

Energy Dissipation and Information Flow in Coupled Markovian Systems

by

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Abstract

A stochastic system under the influence of a stochastic environment will become correlated with both present and future states of the environment. Such a system can be seen as a predictive model of future environmental states. The non-predictive model complexity in such a model has been shown in a recent paper to be fundamentally equivalent to thermodynamic dissipation. In this dissertation, this abstract result is explored in concrete models in order to illustrate how it emerges in realistic systems. In steady-state, this model complexity is found to be the dominant form of dissipation when the system is strongly driven and quick to relax back to equilibrium. Model complexity being the dominant form of dissipation is shown to be equivalent to the rate at which the system learns about its environment being large compared to the heat dissipation.

Keywords: Mutual information; Dissipation; Efficiency; Markovian system; Environment

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Chapter 1

Introduction

Information theory is becoming increasingly relevant in the field of statistical mechanics. In particular, Landauer showed that information processing can have dissipative costs associated with it [1]. In particular, it seems that performing logically irreversible operations, such as erasure, requires that some free energy be dissipated.

Stochastic systems will process information, merely by interacting with their environment. Through their interaction, the system gains information about the environment, such that its state exhibits some correlation with that of the environment. If this environment exhibits temporal correlations, not only does the system carry information about the present state of the environment, it also carries information about future states. In this way, the system implicitly exhibits a predictive model of future states of the environment.

One way to quantify the inefficiency of such a model is through the unnecessary model complexity: additional complexity that does not aid in prediction. In a 2012 paper, Still, Sivak, Bell, and Crooks detailed an equivalence between this intrinsic model efficiency, and thermodynamic efficiency [2]. This result is fascinating for many reasons. First, it connects information theory and statistical mechanics, two fields that superficially appear to have little overlap. It hints at many possible applications, such as a guiding principle for synthetic machines operating in non-equilibrium conditions [3]. As well, it opens up the possibility of addressing more fundamental questions: Can this be used to refine or generalize Landauer's principle? Can techniques for learning environmental statistics from the field of machine learning find application in statistical mechanics?

This result, however, has not yet been illustrated in concrete models in the literature. In this dissertation, this result will be analyzed numerically, calculating the various quantities of interest in some illustrative models. Doing so can help to illuminate this abstract result in a more concrete manner. As well, after calculating these quantities, we will compare them, both analytically and numerically, to other theoretical developments in this area.

These results are applicable to many possible systems. For example, biological molecular machines generally operate far from equilibrium within highly stochastic environments.

For example, ATP synthase, a molecular machine which synthesizes adenosine triphosphate (ATP), is composed of two sub-units. The first, labeled F_0 , behaves like a rotational crankshaft driving the second sub-unit, labeled F_1 . F_1 , in turn, produces the ATP. The rotation of F_0 is largely stochastic, however. Thus, the sub-unit, F_1 , contains an implicit prediction of future rotations of F_0 . In order for this synthesis to be performed efficiently, the implicit model must contain little extraneous model complexity [4]. Examples like this occur throughout the field of biology with organisms learning about statistical patterns in their environment.

Chapter 2

Theoretical Background

2.1 Problem Setup

Consider a stochastic process, $\{X_t\}_{t=0}^{t=\tau}$, representing some environmental variable. Let \mathcal{X} denote the set of all possible states that any X_t can occupy. Throughout this dissertation, t will be seen as indexing discrete time steps, rather than being a quantity with units such as seconds. The time evolution of X_t is governed by the probabilities $p(x_t|\{x_{t'}\}_{t'=0}^{t'=t-1}) := p(X_t = x_t|\{X_{t'} = x_{t'}\}_{t'=0}^{t'=t-1})$ for $x, x_{t'} \in \mathcal{X}$. Let $\{Y_t\}_{t=0}^{t=\tau}$, denoting the system of interest, be some other stochastic process. Let \mathcal{Y} denote the set of all possible states that any Y_t can occupy. Take the dynamics of Y_t to depend on the environmental state via the transition probabilities $p(y|y', x) := p(Y_{t+1} = y|Y_t = y', X_{t+1} = x)$, where $y, y' \in \mathcal{Y}$, where this kernel is independent of t . This dependence on X_{t+1} means that $\{Y_t\}_{t=0}^{t=\tau}$ is not, on its own, a Markov process, although it does have the Markov property in the sense that it depends only on the present state of the system and environment.

In order to model the evolution of these two stochastic processes, an alternating timestep pattern is used. This pattern is illustrated in figure 2.1. One complete timestep is composed of two sub-steps: one environment, or work step, and one system, or relaxation step. For brevity, unless ambiguities arise, probability distributions will be written as the function p , with the argument of the function corresponding to the lowercase version of the random variable to which p corresponds (eg. $p(X = x|Y = y) = p(x|y)$).

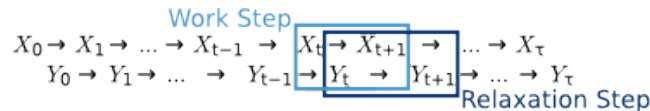


Figure 2.1: In this dissertation, the system and environment alternate steps, with system steps referred to as relaxation steps, and environment steps referred to as work steps.

An additional constraint on the dynamics of Y_t is imposed: namely, the principle of microscopic reversibility is enforced for the system. For our purposes, this gives two constraints

on the transition probabilities:

$$p(y|y', x) > 0 \iff p(y'|y, x) > 0, \quad (2.1a)$$

$$\lim_{t \rightarrow \infty} p(Y_t = y'|x)p(Y_{t+1} = y|Y_t = y', x) = \lim_{t \rightarrow \infty} p(Y_t = y|x)p(Y_{t+1} = y'|Y_t = y, x), \quad (2.1b)$$

for all $x \in \mathcal{X}$, and for all $y, y', y_t, y'_t \in \mathcal{Y}$. Note that the limit in equation (2.1b) is taken with x fixed.

Equation (2.1b) is also known as the condition of detailed balance. This condition can be restated as the requirement that each reaction be balanced by its corresponding reverse reaction at steady-state.

The system is taken to be in contact with a heat bath maintained at temperature T . An energy can then be associated with a joint state of the system and environment, (x, y) , given by:

$$E(x, y) := -k_B T \ln \lim_{t \rightarrow \infty} p(Y_t = y|X_t = x), \quad (2.2)$$

where k_B is Boltzmann's constant. This is equivalent to assuming the system probabilities to be distributed according to a Boltzmann distribution. From this, we can define the equilibrium free energy given an environmental state $x \in \mathcal{X}$:

$$F_{\text{eq}}(x) := -\frac{1}{\beta} \ln \sum_{y \in \mathcal{Y}} e^{-\beta E(x, y)}, \quad (2.3)$$

where $\beta := \frac{1}{k_B T}$. The equilibrium distribution for environment state $x \in \mathcal{X}$ is then:

$$p_{\text{eq}}(y|x) := e^{-\beta(E(x, y) - F_{\text{eq}}(x))}. \quad (2.4)$$

Note that, by construction, we have that the equilibrium distribution is the same before and after a relaxation step:

$$\begin{aligned} p_{\text{eq}}(Y_{t+1} = y|x) &= \sum_{y' \in \mathcal{Y}} p(Y_{t+1} = y|y', x) p_{\text{eq}}(Y_t = y'|x) \\ &= p_{\text{eq}}(Y_t = y|x). \end{aligned} \quad (2.5)$$

As well, we have that $p_{\text{eq}}(y|x) = \lim_{t \rightarrow \infty} p(y_t|x)$, meaning that detailed balance holds in this equilibrium state:

$$p_{\text{eq}}(Y_t = y|x)p(y'|y, x) = p_{\text{eq}}(Y_t = y'|x)p(y|y', x) \quad (2.6)$$

The system is taken to be in equilibrium at time $t = 0$, meaning that:

$$p(x_0, y_0) = p(x_0)p(y_0|x_0) = p(x_0)p_{\text{eq}}(y_0|x_0). \quad (2.7)$$

Thus, a distribution $p(x_0)$ fully defines an initial condition for the model. Thus, we can write the probability of some system trajectory, $\{y_t\}_{t=0}^{t=\tau}$, given some environment protocol, $\{x_t\}_{t=0}^{t=\tau}$, as:

$$p(\{y_t\}_{t=0}^{t=\tau}|\{x_t\}_{t=0}^{t=\tau}) = p_{\text{eq}}(y_0|x_0) \prod_{t=1}^{\tau} p(y_t|y_{t-1}, x_t). \quad (2.8)$$

The corresponding joint probability distribution can then be written:

$$p(\{x_t\}_{t=0}^{t=\tau}, \{y_t\}_{t=0}^{t=\tau}) = p(x_0)p_{\text{eq}}(y_0|x_0) \prod_{t=1}^{\tau} p(y_t|y_{t-1}, x_t)p(x_t|\{x_{t'}\}_{t'=0}^{t'-1}). \quad (2.9)$$

Unless otherwise indicated, all expectation values are taken over this joint distribution.

