CONNECTEDNESS OF CERTAIN RANDOM GRAPHS

by

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ABSTRACT

In this paper we consider the probability of connectedness of a random graph. All of the graphs we examine are labelled graphs with a finite or countable number of vertices.

We will mention the work of various authors on the probability of connectedness of a random graph where various probability spaces have been assumed. We intend to summarize in detail what is known on the probability of connectedness of a random graph with N vertices and the asymptotic behaviour of this probability as $N \to \infty$, when the probability space is the triple $(\Omega, \mathcal{F}, P)$ where: Ω is the set of graphs which may be obtained from the symmetric graph with N vertices or a countable number of vertices by removing some (possibly all or none) of the edges; $\mathcal{F}$ is an appropriate σ-field of subsets of Ω; the probability P is not time dependent and the probability assigned to a graph as a member of $\mathcal{F}$ depends only on the number of edges appearing in the graph. Where these results apply to other sample spaces Ω and probabilities P we make the appropriate observations.

Further, we present work on the probability of connectedness of a random graph with M+N vertices and the asymptotic behaviour of this probability as M and/or $N \to \infty$ when the probability space is the triple $(\Omega, \mathcal{F}, P)$ where: the graphs of the sample space Ω are bipartite graphs with M+N vertices or a countable number of vertices; $\mathcal{F}$ is an appropriate σ-field of subsets of Ω; the probability P is not time dependent and the probability

assigned to a graph depends only on the number of edges appearing in the graph.

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Introduction

The problem we shall consider is most clearly stated after introducing certain notation and definitions.

Our problem concerns labelled graphs. A labelled graph may be described verbally as a number of distinguishable vertices (or points), labelled 1, 2, ... and so on, together with a number of lines joining some of the vertices. Mathematically it may be described as follows. Let I^+ denote the set $\{1,2, \ldots\}$. A set of the form A^B, where $A \subset I^+ \cup \{0\}$, $B \subset I^+ \times I^+$ is called a class of labelled graphs. An element of such a set A^B is a labelled graph.

There are nine different classes of graphs mentioned in this paper; we shall name them here. If $A=\{0,1\}$ and B is $B_{1,N}= \{(i,j) \mid 1\le i< j\le N\}$ or $B_{2,N}= \{(i,j) \mid 1\le i\le j\le N\}$ or $B_{3,N}= \{(i,j) \mid 1\le i\le N,\ 1\le j\le N,\ i\neq j\}$ or $B_{4,N}= \{(i,j) \mid 1\le i\le N,\ 1\le j\le N\}$ respectively, then elements of $A^{B_{i,N}}$ are called labelled graphs with N vertices of class I or class II or class III or class IV respectively. If $A= I^+ \cup 0$ and B is $B_{1,N}$ or $B_{2,N}$ or $B_{3,N}$ or $B_{4,N}$, then elements of $A^{B_{i,N}}$ are called labelled graphs with N vertices of class V or class VI or class VII or class VIII respectively. If $A= \{0,1\}$ and $B= C_{M,N}= \{(i,j) \mid 1\le i\le M,\ M+1\le j\le M+N\}$, then elements of A^B are called bipartite graphs of type M,N.

We will use the symbol Ω_N to denote the set of all class J graphs with N vertices where J is one of I, II, ...VIII, and the particular value of J will be made clear by the context. We will use $\Omega_{M,N}$ to denote the set of all bipartite graphs of type M,N with M+N vertices.

If $\omega = \langle \omega_{(i,j)} \rangle_{(i,j)\varepsilon B}$ is a labelled graph, then $\omega_{(i,j)}$ is the number of edges joining vertex i to vertex j. These edges are considered to be <u>directed</u> from i to j if the graph is an element of Ω_N for J equals III, IV, VII, or VIII, otherwise the edge is considered to be undirected.

Let i,j be vertices of a graph $\omega\varepsilon A^B$. We say i and j are <u>connected</u> if i=j or if i≠j and there is a finite sequence $i = i_0, i_1, \ldots i_n = j$ of vertices such that for each k, $0 \leq k \leq n-1$, $\omega_{(i_k, i_{k+1})} \geq 1$ or $\omega_{(i_{k+1}, i_k)} \geq 1$. A graph ω is <u>connected</u> if any pair of its vertices are connected. For any class of graphs A^B we will denote by $\mathcal{C}$ the set of all connected graphs ω in A^B.

For each of class I through class IV graphs we will define a probability function P_N on the set of all subsets of Ω_N. Also, we will define a probability function $P_{M,N}$ on the set of all subsets of $\Omega_{M,N}$.

For any structural property X (some condition on the graphs ω, normally specified on the $\omega_{(i,j)}$ belonging to ω), one could ask for the value of $P_N(\chi)$ or $P_{M,N}(\chi)$, where χ is the set of all graphs belonging to Ω_N or $\Omega_{M,N}$ respectively, having property X.

The problem we shall consider is: Let the structural property C be that ω is connected. What is $P_N(\mathcal{C})$ and $P_{M,N}(\mathcal{C})$? We shall also examine the asymptotic behaviour of $P_N(\mathcal{C})$ as N becomes unbounded and of $P_{M,N}(\mathcal{C})$ as M and/or N become unbounded.

It is convenient to conceptualize $P_N(\mathcal{C})$ as follows. We shall consider an artificial object we name a random graph. A random graph, we will assume, is a graph belonging to Ω_N but we do not specify exactly which graph; it has probability $P_N(\{\omega\})$ of being ω for each graph ω belonging to Ω_N. We can describe a random graph by: An edge, $\omega_{(i,j)}=1$, appears in the graph with probability $P_N(\{\omega \mid \omega_{(i,j)}=1\}) \equiv p$ and does not appear with probability 1-p. The probability that a random graph is connected is $P_N(\{\omega \mid \omega_{(i,j)}=1\}) = P_N(\mathcal{C})$. (If we are considering $\Omega_{M,N}$ and $P_{M,N}$, a random graph has a similar meaning).

In chapter one we state a few basic definitions and results of probability theory. After these preliminaries we define a probability function P_N for each of classes I, II, III, and IV and a probability function $P_{M,N}$ for each $\Omega_{M,N}$.

In chapter two we discuss some methods of analysis which are useful in examining labelled graphs.

Chapters three and five summarize what is currently known on the probability of connectedness $P_N(\mathcal{C})$ for any P_N defined on classes of graphs I, II, III or IV. Most of the work done pertains to class I graphs. (Graphs of class one are graphs obtained from the symmetric graph on N vertices by removing some, possibly none or all, of the edges). We have restricted the scope of the summary as follows: In chapter three we summarize in detail, results on $P_N(\mathcal{C})$ where Ω_N is one of classes I, II, III or IV and the respective P_N is not time dependent and assigns each edge of a random graph the same probability of appearance. Results of this kind were obtained by E. N. Gilbert[1] and [2]. In chapter five we mention, but do not summarize the work in detail, results obtained by P. Erdos and A. Renyi[4] and [5] on certain conditional probabilities related to $P_N(\mathcal{C})$ where Ω_N is class I, and P_N may or may not be time dependent but still assigns the same probability of appearance to each edge of a random graph. We also mention some results of A. Renyi[3] on the number of connected graphs of class I for a given N. We summarize, but not in detail, results obtained by V. E. Stepanov[6] and [7] on the connectedness of a random graph of class I chosen according to a P_N which is time dependent and assigns different probabilities of appearance to each possible edge at each time t. We mention briefly the work of E. Wright[8] on enumeration of connected graphs from classes I, to VIII (no P_N defined), which is somewhat related to our topic.

In chapter four we present work which is an extension of Gilbert's results. It is an extension in the sense that we examine $P_{M,N}(\mathcal{C})$ for bipartite graphs, which he did not consider, and we use reasoning and methods similar to Gilbert's.

The $P_{M,N}$ we will define is not time dependent and assigns the same probability of appearance to each edge of a random graph.

Chapter I

In this chapter we mention a few basic results from measure and probability theory. We refer the reader to the basic texts of, P.R. Halmos[10] for proof and details of measure theoretic results, and L. Breiman[11] for a modern development of probability theory based on measure theoretic results.

§1 Notations, definitions and results

In a probability theory, the symbol Ω is used to denote a set called the sample space, whose elements ω, are called elementary events or sample points. An event is any subset of Ω. The elements of the set Ω are considered to be the possible outcomes from a given experiment, that is, the ω are minimal events, disjoint, and one is bound to occur in the experiment.

Definition 1.1 A class of subsets $\mathcal{A}$ of a set Ω is a field if it is closed under finite unions, intersections, and complementation. The complement of Ω is the empty set $\emptyset$.

