
\bigoplus_{\langle \boldsymbol{v}_\rho, u\rangle = -1} \tilde{H}^0(\Gamma_\rho(u), \mathbb{C}) \times \bigoplus_{\langle \boldsymbol{v}_{\rho'}, u'\rangle = -1} \tilde{H}^0(\Gamma_{\rho'}(u'), \mathbb{C}) & \dashrightarrow & \bigoplus_{\langle \boldsymbol{v}_\tau, u+u'\rangle = -1} H^1(V_{D_\tau, u+u'}, \mathbb{C}) \\
\downarrow \text{Lemma 6.10} & & \downarrow \\
\bigoplus_{\rho \in \Sigma(1)} H^1(X, \mathcal{O}(D_\rho))_u \times \bigoplus_{\rho' \in \Sigma(1)} H^1(X, \mathcal{O}(D_{\rho'}))_{u'} & & \bigoplus_{\tau \in \Sigma(1)} H^2(X, \mathcal{O}(D_\tau))_{u+u'} \\
\downarrow \text{Theorem 5.6} & & \downarrow \\
H^1(X, \mathcal{T}_X)_u \times H^1(X, \mathcal{T}_X)_{u'} & \xrightarrow{\text{cup product}} & H^2(X, \mathcal{T}_X)_{u+u'}
\end{array}
$$

Figure 7.1: Diagram of overall picture.

elements of a spanning set for both $\tilde{H}^0(\Gamma_\rho(u), \mathbb{C})$, and $\tilde{H}^0(\Gamma_{\rho'}(u'), \mathbb{C})$, and used Lemma 6.10 and Theorem 5.6 to consider their image in $H^1(X, \mathcal{T}_X)_u \times H^1(X, \mathcal{T}_X)_{u'}$. in this section we continue tracing through the diagram, first applying the cup product, then lifting to $H^2(X, \mathcal{O}_X(D_\tau))_{u+u'}$ (for some ray $\tau \in \Sigma(1)$), then lifting again to $H^1(V_{D_\tau, u+u'}, \mathbb{C})$.

7.1 Applying the Cup Product

We are interested in describing the map

$$H^1(X, \mathcal{T}_X) \times H^1(X, \mathcal{T}_X) \to H^2(X, \mathcal{T}_X)$$

on the combinatorial level. The first step is to leverage the grading by M. By applying Theorem 5.2 and using the fact that each $H^1(X, \mathcal{O}_X(D_\rho))$ is graded by M, we see that it suffices to consider

$$H^1(X, \mathcal{T}_X)_u \times H^1(X, \mathcal{T}_X)_{u'} \to H^2(X, \mathcal{T}_X)_{u+u'}$$

for each pair of weights $(u, u') \in M \times M$. Note the image lies in the degree $u + u'$ piece of $H^2(X, \mathcal{T}_X)$, as the multiplication map for $\mathcal{T}_X$ is given by the Lie bracket. Now we apply Theorem 6.3 to conclude that

$$H^1(X, \mathcal{T}_X)_u \simeq \bigoplus_{\langle \boldsymbol{v}_\rho, u\rangle = -1} \tilde{H}^0(\Gamma_\rho(u), \mathbb{C}),$$

and

$$H^1(X, \mathcal{T}_X)_{u'} \simeq \bigoplus_{\langle \boldsymbol{v}_{\rho'}, u'\rangle = -1} \tilde{H}^0(\Gamma_{\rho'}(u'), \mathbb{C}).$$

Therefore we can restrict our attention to the rays ρ and ρ' in each respective direct sum (for which $\langle \boldsymbol{v}_\rho, u\rangle = \langle \boldsymbol{v}_{\rho'}, u'\rangle = -1$), and compute the image for spanning sets of $\tilde{H}^0(\Gamma_\rho(u), \mathbb{C})$ and $\tilde{H}^0(\Gamma_{\rho'}(u'), \mathbb{C})$. Note that we are choosing $\rho \neq \rho'$, since if we have $\langle \boldsymbol{v}_\rho, u\rangle = \langle \boldsymbol{v}_{\rho'}, u'\rangle = -1$, then it is impossible for either of $\boldsymbol{v}_\rho$ or $\boldsymbol{v}_{\rho'}$ to pair to -1 with $u + u'$. This is because Corollary 7.5 tells us exactly which summands the cup product is concentrated in. We have described what these sets are in (6.2), and we can now describe the image of these elements in $\check{H}^2_{\text{alt}}(\mathcal{U}, \mathcal{T}_X)_{u+u'}$.

Lemma 7.1. *Let* $e^c_{\rho,u}(\sigma)$ *and* $e^d_{\rho',u'}(\sigma')$ *be as in* (6.2)*. Then under the cup product map the pair* $(e^c_{\rho,u}(\sigma) \times e^d_{\rho',u'}(\sigma'))$ *is mapped to the Čech representative* $h_{\sigma\tau\gamma} \cdot (\langle \boldsymbol{v}_\rho, u'\rangle \partial(\rho', u + u') - \langle \boldsymbol{v}_{\rho'}, u\rangle \partial(\rho, u + u'))$ *in* $\check{H}^2_{\text{alt}}(\mathcal{U}, \mathcal{T}_X)_{u+u'}$*, where*

$$h_{\sigma\tau\gamma} = \frac{1}{2}\Big(e^c_{\rho,u}(\sigma)(e^d_{\rho',u'}(\tau) - e^d_{\rho',u'}(\gamma)) + e^c_{\rho,u}(\tau)(e^d_{\rho',u'}(\gamma) - e^d_{\rho',u'}(\sigma)) + e^c_{\rho,u}(\gamma)(e^d_{\rho',u'}(\sigma) - e^d_{\rho',u'}(\tau))\Big),$$

and $\check{C}^2_{\text{alt}}(\mathcal{U}, \mathcal{T}_X)_{u+u'}$ *is indexed by the cones* $\sigma, \tau,$ *and* γ*.*

Proof. By Lemma 6.10 the image of $(e^c_{\rho,u}(\sigma) \times e^d_{\rho',u'}(\sigma'))$ in $\check{H}^1(\mathcal{U}, \mathcal{O}_X(D_\rho))_u \times \check{H}^1(\mathcal{U}, \mathcal{O}_X(D'_\rho))_{u'}$ is given by

$$\Big(e^c_{\rho,u}(\sigma) - e^c_{\rho,u}(\tau)\Big) \times \Big(e^d_{\rho',u'}(\sigma') - e^d_{\rho',u'}(\tau')\Big).$$

Apply Theorem 5.6 to conclude that the image of $\left(e^c_{\rho,u}(\sigma) - e^c_{\rho,u}(\tau)\right) \times \left(e^d_{\rho',u'}(\sigma') - e^d_{\rho',u'}(\tau')\right)$ in $\check{H}^1(\mathcal{U}, \mathcal{T}_X)_u \times \check{H}^1(\mathcal{U}, \mathcal{T}_X)_{u'}$ is

$$\left(e^c_{\rho,u}(\sigma) - e^c_{\rho,u}(\tau)\right) \partial(\rho, u) \times \left(e^d_{\rho',u'}(\sigma') - e^d_{\rho',u'}(\tau')\right) \partial(\rho', u').$$

Finally, after applying Theorem 4.12 and Lemma 5.5 we have

$$h_{\sigma\tau\gamma} \cdot (\langle \boldsymbol{v}_\rho, u'\rangle \partial(\rho', u+u') - \langle \boldsymbol{v}_{\rho'}, u\rangle \partial(\rho, u+u')) \in \check{H}^2(\mathcal{U}, \mathcal{T}_X)_{u+u'},$$

where $h_{\sigma\tau\gamma}$ is as above. This comes from the fact that in Theorem 4.12, we are setting

$$a_{\sigma\tau} = e^c_{\rho,u}(\sigma) - e^c_{\rho,u}(\tau) \qquad \text{and} \qquad b_{\sigma'\tau'} = e^d_{\rho,u}(\sigma') - e^d_{\rho,u}(\tau').$$

□

Now the goal is to determine how this cocycle "lifts" to a Čech 2-cocycle for the direct sum of boundary divisor sheaves, via the Euler sequence.

7.2 First Lifting

As before, fix rays $\rho, \rho' \in \Sigma(1)$ and weights $u, u' \in M$ such that $\langle \boldsymbol{v}_\rho, u\rangle = \langle \boldsymbol{v}_{\rho'}, u'\rangle = -1$. Let $z \in \check{C}^2_{\text{alt}}(\mathcal{U}, \mathcal{T}_X)_{u+u'}$ be a Čech representative for the image of $(e^c_{\rho,u}(\sigma) \times e^d_{\rho',u'}(\sigma'))$ as in Lemma 7.1. By Theorem 5.2 the representative z can be lifted to a representative $\bar{z} \in \oplus_{\tau\in\Sigma(1)} \check{C}^2_{\text{alt}}(\mathcal{U}, \mathcal{O}_X(D_\tau))_{u+u'}$ under the isomorphism given by the Euler sequence. Note that z arises as the image of sections related to the rays ρ and ρ'. Naturally, one hopes that $\bar{z}$ lies in only those summands corresponding to ρ and ρ' as well. In fact we will show that if $\bar{z}$ is not equivalent to zero, it is concentrated in exactly one of

$$\check{C}^2_{\text{alt}}(\mathcal{U}, \mathcal{O}_X(D_\rho))_{u+u'} \qquad \text{or} \qquad \check{C}^2_{\text{alt}}(\mathcal{U}, \mathcal{O}_X(D_{\rho'}))_{u+u'}.$$

As in Lemma 7.1, let

$$\begin{aligned} h_{\sigma\tau\gamma} = \frac{1}{2}\Big(& e^c_{\rho,u}(\sigma)(e^d_{\rho',u'}(\tau) - e^d_{\rho',u'}(\gamma)) + e^c_{\rho,u}(\tau)(e^d_{\rho',u'}(\gamma) - e^d_{\rho',u'}(\sigma)) \\ & + e^c_{\rho,u}(\gamma)(e^d_{\rho',u'}(\sigma) - e^d_{\rho',u'}(\tau))\Big), \end{aligned} \tag{7.1}$$

so that z is given by

$$(z_{\sigma\tau\gamma}) = h_{\sigma\tau\gamma} \cdot (\langle \boldsymbol{v}_\rho, u'\rangle \partial(\rho', u+u') - \langle \boldsymbol{v}_{\rho'}, u\rangle \partial(\rho, u+u')).$$

Now we want to lift z. We know the isomorphism $H^p(X, \mathcal{T}_X) \simeq \oplus_{\alpha \in \Sigma(1)} H^p(X, \mathcal{O}_X(D_\alpha))$ from Theorem 5.2 guarantees the existence of a representative

$$\bar{z}_{\sigma\tau\gamma} \in \bigoplus_{\alpha \in \Sigma(1)} \check{C}^2_{\text{alt}}(\mathcal{U}, \mathcal{O}_X(D_\alpha))_{u+u'}$$

which maps to $z_{\sigma\tau\gamma}$. For ease of calculation, we will lift each summand separately. This is okay, since each summand of $z_{\sigma\tau\gamma}$ is regular, which we prove in the next lemma.

