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Online Appendix for Coordination-Free Equilibria in Cheap Talk Games

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Online Appendix A: Relation between Candidate Play and Equilibrium Play in Coordination-Free Equilibrium

First, consider the following example illustrating two ways in which candidate play can fail to be equilibrium play.

Example 6: *Consider candidate play with two senders whose biases satisfy* $b_1(.) < 0$ *and* $b_2(.) > 0$*, and three cells where the equilibrium message vectors are, from left to right,* (L, L)*,* (H, L) *and* (H, H) *(so that* (L, H) *is out-of-equilibrium). It is easy to think of two cases where any location of* $a^{\Gamma}(L, H)$ *induces a deviation:*

a) Sender 1 has a small leftward bias while sender 2 has a large rightward bias, such that the middle interval (H, L) *is very small, and the rightmost interval* (H, H) *is very big (see Figure A.1). Then choosing* $a^{\Gamma}(L, H) < a^{\Gamma}(H, H)$ *induces sender 1 to deviate from* H *to* L *near the left end of the* (H, H) *interval, choosing* $a^{\Gamma}(L, H) > a^{\Gamma}(H, H)$ *induces sender 1 to deviate near the right end of the* (H, H) *interval, while choosing* $a^{\Gamma}(L, H) = a^{\Gamma}(H, H)$ *induces sender 2 to deviate from* L *to* H *near the right end of the* (L, L) *interval.*

Figure A.1: Case a

b) Sender 2 dislikes $a^{\Gamma}(L, L)$ *so much that, at the boundary* θ_1 *between* (L, L) *and* (H, L)*,* $u_2(a, \theta_1) > u_2(a^{\Gamma}(L, L), \theta_1)$ *for all* $a > \theta_1$*. Similarly, sender 1 dislikes* $a^{\Gamma}(H, H)$ *so much that* $u_1(a, \theta_2) > u_1(a^{\Gamma}(H, H), \theta_2)$ *for all* $a < \theta_2$*. Since* $a^{\Gamma}(H, H) > a^{\Gamma}(L, L)$*, it is impossible for* $a^{\Gamma}(L, H)$ *to be simultaneously less than* $a^{\Gamma}(L, L)$ *and greater than* $a^{\Gamma}(H, H)$*, so once again a deviation is always desired.*

The following derives a condition under which candidate play is guaranteed to constitute equilibrium play.

Let $U_i(\theta) = \{u : \exists a_1 \neq a_2 \in \Theta \text{ s.t. } u_i(a_1, \theta) = u_i(a_2, \theta) = u\}$ be the set of utilities achieved for sender i at state θ by two distinct actions. By the single-peakedness of u_i, these actions must be on opposite sides of sender i's ideal action $\theta + b_i(\theta)$. Let $a_i^-(u, \theta) < a_i^+(u, \theta)$ be these actions. Then let $A_i = \max_{\theta \in \Theta} \max_{u \in U_i(\theta)} \max\{\frac{\theta + b_i(\theta) - a_i^-(u,\theta)}{a_i^+(u,\theta) - (\theta + b_i(\theta))}, \frac{a_i^+(u,\theta) - (\theta + b_i(\theta))}{\theta + b_i(\theta) - a_i^-(u,\theta)}\}$ be a measure of how asymmetric sender i's utility function can get around its peak $\theta + b_i(\theta)$: if u_i is perfectly symmetric, as in the quadratic case, then $A_i = 1$, and the more asymmetric it is, the higher A_i.

Given candidate play, call an *i-block* a maximal interval of states where each sender other than i sends a single message. Clearly, every block is a union of cells, and an i-block with more than one cell is formed when sender i's message changes at a boundary. For example, with two senders and three cells numbered 1, 2 and 3 from left to right, if the message pairs are (m_1, m_2), (m_1, m_2') and (m_1', m_2') in cells 1, 2 and 3 respectively, then there are two 1-blocks (cell 1; cells 2 and 3) and two 2-blocks (cells 1 and 2; cell 3). Note that a given i-block and a given j-block can overlap for at most one cell because each boundary can be crossed by only one block.

Proposition 7: Given candidate play Γ, let k_i^Γ denote the size of the largest i-block, and let $x_i^\Gamma = (1 + A_i)(k_i^\Gamma + \max_{\theta \in \Theta} |b_i(\theta)|)$ for each i whose message changes at some boundary. If the sum of the two largest x_i^Γ is less than $\lambda(\Theta) = 1$, then there exists a strictly coordination-free equilibrium where:

- play is described by Γ; and
- each player i's messages can be ordered so that $m_i(\theta)$ is non-decreasing.

Proof of Proposition 7: Given candidate play Γ, assign messages as follows: in the leftmost cell, all senders send 1, and at every boundary where a sender's message changes, that sender's message increases by 1. This message assignment rules out the following scenario: in a cell where the assigned message vector is $m = (m_1, ..., m_n)$, a sender (without loss of generality, sender 1) wants to deviate to m_1', and $m' = (m_1', m_2, ..., m_n)$ occurs on the equilibrium path. To see this, assume without loss of generality that $a^\Gamma(m') > a^\Gamma(m)$. Then it must be that in the cell immediately to the right of the one where m is sent, the message vector is $m'' = (m_1'', m_2, ..., m_n)$ for some m_1'' possibly equal to m_1', so $a^\Gamma(m') \geq a^\Gamma(m'')$. Since within the cell where m is sent, sender 1 prefers $a^\Gamma(m)$ to $a^\Gamma(m'')$, by single-crossing, she also prefers $a^\Gamma(m)$ to $a^\Gamma(m')$ and cannot desire a deviation.

Therefore, the only concern is to place the receiver's actions after off-path message vectors without inducing a deviation. For any off-path message vector m, there are at most two senders whose deviation can induce m. To see this, normalize messages by subtracting a constant to each sender's messages such that $m = (0, ..., 0)$. If a sender i can induce m by deviating from a negative message when all others send 0, then when i sends 0, all other senders' messages must be nonnegative. Thus only one other sender can deviate to m, and must do so from a positive message. The symmetric argument holds as well, so at most one sender can deviate to m from a positive message, and at most one sender can deviate to m from a negative message.

Now suppose sender i can deviate to induce m. The set of states from which she can do this must constitute an i-block, which has size at most k_i^Γ. Denote the left and right endpoints of the i-block by θ_L and θ_R, the leftmost and rightmost inducible actions within the i-block by a_L and a_R, and assume without loss of generality that $b_i(.) > 0$. Then a deviation by i will not be induced if either:

- $a(m) > \theta_R + \max_{\theta\in\Theta} |b_i(\theta)| + A_i(\theta_R + \max_{\theta\in\Theta} |b_i(\theta)| - a_R)$; or
- $a(m) < \min\{a_L, \theta_L + \min_{\theta\in\Theta} |b_i(\theta)| - A_i(a_L - (\theta_L + \min_{\theta\in\Theta} |b_i(\theta)|))\}$.