Suppose we have a sample space Ω and a field $\mathcal{F}_0$ of subsets of Ω. Then we will say

Definition 1.2 The triple $(\Omega, \mathcal{F}_0, P)$ is a *finite probability space* if P is a non-negative, real valued set function on satisfying

i) $P(\Omega) = 1$,

ii) for $A, B \in \mathcal{F}_0$ and disjoint

$P(A \cup B) = P(A) + P(B)$.

Definition 1.3 A class of subsets, $\mathcal{F}$, of Ω is a *σ-field* if it is closed under complementation, and under countable intersections and unions. For any class $\mathcal{D}$ of subsets of Ω denote by $\mathcal{F}(\mathcal{D})$ the smallest σ-field containing $\mathcal{D}$.

Given a sample space Ω and a σ-field $\mathcal{F}$ of subsets of Ω , we will say

Definition 1.4 The triple $(\Omega, \mathcal{F}, P)$ is a *probability space* if P is a non-negative set function defined on satisfying

i) $P(\Omega) = 1$,

ii) for every finite or countable collection $\{B_k\}$ of sets in $\mathcal{F}$ such that B_k is disjoint from B_j, $j \neq k$,

$$P(\bigcup_k B_k) = \sum_k P(B_k) .$$

For a given probability space $(\Omega, \mathcal{F}, P)$ we can define a set function on $\mathcal{F}$ which is related to P and particularly useful, called the conditional probability. To understand the motivation for this conditional probability, suppose we are conducting an experiment and we already have some information about the sample space Ω, for example, that an event A has occurred. This information will probably tell us something about the possible occurence of another event B. We state the following elementary definition of conditional probability, which is all that we need in this paper. (See Breiman [11], chapter IV for a more sophisticated discussion.)

Definition 1.5 Let $(\Omega, \mathcal{F}, P)$ be a probability space, then for sets A, B $\varepsilon\ \mathcal{F}$ such that $P(A) > 0$, the conditional probability of an event B given that A has already occurred is the ratio $\frac{P(B \cap A)}{P(A)}$ and is denoted by $P(B/A)$.

(The same definition can be made for a finite probability space).

In probability theory we often encounter a standard problem of measure theory entitled "extension of measures". Halmos[10] discusses extension in Chapter III. In probability theory the question is (very briefly): suppose we have a given Ω and finite probability space $(\Omega, \mathcal{F}_0, P)$ and suppose further that $\mathcal{F}$ is a σ-field containing $\mathcal{F}_0$; when is it possible to define a unique $\bar{P}$ on $\mathcal{F}$ making $(\Omega, \mathcal{F}, \bar{P})$ a probability space, with $\bar{P}$ and P agreeing on $\mathcal{F}_0$? Breiman[11] (chapters I and II), discusses the coin tossing experiment, how the extension problem arises in analysing coin tossing, and the extension of probability measures in general.

The sample spaces and probabilities we are interested in correspond closely to a biased coin tossing experiment. An analysis of this extension problem gives information which we can adapt easily to our probability spaces.

Let Ω be the sample space consisting of infinite sequences of O's and l's, that is, $\Omega = \{0,1\}^{I^+}$. Denote a point in Ω by $\omega = (\omega_1, \omega_2, \ldots)$. Subsets of Ω are denoted A, B, etc..

For each finite N, let $\Omega_N = \{0,1\}^{\{1,\ldots N\}}$. Let $\mathcal{F}_{0N}$ be the set of all sets $A \subset \Omega$ of the following form: There is a $B \subset \Omega_N$ such that $A = \{\omega = (\omega_1, \omega_2, \ldots) \mid (\omega_1, \omega_2, \ldots \omega_N) \in B\}$. $\mathcal{F}_{0N}$ is easily shown to be a field.. We define P_N on $\mathcal{F}_{0N}$, by

$$P_N(A) = \sum_{\omega \in A} p^{\sum_1^N \omega_j} (1-p)^{N - \sum_1^N \omega_j}$$

where $A \subset \mathcal{F}_{0N}$,

p is a fixed real number $0 < p < 1$.

It is easy to check that $(\Omega, \mathcal{F}_{0N}, P_N)$ is a finite probability space. Let $\mathcal{F}$ be the smallest σ-field containing all the $\mathcal{F}_{0N}$. We would like to determine whether or not $\bar{P}$ can be defined on $\mathcal{F}$ in such a way that $\bar{P}$ agrees with P_N on $\mathcal{F}_{0N}$ for each finite N, and $(\Omega, \mathcal{F}, \bar{P})$ is a probability space. The main difficulty is the following. Let

$$A = \{\omega = (\omega_1, \omega_2, \ldots) \varepsilon\ \Omega \mid (\omega_1, \ldots \omega_N) \varepsilon\ B\} ,$$

where B is a subset of Ω_N, for some $N \geq 1$.

Another way to write A is

$$A = \{\omega = (\omega_1, \omega_2, \ldots) \varepsilon\ \Omega \mid (\omega_1, \ldots \omega_{N+1}) \varepsilon\ C\} ,$$

where

$$C = \{\omega = (\omega_1, \ldots \omega_{N+1}) \varepsilon\ \Omega_{N+1} \mid (\omega_1, \ldots \omega_N) \varepsilon\ B\} .$$

So if $\overline{P}$ is to agree with P_N on each $\mathcal{F}_{o_N}$ we must make sure that P_N and P_{N+1} are consistent, that is we must make sure that $P_N(B) = P_{N+1}(C)$ for the sets B and C as above. We have

$$P_N(B) = \sum_{\omega \varepsilon B} p^{\sum_1^N \omega_j} (1-p)^{N - \sum_1^N \omega_j} ,$$

and

$$P_{N+1}(C) = \sum_{\omega \varepsilon C} p^{\sum_1^{N+1} \omega_j} (1-p)^{N+1 - \sum_1^{N+1} \omega_j}$$

$$= \sum_{\{\omega \varepsilon C \mid \omega_{N+1} = 0\}} p^{\sum_1^N \omega_j} (1-p)^{N+1 - \sum_1^N \omega_j} .$$

$$+ \sum_{\{\omega\varepsilon C|\omega_{N+1}=1\}} p^{(\sum_1^N \omega_j)+1} (1-p)^{N+1-[(\sum_1^N \omega_j)+1]}$$

$$= \sum_{\omega\varepsilon B} p^{\sum_1^N \omega_j} (1-p)^{N-\sum_1^N \omega_j} (p+1-p)$$

(where we have used $\{\omega\varepsilon C|\omega_{N+1}=0\} = \{(\omega_1,\ldots\omega_N,0)| (\omega_1,\ldots\omega_N)\varepsilon B\}$ and $\{\omega\varepsilon C|\omega_{N+1}=1\} = \{(\omega_1,\ldots\omega_N,1)| (\omega_1,\ldots\omega_N)\varepsilon B\}$)

$$= P_N(B) \quad .$$

We can conclude that for all N, the P_N's are consistent on any representation of such a set A. Because of this, the Kolmogorov Extension Theorem, (see Breiman[11], chapter II and appendix) applied to the spaces $(\Omega,\mathcal{F}_{oN},P_N)$ ensures that a unique $\overline{P}$ can be defined on $\mathcal{F}$ so that $\overline{P}$ agrees with P_N on $\mathcal{F}_{o\,N}$ for each N.

§2 Some Probability Spaces referred to in this paper

In the introduction we classified the sample spaces Ω_N and $\Omega_{M,N}$. We shall now define P_N on the field S_N, of all subsets of Ω_N for each of the classes of graphs I,II,III and IV.

We also define $P_{M,N}$ on the field $S_{M,N}$ of all subsets of $\Omega_{M,N}$. We will still refer to the probability spaces as classes of graphs (it will be clear when we mean (Ω_N, S_N, P_N) or just Ω_N, from the context: similarly $(\Omega_{M,N}, S_{M,N}, P_{M,N})$ or just $\Omega_{M,N}$).

Let us denote by Ω, the sets A^{B_i} where $A = \{0,1\}$ and B_i $I^+ x I^+$, $B_1 = \{(i,j) |\ 1 \leq i < j\}$; $B_2 = \{(i,j) |\ 1 \leq i \leq j\}$, $B_3 = \{(i,j) | 1 \leq i,\ 1 \leq j,\ i \neq j\}$ and $B_4 = \{(i,j) |\ 1 \leq i,\ 1 \leq j\}$. Which set A^{B_i}, Ω is to be interpreted as, will be clear from the context. Suppose X and Y are two countable index sets (both equivalent to I^+) and $X \cap Y = \emptyset$; we also let Ω denote the set A^B where $A = \{0,1\}$, and $B = \{(x,y) |\ x \in X,\ y \in Y\}$.