Lemma 7.2. *Let σ, τ, γ be cones in Σ. Then $h_{\sigma\tau\gamma}\langle \boldsymbol{v}_\rho, u' \rangle \partial(\rho', u+u')$ and $h_{\sigma\tau\gamma}\langle \boldsymbol{v}_\rho, u' \rangle \partial(\rho, u+u')$ are regular sections of $\mathcal{T}_X$ over $U_{\sigma\tau\gamma}$.*

Proof. We first show that $h_{\sigma\tau\gamma}\langle \boldsymbol{v}_\rho, u' \rangle \partial(\rho', u+u')$ and $h_{\sigma\tau\gamma}\langle \boldsymbol{v}_\rho, u' \rangle \partial(\rho, u+u')$ are regular sections of $\mathcal{T}_X$ over $U_{\sigma\tau\gamma}$. Since $(z_{\sigma\tau\gamma})$ is the image of the cup product map, it is a regular section of $\mathcal{T}_X$ over $U_{\sigma\tau\gamma}$. Therefore, showing that either summand of $(z_{\sigma\tau\gamma})$ is regular proves the other is as well. So without loss of generality we will show that $h_{\sigma\tau\gamma}\langle \boldsymbol{v}_\rho, u' \rangle \partial(\rho', u+u')$ is regular.

Recall $\partial(\rho', u+u')$ is defined by

$$\partial(\rho', u+u')(\chi^m) = \langle \boldsymbol{v}_{\rho'}, m \rangle \chi^{u+u'+m}.$$

To prove $h_{\sigma\tau\gamma}\langle \boldsymbol{v}_{\rho'}, u \rangle \partial(\rho', u+u')$ is regular on $U_{\sigma\tau\gamma}$ we must show that for all $m \in (\sigma \cap \tau \cap \gamma)^\vee$ either

$$h_{\sigma\tau\gamma}\langle \boldsymbol{v}_\rho, u' \rangle \partial(\rho', u+u')(\chi^m) = 0$$

or, $u+u'+m \in (\sigma \cap \tau \cap \gamma)^\vee$. Recall from the definition of duality for cones that the condition $u+u'+m \in (\sigma \cap \tau \cap \gamma)^\vee$ is equivalent to

$$\langle \boldsymbol{v}_\epsilon, u+u'+m \rangle \geq 0, \qquad \text{for all rays } \epsilon \in \sigma \cap \tau \cap \gamma(1). \tag{7.2}$$

Observe $\langle \boldsymbol{v}_{\rho'}, u \rangle$ is constant with respect to σ, τ, and γ. Therefore we may assume that $\langle \boldsymbol{v}_\rho, u' \rangle \neq 0$, as the zero section is automatically regular. We also assume that m is such that $\langle \boldsymbol{v}_{\rho'}, m \rangle \neq 0$ for the same reason. Finally, assume that $h_{\sigma\tau\gamma} \neq 0$. Observe that instead of writing $h_{\sigma\tau\gamma}$ as in (7.1), we could instead write

$$h_{\sigma\tau\gamma} = \frac{1}{2}\Big(e^d_{\rho',u'}(\sigma)\big(e^c_{\rho,u}(\gamma) - e^c_{\rho,u}(\tau)\big) + e^d_{\rho',u'}(\tau)\big(e^c_{\rho,u}(\sigma) - e^c_{\rho,u}(\gamma)\big) + e^d_{\rho',u'}(\gamma)\big(e^c_{\rho,u}(\tau) - e^c_{\rho,u}(\sigma)\big)\Big).$$

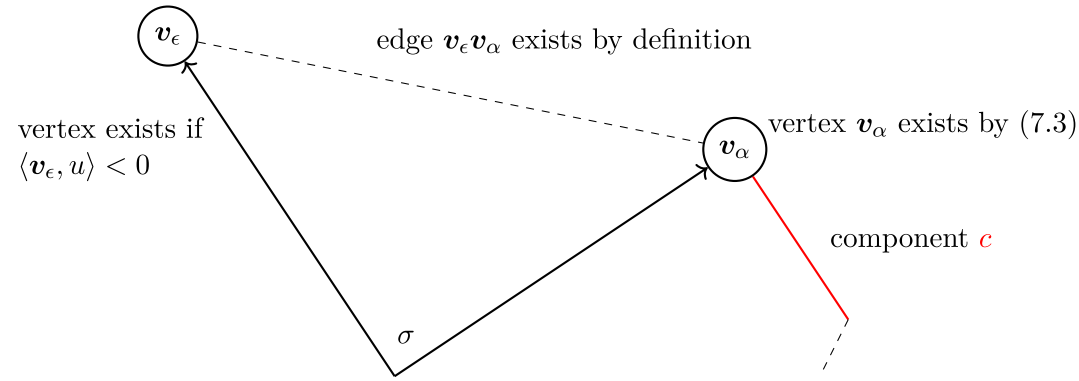


Figure 7.2: Depiction of the graph $\Gamma_\rho(u)$ with component c in red.

From these two descriptions, it is clear that when $h_{\sigma\tau\gamma} \neq 0$ it is not the case that

$$e^d_{\rho',u'}(\sigma) = e^d_{\rho',u'}(\tau) = e^d_{\rho',u'}(\gamma) = 1,$$

or

$$e^c_{\rho,u}(\sigma) = e^c_{\rho,u}(\tau) = e^c_{\rho,u}(\gamma) = 1.$$

Due to the symmetry in the descriptions of $h_{\sigma\tau\gamma}$ it is enough to consider the following two cases. That is, our cases will involve conditions on $e^c_{\rho,u}$. We could have chosen the conditions on $e^d_{\rho',u'}$, but up to sign the arguments work out the same due to the symmetry in $h_{\sigma\tau\gamma}$.

In the first case, assume exactly one of $e^c_{\rho,u}(\sigma), e^c_{\rho,u}(\tau)$, or $e^c_{\rho,u}(\gamma)$ is nonzero. Up to cyclic permutations of σ, τ and γ, suppose that

$$e^c_{\rho,u}(\sigma) = 1, \qquad e^c_{\rho,u}(\tau) = 0, \qquad e^c_{\rho,u}(\gamma) = 0. \tag{7.3}$$

So $h_{\sigma\tau\gamma}$ reduces to

$$\frac{1}{2}\big(e^d_{\rho',u'}(\tau) - e^d_{\rho',u'}(\gamma)\big).$$

Without loss of generality and to keep $h_{\sigma\tau\gamma}$ nonzero, we may assume that $e^d_{\rho',u'}(\tau) = 1$ and $e^d_{\rho',u'}(\gamma) = 0$. To check the condition in (7.2), suppose $\epsilon \in \sigma \cap \tau \cap \gamma(1)$ is neither ρ nor ρ'. If $\langle \boldsymbol{v}_\epsilon, u\rangle < 0$, by definition $\boldsymbol{v}_\epsilon$ is a vertex of the graph $\Gamma_\rho(u)$. By (7.3), we know there is a ray $\alpha \in c \cap \sigma(1)$ such that $\langle \boldsymbol{v}_\alpha, u\rangle < 0$. If $\epsilon = \alpha$ then $c \cap \tau \neq \emptyset$ (since $\epsilon \in \tau(1)$) which contradicts (7.3). But if $\epsilon \neq \alpha$ by Definition 6.2 of the graph $\Gamma_\rho(u)$, we have $\boldsymbol{v}_\alpha\boldsymbol{v}_\epsilon$ as an edge of $\Gamma_\rho(u)$. See Figure 7.2 for a visualization. Thus $\boldsymbol{v}_\epsilon$ lies in the component c, so that $c \cap \tau \neq \emptyset$, which again contradicts (7.3). So it must be the case that $\langle \boldsymbol{v}_\epsilon, u\rangle \geq 0$. One uses a similar argument involving the graph $\Gamma_{\rho'}(u')$ and the component d to conclude that $\langle \boldsymbol{v}_\epsilon, u'\rangle \geq 0$. Since $m \in (\sigma \cap \tau \cap \gamma)^\vee$, it is automatic that $\langle \boldsymbol{v}_\epsilon, m\rangle \geq 0$. Putting this together we see that when $\epsilon \neq \rho, \rho'$ we have