2.2 Entropy and Information

This section establishes some basic background in information theory, taken from chapter 2 of *Elements of Information Theory* by Cover [5].

In order to investigate the information that the system stores about the environment, we must first consider entropy. The entropy of a random variable, X , which can occupy states \mathcal{X} is given by:

$$H[X] := - \sum_{x \in \mathcal{X}} p(x) \ln p(x). \quad (2.10)$$

This entropy is measured in units of nats. Throughout this dissertation, when calculating specific values, units of bits are used. The two are related via:

$$H_{\text{bits}} = H_{\text{nats}} / \ln 2. \quad (2.11)$$

If Y is another random variable with states \mathcal{Y} , the joint entropy is given by:

$$H[X, Y] := - \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} p(x, y) \ln p(x, y). \quad (2.12)$$

The conditional entropy of Y can then be defined as:

$$H[Y|X] := - \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} p(x, y) \ln p(y|x). \quad (2.13)$$

This gives rise to a chain rule for entropies:

$$H[X, Y] = H[Y|X] + H[X]. \quad (2.14)$$

Note that for any random variables X and Y , $H[Y|X] \leq H[Y]$, with equality only if X and Y are independent. The mutual information between the random variables X and Y

is defined to be:

$$I[X, Y] := H[X] + H[Y] - H[X, Y], \quad (2.15)$$

and is symmetric in X and Y . Rewriting this expression with the chain rule, equation (2.14), we find

$$\begin{aligned} I[X, Y] &= H[Y] - H[Y|X] \\ &= H[X] - H[X|Y]. \end{aligned} \quad (2.16)$$

This leads to the following interpretation of mutual information: the mutual information between X and Y represents the reduction in the entropy of Y (X) due to knowing the state of X (Y). Since $H[Y|X] \leq H[Y]$, we have that $I[X, Y] \geq 0$.

Relative entropy is another useful concept from information theory. The relative entropy, also known as the Kullback Leibler (or KL) divergence between distributions $p(x)$ and $q(x)$ for $x \in \mathcal{X}$ is given by:

$$D_{\text{KL}}\left(p(x) \parallel q(x)\right) = \sum_{x \in \mathcal{X}} p(x) \ln \frac{p(x)}{q(x)}, \quad (2.17)$$

subject to the conventions that $0 \ln \frac{0}{q} = 0$ and $p \ln \frac{p}{0} = \infty$. This relative entropy is non-negative, and is 0 if $\forall x \in \mathcal{X}, p(x) = q(x)$. It can thus be loosely interpreted as a sort of distance between distributions $p(x)$ and $q(x)$; however, it is not symmetric in $p(x)$ and $q(x)$, nor does it obey the triangle inequality.

The relative entropy allows for an alternative interpretation of the mutual information. In particular, one can observe that:

$$I[X, Y] = D_{\text{KL}}\left(p(x, y) \parallel p(x)p(y)\right). \quad (2.18)$$

This indicates that the mutual information is a measure of distance between $p(x, y)$ and $p(x)p(y)$, or in other words, it is a measure of how far X and Y are from being independent. The KL divergence is also useful in describing the thermodynamics of a Markovian system. In particular, the free energy can be generalized to non-equilibrium situations as:

$$F_{\text{neq}}[Y|x] = F_{\text{eq}}(x) + F^{\text{add}}[Y|x], \quad (2.19)$$

where:

$$F^{\text{add}}[Y|x] = k_{\text{B}}T D_{\text{KL}}\left(p(y|x) \parallel p_{\text{eq}}(y|x)\right). \quad (2.20)$$

Substituting equations (2.20) and (2.4) into equation (2.19), gives:

$$F_{\text{neq}}[Y|x] = \langle E(x, y) \rangle_{p(y|x)} + k_{\text{B}}T \langle \ln p(y|x) \rangle_{p(y|x)}. \quad (2.21)$$

Averaging this expression over environment states x , gives the expression [2]:

$$\beta \langle F_{\text{neq}}[Y|x] \rangle_{p(x)} = \beta \langle E(x, y) \rangle_{p(x, y)} - H[Y|X]. \quad (2.22)$$

One additional useful result in information theory, is the so-called data processing inequality. Let X , Y , and Z be random variables with allowed states \mathcal{X} , \mathcal{Y} , and \mathcal{Z} , respectively. As well, suppose that X and Z are conditionally independent of one another, meaning that $p(x, z|y) = p(x|y)p(z|y)$. If this condition holds, then the data processing inequality states that:

$$I[X, Y] \geq I[X, Z]. \quad (2.23)$$

This can be understood intuitively as stating that if X depends on Z , only through some intermediate variable Y , then X cannot have more information about Z than that intermediate variable.

One final principle that will be needed from the intersection of thermodynamics and information theory is known as Landauer's principle. Landauer's principle roughly states that erasure of information must coincide with some amount of energy being dissipated. For a single bit of information, this amount of energy would be $k_B T \ln 2$. For a single step of our model, involving a coupled system and environment, Landauer's principle for the system would take the form [2]:

$$\begin{aligned} -\beta \langle Q(t) \rangle - \beta \langle W_{\text{diss}}(t) \rangle &= H[s_t|x_t] - H[s_{t+1}|x_{t+1}] \\ &=: \mathcal{I}_e. \end{aligned} \quad (2.24)$$

2.3 Thermodynamics of Prediction

This section follows the development laid out by Still *et. al* [2].

The work done by the environment on the system over a single work step is given by:

$$W(y_t; x_t \rightarrow x_{t+}) := E(x_{t+1}, y_t) - E(x_t, y_t). \quad (2.25)$$

The heat flow into the system can similarly be defined for each relaxation step to be:

$$Q(x_{t+1}; y_t \rightarrow y_{t+1}) := E(x_{t+1}, y_t) - E(x_{t+1}, y_{t+1}). \quad (2.26)$$

The change in non-equilibrium free energy over a single work step is given by:

$$\Delta F_{\text{neq}}[Y_t; x_t \rightarrow x_{t+1}] := F_{\text{neq}}[Y_t|x_{t+1}] - F_{\text{neq}}[Y_t|x_t], \quad (2.27)$$

while over a relaxation step, it is given by:

$$\Delta F_{\text{neq}}[x_{t+1}; Y_t \rightarrow Y_{t+1}] := F_{\text{neq}}[Y_{t+1}|x_{t+1}] - F_{\text{neq}}[Y_t|x_{t+1}]. \quad (2.28)$$

The average dissipation over a work step is then defined to be:

$$\beta \langle W_{\text{diss}}(y_t; x_t \rightarrow x_{t+1}) \rangle := \beta \langle W(y_t; x_t \rightarrow x_{t+1}) \rangle - \beta \langle \Delta F_{\text{neq}}[Y_t; x_t \rightarrow x_{t+1}] \rangle. \quad (2.29)$$

This can be rewritten as follows:

$$\begin{aligned} \beta \langle W_{\text{diss}}(y_t; x_t \rightarrow x_{t+1}) \rangle &= \beta \langle W(y_t; x_t \rightarrow x_{t+1}) \rangle - \beta \langle F_{\text{neq}}[Y_t|x_{t+1}] - F_{\text{neq}}[Y_t|x_t] \rangle \\ &= \beta \langle W(y_t; x_t \rightarrow x_{t+1}) \rangle - \beta \langle E(x_{t+1}, y_t) \rangle + H[Y_t|X_{t+1}] \\ &\quad + \beta \langle E(x_t, y_t) \rangle - H[Y_t|X_t] \\ &= H[Y_t|X_t] - H[Y_t|X_{t+1}], \end{aligned} \quad (2.30)$$

where the first, second, and third lines follow from equations (2.27), (2.22), and (2.25) respectively. Using equation (2.16), this can be re-written as:

$$\begin{aligned} \beta \langle W_{\text{diss}}(y_t; x_t \rightarrow x_{t+1}) \rangle &= H[Y_t|X_{t+1}] - H[Y_t|X_t] + (H[Y_t] - H[Y_t]) \\ &= (H[Y_t] - H[Y_t|X_t]) - (H[Y_t] - H[Y_t|X_{t+1}]) \\ &= I[X_t, Y_t] - I[X_{t+1}, Y_t]. \end{aligned} \quad (2.31)$$

The first term can be identified as an instantaneous memory, $I_{\text{mem}}(t) := I[X_t, Y_t]$. The second term can be identified as an instantaneous predictive power, $I_{\text{pred}}(t) := I[X_{t+1}, Y_t]$. Since $p(X_{t+1}, Y_t|X_t) = p(X_{t+1}|x_t)p(Y_t|x_t)$, the data processing inequality, equation (2.23), holds, meaning that $I_{\text{mem}}(t) - I_{\text{pred}}(t) \geq 0$. This difference therefore can be interpreted as information that fails to predict future states of the environment. If the system is interpreted as computing a predictive model of future environmental states, this quantity represents unnecessary model complexity. For this reason, $I_{\text{mem}}(t) - I_{\text{pred}}(t)$ will be referred to as the instantaneous nostalgia. Thus, the equation:

$$\beta \langle W_{\text{diss}}(y_t; x_t \rightarrow x_{t+1}) \rangle = I_{\text{mem}}(t) - I_{\text{pred}}(t) \quad (2.32)$$

relates the thermodynamic dissipation to the unnecessary model complexity in the predictive model implicitly calculated by the system's dynamics.