Let $\mathcal{F}_{o_N}$ be the set of all subsets, C, of Ω such that

$$C = \{\omega = \langle \omega_{(i,j)} \rangle_{(i,j) \in B_i} \in \Omega |\ \langle \omega_{(i,j)} \rangle_{(i,j) \in B_{i,N}} \in D \subset S_N\}.$$

It is easy to see that $\mathcal{F}_{o_N}$ is a field. We shall denote by $\mathcal{F}_o$ the smallest sigma field containing $\mathcal{F}_{o_N}$, for each N. Then $\mathcal{F}_o$ is a σ-field of subsets of Ω. Similarly, let $\mathcal{F}_{o_{M,N}}$ be the set of all subsets, C, of Ω such that

$$C = \{\omega = \langle \omega_{(i,j)} \rangle_{(i,j) \in CxD} \in \Omega |\ \langle \omega_{(i,j)} \rangle_{(i,j) \in C_{M,N}} \in D \subset S_{M,N}\}.$$

Again, it is easy to see, $\mathcal{F}_{o_{M,N}}$ is a field. We shall use $\mathcal{F}_o$ to denote the smallest sigma field containing $\mathcal{F}_{o_{M,N}}$ for each M and N. Which $\mathcal{F}_o$ we mean will be clear from the context.

We will define P_N and $P_{M,N}$ for each N or (M,N) respectively, so that they satisfy a consistency condition ensuring a unique $\bar{P}$ can be defined on $\mathcal{F}_0$, which is an extension of each P_N (or each $P_{M,N}$) that is, $\bar{P}$ agrees with P_N on S_N for each N (or agrees with $P_{M,N}$ on $S_{M,N}$ for each M,N).

For any particular graph ω belonging to Ω_N (or $\Omega_{M,N}$) we denote $\sum_{\omega} \omega_{(i,j)}$ by λ and call this the number of edges in the graph. For each Ω_N (or $\Omega_{M,N}$) we can determine the quantity $\max_{\omega\varepsilon\Omega_N} \sum_{\omega} \omega_{(i,j)}$ (or $\max_{\omega\varepsilon\Omega_{M,N}} \sum_{\omega} \omega_{(i,j)}$) which we denote $\bar{\lambda}_N$ (or $\bar{\lambda}_{M,N}$), again we will let the context make clear which class of graphs is being referred to. We will use $\bar{\lambda}_N$ and $\bar{\lambda}_{M,N}$ in defining P_N and $P_{M,N}$ respectively. For class I graphs,

$$\bar{\lambda}_N = \frac{N(N-1)}{2} \qquad . \tag{1.6}$$

To conclude this, count the elements (i,j) of $B_{1,N}$. The number is $\sum_{j=2}^{N} (j-1) = \frac{N(N-1)}{2}$. For graphs of class II, $\bar{\lambda}_N$ is

$$\bar{\lambda}_N = \frac{N(N+1)}{2} \qquad . \tag{1.7}$$

This is because $B_{2,N}$ allows N elements (i,i) in addition to the elements of $B_{1,N}$. The set $B_{3,N}$ contains all the elements (i,j) of $B_{1,N}$ and in addition all the elements (j,i). Therefore we have for class III

$$\bar{\lambda}_N = N(N-1) \quad . \tag{1.8}$$

The set $B_{4,N}$ contains all the elements (i,j) of $B_{3,N}$ and, in addition, N elements (i,i). So, for class IV graphs

$$\bar{\lambda}_N = N^2 \quad . \tag{1.9}$$

For bipartite graphs of type M,N it is easy to see that

$$\bar{\lambda}_{M,N} = MN \quad . \tag{1.10}$$

For class I graphs as Ω_N, we define P_N on S_N for each N as follows. Let p be any fixed real number such that $0 < p < 1$. Then we set

$$P_N(\{\omega | \ \omega_{(i,j)} = 1\} = p$$

and

$$P_N(\{\omega | \ \omega_{(i,j)} = 0\} = 1-p = q \quad ,$$

for each fixed $(i,j) \varepsilon B_1$.

and

$$P_N(\{\omega\}) = \prod_{\omega_{(i,j)}=1} P_N(\{\omega | \ \omega_{(i,j)}=1) \quad .$$

$$\prod_{\omega_{(i,j)}=0} P_N(\{\omega | \ \omega_{(i,j)}=0) \quad . \tag{1.11}$$

The last statement of definition 1.11 is equivalent to the statement: events $\{\omega|\ \omega_{(i,j)}=1\}$, $\{\omega|\ \omega_{(k,l)}=1\}$, $\{\omega|\ \omega_{(m,n)}=0\}$ ... and so on are all independent of each other if $(i,j) \neq (k,l) \neq (m,n)$ ... and so on, and therefore an event ω which is an intersection of independent events of this type, is assigned a probability which is the product of the probabilities of each of the events in the intersection.
The last statement of definition 1.11 can be more compactly written as

$$P_N(\{\omega\}) = p^{\lambda} \quad q^{\overline{\lambda}_N - \lambda}$$

$$\text{where } \lambda = \sum_{\omega} \omega_{(i,j)} \text{ and } \overline{\lambda}_N = \frac{N(N-1)}{2} \quad . \qquad (1.12)$$

Of course, for all $A \subset \Omega_N$

$$P_N(A) = \sum_{\{\omega \varepsilon A\}} P_N(\{\omega\}) \quad .$$

Instead of taking p a constant in the definition of P_N, the p appearing in (1.11) and (1.12) could be taken to be a function of time, for example $p(t) = 1-e^{-\lambda t}$. The resulting P_N is then time dependent. This kind of probability function is mentioned in chapter V where we summarize the results of various authors.

We define P_N for the classes of graphs II, III and IV similarly; also we define $P_{M,N}$ for bipartite graphs of type

M,N similarly, keeping in mind that $\overline{\lambda}_N$ or $\overline{\lambda}_{M,N}$ depends on the class. We note that $P_N(\{\omega\})$ according to (1.11) and (1.12) depends only on the number of edges appearing in ω but not on which edges appear.

For each class of graphs, $(\Omega_N, \mathcal{F}_{o_N}, P_N)$ or $(\Omega_{M,N}, \mathcal{F}_{o_{M,N}}, P_{M,N})$ is a finite probability space. As we have already noted, we would like to ensure that there is a unique $\overline{P}$ defined on the field $\mathcal{F}_o$, of subsets of Ω for each respective class, in such a way that $\overline{P}$ agrees with P_N on Ω_N for each N, (or $\overline{P}$ agrees with $P_{M,N}$ on $\Omega_{M,N}$ for each M and N) for each class respectively. The similarity between this problem and the extension problem arising from an analysis of biased coin tossing is immediately apparent.

We shall show that the necessary consistency condition holds for class I graphs. It holds similarly for the other classes.

Suppose $A \subset \Omega$ is a subset of the form: There is a $B \subset \Omega_N$ such that $A = \{\omega = \langle \omega_{(i,j)} \rangle_{(i,j) \varepsilon B_1} \mid \langle \omega_{(i,j)} \rangle_{(i,j) \varepsilon B_{1,N}} \varepsilon B\}$.

(Recall that $B_1 = \{(i,j) \mid 1 \leq i < j$ and $B_{1,N} = \{(i,j) \mid 1 \leq i < j \leq N\}$.) The set A is also defined by

$$A = \{ \omega = \langle \omega_{(i,j)} \rangle_{(i,j) \varepsilon B_1} \mid \langle \omega_{(i,j)} \rangle_{(i,j) \varepsilon B_{1,N+1}} \varepsilon C \},$$

where $C \subset \Omega_{N+1}$, and $C = \{\omega \varepsilon \Omega_{N+1} \mid \langle \omega_{(i,j)} \rangle_{(i,j) \varepsilon B_{1,N}} \varepsilon B\}$.

The consistency condition is: $P_N(B)$ must equal $P_{N+1}(C)$.