$$\langle \boldsymbol{v}_\epsilon, u + u' + m\rangle \geq 0.$$

Now if $\epsilon = \rho \in \sigma \cap \tau \cap \gamma(1)$, by assumption $\langle \boldsymbol{v}_\epsilon, u\rangle = -1$. Again, it is automatic that $\langle \boldsymbol{v}_\epsilon, m\rangle \geq 0$. So it is enough to show $\langle \boldsymbol{v}_\epsilon, u'\rangle > 0$. We have assumed that $\langle \boldsymbol{v}_\epsilon, u'\rangle \neq 0$, so that $h_{\sigma\tau\gamma} \neq 0$. If $\langle \boldsymbol{v}_\epsilon, u'\rangle < 0$ then $\boldsymbol{v}_\epsilon$ is a vertex of $\Gamma_{\rho'}(u')$. By (7.3), we know there is a ray $\alpha' \in d \cap \tau(1)$ such that $\langle \boldsymbol{v}_{\alpha'}, u'\rangle < 0$. If $\alpha' = \epsilon$, then the vertex $\boldsymbol{v}_\epsilon$ lies in the component d, and $\epsilon \in \gamma(1)$, so that $d \cap \gamma \neq \emptyset$. However this contradicts our assumption in (7.3). But if $\epsilon \neq \alpha'$ by definition 6.2 of the graph $\Gamma_{\rho'}(u')$, we have $\boldsymbol{v}_{\alpha'}\boldsymbol{v}_\epsilon$ as an edge of $\Gamma_{\rho'}(u')$. This also leads to the contradiction $d \cap \gamma \neq \emptyset$. Therefore if $\epsilon = \rho$ we have

$$\langle \boldsymbol{v}_\epsilon, u + u' + m\rangle \geq 0.$$

Finally, if $\epsilon = \rho' \in \sigma \cap \tau \cap \gamma(1)$, we have by assumption that $\langle \boldsymbol{v}_\epsilon, u'\rangle = -1$. Again, it is automatic that $\langle \boldsymbol{v}_\epsilon, m\rangle \geq 0$. Additionally $\langle \boldsymbol{v}_\epsilon, m\rangle \neq 0$ (to make $h_{\sigma\tau\gamma}$ nonzero), which means $\langle \boldsymbol{v}_\epsilon, m\rangle > 0$. Now if $\langle \boldsymbol{v}_\epsilon, u\rangle < 0$, we have that $\boldsymbol{v}_\epsilon$ is a vertex of $\Gamma_\rho(u)$. By (7.3), we know there is a ray $\alpha \in c \cap \sigma(1)$ such that $\langle \boldsymbol{v}_\alpha, u\rangle < 0$. If $\alpha = \rho' = \epsilon$, then $\boldsymbol{v}_\epsilon$ is in component c, forcing $c \cap \tau \neq \emptyset$, a contradiction. Otherwise we form the edge $\boldsymbol{v}_\alpha\boldsymbol{v}_\epsilon$, so that v_ϵ is in component c, forcing $c \cap \tau \neq \emptyset$, a contradiction. So when $\epsilon = \rho'$, we have

$$\langle \boldsymbol{v}_\epsilon, u + u' + m\rangle \geq 0.$$

Putting this all together we see that in the first case, we have a regular section.

In the second case, we assume exactly two of $e^c_{\rho,u}(\sigma), e^c_{\rho,u}(\tau)$, or $e^c_{\rho,u}(\gamma)$ are nonzero. So up to cyclic permutation of σ, τ and γ, suppose that

$$e^c_{\rho,u}(\sigma) = 1, \qquad e^c_{\rho,u}(\tau) = 1, \qquad e^c_{\rho,u}(\gamma) = 0.$$

One can use similar arguments as in case one to show that in this case we have a regular section as well. □

We also need the following lemma, the proof of which is based on the ideas in Lemma 7.2.

Lemma 7.3. *Let σ, τ, γ be cones of the fan Σ. Then $h_{\sigma\tau\gamma}\langle \boldsymbol{v}_\rho, u'\rangle\chi^{u+u'}$ and $h_{\sigma\tau\gamma}\langle \boldsymbol{v}_{\rho'}, u\rangle\chi^{u+u'}$ are regular sections over $U_{\sigma\tau\gamma}$ for $\mathcal{O}_X(D_{\rho'})$ and $\mathcal{O}_X(D_\rho)$ respectively.*

Proof. We will show that $h_{\sigma\tau\gamma}\langle \boldsymbol{v}_\rho, u'\rangle\chi^{u+u'}$ is regular for $\mathcal{O}_X(D_{\rho'})$, the argument to show $h_{\sigma\tau\gamma}\langle \boldsymbol{v}_{\rho'}, u\rangle\chi^{u+u'}$ is regular is almost identical. Recall from Theorem 6.1 that

$$H^0(U_{\sigma\tau\gamma}, \mathcal{O}_X(D_{\rho'}))_{u+u'} = \begin{cases} \mathbb{C} \cdot \chi^{u+u'} & \langle \boldsymbol{v}_\alpha, u + u'\rangle \geq -a_\alpha \text{ for all } \alpha \in \sigma \cap \tau \cap \gamma(1), \\ 0 & \text{otherwise.} \end{cases}$$

Therefore to show $h_{\sigma\tau\gamma}\langle \boldsymbol{v}_\rho, u'\rangle\chi^{u+u'}$ is regular, we need to show that $h_{\sigma\tau\gamma}\langle \boldsymbol{v}_\rho, u'\rangle \neq 0$ implies $\langle \boldsymbol{v}_\alpha, u+u'\rangle \geq -a_\alpha$ for all $\alpha \in \sigma\cap\tau\cap\gamma$. Here we have that $a_{\rho'} = 1$ and $a_\alpha = 0$ for all rays $\alpha \neq \rho'$. We now follow a similar argument as in the proof of the previous lemma. We may assume that $\langle \boldsymbol{v}_\rho, u'\rangle \neq 0$, which means that $\langle \boldsymbol{v}_{\rho'}, u\rangle = 0$. Again, there are two cases if $h_{\sigma\tau\gamma}$ is nonzero.

In the first case we have (up to cyclic permutation), that

$$e^c_{\rho,u}(\sigma) = 1, \qquad e^c_{\rho,u}(\tau) = 0, \qquad e^c_{\rho,u}(\gamma) = 0.$$

So $h_{\sigma\tau\gamma}$ reduces to

$$\frac{1}{2}\big(e^d_{\rho',u'}(\tau) - e^d_{\rho',u'}(\gamma)\big).$$

Without loss of generality and to keep $h_{\sigma\tau\gamma}$ nonzero, we may assume that $e^d_{\rho',u'}(\tau) = 1$ and $e^d_{\rho',u'}(\gamma) = 0$. Now we wish to show that for all $\alpha \in \sigma\cap\tau\cap\gamma(1)$, we have

$$\langle \boldsymbol{v}_\alpha, u+u'\rangle \geq -a_\alpha.$$

If $\alpha = \rho'$, then since $\langle \boldsymbol{v}_{\rho'}, u\rangle = 0$ we have

$$\langle \boldsymbol{v}_\alpha, u+u'\rangle = \langle \boldsymbol{v}_{\rho'}, u+u'\rangle = \langle \boldsymbol{v}_{\rho'}, u\rangle - 1 = -1.$$

So we have $\langle \boldsymbol{v}_{\rho'}, u+u'\rangle \geq -1$, as desired. If $\alpha = \rho$, then

$$\langle \boldsymbol{v}_\alpha, u+u'\rangle = \langle \boldsymbol{v}_\rho, u+u'\rangle = -1 + \langle \boldsymbol{v}_\rho, u'\rangle.$$

If $\langle \boldsymbol{v}_\rho, u'\rangle < 0$, then $\boldsymbol{v}_\rho$ is a vertex of the graph $\Gamma_{\rho'}(u')$. A similar argument as in the proof of Lemma 7.2 shows that $d\cap\gamma(1) \neq \emptyset$, which contradicts our assumption that $e^d_{\rho',u'}(\gamma) = 0$. Therefore it must be the case that $\langle \boldsymbol{v}_\rho, u'\rangle \geq 1$ (since we assumed that $\langle \boldsymbol{v}_\rho, u'\rangle \neq 0$). Thus, $\langle \boldsymbol{v}_\alpha, u+u'\rangle \geq 0$, as desired.

Now suppose α is neither ρ nor ρ'. If $\langle \boldsymbol{v}_\alpha, u\rangle < 0$, then $\boldsymbol{v}_\alpha$ is a vertex of $\Gamma_\rho(u)$. A similar argument as in the proof of Lemma 7.2 shows that $c\cap\tau(1) \neq \emptyset$, which contradicts our assumption that $e^c_{\rho,u}(\tau) = 0$. So it must be the case that $\langle \boldsymbol{v}_\alpha, u\rangle \geq 0$. If $\langle \boldsymbol{v}_\alpha, u'\rangle < 0$, then $\boldsymbol{v}_\alpha$ is a vertex of $\Gamma_{\rho'}(u')$. A similar argument as in the proof of Lemma 7.2 shows that $d\cap\gamma(1) \neq \emptyset$, which contradicts our assumption that $e^d_{\rho',u'}(\gamma) = 0$. So, $\langle \boldsymbol{v}_\alpha, u'\rangle \geq 0$. Putting these together we see that $\langle \boldsymbol{v}_\alpha, u+u'\rangle \geq 0$, as desired. This completes the proof for the first case.

In the second case, we assume exactly two of $e^c_{\rho,u}(\sigma), e^c_{\rho,u}(\tau)$, or $e^c_{\rho,u}(\gamma)$ are nonzero. So up to cyclic permutation of σ, τ and γ, suppose that

$$e^c_{\rho,u}(\sigma) = 1, \qquad e^c_{\rho,u}(\tau) = 1, \qquad e^c_{\rho,u}(\gamma) = 0.$$

Similar arguments show that $\langle \boldsymbol{v}_\rho, u\rangle > 0$, so that $\langle \boldsymbol{v}_\rho, u + u'\rangle \geq 0$, $\langle \boldsymbol{v}_\alpha, u\rangle \geq 0$, and $\langle \boldsymbol{v}_\alpha, u'\rangle \geq 0$. □

This lemma allows us to lift each summand separately. This is done by choosing a representative in $\check{H}^2_{\text{alt}}(\mathcal{U}, \mathcal{T}_X)_{u+u'}$ and showing that it maps to the desired cocycle.