Therefore, letting $D_i = \max_{\theta\in\Theta} |b_i(\theta)| - \min_{\theta\in\Theta} |b_i(\theta)|$, the maximum range where a deviation can be induced is:

$$\begin{aligned} & \max\{(A_i+1)\max_{\theta\in\Theta} |b_i(\theta)| + A_i(\theta_R - a_R) + (\theta_R - a_L), (A_i+1)(k_i^\Gamma + D_i) - A_i(a_R - a_L)\} \\ < \;& (A_i+1)(k_i^\Gamma + \max_{\theta\in\Theta} |b_i(\theta)|) = x_i^\Gamma, \end{aligned}$$

where the inequality follows from $\theta_R - a_R, \theta_R - a_L < k_i^\Gamma$, $D_i < \max_{\theta\in\Theta} |b_i(\theta)|$, and $a_R - a_L \geq 0$.

If the ranges for the two potential deviators do not cover Θ, then it is possible to place $a^\Gamma(m)$ without inducing a deviation. The result follows. ∎

The proof of Proposition 7 shows that if messages are assigned as stated, then no deviation to an on-path message vector is ever desired, and at most two senders, each from one block, can deviate to an out-of-equilibrium message vector. An i-block of size k is associated with an interval of size at most $(1+A_i)(k+\max_{\theta\in\Theta} |b_i(\theta)|)$ where placing an out-of-equilibrium action would cause a deviation by sender i. Therefore, the total area where an out-of-equilibrium vector cannot be placed is at most the sum of the two largest x_i^Γ.

In the uniform-quadratic specification, as shown in the first paragraph of the proof of Proposition 5, cell size changes by $4b_i$ (from left to right) at a boundary where sender i's message changes. Thus, if $b_i > 0$, cells grow from left to right, and vice versa. It follows that:

- cells can be kept small if, in each direction, there is a sender with a small bias; and
- large i-blocks must contain large cells (relative to $|b_i|$) at one end.

Thus, the most informative candidate play must only have small i-blocks if, in each direction, there is a sender with a small bias. In this situation, any sufficiently informative candidate play Γ will have small k_i^Γ, and therefore small $x_i^\Gamma = 2(k_i^\Gamma + |b_i|)$, for all $i \in N$. Proposition 7 implies that such Γ corresponds to play in a strictly coordination-free equilibrium where messages are assigned so that each is used on a connected set of states. As a result, for each $i \in N$, there exists an order on M_i^Γ such that sender i's strategy is *monotonic.*[1]

A similar reasoning can be applied whenever the receiver's preferred action in each cell is not far from its center[2], and A_i is close to 1 for each sender. Therefore, in such settings, Proposition 7 (combined with Theorem 2 motivating strictly coordination-free equilibria) provides a justification for focusing on monotonic strategies when studying the most informative equilibria, if a sender with small bias is available in each direction.

Online Appendix B: Analysis Without Assumption A

This section dispenses with Assumption A, and allows for noise where players have heterogeneous prior, as long as there is common knowledge that noise is small. Then, if there is no state θ and pair of actions between which two senders are both indifferent at θ,[3] the implications of Theorems 1 and 2 about the function $a^\Gamma \circ m^\Gamma$ mapping state to action in a (strongly) robust equilibrium Γ still hold: it must generically correspond to candidate play computed by forward solution (and, for strong robustness, be complete).

Definition: Given a pure-strategy profile Γ, let a *supercell in* Γ be a maximal interval of states throughout which $a^\Gamma \circ m^\Gamma$ remains constant.

Definition: A proper supercell in Γ is *natural** if, denoting its endpoints as $\theta_1 < \theta_2$ and its induced action as a:

- (right-natural*) whenever $\theta_1 \neq 0$, $\exists \theta'$ such that $a^\Gamma(m^\Gamma(\theta')) = a$ and that, for some $i \in N$, $\exists m'_i \in M_i^\Gamma$ such that:

$$\begin{aligned} & (m'_i, m^\Gamma_{-i}(\theta')) = m^\Gamma(\theta) \text{ for some } \theta \in \Theta, \text{ and} \\ u_i(a^\Gamma(m'_i, m^\Gamma_{-i}(\theta')), \theta_1) \quad = \quad & u_i(a, \theta_1) \text{ and } a^\Gamma(m'_i, m^\Gamma_{-i}(\theta')) < a; \text{ and} \end{aligned}$$

- (left-natural*) whenever $\theta_2 \neq 1$, $\exists \theta''$ such that $a^\Gamma(m^\Gamma(\theta'')) = a$ and that, for some

[1] Given a strictly coordination-free equilibrium Γ, it is not always possible to obtain a monotonic strictly coordination-free equilibrium through a reassignment of messages. Consider Example 3, and change m_1 in $(0.51, 1]$ to $z \neq x, y$ so that sender 1's strategy becomes monotonic. Message vector (x, y) is now out-of-equilibrium. If $a^\Gamma(x, y)$ is placed anywhere other than 0.285 and 0.755, then sender 1 would have a profitable deviation to x at some $\theta \in (0.06, 1]$. But placing $a^\Gamma(x, y)$ at 0.285 or 0.755 violates point 3 of the definition of strictly coordination-free equilibrium.

[2] This happens whenever F is not too far from being uniform and u_R is not too asymmetric.

[3] This assumption holds for generic biases (*i.e.* whenever no two biases are exactly equal) within the class of quadratic loss preferences from Section 3.

$j \in N$, $\exists m''_j \in M_i^\Gamma$ such that:

$$\begin{aligned} (m''_j, m^\Gamma_{-j}(\theta'')) &= m^\Gamma(\theta) \text{ for some } \theta \in \Theta, \text{ and} \\ u_j(a^\Gamma(m''_j, m^\Gamma_{-j}(\theta'')), \theta_2) &= u_j(a, \theta_2) \text{ and } a^\Gamma(m''_j, m^\Gamma_{-j}(\theta'')) > a. \end{aligned}$$

Definition: An equilibrium is *natural** if its strategy profile has interval structure, and all of its proper supercells are natural*.

The definition of natural* proper supercell implies that, in Γ, $m^\Gamma(\theta')$ can be sent only at and to the right of θ_1. Since $a^\Gamma(m^\Gamma(\theta')) = a$, we have $\theta_1 \leq a$ and, by a similar argument, $a \leq \theta_2$. It follows that there is at most one proper supercell inducing a. As a result, a left-structure and a right-structure can be defined for a natural* equilibrium Γ in the same way as for natural equilibria, but using supercells rather than cells. The argument in the proof of Proposition 4(b) carries through: if $a^\Gamma \circ m^\Gamma$ does not correspond to candidate play, then the conditions imposed by the structures would be too numerous and thus, generically, would not be satisfied. In this context, that argument implies the following result:

Proposition 4*: Generically, if Γ is natural*, then the endpoints and induced action for all proper supercells in Γ can be computed by forward solution.

The results corresponding to Theorems 1 and 2 in the main text are as follows.

Theorem 1*: Suppose that whenever $u_i(a,\theta) = u_i(a',\theta)$, we have $u_j(a,\theta) \neq u_j(a',\theta)$ for all $j \neq i$ ("no simultaneous indifference," henceforth abbreviated NSI). Then:

(a) Generically, if Γ is strongly robust, then it is complete and corresponds to a forward solution (*i.e.* it has interval structure, and the endpoints and induced action for all proper supercells in Γ can be computed by forward solution.).

(b) If Γ is coordination-free and complete and has finitely many cells, and no cell in Γ is $\{0\}$ or $\{1\}$, then it is strongly robust.

Theorem 2*: Assume NSI. Then:

(a) Generically, if Γ is robust, then it corresponds to a forward solution.

(b) If Γ is strictly coordination-free and has finitely many cells, then it is robust.