Let $\overline{\lambda} = \overline{\lambda}_{N+1} - \overline{\lambda}_N$. Thus for $\omega \varepsilon \Omega_{N+1}$, $\overline{\lambda}$ is the maximum possible number of edges attached to the vertex N+1 of the graph ω. For each j, $0 \leq j \leq \overline{\lambda}$, let

$$S_j = \{\omega \varepsilon\ C \mid \omega \text{ has exactly } j \text{ edges attached to the vertex } N+1\}.$$

Then C is the disjoint union

$$C = \bigcup_{j=0}^{\overline{\lambda}} S_j \quad .$$

Let $\tau \colon \Omega_{N+1} \rightarrow \Omega$ be defined by:

$$\text{for } \omega = \left\langle \omega_{(i,j)} \right\rangle_{(i,j)\varepsilon B_{1,N+1}} ,$$

$$\tau(\omega) = \left\langle \omega_{(i,j)} \right\rangle_{(i,j)\varepsilon B_{1,N}} .$$

Now, for each $\omega \varepsilon$ B, let

$$T(\omega,j) = \{u\ \varepsilon S_j \mid \tau(u) = \omega\} .$$

Then

$$|T(\omega,j)| = \binom{\overline{\lambda}}{j} ,$$

where $\binom{\overline{\lambda}}{j}$ is the binomial coefficient $\dfrac{\overline{\lambda}!}{\overline{\lambda}!\,(\lambda-j)!}$, and for each $u \in T(\omega,j)$

$$P_{N+1}(u) = P_N(\omega)\ p^j\ q^{\bar{\lambda}-j}\ .$$

So,

$$P_{N+1}(T(\omega,j)) = \binom{\bar{\lambda}}{j}\ P_N(\omega)\ p^j\ q^{\bar{\lambda}-j}\ .$$

Also, S_j is the disjoint union

$$S_j = \bigcup_{\omega \in B} T(\omega,j)\ ,$$

So,

$$P_{N+1}(S_j) = \sum_{\omega \in B} \binom{\bar{\lambda}}{j}\ P_N(\omega)\ p^j\ q^{\bar{\lambda}-j}\ .$$

Finally,

$$P_{N+1}(C) = \sum_{j=0}^{\bar{\lambda}} P_{N+1}(S_J)$$

$$= \sum_{j=0}^{\bar{\lambda}} \sum_{\omega \in B} \binom{\bar{\lambda}}{j} P_N(\omega)\ p^j\ q^{\bar{\lambda}-j}$$

$$= \sum_{\omega \in B} P_N(\omega) \sum_{j=0}^{\bar{\lambda}} \binom{\bar{\lambda}}{j}\ p^j\ q^{\bar{\lambda}-j}$$

$$= \sum_{\omega \in B} P_N(\omega)$$

$$= P_N(B)\ .$$

The consistency condition is satisfied.

It is comforting to know that the desired $\bar{P}$ can be defined on $\mathcal{F}_0$, because we are now assured that it is valid to look for any conclusions about the probabilities of events contained in Ω defined in terms of arbitrarily large N. In particular: let $\mathcal{C}(N)$ denote the set of all connected graphs in Ω_N. Since P_N was defined for finite N we cannot legitimately discuss $\lim_{N\to\infty} P_N(\mathcal{C}(N))$. However it makes sense to discuss $\lim_{N\to\infty} \bar{P}(\mathcal{C}(N))$, and we know that $\bar{P}$ agrees with P_N for each N. In subsequent chapters we shall discuss $\lim_{N\to\infty} \bar{P}(\mathcal{C}(N))$ for graphs of class I to IV and bipartite graphs. Please note however, that we continue to use the not quite legitimate notation, $\lim_{N\to\infty} P_N(\mathcal{C}(N))$ when we really mean $\lim_{N\to\infty} \bar{P}(\mathcal{C}(N))$. We also drop the notation $\mathcal{C}(N)$ and use just $\mathcal{C}$.

Chapter II

We discuss here some mathematical methods which will be used in subsequent chapters. Where further elaboration may be desired by the reader, we mention a pertinent reference.

§1

Transform techniques are familiar in many branches of mathematics. We are interested in a technique usually called the "method of generating functions", which is well known. A text by W. Feller[12] contains an informative chapter on the use of generating functions in the theory of probability. We will just briefly outline the method here.

Definition 2.1 Let $a_0, a_1, a_2, \ldots$ be a sequence of real numbers. If

$$A(s) = a_0 + a_1 s + a_2 s^2 + \ldots = \sum_{i=0}^{\infty} a_i s^i \tag{2.1}$$

converges in some interval $-s_0 < s < s_0$, then $A(s)$ is called the generating function of the sequence $\{a_i\}$.

The variable s is a formal variable, so s itself has no significance.

If a formula is known which gives a_i for each i, then $\{a_i\}$ is known. If we can equate the generating function of a sequence $\{a_i\}$ to a known power series or to an analytic function (either expanded or in closed form) then we consider $\{a_i\}$ to be completely known. The fact that in the latter case the expansion coefficients corresponding to our $\{a_i\}$ may actually be difficult to calculate is not of concern.

The advantage to us of considering the generating function of $\{a_i\}$ rather than the sequence itself is that equations relating the elements of sequences when transformed to equations relating power series, can be manipulated more easily. The algebra of power series is well known. The purpose of our manipulations would, of course, be to equate the generating functions of given sequences (where a formula for the a_i is perhaps unknown) to known power series or closed analytic expressions, and in this way extract previously unknown information about the sequences.

For the algebra of sequences we are assuming addition is termwise and multiplication is defined by

$$\{a_n\}_{n=0}^{\infty} \cdot \{b_n\}_{n=0}^{\infty} = \{c_n\}_{n=0}^{\infty}$$

$$\text{where } c_n = \sum_{i=0}^{n} a_i b_{n-i} \quad .$$

The transformation which transforms a sequence for which a generating function can be defined, into the generating function, is then obviously a monomorphism from the algebra onto the usual algebra of convergent power series. (Note: if two power series A and B are convergent for $|x| < r_1$ and $|x| < r_2$ respectively then A+B and AB are convergent for $|x| < \min(r_1, r_2)$).

§2 Formal Power Series

In chapters III and IV we shall perform calculations involving power series which have zero radius of convergence. To justify these calculations, we give briefly, the basic definitions of the algebra of formal (disregarding convergence) power series. The use of formal power series can be justified in detail as in, for example, a paper by E.T. Bell[9]. A very readable account of the algebra of formal power series is given in a text by H. Cartan[13]. (After defining formal power series operations, below, there is no need to restrict definition (2.1) to convergent power series.)

Definition 2.2 A formal power series A, is a series in a formal variable θ, $A = \sum_{i=0}^{\infty} a_i \theta^i$ where the a_i are from a scalar field (we will use $\mathbb{R}$) and the sum may or may not have zero radius of convergence.

Definition 2.3 Two formal power series $A = \sum_{i=0}^{\infty} a_i \theta^i$ and $B = \sum_{i=0}^{\infty} b_i \theta^i$ are equal if and only if $a_i = b_i$ for all

$i = 0,1,\ldots$.

Operations of scalar multiplication, addition of formal power series and multiplication of formal power series are defined by:

<u>Definition</u> Let $A = \sum_{i=0}^{\infty} a_i\theta^i$ and $B = \sum_{i=0}^{\infty} b_i\theta^i$ be formal power series and let α be a scalar, then

$$\alpha\left(\sum_{i=0}^{\infty} a_i\theta^i \right) = \sum_{i=0}^{\infty} (\alpha a_i)\theta^i \quad , \tag{2.4}$$

$$\sum_{i=0}^{\infty} a_i\theta^i + \sum_{i=0}^{\infty} b_i\theta^i = \sum_{i=0}^{\infty} (a_i+b_i)\theta^i \quad , \tag{2.5}$$

$$\left(\sum_{i=0}^{\infty} a_i\theta^i \right)\cdot\left(\sum_{i=0}^{\infty} b_i\theta^i \right) = \sum_{i=0}^{\infty} \left(\sum_{n=0}^{i} a_n b_{i-n} \right)\theta^i \quad . \tag{2.6}$$

Addition can be performed any finite number, m, of times in succession using (2.5) m times. It is easy to see that addition is commutative, associative, has an identity , $0 = \sum_{i=0}^{\infty} 0\theta^i$, and that additive inverses exist. Multiplication can also be performed any finite number, m, of times in succession. For m power series A,B,...X multiplied together we obtain, by using (2.6) m times,

$$A\cdot B\cdot \; \ldots \; \cdot X = \sum_{n=0}^{\infty} \left(\Sigma \, a_{i_1} \cdot b_{i_2} \cdot \; \ldots \cdot x_{i_m} \right) \theta^n ,$$

where the inside sum is over all partitions of n, $i_1+i_2+\ldots+i_m = n$.

Multiplication is commutative, associative, and has identity $1 = \sum_{i=0}^{\infty} a_i\theta^i$ where $a_i = \begin{cases} 1, & i=0 \\ 0 & \text{otherwise} \end{cases}$; further $A = \sum_{i=0}^{\infty} a_i\theta^i$ has an inverse under multiplication if and only if $a_o \neq 0$, (this last is proved in the text by Cartan[13] and the other statements are easily checked).