Theorem 7.4. *The Čech representative $h_{\sigma\tau\gamma}\langle \boldsymbol{v}_\rho, u'\rangle\partial(\rho', u + u') \in \check{C}^2_{\text{alt}}(\mathcal{U}, \mathcal{T}_X)_{u+u'}$ lifts to the Čech representative $h_{\sigma\tau\gamma}\langle \boldsymbol{v}_\rho, u'\rangle\chi^{u+u'} \in \check{C}^2_{\text{alt}}(\mathcal{U}, \mathcal{O}_X(D_{\rho'}))_{u+u'}$, while the representative $h_{\sigma\tau\gamma}\langle \boldsymbol{v}_{\rho'}, u\rangle\partial(\rho, u + u') \in \check{C}^2_{\text{alt}}(\mathcal{U}, \mathcal{T}_X)_{u+u'}$ lifts to the representative $h_{\sigma\tau\gamma}\langle \boldsymbol{v}_{\rho'}, u\rangle\chi^{u+u'} \in \check{C}^2_{\text{alt}}(\mathcal{U}, \mathcal{O}_X(D_\rho))_{u+u'}$.*

Proof. First we see that $h_{\sigma\tau\gamma}\langle \boldsymbol{v}_\rho, u'\rangle\chi^{u+u'}$ lies in the kernel of the differential d^2. Since $\langle \boldsymbol{v}_\rho, u'\rangle$ is constant, we compute $d^2(h_{\sigma\tau\gamma})$. The definition of the Čech differential gives us

$$d^2(h_{\sigma\tau\gamma}) = s_{\sigma\tau\gamma\alpha} \in \check{C}^3(\mathcal{U}, \mathcal{O}_X(D_{\rho'}))_{u+u'} = \bigoplus_{\sigma,\tau,\gamma,\alpha\in\Sigma} H^0(U_{\sigma\tau\gamma\alpha}, \mathcal{O}_X(D_{\rho'}))_{u+u'},$$

where $s_{\sigma\tau\gamma\alpha}$ is given by

$$s_{\sigma\tau\gamma\alpha} = h_{\tau\gamma\alpha} - h_{\sigma\gamma\alpha} + h_{\sigma\tau\alpha} - h_{\sigma\tau\gamma}.$$

Expanding each summand yields the formula

$$\begin{aligned}
\frac{1}{2}\Big(& e^c_{\rho,u}(\tau)(e^d_{\rho',u'}(\gamma) - e^d_{\rho',u'}(\alpha)) + e^c_{\rho,u}(\gamma)(e^d_{\rho',u'}(\alpha) - e^d_{\rho',u'}(\tau)) + e^c_{\rho,u}(\alpha)(e^d_{\rho',u'}(\tau) - e^d_{\rho',u'}(\gamma)) \\
& - e^c_{\rho,u}(\sigma)(e^d_{\rho',u'}(\gamma) - e^d_{\rho',u'}(\alpha)) - e^c_{\rho,u}(\gamma)(e^d_{\rho',u'}(\alpha) - e^d_{\rho',u'}(\sigma)) - e^c_{\rho,u}(\alpha)(e^d_{\rho',u'}(\sigma) - e^d_{\rho',u'}(\gamma)) \\
& e^c_{\rho,u}(\sigma)(e^d_{\rho',u'}(\tau) - e^d_{\rho',u'}(\alpha)) + e^c_{\rho,u}(\tau)(e^d_{\rho',u'}(\alpha) - e^d_{\rho',u'}(\sigma)) + e^c_{\rho,u}(\alpha)(e^d_{\rho',u'}(\sigma) - e^d_{\rho',u'}(\tau)) \\
& - e^c_{\rho,u}(\sigma)(e^d_{\rho',u'}(\tau) - e^d_{\rho',u'}(\gamma)) - e^c_{\rho,u}(\tau)(e^d_{\rho',u'}(\gamma) - e^d_{\rho',u'}(\sigma)) - e^c_{\rho,u}(\gamma)(e^d_{\rho',u'}(\sigma) - e^d_{\rho',u'}(\tau))\Big).
\end{aligned}$$

It is straightforward to verify the cancellation of each term. Therefore $h_{\sigma\tau\gamma}\langle \boldsymbol{v}_\rho, u'\rangle\chi^{u+u'}$ lies in $\ker d^2$, which means it is indeed a cocycle. Further, Lemma 7.3 shows that $h_{\sigma\tau\gamma}\langle \boldsymbol{v}_\rho, u'\rangle\chi^{u+u'}$ is regular. Finally, the argument in Theorem 5.6 shows local sections χ^u are mapped to derivations $\partial(\rho, u)$. Applying this to the representative $h_{\sigma\tau\gamma}\langle \boldsymbol{v}_\rho, u'\rangle\chi^{u+u'}$ gives the desired cocycle.

A similar argument can be made for the representative $h_{\sigma\tau\gamma}\langle \boldsymbol{v}_{\rho'}, u\rangle\partial(\rho, u + u')$ which completes the proof. □

When we combine Theorem 7.4 with Theorem 6.9 we have the following corollary.

Corollary 7.5. *The cup product $\tilde{H}^0(\Gamma_\rho(u), \mathbb{C}) \times \tilde{H}^0(\Gamma_{\rho'}(u'), \mathbb{C}) \to H^2(X, \mathcal{T}_X)_{u+u'}$ is zero unless exactly one of $\boldsymbol{v}_\rho$ or $\boldsymbol{v}_{\rho'}$ pair with $u + u'$ to -1. Then the map is concentrated in the summand corresponding to the ray which pairs to -1 with $u + u'$.*

Proof. Recall that by Theorem 6.9, in order to have any cohomology in the summand $H^2(X, \mathcal{O}_X(D_\tau))_{u+u'}$, we must have $\langle \boldsymbol{v}_\tau, u + u'\rangle = -1$. We have chosen rays ρ, ρ' and weights $u, u' \in M$ such that the pairings are

$$\langle \boldsymbol{v}_\rho, u\rangle = -1 \qquad \text{and} \qquad \langle \boldsymbol{v}_{\rho'}, u'\rangle = -1.$$

So observe that if both

$$\langle \boldsymbol{v}_\rho, u + u'\rangle = -1 \qquad \text{and} \qquad \langle \boldsymbol{v}_{\rho'}, u + u'\rangle = -1,$$

it must be the case that

$$\langle \boldsymbol{v}_\rho, u'\rangle = \langle \boldsymbol{v}_{\rho'}, u\rangle = 0.$$

However this forces the image $h_{\sigma\tau\gamma}(\langle \boldsymbol{v}_\rho, u'\rangle \partial(\rho', u+u') - \langle \boldsymbol{v}_{\rho'}, u\rangle \partial(\rho, u+u'))$ in $\check{H}^2_{\text{alt}}(\mathcal{U}, \mathcal{T}_X)_{u+u'}$ in Lemma 7.1 to be zero.

Further, if neither $\boldsymbol{v}_\rho$ nor $\boldsymbol{v}_{\rho'}$ pair with $u + u'$ to -1, then there is no cohomology in those summands. Therefore if the image is nonzero, it must be concentrated in either $\check{H}^2_{\text{alt}}(\mathcal{U}, \mathcal{O}_X(D_\rho))_{u+u'}$ (corresponding to $\langle \boldsymbol{v}_{\rho'}, u\rangle = 0$) or in $\check{H}^2_{\text{alt}}(\mathcal{U}, \mathcal{O}_X(D_{\rho'}))_{u+u'}$ (corresponding to $\langle \boldsymbol{v}_\rho, u'\rangle = 0$). □

7.3 Second Lifting

We now describe the final "lift", which will complete our combinatorial description of the cup product. First, we will need to define a collection of sets which cover $V_{D,m}$.

Definition 7.6. Fix a torus invariant divisor D on X and a weight $m \in M$. Then define a collection of closed sets which cover $V_{D,m}$ by

$$\mathcal{V}_{D,m} := \{V_{D,m} \cap \sigma \mid \sigma \in \Sigma\}.$$

Note that we will use this closed cover to compute Čech cohomology. It is not true in general that Čech cohomology with respect to a closed cover computes the correct cohomology, but in this case it is allowable since higher cohomology vanishes on intersections. See e.g. [CLO07, C.1.3, §9.0.4] for more details (involving spectral sequences). We use this closed cover in the following theorem.