An analogous expression to equation (2.29) can be defined over relaxation steps. Since no work is done on relaxation steps, this is given by:

$$\beta \langle W_{\text{diss}}(x_{t+1}; y_t \rightarrow y_{t+1}) \rangle := -\beta \langle \Delta F_{\text{neq}}[x_{t+1}; Y_t \rightarrow Y_{t+1}] \rangle. \quad (2.33)$$

The total dissipation over the time step from t to $t + 1$ is given by:

$$\langle W_{\text{diss}}(t) \rangle = \langle W_{\text{diss}}(x_{t+1}; y_t \rightarrow y_{t+1}) \rangle + \langle W_{\text{diss}}(y_t; x_t \rightarrow x_{t+1}) \rangle. \quad (2.34)$$

This can be added to equation (2.32) to obtain the following result for the total dissipation:

$$\beta \langle W_{\text{diss}}(t) \rangle = I_{\text{mem}}(t) - I_{\text{pred}}(t) - \beta \langle \Delta F_{\text{neq}}[x_{t+1}; Y_t \rightarrow Y_{t+1}] \rangle. \quad (2.35)$$

Since the relaxation steps bring the system closer to thermodynamic equilibrium,

$$\begin{aligned} \beta \langle \Delta F_{\text{neq}}^{\text{relax}}(t) \rangle &:= \beta \langle \Delta F_{\text{neq}}[x_{t+1}; Y_t \rightarrow Y_{t+1}] \rangle \\ &\leq 0. \end{aligned} \quad (2.36)$$

Therefore, the following bound is placed on the total dissipation:

$$\beta \langle W_{\text{diss}}(t) \rangle \geq I_{\text{mem}}(t) - I_{\text{pred}}(t). \quad (2.37)$$

The tightness of such a bound can be investigated by studying the quantity:

$$\mu(t) := \frac{I_{\text{mem}}(t) - I_{\text{pred}}(t)}{\beta \langle W_{\text{diss}}(t) \rangle}. \quad (2.38)$$

$\mu(t)$ then ranges from 0 when equation (2.37) places no additional constraints on $\beta \langle W_{\text{diss}}(t) \rangle$, to 1 when the bound is saturated. Using equation (2.32), $\mu(t)$ can be rewritten as:

$$\mu(t) = \frac{\langle W_{\text{diss}}(y_t; x_t \rightarrow x_{t+1}) \rangle}{\langle W_{\text{diss}}(t) \rangle}, \quad (2.39)$$

allowing $\mu(t)$ to be reinterpreted as the fraction of dissipation that occurs during work steps.

Chapter 3

Example Models

The example models to be considered in this dissertation are all of the same form, differing only in their transition probabilities. In each model, the environment can occupy one of two possible states. It is taken to be a Markov process, with a constant probability, γ , of switching states on any given timestep. The environmental transitions are thus governed by:

$$\begin{aligned}\gamma &:= p(X_{t+1} = x^B | X_t = x^A) \\ &= p(X_{t+1} = x^A | X_t = x^B).\end{aligned}\tag{3.1}$$

For simplicity, the γ parameter is mainly kept at a constant value, while the system parameter space is explored.

Similarly, the system can occupy one of two possible states. In general, a two state Markov process has two free parameters, governing its transitions. Since the system has two different transition matrices, one for each environmental state, this gives four free parameters. In this dissertation, two different system models will be explored: one in which all four parameters are free, and one in which additional symmetries have been imposed, resulting in two free parameters.

For behaviour outside of steady-state, initial condition and time parameters must be introduced. This results in a larger dimensional parameter space to explore. For this reason, the initial condition for all calculations has been taken to be that $X_0 = x^A$. To avoid the additional dimension due to time, steady-state behaviour will be emphasized.

3.1 Four-Parameter System

The most general system model given the above constraints is one in which the four parameters are independently assigned. In such a model, the four parameters can be seen

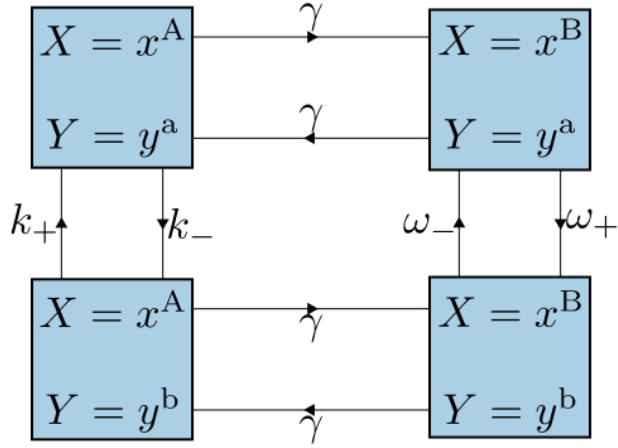


Figure 3.1: This diagram shows the states and transition probabilities for a model in which both the system and environment have two possible states. In this model, there are four parameters required to fully define the system transition probabilities. Probabilities for the state to remain unchanged are not represented, and are given by normalization.

as:

$$\begin{aligned}
 k_+ &:= p(Y_{t+1} = y^a | Y_t = y^b, X_{t+1} = x^A) \\
 k_- &:= p(Y_{t+1} = y^b | Y_t = y^a, X_{t+1} = x^A) \\
 \omega_+ &:= p(Y_{t+1} = y^b | Y_t = y^a, X_{t+1} = x^B) \\
 \omega_- &:= p(Y_{t+1} = y^a | Y_t = y^b, X_{t+1} = x^B).
 \end{aligned} \tag{3.2}$$

The remaining transition probabilities then follow from normalization. Various quantities of interest can then be calculated over this parameter space, in steady-state, or with an added time dimension.

3.2 Two-Parameter Model

The dimensionality of this parameter space can be reduced by imposing additional symmetries on the model. In particular, it will prove fruitful to impose the following:

$$k_+ = \omega_+ \tag{3.3a}$$

$$k_- = \omega_-. \tag{3.3b}$$

These two additional constraints reduce the system parameter space to two dimensions. Conditions (3.3a) and (3.3b) simplify the dependence of the system's dynamics on the

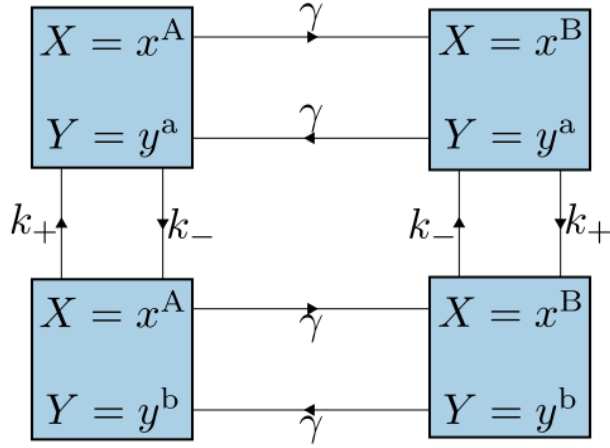


Figure 3.2: This diagram shows the states and transition probabilities for a model in which both the system and environment have two possible states. In this model, there are two parameters required to fully define the transition probabilities. Probabilities for the state to remain unchanged are not represented, and are given by normalization.

environmental state. In particular, flipping the state labels of both the environment and system leaves the transition probabilities unchanged.

3.3 Physical Model

In this section, we describe a simple physical realization of the models discussed above. In this realization, the environment process represents a magnetic field, $\vec{B} = B\hat{z}$. In such a model, the two environment states would correspond to two different discrete values of B , represented as B^A and B^B . In each timestep, the magnetic field has probability γ of switching its state.

