

2-Regular Digraphs on Surfaces

by

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Abstract

A 2-regular digraph is one where every vertex has in-degree and out-degree 2. This thesis focuses on surface embeddings of 2-regular digraphs, one where the underlying graph is embedded in a surface and all faces are bounded by directed closed walks. Immersion acts as a natural minor-like containment relation for embedded 2-regular digraphs and this theory is linked to undirected graph minors by way of the directed medial graph.

We parallel the theory of undirected graphs in surfaces by proving analogues of Whitney's Theorem and Tutte's peripheral cycles theorem for 2-regular digraphs in the sphere. Then, using a notion of branch-width and a 2-regular digraph grid theorem by Johnson, we prove that for each fixed surface S , the 2-regular digraphs embeddable in S are characterized by a finite list of immersion obstructions. We then present the current state of the art with regard to classification of obstructions for surfaces: Johnson characterized the sphere obstructions, we classify all projective plane obstructions, and we have a computer assisted partial list of obstructions for the torus and Klein bottle.

We also consider two open problems in the world of undirected graphs on surfaces and resolve their analogues in the world of 2-regular digraphs on surfaces. The first conjecture by Negami we resolve in the affirmative, that a 2-regular digraph has a finite planar cover if and only if it is projective planar. The second, the strong embedding conjecture, is resolved in the negative and we provide an infinite family of well connected counter-examples.

Keywords: Planar graphs (05C10), Directed graphs (05C20), Eulerian graphs (05C45), Graph structure (05C75), Graph minors (05C83)

Dedication

To Jarin,
without whom there would be no point.

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Chapter 1

Introduction

This thesis centers around *2-regular digraphs*, directed graphs where each vertex has in-degree and out-degree 2, and surface embeddings of these digraphs. We use the term *surface* for a closed 2-manifold without boundary. And we say that a 2-regular digraph *embeds* in a surface S if the *underlying graph* (a graph formed from a digraph by replacing every directed edge with an undirected edge) embeds in S , and additionally if at every vertex v the edges incident to v alternate in-out-in-out around v (see Figure 1.1).

Recall that an *embedding*, of an undirected graph $G = (V, E)$ in S , is a map ϕ that sends vertices in V injectively to points in S , and edges uv in E to pairwise internally disjoint simple curves where $\phi(u)$ and $\phi(v)$ are the endpoints of the curve $\phi(uv)$ and $\phi(uv) \cap \phi(x) = \emptyset$ for all $x \in V \setminus \{u, v\}$. The *faces* of ϕ are the connected components of $S - (\phi(V) \cup \phi(E))$. Each face, F , of ϕ is bounded by a disjoint union of closed walks in G ; accordingly, we may refer to these closed walks as the *face* F as well. An embedding is *2-cell* if each face is homeomorphic to an open disk. Note that in a 2-cell embedding each face is bounded by a single closed walk in G .

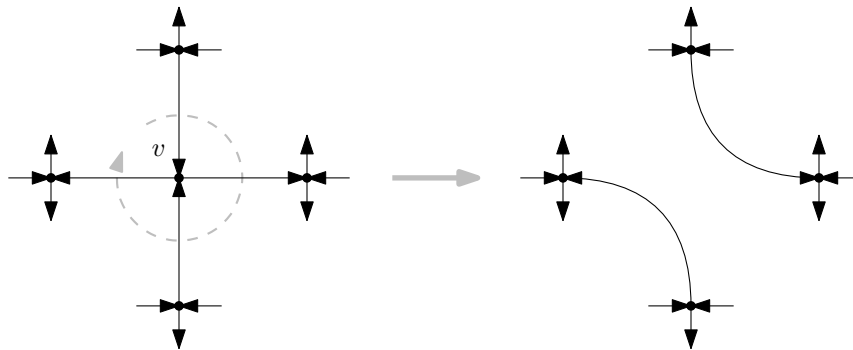


Figure 1.1: A 2-regular digraph, before and after splitting a vertex, embedded in a surface.

Graphs on surfaces form a minor-closed class; that is, if G has an embedding in a surface S , then so does every minor of G . (Recall that an undirected graph H is a *minor* of an undirected graph G if H is isomorphic to a graph obtained from a subgraph G' of G by

contracting a set of edges $A \subseteq E(G')$.) Saying this a different way: If G is embedded in S , then for every $e \in E(G)$ both $G - e$ (G delete e) and G/e (G contract e) are embedded in S . To see this, note that the deletion of e maintains the fact that all curves of $\phi(E(G) \setminus \{e\})$ are pairwise internally disjoint. For contracting, choose $x \in S$ where for each edge f incident to an endpoint of e , one can continuously deform $\phi(f)$ so that all edges remain pairwise internally disjoint and have a new common endpoint x .

The class of 2-regular digraphs embeddable in a surface S is similarly closed under vertex splits, which we now define. Let D be a 2-regular digraph. To *split* a loop-free¹ vertex $v \in V(D)$, with in-edges e_0, e_1 and out-edges f_0, f_1 , is to delete v , add an edge from the tail of e_i to the head of f_j , and add an edge from the tail of e_{i+1} to the tail of f_{j+1} with indices expressed modulo 2. Note that there are two ways to split a single vertex.

The digraph H obtained after splitting v is also 2-regular; thus, the class of 2-regular digraphs is closed under vertex splits. To see that the class of 2-regular digraphs embeddable in a surface S is also closed under vertex splits consider D as an embedded graph with $v \in V(D)$. Since the edges incident to v alternate in-out-in-out, the underlying graph obtained after splitting v is embedded in S , and all neighbors of v have incident edges that alternate in-out-in-out (see Figure 1.1).

The class of 2-regular digraphs under immersion (to be defined in Section 1.2) was studied in Johnson's 2002 thesis [20]. A chief result of his thesis is an rough structure theorem for excluding a fixed 2-regular digraph as an immersion. This is an analogue of some of the work in Robertson and Seymour's Graph Minors project [32, 33, 34]. The main goal of this thesis is to continue the work of Johnson and prove further graph minor analogues for 2-regular digraphs. However, before we embark on this, we wish to motivate the study of the 2-regular digraph. And to do so we examine a naturally occurring instance of a 2-regular digraph, the medial graph of an undirected graph embedded in an orientable surface.

1.1 The medial graph

The class of 2-regular digraphs, equipped with the embedding definition above, is perhaps a strange class to study. As a means to motivate this, we take a quick detour through the medial graph construction. The medial graph has been studied in various different contexts such as voltage and current graphs [6, 3] and graph polynomials/knots [9, 10, 24].

Let G be an undirected graph and let ϕ be an embedding of G into a surface S with faces \mathcal{F} . Form the undirected *medial* G_ϕ as follows:

- Let $V(G_\phi) = E(G)$.

¹Splitting is defined at a more granular level in Section 1.2, so as to include loop edges.

- For $e, f \in V(G_\phi)$, if e and f are consecutive in a facial walk from \mathcal{F} , then $ef \in E(G_\phi)$.

Observe that G_ϕ is a 4-regular graph. This can be seen since each edge $e \in E(G_\phi)$ appears exactly twice in faces of \mathcal{F} and each occurrence of e contributes 2 to the degree of e in G_ϕ . Furthermore, note that G_ϕ has a natural embedding ψ in S where each face corresponds either to a face in \mathcal{F} or a vertex in $V(G)$. In fact, this forms a proper 2-coloring of the faces of G_ϕ .

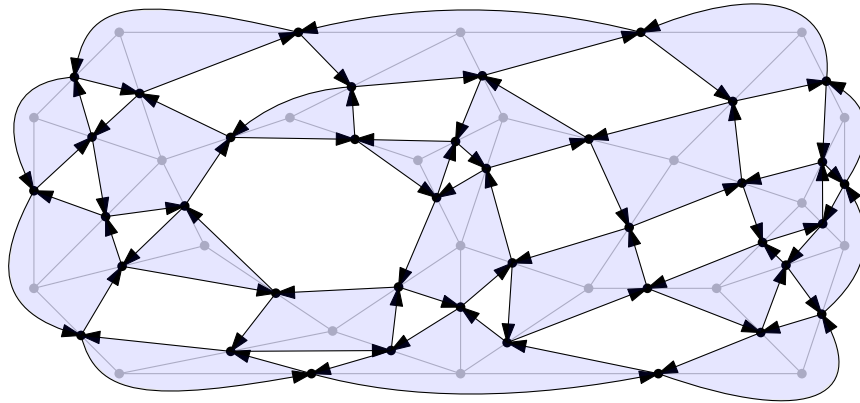


Figure 1.2: The directed medial of a graph embedded in the sphere.

If S is an orientable surface, then we can orient the edges of G_ϕ to form a *directed* medial as follows:

- Choose one of the two color classes of faces of ψ .
- For each face in this color class, orient the edges so that the facial walks are directed clockwise walks in S .
- Observe that the facial walks of the other color class are directed anticlockwise in S .

Let \overrightarrow{G}_ϕ denote a directed medial of G . Observe that ψ is now an embedding of a 2-regular digraph since every vertex of \overrightarrow{G}_ϕ locally alternates in-out-in-out.

Additionally, this construction is reversible: if you take a 2-regular digraph D and an embedding of D into an orientable surface S , one can read off an associated undirected graph by reversing the construction described above.

There is also a strong link between an undirected graph G and an associated directed medial graph D by way of minors of G and vertex splits of D . To see this, consider G and D as embedded graphs on some orientable surface S (see Figure 1.2 for an example). Take an edge $e \in E(G)$ and observe that $G - e$ and G/e have associated medial graphs obtained by considering the two vertex splits of $e \in V(D)$. Of course, this is also reversible: start with $e \in V(D)$ and split e in both ways and observe that there are associated undirected graphs obtained from G by deleting e and contracting e .

1.2 Immersion

Let $D = (V, E)$ be a digraph. Let $e, f \in E$ where $e = xy$ and $f = zw$. If $x = y$ then we say that e is a *loop*. If $e \neq f$ but $x = z$ and $y = w$, then we say that e and f are *parallel* edges. Similarly, if $e \neq f$ but $x = w$ and $y = z$, then we say that e and f form a *digon*, $\{e, f\}$, or a directed cycle of length 2.

For all $v \in V$ we define $E^+(v)$ to be the set of edges whose tail is v and $E^-(v)$ to be the set of edges whose head is v and we define $E(v) = E^+(v) \cup E^-(v)$. We call $E^+(v)$ *out-edges* of v and $E^-(v)$ *in-edges* of v , and define $\deg^+(v) = |E^+(v)|$ as the *out-degree* of v , $\deg^-(v) = |E^-(v)|$ as the *in-degree*, and $\deg(v) = |E(v)|$ is the *degree* of v . If D is connected and $\deg^+(v) = \deg^-(v)$ for all $v \in V$, then D is *Eulerian*. For a vertex $v \in V$, if $\deg^+(v) = \deg^-(v) = k$ then we say that v is *k-regular*. If all vertices in V are *k-regular*, then D is *k-regular*.

For $X \subseteq V$ let $\delta_D^+(X)$ denote the set of edges with a tail in X and a head in $V \setminus X$, let $\delta_D^-(X)$ denote the set of edges with a head in X and a tail in $V \setminus X$, and let $\delta_D(X) = \delta_D^+(X) \cup \delta_D^-(X)$. We call $\delta_D(X)$ an *edge-cut* of X . (When the context is clear we will drop the subscript D .) We use the following notation to denote the sizes of the above sets of edges: $d^+(X) = |\delta^+(X)|$, $d^-(X) = |\delta^-(X)|$, and $d(X) = |\delta(X)|$. For $Y \subseteq V$ disjoint from X , we let $d(X, Y)$ denote the size of a minimum edge cut separating X from Y .

A *walk* in D is an ordered alternating sequence $W = v_1, e_1, v_2, e_2, \dots, e_{k-1}, v_k$ where $e_i = v_i v_{i+1}$ for all $0 < i < k$. The walk W is a *trail* if each e_i is distinct. Let $\text{trails}(D)$ denote the set of trails of D . We say that a walk is *closed* if $v_0 = v_k$. If a walk in D uses every edge in E exactly once, then we call the walk *Eulerian*. If an Eulerian walk is closed, we call it an *Euler tour*.

For a digraph H , we say that H is *immersed* in D if there is a pair of functions (ϕ, ϕ') where $\phi : V(H) \rightarrow V(D)$ is injective and $\phi' : E(H) \rightarrow \text{trails}(D)$ where for $e = xy \in E(H)$ we have $\phi'(e) = u_1, e_1, \dots, e_{k-1}, u_k$ with $\phi(x) = u_1$ and $\phi(y) = u_k$, and for any $f \in E(H)$, with $f \neq e$, $\phi'(e)$ and $\phi'(f)$ are edge disjoint. We say that an edge $e \in E(H)$ and the trail $\phi'(e)$ *correspond*.

Observe that in a digraph a walk, $W = v_1, e_1, v_2, e_2, \dots, e_{k-1}, v_k$, is determined by the sequence of edges $(e_1, e_2, \dots, e_{k-1})$ (v_i is the tail of e_i and v_{i+1} is the head of e_i for $1 \leq i < k$). In light of this, we will abuse notation by treating walks as sequences of directed edges. This treatment is used throughout the thesis. A *subwalk* U of W is a subsequence of consecutive elements of W denoted $U \subseteq W$.

For $v \in V(D)$, we define a *transition* at v as an ordered pair of edges, $t = (e, f)$, where e is an in-edge of v and f is an out-edge of v . If $e = uv$ and $f = vw$ then we say that u is the *tail* of t and that w is the *head* of t and $\{u, w\}$ are the *endpoints* of t . Let $T(v)$ denote the set of transitions at v . Let $\mathcal{T}(D) = \bigcup_{v \in V(D)} T(v)$ denote the set of all transitions of D . Because of our notational abuse above, one may consider a transition as a walk of size 2.

1.2.1 2-regular digraph immersion

A 2-regular digraph D can have loops, digons, and parallel edges; D can also include a somewhat unusual object, an edge with no endpoint. Such an edge should be viewed as a directed circle, and it forms a component of the digraph. We call such an edge *pointless*. Although this is a strange object, it appears naturally in the theory. (It is found in [10], where it is called a “free loop”.) Given a vertex v with $\deg^+(v) = \deg^-(v) = 1$ and in-edge $e = uv$ and out-edge $f = vw$, to *suppress* v is to delete v and add an edge from u to w . If $\{e, f\}$ is a digon, then suppressing v creates a loop. If $e = f$, then we have a loop at v and suppressing v creates a pointless edge.

Observe that since D is 2-regular, $|T(v)| = 4$ for all $v \in V(D)$. Note that if e is a loop at v , then $t = (e, e)$ is a transition at v . If $s, t \in T(v)$ with $s = (e_1, f_1)$ and $t = (e_2, f_2)$ we say that s and t are *complementary* transitions at v if $e_1 \neq e_2$ and $f_1 \neq f_2$, (as splitting either of them yields an isomorphic graph).

Let $W = e_1, \dots, e_k$ be a directed walk in D . Thanks to our (above) abuse of notation, when a transition $t = (e_i, e_{i+1}) \subseteq W$ then we say that t is *contained* in W . We may also replace the edges from t and write $W = e_1, \dots, e_{i-1}, t, e_{i+2}, \dots, e_k$. If W is closed, then the transition (e_k, e_1) is also contained in W . And in the extreme case when $k = 1$ and W is a closed walk (W is just a loop), we consider the transition (e_1, e_1) to be contained in W . For a graph with a component that is a pointless edge e , the possible walks using e are just (e, e, \dots, e) with e appearing k times to indicate the number of times that e has been traversed in the walk. Note that as a pointless edge had no vertex, no transitions are associated with a walk using a pointless edge.

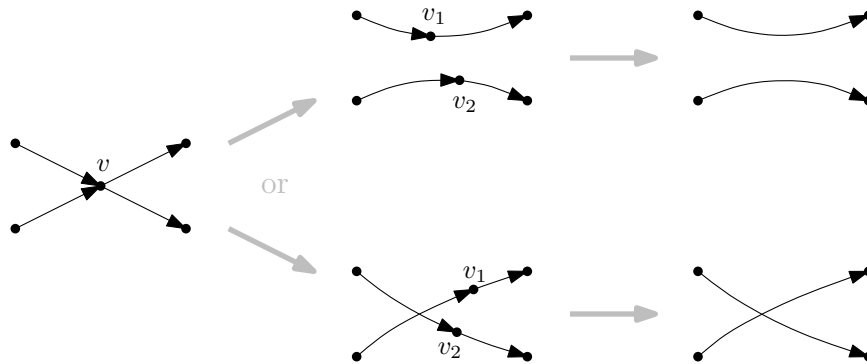


Figure 1.3: Splitting a vertex with a transition.

Let $t = (e_1, f_1)$ and $s = (e_2, f_2)$ be complementary transitions from $T(v)$. To *split* v with t (or s) is to, for $i \in \{1, 2\}$:

- i. Create *children* vertices v_1 and v_2 .
- ii. Change the head of e_i to v_i .

- iii. Change the tail of f_i to v_i .
- iv. Delete the now isolated v .
- v. Suppress children vertices v_1 and v_2 .

We call the new graph formed D/t (or D/s). For any two complementary transitions t and t' , we have that $D/t = D/t'$. This implies that for every $v \in V(D)$ at most 2 distinct digraphs, up to isomorphism, can be obtained from splitting v .

It is helpful to have multiple ways of thinking about immersions of 2-regular digraphs. Propositions 1.1 and 1.2 relate 2-regular digraph immersion to the vertex split operation. We provide proof sketches of the “only if” direction for each proposition.

Proposition 1.1. *Let H and D be 2-regular digraphs. H is immersed in D if and only if a graph isomorphic to H can be obtained from D by the following sequence of operations:*

- i) *Delete the edges of an Eulerian subgraph of D .*
- ii) *Split a set of vertices of D .*
- iii) *Suppress vertices of in-degree and out-degree 1.*
- iv) *Delete isolated vertices and pointless edges.*

Proof sketch. If H is immersed in D with (ϕ, ϕ') , then $D - \phi'(E(H))$ breaks into Eulerian components. Form D' from D by deleting the edges of these components. Observe that every vertex of D' is either isolated, 1-regular, or 2-regular. For a 2-regular vertex $v \in V(D')$, if v is not the endpoint of any trail from $\phi'(E(H))$, then split v with the transition (or its complement) that appears in a trail, from the immersion, through v . The 2-regular vertices of D' that are endpoints of trails of $\phi'(E(H))$ come from $\phi(V(H))$. \square

Proposition 1.2. *Let H and D be connected 2-regular digraphs. H is immersed in D if and only if a graph isomorphic to H can be obtained from D by a sequence of vertex splits of D .*

Proof. The “if” direction follows immediately from Proposition 1.1. For the “only if” direction, let H be immersed in D with the sequence of operations from Proposition 1.1. We will split all vertices as instructed by Proposition 1.1 and replace deletion by splitting in the following way: Let D' be the union of the Eulerian subgraphs of D whose edges are to be deleted. Consider a connected component $D'' \subseteq D'$. Observe that D'' contains at least one 1-regular vertex v . Take an Euler tour W of D'' . Instead of deleting the edges of D'' , split each vertex $u \in V(D'')$, where $u \neq v$, with one of the transitions from $T(u)$ that are contained in W . This results in a loop at v which can be split so as to not create a pointless edge. Repeat for other connected components of D' . \square

1.3 Combinatorial embeddings of 2-regular digraphs

Let $\mathbb{S}_0 = \{\vec{x} \in \mathbb{R}^3 : \|\vec{x}\| = 1\}$ denote the 2-sphere (or *sphere*).² Let \mathbb{S}_h denote the surface obtained from \mathbb{S}_0 by adding h handles to it, and let \mathbb{N}_k denote the surface obtained from \mathbb{S}_0 by adding k cross-caps to it. We refer the reader to [26] for a proof of the fact that every surface is homeomorphic to one of \mathbb{S}_h or \mathbb{N}_k with $h \geq 0$ and $k \geq 1$. We call \mathbb{S}_h the *orientable* surface of genus h and \mathbb{N}_k the *nonorientable* surface of genus k . We define the *Euler genus* of a surface to be $\text{eg}(\mathbb{S}_h) = 2h$ and $\text{eg}(\mathbb{N}_k) = k$.

In 1963, using what he calls “the capping operation”, Youngs shows that embeddings of undirected graphs in surfaces of minimal genus are necessarily 2-cell. We inherit the following proposition from his work.

Proposition 1.3 (Youngs [49]). *Every embedding of a 2-regular digraph in a surface of minimum Euler genus is a 2-cell embedding.*

Proof sketch. Let ϕ be an embedding of a 2-regular digraph in a surface S of minimal Euler genus. Suppose towards a contradiction that ϕ is not 2-cell. Thus, there exists a face $F \subseteq S$ that is not homeomorphic to a disk where F contains a 2-manifold with boundary that has possibly many boundary components. Modify S by capping these boundary components with a disks from \mathbb{R}^2 . This yields an embedding of D in a surface with lower Euler genus, contradicting the assumption. \square

In light of Proposition 1.3, we introduce the following combinatorial description of an embedding of a 2-regular digraph that is used extensively in the thesis. A 2-cell embedding of a 2-regular digraph is *defined* as follows: For a 2-regular digraph D we define a *2-cell embedding* of D , denoted Ω , to be a collection of directed closed walks where each edge of D appears twice and where each transition in $\mathcal{T}(D)$ is contained in exactly one closed walk from Ω . (An analogous embedding definition also holds for undirected cubic graphs, where transitions are defined as unordered adjacent edge pairs.) Each directed closed walk $F \in \Omega$ is called a *face* of Ω . Given D and Ω , we can construct a surface S by sewing an open disk onto the boundary of each face, i.e. “capping”. As such, this embedding necessarily corresponds to a 2-cell embedding of D . We say that Ω is *orientable* (*nonorientable*) if S is orientable (nonorientable).

Given a 2-cell embedding Ω of D , we say that a *bipartition* $\{\mathcal{A}, \mathcal{B}\}$ of Ω is a partition of Ω where $\bigcup_{F \in \mathcal{A}} E(F) = E(D) = \bigcup_{F \in \mathcal{B}} E(F)$. Note that in a bipartition of Ω , every edge $e \in E(D)$ appears exactly once in both \mathcal{A} and \mathcal{B} .

²Throughout the thesis we predominately work with the sphere, but we also occasionally work with the plane, \mathbb{R}^2 , (recalling that the sphere is the one point compactification of the plane) and the disk. 2-regular digraphs embed in the sphere if and only if they embed in the plane if and only if they embed in a disk. The reason for making the distinction is to occasionally rely on the distinguished *outer* (*unbounded*) *face* of the plane and to rely on the boundary of the disk.

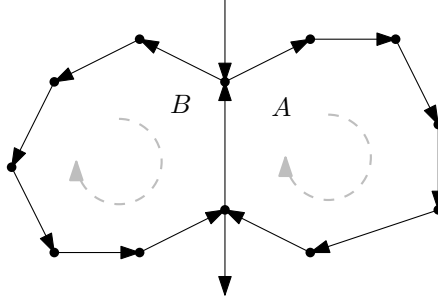


Figure 1.4: An orientable embedding with bipartition $\{\mathcal{A}, \mathcal{B}\}$ and faces $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

Proposition 1.4. *Let D be a 2-regular digraph. A 2-cell embedding Ω of D is orientable if and only if Ω has a bipartition.*

Proof sketch. Let Ω be an orientable embedding in a surface S . Since S is orientable, fix a global clockwise orientation of S . Define $\mathcal{A}, \mathcal{B} \subseteq \Omega$ by: $F \in \mathcal{A}$ ($F \in \mathcal{B}$) if the directed walk in D is clockwise (counter clockwise). The partition $\{\mathcal{A}, \mathcal{B}\}$ is a bipartition of Ω , because for every edge $e \in E(D)$, e is incident to one face in \mathcal{A} and one in \mathcal{B} .

Conversely, let $\{\mathcal{A}, \mathcal{B}\}$ be a bipartition of Ω . We follow the construction of a surface S , but now we equip each open disk with an orientation. For every $F \in \mathcal{A}$ ($F \in \mathcal{B}$), we orient the associated disk so that the directed closed walk F is clockwise (counter clockwise). This gives S a consistent orientation. \square

We define the Euler genus of a 2-regular digraph in analogy with undirected graphs. Let $D = (V, E)$ be a connected 2-regular digraph with $n = |V| \geq 1$ and $e = |E|$. Let Ω be an embedding of D in some surface with $f = |\Omega|$. The *Euler characteristic* of Ω is

$$\chi(\Omega) = n - e + f,$$

and the *Euler genus* of Ω is

$$\text{eg}(\Omega) = 2 - \chi(\Omega).$$

The *Euler genus* of D , denoted $\text{eg}(D)$, is the minimum $\text{eg}(\Omega)$ over all embeddings Ω of D .

For $n = 0$, D consists solely of a pointless edge e . In this case, recall that $\mathcal{T}(D) = \emptyset$; however, according to our definition of a 2-cell embedding, there must exist closed walks that cover e twice. This can be accomplished in two distinct ways: $\Omega_1 = \{(e), (e)\}$ and $\Omega_2 = \{(e, e)\}$. Taking S_i to be the surface formed by sewing disks onto the face(s) of Ω_i , one can see that S_1 is homeomorphic to the sphere and S_2 is homeomorphic to the projective plane. If e contributes zero to the edge count then this fact is verified via Euler characteristic. For simplicity of presentation we generally omit pointless edges from our analysis; nonetheless, the results in the thesis extend naturally to include it.

Chapter 2

2-Regular Digraphs in the Sphere

In 1930 Kuratowski proved a famous result in topological graph theory classifying the graphs that embed in the sphere. (An undirected graph H is a *topological minor* of an undirected graph G if G contains a subdivision of H as a subgraph.)

Theorem 2.1 (Kuratowski [22]). *A graph has an embedding in the sphere if and only if it does not contain K_5 or $K_{3,3}$ as a topological minor.*

Let ϕ_1 and ϕ_2 be 2-cell embeddings of an undirected graph G in surfaces S_1 and S_2 . We say that ϕ_1 and ϕ_2 are *equivalent* if their sets of facial walks in G are equal. If all 2-cell embeddings of G in a surface S are equivalent to ϕ , then we say that ϕ is the *unique* embedding of G in S . If G is embedded in the sphere and G is not suitably connected, then it is possible to modify the embedding to obtain an inequivalent embedding of G in the sphere. One such modification is the Whitney flip (see Figure 2.1).

Let ϕ be a 2-cell embedding of G in a surface S . Suppose that $D \subseteq S$ is homeomorphic to a closed disk, and suppose that the boundary of D meets G in exactly two points, say x and y . Then there is a homeomorphism α mapping D to a closed unit disk $D' \subseteq \mathbb{S}_0$ where $\alpha(x)$ and $\alpha(y)$ are antipodes of D' . Applying the map α , followed by the mirror reflection of D' through the line containing $\alpha(x)$ and $\alpha(y)$, and then applying α^{-1} gives us a new embedding ϕ' of G in the same surface S . We call this operation a *Whitney flip* of the disk D . In 1933, Whitney proved the following theorem describing embeddings of 2-connected undirected graphs in the sphere.

Theorem 2.2 (Whitney [48]). *If ϕ_1 and ϕ_2 are embeddings of a 2-connected graph G in the sphere, then by applying a sequence of Whitney flips, ϕ_1 can be transformed into an embedding equivalent to ϕ_2 .*

For a 3-connected graph no meaningful Whitney flip can be performed, so by Theorem 2.2 we obtain the following corollary.

Corollary 2.3 (Whitney [48]). *Any two embeddings of a 3-connected graph in the sphere are equivalent.*

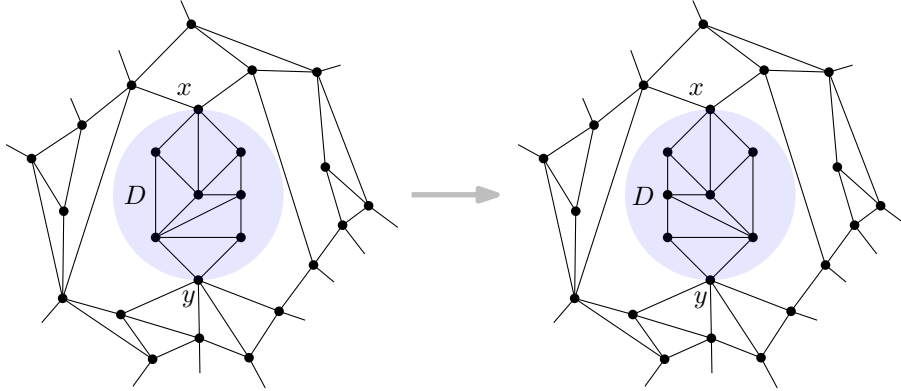


Figure 2.1: A Whitney flip of an undirected graph.

Another way to obtain Corollary 2.3 is through Tutte's notion of peripheral cycles. In 1963, Tutte defines a peripheral cycle, of a connected graph G , as an induced cycle C where $G - V(C)$ is connected. Observe that for a connected graph G embedded in the sphere, every peripheral cycle of G must be the boundary of a face by the Jordan Curve Theorem.

Theorem 2.4 (Jordan Curve Theorem - Veblen [46]). *Any simple closed curve C in the plane divides the plane into exactly two arcwise connected components. Both of these regions have C as the boundary.*

Tutte then proves the following theorem which, equipped with the above observation, results in Corollary 2.3.

Theorem 2.5 (Tutte [45]). *Every edge in a 3-connected undirected graph is contained in at least two peripheral cycles.*

In this chapter we prove 2-regular digraph analogues of the results listed above.