Theorem 7.7. *Let $h_{\sigma\tau\gamma}$ be as in Lemma 7.1. Then the Čech representative*

$$h_{\sigma\tau\gamma}\langle \boldsymbol{v}_\rho, u'\rangle \chi^{u+u'} \in \check{C}^2_{\text{alt}}(\mathcal{U}, \mathcal{O}_X(D_{\rho'}))_{u+u'}$$

lifts to the representative

$$b_{\sigma\tau}\langle \boldsymbol{v}_\rho, u'\rangle \in \check{C}^1_{\text{alt}}(\mathcal{V}_{D_{\rho'},u+u'}, \mathbb{C}),$$

where

$$b_{\sigma\tau} = \begin{cases} \frac{1}{2}\left(e^c_{\rho,u}(\sigma)e^d_{\rho',u'}(\tau) - e^c_{\rho,u}(\tau)e^d_{\rho',u'}(\sigma)\right) & \textit{if } V_{D_{\rho'},u+u'} \cap \sigma \cap \tau \neq \emptyset, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. To prove this we trace out the isomorphism explicitly, which arises from a diagram chase. The diagram in question comes from the proof of Lemma 6.10, which we reproduce here:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \check{C}^1(\mathcal{U}, \mathcal{O}_X(D_{\rho'}))_{u+u'} & \longrightarrow & B^1 & \xrightarrow{\varphi} & D^1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d^1} & & \downarrow{\scriptstyle d^1_B} & & \downarrow{\scriptstyle d^1_D} & & \\
0 & \longrightarrow & \check{C}^2(\mathcal{U}, \mathcal{O}_X(D_{\rho'}))_{u+u'} & \longrightarrow & B^2 & \longrightarrow & D^2 & \longrightarrow & 0
\end{array}$$

The map φ is the inclusion when nonzero, so $e^c_{\rho,u}(\sigma)e^d_{\rho',u'}(\tau) - e^c_{\rho,u}(\tau)e^d_{\rho',u'}(\sigma) \in B^1$ maps to $b_{\sigma\tau}$ under φ. Applying d^1_B (which is analogous to the Čech differential) then a straightforward simplification reveals that

$$d^1_B(e^c_{\rho,u}(\sigma)e^d_{\rho',u'}(\tau) - e^c_{\rho,u}(\tau)e^d_{\rho',u'}(\sigma)) = h_{\sigma\tau\gamma}.$$

This implies that $h_{\sigma\tau\gamma} \in \check{C}^2_{\text{alt}}(\mathcal{U}, \mathcal{O}_X(D_{D_{\rho'}}))_{u+u'}$.

Since the bottom row is exact and the diagram commutes, we see that $b_{\sigma\tau}$ lies in the kernel of d^1_D. Therefore $b_{\sigma\tau}$ is indeed a Čech cocycle, and $b_{\sigma\tau}$ maps to $h_{\sigma\tau\gamma}$ under the isomorphism $H^1(V_{D_{\rho'},u+u'}, \mathbb{C}) \simeq H^2(X, \mathcal{O}_X(D_{\rho'}))_{u+u'}$. □

7.4 Example: Obstructed Toric 3-fold

We will conclude by producing a toric 3-fold for which the cup product does not vanish. To be precise we produce a complete fan $\Sigma \subset N_{\mathbb{Q}} \simeq \mathbb{Q}^3$, and show that the cup product is nonzero for certain choices of rays ρ and ρ'. To this end, let

$$u = (-1, 0, 0) \qquad \text{and} \qquad u' = (0, -1, 0)$$

be elements of the dual lattice $M \simeq \mathbb{Q}^3$. Let $\{\rho_1, \ldots, \rho_9\} \subset \Sigma(1)$ be rays with corresponding ray generators

$$\begin{gathered}
\boldsymbol{v}_{\rho_1} = (1,0,0), \qquad \boldsymbol{v}_{\rho_2} = (1,0,-1), \qquad \boldsymbol{v}_{\rho_3} = (1,0,1), \\
\boldsymbol{v}_{\rho_4} = (2,-1,0), \qquad \boldsymbol{v}_{\rho_5} = (1,-1,0), \qquad \boldsymbol{v}_{\rho_6} = (1,1,0), \\
\boldsymbol{v}_{\rho_7} = (0,1,-1), \qquad \boldsymbol{v}_{\rho_8} = (0,1,1), \qquad \boldsymbol{v}_{\rho_9} = (-1,0,0).
\end{gathered}$$

Additionally in keeping with our notation, set $\rho = \rho_1$ and $\rho' = \rho_6$, so that

$$\langle \boldsymbol{v}_\rho, u \rangle = -1 \qquad \text{and} \qquad \langle \boldsymbol{v}_{\rho'}, u' \rangle = -1.$$

We can turn this data into a fan Σ by specifying the maximal cones. in this case there are 14 maximal cones:

$$\begin{gathered}
\sigma_1 = \operatorname{Cone}(\boldsymbol{v}_{\rho_1}, \boldsymbol{v}_{\rho_2}, \boldsymbol{v}_{\rho_4}), \qquad \sigma_2 = \operatorname{Cone}(\boldsymbol{v}_{\rho_1}, \boldsymbol{v}_{\rho_2}, \boldsymbol{v}_{\rho_7}), \qquad \sigma_3 = \operatorname{Cone}(\boldsymbol{v}_{\rho_1}, \boldsymbol{v}_{\rho_3}, \boldsymbol{v}_{\rho_4}), \\
\sigma_4 = \operatorname{Cone}(\boldsymbol{v}_{\rho_1}, \boldsymbol{v}_{\rho_3}, \boldsymbol{v}_{\rho_8}), \qquad \sigma_5 = \operatorname{Cone}(\boldsymbol{v}_{\rho_1}, \boldsymbol{v}_{\rho_6}, \boldsymbol{v}_{\rho_7}), \qquad \sigma_6 = \operatorname{Cone}(\boldsymbol{v}_{\rho_1}, \boldsymbol{v}_{\rho_6}, \boldsymbol{v}_{\rho_8}), \\
\sigma_7 = \operatorname{Cone}(\boldsymbol{v}_{\rho_2}, \boldsymbol{v}_{\rho_4}, \boldsymbol{v}_{\rho_5}), \qquad \sigma_8 = \operatorname{Cone}(\boldsymbol{v}_{\rho_2}, \boldsymbol{v}_{\rho_5}, \boldsymbol{v}_{\rho_9}), \qquad \sigma_9 = \operatorname{Cone}(\boldsymbol{v}_{\rho_2}, \boldsymbol{v}_{\rho_7}, \boldsymbol{v}_{\rho_9}), \\
\sigma_{10} = \operatorname{Cone}(\boldsymbol{v}_{\rho_3}, \boldsymbol{v}_{\rho_4}, \boldsymbol{v}_{\rho_5}), \qquad \sigma_{11} = \operatorname{Cone}(\boldsymbol{v}_{\rho_3}, \boldsymbol{v}_{\rho_5}, \boldsymbol{v}_{\rho_9}), \qquad \sigma_{12} = \operatorname{Cone}(\boldsymbol{v}_{\rho_3}, \boldsymbol{v}_{\rho_8}, \boldsymbol{v}_{\rho_9}), \\
\sigma_{13} = \operatorname{Cone}(\boldsymbol{v}_{\rho_6}, \boldsymbol{v}_{\rho_7}, \boldsymbol{v}_{\rho_9}), \qquad \sigma_{14} = \operatorname{Cone}(\boldsymbol{v}_{\rho_6}, \boldsymbol{v}_{\rho_8}, \boldsymbol{v}_{\rho_9}).
\end{gathered}$$

One needs to verify that the fan Σ is smooth and complete. Both can be verified by constructing Σ in Macaulay2 and running the commands `isSmooth` and `isComplete`, however smoothness can also be checked "by hand". This is done by checking each cone σ_i is smooth, which means the ray generators form part of a $\mathbb{Z}$-basis for the lattice N. If $\sigma_j = \operatorname{Cone}(\boldsymbol{v}_{i_1}, \boldsymbol{v}_{i_2}, \boldsymbol{v}_{i_3})$, then the ray generators form part of $\mathbb{Z}$-basis if the determinant of

$$\begin{pmatrix} \boldsymbol{v}_{i_1}^T & \boldsymbol{v}_{i_2}^T & \boldsymbol{v}_{i_3}^T \end{pmatrix}$$

is ± 1. We leave these verifications to the reader.

Then one can construct the graphs $\Gamma_\rho(u)$ and $\Gamma_{\rho'}(u')$, and verify that both $\tilde{H}^0(\Gamma_\rho(u), \mathbb{C})$ and $\tilde{H}^0(\Gamma_{\rho'}(u'), \mathbb{C})$ are nonzero. In fact both are isomorphic to $\mathbb{C}$ since both graphs consist of two connected components, as illustrated in Figure 7.3 and Figure 7.4. Similarly, one can look at $H^1(V_{D_{\rho', u+u'}}, \mathbb{C})$ and see that it is also one-dimensional (one connected component consisting of a cycle), as illustrated in Figure 7.5.

In fact, $\Gamma_\rho(u)$ has components $\{c_1, c_2\}$ where c_1 has vertices $\{\boldsymbol{v}_{\rho_2}, \boldsymbol{v}_{\rho_3}, \boldsymbol{v}_{\rho_4}, \boldsymbol{v}_{\rho_5}\}$ and c_2 consists of a single vertex $\{\boldsymbol{v}_{\rho_6}\}$. On the other hand, the graph $\Gamma_{\rho'}(u')$ has components $\{d_1, d_2\}$ where d_1 consists of a single vertex $\{\boldsymbol{v}_{\rho_8}\}$ and d_2 consists of a single vertex $\{\boldsymbol{v}_{\rho_7}\}$.

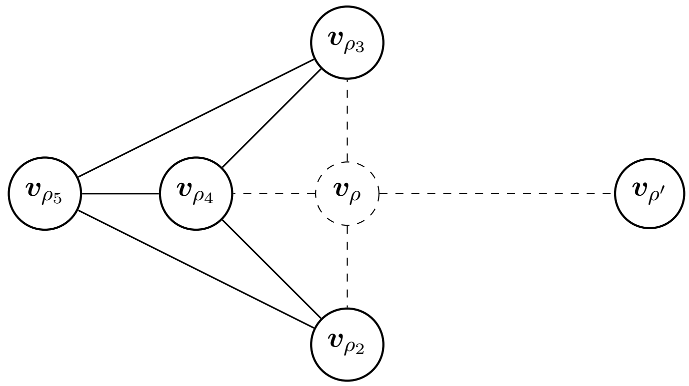


Figure 7.3: The graph $\Gamma_\rho(u)$ as viewed in the half-space $\langle x, u\rangle \leq -1$. Note the removal of the vertex $\boldsymbol{v}_\rho$.