The system can then be represented as a quantum mechanical spin, with dipole moment, $\vec{\mu} = \mu\hat{z}$, that can be either aligned or anti-aligned with the \hat{z} direction. The two system states would correspond to these aligned and anti-aligned states, given by $\mu = \mu^+ = +|\mu|$ and $\mu = \mu^- = -|\mu|$. The energies of these states would given by $E(B, \mu) = -\vec{\mu} \cdot \vec{B} = -\mu B$. At equilibrium, the occupation probabilities for each spin state, $\vec{B} = B^X\hat{z}$, in a given external field, are given by the Boltzmann distribution:

$$p(\mu = \mu^\pm | B = B^X) = \frac{e^{-\beta\mu^\pm B^X}}{e^{-\beta\mu^+ B^X} + e^{-\beta\mu^- B^X}}. \quad (3.4)$$

From this, one can write:

$$\begin{aligned}
 k_+ &= e^{\beta|\mu|B^A} \\
 k_- &= e^{-\beta|\mu|B^A} \\
 \omega_+ &= e^{-\beta|\mu|B^B} \\
 \omega_- &= e^{\beta|\mu|B^B}.
 \end{aligned}
 \tag{3.5}$$

From this, we can see that this system can be described by the four state system defined above. The symmetries imposed in equation (3.3) are then seen to be equivalent to imposing that $B^A = -B^B$, reducing the system to the two state system described above. This setup is shown in figure 3.3.

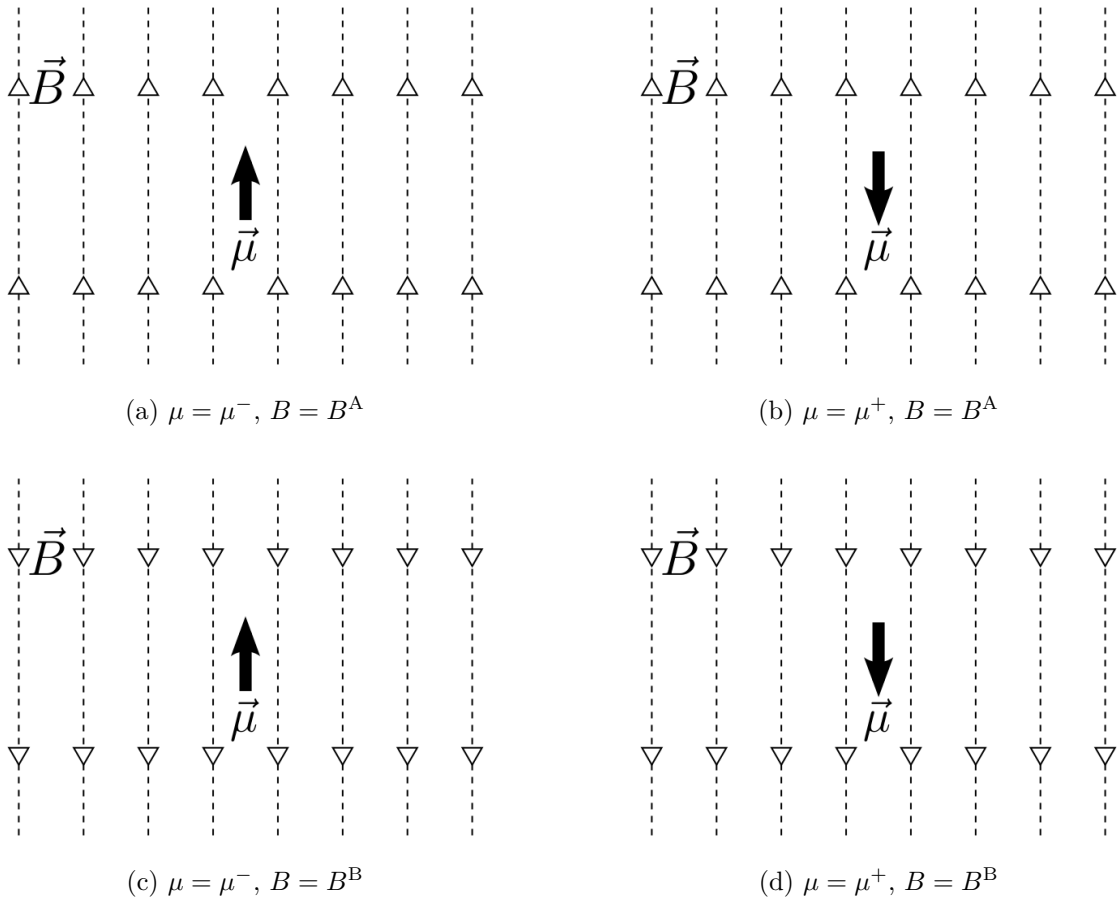


Figure 3.3: The different possible configurations of the spin system described in the text, with two system parameters. The spin's dynamics depend on the external field's direction, causing the two to become correlated.

Chapter 4

Numerical Results

In order to elucidate equations (2.32) and (2.35), we will first investigate the behaviour of the relevant quantities. While this abstract result has been derived, demonstrating its behaviour in concrete models can provide physical intuition, currently absent from the literature. These quantities can be calculated numerically in both the four state and two state models described above.

4.1 Two-Parameter Model

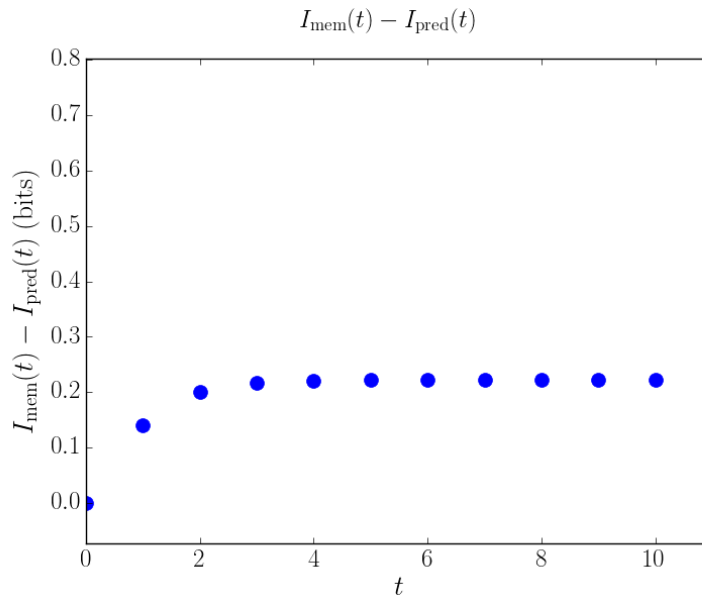


Figure 4.1: $I_{\text{mem}}(t) - I_{\text{pred}}(t)$ in bits versus time with $k_+ = 2^{-1}$, $k_- = 2^{-5}$, and $\gamma = 0.25$.

The instantaneous nostalgia is plotted versus time in figure 4.1, for some particular choice of parameter values, within the two-parameter model. Some immediate observations can be made. First of all, $I_{\text{mem}}(0) - I_{\text{pred}}(0) = 0$. This is expected, since initially, $X_0 = x^A$, meaning

that $H[X_0] = 0$. Therefore, we have that $0 \leq I_{\text{mem}}(0) - I_{\text{pred}}(0) \leq I_{\text{mem}}(0) \leq H[X_0] = 0$. Therefore, for any parameters, $I_{\text{mem}}(0) - I_{\text{pred}}(0) = 0$. As well, the instantaneous nostalgia seems to saturate to some steady-state value.

In order to investigate the dependence of these properties on parameter values, this plot can be reproduced over an array of system parameter values. Keeping the environmental parameter, γ , fixed allows this parameter space to be two dimensional, making this simpler to represent on a two dimensional image. Such an image is shown in figure 4.2. In this figure, each individual subplot has the same axes as figure 4.1, but with the parameter values varied. k_+ and k_- increase logarithmically from subplot to subplot vertically and horizontally respectively.

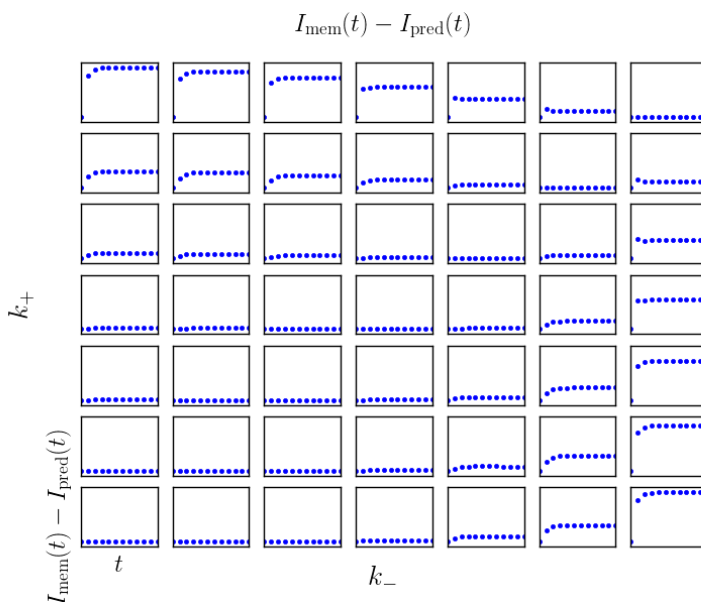


Figure 4.2: $I_{\text{mem}}(t) - I_{\text{pred}}(t)$ versus time for some parameter values, k_+ and k_- . The top right subplot uses $k_+ = k_- = 1$. k_+ is reduced by a factor of 2 in each subsequent subplot to the left, while k_- is reduced by a factor of 2 in each subsequent subplot downwards. $\gamma = 0.25$ is used for every subplot.