2.1 The sphere obstruction

The first 2-regular digraph analogue that we prove is the classification of the immersion-minimal sphere obstructions. There turns out to be a single obstruction for the sphere, C_3^1 . For now, let C_3^1 be the 2-regular digraph depicted in Figure 2.2. (We will provide a general definition for the notation of C_3^1 in Chapter 4.)

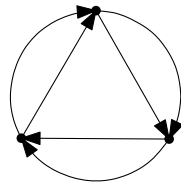


Figure 2.2: C_3^1 the unique planar obstruction.

Theorem 2.6 (Johnson). *A 2-regular digraph has an embedding in the sphere if and only if it has no C_3^1 as an immersion.*

Theorem 2.6 is attributed to Johnson [20], although its proof was never published. The first step in the proof is to show that C_3^1 has no embedding in the sphere. To do so we prove the following more general fact about minimal *feedback edge sets* (set of edges when removed from a digraph, leaves the digraph acyclic) of a 2-regular digraph.

Proposition 2.7. *If $D = (V, E)$ is a 2-regular digraph with a minimum feedback edge set A , then $\text{eg}(D) \geq 2 + |V| - 2|A|$.*

Proof. Consider a minimum Euler genus 2-cell embedding Ω of D . Since faces in Ω are directed closed walks and $D - A$ is acyclic, this implies that every face from Ω uses an edge from A . Therefore, $|\Omega| \leq 2|A|$ since every edge is in at most 2 faces. Since D is 2-regular, we have that $2|V| = |E|$, and applying Euler's formula we obtain

$$\text{eg}(\Omega) = 2 + |V| - |\Omega| \geq 2 + |V| - 2|A|.$$

□

Let D be a 2-regular digraph with $v \in V(D)$ and $t \in T(v)$. We define a *marked split* of t as a split of v with t (see the definition in Section 1.2.1) except the last step v . (where you suppress the children vertices v_1 and v_2) is omitted and instead an undirected edge or *chord* is placed between the children vertices v_1 and v_2 . Note that performing a marked split results in a *mixed* graph (one with both directed and undirected edges). This mixed graph representation is convenient for immersion, as it serves as an intermediate step for splitting a transition. Contract the chord v_1v_2 to undo the split at v , and delete the chord and suppress v_1 and v_2 to complete the split at v . We used mixed graphs quite heavily in the thesis, and their first appearance is here, in the proof of Johnson's Theorem.

Proof of Theorem 2.6. First, see that C_3^1 has no embedding in the sphere since a pair of parallel edges form a minimum feedback edge set. So, $\text{eg}(C_3^1) \geq 1$ by Proposition 2.7.

Next we show that a 2-regular digraph with no embedding in the sphere contains C_3^1 as an immersion. Let $D = (V, E)$ be a 2-regular digraph. We may assume that D is connected (else restrict the proof to a connected component of D). Choose an Euler tour W of D . Let $v \in V$ and consider the behavior of the tour W at v . The tour W must pass through v twice, say using the complementary transitions (e_1, f_1) and (e_2, f_2) . Perform the marked split of these transitions forming chord v_1v_2 so v_i lies on the directed path with edge sequence e_i, f_i . We may unambiguously refer to this chord as v as each chord of H corresponds to a vertex of D .

If we do this at every vertex of D , we obtain a mixed graph H where U is the set of chords, and E (the original edge set of D) is the set of directed edges of H . Note that (the

edge sequence) W forms a directed cycle in H . We shall view H drawn with W as a circle and all other edges as chords. We say that two chords *cross* if their endpoints interleave on W . That is, $u, v \in U$ with endpoints u_1, u_2 and v_1, v_2 respectively, *cross* if (u_1, v_1, u_2, v_2) occurs in this order on W .

Based on this, we construct an auxiliary graph K with vertex set U and an edge between $u, v \in U$ if u and v cross. We call this graph a *circle graph*. We now split into cases depending on whether K is bipartite. If K is a bipartite graph, then we may partition our chords into two sets $\{A, B\}$ so that no two chords in the same set cross. Based on this we can embed the underlying graph of H in the sphere by first embedding the undirected cycle W in the sphere and then embedding the chords from A and B in separate faces of $\mathbb{S}_0 - W$. Observe that contracting all chords from U yields our original 2-regular digraph D embedded in \mathbb{S}_0 .

The remaining possibility is that K is not bipartite, and in this case we may choose an induced odd cycle $C \subseteq K$. For every chord $v \in U$ that is not in $V(C)$, delete v and suppress the 1-regular vertices formed in H . Let H' be the mixed graph obtained by doing this for every chord not in $V(C)$, and let K' be its corresponding circle graph. Observe that the H' is immersed in D and that K' is precisely C .

If our cycle $C = K'$ has length > 3 then we will modify it to make it shorter by two. To do this, we choose two consecutive vertices u, v on C . Recall that u and v are chords coming from some splits of respective transitions t_u and t_v in $\mathcal{T}(D)$. Consider the mixed graph H'' formed from H' by contracting u and v , then splitting u and v with respective transitions t'_u and t'_v where $t'_u \neq t_u$ and $t'_v \neq t_v$ and t'_u is not a complement of t_u and t'_v is not a complement of t_v .

Note that H'' is immersed in D and that its associated circle graph is still a cycle but is now two vertices shorter. By repeating this process, we may obtain a mixed graph H''' immersed in D with the property that H''' has exactly 3 chords and its circle graph is a triangle. It follows that after contracting the chords of H''' , the 2-regular digraph C_3^1 is formed. \square

One can also view the proof of Theorem 2.6 through the lens of the pointless edge. Taking an Euler tour of a 2-regular digraph is equivalent to choosing an immersion of the pointless edge. Therefore, the proof essentially asks (in language we will introduce in Section 3.2.3) whether or not the chord model of a pointless edge immersion has an embedding in the sphere.

2.2 Tutte's peripheral cycles

Let Ω_1 and Ω_2 be 2-cell embeddings of a 2-regular digraph D in surfaces S_1 and S_2 . As in the case of undirected graphs, we say that Ω_1 and Ω_2 are *equivalent* if $\Omega_1 = \Omega_2$. If all 2-cell embeddings of D in a surface S are equivalent to Ω , then we say that Ω is the *unique* embedding of D in S . By Corollary 2.3, undirected graphs have unique embeddings in the

sphere if they are 3-connected. For 2-regular digraphs in the sphere, we will show that strong 2-edge-connectivity guarantees unique embeddings. A digraph D is *strongly k -edge-connected* if $D - F$ is strongly connected (there is a path to and from every pair of vertices in D) for any $F \subseteq E(D)$ where $|F| < k$.

Next, we motivate the definition of a peripheral cycle for 2-regular digraphs. Consider a 2-regular digraph H and an embedding ϕ of H in \mathbb{S}_0 . Let $C \subseteq H$ be a directed cycle, and let D_1 and D_2 be the components of $\mathbb{S}_0 - \phi(C)$ (since ϕ is an embedding in the sphere, both D_1 and D_2 are homeomorphic to a disk). Since the local rotation at each $v \in V(C)$ alternates in-out-in-out, it follows that the two edges in $E(H) \setminus E(C)$ incident to v are either both contained in the closure of D_1 or the closure of D_2 . This implies that H is disconnected unless one of the D_i was a face in ϕ . Thus, we define C to be *peripheral* if $H - E(C)$ is connected.

Theorem 2.8. *Every edge in a strongly 2-edge-connected Eulerian digraph is contained in at least two peripheral cycles.*

Proof. Let D be a strongly 2-edge-connected Eulerian digraph with $e = uv$ an edge of D . Our first goal will be to find one peripheral cycle through e . To do this, we choose a directed path P from v to u so as to lexicographically maximize the sizes of the components of $D' = D - (E(P) \cup \{e\})$. That is, we choose the path P so that the largest component of D' is as large as possible, and subject to this the second largest is as large as possible, and so on.

We claim that D' is connected. Suppose towards a contradiction that D' has components D_1, D_2, \dots, D_k with $k > 1$ where D_k is a smallest component. Let $P' \subseteq P$ be the shortest directed path that contains all vertices of $V(D_k) \cap V(P)$. Let x be the first vertex of P' and y the last (P' is directed). By construction, D_k contains both x and y . Furthermore, since D_k is Eulerian, we may choose a directed path P'' in D_k from x to y . If there is a component D_i with $i < k$ which contains a vertex in the interior of P' , then we get a contradiction to our choice of P , since we can reroute P along P'' instead of P' and get a new path which improves our lexicographic ordering. Therefore, all vertices in the interior of P' must also be in D_k . However, in this case $D_k \cup P'$ is a subgraph which is separated from the rest of the graph by just two edges, and we have a contradiction to the strong 2-edge-connectivity of D . It follows that $k = 1$, so the cycle $C = P, e$ is indeed peripheral.

Since C is peripheral, there exists a directed path from v to u with no edges in common to P . Choose Q to be such a path so that the unique component of $D - (E(Q) \cup \{e\})$ which contains P is as large as possible, and subject to that we lexicographically maximize the sizes of the remaining components. By the same argument as above, this choice will result in another peripheral cycle. \square

Similar to the undirected case, given an embedding of a 2-regular digraph in the sphere, every peripheral cycle must bound a face by the Jordan Curve Theorem 2.4. This fact together with Theorem 2.8 implies a corollary analogous to Corollary 2.3.

Corollary 2.9. *Any two embeddings of a strongly 2-edge-connected 2-regular digraph in the sphere are equivalent.*

2.3 Whitney flips

A *Whitney flip* of a 2-regular digraph H embedded in a surface S , is a Whitney flip of a closed disk $D \subseteq S$ in the underlying graph where we insist that D intersects H in points interior to edges of H . In this section, we prove the following analogue to Theorem 2.2.

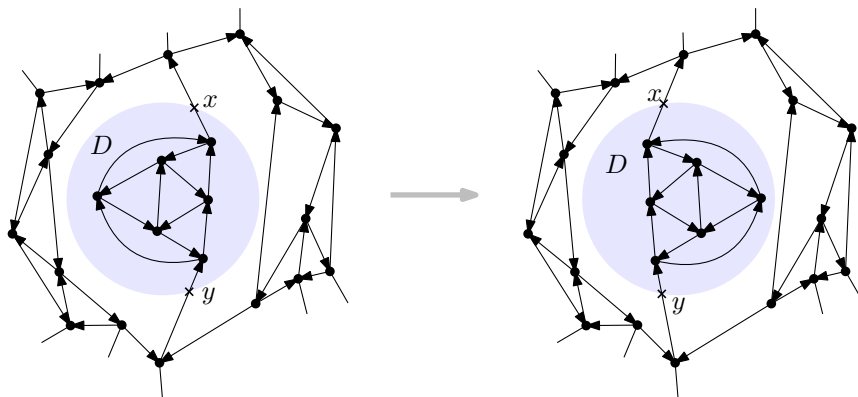


Figure 2.3: A Whitney flip of a 2-regular digraph.

Theorem 2.10. *If Ω and Ω' are embeddings of a connected 2-regular digraph D in the sphere, then by applying a sequence of Whitney flips, Ω can be transformed into an embedding equivalent to Ω' .*

Our proof of Theorem 2.10 utilizes the notion of an induced embedding for 2-regular digraphs. Let D be a connected 2-regular digraph with an embedding ϕ into a surface S . Let H be a 2-regular digraph immersed in D . By Proposition 1.1, H can be obtained from D by deleting edges of Eulerian subgraphs, splitting some set of vertices $X \subseteq V(D)$, suppressing 1-regular vertices, and deleting isolated vertices and pointless edges. Consider the embedded graph $\phi(D)$, and observe that the above operations can be performed in such a way as to yield an embedded graph $\phi(H)$ in S . That is, the underlying graph of H remains embedded in S , and the local rotation of curves with a common endpoint $x \in \phi(H)$ alternate in-out-in-out around x . We call ϕ the *induced embedding* of H . In general, if ϕ is a 2-cell embedding of D , ϕ may not be a 2-cell embedding of H . However, if ϕ is an embedding in the sphere, then the induced embedding is 2-cell as long as H is connected.

Given a 2-regular digraph $D = (V, E)$ and $X \subseteq V$ with $d^+(X) = d^-(X) = 1$, we let D_X denote the 2-regular digraph formed from D by deleting $V \setminus X$ and adding an edge from the

tail of $\delta^+(X)$ to the head of $\delta^-(X)$. Note that the assumption that D is Eulerian implies that D_X must be immersed in D .

Proof. We proceed by induction on $|V(D)|$. If D is strongly 2-edge-connected then Ω is equivalent to Ω' by Corollary 2.9, thus we may assume that there exists $X \subset V(D)$ such that $d_D^-(X) = d_D^+(X) = 1$, subject to this take X minimal. Consider D_X and let e be the edge from $E(D_X) \setminus E(D)$. Observe that D_X is strongly 2-edge-connected. Suppose towards a contradiction that it were not, then there would be $X' \subset V(D_X)$ such that $d_{D_X}^-(X') = d_{D_X}^+(X') = 1$. But since $d_{D_X}(X') \geq d_D(X')$ this would contradict the minimality of X . Therefore, all embeddings of D_X in the sphere are equivalent by Corollary 2.9.

Next, let $Y = V(D) \setminus X$ and consider D_Y with edge f from $E(D_Y) \setminus E(D)$. Let Ω_Y and Ω'_Y be induced embeddings of D_Y obtained from Ω and Ω' respectively. By induction Ω_Y can be transformed into an embedding equivalent to Ω'_Y by a sequence of Whitney flips. Moreover, these disks can be chosen such that f is not contained in any of them (if $D \subseteq \mathbb{S}_0$ is a disk to be flipped that contains f , then instead flip the closure of the disk $D' = \mathbb{S}_0 - D$).

Apply the same sequence of Whitney flips to Ω of D , note that the disks may need to be modified so they do not intersect the subgraph of D induced on X . After this sequence of flips, because all embeddings of D_X are equivalent in the sphere, the resulting embedding of D is either equivalent to Ω' or one last Whitney flip (on a disk whose boundary intersects D in exactly two points, the interior of $\delta_D^+(X)$ and $\delta_D^-(X)$) needs to be performed. \square

Chapter 3

Kuratowski Theorem for General Surfaces

Given a surface S and an undirected graph G , if G has no embedding in S but every proper minor of G has an embedding in S then we say that G is a *minor obstruction* for S . Let $\text{Forb}_M(S)$ denote the set of minor obstructions for S . Similarly, let $\text{Forb}_T(S)$ denote the set of *topological obstructions* for S . In the 1930's Erdős and König [21] conjectured that for each fixed surface S , $\text{Forb}_T(S)$ was finite. In 1989 Archdeacon and Huneke [5] proved the conjecture to be true for nonorientable surfaces, and a year later Robertson and Seymour [37] proved that both $\text{Forb}_M(S)$ and $\text{Forb}_T(S)$ are finite for all surfaces S .

Analogously, a 2-regular digraph that has no embedding in a surface S , but every proper immersion of it does, is called an *immersion obstruction* (or simply an *obstruction*). We let $\text{Forb}_I(S)$ denote the set of obstructions for S . In this chapter we prove a result analogous to Robertson and Seymour's.

Theorem 3.1. *$\text{Forb}_I(S)$ is finite for each surface S .*

Theorem 3.1 is the main result of this chapter, and naturally, we follow the road map laid out by Robertson and Seymour in the Graph Minors project. Their proof involves tree-decompositions (see Diestel [7] for details on tree-decompositions) and at a high level breaks into three parts: the first proves that graphs of bounded tree-width are well-quasi-ordered [36]; the second proves that graphs of large tree-width contain a large grid minor [35]; and the third proves that a graph with a large grid minor cannot be a minor minimal obstructions for a fixed surface [44].

Our proof of Theorem 3.1 follows Robertson and Seymour's road map but relies on more modern techniques. In 1991 [38] Robertson and Seymour introduce branch-decompositions for graphs, and in 2002 Geelen, Gerards, and Whittle [13] establish a general theory of branch-decompositions using symmetric submodular functions. The first theorem that we prove in this chapter is the following well-quasi-ordering result adapted from [13] and found in Section 3.1.

Theorem 3.2. *Let n be an integer. Then each infinite set of 2-regular digraphs with branch-width at most n has two members such that one is isomorphic to an immersion of the other.*

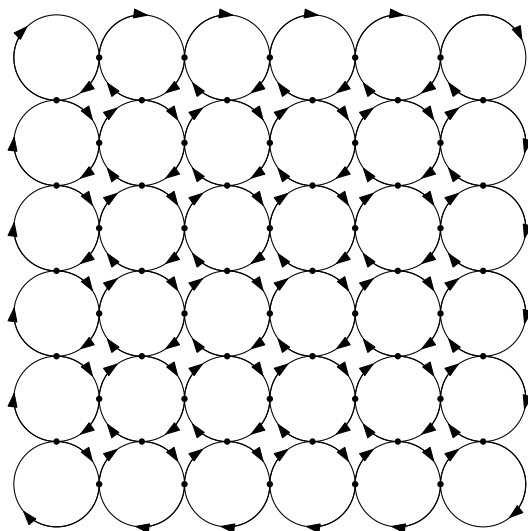


Figure 3.1: The medial grid of size 6.

Next in the road map, Johnson [20] proves an analogue of Robertson and Seymour’s grid theorem for 2-regular digraphs under immersion. We state his theorem below and include it without proof. (We will formally define the medial grid in Section 3.3.)

Theorem 3.3 (Johnson [20]). *Let D be a 2-regular digraph. For any $k \in \mathbb{N}$ there exists a $n \in \mathbb{N}$ such that if the branch-width of D is at least n , then D immerses a medial grid of size k .*

To complete the proof of Theorem 3.1 we follow an elegant argument laid out by Thomassen [44] and prove its analogue for 2-regular digraphs under immersion.

Theorem 3.4. *For every surface S , there exists $k \in \mathbb{N}$ such that no $D \in \text{Forb}_I(S)$ immerses a medial grid of size k .*

3.1 Well-quasi-order: bounded branch-width

The main purpose of this section is to prove Theorem 3.2, which is a well-quasi-ordering result for 2-regular digraphs with bounded branch-width. Subsection 3.1.1 is used to introduce the technical tools needed, and the proof appears in Subsection 3.1.2. We follow the notation, terminology, and proof technique from [13].

3.1.1 Branch-decompositions

A *quasi-ordering* (X, \preceq) is a set X and a relation \preceq that is reflexive and transitive. We say that (X, \preceq) is *well-quasi-ordered* if for every infinite sequence (x_1, x_2, \dots) of elements

from X , there exist $i, j \in \mathbb{N}$ such that $i < j$ and $x_i \preceq x_j$. There are two ways that this can fail to exist: either an infinite *antichain* exists (a set of pairwise incomparable elements) or there is an infinite *strictly descending chain*, a sequence (x_1, x_2, \dots) such that $x_{i+1} \preceq x_i$ and $x_i \not\preceq x_{i+1}$ for all $i \in \mathbb{N}$. Observe that 2-regular digraphs with the relation “is isomorphic to an immersion of” forms a quasi-order with no infinite strictly descending chains.

A function λ defined on the collection of subsets of a finite ground set S is called *submodular* if $\lambda(A) + \lambda(B) \geq \lambda(A \cap B) + \lambda(A \cup B)$ for all $A, B \subseteq S$. And it is called *symmetric* if $\lambda(A) = \lambda(S \setminus A)$ for all $A \subseteq S$.

A branch-decomposition is defined abstractly in [13] for a ground set S and a symmetric submodular function λ . However, not needing this level of generality, we will define it on the vertex set of a 2-regular digraph with the symmetric submodular edge-cut function defined in Section 1.2.

Let $D = (V, E)$ be a 2-regular digraph. A *branch-decomposition* of D is a cubic tree (a tree where every vertex either has degree 1 or 3) T together with an injective mapping from V to the leaves of T . A *set displayed by* a subtree of T is the set of elements of V in that subtree. A set of elements $X \subseteq V$ is *displayed by* an edge $e \in E(T)$ if X is displayed by one of the two components of $T - e$. Let $X \subseteq V$ be a set of elements displayed by an edge $e \in E(T)$, then e corresponds to an edge-cut of D that separates X and $V \setminus X$. So, we define the *width* of e , denoted $d(e)$, as the size of the corresponding edge-cut in D (see Section 1.2), $d(X) = |\delta(X)|$. We further define the *width* of T as the maximum width of all edges in $E(T)$, and the *branch-width* of D as the minimum width of a branch-decomposition of D .

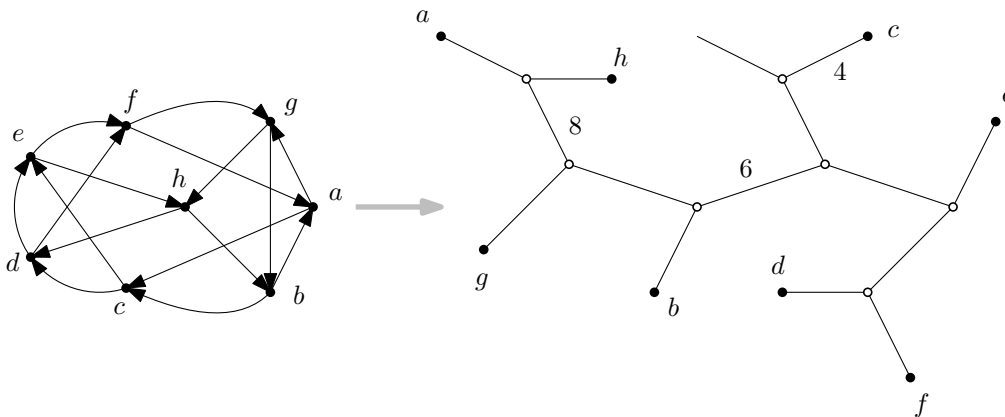


Figure 3.2: A branch decomposition of width 8. Switching the leaves a and b in the branch decomposition yields a decomposition with width 6.

Note that V may be mapped to a proper subset of the leaves of T . In such a case, we call the leaves with no vertex assigned to them *unlabeled*. If T has unlabeled leaves, we can transform T into a branch-decomposition with no unlabeled leaves by deleting the unlabeled leaves and suppressing degree 2 vertices. Conversely, if T has no unlabeled leaves, then we may subdivide an edge of T and add a pendant leaf.

Let T be a branch-decomposition of D . Given distinct edges $f, g \in E(T)$, let $F \subseteq V$ be the set displayed by the component of $T - f$ not containing g , and similarly, let $G \subseteq V$ be the set displayed by the component of $T - g$ not containing f . If P is the shortest path between f and g in T , notice that each edge on P displays a partition of V with F in one part and G in the other. Thus, for every edge $e \in E(P)$ we have that $d(F, G) \leq d(e)$ (recall $d(F, G)$ is the minimum size of an edge-cut separating F from G). We say that f and g are *linked* if $d(F, G)$ appears as the width of an edge in P . We say that the branch-decomposition T is *linked* if all distinct edge pairs of $E(T)$ are linked. The following theorem (stated for 2-regular digraphs) is an analogue of a result from Thomas [43].

Theorem 3.5 (Geelen, Gerards, and Whittle [13]). *Let n be an integer. A 2-regular digraph with branch-width n has a linked branch-decomposition of width n .*

A *rooted digraph*, denoted (D, r) , is a digraph D with a distinguished *root* vertex $r \in V(D)$. We call $\delta(r)$ the *root edges* of (D, r) .

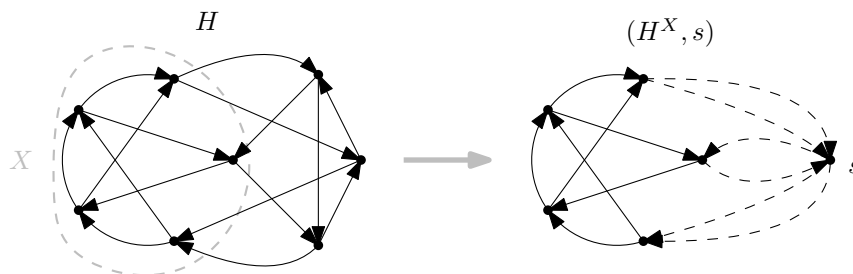


Figure 3.3: A digraph, a subset of vertices, and an associated rooted digraph.

Let $H = (V, E)$ be a digraph with $X \subseteq V$. Form a new digraph by identifying $V \setminus X$ to a single vertex s and deleting any loops formed at s . We denote the graph formed by this operation H^X . This forms a natural rooted digraph (H^X, s) .

Given two rooted digraphs (H, s) and (D, r) , we say that (H, s) is *immersed* in (D, r) if H is immersed in D with (ϕ, ϕ') , where additionally we have that $\phi(s) = r$ and for an edge $e \in \delta_H(s)$, there exists $f \in E(D)$ such that $\phi'(e) \cap \delta_D(r) = \{f\}$. We abbreviate this and say that e *maps to* f under ϕ' .

Lemma 3.6. *Let $D = (V, E)$ be a 2-regular digraph with $X_1 \subseteq X_2 \subseteq V$. If $d(X_1) = d(X_1, V \setminus X_2) = d(X_2)$, then (D^{X_1}, r_1) is immersed in (D^{X_2}, r_2) .*

Proof. Let $H = D^{X_1} - r_1$. Because H is a subgraph of D^{X_2} , we have that H is immersed in D^{X_2} . Now, we extend this immersion. Consider $H' = D^{X_2} - E(H)$. By Menger's theorem, there are $d(X_1, V \setminus X_2)$ edge-disjoint trails in H' between r_2 and X_1 . Mapping r_1 to r_2 and mapping the root edges of r_1 to these trails in H' gives the rooted immersion of (D^{X_1}, r_1) in (D^{X_2}, r_2) . \square

A *rooted tree* is a finite directed tree with exactly one vertex with in-degree equal to 0 called the *root*. Note that in this case, all other vertices must have in-degree equal to 1 (since $|E| = |V| - 1$). The *leaves* are the vertices with out-degree 0. The edges incident to the root are called *root edges*, and similarly, an edge incident to a leaf is called a *leaf edge*. A *rooted forest* is a countable collection of disjoint rooted trees.

An *n-edge labeling* of a graph G as a map from $E(G)$ to $\{0, 1, \dots, n\}$. Let F be a rooted forest. If d is an n -edge labeling of F and $e, f \in E(F)$, then we say that e is *d-linked* to f if F has a directed path P starting with e and ending with f such that $d(g) \geq d(e) = d(f)$ for all $g \in E(P)$. Observe that being d -linked is not necessarily a symmetric property.

A *binary forest* (F, l, r) is a rooted forest together with functions l and r with the following properties:

- Every tree T in F is a cubic tree whose root has out-degree 1.
- For every nonleaf edge $e = uv$ of T , v has exactly two out-edges, a left edge $l(e)$ and a right edge $r(e)$.

We state without proof, the following specialization of Robertson and Seymour’s “lemma on trees” as it appears in [13]. Lemma 3.7 is used exclusively in the proof of Theorem 3.2. In this proof, the binary forest (F, l, r) consists of finite binary trees, where each tree is a linked branch-decomposition of a 2-regular digraph, and the quasi-order on the edges of the forest is the immersion quasi-order on the rooted digraphs displayed by the edges of the trees.

Lemma 3.7 (Robertson, Seymour [36]). *Let (F, l, r) be an infinite binary forest with an n -edge labeling d . Let $(E(F), \preceq)$ be a quasi-order with no infinite strictly descending chains and where f d -linked to e implies that $e \preceq f$. If the leaf edges of F are well-quasi-ordered by \preceq but the root edges are not, then F contains an infinite sequence of nonleaf edges (e_0, e_1, \dots) such that:*

- i) $\{e_0, e_1, \dots\}$ is an antichain in $(E(F), \preceq)$;*
- ii) $l(e_0) \preceq l(e_1) \preceq \dots \preceq l(e_{i-1}) \preceq l(e_i) \preceq l(e_{i+1}) \preceq \dots$; and*
- iii) $r(e_0) \preceq r(e_1) \preceq \dots \preceq r(e_{i-1}) \preceq r(e_i) \preceq r(e_{i+1}) \preceq \dots$.*

3.1.2 Proof of Theorem 3.2

With the necessary tools in place we now prove Theorem 3.2, that each infinite set of 2-regular digraphs with bounded branch-width has two members such that one is isomorphic to an immersion of the other.

Proof. Let \mathcal{D} be an infinite set of 2-regular digraphs with branch-width at most n , and assume towards a contradiction that it is not well-quasi-ordered by immersion. For each

$D \in \mathcal{D}$, by Theorem 3.5 let T_D be a linked branch-decomposition with width at most n . For each T_D fix an unlabeled leaf as the root r_D (if no unlabeled leaf exists, create one by subdividing an edge of $T(D)$ and attaching a pendant edge) and orient T_D such that it is a rooted cubic tree. For an edge $e \in E(T_D)$, let V^e denote the set of vertices of D displayed by the component of $T_D - e$ not containing r_D . Let (D^e, s^e) be a rooted digraph where D^e is shorthand for D^{V^e} .

Taking (F, l, r) as the rooted cubic forest with trees T_D for each $D \in \mathcal{D}$, define the quasi-order $(E(F), \preceq)$ as follows: For $e, f \in E(F)$, we define $e \preceq f$ if (D^e, s^e) is immersed in (D^f, s^f) .

We now check that the assumptions required for Lemma 3.7 hold. Observe that if $e, f \in E(F)$ and f is d -linked to e , then (D^e, s^e) is an immersion of (D^f, s^f) by Lemma 3.6, which implies that $e \preceq f$. Next, observe that $(E(F), \preceq)$ has no strictly descending chains since only finitely many proper immersions of a fixed graph exist. Lastly, the leaf edges are well-quasi-ordered as they all induce a rooted isolated vertex as a graph and the root edges are not well-quasi-ordered since \mathcal{D} is not by assumption.