Let $A = \sum_{i=0}^{\infty} a_i\theta^i$, the Order $w(A)$, of this formal power series is defined, only if $A \neq 0$, to be the smallest number i for which $a_i \neq 0$. A formal series A is of order $\geq k$ if it is 0 or if $w(A) \geq k$. Even though $w(0)$ is not strictly speaking defined we write $w(0) \geq k$ since we can think of $w(0)$ as being $+\infty$.

Definition 2.7 A family of formal power series $(A_i)_{i\varepsilon I}$, where each A_i is a formal series $\sum_{n=0}^{\infty} a_{n,i}\theta^n$, and I is a countable index set, is called summable if: for every k, $w(A_i) \geq k$ except for a finite number of indices i. We define the sum of a summable family of formal series $(A_i)_{i\varepsilon I}$ to be

$$\sum_{i \in I} A_i = \sum_{n \geq 0} a_n \theta^n \quad ,$$

where for each n, $a_n = \sum_i a_{n,i}$.

The above definition is reasonable because the condition of summability ensures that for a given n, the $a_{n,i}$ are all zero except for a finite number of indices i. The operation of addition of a summable family of power series generalizes the finite addition defined in (2.6). This generalized addition can be readily seen to be commutative and associative in the sense that, for each n the sum $a_n = \sum_{i \in I} a_{n,i}$ may be rearranged in any order or associated in any way since the $a_{n,i}$ are from a field and the sum is actually finite (only a finite number of non-zero $a_{n,i}$'s). This is the same as saying that $\sum_{i \in I} A_i$ can be performed in any order over $i \in I$, or with any grouping of terms.

It is convenient to name a formal series $\sum_{i=0}^{\infty} a_i \theta^i$ where $a_i = 0$ for $i \neq p$; we shall call such a series a <u>single term series of degree p</u>. The family of single term series $(a_n \theta^n)_{n \in N}$ is obviously summable and its sum is the formal power series $\sum_{i=0}^{\infty} a_i \theta^i$. We note that the product of two formal power series $A = \sum_{i=0}^{\infty} a_i \theta^i$ and $B = \sum_{i=0}^{\infty} b_i \theta^i$, is the sum of the summable family formed by all the products of single term series $(a_p \theta^p) \cdot (b_p \theta^p) = a_p b_q \theta^{p+q}$. If we let

I be NxN (so the indexing set is a set of ordered pairs), then the product of A and B may be written as the sum of the summable family $(A_{(i)})_{i\varepsilon I}$ where $A_{(p,q)} = a_p b_q \, \theta^{p+q}$ ..

Proposition 2.8 Let $(A_i)_{i\varepsilon I}$ be a summable family of formal power series, where $A_i = \sum_{n=0}^{\infty} a_{n,i}\theta^n$, and let C be a formal series $C = \sum_{i=0}^{\infty} c_i\theta^i$, then

$$(\sum_{i\varepsilon I} A_i)\cdot C = \sum_{i\varepsilon I} (A_i\cdot C) \quad .$$

Proof: $(\sum_{i\varepsilon I} A_i)\cdot C = (\sum_{n=0}^{\infty} a_n\theta^n)\cdot C \qquad \text{where } a_n = \sum_{i\varepsilon I} a_{n,i}$

$$= (\sum_{n=0}^{\infty} (\sum_{i\varepsilon I} a_{n,i})\theta^n)\cdot \sum_{n=0}^{\infty} c_n\theta^n \ .$$

This can be written as the sum of the summable family

$$((\sum_{i\varepsilon I} a_{p,i})c_q \; \theta^{p+q})_{(p,q)\varepsilon N\times N} \quad .$$

Now the sum $\sum_{i\varepsilon I} a_{p,i}$ is actually a finite sum for each p, and we can write this as

$$(\sum_{i\varepsilon I} a_{p,i}c_q \; \theta^{p+q})_{(p,q)\varepsilon N\times N} \quad .$$

This we can again rewrite as the sum of the summable family

$$(a_{p,i}c_q\theta^{p+q})_{(p,q,i)\ NxNxN} .$$

It is readily seen that this family is still summable. Now consider

$$\sum_{i\varepsilon I}(A_iC) = \sum_{i\varepsilon I}[(\sum_{n=0}^{\infty} a_{n,i}\theta^n)(\sum_{n=0}^{\infty} c_n\theta^n)] .$$

The square bracket can be rewritten as the sum of the summable family

$$(a_{p,i}c_q\theta^{p+q})_{(p,q)\varepsilon NxN} .$$

Now consider

$$\sum_{i\varepsilon I}(\sum_{(p,q)\varepsilon NxN} a_{p,i}c_q\theta^{p+q}) ,$$

since for each (p,q) the sum over $i\varepsilon I$ is finite, it is easy to see that we can also write this as the sum of the summable family

$$(a_{p,i}c_q\theta^{p+q})_{(p,q,i)\varepsilon NxNxN} .$$

We therefore have $(\sum_{i\varepsilon I} A_i)\cdot C = \sum_{i\varepsilon I}(A_i\cdot C)$ since they are both the sum of the same summable family.

Consider two formal power series $A = \sum_{i=0}^{\infty} a_i\theta^i$ and $B = \sum_{i=0}^{\infty} b_i\theta^i$. Suppose $b_0 = 0$ or, $w(B) \geq 1$. For each i, the expression $a_i B^i$ is defined and the order of $a_i B^i$ is $\geq i$ (since $b_0 = 0$). The family $(a_i B^i)_{i \in \{0,1,\ldots\}}$ is easily seen to be summable. Therefore it is reasonable to make the following definition.

<u>Definition 2.9</u> Let $A = \sum_{i=0}^{\infty} a_i \theta^i$ and $B = \sum_{i=0}^{\infty} b_i \theta^i$, be two formal power series and let $b_0 = 0$. Then we define the <u>composition</u> of A and B, denoted $A \circ B$, by

$$A \circ B = \sum_{i=0}^{\infty} a_i B^i \quad ,$$

which is a formal power series with the terms in θ^i rearranged.

Various propositions can be proved regarding the composition of formal power series. We list a few below. For proofs of these we refer the reader to the text by H. Cartan[13]. If A and B are two formal power series as in (2.9) and 1 is the multiplicative identity, and C is the formal power series $C = \sum_{i=0}^{\infty} c_i \theta^i$, then

$$1 \circ B = 1 \quad ,$$

$$(A+C) \circ B = A \circ B + C \circ B \quad ,$$

$$AC \circ B = (A \circ B).(C \circ B) \quad ,$$

and if $c_0 = 0$, then

$$(A \circ B) \circ C = A \circ (B \circ C) \quad .$$

If $(A_i)_{i \varepsilon I}$ is a summable family as in (2.8) and B is again as in (2.9), then

$$(\sum_{i \varepsilon I} A_i) \circ B = \sum_{i \varepsilon I} (A_i \circ B) \quad .$$

<u>Definition 2.10</u> The derivative with respect to θ of a formal power series $A = \sum_{i=0}^{\infty} a_i \theta^i$ is given by:

$$\frac{d}{d\theta} A = \sum_{n=0}^{\infty} (n+1) a_{n+1} \theta^n \quad .$$

<u>Proposition 2.11</u> The derivative with respect to θ of a sum of two formal power series $A = \sum_{i=0}^{\infty} a_i \theta^i$ and $B = \sum_{i=0}^{\infty} b_i \theta^i$ is given by $\frac{d}{d\theta}A + \frac{d}{d\theta}B$.

Proof: $A + B = \sum_{i=0}^{\infty} (a_i + b_i) \theta^i$,

$$\frac{d}{d\theta}(A + B) = \sum_{n=0}^{\infty} (n+1)(a_{n+1} + b_{n+1}) \theta^n \quad ,$$

whereas

$$\frac{d}{d\theta}A + \frac{d}{d\theta}B = \sum_{n=0}^{\infty} (n+1) a_{n+1} \theta^n + \sum_{n=0}^{\infty} (n+1) b_{n+1} \theta^n$$

$$= \sum_{n=0}^{\infty} (n+1)(a_{n+1} + b_{n+1})\theta^n \quad .$$

We apply definition (2.3) to get the desired result.

It is easy to extend proposition (2.11) to any number of summands by induction. It is also easy to show $\frac{d}{d\theta}\alpha A = \alpha\frac{d}{d\theta} A$.

<u>Proposition 2.12</u> The derivative with respect to θ of the sum of a summable family $(A_i)_{i\varepsilon I}$, $A_i = \sum_{n=0}^{\infty} a_{n,i}\theta^n$, is equal to the sum of the summable family $\left(\frac{d}{d\theta} A_i\right)_{i\varepsilon I}$.