There are a few observations that can be made from figure 4.2. First, along the diagonal, where $k_+ = k_-$, $I_{\text{mem}}(t) - I_{\text{pred}}(t) = 0$. This follows from the fact that when $k_+ = k_-$, the dynamics of Y are independent of X . This decoupling of dynamics implies that X and Y are independent stochastic processes, meaning that $I_{\text{mem}}(t)|_{k_+=k_-} = 0$ and thus, $I_{\text{mem}}(t) - I_{\text{pred}}(t)|_{k_+=k_-} = 0$.

The steady-state values achieved in figure 4.2 can also be investigated. This data can be represented by a heatmap on a grid of k_+ and k_- points. Such a heat map is shown in figure 4.3. From this, some clear patterns emerge. First, $I_{\text{mem}}(t) - I_{\text{pred}}(t)$ tends to increase as the product k_+k_- is increased. As well, holding k_+k_- constant, and taking $\frac{k_+}{k_-}$ to move

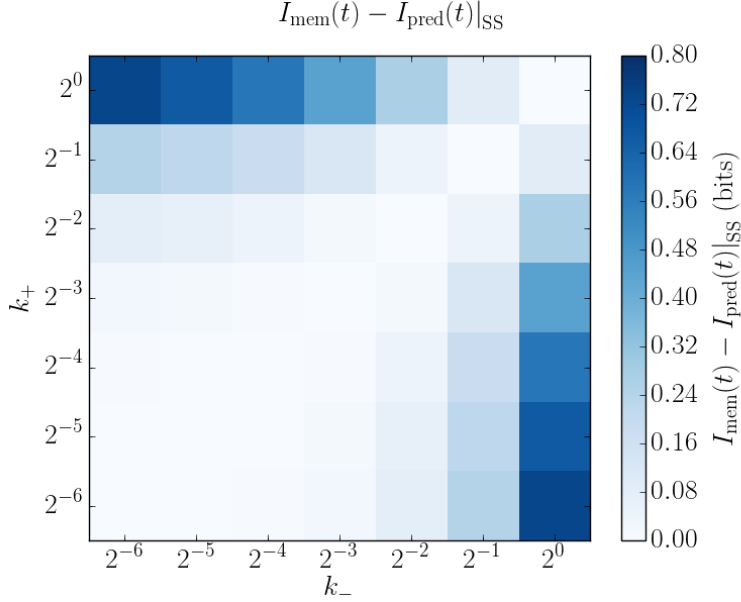


Figure 4.3: Steady-state value of $I_{\text{mem}}(t) - I_{\text{pred}}(t)$ in bits over a range of values for parameters k_+ and k_- . k_+ increases logarithmically upwards, while k_- increases logarithmically to the right. $\gamma = 0.25$ is used for this plot.

away from 1 also tends to increase $I_{\text{mem}}(t) - I_{\text{pred}}(t)$. This second dependence seems to be stronger, meaning that $I_{\text{mem}}(t) - I_{\text{pred}}(t)$ tends to be maximal for $1 = \max(k_+, k_-) \gg \min(k_+, k_-)$.

A similar heatmap to figure 4.3 can be produced for the other terms in equation (2.35) as well. In particular, the heatmap for $-\beta \langle \Delta F_{\text{neq}}^{\text{relax}}(t) \rangle$ is shown in figure 4.4. One immediate observation is that like $I_{\text{mem}}(t) - I_{\text{pred}}(t)$, $-\beta \langle \Delta F_{\text{neq}}^{\text{relax}}(t) \rangle$ also appears to increase from 0 as $\frac{k_+}{k_-}$ is varied away from 1, holding k_+k_- constant. However, comparing the two plots there are clear differences in behaviour. It appears that $-\beta \langle \Delta F_{\text{neq}}^{\text{relax}}(t) \rangle$ is not strictly increasing or decreasing with k_+k_- .

From $I_{\text{mem}}(t) - I_{\text{pred}}(t)$ and $-\beta \langle \Delta F_{\text{neq}}^{\text{relax}}(t) \rangle$, the ratio, $\mu(t)$, defined in equation (2.38) can be calculated. This is shown in figure 4.5. $\mu(t)$ is not well defined at the black squares along the diagonal where $k_+ = k_-$. This ratio represents the tightness of the bound given in equation (2.37). Because $I_{\text{mem}}(t) - I_{\text{pred}}(t)$ increases with k_+k_- , while $-\beta \langle \Delta F_{\text{neq}}^{\text{relax}}(t) \rangle$ does not, this increase is the dominant behaviour of $\mu(t)$. It also increases as $\frac{k_+}{k_-}$ is varied away from 1, with k_+k_- held constant.

In this non-equilibrium steady-state picture, the condition that $\frac{k_+}{k_-}$ be far from 1 requires that the system be driven far from equilibrium over each work step. Large k_+k_- , on the other hand, means that the system relaxes farther towards equilibrium on relaxation steps. Therefore, the ideal case for maximal $\mu(t)$ is that the system is forced far from equilibrium on each work step, before relaxing back to equilibrium on each relaxation step.

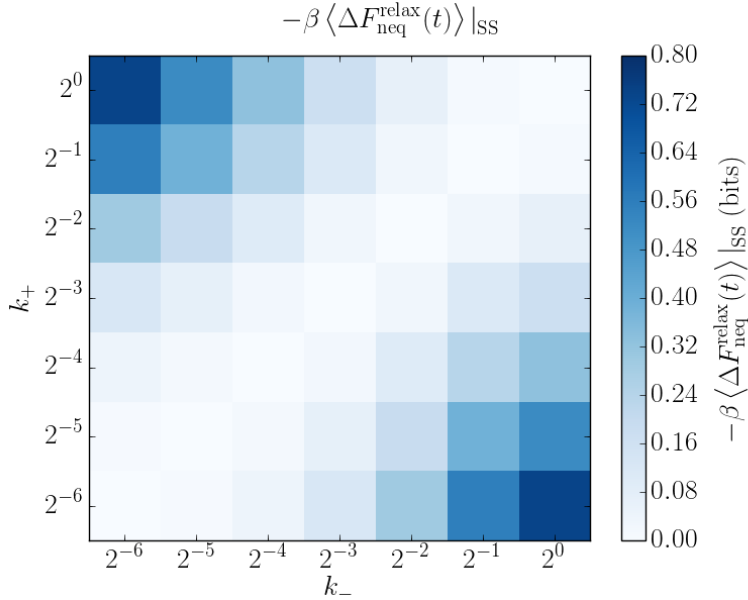


Figure 4.4: Steady-state value of $-\beta \langle \Delta F_{\text{neq}}^{\text{relax}}(t) \rangle$ in bits over a range of values for parameters k_+ and k_- . k_+ increases logarithmically upwards, while k_- increases logarithmically to the right. $\gamma = 0.25$ is used for this plot.

Outside of steady-state, $\mu(t)$ tends to be smaller, as shown in figure 4.6. Since the system starts in equilibrium, this could be attributed to the fact that the system needs to be driven away from this equilibrium in order to gain information about its environment. This can be seen as the system building up nostalgia as it is influenced by the environment. These effects could be studied by modifying the initial conditions.

4.2 Four-Parameter Model

The analysis of the previous section can be extended to the four-parameter system model described in section 3.1. Under the assumption of steady state, this gives a four dimensional parameter space to explore. The instantaneous nostalgia, $I_{\text{mem}}(t) - I_{\text{pred}}(t)$, is shown in figure 4.7, as a heatmap over this parameter space. The generalization from the previous section appears quite natural: $I_{\text{mem}}(t) - I_{\text{pred}}(t)$ increases with the product, $\omega_+ k_+ \omega_- k_-$. In addition, similarly to the two-parameter case, the nostalgia increases from 0 as $\frac{k_+ \omega_+}{k_- \omega_-}$ is varied away from 1.