The proofs of Theorems 1(b) and 2(b) remain valid for Theorems 1*(b) and 2*(b). These arguments rely on the number of cells being finite, which needs to be assumed here: while coordination-freeness still guarantees a finite number of supercells, within a given supercell, there could now be infinitely many cells. (For example, if $a^\Gamma(m_i, m_j, m_{-ij}) =$

$a^\Gamma(m_i', m_j, m_{-ij}) = a^\Gamma(m_i, m_j', m_{-ij})$, then there could be a supercell where the message vector sent switches infinitely many times between (m_i, m_j, m_{-ij}), (m_i', m_j, m_{-ij}) and (m_i, m_j', m_{-ij}). This can pose problems if $a^\Gamma(m_i, m_j, m_{-ij}) \neq a^\Gamma(m_i', m_j', m_{-ij})$.)

By Proposition 4*, to prove Theorem 1*(a), it suffices to show the following lemmata.

Lemma 1*: If Γ is strongly robust, then it is complete, has interval structure, and $\{m : m^\Gamma(\theta) = m \text{ for some } \theta \in \Theta\}$ is finite.

Lemma 2*: If Γ is strongly robust and NSI holds, then Γ is natural*.

Proof of Lemma 1*: The proof of Lemma 1, which shows $\lambda(\theta^\Gamma(m)) > 0$ for all $m \in \times_{i=1}^n M_i^\Gamma$ and interval structure, carries over.

Fix any $\delta \in (0, \frac{1}{4})$, and suppose instead that $\{m : m^\Gamma(\theta) = m \text{ for some } \theta \in \Theta\}$ is infinite. Then, for any $\varepsilon > 0$, $\exists m^0$ such that $\lambda(\theta^\Gamma(m^0)) \in (0, \varepsilon)$. Fix such m^0, and let $\Theta_0 = \theta^\Gamma(m^0)$ and $\Theta_i = \{\theta \in \Theta \backslash \Theta_0 : m_i^\Gamma(\theta) = m_i^0\}$. Let $\Theta' \subset \Theta \backslash \Theta_0$ be a nontrivial set of states such that the receiver's best response conditional on $\theta \in \Theta'$, denoted a', is outside $(a^\Gamma(m^0) - 2\delta, a^\Gamma(m^0) + 2\delta)$. ($\Theta'$ exists for ε sufficiently small.) Denote the *ex ante* probability that $\theta \in S$ by $F(S)$.

Since $\lambda(\Theta_0) < \varepsilon$, for any $\theta \in \Theta_0$, $\exists \theta_0(\theta) \in (\theta - \varepsilon, \theta + \varepsilon)$ such that for some $i \in N$, $m_i^\Gamma(\theta) \neq m_i^\Gamma(\theta_0(\theta))$. Let $i_\varepsilon^\Gamma(\theta)$ be some such i.

Consider noise Ξ where:

(i) at states $\theta \in \Theta'$, for each $i \in N$, with probability $\varepsilon \min\{\frac{1}{n}, \frac{F(\Theta_i)}{F(\Theta')}\}$, sender i observes $s_i \in \Theta_i$ according to density proportional to the prior, and with the remaining probability, $s_i = \theta$; observations are independent across senders;

(ii) at states $\theta \in \Theta_0$, consider a random variable X distributed according to a continuous density g, where $g(0) > 0$; if the realization of X is 0, sender $i_\varepsilon^\Gamma(\theta)$ observes $s_i = \theta$, while if not, sender $i_\varepsilon^\Gamma(\theta)$ observes $s_i = \theta_0(\theta)$;

(iii) if neither (i) or (ii) applies, the true state is observed.

It is straightforward to check that, by construction, Ξ has size at most ε. With *ex ante* probability $F(\Theta')\varepsilon^n \prod_{j=1}^n \min\{\frac{1}{n}, \frac{F(\Theta_j)}{F(\Theta')}\}$, the receiver observes m^0 and the state is in Θ'; with *ex ante* probability 0, the receiver observes m^0 and the state is in Θ_0; and with the remainder probability, the receiver does not observe m^0. Therefore, for ε sufficiently small, $a^\Xi(m^0) = a'$ is more than δ away from $a^\Gamma(m^0)$. By step 1 of the proof of Lemma 1, Γ is then not strongly robust. ■

Proof of Lemma 2*: Like for Lemma 2, we proceed by contradiction. Suppose instead, without loss of generality, that a proper supercell C in Γ with right endpoint θ_b is not left-natural*. Consider the following classes of noise:

Noise $(\varepsilon, N_1, ..., N_n, -)$

- Each sender i believes that $s_j = \max\{\theta - \varepsilon, 0\}$ for $j \in N_i \subseteq N\backslash\{i\}$, and that $s_j = \theta$ for $j \notin N_i$.

- The receiver believes that all senders observe the true state.

- These beliefs are common knowledge.

Noise $(\varepsilon, N_1, ..., N_n, +)$

- Each sender i believes that $s_j = \min\{\theta + \varepsilon, 1\}$ for $j \in N_i \subseteq N\backslash\{i\}$, and that $s_j = \theta$ for $j \notin N_i$.

- The receiver believes that all senders observe the true state.

- These beliefs are common knowledge.

Case A: $\theta_b \in C$

Fix an arbitrary $\varepsilon > 0$, and denote $m^\Gamma(\theta_b) = m$ and $m^\Gamma(\theta_b + \varepsilon) = m^\varepsilon$. Since $\theta_b \in C$, we have $a^\Gamma(m) = a$.

Since Γ is strongly robust, for any $\delta > 0$, there exists ε sufficiently small so that sender i's δ-optimality at $s_i = \theta_b$ under noise $(\varepsilon, N_i, N_{-i}, +)$ implies

$$u_i(a^\Gamma(m_i, m^*_{-i}), \theta_b) \geq u_i(a^\Gamma(m^\varepsilon_i, m^*_{-i}), \theta_b) - \delta,$$

where $m^*_j = m_j$ if $j \notin N_i$ and $m^*_j = m^\varepsilon_j$ if $j \in N_i$. Moreover, sender i's δ-optimality at $s_i = \theta_b + \varepsilon$ under noise $(\varepsilon, N\backslash(N_i \cup \{i\}), N_{-i}, -)$ implies

$$u_i(a^\Gamma(m^\varepsilon_i, m^*_{-i}), \theta_b + \varepsilon) \geq u_i(a^\Gamma(m_i, m^*_{-i}), \theta_b + \varepsilon) - \delta.$$

Therefore, we must have

$$u_i(a^\Gamma(m^\varepsilon_i, m^*_{-i}), \theta_b) \to u_i(a^\Gamma(m_i, m^*_{-i}), \theta_b) \text{ as } \varepsilon \to 0. \tag{1}$$

By Lemma 1*, $|M^\Gamma_i|$ is finite for all $i \in N$, so (1) implies that for any ε sufficiently small,

$$u_i(a^\Gamma(m^\varepsilon_i, m^*_{-i}), \theta_b) = u_i(a^\Gamma(m_i, m^*_{-i}), \theta_b). \tag{2}$$

Fix $\varepsilon > 0$ such that $a^\Gamma(m^\varepsilon) \neq a$ and (2) hold; such ε must exist, or else θ_b would not be an endpoint of C.