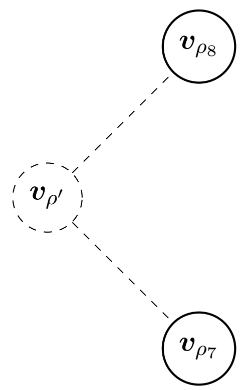


Figure 7.4: The graph $\Gamma_{\rho'}(u')$ as viewed in the half-space $\langle x, u'\rangle \leq -1$. Note the removal of $\boldsymbol{v}_{\rho'}$.

From this data and our description of the cup product, it is possible to verify the nonvanishing for the Čech representations $e^{c_1}_{\rho,u}(\sigma)$ and $e^{d_1}_{\rho',u'}(\sigma')$. The author has written a small Macaulay2 script to facilitate the computations, which is included in the appendix.

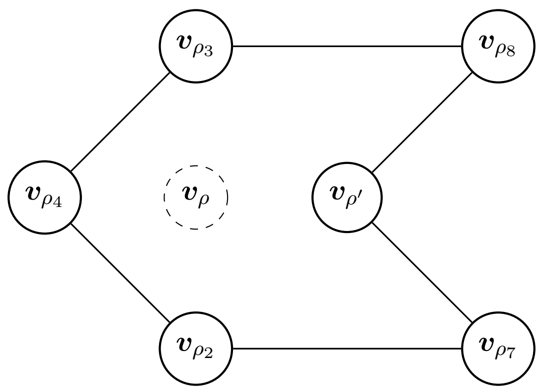


Figure 7.5: The object $V_{D_\rho,u+u'}$ as viewed in the half-space $\langle x, u' + u' \rangle \leq -1$.

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Appendix A

Macaulay2 Example

Contact the author for the source files. To run the script, make sure `header.m2` and `main-1.m2` are in the same directory, then run `main-1.m2`. See

`http://www2.macaulay2.com/Macaulay2/`

for more information about using Macaulay2.

Note that this example is computed using the ordered Čech complex, so that the description of the cup product differs from Chapter 7. This is accounted for by the different choice of q, the homomorphisms of complexes found in Theorem 4.10.