Proof: $\frac{d}{d\theta} \sum_{i\varepsilon I} \sum_{n=0}^{\infty} a_{n,i}\theta^n = \frac{d}{d\theta} \sum_{n=0}^{\infty} \left(\sum_{i\varepsilon I} a_{n,i} \right)\theta^n$

by definition of the sum of a summable family. Now we find the derivative on the right hand side above and we obtain

$$\sum_{n=0}^{\infty} (n+1) \sum_{i\varepsilon I} a_{n+1,i}\theta^n \quad .$$

On the other hand

$$\sum_{i\varepsilon I} \frac{d}{d\theta} A_i = \sum_{n=0}^{\infty} \sum_{i\varepsilon I} a_{n+1,i}(n+1)\ \theta^n$$

$$= \sum_{n=0}^{\infty} (n+1) \sum_{i\varepsilon I} a_{n+1,i}\ \theta^n \quad .$$

Therefore

$$\sum_{i\in I} \frac{d}{d\theta} A_i = \frac{d}{d\theta} \sum_{i\in I} \sum_{n=0}^{\infty} a_{n,i}\theta^n$$

$$= \frac{d}{d\theta} \sum_{i\in I} A_i \quad .$$

Proposition 2.13 The derivative with respect to θ of a product of two formal power series A and B as in (2.11), is given by $\frac{d}{d\theta} AB = A \frac{d}{d\theta} B + B \frac{d}{d\theta} A$.

Proof: $$\frac{d}{d\theta} AB = \frac{d}{d\theta} \sum_{n=0}^{\infty} (\sum_{j=0}^{n} a_j b_{n-j})\theta^n$$

$$= \sum_{n=0}^{\infty} (n+1) (\sum_{j=0}^{n+1} a_j b_{n+1-j})\theta^n \ .$$

Whereas

$$A \frac{d}{d\theta} B + B \frac{d}{d\theta} A = \sum_{n=0}^{\infty} a_n\theta^n \cdot \sum_{n=0}^{\infty} (n+1) b_{n+1}\theta^n$$

$$+ \sum_{n=0}^{\infty} b_n\theta^n \cdot \sum_{n=0}^{\infty} (n+1) a_{n+1}\theta^n$$

$$= \sum_{n=0}^{\infty} (\sum_{j=0}^{n} (j+1) b_{j+1} a_{n-j})\theta^n$$

$$+ \sum_{n=0}^{\infty} (\sum_{j=0}^{n} (n-j+1) b_j a_{n-j+1})\theta^n$$

$$= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} (j+1) b_{j+1} a_{n-j} + \sum_{j=0}^{n} (n-j+1) b_j a_{n-j+1} \right) \theta^n$$

$$= \sum_{n=0}^{\infty}{}' \left(\sum_{j=0}^{n+1} (n+1) b_j \; a_{n+1-j} \right) \theta^n \quad .$$

We apply definition (2.3) to get the desired conclusion.

Proposition (2.13) can be extended to a product of N power series for any N, by induction. A particularly useful case is the derivative of A^N.

<u>Proposition 2.14</u> The derivative with respect to θ, of A^N where $A = \sum_{i=0}^{\infty} a_i \theta^i$, is given by $\frac{d}{d\theta} A^N = (NA^{N-1}) \left(\frac{d}{d\theta} A \right)$.

Proof: By proposition (2.13), (2.14) is true for N = 2.

Let us suppose that $\frac{d}{d\theta} A^N = NA^{N-1} \frac{d}{d\,\theta} A$. Then,

$A^{N=1} = (A) \cdot (A^N)$ so,

$$\frac{d}{d\theta} A^{N+1} = A \frac{d}{d\theta} A^N + A^N \frac{d}{d\theta} A$$

$$= A(NA^{N-1}) \frac{d}{d\theta} A + A^N \frac{d}{d\theta} A$$

$$= \left(\frac{d}{d\theta} A \right) (A^N N + A^N)$$

$$= (N+1)A^N \frac{d}{d\theta}A \quad .$$

Proposition 2.15 The derivative with respect to θ, of $A \circ B$ where $A = \sum_{i=0}^{\infty} a_i \theta^i$ and $B = \sum_{i=0}^{\infty} b_i \theta^i$, $b_0 = 0$, is

$$\left(\frac{d}{dB}(A \circ B)\right)\frac{d}{d\theta}B \quad .$$

Proof: $\frac{d}{d\theta}(A \circ B) = \frac{d}{d\theta} \sum_{i=0}^{\infty} a_i B^i$

$$= \sum_{i=0}^{\infty} \frac{d}{d\theta} a_i B^i \quad \text{(where we have used (2.12))}$$

$$= \sum_{i=0}^{\infty} a_i \frac{d}{d\theta} B^i$$

$$= \sum_{i=0}^{\infty} a_i (i) \; B^{i-1} \frac{d}{d\theta} B \quad \text{(where we have used (2.14))}$$

$$= \left(\sum_{i=0}^{\infty} a_{i+1}(i+1)B^i\right) \frac{d}{d\theta} B \quad \text{(where we have used (2.8))}$$

$$= \left(\frac{d}{dB} A \circ B\right) \cdot \frac{d}{d\theta} B \quad .$$

Higher order derivatives are found by successive applications of the above rules.

We shall now define an operation we call integration of formal power series.

<u>Definition 2.16</u>
$$\int_0^\theta \left(\sum_{n=0}^{\infty} t_n\theta^n\right)d\theta = \sum_{n=0}^{\infty} \frac{t_n\theta^{n+1}}{n+1} - \sum_{n=0}^{\infty} \frac{t_n 0^{n+1}}{n+1}$$
$$= \sum_{i=0}^{\infty} \frac{t_{i-1}\theta^i}{i} \quad .$$

Then also,
$$\frac{d}{d\theta}\int_0^\theta \left(\sum_{n=0}^{\infty} a_n\theta^n\right)d\theta = \sum_{n=0}^{\infty} a_n\theta^n \quad ,$$

and
$$\int_0^\theta \frac{d}{d\theta}\left(\sum_{n=0}^{\infty} a_n\theta^n\right)d\theta = \sum_{n=1}^{\infty} a_n\theta^n \quad .$$

These are easily checked.

It is easy to apply the definition to special cases of formal power series, for example, $\int_0^\theta \frac{d}{d\theta}(A\circ B)d\theta = A\circ B$,

and so on.

Certain compositions of formal power series have properties formally analagous to well known scalar functions and we name these power series accordingly. For example, if $A = \sum_{i=0}^{\infty} \frac{a^i}{i!} \theta^i$ and B is any appropriate formal power series, that is, $w(B) \geq 1$, then $A \circ B = \sum_{i=0}^{\infty} \frac{a^i}{i!} B^i$ is named exponent(aB). It is possible to check that exponent(aB) has the usual properties $\frac{d}{dB} \exp(aB) = a \exp(aB)$, and so on. One of these formal power series which we shall use later is named $\log_e(1+B)$.

<u>Definition 2.17</u> We denote the formal power series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \theta^n$ by $\log_e(1+\theta)$. If A is any formal power series $\sum_{n=0}^{\infty} a_n\theta^n$ with $w(A) \geq 1$ (that is, at least, $a_0=0$) then $\log(A+1) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} A^n$.

Since we took $w(A) \geq 1$, the family $(A^n)_{n \in N}$ is summable and the above definition of log(A+1) makes sense.

<u>Proposition 2.18</u> Let A, and log(A+1) be as in definition (2.17). Then $\frac{d}{d\theta} \log(A+1) = \frac{1}{A+1} \frac{dA}{d\theta}$.

Proof: $\frac{d}{d\theta} \log(A+1) = \frac{d}{d\theta} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} A^n$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \frac{d}{d\theta} A^n$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} nA^{n-1} \frac{d}{d\theta} A$$

$$= \left(\sum_{n=1}^{\infty} (-1)^{n+1} A^{n-1} \right) \frac{d}{d\theta} A \quad .$$

The formal series A+1 has first term = 1 ≠ 0, and therefore as we remarked previously, A+1 has a multiplicative inverse, $\frac{1}{A+1}$. We can calculate $\frac{1}{A+1}$ easily by observing that,

$(1-(-A))(1+(-A)+(-A)^2+\ldots) = 1.$ So,

$\frac{1}{A+1} = \sum_{n=1}^{\infty} (-1)^{n+1} A^{n-1}$. We conclude that

$$\frac{d}{d\theta}(\log(A+1)) = \frac{1}{A+1} \frac{d}{d\theta} A \quad .$$

<u>Proposition 2.19</u> Let A be as in (2.17). Then

$$\int_0^{\theta} \left(\left(\frac{1}{A+1} \right) \frac{d}{d\theta} A \right) d\theta = \log(A+1) \quad .$$

Proof: $\int_0^{\theta} \left(\left(\frac{1}{A+1} \frac{d}{d\theta} A \right) d\theta = \int_0^{\theta} \sum_{n=1}^{\infty} (-1)^{n+1} A^{n-1} \frac{d}{d\theta} A \, d\theta$

$$= \int_0^\Theta \frac{d}{d\Theta} \left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} A^n \right) d\Theta$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} A^n$$

$$= \log(1+A) \quad .$$

The development of the theory of formal power series we have given is, of course, not complete, but it is sufficient to explain our use of formal power series in later chapters.