Observation 1:

(i) Suppose (2) holds, $m_i \neq m_i^\varepsilon$, $m_j \neq m_j^\varepsilon$, and $a^\Gamma(m_i, m_j, m_{-ij}^*) = a$ for some $m_{-ij}^* \in \times_{k \neq i,j}\{m_k, m_k^\varepsilon\}$. Then either $a^\Gamma(m_i^\varepsilon, m_j, m_{-ij}^*) = a^\Gamma(m_i, m_j^\varepsilon, m_{-ij}^*) = a^\Gamma(m_i^\varepsilon, m_j^\varepsilon, m_{-ij}^*) \neq a$, or at least one of these three actions is equal to a.

(ii) If NSI additionally holds, then we have $a^\Gamma(m_i^\varepsilon, m_j, m_{-ij}^*) = a$, $a^\Gamma(m_i, m_j^\varepsilon, m_{-ij}^*) = a$, or both.

Proof of Observation 1: (i) Suppose $a^\Gamma(m_i^\varepsilon, m_j, m_{-ij}^*) > a$. Then $u_i(a^\Gamma(m_i, m_j, m_{-ij}^*), \theta_b) = u_i(a^\Gamma(m_i^\varepsilon, m_j, m_{-ij}^*), \theta_b)$ implies $\theta_b + b_i(\theta_b) \in (a^\Gamma(m_i, m_j, m_{-ij}^*), a^\Gamma(m_i^\varepsilon, m_j, m_{-ij}^*))$. We also have $u_j(a^\Gamma(m_i^\varepsilon, m_j, m_{-ij}^*), \theta_b) = u_j(a^\Gamma(m_i^\varepsilon, m_j^\varepsilon, m_{-ij}^*), \theta_b)$. If $a^\Gamma(m_i^\varepsilon, m_j^\varepsilon, m_{-ij}^*) \neq a$, then either:

a) $a^\Gamma(m_i^\varepsilon, m_j^\varepsilon, m_{-ij}^*) > a^\Gamma(m_i^\varepsilon, m_j, m_{-ij}^*)$, so that $\theta_b + b_j(\theta_b) > a^\Gamma(m_i^\varepsilon, m_j, m_{-ij}^*)$. In this case, since $a^\Gamma(m_i^\varepsilon, m_j, m_{-ij}^*) > a^\Gamma(m_i, m_j, m_{-ij}^*)$ and $u_j(a^\Gamma(m_i, m_j, m_{-ij}^*), \theta_b) = u_j(a^\Gamma(m_i, m_j^\varepsilon, m_{-ij}^*), \theta_b)$, we have either $a^\Gamma(m_i, m_j^\varepsilon, m_{-ij}^*) = a^\Gamma(m_i, m_j, m_{-ij}^*)$, or $a^\Gamma(m_i, m_j^\varepsilon, m_{-ij}^*) > a^\Gamma(m_i^\varepsilon, m_j^\varepsilon, m_{-ij}^*)$. The latter is not possible since both $a^\Gamma(m_i, m_j^\varepsilon, m_{-ij}^*)$ and $a^\Gamma(m_i^\varepsilon, m_j^\varepsilon, m_{-ij}^*)$ would be to the right of $\theta_b + b_i(\theta_b)$, and sender i must be indifferent between these actions at θ_b. Therefore, $a^\Gamma(m_i, m_j^\varepsilon, m_{-ij}^*) = a^\Gamma(m_i, m_j, m_{-ij}^*) = a$.

b) $a^\Gamma(m_i^\varepsilon, m_j^\varepsilon, m_{-ij}^*) \in (a, a^\Gamma(m_i^\varepsilon, m_j, m_{-ij}^*))$, so that $\theta_b + b_j(\theta_b) \in (a^\Gamma(m_i^\varepsilon, m_j^\varepsilon, m_{-ij}^*), a^\Gamma(m_i^\varepsilon, m_j, m_{-ij}^*))$. It follows that either $a^\Gamma(m_i, m_j^\varepsilon, m_{-ij}^*) = a$, or $a^\Gamma(m_i, m_j^\varepsilon, m_{-ij}^*) > a^\Gamma(m_i^\varepsilon, m_j, m_{-ij}^*)$. The latter is ruled out since sender i cannot be simultaneously indifferent between a and $a^\Gamma(m_i^\varepsilon, m_j, m_{-ij}^*)$, as well as between $a^\Gamma(m_i^\varepsilon, m_j^\varepsilon, m_{-ij}^*) > a$ and $a^\Gamma(m_i, m_j^\varepsilon, m_{-ij}^*) > a^\Gamma(m_i^\varepsilon, m_j, m_{-ij}^*)$.

c) $a^\Gamma(m_i^\varepsilon, m_j^\varepsilon, m_{-ij}^*) < a$, so that $\theta_b + b_j(\theta_b) \in (a^\Gamma(m_i^\varepsilon, m_j^\varepsilon, m_{-ij}^*), a^\Gamma(m_i^\varepsilon, m_j, m_{-ij}^*))$. It follows that $a^\Gamma(m_i, m_j^\varepsilon, m_{-ij}^*) < a^\Gamma(m_i^\varepsilon, m_j, m_{-ij}^*)$. Then, $a^\Gamma(m_i, m_j^\varepsilon, m_{-ij}^*) \neq a$ is not possible since sender i cannot be simultaneously indifferent between a and $a^\Gamma(m_i^\varepsilon, m_j, m_{-ij}^*)$, as well as between $a^\Gamma(m_i^\varepsilon, m_j^\varepsilon, m_{-ij}^*) < a$ and $a^\Gamma(m_i, m_j^\varepsilon, m_{-ij}^*) < a^\Gamma(m_i^\varepsilon, m_j, m_{-ij}^*)$.

d) $a^\Gamma(m_i^\varepsilon, m_j^\varepsilon, m_{-ij}^*) = a^\Gamma(m_i^\varepsilon, m_j, m_{-ij}^*)$. Since sender i must be simultaneously indifferent between a and $a^\Gamma(m_i^\varepsilon, m_j, m_{-ij}^*)$, as well as $a^\Gamma(m_i^\varepsilon, m_j^\varepsilon, m_{-ij}^*) = a^\Gamma(m_i^\varepsilon, m_j, m_{-ij}^*)$ and $a^\Gamma(m_i, m_j^\varepsilon, m_{-ij}^*)$, we have either $a^\Gamma(m_i, m_j^\varepsilon, m_{-ij}^*) = a^\Gamma(m_i^\varepsilon, m_j^\varepsilon, m_{-ij}^*)$, or $a^\Gamma(m_i, m_j^\varepsilon, m_{-ij}^*) = a$.

A symmetric argument applies if $a^\Gamma(m_i^\varepsilon, m_j, m_{-ij}^*) < a$.