A.1 header.m2

```
printWidth = 0;--remove line wrapping

loadPackage "Polyhedra"
loadPackage "ToricVectorBundles"

------------------------------------------------------------------------
-- FUNCTIONS
------------------------------------------------------------------------
printCones = (VECTOR, PAIRSOFCONES) -> (
-- INPUT:   VECTOR        | A 91x1 matrix representing an element in C^1
--          PAIRSOFCONES  | A list whose elements are pairs of lists of
--                        | ray generators.
-- OUTPUT: N/A
-- COMMENTS:
-- Takes a cocycle in C^1 (represented as an element of C^99) whose
-- entries are indexed by pairs of cones, and for every nonzero entry,
-- prints the pair of cones associated to the nonzero entry as well as
-- the value.
    v := VECTOR; P := PAIRSOFCONES;

    for i from 0 to #P-1 do(
                if(v_(i,0) != 0) then(print(flatten(append(P_{i}, v_(i
                    ↪ ,0)))))
                )
    )
```

```
entryInDiffMatrix0 = (PAIRSOFCONES, MAXIMALCONES) -> (
-- INPUT:   PAIRSOFCONES | A list whose elements are {sig', tau'}, where
--                       | sig', tau' are lists whose entries are indices
--                       | of ray generators (generating the respective
--                       | cone).
--          MAXIMALCONES | A list whose elements are {sig}, where sig
--                       | is a list whose entries are indices of ray
--                       | generators that generate sigma.
-- OUTPUT:  -1           | If sig=sig'
--          1            | If sig=tau'
--          0            | Otherwise
-- COMMENTS:
-- The Cech differential C^1 -> C^2 can be written as a 364x91 matrix,
-- and this function computes the entries of that matrix.
    m := MAXIMALCONES; p := PAIRSOFCONES;

    if(m == p#0) then(return -1);
    if(m == p#1) then(return 1);
    return 0;
    )

entryInDiffMatrix = (TRIPLESOFCONES, PAIRSOFCONES) -> (
-- INPUT:   TRIPLESOFCONES | A list whose elements are {sig, tau, gam},
--                         | where sig, tau, gam are lists whose
--                         | entries are indices of ray generators
--                         | (generating the respective cone).
--          PAIRSOFCONES   | A list whose elements are {sig', tau'},
--                         | where sig', tau' are lists whose entries
--                         | are indices of ray generators (generating
--                         | the respective cone).
-- OUTPUT:  1              | If sig=sig' and tau=tau'
--          -1             | If sig=sig' and gam=tau'
--          1              | If tau=sig' and gam=tau'
--          0              | Otherwise
-- COMMENTS:
-- The Cech differential C^1 -> C^2 can be written as a 364x91 matrix,
-- and this function computes the entries of that matrix.
    t := TRIPLESOFCONES; p := PAIRSOFCONES;

    if((p#0 == t#0) and (p#1 == t#1)) then(return 1);
    if((p#0 == t#0) and (p#1 == t#2)) then(return -1);
    if((p#0 == t#1) and (p#1 == t#2)) then(return 1);
    return 0;
    )

entryInDiffMatrix2 = (QUADSOFCONES, TRIPLESOFCONES) -> (
-- INPUT:   QUADSOFCONES   | A list whose elements are {sig, tau, gam,
--                         | alp}, where sig, tau, gam, alp are lists
--                         | whose entries are indices of ray gens
--                         | (generating the respective cone).
--          TRIPLESOFCONES | A list whose elements are {sig', tau',
--                         |  gam'}, where sig', tau', gam' are lists
```

```
--                              | whose entries are indices of ray gens
--                              | (generating the respective cone).
-- OUTPUT: -1                   | If sig=sig' and tau=tau' and gam=gam'
--          1                   | If sig=sig' and tau=tau' and alp=gam'
--         -1                   | If sig=sig' and gam=tau' and alp=gam'
--          1                   | If tau=sig' and gam=tau' and alp=gam'
--          0                   | Otherwise
-- COMMENTS:
-- The Cech differential C^2 -> C^3 can be written as a 1001x364 matrix,
-- and this function computes the entries of that matrix.
    t := TRIPLESOFCONES; q := QUADSOFCONES;

    if((q#0 == t#0) and (q#1 == t#1) and (q#2 == t#2)) then(return -1);
    if((q#0 == t#0) and (q#1 == t#1) and (q#3 == t#2)) then(return 1);
    if((q#0 == t#0) and (q#2 == t#1) and (q#3 == t#2)) then(return -1);
    if((q#1 == t#0) and (q#2 == t#1) and (q#3 == t#2)) then(return 1);
    return 0;
    )

existsCohom3ForTopObj = (SIGMA, TAU, GAMMA, INDEXOFRHO, MATRIXOFRAYS) ->
  ↪ (
-- INPUT:   SIGMA              | List of indices of ray generators that
--                             | determine a maximal dimensional cone SIGMA
--                             | in the fan.
--          TAU                |        ""        "" ... TAU in the fan.
--          GAMMA              |        ""        "" ... GAMMA in the fan.
--          INDEXOFRHO         | The index of the column of MATRIXOFRAYS
--                             | which specifies rho.
--          MATRIXOFRAYS       | A matrix whose columns represent the ray
--                             | generators of the fan.
-- OUTPUT:  1                  | If H^0(V_D_rho \cap sig \cap tau
--                             | \cap gam,u+u', C) exists
--                             | (see CLS p402, ch 9.1)
--          0                  | Otherwise.
   sigma := SIGMA; tau := TAU; gam := GAMMA; rhoIndex := INDEXOFRHO;
   matrixOfRays := MATRIXOFRAYS; v := {-1,-1,0};--this is u+u'

   temp = select(sigma, s->member(s,tau));
   commonRays = select(temp, s->member(s,gam));

   --sigma and tau have only trivial intersection, so no ray gen
   if(length(commonRays) == 0) then(return 0)
   else(--check condition for common rays
       for s in commonRays do(
           --compute the inner product (and convert vec to list)
           temp = innerProduct(v, flatten entries matrixOfRays_s);
           --different condition for rho
           if(s == rhoIndex) then(if (temp < -1) then(return 1))
           --every other ray has same condition
           else(if (temp < 0) then(return 1))
           );
       return 0;--conditions not satisfied
       )
```

```
)

existsCohom2ForTopObj = (SIGMA, TAU, INDEXOFRHO, MATRIXOFRAYS) -> (
-- INPUT:   SIGMA           | List of indices of ray generators that
--                          | determine a maximal dimensional cone SIGMA
--                          | in the fan.
--          TAU             |       ""       ""   ...     TAU in the fan.
--          INDEXOFRHO      | The index of the column of MATRIXOFRAYS
--                          | which specifies rho.
--          MATRIXOFRAYS    | A matrix whose columns represent the ray
--                          | generators of the fan.
-- OUTPUT:  1               | If H^0(V_D_rho \cap sig \cap tau,u+u', C)
--                          | exists (see CLS p402, ch 9.1)
--          0               | Otherwise.
    sigma := SIGMA; tau := TAU; rhoIndex := INDEXOFRHO;
    matrixOfRays := MATRIXOFRAYS; v := {-1,-1,0};--this is u+u'

    commonRays = select(sigma, s->member(s,tau));

    --sigma and tau have only trivial intersection, so no ray gen
    if(length(commonRays) == 0) then(return 0)
    else(--check condition for common rays
        for s in commonRays do(
            --compute the inner product (and convert vec to list)
            temp = innerProduct(v, flatten entries matrixOfRays_s);
            --different condition for rho
            if(s == rhoIndex) then(if (temp < -1) then(return 1))
            --every other ray has same condition
            else(if (temp < 0) then(return 1))
            );
        return 0;--conditions not satisfied
        )
    )

existsCohom1ForTopObj = (SIGMA, INDEXOFRHO, MATRIXOFRAYS) -> (
-- INPUT:   SIGMA           | List of indices of ray generators that
--                          | determine a maximal dimensional cone SIGMA
--                          | in the fan.
--          INDEXOFRHO      | The index of the column of MATRIXOFRAYS which
--                          | specifies rho.
--          MATRIXOFRAYS    | A matrix whose columns represent the ray
--                          | generators of the fan.
-- OUTPUT:  1               | If H^0(V_D_rho \cap sig,u+u', C)
--                          | exists (see CLS p402, ch 9.1)
--          0               | Otherwise.
    sigma := SIGMA; rhoIndex := INDEXOFRHO; matrixOfRays := MATRIXOFRAYS;
    v := {-1,-1,0};--this is u+u'

   --check condition for common rays
    for s in sigma do(
        --compute the inner product (and convert vec to list)
        temp = innerProduct(v, flatten entries matrixOfRays_s);
            --different condition for rho
```

```
        if(s == rhoIndex) then(if (temp < -1) then(return 1))
            --every other ray has same condition
        else(if (temp < 0) then(return 1))
        );
    return 0;--conditions not satisfied
    )

existsCohom2 = (SIGMA, TAU, INDEXOFRHO, MATRIXOFRAYS) -> (
-- INPUT:  SIGMA         | List of indices of ray generators that
--                       | determine a maximal dimensional cone SIGMA
--                       | in the fan.
--         TAU           |       ""      ""   ...     TAU in the fan.
--         INDEXOFRHO    | The index of the column of MATRIXOFRAYS which
--                       | specifies rho.
--         MATRIXOFRAYS  | A matrix whose columns represent the ray
--                       | generators of the fan.
-- OUTPUT: 1             | If H^0(U_sigtau, O_X(D_rho))_u+u'
--                       | exists (checks condition in CLS p399)
--         0             | Otherwise.
   sigma := SIGMA; tau := TAU; rhoIndex := INDEXOFRHO;
   matrixOfRays := MATRIXOFRAYS; v := {-1,-1,0};--this is u+u'

   commonRays = select(sigma, s->member(s,tau));

   --trivial intersection
   if(length(commonRays) == 0) then(return 1)
   else(--check condition for common rays
       for s in commonRays do(
           --compute the inner product (and convert vec to list)
           temp = innerProduct(v, flatten entries matrixOfRays_s);
           --different condition for rho
           if(s == rhoIndex) then(if (temp < -1) then (return 0))
           --every other ray has same condition
           else(if (temp < 0) then (return 0))
           );
       return 1;--all conditions are satisfied!
       )
   )

existsCohom3 = (SIGMA, TAU, GAMMA, INDEXOFRHO, MATRIXOFRAYS) -> (
-- INPUT:  SIGMA         | List of indices of ray generators that
--                       | determine a maximal dimensional cone SIGMA.
--         TAU           |       ""                 ""   ...    TAU.
--         GAMMA         |       ""                 ""   ...    GAMMA.
--         INDEXOFRHO    | The index of the column of MATRIXOFRAYS
--                       | which specifies rho.
--         MATRIXOFRAYS  | A matrix whose columns represent the ray
--                       | generators of the fan.
-- OUTPUT: 1             | If H^0(U_sigtaugam, O_X(D_rho))_u+u'
--                       | exists (checks condition in CLS p399)
--         0             | Otherwise.
   sig := SIGMA; tau := TAU; gam := GAMMA; rhoIndex := INDEXOFRHO;
   matrixOfRays := MATRIXOFRAYS; v := {-1,-1,0};--this is u+u'
```

```
    --find the rays common to both sigma and tau
    temp        = select(sig, s->member(s,tau));
    --find the rays common to sigma, tau, and gamma
    commonRays = select(temp, s->member(s,gam));

    if(length(commonRays) == 0) then(return 1)--trivial intersection
    else(--check condition for common rays
        for s in commonRays do(
            --compute the inner product (and convert vec to list)
            temp = innerProduct(v, flatten entries matrixOfRays_s);
            --different condition for rho
            if(s == rhoIndex) then(if (temp < -1) then (return 0))
            --every other ray has same condition
            else(if (temp < 0) then (return 0))
            );
        return 1;--all conditions are satisfied!
        )
    )

existsCohom4 = (SIGMA, TAU, GAMMA, ALPHA, INDEXOFRHO, MATRIXOFRAYS) -> (
-- INPUT:   SIGMA         | List of indices of ray generators that
--                        | determine a maximal dimensional cone SIGMA
--                        | in the fan.
--          TAU           |     ""            ""     ...   TAU in the fan.
--          GAMMA         |     ""            ""     ...   GAMMA in the fan.
--          ALHPA         |     ""            ""     ...   ALPHA in the fan.
--          INDEXOFRHO    | The index of the column of MATRIXOFRAYS which
--                        | specifies rho.
--          MATRIXOFRAYS  | A matrix whose columns represent the ray
--                        | generators of the fan.
-- OUTPUT:  1             | If H^0(U_sigtaugam, O_X(D_rho))_u+u'
--                        | exists (checks condition in CLS p399)
--          0             | Otherwise.
    sig := SIGMA; tau := TAU; gam := GAMMA; alp := ALPHA;
    rhoIndex := INDEXOFRHO; matrixOfRays := MATRIXOFRAYS;
    v := {-1,-1,0};--this is u+u'

    --find the rays common to both sigma and tau
    temp1       = select(sig, s->member(s,tau));
    --find the rays common to sigma, tau, and gamma
    temp2       = select(temp1, s->member(s,gam));
    --find the rays common to sigma, tau, gamma, and alpha
    commonRays  = select(temp2, s->member(s,alp));

    if(length(commonRays) == 0) then(return 1)--trivial intersection
    else(--check condition for common rays
        for s in commonRays do(
            --compute the inner product (and convert vec to list)
            temp = innerProduct(v, flatten entries matrixOfRays_s);
            --different condition for rho
            if(s == rhoIndex) then(if (temp < -1) then (return 0))
            --every other ray has same condition
```

```
          else(if (temp < 0) then (return 0))
          );
       return 1;--all conditions are satisfied!
       )
   )

innerProduct = (VECTORA, VECTORB) -> (
-- INPUT:   VECTORA    | List of numbers representing a vector in C^n
--          VECTORB    | List of numbers representing a vector in C^n
-- OUTPUT: Euclidean inner product of the two vectors.
     a := VECTORA; b := VECTORB; temp := 0;

     if(#a == #b) then(
         for i from 0 to #a - 1 do(temp = temp + a#i * b#i);
         return temp;
         )
     else(print "Incompatible vector lengths.")
     )

intersectsComponent = (SIGMA, COMPONENT) -> (
-- INPUT:   SIGMA      | List of indices of ray generators that determine
--                     | a maximal dimensional cone SIGMA in the fan.
--          COMPONENT  | List of indices of ray generators whose induced
--                     | subgraph is a component of Gamma_rho(u).
-- OUTPUT: 1           | If the cone SIGMA intersects COMPONENT.
--         0           | Otherwise.
-- COMMENTS:
-- This is the implementation of the functions f_sig and g_sig.
     sig := SIGMA; comp := COMPONENT;
     --return 1 there is a ray in sigma that is also
         --in the component of the graph
     for s in sig do( if(member(s, comp)) then( return 1 ) );
     return 0;--if we haven't returned yet, then it must not intersect
     )

h = (SIGMA, TAU, GAMMA, COMPONENTC, COMPONENTD) -> (
-- INPUT:   SIGMA                              | List of ray generators that
--                                             | determine a maximal
--                                             | dimensional cone SIGMA in
--                                             | the fan.
--          TAU                                |   ""  ""   TAU in the fan.
--          GAMMA                              |   ""  ""   GAMMA in the fan.
--          COMPONENTC                         | List of ray generators
--                                             | whose induced subgraph is
--                                             | a component of Gamma_rho(u).
--          COMPONENTD                         | List of ray generators
--                                             | whose induced subgraph is
--                                             | a component of
--                                             | Gamma_rho'(u').
-- OUTPUT: (f_sig - f_tau)(g_tau - g_gam)      | Living in Cech
--                                             | representation of
--                                             | H^2(X,T_X)_u+u', then lift
--                                             | to H^2(X, O_X(D_rho))_u+u'
```

```
    sig := SIGMA; tau := TAU; gam := GAMMA; c := COMPONENTC;
        d := COMPONENTD;

    return (intersectsComponent(sig, c) - intersectsComponent(tau, c))*(
       ↪ intersectsComponent(tau, d) - intersectsComponent(gam, d));
    )

s = (SIGMA, TAU, COMPONENTC, COMPONENTD) ->(
-- INPUT:  SIGMA                       | List of ray generators that
--                                     | determine a maximal dimensional
--                                     | cone SIGMA in the fan.
--         TAU                         |   ""      ""      TAU in the fan.
--         COMPONENTC                  | List of ray generators whose
--                                     | induced subgraph is a component
--                                     | of Gamma_rho(u).
--         COMPONENTD                  | List of ray generators whose
--                                     | induced subgraph is a component
--                                     | of Gamma_rho'(u').
-- OUTPUT: f_sig * (g_tau - g_sig)     | Living in Cech representation of
--                                     | H^2(X,T_X)_u+u', then lift to
--                                     | H^2(X, O_X(D_rho))_u+u'
-- COMMENTS:
-- This computes a potential preimage for the nonzero cocycle (so that
-- the cocyle is zero modulo the image)
    sig := SIGMA; tau := TAU; c := COMPONENTC; d := COMPONENTD;

    return intersectsComponent(sig, c)*(intersectsComponent(tau, d) -
       ↪ intersectsComponent(sig, d));
    )

checkPreimage = (PREIMAGE, PAIRSOFCONES, INDEXOFRHO, MATRIXOFRAYS) ->(
-- INPUT:  PREIMAGE      | A 91x1 matrix, representing an element of
--                       | C^1 viewed in C^91.
--         PAIRSOFCONES  | A list of lists of the form {A,B}, where A
--                       | and B are lists of ray generators specifying
--                       | maximal cones in the fan.
--         INDEXOFRHO    | The index of the column of MATRIXOFRAYS
--                       | which specifies rho.
--         MATRIXOFRAYS  | A matrix whose columns represent the ray
--                       | generators of the fan.
-- OUTPUT: N/A
-- COMMENTS:
-- This checks if elements of C^1 are allowable. That is, a vector
-- cannot have a nonzero entry for a pair of cones sigma, tau
-- which make H^0(U_sigmatau, O_X(D_rho))_u+u' vanish. It prints the
-- index for which this clash happens.
    preImage := PREIMAGE; P := PAIRSOFCONES, i := INDEXOFRHO;
        M := MATRIXOFRAYS; j := 0;

    for p in P do(
        sig = p#0; tau = p#1;

        temp = existsCohom2(sig, tau, i, M);
```

```
    if(temp != 1) then(if(preImage_(j,0) != 0) then(print j));
    j = j+1;
    );
)
```

