Risk Taking with Background Risk under Recursive Rank-Dependent Utility

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Abstract

This paper examines how background risk affects risk taking under rank-dependent utility. I assume that a decision-maker facing a risk taking decision in the presence of background risk views these risks as composing a compound lottery, and recursively evaluates this compound lottery using rank-dependent utility. I show that adding background risk increases risk aversion whenever the utility-for-wealth function is risk vulnerable (Gollier and Pratt, 1996) in this model.

Keywords: non-expected utility; recursive preferences; risk aversion; background risk.

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1 Introduction

People who face risky decisions almost invariably have uninsurable pre-existing risks, and empirical work suggests that such pre-existing risks tend to make people more risk averse (Guiso et al., 1996; Guiso and Paiella, 2008; Beaud and Willinger, 2015). Previous research has provided, under expected utility (EU), necessary and sufficient conditions on a a decision-maker's (DM) utility-for-wealth function for the DM to exhibit more risk aversion following a deterioration in background risk (Gollier and Pratt, 1996; Eeckhoudt et al., 1996). This paper provides analogous results for a recursive application of rank-dependent utility (RDU) by showing that these conditions remain sufficient under RDU.

It is known that in RDU, a DM's evaluation of risks may depend on how and when they resolve (Segal, 1990). I assume that a DM who is offered the opportunity to take an additional risk (1) views the additional risk and her background risk as forming a two-stage lottery with her background risk being resolved at the second stage, and (2) evaluates two-stage lotteries recursively (consistent with Segal's compound independence axiom).[1] Theorem 2 shows that if the DM has RDU preferences and her utility-for-wealth is "risk vulnerable" in the sense of (Gollier and Pratt, 1996), then she exhibits more risk aversion when faced with an actuarially unfavorable background risk. Theorem 2 provides the analogous result for first- and second- order stochastic dominance (FSD and SSD) deteriorations in background risk.

My results contrast with those of Quiggin (2003) and Safra and Segal (2008), who assume that a DM integrates any additional risk and her background risk into a single lottery using the laws of probability (as required by the reduction of compound lotteries axiom) and find that in RDU and related models, risk aversion due to probability weighting is attenuated by the presence of uninsurable background risk.

[1] The modeling approach to risk taking with background risk under recursive non-expected utility builds on Freeman (2015), who uses this approach to show that many non-expected utility theories, including RRDU, can capture descriptively reasonable small-stakes risk aversion without implying absurd large-stakes risk aversion. See Dillenberger (2010) for another recent application of recursive non-expected utility over multi-stage lotteries.

2 Model

2.1 Preliminaries

The setup follows Freeman (2015). Let $W = [a, b] \subset \mathbb{R}_+$ denote the set of feasible final wealth levels; let $\Delta(W)$ denote the set of all finite-support probability distributions over W, and refer to $\Delta(W)$ as the set of *one-stage lotteries* over W. A one-stage lottery over W can be written as $q = [w_1, q_1; ...; w_m, q_m] \in \Delta(W)$, where q_i denotes the probability of receiving prize w_i; for such lotteries, adopt the convention that $w_1 \leq ... \leq w_m$. Given $q \in \Delta(W)$, let F_q denote the cumulative distribution function (CDF) of q. For a given $q \in \Delta(W)$ and $y \in \mathbb{R}$, let $q + y = [w_1 + y, q_1; ...; w_m + y, q_m]$; the resulting $q + y$ is only in $\Delta(W)$ if $w_1 + y \geq a$ and $w_m + y \leq b$; I omit this caveat in statements below for expositional ease.

Define a *two-stage lottery* as a finite-support lottery over lotteries over final wealth levels. A two-stage lottery can be written as $Q = [q^1, p_1; ...; q^n, p_n]$ where $q^i \in \Delta(W)$ and p_i is the probability of receiving lottery q^i. Let $\Delta(\Delta(W))$ denote the set of two-stage lotteries.