The algebra of double formal power series, A, where A is a double sum in formal variables Θ and Λ,

$$A = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{nm} \Theta^n \Lambda^m \ ,$$ where a_{nm} is from a scalar field (we will use $\mathbb{R}$),

is defined similarly to the algebra of formal power series. All the various definitions and propositions stated for formal power series are similarly stated for double formal power series. The extension of most of the definitions and propositions is obvious. To clarify a few details of the algebra of double formal power series we make the following comments.

Multiplication is defined by: let $A = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{nm} \theta^n \Lambda^m$, $B = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_{nm} \theta^n \Lambda^m$, then

$$AB = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(\sum_{i=0}^{m} \sum_{j=0}^{n} a_{ji} b_{n-j,m-i} \right) \theta^n \Lambda^m .$$

We defined a special composition of formal power series named log(1+A) in definition (2.17). To make the corresponding definition for double formal power series the formal power series defined as log (1+θ) in (2.17) is considered to be a double formal power series. That is, $\sum_{n=1}^{\infty} (-1)^{n+1} \theta^n$ is the same as the double formal power series $C = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_{nm} \theta^n \Lambda^m$, where $c_{nm} = \begin{cases} \frac{(-1)^{n+1}}{n} & \text{if } m=0 \\ 0 & \text{otherwise} \end{cases}$. Then, for A a double formal power series with order ≥ 1, log(1+A) would be defined as

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} A^n$$

, where A^n refers to the n^{th} power of A under the defined multiplication.

Chapter III

In this chapter we will summarize what is currently known about the probability of connectedness, and its asymptotic behaviour, of a random graph from one of the probability spaces (Ω_N, P_N, S_N) where P_N is not time dependent and the sample space Ω_N is the set of class I or II or III or IV graphs.

In 1959, E.N. Gilbert published a paper entitled "Random Graphs"[2]. In this paper, the probability of connectedness of a random graph of class I was investigated. The method used applies equally well to random graphs of classes II, III, and IV and the results differ only slightly for each class so we shall state the results for all four classes. This paper by Gilbert is based largely on an earlier paper (1956) also by Gilbert, entitled "Enumeration of Labelled Graphs"[1]. The earlier paper is referred to by several other authors who afford it some importance. We review both papers in detail here, beginning with some introductory comments.

Consider the probability spaces of Chapter I. The family (A_λ) of subsets of Ω_N, where $A_\lambda = \{\omega | \sum_{\omega} \omega_{(i,j)} = \lambda\}$, satisfies $\bigcup_{\lambda=0}^{\bar{\lambda}_N} A_\lambda = \Omega_N$ and, $A_\lambda \cap A_{\lambda'} = \emptyset$, for $\lambda \neq \lambda'$.

We can therefore write $P_N(\mathscr{C})$ as

$$P_N(\mathcal{C}) = P_N(\mathcal{C} \cap \bigcup_{\lambda=0}^{\bar{\lambda}_N} A_\lambda)$$

$$= P_N(\bigcup_{\lambda=0}^{\bar{\lambda}_N} \mathcal{C} \cap A_\lambda)$$

$$= \sum_{\lambda=0}^{\bar{\lambda}_N} P_N(\mathcal{C} \cap A_\lambda) \quad . \tag{3.1}$$

For the spaces we are discussing, $P_N(\mathcal{C} \cap A_\lambda)$ is just the number of connected graphs ω, with $\sum_\omega \omega_{(i,j)} = \lambda$ edges, multiplied by the probability of one particular graph with λ edges. The number of connected graphs (of whatever class) having $\sum_\omega \omega_{(i,j)} = \lambda$ edges and belonging to Ω_N we will denote by $C_{N\lambda}$; the total number of graphs with N vertices and λ edges we will denote by $T_{N\lambda}$. We can now rewrite (3.1) using (1.11) and this notation, as

$$P_N(\mathcal{C}) = \sum_{\lambda=0}^{\bar{\lambda}_N} C_{N\lambda}\, p^\lambda\, q^{\bar{\lambda}_N - \lambda} \quad . \tag{3.2}$$

If $C_{N\lambda}$ can be determined for all N and λ then we have an explicit expression for $P_N(\mathcal{C})$ for all finite N. (This does not necessarily aid in finding asymptotic behaviour.)

Gilbert's earlier paper presents the generating functions for $C_{N\lambda}$ for graphs of classes I through VIII. Recall that the classes V to VIII are similar to I to IV, but allow any number of multiple edges between vertices. For these graphs

$\overline{\lambda}_N$ does not exist and therefore, we cannot define a probability function on the sample spaces Ω in the same way as we did in (1.11). We state Gilbert's results on $C_{N\lambda}$ for all eight classes of graphs but continue to restrict our summary of results on connectedness of random graphs to the first four classes of graphs.

For each of the eight classes of graphs, $T_{N\lambda}$ can be determined. For classes I through IV, $T_{N\lambda}$ is easily seen to be the binomial coefficient $\binom{\overline{\lambda}_N}{\lambda}$ where $\overline{\lambda}_N$ is different for each class, as determined in chapter I. For class V, notice that $\Omega = A^B$ where B is the same as for class I, that is B_{1N} . It follows that $T_{N\lambda}$ for class V is just the number of combinations of λ lines which can be drawn from $\overline{\lambda}_N$ (for classI) lines with repetitions allowed, that is $\binom{N(N-1)/2 + \lambda - 1}{\lambda}$. Similarly, for class VI, $T_{N\lambda} = \binom{N(N+1)/2 + \lambda - 1}{\lambda}$; for class VII, $T_N = \binom{N(N-1) + \lambda - 1}{\lambda}$; for class VIII, $T_N = \binom{N^2 + \lambda - 1}{\lambda}$.

Now we proceed to counting connected graphs with N labelled vertices and λ edges. First we define a subgraph of a graph $\omega \varepsilon \Omega_N$: given any $\omega = \left\langle \omega_{(i,j)} \right\rangle_{(i,j)\varepsilon B_{i,N}}$,

$\nu = \left\langle \nu_{(i,j)} \right\rangle_{(i,j)\varepsilon C \subset B_{i,N}}$ is a subgraph of ω if $\nu_{(i,j)} = \omega_{(i,j)}$ for all $(i,j)\ \varepsilon C$.

A recursion relation for $T_{N\lambda}$ can be obtained through the following considerations: In a graph with λ edges, belonging to Ω_{N+1}, the vertex labelled N+1 is connected to some number,b, of other vertices (possibly b=0), forming a connected subgraph with b+1 labelled vertices and some number $\mu \leq \lambda$ of edges. The remaining part of the graph has N-b vertices and $\lambda-\mu$ edges. The b vertices connected to N+1 can be chosen in $\binom{N}{b}$ ways. For each of these choices there are $C_{b+1,\mu}$ possible connected subgraphs and $T_{N-b,\lambda-\mu}$ possible remaining parts. If we sum over all of these possible graphs by summing over b and μ, we obtain $T_{N+1,\lambda}$.

$$T_{N+1,\lambda} = \sum_{b=0}^{N} \sum_{\mu=0}^{\lambda} \binom{N}{b} C_{b+1,\mu} \, T_{N-b,\lambda-\mu} \, . \qquad (3.3)$$

We must specify here, that $T_{0,\lambda} = \begin{cases} 1 & \text{if } \lambda=0 \\ 0 & \text{otherwise} \end{cases}$.

We cannot allow $T_{00} = 0$ because in the above sum we would lose the case when vertex N+1 is connected to N other vertices by λ lines.

A compact derivation of a generating function for $C_{N,\lambda}$, can now be obtained using the generating functions, which we define below, and formal power series manipulations. (See chapter II).