Like the nostalgia, $-\beta \langle \Delta F_{\text{neq}}^{\text{relax}}(t) \rangle$ also seems to behave in a fairly natural way compared to the two-parameter system. This behaviour is shown in figure 4.8. $-\beta \langle \Delta F_{\text{neq}}^{\text{relax}}(t) \rangle$ also vanishes when $k_+ = k_- = \omega_+ = \omega_-$, and tends to be increase as $\frac{k_+ \omega_+}{k_- \omega_-}$ is varied away from 1.

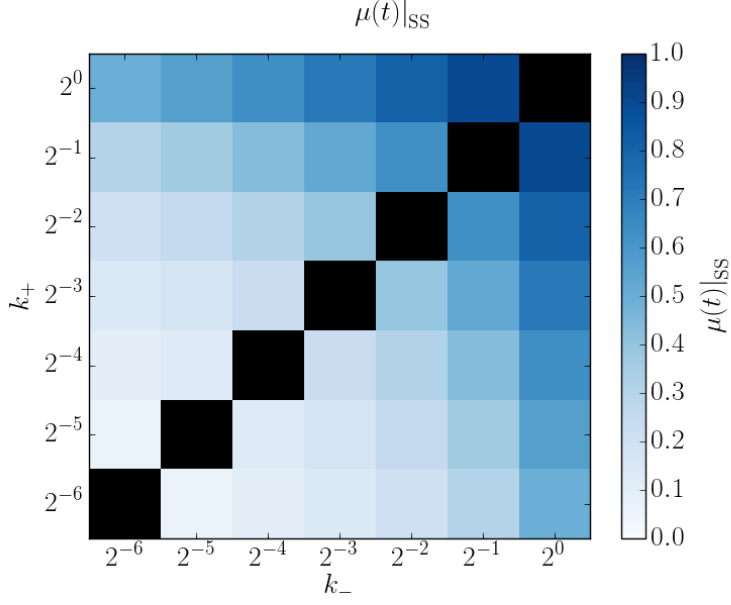


Figure 4.5: Steady-state value of $\mu(t)$ over a range of values for parameters k_+ and k_- . k_+ increases logarithmically upwards, while k_- increases logarithmically to the right. $\gamma = 0.25$ is used for this plot. The black squares are points at which the ratio, $\mu(t)$, is not defined since $\langle W_{\text{diss}}(t) \rangle = 0$.

In order to clarify the differences in behaviour between $-\beta \langle \Delta F_{\text{neq}}^{\text{relax}}(t) \rangle$, and $I_{\text{mem}}(t) - I_{\text{pred}}(t)$, the ratio $\mu(t)$ can also be plotted in this parameter space. $\mu(t)$ is plotted in figure 4.9. The black squares, lying where $k_+ = k_- = \omega_+ = \omega_-$, indicate where $\mu(t)$ is not defined, since $I_{\text{mem}}(t) - I_{\text{pred}}(t) = \beta \langle \Delta F_{\text{neq}}^{\text{relax}}(t) \rangle = 0$. In this plot, looking at the area where $k_+ \gg k_-$, some interesting behaviour emerges. In the top left subplot, maximal values of η appear neither where $k_+\omega_+k_-\omega_- \rightarrow 1$, nor where $\frac{k_+\omega_+}{k_-\omega_-}$ is far from 1. In fact, maximal values appear near where $\frac{k_+\omega_+}{k_-\omega_-} \approx 1$, and quite far from where $\omega_+\omega_-$ is maximal (which would maximize $k_+\omega_+k_-\omega_-$ within this subplot). Thus, there is clearly more subtlety to this behaviour than described in the two-parameter system section.

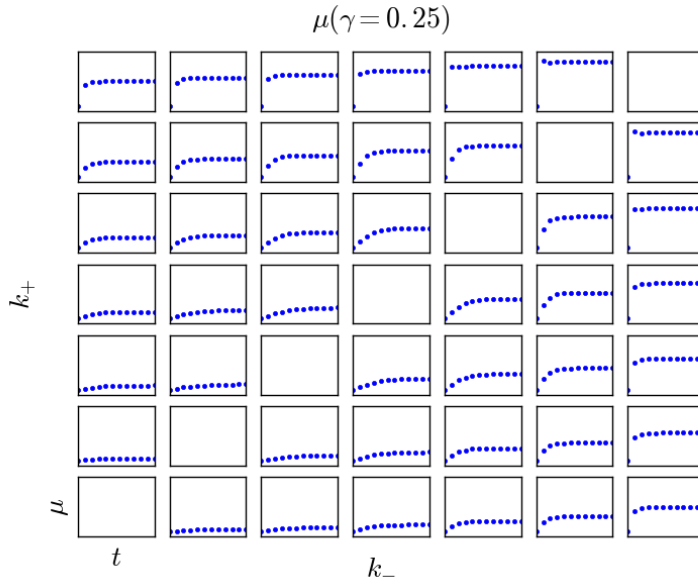


Figure 4.6: $\mu(t)$ versus time for some parameter values, k_+ and k_- . The top right subplot uses $k_+ = k_- = 1$. k_+ is reduced by a factor of 2 in each subsequent subplot to the left, while k_- is reduced by a factor of 2 in each subsequent subplot downwards. $\gamma = 0.25$ is used for every subplot. $\mu(t)$ is not defined along the main diagonal, where there are no points drawn.

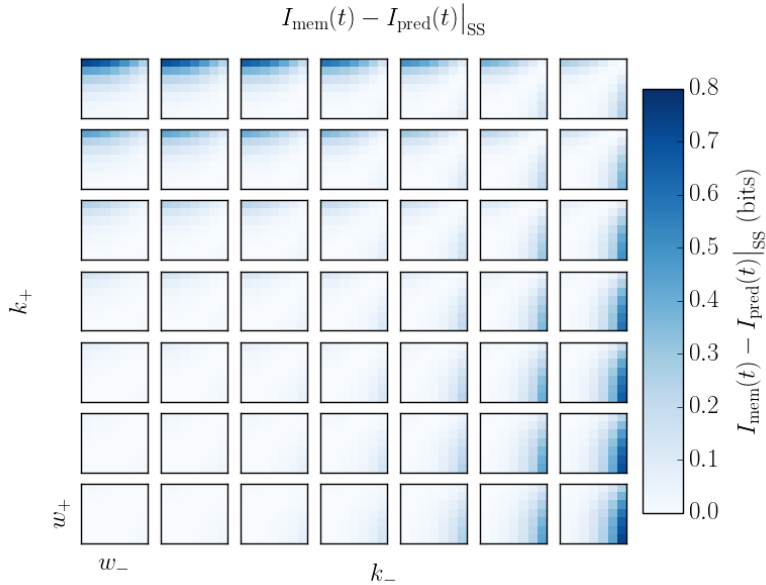


Figure 4.7: Steady-state value of $I_{\text{mem}}(t) - I_{\text{pred}}(t)$ in bits over a range of values for parameters k_+ , k_- , w_+ , and w_- . From subplot to subplot, k_+ increases logarithmically upwards up to 1, while k_- increases logarithmically to the right up to 1. Within each subplot, w_+ increases logarithmically upwards up to 1, while w_- increases logarithmically to the right up to 1. $\gamma = 0.25$ is used for this plot.

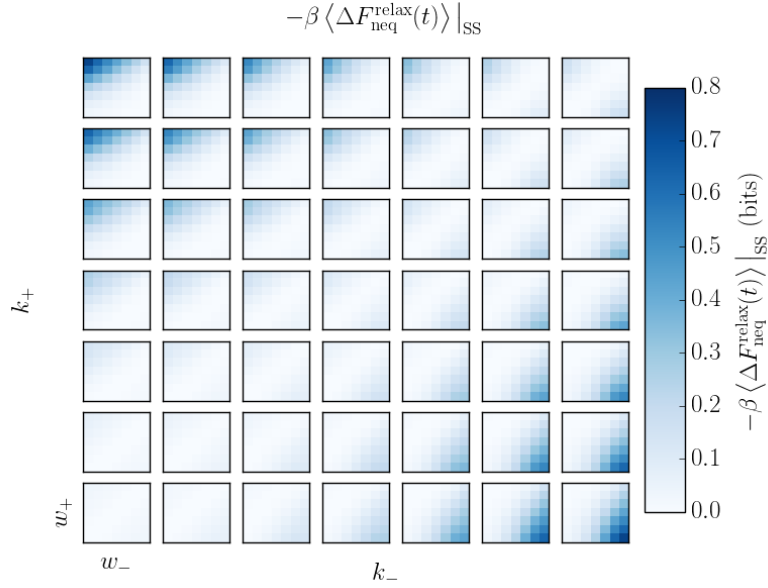


Figure 4.8: Steady-state value of $-\beta \langle \Delta F_{\text{neq}}^{\text{relax}}(t) \rangle$ in bits over a range of values for parameters k_+ , k_- , ω_+ , and ω_- . From subplot to subplot, k_+ increases logarithmically upwards up to 1, while k_- increases logarithmically to the right up to 1. Within each subplot, ω_+ increases logarithmically upwards up to 1, while ω_- increases logarithmically to the right up to 1. $\gamma = 0.25$ is used for this plot.