2.2 Preferences over compound lotteries

I assume that a DM has RDU (Quiggin, 1982; Yaari, 1987) preferences over single-stage lotteries. $V : \Delta(W) \to \mathbb{R}$ is a *rank-dependent utility function* if there exist strictly increasing functions and continuous $u : W \to \mathbb{R}$ and $g : [0, 1] \to [0, 1]$ such that for any $q \in \Delta(W)$, $V(q) = \int u(w)dg(F_q(w)$. The function g is called a *probability weighting function*, and is required to satisfy $g(0) = 0$ and $g(1) = 1$; the function u is called a *utility-for-wealth function*. Notice that EU corresponds to the special case of RDU in which the probability weighting function is linear. Given an RDU function V, let c denote its corresponding certainty equivalent function defined by $c = u^{-1} \circ V$. In the analysis that follows, assume that V is risk averse – that is, averse to mean-preserving spreads. This is equivalent to assuming that g and u are both weakly concave (Chew et al., 1987).

I assume that a DM applies her single-stage lottery preferences to any two-stage lottery recursively. That is, the DM evaluates $Q = [q^1, p_1; ...; q^n, p_n]$ by first applying c

to each q^i to obtain the single-stage lottery $[c(q^1), p_1; ...; c(q^n), p_n]$, to which she applies V. This assumes that the DM applies the same single-stage lottery preferences at each stage of the two-stage lottery, as in Segal's (1990) time neutrality axiom. To capture this criterion, given an RDU function with certainty equivalent function c, define the corresponding *Recursive RDU (RRDU)* function $U : \Delta(\Delta(W)) \to \mathbb{R}$ over two-stage lotteries by $U(Q) = V([c(q^1), p_1; ...; c(q^n), p_n])$.

An normatively-appealing alternative way to apply V to two-stage lotteries is to use the laws of probability to reduce any two-stage lottery Q to a single-stage lottery and apply V to this reduced lottery. Given any two-stage lottery $Q = [q^1, p_1; ...; q^n, p_n]$, define the *reduction of* Q, denoted Q^R, by by the single-stage lottery with CDF $F_{Q^R} = \sum_{i=1}^{n} p_i F_{q^i}$; when each q^i has finite support, this gives $Q^R = [w_1, \sum_{i=1}^{n} p_i q^i_1; ...; w_K, \sum_{i=1}^{n} p_i q^i_K]$. This leads to the following definition: a DM *reduces compound lotteries* if she evaluates the desirability of any two-stage lottery Q according to $V(Q^R)$.

2.3 Risk taking with background risk

Define a *gamble* as a finite-support lottery over gain and loss prizes (as opposed to wealth levels). Consider a DM with wealth level w who faces background wealth risk described by the gamble $\hat{q} = [y'_1, q_1; ...; y'_m, q_m]$, which is not the subject of choice. This DM is offered the gamble over prizes $\hat{p} = (y_1, p_1; ...; y_n, p_n)$. Each $y_i \in \mathbb{R}$ is a monetary prize added to or taken away from the DM's final wealth after lottery $\hat{p}$ resolves and each $y'_i \in \mathbb{R}$ is similarly a monetary gain or loss similarly added to or taken away from the DM's final wealth after $\hat{q}$ resolves.[2]

Let $\hat{p} \oplus \hat{q} + w$ denote the two-stage lottery formed by the simple gamble over prizes $\hat{p}$, which resolves at the first stage, and independent background risk $\hat{q}$, which resolves at the second stage, given initial wealth w. When $\hat{q}$ has finite support, the two-stage lottery $\hat{p} \oplus \hat{q} + w$ is given by

$$\hat{p} \oplus \hat{q} + w = [\hat{q} + y_1 + w, p_1; ...; \hat{q} + y_n + w, p_n] \tag{1}$$

[2]I assume that background risk resolves after the offered gamble, however, if this order were reversed, it would require only straightforward modifications to the proofs of Theorems 1 and 2.

where $\hat{q}+y_i+w=[y'_1+y_i+w, q_1; ...; y'_m+y_i+w_m, q_m]$ denotes the lottery over final wealth states that the DM faces if prize y_i is won in the gamble $\hat{p}$.