Thus, applying Lemma 3.7 there exists an infinite sequence of nonleaf edges $A = (e_0, e_1, \dots)$ such that conditions i), ii), and iii) of Lemma 3.7 hold. Our aim is to work towards a contradiction of i). That is, we show how to find $i < j$ such that (D^{e_i}, s^{e_i}) is an immersion of (D^{e_j}, s^{e_j}) .

Observe that ii) and iii) give that, for all $i < j$, $(D^{l(e_i)}, s^{l(e_i)})$ is immersed in $(D^{l(e_j)}, s^{l(e_j)})$ and $(D^{r(e_i)}, s^{r(e_i)})$ is immersed in $(D^{r(e_j)}, s^{r(e_j)})$. We describe these immersions in more detail below.

Since T_D is a linked branch-decomposition of width at most n we have that $d(s^{l(e_i)}) \leq n$ and $d(s^{r(e_i)}) \leq n$ for all i . Therefore, taking an infinite subsequence of A , $A' = (e'_0, e'_1, \dots)$ we may assume that $d(s^{l(e'_i)}) = d(s^{l(e'_{i+1})})$ and $d(s^{r(e'_i)}) = d(s^{r(e'_{i+1})})$ for all $i \geq 0$. Assign to each edge of $\delta(s^{l(e'_0)})$ a distinct *left color* from $\{1, 2, \dots, n\}$ and to each edge of $\delta(s^{r(e'_0)})$ a distinct *right color* from $\{1, 2, \dots, n\}$. Extend the coloring for all $i \geq 0$:

- $(D^{l(e'_i)}, s^{l(e'_i)})$ is immersed in $(D^{l(e'_{i+1})}, s^{l(e'_{i+1})})$ where edges in $\delta(s^{l(e'_i)})$ are mapped to edges in $\delta(s^{l(e'_{i+1})})$ with the same left color, and
- $(D^{r(e'_i)}, s^{r(e'_i)})$ is immersed in $(D^{r(e'_{i+1})}, s^{r(e'_{i+1})})$ where edges in $\delta(s^{r(e'_i)})$ are mapped to edges in $\delta(s^{r(e'_{i+1})})$ with the same right color.

Observe that edges in $\delta(s^{l(e'_i)}) \cap \delta(s^{r(e'_i)})$ get colored with both a right and left color. Moreover, since the subsequence A' is infinite and there are finitely many colors $\{1, 2, \dots, n\}$ there must exist some $i < j$ such that

- edges in $\delta(s^{l(e'_i)}) \cap \delta(s^{r(e'_i)})$ and $\delta(s^{l(e'_j)}) \cap \delta(s^{r(e'_j)})$ have the same left and right colors, and
- the set of colors in $\delta(s^{e'_i})$ is the same as the set of colors in $\delta(s^{e'_j})$.

Obtain $(D^{e'_i}, s^{e'_i})$ from $(D^{l(e'_i)}, s^{l(e'_i)})$ and $(D^{r(e'_i)}, s^{r(e'_i)})$ by identifying edges in $\delta(s^{l(e'_i)}) \cap \delta(s^{r(e'_i)})$. A similar identification yields $(D^{e'_j}, s^{e'_j})$. However, this implies that $(D^{e'_i}, s^{e'_i})$ is immersed in $(D^{e'_j}, s^{e'_j})$ which contradicts the fact that A was an antichain. \square

3.2 Some helpful tools towards excluding a grid

The main purpose of this section is to establish the machinery necessary to prove Theorem 3.4. We introduce the induced 2-cell embedding and prove some genus decomposition results. Then we introduce the concept of an immersion model.

3.2.1 Induced embedding

As mentioned in Section 2.3, if D is a 2-regular digraph embedded in a surface S , and H is a 2-regular digraph immersed in D , then the immersion operations from Proposition 1.1 can be performed in such a way that H is also embedded in S . However, it is possible for D to be 2-cell embedded in S and H to not be. Therefore, simply as a means of bookkeeping, we introduce a notion of an induced embedding that preserves the property of an embedding being 2-cell.

Let D be a connected 2-regular digraph with a 2-cell embedding Ω . We define the *induced 2-cell embedding* to be the 2-cell embedding obtained from splitting vertices of D . Note that defining the induced embedding only for a vertex split is sufficient for the class of 2-regular digraphs by Proposition 1.2.

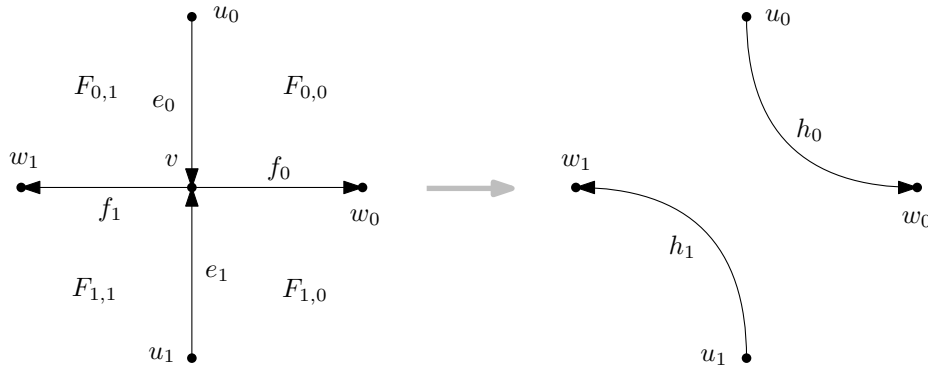


Figure 3.4: Inducing an embedding from a vertex split.

Proposition 3.8. *Let D be a connected 2-regular digraph. Let Ω be a 2-cell embedding of D . Let $t \in \mathcal{T}(D)$ and let Ω' be the induced 2-cell embedding of D/t , then*

$$\text{eg}(\Omega') \in \{\text{eg}(\Omega), \text{eg}(\Omega) - 2\}.$$

Proof. Let $v \in V(D)$ be a loop-free vertex with $E^-(v) = \{e_0, e_1\}$ and $E^+(v) = \{f_0, f_1\}$. Recall that each transition of $T(v)$ is contained in exactly one face in Ω . That is, for

$i, j \in \{0, 1\}$ let $t_{i,j} = (e_i, f_j)$ and consider the four, not necessarily distinct, faces $F_{i,j}$ that contain each transition. For notational clarity, let $t = t_{0,0}$.

Consider D/t , and let h_0 be the edge from the tail of t to the head of t and h_1 be the edge from the tail to the head of $t_{1,1}$ (see Figure 3.4). Modify the edge sequence of each $F_{i,j}$ by replacing every occurrence of (e_0, f_0) with h_0 and every occurrence of (e_1, f_1) (the complementary transition) with h_1 to form $F'_{i,j}$. Note that $F'_{i,j}$ is not necessarily a face, but rather a formal sequence of edges from $E(D) \cup \{h_0, h_1\}$. We break into the following two cases:

Case 1: $F_{0,1} \neq F_{1,0}$

There exist walks W and U such that $F'_{0,1} = W, t_{0,1}$ and $F'_{1,0} = U, t_{1,0}$. In the induced embedding $F'_{0,1}$ and $F'_{1,0}$ *merge* to form the new face (W, h_0, U, h_1) .

Case 2: $F_{0,1} = F_{1,0}$

There exist walks W and U such that $F'_{0,1} = F'_{1,0} = W, t_{0,1}, U, t_{1,0}$. In the induced embedding, $F'_{0,1}$ will *divide* to form the new faces (W, h_0) and (U, h_1) .

In both cases, Ω has been modified to a collection of directed closed walks in D/t where each transition in $\mathcal{T}(D/t)$ is contained in exactly one closed walk; i.e. a 2-cell embedding Ω' of D/t . Note that $|V(D/t)| = |V(D)| - 1$, $|E(D/t)| = |E(D)| - 2$, and either $|\Omega'| = |\Omega| - 1$ (Case 1) or $|\Omega'| = |\Omega| + 1$ (Case 2) yields the result via Euler's formula.

If there is a loop edge incident to v , then a similar procedure will produce the induced embedding of D/t .

□

3.2.2 Genus decompositions

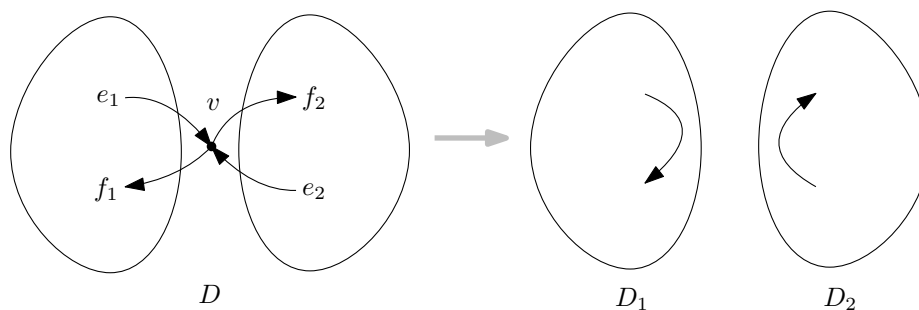


Figure 3.5: The configuration from Proposition 3.9.

Proposition 3.9. *Let D be a connected 2-regular digraph with a 2-cell embedding Ω . If D has a cut-vertex v with $t \in T(v)$ such that D/t disconnects D into components D_1 and D_2 , then letting Ω_1 and Ω_2 be the induced 2-cell embeddings of D_1 and D_2 respectively,*

$$\text{eg}(\Omega) = \text{eg}(\Omega_1) + \text{eg}(\Omega_2).$$

Proof. Let $E^-(v) = \{e_1, e_2\}$, $E^+(v) = \{f_1, f_2\}$, and $t_{i,j} = (e_i, f_j)$ for all $i, j \in \{1, 2\}$ such that splitting $t_{1,1}$ disconnects D and the endpoints of $t_{1,1}$ are in D_1 . Observe that since v is a cut-vertex of D , there exists a face $F \in \Omega$ such that both the transitions $t_{1,2}$ and $t_{2,1}$ are contained in F . Notice that F splits into two faces $F_1 \in \Omega_1$ and $F_2 \in \Omega_2$ in the induced embedding of $D/t_{1,1}$.

Recall that the number of edges is twice the number of vertices in a 2-regular digraph. Thus, if we let $n = |V(D)|$, $n_i = |V(D_i)|$, $q = |\Omega|$, and $q_i = |\Omega_i|$ we obtain via Euler genus

$$\begin{aligned} \text{eg}(\Omega) &= 2 + n - q \\ &= 2 + (n_1 + n_2 + 1) - (q_1 + q_2 - 1) \\ &= (2 + n_1 - q_1) + (2 + n_2 - q_2) \\ &= \text{eg}(\Omega_1) + \text{eg}(\Omega_2). \end{aligned}$$

□

It's important to note that the above result holds for every embedding of D ; whereas Proposition 3.10 is a result that pertains to the Euler genus of a 2-regular digraph.

For a digraph D , we say that a pair of subdigraphs (H_1, H_2) is a k -separation if $E(H_1) \cap E(H_2) = \emptyset$, $E(H_1) \cup E(H_2) = E(D)$, and $|V(H_1) \cap V(H_2)| = k$. Given a k -separation, (H_1, H_2) , of a 2-regular digraph D we say that the k -separation is *balanced* if every $v \in V(H_1) \cap V(H_2)$ satisfies $|E^+(v) \cap E(H_i)| = |E^-(v) \cap E(H_i)| = 1$ for $i \in \{1, 2\}$.

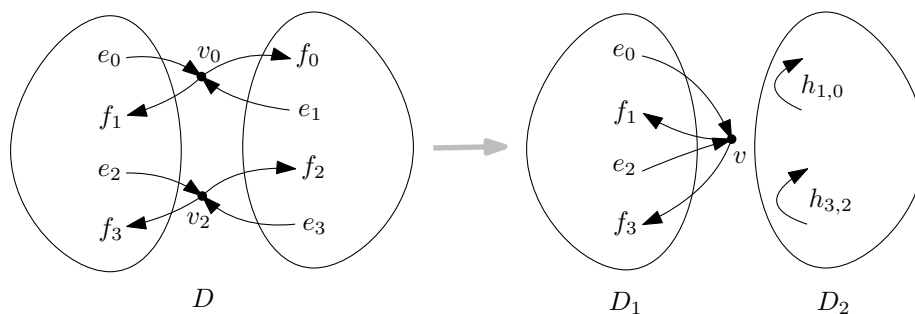


Figure 3.6: The balanced 2-separation from Proposition 3.10.

Proposition 3.10. *Let D be a connected 2-regular digraph with a balanced 2-separation (H_1, H_2) . Let D_1 be obtained from H_1 by identifying the vertices in $V(H_1) \cap V(H_2)$, and let D_2 be obtained from H_2 by suppressing the vertices in $V(H_1) \cap V(H_2)$. Then*

$$\text{eg}(D) \geq \text{eg}(D_1) + \text{eg}(D_2).$$

Proof. Let $V(H_1) \cap V(H_2) = \{v_0, v_2\}$ and for $k \in \{0, 2\}$ let $E^-(v_k) = \{e_k, e_{k+1}\}$ and $E^+(v_k) = \{f_k, f_{k+1}\}$. For $i, j \in \{0, 1, 2, 3\}$, let $t_i = (e_i, f_i)$, and for valid $i \neq j$ pairs let

$t_{i,j} = (e_i, f_j)$ where the endpoints of $t_{0,1}$ and $t_{2,3}$ lie in H_1 . Let $T = \{t_0, t_1, t_2, t_3\}$ and let Ω be a minimal genus 2-cell embedding of D . Consider the faces

$$\mathcal{F} = \{F \in \Omega : \exists t \in T, t \subseteq F\}.$$

Note that $|\mathcal{F}| \leq 2$ since every face in \mathcal{F} has at least two t_i contained in it.

Let F be the face containing t_0 and observe that $F \in \mathcal{F}$ and F must also contain either t_1 or t_3 . We say that H_2 *switches (repeats)* if in F after the transition t_0 , then next transition from T is t_3 (t_1). Note that if H_2 switches (repeats), then in the face with transition t_2 , after t_2 the next transition from T is t_1 (t_3). Similarly, if F' is the face containing t_1 we say that H_1 *switches (repeats)* if in F' after the transition t_1 the next transition from T is t_2 (t_0).

Let $v \in V(D_1)$ be the vertex after identifying v_0 and v_2 . Since $\{e_0, f_1, e_2, f_3\} \subseteq E(D_1)$ are the same edges as from $E(D)$, we abuse notation and label the transitions $T(v) \subseteq \mathcal{T}(D_1)$ with the same notation, $t_{i,j} = (e_i, f_j)$, as from $\mathcal{T}(D)$. Let $h_{1,0}$ and $h_{3,2}$ be the edges in $E(D_2)$ obtained after suppressing v_0 and v_2 respectively. We examine the following four cases for the faces in \mathcal{F} , and define \mathcal{F}_1 and \mathcal{F}_2 to be the set of closed walks, in D_1 and D_2 respectively, that are ‘‘inherited’’ from \mathcal{F} (in the following, $W_{j,i}$ denotes a walk contained in either D_1 or D_2 from the head of f_j to the tail of e_i , or from the head of $h_{i,j}$ to the tail of $h_{i,j}$, with possibly empty edge sequence):

Case 1: Repeat in H_1 and H_2 .

$$\begin{aligned} \mathcal{F} &= \{(W_{1,0}, t_0, W_{0,1}, t_1), (W_{3,2}, t_2, W_{2,3}, t_3)\} \\ \mathcal{F}_1 &= \{(W_{1,0}, t_0, W_{3,2}, t_2, t_1)\} & \mathcal{F}_2 &= \{(W_{0,1}, h_{1,0}), (W_{2,3}, h_{3,2})\} \end{aligned}$$

Case 2: Repeat in H_1 and switch in H_2

$$\begin{aligned} \mathcal{F} &= \{(W_{1,0}, t_0, W_{0,3}, t_3, W_{3,2}, t_2, W_{2,1}, t_1)\} \\ \mathcal{F}_1 &= \{(W_{1,0}, t_0, W_{3,2}, t_2, t_1)\} & \mathcal{F}_2 &= \{(W_{0,3}, h_{3,2}, W_{2,1}, h_{1,0})\} \end{aligned}$$

Case 3: Switch in H_1 and repeat in H_2

$$\begin{aligned} \mathcal{F} &= \{(W_{1,2}, t_2, W_{2,3}, t_3, W_{3,0}, t_0, W_{0,1}, t_1)\} \\ \mathcal{F}_1 &= \{(W_{3,0}, t_0, W_{1,2}, t_2, t_1)\} & \mathcal{F}_2 &= \{(W_{0,1}, h_{1,0}), (W_{2,3}, h_{3,2})\} \end{aligned}$$

Case 4: Switch in H_1 and H_2

$$\begin{aligned} \mathcal{F} &= \{(W_{1,2}, t_2, W_{2,1}, t_1), (W_{3,0}, t_0, W_{0,3}, t_3)\} \\ \mathcal{F}_1 &= \{(W_{3,0}, t_0, W_{1,2}, t_2, t_1)\} & \mathcal{F}_2 &= \{(W_{0,3}, h_{3,2}, W_{2,1}, h_{1,0})\} \end{aligned}$$

For $i \in \{1, 2\}$, let $\Omega|_{D_i}$ denote the facial walks from $\Omega \setminus \mathcal{F}$ contained in D_i . Observe that in all four cases above, $\Omega_i = \mathcal{F}_i \cup \Omega|_{D_i}$ forms a 2-cell embedding of D_i . Let $n = |V(D)|$, $q = |\Omega|$, $n_i = |V(D_i)|$, $q_i = |\Omega_i|$, and notice that in all cases we have $q = q_1 + q_2 - c$ where $c \in \{1, 3\}$. We obtain via Euler genus

$$\begin{aligned}
 \text{eg}(D) &= \text{eg}(\Omega) \\
 &= 2 + n - q \\
 &= 2 + (n_1 + n_2 + 1) - (q_1 + q_2 - c) \\
 &= (2 + n_1 - q_1) + (1 + c + n_2 - q_2) \\
 &\geq \text{eg}(\Omega_1) + \text{eg}(\Omega_2) \\
 &\geq \text{eg}(D_1) + \text{eg}(D_2).
 \end{aligned}$$

□

3.2.3 Immersion model

We now define a helpful tool that we call an immersion model. A model of H in D is a description of an immersion of H in D that encodes the information needed to move between the two graphs.

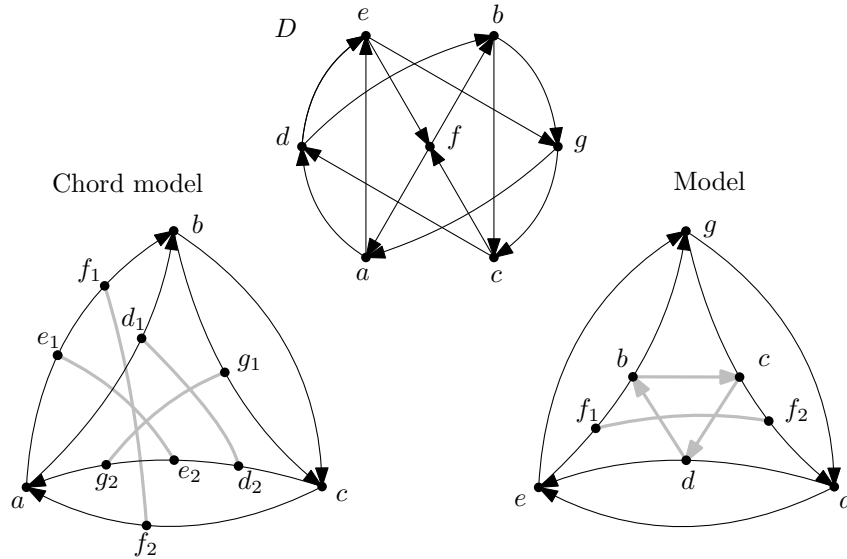


Figure 3.7: Two models of C_3^1 immersed in D .

If $D = (V, E)$ is a 2-regular digraph and H is an Eulerian digraph immersed in D , then by Proposition 1.1, H can be obtained from D by deleting edges, $F \subseteq E(D)$, of an Eulerian subgraph, splitting vertices $X \subseteq V(D)$ and deleting isolated vertices and pointless edges. If we omit the deletion of F and perform marked splits of transitions of X , and let V_X denote the set of children vertices from the marked splits, then we obtain a mixed

digraph $D^* = (V^*, E^*, U)$ where $V^* = (V \setminus X) \cup V_X$, $E^* = E$, and U is the set of chords. Form a second digraph $H^* = (V', E')$ from D^* by deleting U , deleting F , and deleting isolated vertices and pointless edges. Note that, H^* is a subdivision of H where $E' \subseteq E^*$ and $V' \subseteq V^*$. We say that (D^*, H^*) is a *model* of H immersed in D ; however, for readability, we often drop D^* and instead refer to just H^* as the model.

Next we define an H^* -bridge, which is another analogue of a concept from the world of undirected graphs (see [26]). An H^* -bridge is a nontrivial component of $D^* - (E \setminus F)$. Note that a chord may be (and often is) an H^* -bridge. When the model is clear we will simply refer to an H^* -bridge as a bridge.

For an H^* -bridge, X , if $v \in V(H^*) \cap V(X)$ we say that v is an *attachment* vertex for X . Given a non-chord bridge X and an attachment vertex $v \in V(X)$, if you suppress v in X we say that the newly formed edge in X is *attached* to D . For $u \in V(H^*)$, if u is 2-regular then we call u is a *branch endpoint* or a *branch vertex*, and the directed walks in H^* between branch endpoints are called *branches*. Observe that H can be obtained from H^* by suppressing all attachment vertices, and that D can be obtained from D^* by contracting all chords.

Definition 3.11 (Chord model). If H^* is a model of H immersed in D where every bridge is a chord, then we say that H^* is a *chord model*.

If H and D are connected and H is also a 2-regular digraph, then by Proposition 1.2 H can be obtained from D by a sequence of vertex splits. Such a model is necessarily a chord model. We use the chord model quite heavily in Section 3.3.

Consider a subgraph $J \subseteq H^*$. We use \mathcal{B}_J to denote the set of H^* -bridges with all attachments on J . Let $C \subseteq H^*$ be a closed walk and let $X, Y \in \mathcal{B}_C$. We say that X and Y *cross* (relative to C) if you can find an alternation of attachment vertices when walking along C ; i.e. there exist attachment vertices x, x' of X and y, y' of Y occurring in the order (x, y, x', y') on C .

Definition 3.12 (Mixed graph embedding). Let D' be a mixed graph where for every vertex $v \in V(D')$, v is either 2-regular, 1-regular, or is incident to 1 in-edge, 1 out-edge, and 1 undirected edge. We say that D' *embeds* in a surface S if the 2-regular digraph obtained by contracting all undirected edges and suppressing all 1-regular vertices embeds in S .

Given a model (D^*, H^*) , one can think of the D' from Definition 3.12 as a subgraph of D^* after “removing” some H^* -bridges from D^* , but leaving the attachment vertices of these bridges. This leads to a natural combinatorial description of embedding D' itself. If we treat the undirected edges as bidirectional edges, then the notion of a transition (as defined in Section 1.2) can be extended to include edges of this type. Therefore, we call Ω a 2-cell embedding of D' if Ω is a collection of closed walks where every transition of D' appears exactly once and each edge of D' appears exactly twice.

3.3 Excluding a grid: unbounded branch-width

Equipped with the necessary tools from Section 3.2, we now set out to prove Theorem 3.4, which states that an obstruction for a surface cannot immerse a large medial grid. We follow the notation, terminology, and proof technique from [44].

A *medial grid* M of size n is a directed medial graph of $P_n \square P_n$ embedded in the plane; where, as a convention, the clockwise cycles of M correspond to the vertices of $P_n \square P_n$. Note that M is unique if $n \geq 3$, since $P_n \square P_n$ is uniquely embedded in the plane by Corollary 2.3. Moreover, M has a unique embedding in the plane by Corollary 2.9. Define S_n as a finite subgraph of the square tiling of the Euclidean plane. Let S_1 be a cycle of length 4. For $n \geq 2$ we define S_n as the union of S_{n-1} and the 4-cycles in the square tiling that intersect S_{n-1} . For all $n \geq 1$, let J_n denote the medial grid of S_n . Note that each J_n is a medial grid of size $2n$. Consider an embedding Ω of J_n in the plane. The *outer cycle* is the cycle $C \in \Omega$ that is the boundary of the outer face in \mathbb{R}^2 . By the Jordan Curve Theorem 2.4, $\mathbb{R}^2 - C$ divides the plane into the outer face and the *interior of C* .

Let J_k^* be a model of a medial grid J_k immersed in a connected 2-regular digraph D , and let C be the outer cycle of J_k^* . We call the intersection of J_k^* and the interior of C , the *interior of J_k^** . We say that J_k^* is *good* in D if every J_k^* -bridge with an attachment in the interior of J_k^* has all attachments in a facial cycle of J_k^* embedded in the plane, and the union of J_k^* and all such bridges is planar.

For the proof of Proposition 3.13 we fix some notational conventions: If a and b are attachment vertices in J_k^* , then we will always use the symbol $R_{a,b}$ to denote a chord with attachment vertices a and b . Similarly, we will always use the symbol $P_{a,b}$ to denote a pair of edge-disjoint paths to and from a and b in J_k^* .

Proposition 3.13. *Let D be a connected 2-regular digraph of Euler genus g that immerses the medial grid J_m . If $k \in \mathbb{N}$ and $m > 100k\sqrt{g}$, then J_m immerses J_k with a good model in D .*

Proof. If $k = 1$, then all J_1 models are vacuously good in D since the interior of J_1 is empty. Therefore, we may assume that $k \geq 2$. Since J_m is 2-regular and D connected, let J_m^* be a chord model of the immersion by the paragraph following Definition 3.11. Since $m > 100k\sqrt{g}$ we may choose $Q_1, Q_2, \dots, Q_{2g+2}$ disjoint subgraphs of J_m^* such that after suppressing attachment vertices, each Q_j is isomorphic to J_k , and where additionally for $i \neq j$, $1 \leq i, j \leq 2g+2$ the following is true: if x_i is on the outer cycle of Q_i and x_j is on the outer cycle of Q_j and x_i has an in-neighbor and out-neighbor in $J_m^* - Q_i$ and x_j has an in-neighbor and out-neighbor in $J_m^* - Q_j$, then there exists two edge disjoint paths in J_m^* from x_i to x_j and back having only x_i, x_j in common with $Q_1 \cup Q_2 \cup \dots \cup Q_{2g+2}$.

We'll show that at least one of these Q_j 's are good in D . Suppose towards a contradiction that none of the Q_j are good in D . We construct a sequence M_1, M_2, \dots, M_{g+2} of connected immersions of J_m adhering to the following conditions:

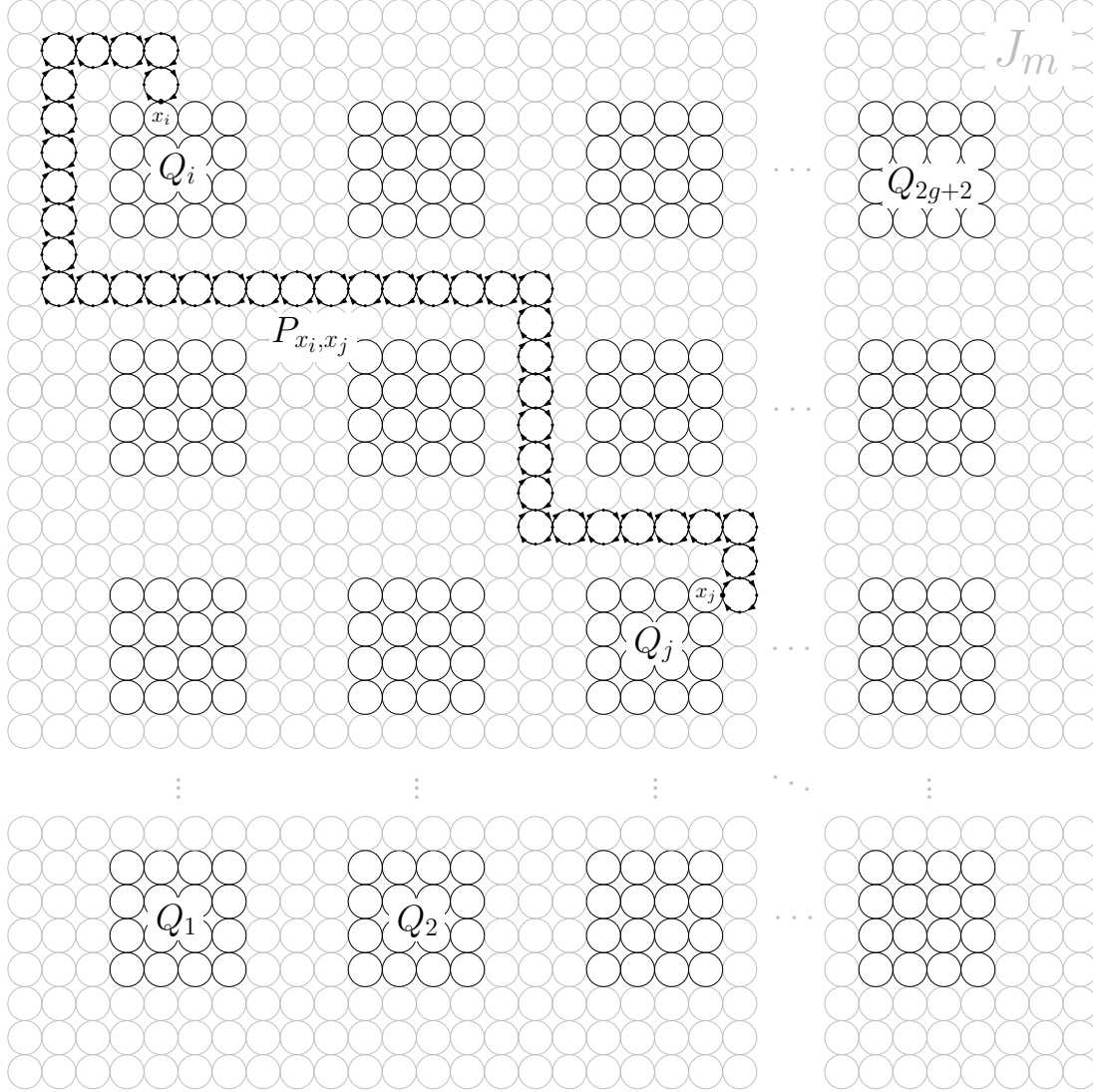


Figure 3.8: Disjoint copies of J_k immersed in J_m with a pair of edge-disjoint paths P_{x_i, x_j} .