(ii) By NSI, i and j cannot be indifferent between $a^\Gamma(m_i, m_j, m_{-ij}^*) = a$ and the same other action at θ_b, so by (2), we cannot have $a^\Gamma(m_i^\varepsilon, m_j, m_{-ij}^*) = a^\Gamma(m_i, m_j^\varepsilon, m_{-ij}^*) \neq a$. Thus, if $a^\Gamma(m_i^\varepsilon, m_j, m_{-ij}^*), a^\Gamma(m_i, m_j^\varepsilon, m_{-ij}^*) \neq a$, by part (i), we have $a^\Gamma(m_i^\varepsilon, m_j^\varepsilon, m_{-ij}^*) = a$. But then, at θ_b, i is indifferent between $a^\Gamma(m_i, m_j, m_{-ij}^*) = a$ and $a^\Gamma(m_i^\varepsilon, m_j, m_{-ij}^*)$, while j is indifferent between $a^\Gamma(m_i^\varepsilon, m_j^\varepsilon, m_{-ij}^*) = a$ and $a^\Gamma(m_i^\varepsilon, m_j, m_{-ij}^*)$. This again cannot occur by NSI. □

Therefore, if $m_i \neq m_i^\varepsilon$, $m_j \neq m_j^\varepsilon$, and $a^\Gamma(m_i, m_j, m_{-ij}^*) = a$, then it is possible to change a component of (m_i, m_j, m_{-ij}^*) from its value in m to its value in m^ε without changing the induced action. Doing so and iterating the process yields $a^\Gamma(m_i, m^\varepsilon_{-i}) = a$ for some $i \in N$. Substituting this into (2) with $m^*_{-i} = m^\varepsilon_{-i}$ gives

$$u_i(a, \theta_b) = u_i(a^\Gamma(m^\varepsilon), \theta_b). \tag{3}$$

Moreover, by i's optimality at $\theta_b + \varepsilon$ in the noiseless game and single-crossing, we cannot have $a > a^\Gamma(m^\varepsilon)$. It follows that $a < a^\Gamma(m^\varepsilon)$, which, together with (3), implies that θ_b is left-natural* after all.

Case B: $\theta_b \notin C$

Now denote $m^\Gamma(\theta_b) = m'$ and $m^\Gamma(\theta_b - \varepsilon) = m^\varepsilon$. Since $\theta_b \notin C$, we have $a^\Gamma(m') \neq a$.

Since Γ is strongly robust, for any $\delta > 0$, there exists ε sufficiently small so that sender i's δ-optimality at $s_i = \theta_b$ under noise $(\varepsilon, N_i, N_{-i}, -)$ implies

$$u_i(a^\Gamma(m'_i, m^*_{-i}), \theta_b) \geq u_i(a^\Gamma(m^\varepsilon_i, m^*_{-i}), \theta_b) - \delta,$$

where $m^*_j = m'_j$ if $j \notin N_i$ and $m^*_j = m^\varepsilon_j$ if $j \in N_i$. Moreover, sender i's δ-optimality at $s_i = \theta_b - \varepsilon$ under noise $(\varepsilon, N \backslash (N_i \cup \{i\}), N_{-i}, +)$ implies

$$u_i(a^\Gamma(m^\varepsilon_i, m^*_{-i}), \theta_b - \varepsilon) \geq u_i(a^\Gamma(m'_i, m^*_{-i}), \theta_b - \varepsilon) - \delta.$$

By a similar reasoning as in Case A, we have that for any ε sufficiently small,

$$u_i(a^\Gamma(m^\varepsilon_i, m^*_{-i}), \theta_b) = u_i(a^\Gamma(m'_i, m^*_{-i}), \theta_b). \tag{4}$$

Fix $\varepsilon > 0$ such that (4) holds, and note that $a^\Gamma(m^\varepsilon) = a \neq a^\Gamma(m')$. The remainder of the proof is symmetric to the argument in Case A. ■

Similarly, by Proposition 4*, to prove Theorem 2*(a), it suffices to prove the following Lemma.

Lemma 3*: If Γ is robust and NSI holds, then Γ is natural*.

Proof of Lemma 3*: Steps 1 to 4 of the proof of Lemma 3 carry over to show interval structure. The following observation can be obtained by strengthening step 2 of the proof of Lemma 3:

Observation 2: For any $\delta > 0$, there exists $\varepsilon(\delta) > 0$ such that if $\theta^\Gamma(m) \neq \varnothing$ and $\lambda(\theta^\Gamma(m)) < \varepsilon(\delta)$, then $\sup \theta^\Gamma(m) - \inf \theta^\Gamma(m) < 3\delta$.

Proof of Observation 2: Suppose not, so that for any $\bar{\varepsilon} > 0$, $\exists \varepsilon \in (0, \bar{\varepsilon})$ such that $\theta^{\Gamma}(m) \neq \varnothing$, $\lambda(\theta^{\Gamma}(m)) < \varepsilon$, and $\sup \theta^{\Gamma}(m) - \inf \theta^{\Gamma}(m) \geq 3\delta$. For such m, there exists $\theta^* \in \theta^{\Gamma}(m)$ such that $|\theta^* - a^{\Gamma}(m)| > \delta$.

Since $\lambda(\theta^{\Gamma}(m)) < \varepsilon$, for any θ where $m^{\Gamma}(\theta) = m$, $\exists \theta_0(\theta) \in [\theta - \varepsilon, \theta + \varepsilon]$ such that for some $i \in N$, $m_i^{\Gamma}(\theta_0(\theta)) \neq m_i$. Let $i_{\varepsilon}^{\Gamma}(\theta)$ be any such i, and consider the following noise Ξ:

(i) at states $\theta \in \theta^{\Gamma}(m)\backslash\{\theta^*\}$, consider a random variable $X \sim U[0,1]$; if the realization of X is θ, sender $i_{\varepsilon}^{\Gamma}(\theta)$ observes $s_i = \theta$, while if not, sender $i_{\varepsilon}^{\Gamma}(\theta)$ observes $s_i = \theta'$ for some $\theta' \in [\theta - \varepsilon, \theta + \varepsilon]$ where $m_i^{\Gamma}(\theta') \neq m_i$;

(ii) for all other senders, and for $i_{\varepsilon}^{\Gamma}(\theta)$ at all other states, the true state is observed.

Clearly, Ξ has size at most ε, and $a^{\Xi}(m) = \theta^*$. By step 1 of the proof of Lemma 3, Γ is not robust. $\square$

In the remainder of this proof, adopt the notation from the proof of Lemma 2*.

Case A: $\theta_b \in C$, $a \neq \theta_b$

Note that (1) still holds. If (2) still holds for ε sufficiently small, then the argument in Lemma 2* carries through. For (2) not to hold for ε sufficiently small, it must be that for any $\bar{\varepsilon} > 0$, there are infinitely many distinct m^{ε} for $\varepsilon \in (0, \bar{\varepsilon})$. Observation 2 implies that this can only be the case if there exists a sequence $\{\varepsilon_k\}_{k=1}^{\infty}$ converging to 0 as $k \to \infty$ such that $a^{\Gamma}(m^{\varepsilon_k}) \to \theta_b$ as $k \to \infty$.

An approximate version of Observation 1(i) that converges to Observation 1(i) as $\varepsilon \to 0$ can be obtained by making a similar argument and using (1). If at θ_b, i (resp. j) is indifferent between a and some action a_i (resp. a_j), then by NSI and the finiteness of N, $\min_{j \neq i} |a_i - a_j| > 0$. A similar argument as in the proof of Observation 1(ii) thus shows that if NSI holds, then as $\varepsilon \to 0$, we have $a^{\Gamma}(m_i^{\varepsilon}, m_j, m_{-ij}^*) \to a$, $a^{\Gamma}(m_i, m_j^{\varepsilon}, m_{-ij}^*) \to a$, or both.