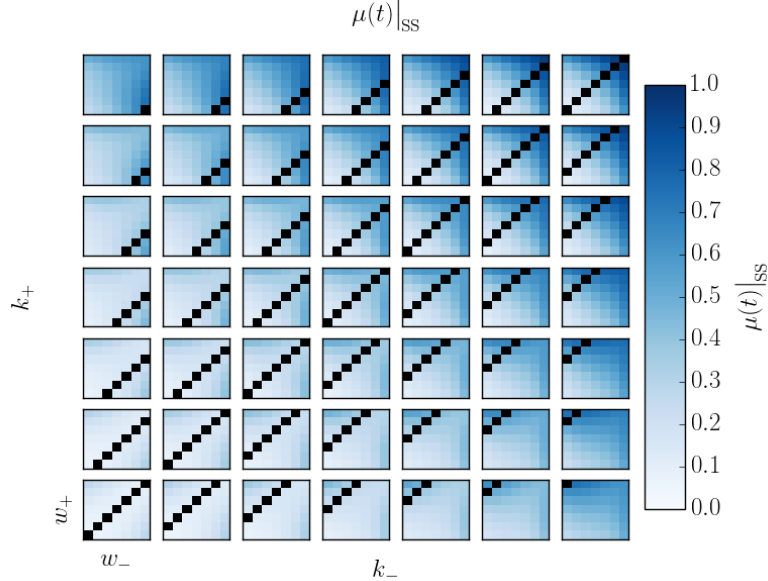


Figure 4.9: Steady-state value of $\mu(t)$ over a range of values for parameters k_+ , k_- , ω_+ , and ω_- . From subplot to subplot, k_+ increases logarithmically upwards up to 1, while k_- increases logarithmically to the right up to 1. Within each subplot, ω_+ increases logarithmically upwards up to 1, while ω_- increases logarithmically to the right up to 1. $\gamma = 0.25$ is used for this plot. The black squares are points at which the ratio, $\mu(t)$, is not defined since $\langle W_{\text{diss}}(t) \rangle = 0$.

Chapter 5

Analytic Results

5.1 Identification of $\langle \Delta F_{\text{neq}}^{\text{relax}}(t) \rangle$

In order to further examine the inequality given in equation (2.37), it would be helpful to illuminate the term, $\langle \Delta F_{\text{neq}}^{\text{relax}}(t) \rangle$. In this section, $\langle \Delta F_{\text{neq}}^{\text{relax}}(t) \rangle$ will be related to the information erasure over relaxation steps, $\mathcal{I}_e^{\text{relax}}(t)$, and thermodynamic entropy production due to relaxation steps, $\sigma_Y(t)$. This thermodynamic entropy production due to relaxation steps has been studied in comparable systems in the literature [6, 7, 8, 9].

By equation (2.19), the definition of $\langle \Delta F_{\text{neq}}^{\text{relax}}(t) \rangle$ given in equation (2.28) becomes:

$$\begin{aligned} \langle \Delta F_{\text{neq}}^{\text{relax}}(t) \rangle &= \langle (F_{\text{eq}}(x_{t+1}) + F^{\text{add}}[Y_{t+1}|x_{t+1}]) \rangle \\ &\quad - \langle (F_{\text{eq}}(x_{t+1}) + F^{\text{add}}[Y_t|x_{t+1}]) \rangle \\ &= \langle F^{\text{add}}[Y_{t+1}|x_{t+1}] - F^{\text{add}}[Y_t|x_{t+1}] \rangle. \end{aligned} \tag{5.1}$$

Using the definition of this non-equilibrium free energy contribution, this expression becomes:

$$\begin{aligned} \beta \langle \Delta F_{\text{neq}}^{\text{relax}}(t) \rangle &= \left\langle D_{\text{KL}} \left(p(y_{t+1}|x_{t+1}) \middle\| p_{\text{eq}}(y_{t+1}|x_{t+1}) \right) \right\rangle \\ &\quad - \left\langle D_{\text{KL}} \left(p(y_t|x_{t+1}) \middle\| p_{\text{eq}}(y_t|x_{t+1}) \right) \right\rangle \\ &= \left\langle \left\langle \ln \frac{p(y_{t+1}|x_{t+1})}{p_{\text{eq}}(y_{t+1}|x_{t+1})} \right\rangle_{p(y_{t+1}|x_{t+1})} \right\rangle \\ &\quad - \left\langle \left\langle \ln \frac{p(y_t|x_{t+1})}{p_{\text{eq}}(y_t|x_{t+1})} \right\rangle_{p(y_t|x_{t+1})} \right\rangle, \end{aligned} \tag{5.2}$$

where the second line follows from the definition of the KL divergence. Writing out the expectation values explicitly, along with some algebraic manipulation allows this to be

brought into the following form:

$$\beta \left\langle \Delta F_{\text{neq}}^{\text{relax}}(t) \right\rangle = \sum_{y_{t+1}, y_t} \sum_{x_{t+1}} p(x_{t+1}, y_{t+1}, y_t) \ln \frac{p(y_{t+1}|x_{t+1})}{p(y_t|x_{t+1})} - p(x_{t+1}, y_{t+1}, y_t) \ln \frac{p_{\text{eq}}(y_{t+1}|x_{t+1})}{p_{\text{eq}}(y_t|x_{t+1})}. \quad (5.3)$$

The information erasure and thermodynamic entropy production over relaxation steps and can now be introduced. The erasure over relaxation steps is given by:

$$\mathcal{I}_e^{\text{relax}}(t) := \sum_{y_{t+1}, y_t} \sum_{x_{t+1}} p(x_{t+1}, y_{t+1}, y_t) \ln \frac{p(y_{t+1}|x_{t+1})}{p(y_t|x_{t+1})}, \quad (5.4)$$

while the thermodynamic entropy production is given by:

$$\sigma_Y(t) := \sum_{y_{t+1}, y_t} \sum_{x_{t+1}} p(x_{t+1}, y_{t+1}, y_t) \ln \frac{p_{\text{eq}}(y_{t+1}|x_{t+1})}{p_{\text{eq}}(y_t|x_{t+1})}. \quad (5.5)$$

This allows equation (5.3) to be written simply as:

$$\beta \left\langle \Delta F_{\text{neq}}^{\text{relax}}(t) \right\rangle = \mathcal{I}_e^{\text{relax}}(t) - \sigma_Y(t). \quad (5.6)$$

The usefulness of the definitions in equations (5.4) and (5.5) will now be shown, starting with (5.4). A straightforward rearrangement of this expression gives:

$$\begin{aligned} \mathcal{I}_e^{\text{relax}}(t) &= \sum_{x_{t+1}, y_{t+1}} p(x_{t+1}, y_{t+1}) \ln p(y_{t+1}|x_{t+1}) - \sum_{x_{t+1}, y_t} p(x_{t+1}, y_t) \ln p(y_t|x_{t+1}) \\ &= H[Y_t|X_{t+1}] - H[Y_{t+1}|X_{t+1}], \end{aligned} \quad (5.7)$$

which is identified as the information erasure over the relaxation step from t to $t + 1$.

The σ_Y expression in equation (5.5) can be simplified by explicitly writing the random variables X_{t+1} , Y_t , and Y_{t+1} :

$$\sigma_Y(t) = \sum_{y_{t+1}, y_t} \sum_{x_{t+1}} p(X_{t+1} = x_{t+1}, Y_{t+1} = y_{t+1}, Y_t = y_t) \ln \frac{p_{\text{eq}}(Y_{t+1} = y_{t+1}|X_{t+1} = x_{t+1})}{p_{\text{eq}}(Y_t = y_t|X_{t+1} = x_{t+1})}. \quad (5.8)$$

To illustrate that x_{t+1} , y_t , and y_{t+1} are merely dummy indices, we will make the following substitutions: $x_{t+1} \rightarrow x$, $y_t \rightarrow y'$, $y_{t+1} \rightarrow y$. The above expression is then:

$$\sigma_Y(t) = \sum_{y, y'} \sum_x p(X_{t+1} = x, Y_{t+1} = y, Y_t = y') \ln \frac{p_{\text{eq}}(Y_{t+1} = y|X_{t+1} = x)}{p_{\text{eq}}(Y_t = y'|X_{t+1} = x)}. \quad (5.9)$$

From here, equation (2.5) can be applied in the argument of the logarithm, giving:

$$\sigma_Y(t) = \sum_{y,y'} \sum_x p(X_{t+1} = x, Y_{t+1} = y, Y_t = y') \ln \frac{p_{\text{eq}}(Y_t = y | X_{t+1} = x)}{p_{\text{eq}}(Y_t = y' | X_{t+1} = x)}. \quad (5.10)$$

Detailed balance, equation (2.6) can then be applied inside the logarithm to obtain:

$$\begin{aligned} \sigma_Y(t) &= \sum_{y,y'} \sum_x p(X_{t+1} = x, Y_{t+1} = y, Y_t = y') \ln \frac{p(y|y', x)}{p(y'|y, x)} \\ &= \sum_{y,y'} \sum_x p(X_{t+1} = x, Y_t = y') p(y|y', x) \ln \frac{p(y|y', x)}{p(y'|y, x)}. \end{aligned} \quad (5.11)$$

Equation (5.11) is simply the thermodynamic entropy production over relaxation steps, as defined by Barato, Hartich and Seifert, which has been investigated at steady-state in similar models [7].