- (i) $M_1 = Q_1$,
- (ii) $M_i \subseteq M_{i+1}$ for all $1 \leq i \leq g + 1$,
- (iii) M_i intersects at most $2i - 1$ many Q_j 's, and if M_i intersects a Q_j then $Q_j \subseteq M_i$ for all $1 \leq i \leq g + 1$, and $1 \leq j \leq 2g + 2$,
- (iv) $\text{eg}(M_i) \geq i - 1$ for all $1 \leq i \leq g + 2$.

Suppose that we have constructed M_i for $1 \leq i \leq g + 1$ that satisfies (i), (ii), (iii), and (iv). We show how to construct M_{i+1} that satisfy the above conditions. This yields a contradiction when $i = g + 2$ via (iv).

Take M_i . By (iii) there exists a Q_j that does not intersect M_i . First suppose that every chord with an attachment in the interior of Q_j has its other attachment also in Q_j , call

this set of chords \mathcal{I}_j . Let y be a vertex on the outer cycle of Q_j where y has an in and out-neighbor in $J_m^* - Q_j$. Choose $y' \in M_i$ such that there exists $P_{y,y'} \subseteq J_m^*$ where the only vertices from $P_{y,y'}$ to intersect $M_i \cup Q_1 \cup \dots \cup Q_{2g+2}$ are y and y' . Let $Q_j^+ = Q_j \cup \mathcal{I}_j$. Since Q_j is not good in D , Q_j^+ is nonplanar. Therefore, apply Proposition 3.9 to $G = Q_j^+ \cup (M_i \cup P_{y,y'})$ (after contracting all chords in G) to obtain $\text{eg}(G) \geq \text{eg}(Q_j^+) + \text{eg}(M_i \cup P_{y,y'}) \geq 1 + \text{eg}(M_i)$ and we can take $M_{i+1} = G$.

Now suppose that there exists a chord $R_{x,x'}$ where x is in the interior of Q_j and x' is not in Q_j . Let y be on the outer cycle of Q_j so that y is not cofacial (in the unique planar embedding of Q_j) with x , and y has an in and out-neighbor in $J_m^* - Q_j$. Once again, choose $y' \in M_i$ such that there exists $P_{y,y'} \subseteq J_m^*$ where the only vertices from $P_{y,y'}$ to intersect $M_i \cup Q_1 \cup \dots \cup Q_{2g+2}$ are y and y' . We break into cases based on the location of x' .

First assume that x' is in M_i . Let $G' = Q_j \cup R_{x,x'}$. Observe that G' is nonplanar by Corollary 2.9. Therefore, apply Proposition 3.10 to $G = (M_i \cup P_{y,y'}) \cup (Q_j \cup R_{x,x'})$ to obtain $\text{eg}(G) \geq \text{eg}(G') + \text{eg}(M_i \cup P_{y,y'}) \geq 1 + \text{eg}(M_i)$. So, take $M_{i+1} = G$.

Now assume that x' is in $J_m^* - (M_i \cup Q_j)$. Choose $x'' \in M_i \cup P_{y,y'}$ such that there exists $P_{x',x''} \subseteq J_m^*$ where the only vertices from $P_{x',x''}$ to intersect $M_i \cup P_{y,y'}$ are x' and x'' , and additionally, $P_{x',x''}$ intersects at most one Q_r for $1 \leq r \leq 2g+2$, $r \neq j$. Let $G' = Q_j \cup R_{x,x'}$. Again, observe that G' is nonplanar by Corollary 2.9. Apply Proposition 3.10 to $G = (M_i \cup P_{y,y'} \cup P_{x',x''}) \cup (Q_j \cup R_{x,x'})$ (or include Q_r if it exists) to obtain $\text{eg}(G) \geq \text{eg}(G') + \text{eg}(M_i \cup P_{y,y'} \cup P_{x',x''}) \geq 1 + \text{eg}(M_i)$ (or an analogous result if Q_r exists). So, take $M_{i+1} = G$. \square

The next two results concern embeddings of digraphs containing good grids. In preparation for this it will be helpful to consider planar embeddings of digraphs containing (good) grids. Let $k \geq 2$ and let D be a planar digraph immersing a grid J_k with model J_k^* . Let X be a J_k^* -bridge. We say that X is *strong* if it attaches on at least 2 distinct branches of J_k^* . Otherwise, X attaches on exactly one branch of J_k^* , and we call it *weak*.

Consider an embedding Ω of D in the plane. First, note that Ω induces an embedding of J_k where, by Corollary 2.9, each 4-cycle of J_k bounds a face (as does the outer cycle of J_k); additionally, Ω induces an embedding of J_k^* . Every J_k^* -bridge is planarly embedded in a disk whose boundary contains all of its attachment vertices. Moreover, the J_k^* -bridges satisfy the following:

- i) If X, Y are weak bridges attaching on the same branch of J_k^* and cross relative to one of their facial walks, then they embed in distinct faces.
- ii) If X is a strong bridge attaching on a facial walk $C \subseteq J_k^*$ and Y is a weak bridge such that X and Y cross relative to C , then Y does not embed in the face bounded by C .

Conversely, if we've assigned each J_k^* -bridge to a face of Ω containing all its attachment vertices so that i) and ii) are satisfied, then there exists a planar embedding of D such that

each J_k^* -bridge is embedded in a disk bounded by the assigned facial walk. We call such an embedding an *assigned embedding*.

Proposition 3.14. *Let D be a connected 2-regular digraph of Euler genus g that immerses a medial grid J_k whose model is good in D . Let Ω be an embedding of D with genus g . If $k \geq 4g + 6$, then the induced embedding of the medial grid J_{k-4g-4} in J_k has genus zero.*

Proof. Let J_k^* be a good model of the J_k immersion in D . For $1 \leq j \leq k$, consider the subgraphs $J_j^* \subseteq J_k^*$ and let \mathcal{I}_j denote the set of J_j^* -bridges with at least one attachment in the interior of J_j^* . Let $J_j^+ = J_j^* \cup \mathcal{I}_j$. Note that J_j^+ is planar since J_k^* is good in D .

Let J_j denote the medial grid obtained from J_j^* by suppressing all attachment vertices. For all j , Ω induces an embedding of J_j (and one of J_j^*) which we also denote by Ω . For $2 \leq j \leq k - 1$ let W_j denote the outer cycle of J_j (where W_j^* denotes the outer cycle of J_j^*) and let \mathcal{F}_j denote the set of 4-cycles from J_k that contain an edge from W_j . For every edge $e \in W_j$, there exist two 4-cycles $I_e, O_e \in \mathcal{F}_j$ that contain e where, I_e is the one with nonempty intersection with the interior of J_j , and O_e is the other one. Let O_e^* and I_e^* denote the cycles in J_j^* corresponding to O_e and I_e and let $O_e^+ = O_e^* \cup \mathcal{B}_{O_e^*}$ and $I_e^+ = I_e^* \cup \mathcal{B}_{I_e^*}$. We claim that there exists an i , $1 \leq i \leq g + 1$, such that for every edge $e \in W_{k-4i+3}$:

- i) O_e and I_e are a faces in Ω and
- ii) $O_e^+ \cup I_e^+$ is planar in the induced embedding Ω .

Given the existence of such an i , we call the union of $O_e^+ \cup I_e^+$ for all $e \in W_{k-4i+3}$ a *facial ring*.

Suppose that this were not the case. Then for every $1 \leq i \leq g + 1$ you can find an edge e_i in W_{k-4i+3} so that I_{e_i} or O_{e_i} is not a face in Ω or $O_{e_i}^+ \cup I_{e_i}^+$ is not planar in Ω . For each i , let R'_i be the 2-regular digraph obtained from the union of I_{e_i} and the collection of 4-cycles in J_k that surround I_{e_i} after suppressing the 1-regular vertices. Observe that each R'_i is isomorphic to the medial grid of $P_3 \square P_3$ which has a unique embedding in the plane by Corollary 2.9. If I_{e_i} or O_{e_i} is not a face in Ω , then R'_i has genus > 0 in the induced embedding. Form R_i from R'_i by taking the union of R'_i and the bridges from $\mathcal{B}_{O_{e_i}^*} \cup \mathcal{B}_{I_{e_i}^*}$. If $O_{e_i}^+ \cup I_{e_i}^+$ is not planar in Ω , then the induced embedding of R_i has genus > 0 .

We extend $R_1 \cup R_2 \cup \dots \cup R_{g+1}$ to a connected 2-regular digraph R by the following procedure:

- Start with $R = R_1$.
- For each fixed $1 < i \leq g + 1$ choose a pair of edge-disjoint paths, $P_i \subseteq (J_k - R)$, from R_i to R .
- Add $R_i \cup P_i$ to R .

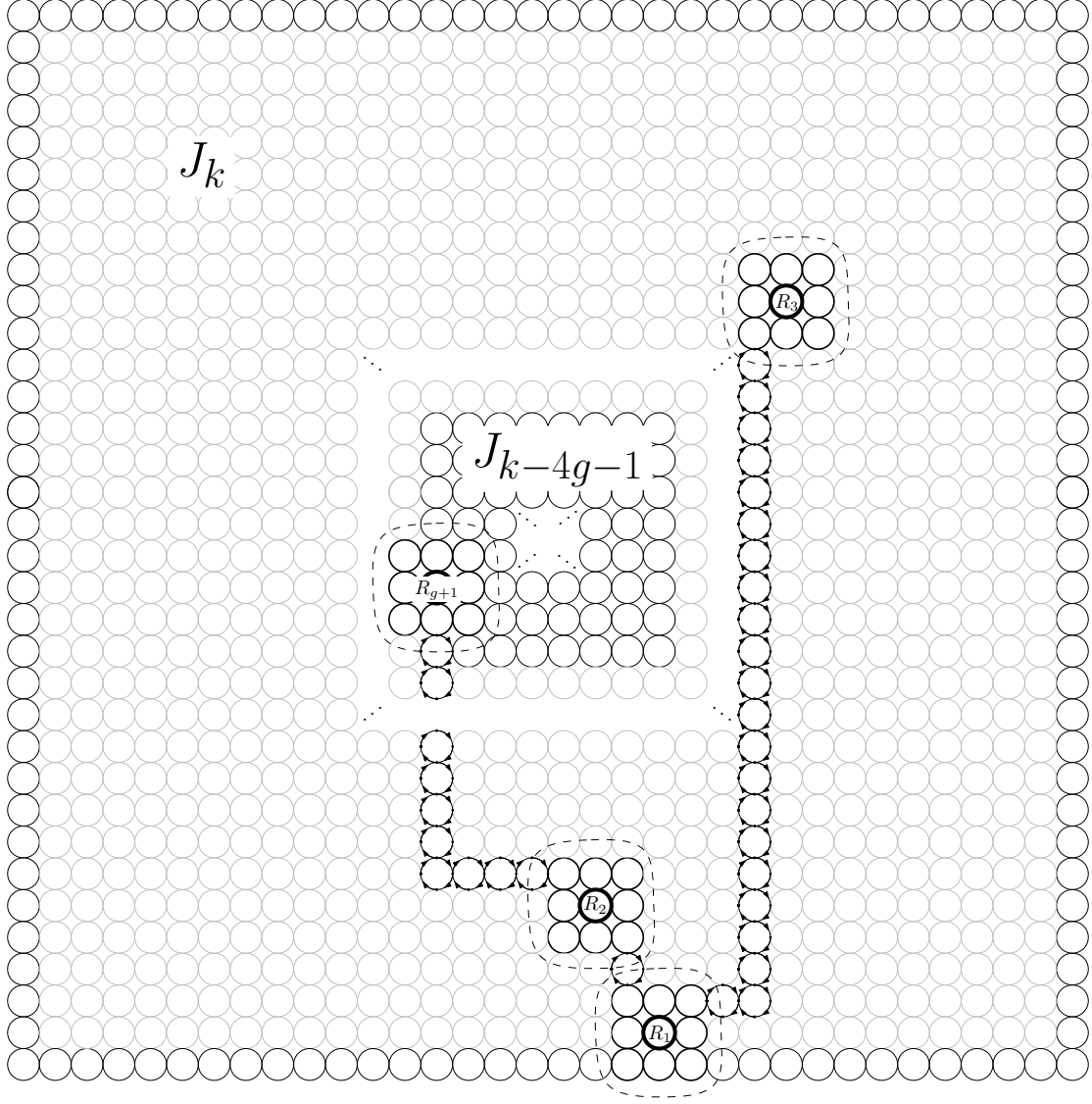


Figure 3.9: Finding the planar grid in Proposition 3.14.

- To see that the genus of the induced embedding of R has increased by at least one apply Proposition 3.9.

Since there are $g + 1$ many R_i 's the induced genus of R is at least $g + 1$ contradicting the genus of D . Thus, such an i exists.

Note that as stated above, since J_k^* is good in D , a bridge in \mathcal{I}_k must attach in a cycle of J_k^* that corresponds to a 4-cycle of J_k . In particular, we can partition the bridges $\mathcal{B}_{J_{k-4i+3}^*}$ into the following classes. Let $X \in \mathcal{B}_{J_{k-4i+3}^*}$, then either:

- $X \in \mathcal{I}_{k-4i+3}$ and we say that X is on the *interior* of J_{k-4i+3}^* or
- X is weakly attached to W_{k-4i+3} and we say that X is *on the boundary* of J_{k-4i+3}^* .

We define M to be the mixed graph obtained from D by deleting the the interior of J_{k-4i+3}^* and deleting all bridges in $\mathcal{B}_{J_{k-4i+3}^*}$ (but leaving the attachment vertices of the weak bridges on the boundary of J_{k-4i+3}^*). By Proposition 1.1 M is immersed in D ; moreover, M is connected, and therefore the genus of the induced embedding of M is at most g . We can extend the embedding of M to one of D in the following way (implying that the genus of M is g):

First, observe that W_{k-4i+3}^* is a face in the induced embedding of M , since I_e is a face in Ω , for every $e \in W_{k-4i+3}$. Second, take an assigned planar embedding of $J_{k-4i+3}^+ = J_{k-4i+3}^* \cup \mathcal{B}_{J_{k-4i+3}^*}$ where every weak bridge on the boundary of J_{k-4i+3}^* embeds in the face assigned to it from the facial ring. That is, for every $e \in W_{k-4i+3}$, if X is a weak bridge attaching on the branch of J_{k-4i+3}^* corresponding to e , then

- if X embeds in O_e^* in the facial ring, then embed X in the unbounded face of J_{k-4i+3}^* in the plane, and
- if X embeds in I_e^* in the facial ring, then embed X in I_e^* in J_{k-4i+3}^* in the plane.

Identifying W_{k-4i+3}^* in M and J_{k-4i+3}^+ yields an embedding of D .

To see that there exists a medial grid with whose induced embedding Ω has genus zero, observe that every 4-cycle of J_{k-4i} must be a face in Ω . Otherwise, (as before) you could take a nonfacial 4-cycle I of J_{k-4i} and union that with its surrounding 4-cycles to form R' and find a pair of edge-disjoint paths to and from R' and M , which by Proposition 3.9 implies that $M \cup R'$ union the paths has genus $> g$, contradicting the assumption. Therefore, J_{k-4i} has genus 0 in the induced embedding, and so taking the maximum possible value for i , we obtain the smallest grid J_{k-4g-4} in J_k that is necessarily planar in an induced embedding. \square

We now prove Theorem 3.4. We restate it here with more exact parameters than when mentioned before.

Theorem. *Let D be a connected 2-regular digraph with no cut-vertex and let $\text{eg}(D) = g$ but $\text{eg}(D/t) < g$ for any transition $t \in \mathcal{T}(D)$. Then D does not immerse J_m for $m = \lceil 800g^{3/2} \rceil$.*

Proof. Suppose towards a contradiction that D contained J_m as an immersion. Let $k \geq 4g + 4$. By Proposition 3.13, J_m immerses J_k with a model that is good in D . Take a vertex v in J_1 (of this J_k) and split v with transition t , forming edges $h_1, h_2 \in E(D/t)$. By assumption D/t has genus $< g$. Let Ω be a 2-cell embedding of D/t of genus $< g$. We may assume $\text{eg}(\Omega) = g' = g - 1$.

Notice that after splitting t , D/t immerses a medial grid J_{k-1} whose model is good in D/t where edges h_1 and h_2 are contained in the same 4-cycle of J_{k-1} . Since $k-1 \geq 4g+3 \geq 4g'+7$, by Proposition 3.14, the induced embedding of the medial grid $J_{(k-1)-4g'-4}$ in J_{k-1} has genus zero. Therefore, all the 4-cycles in $J_{(k-1)-4g'-4}$ are faces in Ω . Thus, there exists

a planar embedding of $J_{(k-1)-4g'-4}$ together with all bridges that attach to the model of $J_{(k-1)-4g'-4}$ where subdividing h_1 and h_2 , adding a chord, and contracting this chord yields an embedding of D with genus g' contradicting the assumption. \square

Now, we have the necessary tools to prove Theorem 3.1, that $\text{Forb}_I(S)$ is finite for each fixed S .

Proof of Theorem 3.1. Let S be a fixed surface and consider $\text{Forb}_I(S)$. It follows from 3.3 and 3.4 that there exists a constant $N \in \mathbb{N}$ such that every graph in $\text{Forb}_I(S)$ has branch-width $\leq N$. Now, Theorem 3.2 implies that $\text{Forb}_I(S)$ is finite. \square

Chapter 4

Obstructions for Small Surfaces

As stated in Chapter 2, in 1930 Kuratowski [22] proved that $\text{Forb}_T(\mathbb{S}_0) = \{K_5, K_{3,3}\}$, and in 1937 Wagner [47] showed that $\text{Forb}_M(\mathbb{S}_0) = \{K_5, K_{3,3}\}$. In 1979 Glover, Huneke, and Wang [14] found a list of 103 topological obstructions for the projective plane, and in 1981 Archdeacon [2] proved that their list was complete. Unlike the sphere, after filtering this list through the minor relation [12] we obtain a more compact list; that is, $|\text{Forb}_T(\mathbb{N}_1)| = 103$ and $|\text{Forb}_M(\mathbb{N}_1)| = 35$. To date, the sphere and the projective plane are the only surfaces whose obstructions have been fully classified, but this is not for lack of trying. A 2018 paper by Myrvold and Woodcock [27] has the current list of graphs so that $|\text{Forb}_T(\mathbb{S}_1)| \geq 250, 815$ and $|\text{Forb}_M(\mathbb{S}_1)| \geq 17, 535$.

We showed in Chapter 3 that for a fixed surface S , $\text{Forb}_I(S)$ is finite. And in Chapter 2 we showed that $\text{Forb}_I(\mathbb{S}_0) = \{C_3^1\}$. The purpose of this chapter is to provide the full classifications of obstructions for the projective plane as well as provide a partial classification of obstructions for surfaces with Euler genus 2.

Theorem 4.1. $\text{Forb}_I(\mathbb{N}_1) = \{C_4^1, C_6^2, C_3^1 \cdot C_3^1, C_3^1 \cup C_3^1\}$.

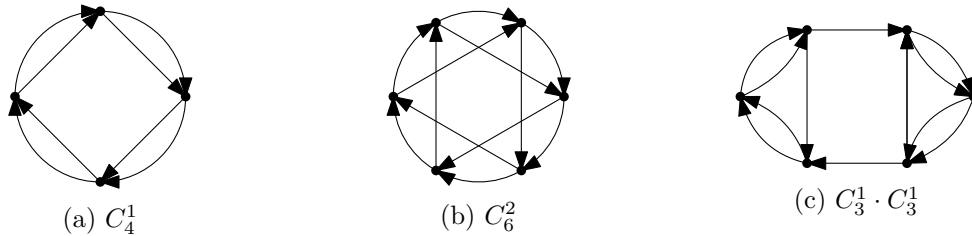


Figure 4.1: The connected obstructions for the projective plane.

Proposition 4.2.

$$\text{Forb}_I(\mathbb{S}_1) \supseteq \{C_5^1, C_7^2, D_6, D_{10}, C_3^1 \cdot C_3^1, C_3^1 \cup C_3^1\}.$$

We define the sets: $C_3^1 \cdot \text{Forb}_I(S) = \{C_3^1 \cdot D : D \in \text{Forb}_I(S)\}$ and $C_3^1 \cup \text{Forb}_I(S) = \{C_3^1 \cup D : D \in \text{Forb}_I(S)\}$. With this, we can compactly state the partial results for the classification of obstructions for the Klein bottle.

Proposition 4.3.

$$\text{Forb}_I(\mathbb{N}_2) \supseteq \{C_5^1, C_7^2, \underline{C_6^2}, D_7, D_9\} \cup (C_3^1 \cdot \text{Forb}_I(\mathbb{N}_1)) \cup (C_3^1 \cup \text{Forb}_I(\mathbb{N}_1))$$

Now let us explain some of the naming conventions used above. For any digraph D and $i \in \mathbb{N}$ with $0 \leq i < |D|$, let D^i denote the digraph obtained from D where for all $u, v \in V(D)$, the edge uv is added to D if the directed distance from u to v is i . Let C_k denote the directed cycle of length k . Observe that C_k^i is a 2-regular digraph for $0 \leq i \leq k - 1$. Let D_1 and D_2 be 2-regular digraphs, then $D_1 \cdot D_2$ is a 2-regular digraph obtained by deleting an edge $u_i v_i$ from D_i , for $i \in \{1, 2\}$, and adding edges $u_1 v_2$ and $u_2 v_1$. Given D_1 and D_2 there may be several nonisomorphic 2-regular digraphs that may be obtained by the above operation; however, when we write $D_1 \cdot D_2$ we are invoking the instantiation of one of these graphs. In the case where D_1 and D_2 are *edge-transitive* (there is an automorphism mapping any edge of a graph to any other edge of the graph), then $D_1 \cdot D_2$ unambiguously defines a single graph up to isomorphism.

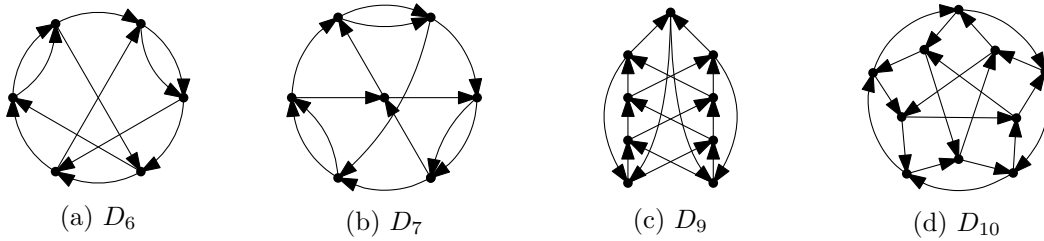


Figure 4.2: Some obstructions for Euler genus 2.

Let D be a 2-regular digraph with $v \in V(D)$ such that $e_1, e_2 \in E^-(v)$ and $f_1, f_2 \in E^+(v)$. The 2-regular digraph \underline{D} is obtained from D by the following operations:

- i) Add new vertices v_1 and v_2 ,
- ii) change the head of e_i to v_1 and the tail of f_i to v_2 for $i \in \{1, 2\}$,
- iii) add a pair of parallel edges $\{v_1 v_2, v_1 v_2\}$, and
- iv) delete the now isolated vertex v .

Given D there may be several nonisomorphic 2-regular digraphs that may be obtained by the above operation; however, when we write \underline{D} we are invoking the instantiation of one of these graphs. In the case where D is *vertex-transitive* (there is an automorphism mapping any vertex of a graph to any other vertex of the graph), then \underline{D} unambiguously defines a

single graph up to isomorphism. Observe that, for instance, \underline{C}_n^1 is isomorphic to C_{n+1}^1 for $n \geq 1$. We call \underline{D} an *inflation* of D .

Lastly, the graphs D_6 , D_7 , D_9 , and D_{10} are as shown in Figure 4.2. It is perhaps curious to note that D_7 is isomorphic to \underline{D}_6 .

4.1 Surface independent embedding results

In this section we prove some 2-regular digraph embedding results that do not rely on any particular surface. Chief among them is the existence of two infinite families of 2-regular digraphs where for each surface S , there exists a graph from the family that is a member of $\text{Forb}_I(S)$. Another is the disk embedding extension problem, which is used quite heavily in the proof of Theorem 4.1.

4.1.1 Obstructions for higher surfaces

In this section we provide two well-connected infinite families of obstructions that persists throughout genera. Additionally, we provide some tools to help assist in determining the Euler genus of a given 2-regular digraph. We start with the following observation. Let D be a 2-regular digraph with a 2-cell embedding Ω in a surface with Euler genus g . To *extend* Ω to a 2-cell embedding $\underline{\Omega}$ of \underline{D} is to continue the facial walks of Ω across the new parallel edges of \underline{D} . The extension $\underline{\Omega}$ is unique since every transition of $\mathcal{T}(\underline{D})$ must appear exactly once in $\underline{\Omega}$. Since no new faces were created, $\underline{\Omega}$ has Euler genus $g + 1$. Observe that $\underline{\Omega}$ is necessarily an embedding in a nonorientable surface. We collect this fact below.

Observation 4.4. *Let D be a 2-regular digraph. If D has an embedding in a surface with Euler genus g , then \underline{D} has an embedding in the nonorientable surface of Euler genus $g + 1$.*

Recall, that if D is a 2-regular digraph with an embedding Ω in an orientable surface, then there exists a bipartition $\{\mathcal{A}, \mathcal{B}\}$ of Ω where every edge $e \in E(D)$ appears exactly once in both $\bigcup_{F \in \mathcal{A}} E(F)$ and $\bigcup_{F \in \mathcal{B}} E(F)$. Moreover, notice that given say \mathcal{A} , the faces of \mathcal{B} are uniquely determined since every transition in $\mathcal{T}(D)$ is contained in exactly one face of Ω .

Lemma 4.5. *For $n \geq 2$, C_n^1 has a 2-cell embedding in all surfaces with Euler genus $n - 2$.*

Proof. We proceed by induction on n . If $n = 2$, then C_2^1 has a 2-cell embedding in \mathbb{S}_0 . Therefore, we may assume $n > 2$. Let $t \in \mathcal{T}(C_n^1)$ and observe that splitting t yields a graph isomorphic to C_{n-1}^1 . By induction C_{n-1}^1 has a 2-cell embedding Ω in a surface with Euler genus $n - 3$. Take the extension $\underline{\Omega}$ of $\underline{C_{n-1}^1}$ (which is isomorphic to C_n^1) to obtain an embedding in the nonorientable surface of genus $n - 2$.

If n is even, then we construct an embedding Ω' of C_n^1 in the orientable surface of genus $n - 2$. Take two disjoint n cycles of C_n^1 as one part of the bipartition of Ω' . Given these facial walks, Ω' completes uniquely to a 2-cell embedding of C_n^1 that is in the orientable surface with Euler genus $n - 2$. \square

Proposition 4.6. For $n \geq 3$, C_n^1 is an obstruction for each surface with Euler genus $n - 3$.

Proof. Let $n \geq 3$. Observe that a minimum feedback edge set of C_n^1 has size 2. Therefore, $\text{eg}(C_n^1) \geq n - 2$ by Proposition 2.7, which implies that C_n^1 cannot embed in a surface with Euler genus $n - 3$. To see that C_n^1 is minimal, let $t \in \mathcal{T}(C_n^1)$ and observe that splitting t yields a graph isomorphic to C_{n-1}^1 which has an embedding in all surfaces with Euler genus $n - 2$ by Lemma 4.5. \square

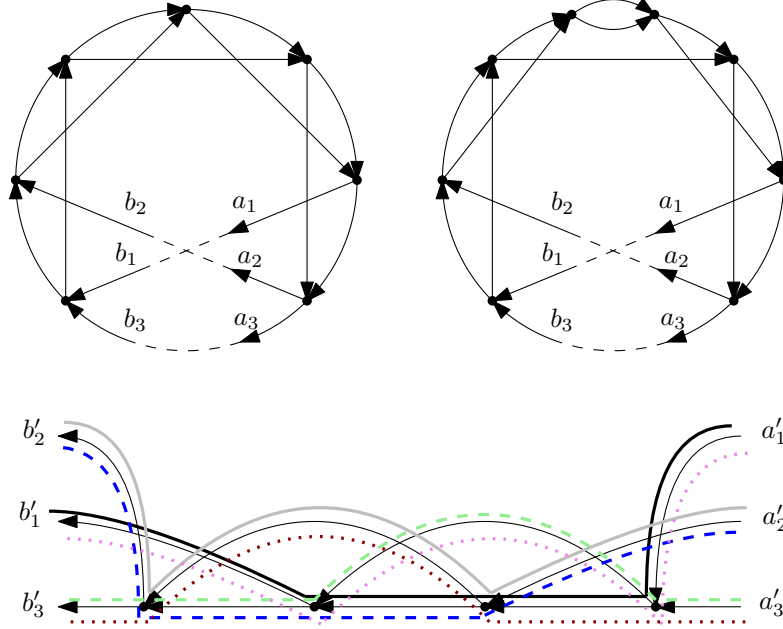


Figure 4.3: A minimum feedback edge set of C_7^2 and C_7^2 , whose edges are cut. Below is the gadget G^2 that is used to extend the embeddings to one of C_{11}^2 and C_{11}^2 .