Now consider $a^{\Gamma}(m_i^{\varepsilon}, m_{-i})$. By point 2 in the definition of robustness, if $(m_i^{\varepsilon}, m_{-i})$ were off-path, we could not have $u_i(a', \theta_b) = u_i(a, \theta_b)$ whenever $|a' - a^{\Gamma}(m_i^{\varepsilon}, m_{-i})| < \gamma$. This implies, by (1), that $(m_i^{\varepsilon}, m_{-i})$ must occur on path for all sufficiently small ε. Because any two distinct on-path actions induced by message vectors differing in only one component are separated by at least η (see the second-to-last paragraph of the proof of Theorem 1a), we have that for all i and sufficiently small ε, $u_i(a^{\Gamma}(m_i^{\varepsilon}, m_{-i}), \theta_b) = u_i(a, \theta_b)$. Taking $m_{-ij}^* = m_{-ij}$ in the previous paragraph implies that for at least $n-1$ senders i, we have $a^{\Gamma}(m_i^{\varepsilon}, m_{-i}) = a$ for sufficiently small ε.

(i) If $a^{\Gamma}(m_i^{\varepsilon}, m_{-i}) = a$ for sufficiently small ε for all i, then as $\varepsilon \to 0$, we must have $a^{\Gamma}(m_i^{\varepsilon}, m_j^{\varepsilon}, m_{-i}) \to a$ for all pairs (i, j). To see this, note that, at θ_b, i (resp. j) must be nearly indifferent between $a^{\Gamma}(m_i^{\varepsilon}, m_j^{\varepsilon}, m_{-i})$ and $a^{\Gamma}(m_i, m_j^{\varepsilon}, m_{-i}) = a$ (resp. $a^{\Gamma}(m_i^{\varepsilon}, m_j, m_{-i}) = a$), which, by NSI and the finiteness of N, can occur only if $a^{\Gamma}(m_i^{\varepsilon}, m_j^{\varepsilon}, m_{-i})$ is near a. Iterating this reasoning (which also applies when $a^\Gamma(m_i^\varepsilon, m_{-i}) \to a$ as $\varepsilon \to 0$) yields $a^\Gamma(m^\varepsilon) \to a \neq \theta_b$, which contradicts the existence of $\{\varepsilon_k\}_{k=1}^\infty$ noted earlier.

(ii) If instead there exist arbitrarily small ε such that $a^\Gamma(m_1^\varepsilon, m_{-1}) \neq a$, the above iterative reasoning can still be used with senders $2, ..., n$, so that $a^\Gamma(m_1, m_{-1}^\varepsilon) \to a$ as $\varepsilon \to 0$. Since $u_1(a^\Gamma(m_1, m_{-1}^\varepsilon), \theta_b) - u_1(a^\Gamma(m^\varepsilon), \theta_b) \to 0$ as $\varepsilon \to 0$, for small ε, $a^\Gamma(m^\varepsilon)$ must be near either a or $a' \neq a$, where $u_1(a', \theta_b) = u_1(a, \theta_b)$. The existence of $\{\varepsilon_k\}_{k=1}^\infty$, combined with $a \neq \theta_b$, implies that we must have $a' = \theta_b$. Since $u_1(a^\Gamma(m_1^\varepsilon, m_{-1}), \theta_b) = u_1(a, \theta_b)$ for sufficiently small ε, and there exist arbitrarily small ε such that $a^\Gamma(m_1^\varepsilon, m_{-1}) \neq a$, there also exist arbitrarily small ε such that $a^\Gamma(m_1^\varepsilon, m_{-1}) = a' = \theta_b$. Because, for sufficiently small ε, $(m_1^\varepsilon, m_{-1})$ occurs on path, if $a^\Gamma(m_1^\varepsilon, m_{-1}) = \theta_b \neq a$, $(m_1^\varepsilon, m_{-1})$ must occur both to the left and to the right of θ_b. This is not possible: because $u_1(a^\Gamma(m_1^\varepsilon, m_{-1}), \theta_b) = u_1(a, \theta_b)$, in one of the two cases, by single-crossing, sender 1 strictly prefers inducing $a^\Gamma(m_1, m_{-1}) = a$ to $a^\Gamma(m_1^\varepsilon, m_{-1}) \neq a$.

Case B: $\theta_b \notin C$

Adopt the notation of case B of the proof of Lemma 2*. Like in case A of this proof, if (4) holds for sufficiently small ε, then we are done. Once again, by Observation 2, if there is no ε sufficiently small such that (4) holds, there must exist a sequence $\{\varepsilon_k\}_{k=1}^\infty$ converging to 0 as $k \to \infty$ such that $a^\Gamma(m^{\varepsilon_k}) \to \theta_b$ as $k \to \infty$. Here, since m^ε are sent inside C for small ε, we have $a^\Gamma(m^\varepsilon) = a$ for small ε. Thus, $a = \theta_b$.

Proceeding like in case A (with $m^\Gamma(\theta_b)$ and $a^\Gamma(m^\Gamma(\theta_b))$ taking the place of m and a, respectively), in subcase (i), we have $a^\Gamma(m^\varepsilon) \to a^\Gamma(m^\Gamma(\theta_b))$. Here, this implies $a = a^\Gamma(m^\Gamma(\theta_b))$, which contradicts $\theta_b \notin C$.

In subcase (ii), we have that a must be equal to either $a^\Gamma(m^\Gamma(\theta_b))$ or $a' \neq a^\Gamma(m^\Gamma(\theta_b))$, where $u_1(a', \theta_b) = u_1(a^\Gamma(m^\Gamma(\theta_b)), \theta_b)$. Since $a \neq a^\Gamma(m^\Gamma(\theta_b))$ (because $\theta_b \notin C$), we have $a' = a = \theta_b$. The remainder of the argument is analogous to case A.

Case C: $\theta_b \in C$, $a = \theta_b$

Because C is proper with right endpoint θ_b, the receiver's optimality implies that there exists another proper supercell C' with left endpoint $\theta'_b > \theta_b$ where the induced action is also θ_b. Moreover, θ'_b cannot be right-natural: otherwise, some message vector inducing action θ_b would be sent only to the right of θ'_b, which cannot be the case.

If $\theta'_b \in C'$, then the situation is symmetric to case A (now the endpoint θ'_b and induced action θ_b cannot be equal), so we are done. Moreover, we cannot have $\theta'_b \notin C'$: by the first paragraph of case B, this is possible only if θ'_b is the action induced in C', which is not the case here. ■

Online Appendix C: Near Robustness

This section introduces weaker robustness concepts, near robustness and strong near robustness, that require a "nearby" strategy profile, rather than the exact original strategy profile, to be approximately optimal under noise. As stated in the main paper, Theorems 1 and 2 remain true provided that heterogeneous priors about the noise are allowed.

The closeness of strategy profiles is defined as follows.

Definition: Given a profile Γ, messages m_i and m_i' are (Γ, δ)-*close* if for any $m_{-i} \in \times_{j \neq i} M_j^{\Gamma}$, $|a^{\Gamma}(m_i, m_{-i}) - a^{\Gamma}(m_i', m_{-i})| < \delta$.