We have thus shown that the change in non-equilibrium free energy of a relaxation step is given by the difference between the information erasure and the thermodynamic entropy production.

5.2 Relationship to Landauer's Principle

This section will illustrate the connections between (2.24), (2.32), and (5.6). To start, $\mathcal{I}_e^{\text{work}}(t) := H[s_t|x_t] - H[s_t|x_{t+1}] = -(I_{\text{mem}}(t) - I_{\text{pred}}(t))$ is identified as the information erasure over work steps. The net erasure, $\mathcal{I}_e(t)$, is given by $\mathcal{I}_e(t) := \mathcal{I}_e^{\text{work}}(t) + \mathcal{I}_e^{\text{relax}}(t)$. Thus, combining this with equations (2.32) and (5.6) gives:

$$\begin{aligned} \mathcal{I}_e(t) &= \sigma_Y(t) + \beta \langle \Delta F_{\text{neq}}[x_{t+1}; Y_t \rightarrow Y_{t+1}] \rangle - \beta \langle W_{\text{diss}}(y_t; x_t \rightarrow x_{t+1}) \rangle \\ &= \sigma_Y(t) - \beta \langle W_{\text{diss}}(t) \rangle, \end{aligned} \quad (5.12)$$

where the second line follows from equation (2.33). This bears strong similarities to Landauer's principle, equation (2.24). Comparing the two equations leads to the following identification:

$$\sigma_Y(t) = -\beta \langle Q(t) \rangle. \quad (5.13)$$

With this identification in hand, analogous expressions to Landauer's principle can be written for the individual substeps. For relaxation steps, we have:

$$\mathcal{I}_e^{\text{relax}}(t) = -\beta \langle Q(t) \rangle - \beta \langle W_{\text{diss}}(x_{t+1}; y_t \rightarrow y_{t+1}) \rangle, \quad (5.14)$$

while for work steps, we have:

$$\mathcal{I}_e^{\text{work}}(t) = -\beta \langle W_{\text{diss}}(y_t; x_t \rightarrow x_{t+1}) \rangle. \quad (5.15)$$

This novel result allows Landauer’s principle to be split into erasure due to system and environmental dynamics. Since $\mathcal{I}_e^{\text{work}}(t) \leq 0$, this means that the Landauer limit can be refined by considering only erasure due to the system’s dynamics.

5.3 Identification of $\mu(t)$ at Steady-State

Barato, Hartich and Seifert introduced an informational efficiency, $\eta(t)$, which quantifies the rate at which the system learns about its environment, relative to the rate at which it dissipates energy [7, 9]. This efficiency bears close resemblance to the ratio $\mu(t)$ that we have introduced to study equations (2.35) and (2.37). By restricting our investigation to steady-state behaviour, the two can be shown to be equal. This is demonstrated in this section.

We can begin by defining a learning rate,

$$l_Y(t) = I[X_{t+1}, Y_{t+1}] - I[X_{t+1}, Y_t], \quad (5.16)$$

in analogy with Barato, Hartich, and Seifert [7]. By restricting to steady-state, we have that $I[X_{t+1}, Y_{t+1}] = I[X_t, Y_t]$. Therefore, we have:

$$I_{\text{mem}}(t) - I_{\text{pred}}(t) \Big|_{\text{SS}} = l_Y(t) \Big|_{\text{SS}}. \quad (5.17)$$

The information erasure at steady-state becomes:

$$\begin{aligned} \mathcal{I}_e(t) \Big|_{\text{SS}} &= (H[Y_t|X_t] - H[Y_{t+1}|X_{t+1}]) \Big|_{\text{SS}} \\ &= 0. \end{aligned} \quad (5.18)$$

Thus, equation (5.12) becomes the following at steady-state:

$$\sigma_Y(t) \Big|_{\text{SS}} = \beta \langle W_{\text{diss}}(t) \rangle \Big|_{\text{SS}}. \quad (5.19)$$

Therefore, using equations (5.17) and (5.19), at steady-state, $\mu(t)$ can be written as:

$$\mu(t) \Big|_{\text{SS}} = \frac{l_Y(t)}{\sigma_Y(t)} \Big|_{\text{SS}}. \quad (5.20)$$

This form allows for $\mu(t) \Big|_{\text{SS}}$ to be identified with the informational efficiency, η , which is studied by Barato, Hartich, and Seifert [7, 9].

Thus, we have shown that:

$$\frac{I_{\text{mem}}(t) - I_{\text{pred}}(t)}{\langle W_{\text{diss}}(t) \rangle} \Big|_{\text{SS}} = \frac{l_Y(t)}{\sigma_Y(t)} \Big|_{\text{SS}}, \quad (5.21)$$

which the author has not seen before in the literature. These two sides lead to different interpretations of the same quantity. On the left, this quantity is the fraction of dissipation that is due to model inefficiency. On the right, this quantity is the rate at which the system learns about the environment, relative to the thermodynamic entropy production over relaxation steps. The thermodynamic entropy production over relaxation steps has previously been shown to provide an upper bound on this learning rate at steady-state [7].

Chapter 6

Discussion

6.1 Interpretation

The result in section 5.3 allows for new interpretations of the results described in sections 4.1 and 4.2. Knowing that $\mu(t)\Big|_{\text{SS}} = \eta(t)$, the rate at which the system learns about the environment can be seen to approach the heat dissipation as the system approaches equilibrium. Outside of steady-state, the two quantities are not in general equal.

The results of Barato, Hartich and Seifert can also be re-interpreted in the context of equation (2.35) [7]. In their analysis, time was taken to be continuous, rather than discrete as in our framework. As well, only the steady-state case was considered. In this framework, using the four-parameter model discussed in this paper, η was found to increase as $\frac{k_+\omega_+}{k_-\omega_-}$ was increased away from 1, or as γ was decreased. In a continuous time framework, decreasing γ is equivalent to increasing the four system transition rates, $k_+\omega_+k_-\omega_-$. This is consistent with our findings that the bound, equation (2.37), is maximal in the same case. Barato et. al also investigate a more complex four state system, coupled again to a two state environment. In this model, η is again found to be maximal when the environment transitions are slow compared to those of the system.

6.2 Conclusion

Thermodynamics of Prediction, by Still, Sivak, Bell, and Crooks described a relationship between dissipation and a novel and abstract information-theoretical concept, here referred to as the nostalgia [2]. This nostalgia represents information that the system stores about its environment that fails to be predictive of the future states of the environment. In order to perform efficiently, this nostalgia must be avoided. This framework finds applications throughout biology, as all living things are influenced by, and thus learn about, their environments.

While fascinating, this development has remained rather mysterious and unexplored. In order to remedy this, in this work the nostalgia has been calculated in some example systems, alongside the other variables of interest in the original paper. Mapping these variables out over the system parameter space helps to establish an intuitive picture. It has been found that in simple systems, the nostalgia provides a tight lower bound on the dissipation at steady state when the system is strongly driven and quick to relax. It has also been shown to be equivalent to the statement that when the system is strongly driven and slow to relax, it learns more about the environment per unit heat dissipated.

6.3 Future Work

There are many future directions to pursue on this topic. Among the most promising is to study the system resulting from taking the continuous time limit. In this limit, the transition probabilities k_+ , k_- , ω_+ , and ω_- would be replaced with corresponding transition rates. Taking time to be continuous would allow for better modeling of many real-world systems. As well, it would allow for a wider range of behaviour, since the current discrete system places an upper bound on the effective transition rates.

Another area to be explored would be more complex models than the simple two-state systems and environments described here. Barato, *et al.* take the approach of expanding the system to four states, with the environment remaining a simple two-state system [7]. One could also expand the environmental behaviour, whether through additional states, or through non-Markovian dynamics, since this framework does not restrict the form of these transitions.

Studies such as these would help to further elucidate the work done by Still, *et al.*, described in section 2.3 [2].

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