Take an RDU certainty equivalent function c and its associated RRDU function U. Consider the following behavioral properties of c. Define that *adding actuarially unfavorable background risk increases risk aversion* if for any gamble $\hat{p} \in \Delta(Y)$, any gamble $\hat{q} \in \Delta(Y)$ with a weakly negative expected value, and any admissible $w \in W$, $c(\hat{p}+w) \leq w$ implies that $U(\hat{p} \oplus (w+\hat{q})) \leq V(w+\hat{q})$. Define that *an FSD (SSD) deterioration in background risk increases risk aversion* if for any gambles $\hat{p}, \hat{q}, \hat{r} \in \Delta(Y)$ for which $\hat{q}$ first-order (second-order) stochastically dominates $\hat{r}$, and any admissible $w \in W$, $c(\hat{p} \oplus \hat{q}+w) \leq c(\hat{q}+w)$ implies that $U(\hat{p} \oplus \hat{r}+w)) \leq V(\hat{r}+w)$.

3 The effect of background risk on risk taking in RRDU

Following Gollier and Pratt (1996) , define that a utility-for-wealth function u is *risk vulnerable* if for any $\hat{q} \in \Delta(Y)$ with a weakly negative expected value and for any w,

$$-\frac{\int u''(w+y)dF_{\hat{q}}(y)}{\int u'(w+y_i)dF_{\hat{q}}(y)} \geq -\frac{u''(w)}{u'(w)}.$$

Gollier and Pratt (Proposition 1) show that risk vulnerability of u is a necessary and sufficient condition for the addition of an unfavorable background risk to increase risk aversion over offered gambles in EU – their result is presented as Proposition 1 below.

Proposition 1. *(Gollier and Pratt, 1996) Under EU, adding an actuarially unfavorable background risk increases risk aversion if and only if u is risk vulnerable.*

Theorem 1 shows that if the utility-for-wealth function u is risk vulnerable, then any RRDU function with utility-for-wealth function u and a concave g that evaluates offered gambles according to (1) also has the property that adding unfavorable background risk increases risk aversion.

Theorem 1. *Suppose U is an RRDU function with a concave probability weighting function g and a concave utility-for-wealth function u. If u is risk vulnerable, then adding an actuarially unfavorable background risk increases risk aversion.*

Proof. Suppose U captures a DM's RRDU preferences, with corresponding probability weighting function g and a risk vulnerable utility-for-wealth function u. Let $\hat{p}, \hat{q} \in \Delta(Y)$ with $\int y d\hat{q}(y) \leq 0$.

Suppose c turns down $\hat{p}$ at wealth level w. Then,

$$\int u(w+y) dg(F_{\hat{p}}(y)) < u(w)\,. \tag{2}$$

Since g is a probability weighting function, $g \circ F_{\hat{p}}$ and $g \circ F_{\hat{q}}$ are CDFs corresponding to gambles in $\Delta(Y)$. Equation (2) is equivalent to an EU maximizer with utility-for-wealth function u turning down the gamble with CDF $g \circ F_{\hat{p}}$.

Since $\int y dF_{\hat{q}}(y) \leq 0$ and g is concave, it follows that $\int y dg(F_{\hat{q}}(y)) \leq \int y dF_{\hat{q}}(y) \leq 0$. Thus $g \circ F_{\hat{q}}$ is the CDF of an actuarially unfair gamble.

Since u is risk vulnerable and an EU maximizer with utility-for-wealth function u turns down the gamble with CDF $g \circ F_{\hat{p}}$ at wealth w, by Proposition 1, an EU-maximizer with utility-for-wealth function u, wealth w, and background risk $g \circ F_{\hat{q}}$ also turns down $\hat{p}$, that is,

$$\int\int u(w+y+x) dg(F_{\hat{p}}(y)) dg(F_{\hat{q}}(x)) < \int u(w+x) dg(F_{\hat{q}}(x))\,.$$