A.2 main-1.m2

```
restart
load "header.m2"

-----------------------------------------------------------------------
-- MAIN
-----------------------------------------------------------------------

-- Some useful constants.
u        = {-1,  0, 0};
uPrime   = { 0, -1, 0};
rho      = { 1,  0, 0};
rhoPrime = { 1,  1, 0};

-- Specify ray generators.
M = transpose(matrix({ rho, {1,0,-1}, {1,0,1}, {2,-1,0}, {1,-1,0},
  ↪ rhoPrime, {0,1,-1}, {0,1,1}, {-1,0,0} }));

-- Specify all the maximal cones of the fan via ray generators.
rayGenerators = { {0,1,3}, {0,1,6}, {0,2,3}, {0,2,7}, {0,5,6}, {0,5,7},
  ↪ {1,3,4}, {1,4,8}, {1,6,8}, {2,3,4}, {2,4,8}, {2,7,8}, {5,6,8},
  ↪ {5,7,8} };

-- Turn this data into actual cones for macaulay2.
coneList = apply(rayGenerators, u->coneFromVData(M_u));

-- Construct the fan.
F = fan(coneList)

-- The fan is complete.
isComplete(F)

-- The fan is also smooth.
isSmooth(F)

-- Sanity check, H^1 for T_X is nonvanishing.
HH^1 tangentBundle(F)

-- Determine the indices of each ray (index of rho is 1).
matrixOfRays = rays(F)

-- Maximal cones of F specified by indices of rays.
maximalCones = faces(0, F);

-- Specify connected components by indices of rays.
componentC1 = {2,3,5,7};-- C1 for Gamma_rho(u).
componentC2 = {4};-- C2 for Gamma_rho(u).
componentD1 = {8};-- D1 for Gamma_rho'(u').
componentD2 = {6};-- D2 for Gamma_rho'(u').

quadsOfCones   = subsets(maximalCones, 4);
triplesOfCones = subsets(maximalCones, 3);
pairsOfCones   = subsets(maximalCones, 2);
```

```
-- Compute the image of the cup product for C1 x D1.
hSigmaTauGammaC1D1 = {};
for T in triplesOfCones do(hSigmaTauGammaC1D1 = append(
    ↪ hSigmaTauGammaC1D1, h(T#0, T#1, T#2, componentC1, componentD1)))

-- Compute the image of the cup product for C1 x D2.
hSigmaTauGammaC1D2 = {};
for T in triplesOfCones do(hSigmaTauGammaC1D2 = append(
    ↪ hSigmaTauGammaC1D2, h(T#0, T#1, T#2, componentC1, componentD2)))

-- Compute the image of the cup product for C2 x D1.
hSigmaTauGammaC2D1 = {};
for T in triplesOfCones do(hSigmaTauGammaC2D1 = append(
    ↪ hSigmaTauGammaC2D1, h(T#0, T#1, T#2, componentC2, componentD1)))

-- Compute the image of the cup product for C2 x D2.
hSigmaTauGammaC2D2 = {};
for T in triplesOfCones do(hSigmaTauGammaC2D2 = append(
    ↪ hSigmaTauGammaC2D2, h(T#0, T#1, T#2, componentC2, componentD2)))

-- Build the matrices D0, D1, D2 which encode the Cech differentials.
listOfRows = {};
for P in pairsOfCones do(
     row_P = {};
     for M in maximalCones do(row_P = append(row_P, entryInDiffMatrix0(P,
         ↪  M)));
     listOfRows = append(listOfRows, row_P);
     )

-- Matrix encoding Cech differential map from C^0 to C^1, viewed as C^14
    ↪  -> C^91.
D0  = sub(matrix(listOfRows), QQ);

-- Matrix of generators for the kernel of D0: C^14 -> C^91.
KD0 = gens(ker(D0));

listOfRows = {};
for T in triplesOfCones do(
     row_T = {};
     for P in pairsOfCones do(row_T = append(row_T, entryInDiffMatrix(T,
         ↪ P)));
     listOfRows = append(listOfRows, row_T);
     )

-- Matrix encoding Cech differential map from C^1 to C^2, viewed as C^91
    ↪  -> C^364.
D1  = sub(matrix(listOfRows), QQ);

-- Matrix of generators for the kernel of D1: C^91 -> C^364.
KD1 = gens(ker(D1));

listOfRows = {};
```

```
for Q in quadsOfCones do(
    row_Q = {};
    for T in triplesOfCones do(row_Q = append(row_Q, entryInDiffMatrix2(
        ↪ Q, T)));
    listOfRows = append(listOfRows, row_Q);
    )

-- Matrix encoding Cech differential map from C^2 -> C^3, viewed as C
    ↪ ^364 -> C^1001.
D2 = sub(matrix(listOfRows), QQ);

-- Matrix of generators for the kernel of D2: C^364 -> C^1001.
KD2 = gens(ker(D2));

-- Build matrices whose image is C^i (allowable)
m = {};
for P in pairsOfCones do(m = append(m, existsCohom2(P#0, P#1, 1,
    ↪ matrixOfRays)))
I1 = sub(diagonalMatrix(m), QQ);-- Matrix giving allowable elements of C
    ↪ ^1.

b = {};
for T in triplesOfCones do(b = append(b, existsCohom3(T#0, T#1, T#2, 1,
    ↪ matrixOfRays)))
I2 = sub(diagonalMatrix(b), QQ);-- Matrix giving allowable elements of C
    ↪ ^2.

c = {};
for Q in quadsOfCones do(c = append(c, existsCohom4(Q#0, Q#1, Q#2, Q#3,
    ↪ 1, matrixOfRays)))
I3 = sub(diagonalMatrix(c), QQ);-- Matrix giving allowable elements of C
    ↪ ^2.

t = {};-- This is for the topological object
for P in pairsOfCones do(t = append(t, existsCohom2ForTopObj(P#0, P#1,
    ↪ 1, matrixOfRays)))
J1 = sub(diagonalMatrix(t), QQ);-- Matrix giving allowable elements of C
    ↪ ^1.

t = {};-- This is for the topological object
for M in maximalCones do(t = append(t, existsCohom1ForTopObj(M, 1,
    ↪ matrixOfRays)))
J0 = sub(diagonalMatrix(t), QQ);-- Matrix giving allowable elements of C
    ↪ ^1.

y = {};-- This is for the topological object
for T in triplesOfCones do(y = append(y, existsCohom3ForTopObj(T#0, T#1,
    ↪ T#2, 1, matrixOfRays)))
J2 = sub(diagonalMatrix(y), QQ);-- Matrix giving allowable elements of C
    ↪ ^1.

hSTGC1D1 = transpose(sub(matrix({hSigmaTauGammaC1D1}), QQ));-- Potential
    ↪ nonzero element in H^2, interpreted as an element of C^2.
```

```
hSTGC1D1%I2 -- Check it is allowable.
hSTGC1D1%KD2 -- Check it lies in H^2.
hSTGC1D1%D1I1 -- Doesn't lie in image.

hSTGC1D2 = transpose(sub(matrix({hSigmaTauGammaC1D2}), QQ));-- Potential
    ↪ nonzero element in H^2, interpreted as an element of C^2.
hSTGC1D2%I2 -- Check it is allowable.
hSTGC1D2%KD2 -- Check it lies in H^2.
hSTGC1D2%D1I1 -- Doesn't lie in image.

hSTGC2D1 = transpose(sub(matrix({hSigmaTauGammaC2D1}), QQ));-- Potential
    ↪ nonzero element in H^2, interpreted as an element of C^2.
hSTGC2D1%I2 -- Check it is allowable.
hSTGC2D1%KD2 -- Check it lies in H^2.
hSTGC2D1%D1I1 -- Doesn't lie in image.

hSTGC2D2 = transpose(sub(matrix({hSigmaTauGammaC2D2}), QQ));-- Potential
    ↪ nonzero element in H^2, interpreted as an element of C^2.
hSTGC2D2%I2 -- Check it is allowable.
hSTGC2D2%KD2 -- Check it lies in H^2.
hSTGC2D2%D1I1 -- Doesn't lie in image.

K = intersect(ker(I3*D2*I2), image I2);
I = image(I2*D1*I1);
C = K/I;
prune C --H^2(X, O(D)) is 1-dimensional, as expected.

K = intersect(ker(J2*D1*J1), image J1);
I = image(J1*D0*J0);
C = K/I;
prune C-- H^1(V,C) is 1-dimensional, as expected.

W = I2*D1*J1;
vC1D1 = W\\hSTGC1D1;
printCones(vC1D1, pairsOfCones)

vC1D2 = W\\hSTGC1D2;
printCones(vC1D2, pairsOfCones)

vC2D1 = W\\hSTGC2D1;
printCones(vC2D1, pairsOfCones)

vC2D2 = W\\hSTGC2D2;
printCones(vC2D2, pairsOfCones)
```