Definition: Given a profile Γ, profile Γ' is δ-*close to* Γ if:

1. $M_i^{\Gamma'} = M_i^{\Gamma}$;

2. for any proper cell C in Γ or Γ', $m_i^{\Gamma'}(s_i)$ and $m_i^{\Gamma}(s_i)$ are (Γ, δ)-close for all $s_i \in [\inf C + \delta, \sup C - \delta]$;

3. letting a^{Ξ} and $a^{\Xi\prime}$ be the receiver's best responses given noise Ξ to the senders' strategies in Γ and Γ' respectively, $\exists \varepsilon > 0$ such that, for all $m \in \times_{i=1}^{n} M_i^{\Gamma}$, $|a^{\Xi}(m) - a^{\Xi\prime}(m)| < \delta$ whenever the size of Ξ is less than ε, and $a^{\Xi}(m)$ and $a^{\Xi\prime}(m)$ exist; and

4. $|a^{\Gamma}(m) - a^{\Gamma'}(m)| < \delta$ for all $m \in \times_{i=1}^{n} M_i^{\Gamma}$.

Points 1 and 4 in the definition of δ-closeness simply require that the senders use the same messages in Γ' as in Γ, and that the receiver takes a nearby action after every message vector. Point 2 restricts the senders' strategies by requiring the use of the similar messages in Γ and Γ' in proper cells at least δ away from boundaries.[4] However, this condition has no power when dealing with sender strategies that do not feature intervals: it is difficult to directly determine whether two sender profiles with complicated structures are "close."[5] Point 3 addresses this issue by using the receiver's best response to evaluate how close sender profiles are to each other. Noise is used because, in some cases, two sender profiles could generate the same receiver actions without noise while generating far apart actions with small noise; such profiles ought to be considered distant.

[4] The requirement can be weakened to allow a small probability of deviation and/or deviation on a small set of states.

[5] For example, suppose that within some interval, strategy m_i^{Γ} assigns m_i within the set of irrational numbers and m_i' elsewhere. Strategy $m_i^{\Gamma'}$ is identical to m_i^{Γ} everywhere except on the said interval, where it assigns m_i within the set of transcendental numbers and m_i' elsewhere. It is unclear by simple inspection how "close" m_i^{Γ} and $m_i^{\Gamma'}$ should be considered.

For instance, consider Example 2, and shift all cell boundaries and receiver actions by less than δ. The resulting profile is δ-close to the original one: points 1, 2 and 4 are clearly satisfied, and point 3 is as well because as $\varepsilon \to 0$, we must have $a^{\Xi}(m) \to a^{\Gamma}(m)$ and $a^{\Xi\prime}(m) \to a^{\Gamma'}(m)$.

The definitions for strong near robustness and near robustness parallel the ones for strong robustness and robustness.

Definition: An equilibrium Γ in the noiseless game is *strongly near-robust* if, for every $\delta > 0$, there exists $\varepsilon > 0$ such that whenever there is common knowledge that noise has size less than ε, there exists a δ-close strategy profile Γ' where each player's strategy $r_i^{\Gamma'}$ is a δ-best response to $r_{-i}^{\Gamma'}$ evaluated under sender i's belief about the noise.

Definition: An equilibrium Γ in the noiseless game is *near-robust* if:

1. for every $\delta > 0$, there exists $\varepsilon > 0$ such that whenever there is common knowledge that noise has local size less than ε, there exists a δ-close strategy profile Γ' where each player's strategy $r_i^{\Gamma'}$ is an on-path δ-best response to $r_{-i}^{\Gamma'}$ evaluated under sender i's belief about the noise, and

2. in the noiseless game, there exists $\gamma > 0$ such that whenever the perturbation on the receiver's off-path beliefs has size less than γ, every sender's strategy m_i^{Γ} is a best response to m_{-i}^{Γ} and $a^{\Gamma *}$, where $a^{\Gamma *}$ denotes the receiver's best-response to m^{Γ} and her perturbed off-path beliefs.[6]

A profile with the characteristics of Γ' will be called a *δ-supporting profile.* Γ' is interim δ-optimal, where each player's payoffs are evaluated under her own beliefs.

With these definitions, Theorems 1 and 2 hold with no change. The proofs of Theorems 1b and 2b still apply: they allow for heterogeneous priors, and Γ is δ-close to itself for all $\delta > 0$. The proofs of Lemmata 1 to 3, which imply Theorems 1a and 2a, are modified as follows.

Modified proof of Lemma 1: Suppose Γ is strongly near-robust. Given Γ and Γ', let a^{Ξ} and $a^{\Xi\prime}$ denote the receiver's best response to $\{m_j^{\Gamma}\}_{j=1}^n$ and $\{m_j^{\Gamma'}\}_{j=1}^n$, respectively, given noise Ξ.

[6]Point 2 is the same as in the definition of robustness.

Step 1: For any $\delta > 0$, $\exists \varepsilon > 0$ such that for all noise Ξ with size less than ε, $|a^{\Gamma}(m) - a^{\Xi}(m)| < \delta$ for all $m \in \times_{i=1}^{n} M_i^{\Gamma}$.

By the definitions of strong near robustness and δ-closeness, we know that for any $\delta > 0$, $\exists \varepsilon > 0$ such that for all noise Ξ with size less than ε, $\exists \Gamma'$ such that:

- $a^{\Gamma'}$ is a δ-best response to $\{m_j^{\Gamma'}\}_{j=1}^{n}$ under Ξ;
- $|a^{\Gamma}(m) - a^{\Gamma'}(m)| < \delta$ for all $m \in \times_{i=1}^{n} M_i^{\Gamma}$; and
- $|a^{\Xi}(m) - a^{\Xi\prime}(m)| < \delta$ for all $m \in \times_{i=1}^{n} M_i^{\Gamma}$.

Because u_R is continuous and strictly concave in a, and Θ is compact, the first point implies that $\exists \gamma(\delta)$ such that, for all $m \in \times_{i=1}^{n} M_i^{\Gamma}$, $|a^{\Gamma'}(m) - a^{\Xi\prime}(m)| < \gamma(\delta)$, with $\lim_{\delta \to 0} \gamma(\delta) = 0$. Therefore, $|a^{\Gamma}(m) - a^{\Xi}(m)| < 2\delta + \gamma(\delta)$ for all $m \in M_i^{\Gamma}$ and Ξ with size less than ε.

Rewriting δ in lieu of $2\delta + \gamma(\delta)$ yields the result. $\lozenge$

The remainder of the proof (steps 2 to 4) is unchanged. ■

Lemma 2 is now proved in two steps, numbered 5 and 6 (numbering continued from the proof of Lemma 1). Suppose a boundary θ_b in Γ is not left-natural, such as the boundary between the first two cells in Example 2, and consider the following beliefs about noise: each sender believes that she observes the true state while all other senders observe $s_i = \max\{\theta - \varepsilon, 0\}$, the receiver believes that all senders observe $s_i = \theta$, and these beliefs are common knowledge. Let m be the message vector sent to the left of the boundary - $(1,1)$ in our example. The proof applies the definition of δ-closeness to show that in a δ-supporting profile Γ', for δ small enough, m must be sent in a neighborhood to the left of $\theta_b - \delta$. Then, for ε small enough, m must also be sent between $\theta_b - \delta$ and $\theta_b - \delta + \varepsilon$: upon observing a signal in that range, each sender believes opponents will send m_{-i}, and in turn must send m_i, which gives i expected payoff at least δ higher than any other message, for δ small enough. Because θ_b is not left-natural, this argument can be iterated past $\theta_b + \delta$, which means that no δ-supporting profile can exist. Therefore, Γ must be natural. This intuition bears parallels to the global games contagion argument (except for the heterogeneous prior).