Next, we introduce digraphs with *half-edges* (edges with either a head or a tail but not both). Half-edges with just a head are called *in-half-edges* and those with just a tail are called *out-half-edges*. We call a digraph with half-edges a *gadget*. Let D be a digraph with half-edges. For every vertex $v \in V(D)$ we modify $E(v)$ so as to include half-edges, and then $\text{deg}(v)$ is the sum of the number of edges and half-edges incident to v ($\text{deg}^+(v)$ and $\text{deg}^-(v)$ extend analogously). Given an edge $e \in E(D)$, to *cut* e is to delete e and add an out-half-edge at the tail of e and an in-half-edge at the head of e . Given an out-half-edge $a \in E(D)$ and an in-half-edge $b \in E(D)$, to *glue* a and b is to delete a and b and add an edge from the tail of a to the head of b . If every $v \in V(D)$ has $\text{deg}^+(v) = \text{deg}^-(v) = 2$ then we say that D is a *2-regular gadget*. We call *walks* in D sequences of edges or half-edges and *transitions* are walks of size 2. We say that a 2-regular gadget D has a 2-cell *embedding* Ω , if Ω consists of a set of walks \mathcal{W} , starting and ending with the half-edges and a set of closed walks \mathcal{C} , where every transition appears exactly once in Ω . We call the members of \mathcal{W} and \mathcal{C} *faces* of Ω .

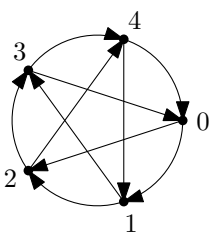
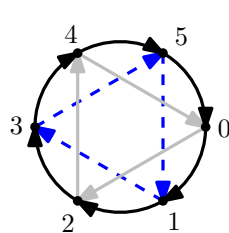
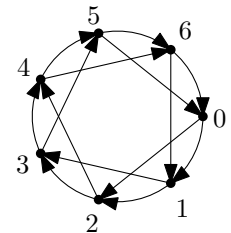
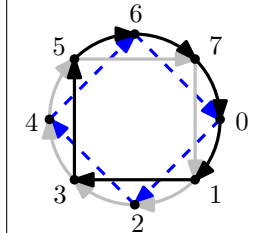
C_5^2	C_6^2	C_7^2	C_8^2
			
(0, 1, 3, 0) (0, 2, 3, 0) (0, 2, 4, 0) (1, 2, 4, 1) (1, 3, 4, 1) (0, 1, 2, 3, 4, 0)	(0, 1, 2, 4, 0) (0, 1, 3, 5, 0) (0, 2, 3, 4, 0) (0, 2, 4, 5, 0) (1, 2, 3, 5, 1) (1, 3, 4, 5, 1)	(0, 1, 3, 5, 0) (0, 2, 4, 5, 0) (1, 2, 4, 6, 1) (0, 2, 3, 4, 6, 0) (1, 3, 4, 5, 6, 1) (0, 1, 2, 3, 5, 6, 0)	(0, 2, 4, 6, 0) (1, 3, 5, 7, 1) (0, 1, 2, 3, 5, 6, 0) (0, 1, 3, 4, 5, 7, 0) (0, 2, 3, 4, 6, 7, 0) (1, 2, 4, 5, 6, 7, 1)

Figure 4.4: Each column represents C_n^2 embedded in each surface of Euler genus $n - 4$. When n is even, one part of the bipartition of faces of the orientable embedding is shown with edge colorings. The nonorientable embeddings are given by the rows, where each row is a facial walk given by its vertex sequence.

Lemma 4.7. *For $n \geq 5$, C_n^2 has a 2-cell embedding in each surface with Euler genus $n - 4$.*

Proof. Let S be a surface with Euler genus $\text{eg}(S) = n - 4$, and suppose Ω was an embedding of C_n^2 in S . We show how to extend this embedding to one of C_{n+4}^2 in S . Form the gadget G_n from C_n^2 by taking a minimum feedback arc set $A = \{e_1, e_2, e_3\}$ in C_n^2 and cutting each edge in A to obtain 3 out-half-edges $\{a_1, a_2, a_3\}$ and 3 in-half edges $\{b_1, b_2, b_3\}$, where a_i and b_i are obtained from cutting e_i for $i \in \{1, 2, 3\}$ (see Figure 4.3). Note that the embedding Ω is also an embedding of G_n .

Consider the 4 vertex gadget G^2 as seen in Figure 4.3. Observe that gluing a_i and a'_i and gluing b_i and b'_i for $i \in \{1, 2, 3\}$ yields a graph isomorphic to C_{n+4}^2 where the embedding Ω has been extended to an embedding of C_{n+4}^2 in S (note that extending this embedding can be done in such a way as to preserve orientability of Ω). By Figure 4.4 we have that C_5^2 , C_6^2 , C_7^2 , and C_8^2 each embed in every surface of the requisite Euler genus; therefore, we obtain the result. \square

Lemma 4.8. *For $n \geq 4$, \underline{C}_n^2 has a 2-cell embedding in each surface with Euler genus $n - 3$.*

Proof. By Lemma 4.7, we have that C_n^2 embeds in all surfaces of Euler genus $n - 4$. Therefore, by Observation 4.4, this implies that \underline{C}_n^2 embeds in all nonorientable surfaces of Euler genus $n - 3$. It remains to show that for odd n , \underline{C}_n^2 embeds in the orientable surface with Euler genus $n - 3$.

Similar to the proof above, let S be the orientable surface with Euler genus $\text{eg}(S) = n - 3$, and suppose Ω was an embedding of \underline{C}_n^2 in S . We show how to extend this embedding to one of \underline{C}_{n+4}^2 in S . Form the gadget \underline{G}_n from \underline{C}_n^2 by taking a minimum feedback arc set

$A = \{e_1, e_2, e_3\}$ in \underline{C}_n^2 (not containing the parallel edges of \underline{C}_n^2) and cutting each edge in A to obtain 3 out-half-edges $\{a_1, a_2, a_3\}$ and 3 in-half edges $\{b_1, b_2, b_3\}$, where a_i and b_i are obtained from cutting e_i for $i \in \{1, 2, 3\}$ (see Figure 4.3). Note that the embedding Ω is also an embedding of G_n .

As before, take the 4 vertex gadget G^2 from Figure 4.3. Gluing a_i and a'_i and gluing b_i and b'_i for $i \in \{1, 2, 3\}$ yields a graph isomorphic to \underline{C}_{n+4}^2 where the embedding Ω has been extended to an embedding of \underline{C}_{n+4}^2 in S . By Figure 4.5 we have that \underline{C}_5^2 and \underline{C}_7^2 embed in \mathbb{S}_1 and \mathbb{S}_2 respectively, and this yields the result. \square

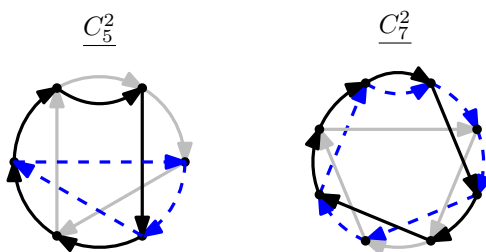


Figure 4.5: Embeddings of \underline{C}_5^2 and \underline{C}_7^2 in \mathbb{S}_1 and \mathbb{S}_2 respectively where one part of the bipartition of faces is shown with edge colorings.

Proposition 4.9. *For $n \geq 6$, \underline{C}_n^2 is an obstruction for each surface with Euler genus $n - 5$.*

Proof. Let $n \geq 6$. Observe that a minimum feedback edge set of \underline{C}_n^2 has size 3. Therefore, $\text{eg}(\underline{C}_n^2) \geq n - 4$ by Proposition 2.7, which implies that \underline{C}_n^2 cannot embed in a surface with Euler genus $n - 5$. To see that \underline{C}_n^2 is minimal, let $t \in \mathcal{T}(\underline{C}_n^2)$ and observe that splitting t yields either a graph isomorphic to \underline{C}_{n-1}^2 or one isomorphic to \underline{C}_{n-2}^2 . By Lemma 4.7 \underline{C}_{n-1}^2 embeds in each surface with Euler genus $n - 5$ and by Lemma 4.8 \underline{C}_{n-2}^2 embeds in each surface with Euler genus $n - 5$. \square

Let $D = (V, E)$ be a 2-regular digraph with $e, f \in E$ and a digon $\{e, f\}$. To *contract* $\{e, f\}$ is to delete edges e and f and identify the endpoints of e and f to a single vertex. Observe that the resulting digraph is also 2-regular. To *replace* $v \in V$ with a digon is to perform a marked split at v and instead of putting a chord with endpoints the children of v , put a digon with endpoints the children of v . As with splitting, replacing a vertex with a digon can be performed in two ways, possibly yielding nonisomorphic graphs.

Proposition 4.10. *Let D be a 2-regular digraph and let D' be the 2-regular digraph obtained from D after replacing a vertex of D with a digon. If D has a 2-cell embedding in a surface S , then D' has a 2-cell embedding in S .*

Proof. Let Ω be a 2-cell embedding of D . Let $\{e, f\}$ be the digon in D' that replaced $v \in V(D)$. We extend Ω to an embedding Ω' of D' by the following: Consider the facial walks in Ω containing a transition from $T(v)$. Two of these facial walks remain closed walks

in D' and two are no longer closed. Close the two open walks by adding one edge from $\{e, f\}$ to each one. Lastly, add the face $(e, f) \in \Omega'$. Therefore, we have an embedding of D' in a surface with the same Euler genus as S . To see that orientability (or nonorientability) of the embedding is preserved, observe that modifying Ω to form Ω' does not alter the existence (or nonexistence) of a bipartition. \square

In light of Proposition 4.10 we obtain the following corollary.

Corollary 4.11. *If D' is a 2-regular digraph with a digon $\{e, f\}$ and D is the 2-regular digraph obtained after contracting $\{e, f\}$, then $\text{eg}(D) = \text{eg}(D')$. Moreover, if $H \in \text{Forb}_I(S)$, then H does not contain a digon.*

Proof sketch. Proposition 4.10 tells us that $\text{eg}(D') \leq \text{eg}(D)$. To see that $\text{eg}(D) \leq \text{eg}(D')$, observe that D is immersed in D' . \square

Corollary 4.11 essentially tells us that when we are trying to determine the Euler genus of a 2-regular digraph, we may freely contract all digons.

Let Ω be a 2-cell embedding of a 2-regular digraph D . Each edge $e \in E(D)$ either appears in two distinct faces of Ω or e appears twice in a single face of Ω . In the latter case, we say that e is *singular*.

Lemma 4.12. *A minimum genus 2-cell embedding of a 2-regular digraph has no singular edges.*

Proof. Let Ω be a minimum genus embedding of a 2-regular digraph D . Suppose towards a contradiction that there was an edge $e \in E(D)$ appearing twice in $F \in \Omega$. Therefore, there exist walks W and U such that $F = e, W, e, U$. Observe that declaring $F_1 = e, W$ and $F_2 = e, U$ yields an embedding $\Omega' = (\Omega \setminus \{F\}) \cup \{F_1, F_2\}$ with smaller Euler genus, contradicting the assumption. \square

Proposition 4.13. *If D_1 and D_2 are connected 2-regular digraphs, then*

$$\text{eg}(D_1 \cdot D_2) = \text{eg}(D_1) + \text{eg}(D_2).$$

Proof. Let Ω be a minimum genus 2-cell embedding of $D_1 \cdot D_2$. Form the induced 2-cell embeddings Ω_1 and Ω_2 of D_1 and D_2 respectively. Observe that there are two distinct faces $F, G \in \Omega$ containing edges from $E(D_1 \cdot D_2) \setminus (E(D_1) \cup E(D_2))$ by Lemma 4.12. These faces appear as distinct “truncated” faces $F_i, G_i \in \Omega_i$ for $i \in \{1, 2\}$. Therefore, for $i \in \{1, 2\}$, let

$n = |V(D)|$, $n_i = |V(D_i)|$, $q = |\Omega|$, $q_i = |\Omega_i|$, and we obtain via Euler genus

$$\begin{aligned} \text{eg}(D_1 \cdot D_2) &= 2 + n - q \\ &= 2 + (n_1 + n_2) - (q_1 + q_2 - 2) \\ &= (2 + n_1 - q_1) + (2 + n_2 - q_2) \\ &\geq \text{eg}(D_1) + \text{eg}(D_2). \end{aligned}$$

Similarly, let Ω_1 and Ω_2 be minimum genus 2-cell embeddings of D_1 and D_2 respectively. For $i \in \{1, 2\}$, observe that there are two distinct faces $F_i, G_i \in \Omega_i$ containing the edge in $E(D_i) \setminus E(D_1 \cdot D_2)$ by Lemma 4.12. These closed walks can be concatenated to form closed walks F and G in $D_1 \cdot D_2$. Let $\Omega = (\Omega_1 \setminus \{F_1, G_1\}) \cup (\Omega_2 \setminus \{F_2, G_2\}) \cup \{F, G\}$ and observe that this is an embedding of $D_1 \cdot D_2$ where (using the same notation as above) by Euler genus

$$\begin{aligned} \text{eg}(D_1) + \text{eg}(D_2) &= (2 + n_1 - q_1) + (2 + n_2 - q_2) \\ &= 2 + (n_1 + n_2) - (q_1 + q_2 - 2) \\ &= 2 + n - q \\ &\geq \text{eg}(D_1 \cdot D_2). \end{aligned}$$

□

Proposition 4.14. *For $k \geq 2$, if $D \in \text{Forb}_I(\mathbb{N}_{k-1})$ with $\text{eg}(D) = k$, then $C_3^1 \cdot D \in \text{Forb}_I(\mathbb{N}_k)$.*

Proof. We have that $\text{eg}(C_3^1 \cdot D) = k + 1$ by Proposition 4.13. This implies that $C_3^1 \cdot D$ cannot embed in \mathbb{N}_k . To see that $C_3^1 \cdot D$ is minimal, let $t \in \mathcal{T}(C_3^1 \cdot D)$ and consider $(C_3^1 \cdot D)/t$. Observe that there exists either a $t' \in \mathcal{T}(C_3^1)$ or a $t' \in \mathcal{T}(D)$ such that $(C_3^1 \cdot D)/t$ is isomorphic to either $(C_3^1/t') \cdot D$ or $C_3^1 \cdot (D/t')$. In the first case, $\text{eg}((C_3^1 \cdot D)/t) = 0 + \text{eg}(D) = k$ and in the second $\text{eg}((C_3^1 \cdot D)/t) = 1 + \text{eg}(D/t') \leq k$. □

4.1.2 Embedding in a disk

In this section we prove an analogue of the disk embedding extension problem (see [26]). We frame the problem in the following way: let H be a 2-regular digraph with a 2-cell embedding into a surface S and $F \subseteq S$ a face of this embedding with corresponding facial walk C in D . Consider a “bridge” B . What are the obstructions to embedding B in F so that B attaches to C and the resulting 2-regular digraph is embedded in S ?

We first examine C_2^1 . Observe that it is planar and strongly 2-edge-connected, so by Corollary 2.9 it has a unique planar embedding. However, one can see that in a planar embedding of C_2^1 , pairs of parallel edges are not cofacial. Thus, C_2^1 has no embedding in the plane with parallel edges bounding a common face. Therefore, a “bridge” containing

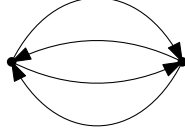


Figure 4.6: The essentially unique planar embedding of C_2^1 . Parallel edges are not cofacial.

C_2^1 as an immersion cannot embed in a disk where a pair of its parallel edges attach to the boundary of the disk. We formalize this in Lemma 4.16.

Let D be a 2-regular digraph with an embedding ϕ into a surface S . Let $\mathcal{F} \subseteq S$ be the set of faces of $S - \phi(D)$. We define the *dual* of D (with respect to ϕ), denoted \hat{D} , as the geometric dual of the embedding of the underlying graph. That is:

- For every face $f \in \mathcal{F}$, associate a vertex $\hat{f} \in V(\hat{D})$.
- For every edge $e \in E(D)$, let $f, g \in \mathcal{F}$ be the (not necessarily distinct) faces whose boundary walks in D contain e . Add the edge $\hat{e} \in E(\hat{D})$ so that \hat{e} has endpoints \hat{f}, \hat{g} .

Note that \hat{D} is an abstract graph, but there exists a natural embedding of \hat{D} in S .

Lemma 4.15. *Let D be a connected planar 2-regular digraph with a distinguished set of edges $F \subseteq E(D)$. Let $\{e, f\}$ be a pair of parallel edges from C_2^1 . Either there exists an embedding of D in the plane where all edges F lie on the outer face, or there exists an immersion (ψ, ψ') of C_2^1 in D where $\psi'(e) \cap F \neq \emptyset$ and $\psi'(f) \cap F \neq \emptyset$.*

Proof. We induct on $|V(D)|$. First, assume that D is strongly 2-edge-connected. Let Ω be the unique embedding of D in the plane by Corollary 2.9. We claim that if every pair of edges of F lie in a common face, then all edges of F lie in a common face.

Suppose that every pair of edges of F was in a common face. Consider the dual graph \hat{D} and let $\hat{F} = \{\hat{e} \in E(\hat{D}) : e \in F\}$. Let \hat{H} be the subgraph of \hat{D} with edge set \hat{F} . Observe that every pair of edges from \hat{H} have an endpoint in common. Therefore \hat{H} is either isomorphic to a star (a complete bipartite graph $K_{1,k}$) or a triangle (a cycle of length 3). If \hat{H} is a star, then all edges of F lie on a common face. In the latter case, let \hat{H} be the triangle $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$. Consider the corresponding edges $e_1, e_2, e_3 \in F$. Since Ω is an embedding in the plane, $\{e_1, e_2, e_3\}$ forms a 3 edge-cut in the underlying graph of D . However, this cannot exist since D is Eulerian and every edge-cut has even size. Thus, the claim is verified.

Now, we may assume that there exists $e_1, e_2 \in F$ such that they do not lie in a common face. Let $e_1 = u_1v_1$ and consider two peripheral cycles, from Theorem 2.8, F_1 and F_2 containing e_1 . It follows that F_1 and F_2 are faces in Ω . Note that the only edge in common to both F_1 and F_2 is e_1 since D is strongly 2-edge-connected (consider the dual).

Let H be the subgraph obtained from D by deleting the edges of $F_1 \cup F_2$. Since e_2 was assumed not cofacial with e_1 we get that e_2 is an edge of H . We claim that H is connected. Since F_1 and F_2 are peripheral in D , every component of H has at least one vertex in $V(F_1)$

and one vertex in $V(F_2)$. Now, since every $w \in V(H) \setminus \{u_1, v_1\}$ has $\deg_H^+(w) = \deg_H^-(w)$ and $\deg_H^+(u_1) = \deg_H^-(v_1) = 1$ we get that there exists a component H' of H with a path from u_1 to v_1 . But since the underlying graph of H is embedded in the plane, we obtain that $H' = H$ by the Jordan Curve Theorem 2.4. Thus H is connected. Let T be an Eulerian walk of H from u_1 to v_1 , we obtain an immersion (ϕ, ϕ') of C_2^1 in D where $\phi'(e) = e_1$ and $\phi'(f) = T$.

If D is not strongly 2-edge-connected then choose $X \subseteq V(D)$ such that $\delta^+(X) = \{x_1y_1\}$ and $\delta^-(X) = \{y_2x_2\}$. Let $Y = V(D) \setminus X$, and form D_X and D_Y (as defined in Section 2.3). If all edges of F have both end-points in X , then applying induction to D_X we either get that D_X has an embedding Ω_X in the plane with edges of F on the outer face, or (ϕ, ϕ') is an immersion of C_2^1 in D_X where $\phi'(e) \cap F \neq \emptyset$ and $\phi'(f) \cap F \neq \emptyset$.

In the latter case, D_X is immersed in D and by transitivity of immersion we obtain the obstruction. In the former case, since D_Y is immersed in D , $\text{eg}(D_Y) = 0$. Let Ω_Y be an embedding of D_Y in the plane where edge y_2y_1 appears on the outer face. Obtain an embedding of D in the plane with edges of F on the outer face by embedding Ω_Y in a face of $\mathbb{R}^2 - \Omega_X(D_X)$ that contains the edge x_1x_2 . Delete x_1x_2 and y_2y_1 and add back edges x_1y_1 and y_2x_2 . A similar argument holds if all end-points of F are contained in Y .

In the remaining case, there exists an edge $e \in F$ such that one endpoint is in X and the other is in Y . Define $F_X = (F \cap E(D_X)) \cup \{x_1x_2\}$ and $F_Y = (F \cap E(D_Y)) \cup \{y_2y_1\}$ to be the new sets of distinguished edges. By induction either D_X embeds with F_X on the outer face, or (ϕ_X, ϕ'_X) is an immersion of C_2^1 in D_X where $\phi'_X(e) \cap F_X \neq \emptyset$ and $\phi'_X(f) \cap F_X \neq \emptyset$. By induction, a similar conclusion holds for D_Y and F_Y .

If both D_X embeds with F_X on the outer face and D_Y embeds with F_Y on the outer face, then embed both graphs in the plane, delete x_1x_2 and y_2y_1 and add edges x_1y_1 and y_2x_2 to obtain D embedded in the plane with F on the outer face. Otherwise, (ϕ_X, ϕ'_X) works as an immersion of C_2^1 in D with $\phi'_X(e) \cap F \neq \emptyset$ and $\phi'_X(f) \cap F \neq \emptyset$ since D_X is immersed in D by taking an Euler trail from x_1 to x_2 of the nontrivial component of $D - E(D_X)$. \square

To state the next obstructions, we use the language of the immersion model. Let H^* be a model of H immersed in D and let C be a directed closed walk in H^* (we are implicitly treating C as the boundary of our disk). One obstruction is a bridge $X \in \mathcal{B}_C$ whose ‘‘orientation agrees with C ’’; we formalize this in Lemma 4.16. The last obstruction is when two bridges $X, Y \in \mathcal{B}_C$ cross each other, relative to C .

Lemma 4.16. *Let D and H be 2-regular digraphs and let H^* be a model of H immersed in D . Let C be a closed walk in H^* . The mixed graph $C^+ = C \cup \mathcal{B}_C$ either embeds in a closed disk with C as the boundary, or there exist an $X \in \mathcal{B}_C$ where:*

- i) X is nonplanar, or
- ii) X contains an immersion of C_2^1 with a pair of parallel edges attaching to C , or

- iii) X has attachment vertices (a, b, c) , appearing in that order on C , and a closed walk in X encountering the vertices (a, b, c) in that order, or
- iv) there exists a $Y \in \mathcal{B}_C$ such that X and Y cross relative to C .

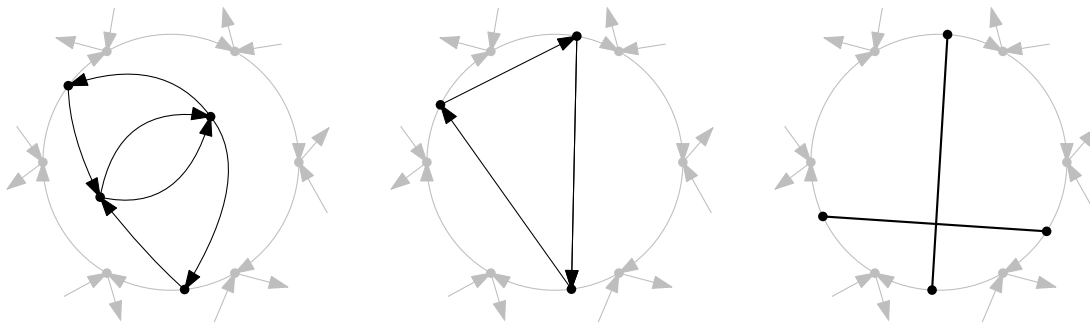


Figure 4.7: Different types of obstructions to embedding in a disk.

Proof. First, we can see that if either i), ii), or iii) hold then C^+ immerses C_3^1 and is therefore nonplanar by Theorem 2.6. Next, observe that if iv) holds, then C^+ cannot be embedded in a disk with C as the boundary by the Jordan Curve Theorem 2.4. Thus, we now show that if iv), i), ii), and iii) do not hold, then C^+ embeds in a disk with C as the boundary. We prove this by induction on $|\mathcal{B}_C|$.

If $|\mathcal{B}_C| = 0$ then C^+ is a subdivision of a cycle, which has an embedding in a disk with C as the boundary. Let $X \in \mathcal{B}_C$. Consider the nontrivial component J obtained from C^+ by deleting $E(X)$. By induction, J has an embedding Ω in a disk with C as the boundary.

Since iv) does not hold, there exists a face $C' \in \Omega$ that contains all attachments of X . If X was a chord, we can extend Ω to an embedding of C^+ in a disk with C as the boundary. If X was not a chord, since i) does not hold X is planar. Let $F \subseteq E(X)$ be the set of edges of X incident to the attachment vertices of X . By Lemma 4.15 either there exists a planar embedding of X with F on the outer face, or X contains C_2^1 as an immersion where the trails corresponding to a pair of parallel edges has nonempty intersection with F . However, the latter cannot happen since ii) does not hold. Thus, consider the planar embedding Ω_X of X whose outer face contains all edges from F . Observe that C' contains the edges from C that are incident to the attachment vertices of X . Therefore, since iii) does not hold for C , it also does not hold for C' and we can extend Ω to an embedding of C^+ in a disk with C as the boundary. \square

4.2 Immersion tools

In this section we introduce what we call the efficient immersion model. This model is quite helpful in classifying the projective plane obstructions. We also prove additional tools for immersing non-Eulerian digraphs.

4.2.1 The efficient model

Let H and D be 2-regular digraphs where H is immersed in D with a model (D^*, H^*) . If every chord is an H^* -bridge (i.e. for all $v_1 v_2 \in U(D^*)$ we have $v_1, v_2 \in V(H^*)$), then we say that H^* is a *tidy* model. We call a model H^* *efficient* if:

- i) H^* is tidy and
- ii) every H^* -bridge has attachment vertices on ≥ 2 branches of H^* .

The proof of Lemma 4.17 is reminiscent of the proof of Theorem 2.8.

Lemma 4.17. *If H is immersed in D and D is strongly 2-edge-connected, then there exists an efficient model of this immersion.*

Proof. Given a model J^* of H immersed in D , we say that a J^* -bridge is *rich* if it has vertices on ≥ 2 branches of J^* and *poor* otherwise. If J^* is not tidy, then we may form a tidy model of H immersed in D by contracting any chord that is not a J^* -bridge.¹

Among all tidy models, choose an immersion model H^* so as to lexicographically maximize the sizes of the rich H^* -bridges, and subject to this, lexicographically maximize the sizes of the poor H^* -bridges. That is, the largest rich H^* -bridge is as large as possible, and subject to this the second largest rich H^* -bridge is as large as possible, and so on until all rich H^* -bridges are accounted for. Then the largest poor H^* -bridge is as large as possible, and so on. We claim that H^* is efficient.

Suppose for a contradiction that there exists a poor H^* -bridge. According to our ordering, choose the smallest one X . Since X is poor its attachment vertices lie on only one branch T of H^* . Let P be the shortest subpath of T that contains all attachment vertices of X and suppose that P starts at vertex x and ends at vertex y . If there is another H^* -bridge with an attachment in the interior of P , then we break into cases depending on X .

If X is a chord, contracting X yields a model of H in D that merges all H^* -bridges with attachments on P and the edges of P into a new bridge. Note that this model can be made tidy by contracting every chord that is not a bridge. This improves the lexicographic ordering of the bridges as X was the smallest poor bridge. If X not a chord, since X is Eulerian, take a directed path Q in X from x to y . Switching P for Q on T yields a model of H in D that merges all H^* -bridges with attachments on P and the edges of P into a new bridge. Once again, this model can be made tidy, and after doing so, this improves the lexicographic ordering of the bridges. Thus, all attachment vertices on P belong to X , but then X is separated from the rest of D by two edges, contradicting the fact that D is strongly 2-edge connected. It follows that all bridges are rich and thus H^* is efficient. \square

¹This amounts to prioritizing the deletion of Eulerian subgraphs of D over vertex splits when immersing H (see Proposition 1.1).

Given a model H^* of H immersed in D and an H^* -bridge that is not a chord X , we can move to a “simpler” model by reducing bridge sizes. We say that we *replace X with a chord* (replace X with a cycle of length k) when we do the following:

- select distinct attachment vertices x_1, x_2 (vertices x_1, x_2, \dots, x_k) of X ,
- delete $E(X)$ from X ,
- add chord x_1x_2 (add cycle C_k with $E(C_k) = \{v_1v_2, v_2v_3, \dots, v_kv_1\}$) to X , and
- delete isolated vertices of X and suppress the remaining attachment vertices in H^* .

Keep in mind that if $k \geq 3$ one can replace X by a cycle in nonisomorphic ways. Moreover, note that replacing X with a chord or a cycle are immersion closed operations (the replacement can be done via vertex splits).