Equivalently, $$U(\hat{p} \oplus (w+\hat{q})) < V(w+\hat{q})$$

Thus such a DM will also turn down $\hat{p}$ when she faces actuarially unfavorable background risk $\hat{q}$. Since the choices of $\hat{p}$ and $\hat{q}$ were arbitrary, conclude that adding an unfavorable background risk increases risk aversion for an RRDU DM with a risk vulnerable u. □

The main step of the proof shows that in this approach to modeling background risk under RRDU, probability weighting is separately applied to the background risk and to the additional risk. Because of this, an RRDU risk taking decision with a given background risk is equivalent to an EU risk taking decision involving transformed risks obtained by applying the RDU probability weighting function. This provides a way to port existing results on the impact of background risk on risk taking in EU to the RRDU model, and I provide two additional results that follow from this insight.

One might notice that in Proposition 1 (from Gollier and Pratt 1996), risk vulnerability of u is a necessary and sufficient condition for adding an unfavorable background risk to increase risk aversion, whereas under RRDU, it is only a sufficient condition. The gap between the two results stems from the fact that for any actuarially unfavorable gamble with CDF F, the transformed gamble $g \circ F$ is also actuarially unfavorable when g is a concave probability weighting function. Corollary 1 provides the analogous necessary and sufficient behavioral condition for u to be risk vulnerable, and given Proposition 1, follows from the proof of Theorem 1.

Corollary 1. *Suppose U is an RRDU function with a concave probability weighting function g and a concave utility-for-wealth function u. The utility-for-wealth function u is risk vulnerable if and only if adding any background risk $\hat{q}$ for which $\int y dg(F_q(y)) \leq 0$ increases risk aversion.*

A remaining question is how these results could be generalized to provide a condition for g and u for which, under RRDU, an increase in background risk, from $\hat{q}$ to the riskier $\hat{r}$, raises risk aversion. I answer that question for the cases in which $\hat{r}$ is related to $\hat{q}$ by FSD or SSD.

Eeckhoudt et al. (1996) provide two conditions that, under EU, they show are necessary and sufficient for FSD and SSD deteriorations in background risk to increase risk aversion. Theorem 2 below shows that these conditions on u are also sufficient in RRDU.

Theorem 2. *Suppose U is an RRDU function with a concave probability weighting function g and a concave utility-for-wealth function u. If there exists a $\lambda \in \mathbb{R}$ such that $-\frac{u'''(w')}{u''(w')} \geq \lambda \geq -\frac{u''(w)}{u'(w)}$ $\forall w, w' \in [a, b]$, then an FSD deterioration in background risk increases risk aversion. If, in addition, there exists a $\kappa \in \mathbb{R}$ such that $-\frac{u''''(w')}{u'''(w')} \geq \kappa \geq -\frac{u''(w)}{u'(w)}$ $\forall w, w' \in [a, b]$, then an SSD deterioration in background risk increases risk aversion.*

Proof. If $\hat{q}$ first- (second-) order stochastically dominates $\hat{r}$, then $g \circ F_{\hat{q}}$ first- (second-) order stochastically dominates $g \circ F_{\hat{r}}$ (Chew et al., 1987).

The results then follow by applying this fact, the proof of Theorem 1, and Eeckhoudt et al.'s Proposition 2 (and 3). □

Quiggin (2003) shows that very different results hold under RDU with reduction of compound lotteries. For comparison, I summarize his result in the setting of this paper as Proposition 2.[3]

Proposition 2. *(Quiggin, 2003) If a DM has RDU preferences with a linear u and she reduces compound lotteries, then adding an actuarially unfavorable background risk reduces risk aversion.*

4 Discussion

The RRDU model here parsimoniously accommodates violations of the independence axiom and reduction of compound lotteries as well as small-stakes risk aversion. Theorem 1 showed that when u is risk vulnerable RRDU is also consistent with evidence that background risk increases risk aversion. Thus the model here provides a tractable and descriptively-motivated way to apply RDU in the presence of background risk that avoids the descriptively problematic predictions that follow under reduction of compound lotteries.

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[3] Quiggin (2003) works on the space of Savage acts with a known probability of each state. The stated result is an implication of Quiggin's result translated onto the space of objective lotteries here.

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