Like for steps 2 to 4, the noise distribution used for steps 5 and 6 does not have to be atomic. For example, the argument carries through if each sender instead believes that other senders' signals are distributed according to $U[\max\{\theta - \varepsilon, 0\}, \theta]$.[7] Unlike for steps 2 to 4, the argument uses noise where the prior is heterogeneous.

[7] Point 2 of the definition of closeness can also be relaxed: if a message vector close to m must be sent in Γ' with probability near 1 in some interval I_m to the left of $\theta_b - \delta$, then the unraveling reasoning remains valid for sufficiently small ε (in particular, $\varepsilon < |I_m|$).

Modified proof of Lemma 2: We proceed by contradiction. Suppose instead, without loss of generality, that the right endpoint $\theta_b \neq 1$ of a proper cell C in Γ where $m = (m_1, ..., m_n)$ is sent is not left-natural. Denote the measure of this proper cell by λ.

Step 5: $\exists\overline{\delta} > 0$ such that for all $i \in N$ and $m_i' \in M_i^\Gamma \backslash \{m_i\}$, $u_i(a', \theta) + \overline{\delta} < u_i(a, \theta)$ for all $\theta \in [\theta_b - \overline{\delta}, \theta_b + \overline{\delta}]$, $a' \in [a^\Gamma(m_i', m_{-i}) - \overline{\delta}, a^\Gamma(m_i', m_{-i}) + \overline{\delta}]$, and $a \in [a^\Gamma(m_i, m_{-i}) - \overline{\delta}, a^\Gamma(m_i, m_{-i}) + \overline{\delta}]$.

By the definition of "left-natural," for any i and any $m_i' \in M_i^\Gamma \backslash \{m_i\}$, either:

(i) $u_i(a^\Gamma(m_i', m_{-i}), \theta_b) < u_i(a^\Gamma(m_i, m_{-i}), \theta_b)$,

(ii) (m_i', m_{-i}) is sent on path in Γ and $a^\Gamma(m_i, m_{-i}) = a^\Gamma(m_i', m_{-i})$, or

(iii) (m_i', m_{-i}) is not sent at any state in Γ and $u_i(a^\Gamma(m_i', m_{-i}), \theta_b) = u_i(a^\Gamma(m_i, m_{-i}), \theta_b)$.

Because u_i is continuous, if (i) holds, then $u_i(a', \theta) < u_i(a, \theta)$ for all θ in a non-degenerate interval around θ_b, and all a' and a sufficiently near $a^\Gamma(m_i', m_{-i})$ and $a^\Gamma(m_i, m_{-i})$ respectively. Therefore, it suffices to show that for any m_{-i}, there are finitely many $a^\Gamma(m_i', m_{-i})$ occurring on the equilibrium path. This must be true since any two such actions must be separated by at least η (see the second-to-last paragraph of the proof of Theorem 1a).

Case (ii) cannot arise by Assumption A.

Case (iii) cannot arise by step 2. $\Diamond$

Step 6: Let $\delta < \min\{\overline{\delta}, \frac{\lambda}{2}, \eta\}$. Then, for any $\varepsilon \in (0, \lambda - 2\delta)$, there is no δ-supporting profile for Γ under the following beliefs about the noise:

- Each sender believes that they observe the true state and that other senders observe $\max\{\theta - \varepsilon, 0\}$.

- The receiver believes that all senders observe the true state. (For the sake of completeness - this will not matter.)

- These beliefs are common knowledge.

By the definition of δ-closeness, in any δ-supporting profile Γ', it must be that for sufficiently small ε, for all i, and for all $s_i \in [\theta_b - \delta - \varepsilon, \theta_b - \delta)$, $m_i^{\Gamma'}(s_i)$ is (Γ, δ)-close to m_i. Now suppose $m_i^{\Gamma'}(s_i) \neq m_i$. Then, for some m_{-i}', $a^\Gamma(m_i^{\Gamma'}(s_i), m_{-i}') \neq a^\Gamma(m_i, m_{-i}')$. By completeness, both $(m_i^{\Gamma'}(s_i), m_{-i}')$ and (m_i, m_{-i}') occur on path in Γ, so by the same reasoning used at the end of case (i) of step 5, we must have $|a^\Gamma(m_i^{\Gamma'}(s_i), m_{-i}') - a^\Gamma(m_i, m_{-i}')| > \eta$. This contradicts $m_i^{\Gamma'}(s_i)$ being (Γ, δ)-close to m_i since $\delta < \eta$. Thus $m_i^{\Gamma'}(s_i) = m_i$ for all $s_i \in [\theta_b - \delta - \varepsilon, \theta_b - \delta)$.

Now suppose sender j observes $s_j \in [\theta_b - \delta, \theta_b - \delta + \varepsilon)$. She believes that all senders $i \neq j$ observed $s_i \in [\theta_b - \delta - \varepsilon, \theta_b - \delta)$, and therefore will send m_i. By step 5, her unique

δ-best response is m_j. Since this holds for all senders, we have that for all i and for all $s_i \in [\theta_b - \delta, \theta_b - \delta + \varepsilon)$, $m_i^{\Gamma'}(s_i) = m_i$.

Iterating the above argument, it follows by step 5 that for all i and for all $s_i \in [\theta_b - \delta, \theta_b + \overline{\delta}]$, $m_i^{\Gamma'}(s_i) = m_i$. By definition, Γ' can be δ-close to Γ only if, for all i, $m_i^{\Gamma}(s_i) = m_i$ for all $s_i \in [\theta_b, \theta_b + \overline{\delta} - \delta]$, which contradicts θ_b being the right endpoint of C. Therefore, Γ' is not a δ-supporting profile of Γ. $\Diamond$

Under the beliefs about noise in step 6, there is common knowledge that noise is less than ε. We therefore conclude that Γ is, in fact, not strongly near-robust. ■

Modified proof of Lemma 3: Modify step 1 as in the proof of Lemma 1. Steps 2 to 4 are unchanged. Step 5 and 6 follow the modified proof of Lemma 2, as adjusted below.

Step 5: Same statement as step 5 in the proof of Lemma 2, and same argument in cases (i) and (ii).

Case (iii) is ruled out by point 2 in the definition of near robustness and the continuity of u_i. $\Diamond$

Step 6: Same statement as step 6 in the proof of Lemma 2, except that δ is chosen to be also less than γ from the point 2 in the definition of near robustness. Then, to show that, for small enough ε, $m_i^{\Gamma'}(s_i) = m_i$ for all $s_i \in [\theta_b - \delta - \varepsilon, \theta_b - \delta)$, proceed again by contradiction. The argument is the same if, for some m'_{-i}, both $(m_i^{\Gamma'}(s_i), m'_{-i})$ and (m_i, m'_{-i}) occur on path but induce different actions in Γ. If not, then $(m_i^{\Gamma'}(s_i), m_{-i})$ is off path in Γ, and point 2 in the definition of near robustness implies $|a^{\Gamma}(m_i^{\Gamma'}(s_i), m_{-i}) - a^{\Gamma}(m_i, m_{-i})| \geq \gamma$. This again contradicts $m_i^{\Gamma'}(s_i)$ being (Γ, δ)-close to m_i since $\delta < \gamma$.

The remainder of the proof is identical to the analogous part of the modified proof of Lemma 2. ■