Lemma 4.18. *Let H^* be an efficient model of H immersed in D and let $C \subseteq H^*$ be a directed closed walk. If $X, Y \in \mathcal{B}_C$ cross relative to C , then X and Y can be replaced by chords, each with attachment vertices on 2 distinct branches of H^* , that cross relative to C .*

Proof. Let x, x' be attachment vertices of X and y, y' be attachment vertices of Y such that they appear in the order (x, y, x', y') when walking along C . If all attachment vertices are on the same branch B of H^* , then since H^* is efficient, there must be another attachment vertex y'' of Y on another branch $B' \neq B$ and another attachment vertex x'' of X on a (possibly the same) branch $B'' \neq B$.

The attachment vertices either appear in the order (x, y, x', y', x'', y'') or they appear in the order (x, y, x', y', y'', x'') when walking around C . In the first case, replacing X by a chord with endpoints x' and x'' and replacing Y by a chord with endpoints y' and y'' yields one such desired model. In the second case, replacing X by a chord with endpoints x' and x'' and replacing Y by a chord with endpoints y and y'' yields a desired model. \square

4.2.2 Immersing non-Eulerian digraphs

Observation 4.19. *Let D be a non-Eulerian digraph with vertex set V . For every $u \in V$ with $\deg^+(u) > \deg^-(u)$ there exists a trail from u to v where $v \in V$ satisfies $\deg^-(v) > \deg^+(v)$.*

Proof sketch. Take a maximal trail starting at u . \square

Let D be an Eulerian digraph and let H be a digraph immersed in D . If H is not Eulerian, then since D is Eulerian one can “complete” H to an Eulerian immersion. The following lemma formalizes this and defines the term *Eulerian completion*.

Lemma 4.20. *If D is an Eulerian digraph immersing a digraph H , then there exists an Eulerian digraph H' immersed in D such that:*

- i) H is a subgraph of H' ,
- ii) $V(H) = V(H')$, and
- iii) for every $v \in V(H)$, $\deg_{H'}^+(v) = \deg_{H'}^-(v) = \max \{ \deg_H^+(v), \deg_H^-(v) \}$.

We call H' an Eulerian completion of H in D .

Proof. Let H be immersed in D with maps (ϕ, ϕ') . Define the *deficit* of $v \in V(H)$ as $\text{def}(v) = |\deg_H^+(v) - \deg_H^-(v)|$ and the *deficit* of H as

$$\text{def}(H) = \sum_{v \in V(H)} \text{def}(v).$$

We induct on $\text{def}(H)$.

If $\text{def}(H) = 0$ then H is Eulerian and we can take $H' = H$. Otherwise, consider $D' = D - E(H)$. Since $\text{def}(H) > 0$, there exists a vertex $u \in V(D')$ such that $\deg_{D'}^+(u) > \deg_{D'}^-(u)$. Observation 4.19 yields a trail T from u to v where $v \in V(D')$ and $\deg_{D'}^-(v) > \deg_{D'}^+(v)$. Create a super graph J of H by adding edge uv to J . Notice that J is immersed in D by modifying the immersion so that $\phi'(uv) = T$. Since $\text{def}(J) < \text{def}(H)$ by induction there exists an H' meeting the conditions of the lemma. This same H' works for H as well. \square

4.3 The projective plane

It is quite easy to show one direction of Theorem 4.1, that C_4^1 , C_6^2 , $C_3^1 \cdot C_3^1$, and $C_3^1 \cup C_3^1$ are all obstructions for the projective plane; however, showing that this list is complete needs some additional tools. This section builds towards the proof of Theorem 4.1. First, we deal with the poorly connected obstructions, and then we show that C_5^2 (not an obstruction) is an important intermediate graph to consider.

4.3.1 Poorly connected obstructions

Proposition 4.21. *If a connected 2-regular digraph contains $C_3^1 \cup C_3^1$ as an immersion, then it contains $C_3^1 \cdot C_3^1$ as an immersion.*

Proof. Let D be a 2-regular digraph immersing $C_3^1 \cup C_3^1$ and let H^* be a model of this immersion. Let H_1 and H_2 be distinct subgraphs of H^* each isomorphic to a subdivision of C_3^1 . Since D is connected there exists a bridge X with attachment vertices $v_1 \in V(H_1)$ and $v_2 \in V(H_2)$. Replace X with a chord whose endpoints are v_1, v_2 . Observe that this mixed graph immerses $C_3^1 \cdot C_3^1$. \square

The following is a well-known fact in the world of digraph immersion. We prove a version for 2-regular digraphs.

Proposition 4.22 (Lovász [23]). *Let $D = (V, E)$ be a strongly 2-edge-connected 2-regular digraph. For every $v \in V$ there exists a transition $t \in T(v)$ such that D/t is strongly 2-edge-connected.*

Proof. Let $v \in V$ with in-edges $\{av, bv\}$ and out-edges $\{vc, vd\}$. Let $t = (av, vc)$ and $t' = (av, vd)$ be non-complementary transitions at v . Suppose towards a contradiction that both D/t and D/t' are not strongly 2-edge connected. This implies that there exists $X \subset V$ where $\{a, c\} \subseteq X$ and $\{b, d\} \subseteq V \setminus X$ and there exists $Y \subset V$ where $\{a, d\} \subseteq Y$ and $\{b, c\} \subseteq V \setminus Y$ and $d_D(X) = d_D(Y) = 4$.

Let $D' = D - v$, $X' = X \setminus \{v\}$, and $Y' = Y \setminus \{v\}$. Observe that $d_{D'}(X') = d_{D'}(Y') = 2$. By submodularity, one of $d_{D'}(X' \cap Y')$ and $d_{D'}(X' \cup Y')$ is at most 2; assume without loss that $d_{D'}(X' \cap Y') \leq 2$. But this implies that $d_D(X' \cap Y') \leq 3$ contradicting the assumption that D is strongly 2-edge-connected. \square

To aid in the proof of the following proposition let C_3^{1-} denote $C_3^1 - e$ for $e \in E(C_3^1)$. Since C_3^1 is edge transitive, C_3^{1-} is unique up to isomorphism.

Proposition 4.23. *If a strongly 2-edge-connected 2-regular digraph contains $C_3^1 \cdot C_3^1$ as an immersion, then it contains C_5^1 or D_6 as an immersion.*

Proof. Let D be a strongly 2-edge-connected 2-regular digraph immersing $C_3^1 \cdot C_3^1$. Observe that D also immerses $C_3^{1-} \cup C_3^{1-}$. We proceed by induction on $|V(D)|$.

Let H^* be a model of $C_3^1 \cdot C_3^1$ in D where W_1 and W_2 are the branches of H^* corresponding to the underlying 2-edge-cut in $C_3^1 \cdot C_3^1$, and let A and B be the two nontrivial components of $H^* - (E(W_1) \cup E(W_2))$, each isomorphic to a subdivision of C_3^{1-} . For $i \in \{1, 2, 3\}$ let A_i and B_i denote the branches of A and B shown in Figure 4.8.

Suppose that there was a bridge X with attachments $a \in V(A)$ and $b \in V(B)$. Replace X with a chord whose attachments are a and b . There are four possible configurations of X (up to symmetry):

- | | |
|---------------------------------------|--|
| i) $a \in V(A_1)$ and $b \in V(B_1)$ | iii) $a \in V(A_3)$ and $b \in V(B_2)$ |
| ii) $a \in V(A_1)$ and $b \in V(B_2)$ | iv) $a \in V(A_3)$ and $b \in V(B_3)$ |

In configurations i) and ii) observe that C_5^1 is immersed, and in configurations iii) and iv) D_6 is immersed. Thus, we may assume that no such bridge exists.

Next, we claim that any bridge attaching on either W_1 or W_2 must be a chord whose other attachment is in either A or B . Suppose, without loss, that X was a bridge attached to W_1 . If X is a chord, contract X to a vertex x , otherwise, let $x \in V(W_1)$ be the name of this attachment vertex. By Theorem 4.22, there exists a transition $t \in T(x)$ so that D/t is strongly 2-edge-connected. Notice that $C_3^{1-} \cup C_3^{1-}$ is still immersed in D/t . Therefore, we can apply Lemma 4.20 to obtain either $C_3^1 \cdot C_3^1$ or $C_3^1 \cup C_3^1$ as an Eulerian completion of $C_3^{1-} \cup C_3^{1-}$

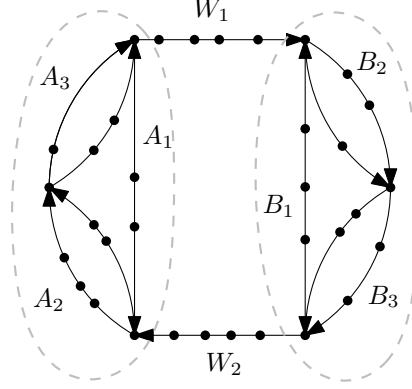


Figure 4.8: A labeled model of $C_3^1 \cdot C_3^1$.

in D/t ; however, since D/t is connected, if we obtain $C_3^1 \cup C_3^1$ as the Eulerian completion, then apply Proposition 4.21 to get a $C_3^1 \cdot C_3^1$ immersion in D/t . Then, by induction D/t contains either C_5^1 or D_6 as an immersion and thus so does D .

Call a bridge an A -bridge (B -bridge) if either all its attachments are in A (B) or it's a chord with exactly one attachment in A (B) and the other attachment on either W_1 or W_2 . Fix $i \in \{1, 2\}$ and let X be an A -bridge chord with attachments $a \in V(A)$ and $x \in V(W_i)$, and let Y be a B -bridge chord with attachments $b \in V(B)$ and $y \in V(W_i)$. We say that X and Y *cross on* W_i if when $i = 1$, y appears before x on W_1 , or when $i = 2$, x appears before y on W_2 . If there are no bridges that cross on W_1 or W_2 then there is a 2-edge cut separating A from B , contradicting the assumption. Thus, without loss, we consider the possible configurations of X and Y crossing on W_1 (up to symmetry):

- | | |
|--|---------------------------------------|
| i) $a \in V(A_1)$ and $b \in V(B_1)$ | iv) $a \in V(A_2)$ and $b \in V(B_2)$ |
| ii) $a \in V(A_1)$ and $b \in V(B_2)$ | v) $a \in V(A_2)$ and $b \in V(B_3)$ |
| iii) $a \in V(A_1)$ and $b \in V(B_3)$ | vi) $a \in V(A_3)$ and $b \in V(B_2)$ |

In all configurations D_6 is immersed. Also in configurations i), ii), and iii) C_5^1 is immersed. \square

Proposition 4.23 is stated so that the graphs C_5^1 and D_6 are the target because these (as well as $C_3^1 \cup C_3^1$ and $C_3^1 \cdot C_3^1$) are toridal obstructions. Since C_4^1 is immersed in both C_5^1 and D_6 , we obtain the following corollary for the projective plane.

Corollary 4.24. *If a 2-regular digraph is strongly 2-edge-connected and immerses either $C_3^1 \cup C_3^1$ or $C_3^1 \cdot C_3^1$, then it also immerses C_4^1 .*

4.3.2 Running into a nice graph

Any obstruction in $\text{Forb}_I(\mathbb{N}_1)$ must contain C_3^1 as an immersion, otherwise it embeds in the sphere (Theorem 2.6). We look at “extending” immersions of C_3^1 . Let D be a 2-regular digraph with H immersed in D with model H^* . We say that an H^* -bridge X *attaches on a transition* $t \in \mathcal{T}(H)$ if all attachment vertices of X lie on branches corresponding to t .

Lemma 4.25. *Let D be a strongly 2-edge-connected 2-regular digraph that immerses C_3^1 with efficient model H^* . One of the following holds:*

- i) D immerses either C_4^1 or C_5^2 or
- ii) every H^* -bridge attaches on a transition of C_3^1 and for every cycle $C \subseteq H^*$, $C \cup \mathcal{B}_C$ embeds in a disk with C as the boundary.

Proof. We may assume there exists at least one H^* -bridge, otherwise ii) is satisfied. If there exists a bridge with attachment vertices on parallel branches (branches that correspond to a pair of parallel edges of C_3^1), then it suffices to replace the bridge with a chord with these attachment vertices. This yields an immersion of C_4^1 (contract the chord). Therefore, we may now assume that no bridge attaches on parallel branches of H^* . This implies that for every H^* -bridge, there is a cycle of H^* that contains all of the attachment vertices of the bridge. All such cycles correspond to directed triangles of C_3^1 . Let C be such a cycle and let $X \in \mathcal{B}_C$. If X has attachment vertices on 3 distinct branches of C , then replacing X with a C_3 on these attachment vertices (this can be done in two distinct ways) yields a C_5^2 immersion. Therefore, we may now assume that every bridge in \mathcal{B}_C attaches on exactly two distinct branches of C .

By Lemma 4.16, either $C \cup X$ embeds in a disk with C as the boundary or one of the first three cases from the lemma occurs. In the latter case, first suppose that Case i) holds. Then X contains C_3^1 as an immersion and hence D contains $C_3^1 \cup C_3^1$ as an immersion. Since D is strongly 2-edge-connected we apply Corollary 4.24 to obtain C_4^1 as an immersion. Thus, we now assume that every bridge in \mathcal{B}_C is planar.

If Case ii) holds then X immerses C_2^1 where a pair of parallel edges of C_2^1 attach to C . It suffices to consider the various ways of attaching a copy of C_2^1 to C where each edge in a parallel class has been subdivided once (see Figure 4.7). Observe that all such configurations yield an immersion of either $C_3^1 \cdot C_3^1$ or C_4^1 . In the former case, since D is strongly 2-edge-connected we apply Corollary 4.24 to obtain C_4^1 as an immersion. Thus, we now assume that Case ii) does not hold for any bridge in \mathcal{B}_C .

Lastly, if Case iii) holds then it suffices to replace X with C_3 whose orientation agrees with C . Observe that all such configurations yield an immersion of either $C_3^1 \cdot C_3^1$ or C_4^1 . In the former case, since D is strongly 2-edge-connected we apply Corollary 4.24 to obtain C_4^1 as an immersion. Thus, we now assume that Case iii) does not hold for any bridge in \mathcal{B}_C .

This implies that for any $X \in \mathcal{B}_C$, $C \cup X$ embeds in a disk with C as the boundary. This leaves Case iv) of Lemma 4.16. Let $X, Y \in \mathcal{B}_C$ where X and Y cross with respect to C . Using Lemma 4.18 we can replace X and Y by crossing chords X' and Y' such that X' and Y' cross with respect to C . If the attachment vertices of X' and Y' lie on exactly two branches of C , then this yields a C_4^1 immersion. If X' and Y' have attachment vertices on all three branches of C , then this yields a C_5^2 immersion. Thus, we may now assume that Case iv) does not hold for any pair of bridges from \mathcal{B}_C . But this implies that $C \cup \mathcal{B}_C$ embeds in a disk with C as the boundary. As C was chosen arbitrarily ii) holds. \square

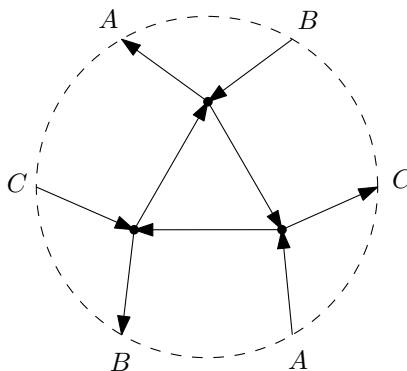


Figure 4.9: C_3^1 embedded in the projective plane.

Observe that by Euler characteristic and Proposition 1.3 any embedding of C_3^1 in the projective plane consists of exactly four triangular faces arranged as in Figure 4.9. This embedding is unique up to edge relabeling. (To see this, choose one triangular face and see that the remaining three triangular faces are forced.) Therefore, by Lemma 4.25 we obtain the following corollary.

Corollary 4.26. *A strongly 2-edge-connected 2-regular digraph either embeds in the projective plane or it contains C_4^1 or C_5^2 as an immersion.*

Lemma 4.27. *C_5^2 has a unique embedding in the projective plane.*

Proof. Let Ω be an embedding of C_5^2 in the projective plane. By Proposition 1.3 and Euler characteristic we see that $|\Omega| = 6$. Furthermore, a counting argument gives that Ω must contain either four or five faces of length 3 (every edge must appear twice faces of Ω). For reference, color the edges of C_5^2 black and grey as in Figure 4.10 and observe that up to automorphism, C_5^2 has one type of 3-cycle. It consists of two grey edges and one black edge.

Suppose that Ω had exactly four faces of length 3. Since there are only five 3-cycles in C_5^2 , there is a unique way up to automorphism of selecting these 3-cycles as faces. However, doing so forces a face in Ω that is longer than length 4. This contradicts the fact that Ω was an embedding on the projective plane. Suppose that Ω had exactly five faces of length 3. Taking each 3-cycle as a face in Ω uniquely forces the last face of length 5 in Ω . \square

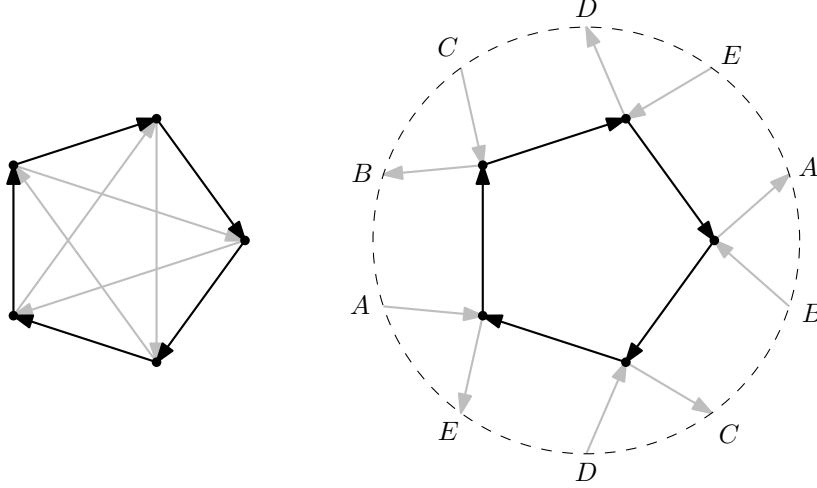


Figure 4.10: An edge coloring of C_5^2 and the unique embedding of C_5^2 in the projective plane.

Lemma 4.28. *A strongly 2-edge-connected 2-regular digraph immersing C_5^2 either embeds in the projective plane or contains C_4^1 or C_6^2 as an immersion.*

Proof. Let D be a strongly 2-edge-connected 2-regular digraph immersing C_5^2 , and let H^* be an efficient model of this immersion given by Lemma 4.17. By Lemma 4.27 let Ω be the unique embedding of H^* in \mathbb{N}_1 . We may assume that there exists at least one H^* -bridge, otherwise D is isomorphic to C_5^2 and thus projective planar. We consider the various attachment points for bridges. For reference, color the edges of C_5^2 black and grey as in Figure 4.10 and let E_B be the set of branches of H^* corresponding to the black edges and E_G be the set of branches of H^* corresponding to the grey edges. Observe that there is an automorphism of order 5 that permutes the edges of C_5^2 contained in each color class.

We first consider when an H^* -bridge attaches on branches that do not have a face in common in Ω . Let X be such an bridge. If X attaches to a branch B in E_B then there are three branches in E_G that are not cofacial with B . If X also attaches to any of these three branches, then in all three cases it suffices to replace X with a chord on these attachment vertices as this yields a C_4^1 immersion. Therefore, if X attaches on two distinct branches that are not cofacial in Ω , we may assume they are two branches from E_G . Suppose that X attaches to a branch B in E_G . There are two branches in E_G that are not cofacial with B . If X attaches to either of these, then in both cases it suffices to replace X with a chord on these attachment vertices as this yields a C_6^2 immersion. Therefore, we may assume that every pair of branches that X attaches on are cofacial.

Suppose that X attached on two branches B and B' that share a face, but still not all branches that X attached on were cofacial. As we have considered all pairs of branches that do not share a face, there must exist another bridge B'' that X attaches on where B'' shares a face with B and one with B' but there is no face containing all three. We claim there is

only one such configuration, up to symmetry, where without loss B and B' are nonadjacent branches (branches that do not share a branch endpoint) from E_B and $B'' \in E_G$.

To prove the claim, suppose that B and B' are not nonadjacent branches from E_B . This means that since B and B' are cofacial, they are either adjacent branches of E_B or they are branches corresponding to a triangle of C_5^2 where one is in E_B and the other in E_G , or both are in E_G . Observe that in all cases there is no suitable choice for B'' .

Thus, consider B and B' as nonadjacent branches from E_B . There is one choice for B'' in E_G up to symmetry. It suffices to replace X with a C_3 (in both possible ways) attached to these same branches. Observe that both orientations of C_3 yield C_6^2 as an immersion in D . Therefore, we may now assume that every H^* -bridge attaches on branches that have a common face.

Up to automorphism, there are two types of faces in Ω . There is the 5-cycle from C_5^2 consisting of branches from E_B , and there is a 3-cycle from C_5^2 consisting of branches from both E_B and E_G . Let F_5 denote the 5-cycle face and let F_3 be one of the 3-cycle faces. Applying Lemma 4.16 to F_5 we get that either $F_5 \cup \mathcal{B}_{F_5}$ embeds in a disk with F_5 as the boundary or one of i), ii), iii), iv) from the lemma occurs. A similar conclusion holds for F_3 . In both cases, if i), ii), or iii) occurs, then by splitting the branch vertices of H^* , we can move to a model of C_3^1 immersed in D without affecting any H^* -bridges. By the proof of Lemma 4.25, we obtain C_4^1 as an immersion. Therefore, we consider what happens when iv) occurs. By Lemma 4.18 this amounts to considering all configurations of crossing chords in \mathcal{B}_{F_5} and \mathcal{B}_{F_3} where each chord has its endpoints on distinct branches.

Observe that reversing the direction of all edges of C_5^2 yields an isomorphic graph. Therefore, to cut down on the case analysis we consider the chord crossings up to this additional symmetry. There are 11 such cases: 7 of them are crossing chords in \mathcal{B}_{F_5} , and the remaining 4 are crossing chords in \mathcal{B}_{F_3} . These configurations are shown in Figure 4.11. All such configurations yield either an immersion of C_4^1 or C_6^2 . \square

The proof of Theorem 4.1 is below. It follows a similar format to that of Theorem 2.6 in that we first argue that C_4^1 , C_6^2 , $C_3^1 \cdot C_3^1$, $C_3^1 \cup C_3^1$ are indeed obstructions for the projective plane, then we prove that an arbitrary 2-regular digraph either embeds in the projective plane or has one of these four obstructions as an immersion.

4.3.3 Proof of Theorem 4.1

Proof of Theorem 4.1. By Propositions 4.6 and 4.9, we have that $C_4^1 \in \text{Forb}_I(\mathbb{N}_1)$ and $C_6^2 \in \text{Forb}_I(\mathbb{N}_1)$. Next, we have $C_3^1 \cdot C_3^1$ who also has a feedback edge set of size 3, thus apply Proposition 1.1. Then notice that every vertex split of $C_3^1 \cdot C_3^1$, after contracting digons, results in a graph isomorphic to C_3^1 . Lastly, $C_3^1 \cup C_3^1$ is not projective planar because every embedding of C_3^1 in \mathbb{N}_1 is a 2-cell embedding. Thus, after embedding the first copy of C_3^1 , we would require that the second C_3^1 embeds in a disk, but C_3^1 is not planar by

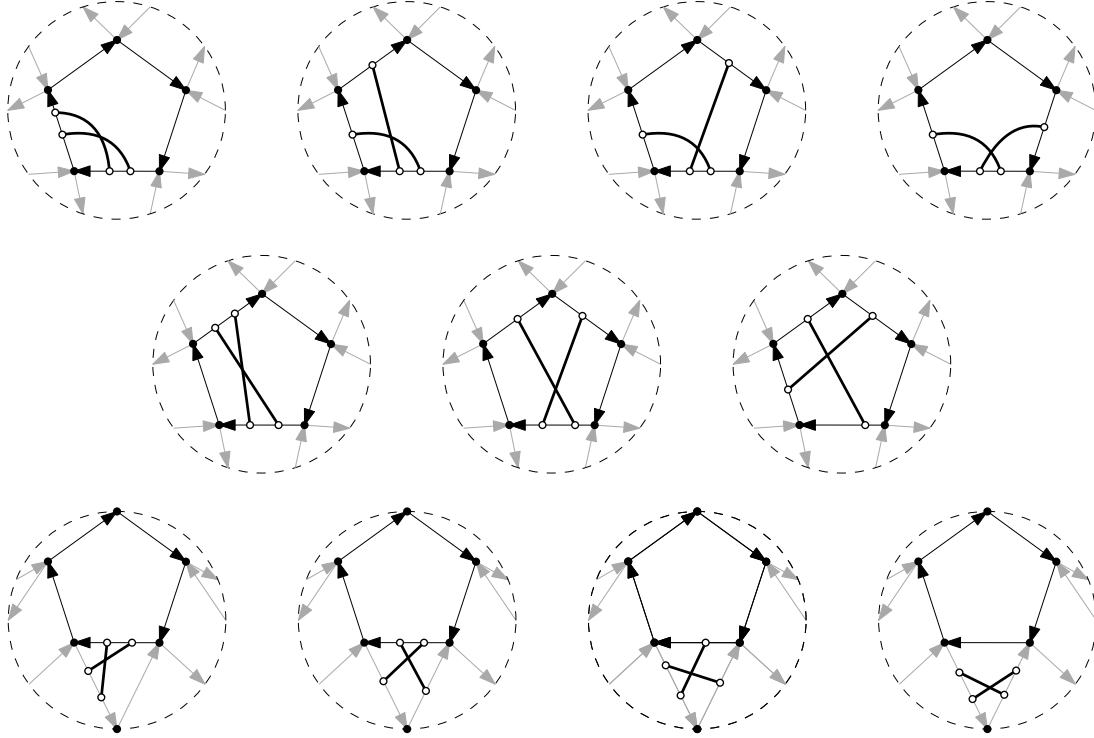


Figure 4.11: Distinct pairs of crossing chords in C_5^2 .

Theorem 2.6. See that it is minimal since any split of $C_3^1 \cup C_3^1$ gives $C_3^1 \cup H$ where H is planar. Therefore, $C_3^1 \cup H$ is projective planar.

Now we show that an arbitrary 2-regular digraph $D = (V, E)$ either embeds in the projective plane, or D contains one of the obstructions as an immersion. If D is not connected, consider two distinct components D_1 and D_2 of D . Suppose without loss that D_1 did not embed in \mathbb{N}_1 . Then it suffices to only consider D_1 , because an obstruction immersed in D_1 is also immersed in D . Thus, we may assume that both D_1 and D_2 embed in \mathbb{N}_1 . Now suppose without loss that D_1 was planar. Then embedding D_2 in \mathbb{N}_1 and embedding D_1 in a face of the embedding of D_2 yields an embedding of D . Thus it must be the case that both D_1 and D_2 are nonplanar and thus D immerses $C_3^1 \cup C_3^1$.

If D is connected but not strongly 2-edge-connected, then consider $X \subseteq V$ with $d^+(X) = d^-(X) = 1$ and let $Y = V \setminus X$. Recall the construction of D_X and D_Y . Suppose without loss that D_X did not embed in \mathbb{N}_1 . Then as before, it suffices to only consider D_X , because an obstruction immersed in D_X is also immersed in D . Thus, we may assume that both D_X and D_Y embed in \mathbb{N}_1 . Now suppose without loss that D_X was planar. In this case, let Ω_Y be a 2-cell embedding of D_Y in \mathbb{N}_1 with face $F \subseteq \mathbb{N}_1$ whose boundary contains the edge from $E(D_Y) \setminus E$. Take an embedding Ω_X of D_X in the plane with the edge from $E(D_X) \setminus E$ on the other face. Embed D_X in F with Ω_X and reconstruct D to obtain an embedding of D in \mathbb{N}_1 . Thus, we may assume that both D_X and D_Y are nonplanar. In this case, both contain C_3^1 as an immersion. But this implies that D contains $C_3^1 \cup C_3^1$ as an immersion.

Now, suppose that D is strongly 2-edge-connected. If D does not contain C_3^1 as an immersion, then D embeds in the projective plane. Otherwise, apply Lemma 4.25. Therefore, either D contains C_4^1 or C_5^2 as an immersion or embeds in the projective plane. If D immerses C_5^2 then apply Lemma 4.28 to obtain the result. \square

4.4 The torus and Klein bottle

In this section we discuss computational results that culminate in the partial classification of obstructions for Euler genus 2 surfaces (Propositions 4.2 and 4.3):

$$\begin{aligned} \text{Forb}_I(\mathbb{S}_1) &\supseteq \{C_5^1, D_6, C_7^2, D_{10}, C_3^1 \cdot C_3^1, C_3^1 \cup C_3^1\} \\ \text{Forb}_I(\mathbb{N}_2) &\supseteq \{C_5^1, C_7^2, \underline{C_6^2}, D_7, D_9\} \cup (C_3^1 \cdot \text{Forb}_I(\mathbb{N}_1)) \cup (C_3^1 \cup \text{Forb}_I(\mathbb{N}_1)) \end{aligned}$$

We also provide a human readable proofs of the above propositions.

4.4.1 Computational results

The first stage in computation was exhaustive 2-regular digraph generation. Let \mathcal{D}_n denote the set of connected 2-regular digraphs with order $\leq n$. Given a connected 2-regular digraph D with n vertices, we can create a 2-regular digraph with $n + 1$ vertices, that contains D as an immersion, by attaching a single chord on edges of D and contracting this chord. If this is done in all possible ways, then we obtain all graphs of order $n + 1$ containing D as an immersion by Proposition 1.2. Starting this procedure with the pointless edge, we can generate \mathcal{D}_n (see Algorithm 1).

Algorithm 1 2-Regular Digraph Generation

```

1: procedure GENERATE( $n$ )
2:    $\mathcal{H}_0 \leftarrow \{D_0\}$  ▷ Initialize  $\mathcal{H}_0$  as a set containing  $D_0$ , the pointless edge.
3:   for  $1 \leq k \leq n$  do
4:      $\mathcal{H}_k \leftarrow \{ \}$  ▷ Initialize  $\mathcal{H}_k$  as an empty set.
5:     for  $D \in \mathcal{H}_{k-1}$  do
6:       for  $e, f \in E(D)$  do ▷  $e$  and  $f$  may be the same edge.
7:          $H \leftarrow \text{D.COPY}( )$ 
8:          $x, y \leftarrow \text{H.SUBDIVIDE}(e, f)$  ▷ Subdivide  $e$  and  $f$  with vertices  $x$  and  $y$ .
9:          $\text{H.IDENTIFY}(x, y)$  ▷ Vertex identification.
10:         $\mathcal{H}_k.\text{ADD}(H)$ 
11:      end for
12:    end for
13:  end for
14:  return  $\mathcal{D}_n \leftarrow \mathcal{H}_0 \cup \mathcal{H}_1 \cup \dots \cup \mathcal{H}_n$ 
15: end procedure

```

Before the combinatorial framework that is used in the thesis was formalized (treating 2-cell embeddings as a collection of closed walks where each transition appears exactly once) we used the more standard combinatorial embedding description, the rotation system (see [26] for details). Let D be a 2-regular digraph and let $v \in V(D)$. Observe that there are only two distinct rotation systems at v , since edges incident to v must alternate in-out-in-out. Therefore, generating all possible rotation schemes and using the standard face tracing procedure (see [26]), we were able to exhaustively check \mathcal{D}_{10} for members of $\text{Forb}_I(\mathbb{S}_1)$ and \mathcal{D}_9 for members of $\text{Forb}_I(\mathbb{N}_2)$.

4.4.2 Proofs for humans

For the proof of Proposition 4.2, we rely heavily on the fact that \mathbb{S}_1 is an orientable surface. Recall, that if D is a 2-regular digraph with an embedding Ω in an orientable surface, then there exists a bipartition $\{\mathcal{A}, \mathcal{B}\}$ of Ω where every edge $e \in E(D)$ appears exactly once in both $\bigcup_{F \in \mathcal{A}} E(F)$ and $\bigcup_{F \in \mathcal{B}} E(F)$. Moreover, notice that given say \mathcal{A} , the faces of \mathcal{B} are uniquely determined since every transition in $\mathcal{T}(D)$ is contained in exactly one face of Ω . We refer to this as *extending* \mathcal{A} to Ω .

Given an integer vector $\vec{v} = (n_1, n_2, \dots, n_k)$, we say that \vec{v} is a *configuration* of D if for $1 \leq i < k$, $1 \leq n_i \leq n_{i+1}$ and $\sum_{1 \leq i \leq k} n_i = |E(D)|$. We say that \vec{v} is *realizable* if there exists closed walks $F_1, F_2, \dots, F_k \subseteq D$ such that $\text{len}(F_i) = n_i$ for all $1 \leq i \leq k$ and each edge $e \in E(D)$ appears exactly once in $\bigcup_{1 \leq i \leq k} E(F_i)$ and we say that (F_1, F_2, \dots, F_k) is a *realization* of \vec{v} . A realization of \vec{v} forms one side of a bipartition of an orientable 2-cell embedding of D .

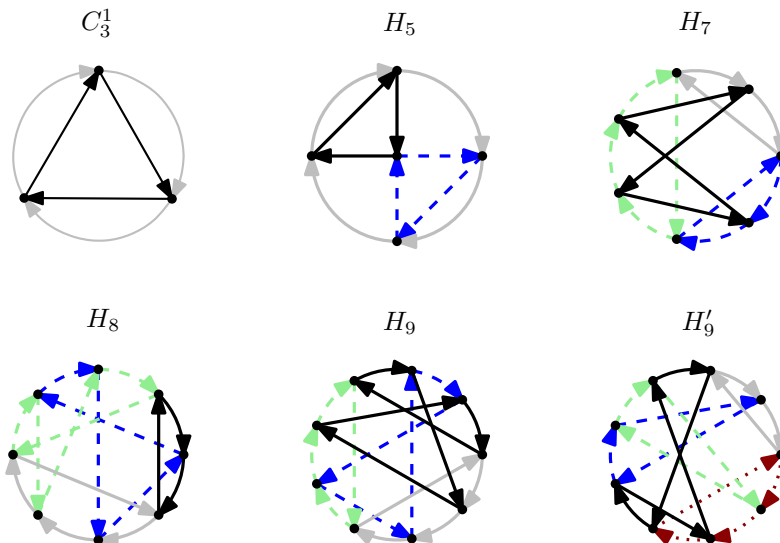


Figure 4.12: Each color class represents one face from one part of the bipartition of the embedding in \mathbb{S}_1 .

To cut down on case analysis in the proofs below, we talk about “essentially unique” realizations of configurations. By this, we mean realizations up to automorphism of D as well as realizations that take into account the action of reversing all edge directions of D , if such an action yields a digraph isomorphic to D .

Proof of Proposition 4.2. By Propositions 4.6 and 4.9, $C_5^1 \in \text{Forb}_I(\mathbb{S}_1)$ and $C_7^2 \in \text{Forb}_I(\mathbb{S}_1)$. Next, consider D_6 . Suppose towards a contradiction that Ω was an embedding of D_6 in \mathbb{S}_1 with bipartition $\{\mathcal{A}, \mathcal{B}\}$, each part containing a copy of $E(D_6)$. By Euler genus, $|\Omega| = 6$; therefore, we may assume that $|\mathcal{A}| = |\mathcal{B}| = 3$ since D_6 does not contain four edge-disjoint triangles or digons. By edge count, the possible realizable configurations of D_6 are $(3, 3, 6)$ and $(3, 4, 5)$. Each vector is essentially uniquely realizable as \mathcal{A} . In both cases, extending \mathcal{A} to Ω yields $|\Omega| < 6$. To see that D_6 is minimal, let $t \in \mathcal{T}(D_6)$ and observe that after contracting digons (Corollary 4.11) D_6/t is isomorphic to either C_3^1 or H_5 . Both of these graphs embed in \mathbb{S}_1 (see Figure 4.12).

Next, consider D_{10} . Suppose towards a contradiction that Ω was an embedding of D_{10} in \mathbb{S}_1 with bipartition $\{\mathcal{A}, \mathcal{B}\}$. By Euler genus, $|\Omega| = 10$; therefore, we may assume that either $|\mathcal{A}| = 6$ and $|\mathcal{B}| = 4$ or $|\mathcal{A}| = |\mathcal{B}| = 5$ since D_{10} does not contain any digons. First, suppose that $|\mathcal{A}| = 6$. By edge count, the possible realizable configurations of D_{10} are $(3, 3, 3, 3, 3, 5)$ and $(3, 3, 3, 3, 4, 4)$. The first vector is essentially uniquely realizable while the second is not realizable. Taking the first and extending \mathcal{A} to Ω yields $|\Omega| < 10$. Now, suppose that $|\mathcal{A}| = |\mathcal{B}| = 5$. Once again, by edge count, the possible realizable configurations of D_{10} are $(3, 3, 3, 3, 8)$, $(3, 3, 3, 5, 6)$, $(3, 3, 3, 4, 7)$, $(3, 3, 4, 5, 5)$, and $(3, 3, 4, 4, 6)$. The configuration $(3, 3, 4, 5, 5)$ is not realizable and all others are essentially uniquely realizable as \mathcal{A} . However, in all cases extending \mathcal{A} to Ω yields $|\Omega| < 10$. To see that D_{10} is minimal, let $t \in \mathcal{T}(D_{10})$ and observe that after contracting digons D_{10}/t is isomorphic to one of H_7 , H_8 , H_9 , or H_9' all of which embed in \mathbb{S}_1 (see Figure 4.12).

Next, consider $C_3^1 \cdot C_3^1$. Suppose towards a contradiction that Ω was an embedding of $C_3^1 \cdot C_3^1$ in \mathbb{S}_1 with bipartition $\{\mathcal{A}, \mathcal{B}\}$. By Euler genus, $|\Omega| = 6$; therefore, we may assume that $|\mathcal{A}| = |\mathcal{B}| = 3$ since $C_3^1 \cdot C_3^1$ contains only two edge-disjoint triangles and no digons. By edge count, the possible realizable configurations of $C_3^1 \cdot C_3^1$ are $(3, 3, 6)$ and $(3, 4, 5)$. Since $C_3^1 \cdot C_3^1$ does not contain 4-cycles, $(3, 4, 5)$ is not realizable and $(3, 3, 6)$ is essentially uniquely realizable as \mathcal{A} . However, extending \mathcal{A} to Ω yields $|\Omega| < 6$. To see that $C_3^1 \cdot C_3^1$ is minimal, let $t \in \mathcal{T}(C_3^1 \cdot C_3^1)$ and observe that after contracting digons, $C_3^1 \cdot C_3^1/t$ is isomorphic to C_3^1 which embeds in \mathbb{S}_1 (see Figure 4.12).

Lastly, consider $C_3^1 \cup C_3^1$. By the proof of Proposition 1.3, every embedding of C_3^1 in \mathbb{S}_1 must be 2-cell; otherwise, the capping operation would result in C_3^1 embedded in \mathbb{S}_0 contradicting Theorem 2.6. This implies that $C_3^1 \cup C_3^1$ has no embedding in \mathbb{S}_1 . To see that $C_3^1 \cup C_3^1$ is minimal, let $t \in \mathcal{T}(C_3^1 \cup C_3^1)$ and observe that $C_3^1 \cdot C_3^1/t$ is isomorphic to $C_3^1 \cup H$ where H is planar. \square

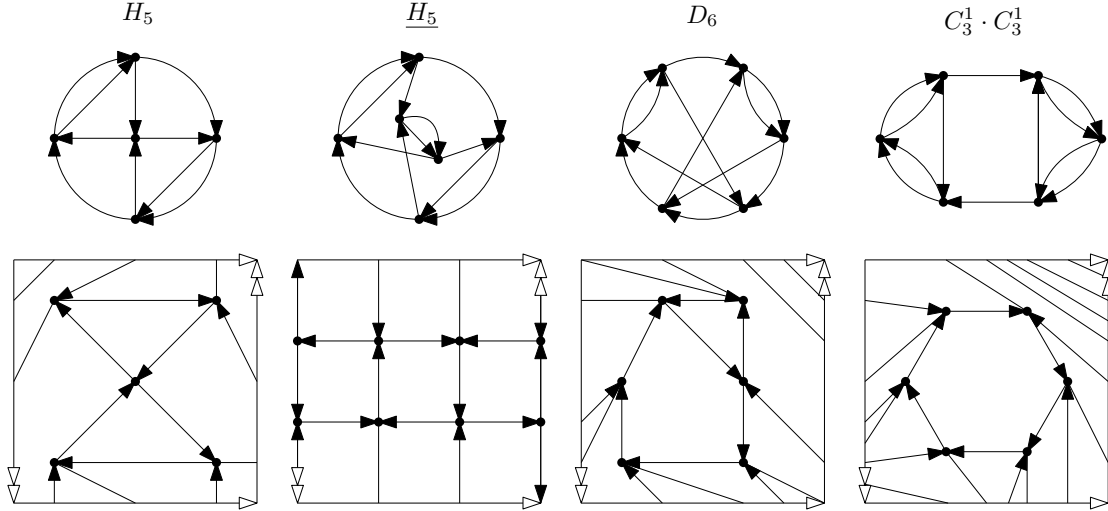


Figure 4.13: Embeddings of graphs in the Klein bottle.

Proof of Proposition 4.3. By Propositions 4.6 and 4.9, $C_5^1 \in \text{Forb}_I(\mathbb{N}_2)$ and $C_7^2 \in \text{Forb}_I(\mathbb{N}_2)$. Next, consider D_7 . Observe that a minimum feedback edge set of D_7 has size 3. Therefore, $\text{eg}(D_7) \geq 3$ by Proposition 2.7. To see that D_7 is minimal, let $t \in \mathcal{T}(D_7)$ and observe that after contracting digons D_7/t is isomorphic to one of C_4^1 , H_5 , D_6 or \underline{H}_5 . All of these graphs embed in \mathbb{N}_2 by Lemma 4.5 and Figure 4.13.

Next, consider \underline{C}_6^2 . Observe that a minimum feedback edge set of \underline{C}_6^2 has size 3. Therefore, $\text{eg}(\underline{C}_6^2) \geq 3$ by Proposition 2.7. To see that \underline{C}_6^2 is minimal, let $t \in \mathcal{T}(\underline{C}_6^2)$ and observe that after contracting digons \underline{C}_6^2/t is isomorphic to one of C_4^1 , H_5 , D_6 , \underline{C}_6^2 , or \underline{C}_5^2 . All of these graphs embed in \mathbb{N}_2 by Lemmas 4.5, 4.7, and 4.8 and Figure 4.13.

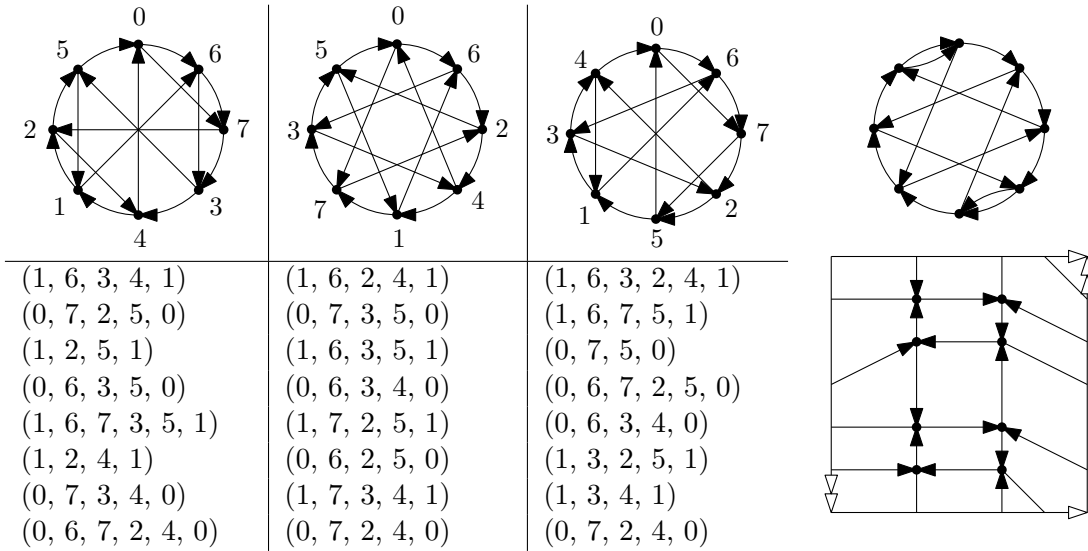


Figure 4.14: On the left, each column represents an embedding of the graph in the Klein bottle, where each row is a facial walk given by its vertex sequence.

Next, consider D_9 . Observe that a minimum feedback edge set of D_9 has size 4. Therefore, $\text{eg}(D_9) \geq 3$ by Proposition 2.7. To see that D_9 is minimal, let $t \in \mathcal{T}(D_9)$ and observe that after contracting digons D_9/t is isomorphic to one of four graphs, all of which embed in \mathbb{N}_2 (see Figure 4.14).

To see that every graph in $C_3^1 \cdot \text{Forb}_I(\mathbb{N}_1)$ is an obstruction for the Klein bottle, observe that by Lemmas 4.5, 4.7 and Figure 4.13, each connected graph $D \in \text{Forb}_I(\mathbb{N}_1)$ has an embedding in the Klein bottle, and apply Proposition 4.14. Similarly, every graph in $C_3^1 \cup \text{Forb}_I(\mathbb{N}_1)$ is an obstruction for the Klein bottle since every embedding of $D \in \text{Forb}_I(\mathbb{N}_1)$ in \mathbb{N}_2 is a 2-cell embedding by Proposition 1.3. \square

Chapter 5

Negami's Conjecture

This chapter discusses a conjecture, by Negami, on undirected graphs. We give a brief overview of the problem and its current state but refer the reader to an excellent survey by Hliněný [17] to fill in the gaps in this exposition.

A graph H is a *cover* of a graph G if there exists a pair of onto mappings (ϕ, ψ) , $\phi : V(H) \rightarrow V(G)$ and $\psi : E(H) \rightarrow E(G)$, such that ψ maps the edges incident to each vertex $v \in V(H)$ bijectively onto the edges incident to $\phi(v)$. If H and G are directed graphs, then for $uv \in E(H)$ we also insist that $\psi(uv) = \phi(u)\phi(v)$; i.e. ψ preserves edge direction.

Given a map $f : X \rightarrow Y$ we call $f^{-1}(y)$ the *fiber* of f over $y \in Y$. Observe that for all $v \in V(G)$, $\deg_H(\phi^{-1}(v)) = \deg_G(v)$. If G is connected then all fibers $\phi^{-1}(v)$, have a fixed size k , and we say that H is a k -fold cover. When $k = 2$ we say that H is a *double cover* of G . In 1986 Negami proved the following result about undirected graphs.

Theorem 5.1 (Negami [28]). *A connected undirected graph G has a double planar cover if and only if G embeds in the projective plane.*

Two years later Negami extended this result for a type of cover called a regular cover. A cover H of G with maps (ϕ, ψ) is *regular* if there is a subgroup A of the automorphism group of H such that for $u, v \in V(H)$, $\phi(u) = \phi(v)$ if and only if there exists some $\tau \in A$ such that $\tau(u) = v$.

Theorem 5.2 (Negami [31]). *A connected undirected graph G has a finite regular planar cover if and only if G embeds in the projective plane.*

In this same paper Negami made the conjecture that the regular condition could be relaxed.

Conjecture 5.3 (Negami [31]). *A connected undirected graph G has a finite planar cover if and only if G embeds in the projective plane.*

By Theorem 5.1 (or by the fact that the sphere is a double cover of the projective plane), the “if” direction of Conjecture 5.3 is true; however, the “only if” direction has

eluded capture despite years of research and a multitude of papers. Combining results from 1988 to 2002 we arrive at the following theorem.

Theorem 5.4 (Archdeacon, Fellows, Hliněný, Hueneke, Negami [4, 11, 16, 18, 29, 31]). *If $K_{1,2,2,2}$ has no finite planar cover, then Conjecture 5.3 is true.*

In this chapter we prove an analogue of Conjecture 5.3 in the setting of 2-regular digraphs on surfaces. The main theorem is the following:

Theorem 5.5. *A connected 2-regular digraph D has a finite planar cover if and only if D has an embedding in the projective plane.*

5.1 Projective plane obstructions have no finite planar covers

One method of attacking Conjecture 5.3, is to take the classified list of minor-obstructions for the projective plane [5, 14] and show that each graph in the list has no finite planar cover. Our proof of Theorem 5.5 uses a similar technique. We lean on the classification theorem for 2-regular digraph projective plane obstructions from Section 4.3 and show that each connected graph appearing in the list has no finite planar cover.

Recall that $\text{Forb}_I(\mathbb{N}_1) = \{C_4^1, C_6^2, C_3^1 \cdot C_3^1, C_3^1 \cup C_3^1\}$.

5.1.1 Euler's formula

Lemma 5.6. C_4^1 has no finite planar cover.

Proof. Suppose for a contradiction that $H = (V, E)$ was a finite planar cover of C_4^1 . Let Ω be a planar embedding of H and let $v = |V|$, $e = |E|$, and $f = |\Omega|$. Recall that faces of Ω are directed closed walks and observe that each face in Ω is a walk of size at least 4. Therefore, we have that $2e \geq 4f$. Thus, using Euler's formula we obtain $4v - 2e \geq 8$. But by the handshake lemma, $2e = 4v$ which is a contradiction. \square

5.1.2 Combinatorial curvature

For the next result, we extend the idea of Euler's characteristic formula with the concept of combinatorial curvature as defined by Ishida [19]. Given a 2-regular digraph $D = (V, E)$ and an embedding Ω , let $v \in V$ and let $F \in \Omega$. If v appears in the walk F then we say that F and v are *incident* and write $F \sim v$. Let $\#\text{inc}(v, F)$ denote the number of times v is incident to F . The *combinatorial curvature* is the function $\Phi : V \rightarrow \mathbb{R}$ where for every $v \in V$:

$$\Phi(v) = 1 - \frac{\deg(v)}{2} + \sum_{F \in \Omega} \frac{\#\text{inc}(v, F)}{\text{len}(F)}.$$

Summing the combinatorial curvature over all vertices of D yields the Euler characteristic of Ω :

$$\Phi(\Omega) = \sum_{v \in V} \Phi(v) = |V| - |E| + |\Omega| = \chi(\Omega).$$

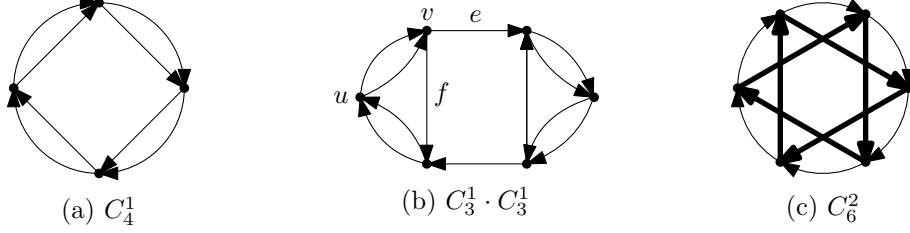


Figure 5.1: The connected obstructions for the projective plane.

Lemma 5.7. $C_3^1 \cdot C_3^1$ has no finite planar cover.

Proof. Suppose for a contradiction that $H = (V, E)$ is a finite planar cover of $C_3^1 \cdot C_3^1$ with planar embedding Ω . First observe that $C_3^1 \cdot C_3^1$ has no closed walks of length 2, 4, or 5. Next, observe that $C_3^1 \cdot C_3^1$ has two types of vertices up to symmetry:

Type 1: $v \in V(C_3^1 \cdot C_3^1)$ is incident to two pairs of parallel edges,

Type 2: $v \in V(C_3^1 \cdot C_3^1)$ is incident to a single pair of parallel edges.

When $v \in V$ and v is mapped to a Type 1 (Type 2) vertex, we say that v is also a Type 1 (Type 2) vertex. We now consider the vertices of H and show that they all have non-positive combinatorial curvature contradicting the fact that Ω was an embedding in the plane.

Let $v' \in V$ be a Type 2 vertex, where without loss it is mapped to the vertex labeled v in Figure 5.1b. Every face in Ω that contains an edge from the fiber over e (where e is labeled in Figure 5.1b) has length ≥ 6 by inspection. Since v' is incident to such an edge, at least two of its face incidences come from faces with length ≥ 6 . Taking the remaining two incidences to come from the shortest possible faces (triangles) we maximize $\Phi(v')$ and obtain $\Phi(v') \leq -1 + \frac{1}{3} + \frac{1}{3} + \frac{1}{6} + \frac{1}{6} = 0$. Hence, all Type 2 vertices have non-positive combinatorial curvature.

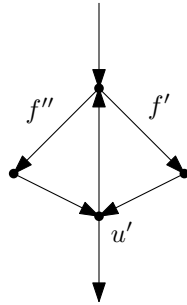


Figure 5.2: An impossible configuration of a Type 1 vertex in H .

Let $u' \in V$ be a Type 1 vertex, where without loss it is mapped to the vertex labeled u in Figure 5.1b. We claim that there are at most two face incidences with u' and faces of length 3. To see this, observe that any triangular face incident to u' contains an edge in the fiber over f (where f is labeled in Figure 5.1b). Suppose that $f', f'' \in E$ are both in

the fiber over f and both contained in triangles with u' . It must be the case that $f' = f''$ in H , else the heads of f' and f'' are distinct, contradicting the fact that a cover is locally bijective (see Figure 5.2). Thus, we have at most two triangular face incidences to u' since each edge (f') is contained in at most 2 faces. Taking the remaining two incidences with u' to come from the shortest possible faces (length ≥ 6) we maximize $\Phi(u')$ and obtain $\Phi(u) \leq -1 + \frac{1}{3} + \frac{1}{3} + \frac{1}{6} + \frac{1}{6} = 0$. \square

5.1.3 Discharging

We extend the idea of Euler's characteristic yet again by way of discharging. However, instead of encoding the information on the vertices, as was done with combinatorial curvature, we now encode the information on the faces of the proposed embedding. More formally, for a 2-regular digraph D and a 2-cell embedding Ω , we define the *charge* of $F \in \Omega$ as

$$\text{charge}(F) = 30(4 - \text{len}(F)).$$

If you sum over all the faces in Ω you obtain the *charge* of Ω

$$\text{charge}(\Omega) = \sum_{F \in \Omega} \text{charge}(F) = 120 \cdot \chi.$$

Our strategy for proving that C_6^2 does not have a finite planar cover is the same as with $C_3^1 \cdot C_3^1$; we will take a cover H and an embedding and show that every face has non-positive charge.

Lemma 5.8. C_6^2 has no finite planar cover.

Proof. We start with some observations about C_6^2 . First, note that the drawing in Figure 5.1c has bold edges and non-bold edges. Next, note that C_6^2 has exactly two directed triangles, comprised of the bold edges. Finally, note that every vertex of C_6^2 is incident with exactly one bold in-edge and one bold out-edge. This bold-edge-coloring extends to any cover of C_6^2 .

Now suppose towards a contradiction, that H was a finite planar cover of C_6^2 with embedding Ω of H . We say that two faces $F, G \in \Omega$ are *adjacent* if $F \neq G$ and F shares a vertex or an edge with G . We refer to these respectively as *vertex-adjacent* or *edge-adjacent* faces, and we denote face adjacency by $F \sim G$.

Let $F \in \Omega$ and notice that $\text{charge}(F) > 0$ if and only if $\text{len}(F) = 3$ (since C_6^2 has no faces of length 2). Thus, our discharging rule is simple, every triangular face sends its charge to its adjacent faces that are "long enough". Before we state the discharging rule explicitly we note that, up to automorphism, C_6^2 has one type of triangular face.

Let $\text{charge}'(F)$ represent the charge on face F after one step of discharging. Let \mathcal{T} be the set of triangular faces in Ω . For $\tau \in \mathcal{T}$ define \mathcal{F}_τ be the set of faces adjacent to τ with length ≥ 5 . Our discharging rule is the following:

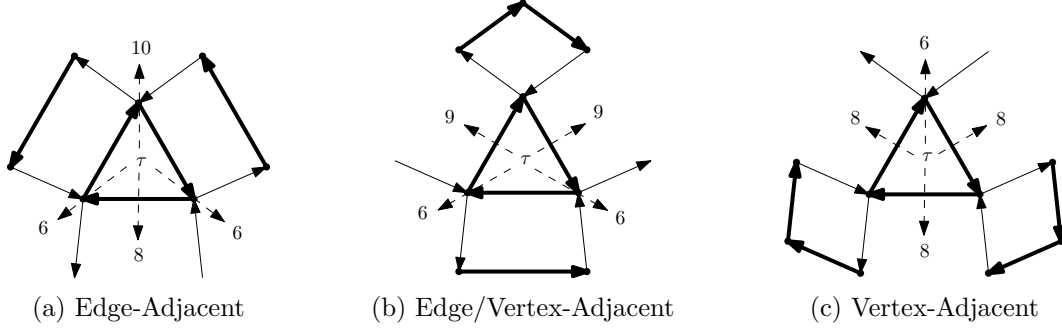


Figure 5.3: Discharging rules for the various configurations of $n_4(\tau) = 2$.

- For all $\tau \in \mathcal{T}$ and $c_i \in \mathbb{Z}^+$

$$\text{charge}'(\tau) = 30 - \sum_{F_i \in \mathcal{F}_\tau} c_i \quad \text{where} \quad \sum_{F_i \in \mathcal{F}_\tau} c_i = 30$$

- and for all $F_i \in \mathcal{F}_\tau$

$$\text{charge}'(F_i) = \text{charge}(F_i) + c_i.$$

We've left the distribution of charge to \mathcal{F}_τ vague, because we will distribute differently based on the number of faces in \mathcal{F}_τ where $\text{len}(F_i) = 4$. Before we nail down these charges, we define another term; for an arbitrary face F , let $n_k(F)$ denote the number of faces of length k adjacent to F . We have the following distribution of charges to \mathcal{F}_τ :

- If $n_4(\tau) = \{0, 1, 3\}$, then distribute equally to $F_i \in \mathcal{F}_\tau$:

$$\text{charge}'(F_i) = \text{charge}(F_i) + \frac{30}{6 - n_4(\tau)}$$

- If $n_4(\tau) = 2$, then we say that τ is either Type Edge-Adjacent, Type Edge/Vertex-Adjacent, or Type Vertex-Adjacent as depicted in Figure 5.3, where c_i is written explicitly for each $F_i \in \mathcal{F}_\tau$ in Figure 5.3.

We claim that this discharging rule is well defined (i.e. for all $\tau \in \mathcal{T}$, all possible configurations of adjacent faces have been considered), that $n_4(\tau) \leq 3$, and if a closed walk of length 4 is adjacent to τ , then we know whether or not it is edge-adjacent or vertex-adjacent to τ .

Proof of claim. First notice that no other directed triangle may be adjacent to τ . This can be seen since the only two directed triangles in C_6^2 share no vertices or edges, so triangular faces cannot share vertices or edges in Ω .

Let W be a closed walk of length 4 in C_6^2 , note that W is a cycle and that, up to automorphism, there are only two types of 4-cycles in C_6^2 and they both have exactly 2 bold edges (see Figure 5.1c):

Type 1: W has two adjacent bold edges,

Type 2: W has two nonadjacent bold edges.

When a face $F \in \Omega$ of length 4 is mapped to a Type 1 (Type 2) 4-cycle in C_6^2 then we say that F is also of Type 1 (Type 2). Now, using the fact that every vertex of H has exactly one bold in-edge and one bold out-edge, we can see that if F is adjacent to τ and Type 1 (Type 2), then it must be vertex-adjacent (edge-adjacent). Next, let F be a Type 2 face (not necessarily adjacent to a triangular face) and consider the following: F cannot have short faces ($\text{len} < 5$) edge-adjacent to its non-bold edges. This is the case since, once again, all vertices of H have exactly one bold in and out edge and triangles are solely comprised of bold edges and there are no 4-cycles with 3 non-bold edges. Using these facts, we conclude that a triangular face, τ , has $n_4(\tau) \leq 3$ which implies the claim. \square

Note that after discharging every triangular face has zero charge. Next, we show that any face with length ≥ 6 does not have positive charge. Let $\gamma \in \Omega$ with $\text{len}(\gamma) \geq 6$. As stated above, no triangle face may be adjacent to another triangle face. Therefore, there are at most $\text{len}(\gamma)$ many triangles adjacent to γ . However, the maximum amount of charge that a triangle can send to γ is 10 (either Figure 5.3a or when $n_4(\tau) = 3$); therefore, γ gains at most a charge of $10 \text{len}(\gamma)$. But

$$\begin{aligned} \text{charge}'(\gamma) &\leq \text{charge}(\gamma) + 10 \text{len}(\gamma) \\ &\leq 30(4 - \text{len}(\gamma)) + 10 \text{len}(\gamma) \\ &\leq 0. \end{aligned}$$

Thus, we have shown that after discharging, if Ω has any positive charge, then it must be on faces of length 5.

Let $\phi \in \Omega$ be a face of length 5. If $n_3(\phi) < 4$ then by the above argument there is no positive charge on ϕ ; therefore, let us assume that $n_3(\phi) \geq 4$. Notice that in C_6^2 there is only one closed walk of length 5, up to automorphism, and it is a cycle that uses exactly one bold edge. This forces the number of adjacent triangular faces to ϕ to be at most four. Therefore, we may assume that $n_3(\phi) = 4$ with adjacent faces as configured in Figure 5.4.

First note that F_2 and F_6 have length ≥ 5 since they are adjacent to triangle F_1 , and they cannot complete to 4-cycles of Types 1 or 2. This implies that $n_4(F_i) \leq 2$ for $i \in \{3, 5\}$. If $n_4(F_i) = 1$ then F_i sends 6 charge to ϕ , and if $n_4(F_i) = 2$, then as per the discharging rules in Figure 5.3, F_i sends ≤ 6 charge to ϕ . Next consider, F_4 which has no restriction and therefore sends ≤ 10 charge to ϕ . Lastly, observe that $n_4(F_1) \leq 2$ since ϕ , F_2 , and F_6 are all of length ≥ 5 . Therefore, F_1 sends ≤ 8 charge to ϕ (it can be of type Edge-Adjacent). Therefore, the total charge on ϕ is non-positive.

$$\text{charge}'(\phi) \leq \text{charge}(\phi) + (6 + 6 + 10 + 8) = 0$$

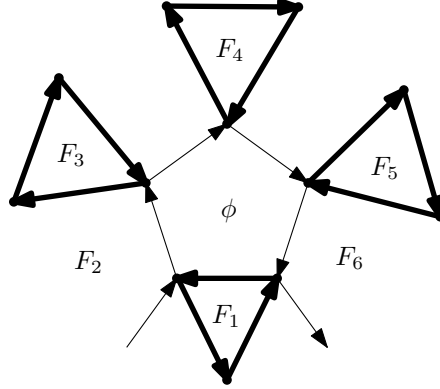


Figure 5.4: A face ϕ of length 5 with a maximum amount of adjacent triangles.

Thus, no face of Ω has positive charge which contradicts the fact that Ω was a planar embedding. Therefore, C_6^2 has no finite planar cover. \square

5.1.4 Proof of Theorem 5.5

Observe that if a 2-regular digraph D has a finite planar cover and H is immersed in D , then H also has a finite planar cover since surface embedding is an immersion closed property. We use the negation of this fact in the proof of Theorem 5.5, if an immersion H of D does not have a finite planar cover, then neither does D .

Proof of Theorem 5.5. For the “if” direction, let D be a connected 2-regular digraph embedded in the projective plane. Since the sphere is a double cover of the projective plane, we have that D has a double cover in the plane.

For the “only if” direction, we prove the contrapositive. Let D be a connected 2-regular digraph with no embedding in the projective plane. Therefore, D contains either C_4^1 , C_6^2 , $C_3^1 \cdot C_3^1$, or $C_3^1 \cup C_3^1$ as an immersion. Applying Lemma 5.6, 5.7, 5.8, or 4.21 where appropriate implies that D does not have a finite planar cover. \square

Chapter 6

Strong Embeddings

Let G be an undirected graph, and let ϕ be a 2-cell embedding of G in a surface S . The embedding ϕ is called *strong* if the closure of each face of $S - \phi(G)$ is homeomorphic to a closed disk. Combinatorially, this is equivalent to each face of ϕ being bounded by a *cycle* (a closed walk with no repeated edges or vertices) in G . (In the literature cycles are sometimes referred to as circuits and strong embeddings are also called closed 2-cell embeddings or circular embeddings.) At the combinatorial level, an obvious obstruction to a graph having a strong embedding is a cut-vertex, as any embedding of a graph with a cut-vertex x must contain a facial walk that encounters x twice. However, it is conjectured that aside from this, every graph should have a strong embedding.

Conjecture 6.1 (Haggard [15]). *Every 2-connected graph has a strong embedding in some surface.*

It is straightforward that Conjecture 6.1 is true for 2-connected planar graphs. And it has been shown, independently, by Negami [30] and Robertson & Vitray [39] that Conjecture 6.1 is true for projective planar graphs. It was also shown by Zha that Conjecture 6.1 is true for doubly toroidal graphs [50] (graphs that embed in \mathbb{S}_2) and 5-crosscap embeddable graphs [51] (those that embed in \mathbb{N}_5). In the vein of minors, it has also been shown by Zhang [52] that Conjecture 6.1 is true for K_5 -free graphs and by Robertson & Zha [40] that the conjecture is true for V_8 -free graphs (where V_8 is the Wagner graph or the 8-vertex Möbius ladder). Many more results are also known [1, 8, 25].

Taking the faces of a strong embedding one obtains a cycle double cover. Thus, Conjecture 6.1 implies the more well known Cycle Double Cover Conjecture.

Conjecture 6.2 (Seymour [41], Szekeres [42]). *For every 2-edge-connected graph, there exists a collection of cycles \mathcal{C} so that every edge appears exactly twice in \mathcal{C} .*

In this thesis, 2-regular digraphs are equipped with a suitable definition of embeddings and so we can ask a question analogous to Conjecture 6.1. Call a 2-cell embedding of a 2-regular digraph *strong* if the closure of each face is homeomorphic to a closed disk. The combinatorial equivalence of this is every face being bounded by a directed cycle.

Question 6.3. *Does every loopless 2-regular digraph with no cut-vertex have a strong embedding in some surface?*

As in the case of undirected graphs, two obvious obstructions to a 2-regular digraph having a strong embedding are a cut vertex and a loop. Let D be a 2-regular digraph and let Ω be a 2-cell embedding of D in some surface S . If D has a cut vertex $v \in V(D)$, then observe that there is a pair of complementary transitions from $T(v)$ that appear in the same face of Ω , which implies that v is repeated in this facial walk and Ω is not strong. For a very similar reason a loop edge obstructs the existence of a strong embedding. Besides these obvious obstructions, one can ask (as in the case of undirected graphs) if a strong embedding exists.

In this chapter, we answer Question 6.3 in the negative. We show that there exist infinitely many “well”-connected 2-regular digraphs that have no strong embedding in any surface. That is, the analogue of Conjecture 6.1 is false for 2-regular digraphs (when equipped with our notion of a strong surface embedding). This chapter culminates in the following result.

Theorem 6.4. *There exist infinitely many internally strongly 3-edge-connected 2-regular digraphs with no strong embedding in any surface.*

6.1 A 2-regular digraph with no strong embedding

Through computer search (see Algorithm 2), a single smallest 2-regular digraph with no cut-vertex and no strong embedding in any surface was found to be of order 7. We will refer to it as G_7 (see Figure 6.1). We give a proof below that G_7 does not have a strong embedding in any surface that does not rely on the algorithm used in the computer search. In this proof we find a vertex in G_7 where the local transitions at this vertex cannot be covered by cycles of G_7 . For a 2-regular digraph D , we say that two cycles *conflict* (or are *incompatible*) if there exists a transition of $\mathcal{T}(D)$ contained in both.

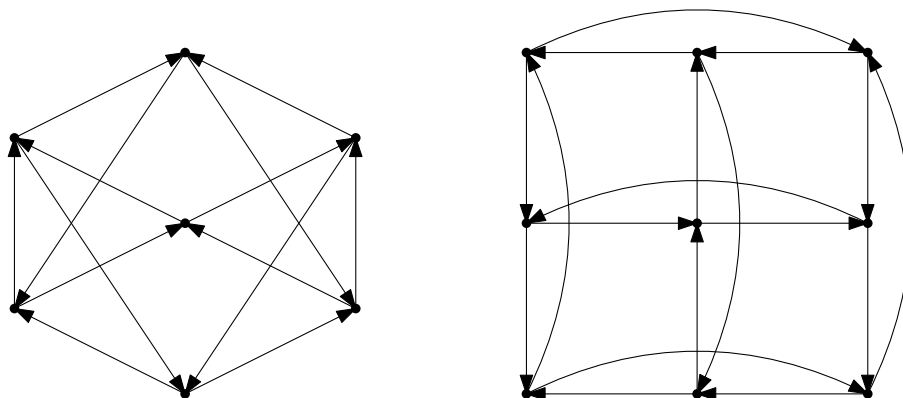


Figure 6.1: The smallest (simple) 2-regular digraphs with no strong embedding.

Proposition 6.5. G_7 has no strong embedding in any surface.

Proof. We begin by making a few observations about the structure of G_7 . First observe that the underlying undirected graph is $K_{4,3}$ with an added 2-edge matching on the independent set of order 4. As per Figure 6.1 we will refer to the two matching edges as the *left* and *right* of G_7 and the bipartition of size 3 as the *middle* of G_7 . The middle has two vertex classes: the *clones* (pictured at the top and bottom of the middle of Figure 6.1) and the *center*. Let v_0 denote the center vertex. Note there is an automorphism of G_7 switching the two clones, and there is another automorphism switching the left and right.

Suppose towards a contradiction that Ω was a strong embedding of G_7 in some surface. Notice that the only directed cycles that use v_0 are of length 4 and length 7. Since Ω is strong, every transition of $T(v_0)$ is contained in a distinct face of Ω , giving four faces containing v_0 . We reach the contradiction by showing that four facial cycles cannot cover $T(v_0)$.

Suppose that Ω had two 7-cycles as faces with transitions in $T(v_0)$ (up to automorphism there is only one such 7-cycle). Choosing one 7-cycle as a face in Ω uniquely determines the other one (recall that each transition in $\mathcal{T}(G_7)$ can only appear once in Ω). Taking these two 7-cycles, we cannot complete Ω to a strong embedding, the 4-cycles necessary to cover the remaining transitions at v_0 are incompatible with the chosen 7-cycles. Thus, Ω cannot contain two 7-cycle faces with transitions in $T(v_0)$.

So suppose Ω contained exactly one 7-cycle as a face with a transition in $T(v_0)$. This uses 1 of the 4 transitions at v_0 ; therefore, three 4-cycles must be used to cover the remaining transitions of $T(v_0)$. However, at most two 4-cycles can be chosen as faces in Ω without conflicting with the chosen 7-cycle. Thus, Ω cannot contain a 7-cycle face with a transition in $T(v_0)$.

Now it must be the case that Ω consists of four 4-cycles that cover the transitions at v_0 . Up to automorphism there is a unique way to choose four such cycles, none of which conflict. However, after these faces have been chosen, the rest of Ω is forced (use the fact that each transition of $\mathcal{T}(G_7)$ appears exactly once in Ω) and there exists a face with a repeated vertex. \square

6.1.1 Computational results

The algorithm mentioned above was developed in SageMath and used to exhaustively search all 2-regular digraphs up to 11 vertices. There are, in total, 103 strongly 2-edge-connected 2-regular digraphs up to order 11 with no strong embedding: 13 of them have underlying simple graphs and 90 have digons. A table of all underlying simple 2-regular digraphs with

no strong embedding is included, for the interested reader, where the graphs are given in the digraph6 format.¹

order	digraph6 string
7	F@c'eB?PE?
9	H?X_?gIKE?o?SB?
10	I?Kc@AACa@GoB??c@O I?KWC0?Ka@CoA_?o?o
11	J?EW?CG0cCGC?XGA_?S?W? J?EH@0?GQ@K?APCA@?D?Q? J?ED?SACACH?GDAA0?@OE? J?EAGAIACACAH_A_?o?W? J?EA'AA?Q@B?E@OA@?H?I? J?EK@@@A?PGCK@@AG?GOI? J?EE@0AAAGCACPCAC?@OI? J?EG'G?G_Gh?K@OA@?D?Q? J?EG'G?G_A'OS@OA_?D?S?

Table 6.1: Underlying simple 2-regular digraphs, up to order 11, with no strong embedding.

Given a 2-regular digraph D , we can create an auxiliary undirected graph \mathcal{C}_D in the following manner. For each directed cycle $C \subseteq D$ there is a vertex $C \in V(\mathcal{C}_D)$ and $C_1C_2 \in E(\mathcal{C}_D)$ if C_1 and C_2 contain a common transition from $\mathcal{T}(D)$. Assign to each vertex $C \in V(\mathcal{C}_D)$ the weight $w(C) = |C|$. A strong embedding, Ω of D , is an independent set $I \subseteq V(\mathcal{C}_D)$ where

$$w(I) = \sum_{C \in I} w(C) = 2|E(D)|.$$

The Euler genus of Ω is found by considering $|I|$.

Algorithm 2 takes a 2-regular digraph D on input and creates this auxiliary *cycle graph* \mathcal{C}_D . It then finds all independent sets of \mathcal{C}_D and returns either an independent set of requisite weight (a strong embedding of D) or returns nothing.

6.2 Infinite families of 2-regular digraphs with no strong embedding

One example of a 2-regular digraph with no strong embedding is nice, but an infinite family is better. We construct one infinite family by taking advantage of the $D \cdot H$ operation introduced in Chapter 4. After this we construct another, more “well-connected”, infinite family.

¹<https://users.cecs.anu.edu.au/~bdm/data/formats.html>

Algorithm 2 Strong Embedding

```
1: procedure STRONGEMBEDDING( $D$ )
2:    $H \leftarrow$  CYCLEGRAPH( $D$ )
3:   for  $i \in H.INDEPENDENTSETS()$  do                                      $\triangleright$  All independent sets of  $H$ 
4:     if WEIGHT( $i$ ) =  $2 \cdot |E(D)|$  then
5:       return  $i$ 
6:     end if
7:   end for
8:   return None
9: end procedure
10:
11: function CYCLEGRAPH( $D$ )                                              $\triangleright$  Builds auxiliary cycle graph of  $D$ 
12:    $H \leftarrow$  GRAPH( )
13:   for  $C \in D.CYCLES()$  do                                          $\triangleright$  All directed cycles of  $D$ 
14:     H.ADDVERTEX( $C$ )
15:   end for
16:   for  $C_1, C_2 \in V(H)$   $C_1 \neq C_2$  do
17:     if D.SHARETRANSITION( $C_1, C_2$ ) then
18:       H.ADDEDGE( $(C_1, C_2)$ )
19:     end if
20:   end for
21:   return  $H$ 
22: end function
```

6.2.1 A poorly connected family

Lemma 6.6. *Let D and H be connected 2-regular digraphs. If D has no strong embedding in any surface, then $D \cdot H$ has no strong embedding in any surface.*

Proof. Suppose towards a contradiction that $D \cdot H$ has a strong embedding Ω . Let $e_0 = uv \in E(D)$ be the edge deleted in forming $D \cdot H$. Let (ϕ, ϕ') be an immersion of D in $D \cdot H$ where for all $e \in E(D)$, $\phi'(e) = e$ and $\phi'(e_0)$ is mapped to a trail in $(D \cdot H) - E(D)$. Let Ω_D be the induced embedding formed by this immersion. We claim that Ω_D is a strong embedding of D .

Every face in $\Omega \cap \Omega_D$ is a cycle since Ω is strong; therefore, it remains to show that faces in $\Omega_D \setminus \Omega$ are cycles. Let $F \in \Omega_D \setminus \Omega$. Since Ω is a strong embedding $D \cdot H$ has no cut-vertex; therefore, there exists a nontrivial path $P \subseteq D$ from v to u so that $F = P, e_0$ is a cycle. Therefore, Ω_D is strong. \square

Using Lemma 6.6 we obtain the following corollary by using copies of G_7 (or one of the other graphs found to have not strong embedding, see Table 6.1).

Corollary 6.7. *There exist infinitely many connected 2-regular digraphs with no strong embedding in any surface.*

6.2.2 A more well connected family

The infinite family found via Corollary 6.7 is nice but poorly connected ($D \cdot H$ has an edge-cut of size 2 in the underlying graph for all D and H). We now work towards showing the existence of infinitely many 2-regular digraphs with no strong embedding and higher edge connectivity by using gadgets, as introduced in Section 4.1.1.

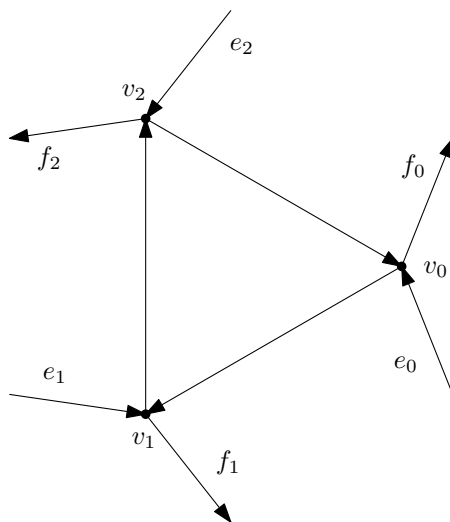


Figure 6.2: A labeling of the directed triangle gadget G_3 from Lemma 6.8.

Let G be a 2-regular gadget. Recall that a 2-cell embedding Ω of G is a collection of walks \mathcal{W} starting and ending with half-edges and a collection of closed walks \mathcal{C} , where each transition appears exactly once in $\mathcal{C} \cup \mathcal{W}$. As before, if two walks in G contain the same transition, then we say that they *conflict* or are *incompatible*. If \mathcal{W} is a set of paths and \mathcal{C} a set of cycles, then Ω is a *strong* embedding of G .

Lemma 6.8. *The directed triangle gadget G_3 has 2 strong embeddings: one where the directed triangle appears as a face and one where it does not.*

Proof. Let G_3 be labeled as in Figure 6.2. Let Ω be a strong embedding of G_3 . Observe that for $i \in \{0, 1, 2\}$, the half-edge path (e_i, f_i) must appear as a face in Ω since it is the only way for the transition (e_i, f_i) to appear.

If the directed triangle $C = v_0v_1, v_1v_2, v_2v_0$ appears as a face in Ω , then there is a unique way to complete Ω to an embedding. Expressing indices modulo 3, the faces $(e_i, v_iv_{i+1}, f_{i+1})$ are forced to be in Ω for all i . If C is not in Ω , then the faces $(e_i, v_iv_{i+1}, v_{i+1}v_{i-1}, f_{i-1})$ are forced to be in Ω for all i . \square

The proof of Lemma 6.8 is slightly more enlightening than the statement. In the proof we show that in a strong embedding of G_3 the edges of the directed triangle C are either covered by C plus 3 other faces, each sharing exactly one edge with C , or the edges of C

are covered by 3 faces each containing exactly 2 edges of C . We use this fact repeatedly in the following lemma.

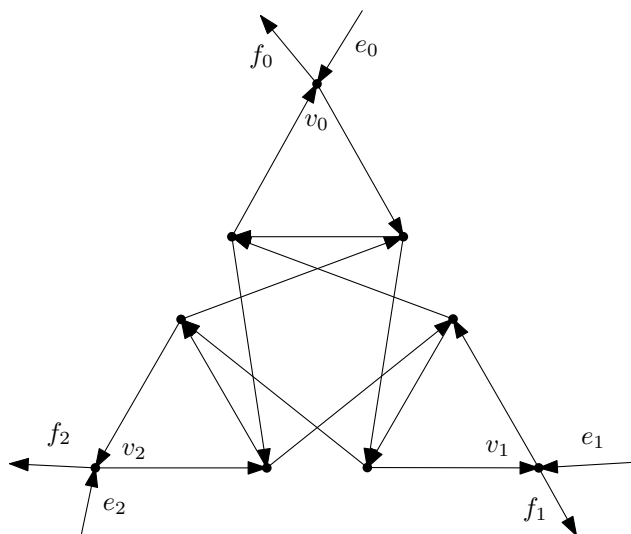


Figure 6.3: A labeling of the gadget G_6 .

Proposition 6.9. *The gadget G_6 has no strong embedding.*

Proof. Let G_6 be labeled as in Figure 6.3 and suppose towards a contradiction that G_6 had a strong embedding Ω . Observe that for $i \in \{0, 1, 2\}$, the half-edge path $P_i = e_i, f_i$ must appear as a face in Ω since it is the only way for the transition (e_i, f_i) to appear. Thus, since each transition must appear exactly once in Ω (and since Ω is strong) we have that either faces $\{P_{0,1}, P_{1,2}, P_{2,0}\} \subseteq \Omega$ or faces $\{P_{0,2}, P_{2,1}, P_{1,0}\} \subseteq \Omega$ where $P_{i,j}$ denotes a path starting on the in-half-edge e_i and ending with the out-half-edge f_j .

Notice that G_6 contains five edge-disjoint directed triangles. If we treat each of these directed triangles separately as a directed triangle gadget (in the obvious way) then since Ω is strong, it must be the case that each of these triangle gadgets has a strong embedding, so we can apply Lemma 6.8 to each one of them.

Suppose that $\{P_{0,1}, P_{1,2}, P_{2,0}\} \subseteq \Omega$. Observe that there are only 2 valid paths $P_{0,1}$ (one uses 4 vertices of G_6 and the other uses 7). Take $P_{0,1}$ to be either one of these and observe that by applying Lemma 6.8 to each triangle in G_6 , we get that $P_{1,2}$ is forced. But now $P_{2,0}$ cannot be a face of Ω because it is either incompatible with the already chosen 2 faces or it violates Lemma 6.8. The argument for $\{P_{0,2}, P_{2,1}, P_{1,0}\} \subseteq \Omega$ is similar. \square

Proof of Theorem 6.4. Let D be an internally strongly 3-edge-connected 2-regular digraph with 3 distinct edges $e_0, e_1, e_2 \in E(D)$, not all incident to the same vertex. Form a new digraph D' by subdividing each e_i . Let $A \subseteq E(G_6)$ be the set of half-edges of G_6 . Let D'' be a 2-regular digraph obtained from $D' \cup (G_6 - A)$ by pairwise identifying the 1-regular vertices in D' and $G_6 - A$. Since D was internally strongly 3-edge-connected, so is D'' . It

follows from Proposition 6.9 that D'' has no strong embedding. Since there exist infinitely many internally strongly 3-edge-connected 2-regular digraphs, we obtain the result. \square

The simplest 2-regular digraph that one can make by “attaching” G_6 to a 2-regular digraph is the 9 vertex graph shown in Figure 6.1. This graph can be realized by subdividing a pointless edge three times, deleting the half-edges of G_6 and pairwise identifying the 1-regular vertices in each graph (there are two ways to do this, but they give isomorphic graphs).

Chapter 7

Open Problems

In this thesis, we studied the class of 2-regular digraphs, immersion, and a sensible notion of embedding in a surface. We were able to prove analogues of many results for undirected graphs on surfaces. In this chapter, we suggest some open problems.

7.1 Classifying obstructions for fixed surfaces

Conjecture 7.1. *If $D \in \text{Forb}_I(\mathbb{N}_k)$, then $\underline{D} \in \text{Forb}_I(\mathbb{N}_{k+1})$ for $k \geq 1$.*

Conjecture 7.1 is true for $k = 1$; however, there may also be something at play with orientable surfaces. Inflating a vertex of the torus obstruction $D_6 \in \text{Forb}_I(\mathbb{S}_1)$ yields \underline{D}_6 which is isomorphic to the obstruction $D_7 \in \text{Forb}_I(\mathbb{N}_2)$. However, this may be anecdotal. As in general, obstructions for orientable and nonorientable surfaces need not be related. For instance, $D_{10} \in \text{Forb}_I(\mathbb{S}_1)$ but D_{10} embeds in the projective plane; i.e. D_{10} is an obstruction for the orientable surface of Euler genus 2 but embeds in the nonorientable surface of Euler genus 1. Perhaps a more general conjecture is in order.

Conjecture 7.2. *If D is a loop-free 2-connected 2-regular digraph, then $\text{eg}(D) < \text{eg}(\underline{D})$.*

Since D is immersed in \underline{D} , we know that $\text{eg}(D) \leq \text{eg}(\underline{D})$. We also know that if Ω is a 2-cell embedding of D in a surface with Euler genus g , then we can extend Ω to a 2-cell embedding $\underline{\Omega}$ of \underline{D} in the nonorientable surface with Euler genus $g + 1$. Therefore, one way to prove Conjecture 7.2 would be to prove that $\underline{\Omega}$ is a minimal genus embedding. Note that both the loop-free and connectivity conditions are necessary. If a vertex with a loop edge $v \in V(D)$ is replaced by a pair of parallel edges, then \underline{D} contains a digon, where after contracting the digon yields a graph isomorphic to D . If (H_1, H_2) is a 1-separation of D , then for $i \in \{1, 2\}$ embedding H_i in its surface of minimal genus S_i , one can obtain an embedding of both D and \underline{D} in a surface with Euler genus $\text{eg}(S_1) + \text{eg}(S_2)$, which is the Euler genus of D .

While we're on the subject of conjectures and obstructions for fixed surfaces. We might as well offer up the partial lists of obstructions for the torus and the Klein bottle.

Conjecture 7.3.

$$\text{Forb}_I(\mathbb{S}_1) = \{C_5^1, C_7^2, D_6, D_{10}, C_3^1 \cdot C_3^1, C_3^1 \cup C_3^1\}.$$

Conjecture 7.4.

$$\text{Forb}_I(\mathbb{N}_2) = \{C_5^1, C_7^2, \underline{C_6^2}, D_7, D_9\} \cup (C_3^1 \cdot \text{Forb}_I(\mathbb{N}_1)) \cup (C_3^1 \cup \text{Forb}_I(\mathbb{N}_1)).$$

The evidence for either Conjecture 7.3 or 7.4 is only that the list was found via exhaustive computer search of 2-regular digraphs up to 9 and 10 vertices respectively, and independently verified in Propositions 4.2 and 4.3. It's not clear why either list would be complete, but it seems remiss to not conjecture a partial list complete.

7.2 Relating 2-regular digraphs to undirected graphs

As explained in Section 1.1, viewing a 2-regular digraph as a medial of an undirected graph is a way to link the two classes of graphs. However, this requires that the undirected graph, as well as a medial counterpart, to be embeddable in the same orientable surface. Is there a way to relate these two classes of graphs without a surface? In particular:

Question 7.5. *Let S be a fixed surface. How are $\text{Forb}_I(S)$ and $\text{Forb}_M(S)$ related?*

surface S	$ \text{Forb}_I(S) $	$ \text{Forb}_M(S) $	$ \text{Forb}_T(S) $
\mathbb{S}_0	1	2	2
\mathbb{N}_1	4	35	103
\mathbb{S}_1	≥ 6	$\geq 17,535$	$\geq 250,815$
\mathbb{N}_2	≥ 16	?	?

Table 7.1: Comparison of sizes of obstructions for fixed surfaces.

Question 7.5 isn't the most well formed; however in light of Table 7.1, a meaningful answer could help in the classification of undirected obstructions for fixed surfaces. A few similar questions are the following:

Question 7.6. *Can Theorem 5.5 be used to prove (or disprove) Negami's Conjecture 5.3 for undirected graphs?*

In the case of Question 7.6, a stumbling point is the fact that Negami's Conjecture 5.3 deals with the projective plane, a nonorientable surface. Therefore, the medial construction (as it appears in Section 1.1) cannot be used. Perhaps a suitable medial graph construction for nonorientable surfaces can be devised.

Question 7.7. *Can Theorem 6.4 be used to disprove (or prove) the strong embedding conjecture 6.1 for undirected graphs?*

Question 7.7 seems especially hard. There also seems to be a disconnect between 2-regular digraphs and undirected graphs with regard to strong embeddings. As stated in Chapter 6, it has been shown that undirected graphs with Euler genus ≤ 5 have strong embeddings. However, some 2-regular digraphs with Euler genus 2 have been shown to have no strong embedding.

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