Let us introduce the generating functions

$$C_N(y) = \sum_{\lambda} C_{N,\lambda}\, y^{\lambda} \quad , \tag{3.4}$$

and

$$T_N(y) = \sum_{\lambda} T_{N,\lambda}\, y^{\lambda} \quad . \tag{3.5}$$

Several comments can be made regarding $C_{N,\lambda}$ and $T_{N,\lambda}$. First, for a given N, $T_{N,\lambda} = 0$ if $\lambda > \overline{\lambda}_N$ for graphs of classes I to IV. We note that $C_{N,\lambda} \leq T_{N,\lambda}$. Further, $C_{N,\lambda} = 0$ if $\lambda < N-1$, for all eight classes of graphs. Also, for classes I to IV, $C_N(y)$ and $T_N(y)$ are finite sums and hence converge for all y. For classes V to VIII we can apply the ratio test to the series $T_N(y)$ to find that $C_N(y)$ and $T_N(y)$ converge at least for $|y|<1$. For classes V to VIII, with the appropriate $\overline{\lambda}_N$ respectively,

$$\lim_{\lambda\to\infty} \left| \frac{T_{N,\lambda+1}\, y^{\lambda+1}}{T_{N,\lambda}\, y^{\lambda}} \right|$$

$$= \lim_{\lambda\to\infty} \left| \frac{(\lambda_{\overline{N}} + \lambda+1-1)!}{(\lambda+1)!\ (\overline{\lambda}_N-1)!} \ \frac{\lambda!\,(\overline{\lambda}_N-1)!}{(\overline{\lambda}_N + \lambda-1)!} \right| \ |y|$$

$$= \lim_{\lambda\to\infty} \left| \frac{(\overline{\lambda}_N + \lambda\ )}{(\lambda+1)} \right| \ |y|$$

$$= |y| \quad . \tag{3.5}$$

We can now simplify equation (3.3). Multiplying both sides by y^λ and summing over λ yields

$$\sum_{\lambda=0}^{\infty} T_{N+1,\lambda} y^\lambda = \sum_{\lambda=0}^{\infty} \sum_{b=0}^{N} \sum_{\mu=0}^{\lambda} \binom{N}{b} C_{b+1,\mu} T_{N-b,\lambda-\mu} y^\lambda . \quad (3.7)$$

This becomes, using (3.4) and (3.5)

$$T_{N+1}(y) = \sum_{b=0}^{N} \binom{N}{b} \sum_{\lambda=0}^{\infty} \left(\sum_{\mu=0}^{\lambda} C_{b+1,\mu} T_{N-b,\lambda-\mu} \right) y^\lambda ,$$

which becomes, according to the definition of a product of two power series

$$= \sum_{b=0}^{N} \binom{N}{b} \sum_{\lambda=0}^{\infty} C_{b+1,\lambda} y^\lambda \sum_{\lambda=0}^{\infty} T_{N-b,\lambda} y^\lambda$$

$$= \sum_{b=0}^{N} \binom{N}{b} C_{b+1}(y) \; T_{N-b}(y) \; . \quad (3.8)$$

Equation (3.8) relates $C_{N+1}(y)$ (the term b=N) to $C_1(y)$,.. ..$C_N(y)$ and the known $T_\nu(y)$. We can, however, proceed to solve (3.8) for $C_{N+1}(y)$ explicitly in terms of $T_\nu(y)$.

We will deviate from Gilbert's paper here, and solve (3.8) for $C_{N+1}(y)$ by using formal power series. Gilbert uses umbral calculus to solve (3.8), a method which is just as concise as the following but probably less familiar. In appendix A we will give the basic definitions of umbral calculus and then show

Gilbert's solution of (3.8).

By multiplying both sides of (3.8) by $\frac{x^N}{N!}$ and summing over N we obtain

$$\sum_{N=0}^{\infty} T_{N+1}(y)\frac{x^N}{N!} = \sum_{N=0}^{\infty}\sum_{b=0}^{N}\binom{N}{b} C_{b+1}(y)\ T_{N-b}(y)\frac{x^N}{N!} . \quad (3.9)$$

Equation (3.9) is an equation in formal power series (see chapter II). For convenience, we drop the y dependence in the notation, writing just T_N rather than $T_N(y)$ for example. We can rewrite (3.9) as

$$\sum_{N=0}^{\infty} T_{N+1} \frac{x^N}{N!} = \sum_{N=0}^{\infty} N!\,(\sum_{b=0}^{N} \frac{1}{b!\ \overline{N-b}!} C_{b+1}T_{N-b})\frac{x^N}{N!} , \quad (3.10)$$

which, using definition (2.6), is the same as

$$\sum_{N=0}^{\infty} T_{N+1} \frac{x^N}{N!} = (\sum_{N=0}^{\infty} C_{N+1} \frac{x^N}{N!})\cdot(\sum_{N=0}^{\infty} T_N \frac{x^N}{N!}) . \quad (3.11)$$

Let T and C be the formal power series:

$$T = \sum_{N=1}^{\infty} T_N \frac{x^N}{N!} ,$$

$$C = \sum_{N=0}^{\infty} C_N \frac{x^N}{N!} . \quad (3.12)$$

(Note that since $T_O(y) = 1$, $1+T = \sum_{N=O}^{\infty} T_N \frac{x^N}{N!}$.).

We can find the derivatives of T and C with respect to x, using definition (2.10):

$$\frac{d\,T}{dx} = \sum_{N=1}^{\infty} NT_N \frac{x^{N-1}}{N!} = \sum_{N=1}^{\infty} T_N \frac{x^{N-1}}{(N-1)!}$$

$$= \sum_{N=0}^{\infty} T_{N+1} \frac{x^N}{N!} \quad .$$

and

$$\frac{d\,C}{dx} = \sum_{N=0}^{\infty} NC_N \frac{x^{N-1}}{N!} = \sum_{N=1}^{\infty} C_N \frac{x^{N-1}}{(N-1)!}$$

$$= \sum_{N=0}^{\infty} C_{N+1} \frac{x^N}{N!} \quad . \tag{3.13}$$

We can now rewrite (3.11) as

$$\frac{d\,T}{dx} = \frac{d\,C}{dx}(T+1) \qquad \text{or}$$

$$\frac{1}{T+1}\frac{d\,T}{dx} = \frac{d\,C}{dx} \quad . \tag{3.14}$$

Proceeding to solve for C, we integrate both sides of (3.14) from 0 to x.

$$\int_0^x \frac{d\,T}{dx}\frac{1}{T+1}\,dx = \int_0^x \frac{d\,C}{dx}\,dx \quad ,$$

using definition (2.16) and proposition (2.19) we perform the integration to get

$$\log(T+1) = C \quad . \tag{3.15}$$

From (3.12) we can write (3.15) as

$$\log\left(\sum_{N=1}^{\infty} T_N \frac{x^N}{N!} + 1\right) = \sum_{N=0}^{\infty} C_N \frac{x^N}{N!} \quad .$$

Since $T_0(y) = 1$ we could write this as

$$\log\left(\sum_{N=0}^{\infty} T_N \frac{x^N}{N!}\right) = \sum_{N=0}^{\infty} C_N \frac{x^N}{N!} \quad . \tag{3.16}$$

In summary, we state the following theorem:

<u>Theorem 3.17</u> Let $T_N(y)$ and $C_N(y)$ be as in (3.14). Then $C_N(y)$ is N! times the coefficient of x^N in the formal power series expansion for $\log \sum_{N=0}^{\infty} T_N(y) \frac{x^N}{N!}$, where $T_0(y) = 1$ by convention. Also, (from (3.11)), $C_N(y)$ is (N-1)! times the coefficient of x^{N+1} in the formal power series expansion for the quotient

$$\frac{\sum_{N=0}^{\infty} T_{N+1}(y) \frac{x^N}{N!}}{\sum_{N=0}^{\infty} T_N(y) \frac{x^N}{N!}} \quad .$$

Recall that in paragraph five of this chapter we have already determined $T_{N\lambda}$ for the eight classes of graphs mentioned. We can now use the expressions for $T_{N\lambda}$ in the definition of $T_N(y)$, (3.5), and obtain $T_N(y)$ for each class of graphs. For example, for class I graphs,

$$T_N(y) = \sum_{\lambda=0}^{\infty} T_{N\lambda}\, y^{\lambda}$$

$$= \sum_{\lambda=0}^{\overline{\lambda}_N} \binom{\overline{\lambda}N}{\lambda} y^{\lambda}$$

$$= \sum_{\lambda=0}^{N(N-1)/2} \binom{N(N-1)/2}{\lambda} y^{\lambda}$$

$$= (1+y)^{N(N-1)/2} \quad .$$

Determining $T_N(y)$ for classes II, III, and IV is similar. For class V graphs,

$$T_N(y) = \sum_{\lambda=0}^{\infty} \binom{N(N-1)/2 + \lambda - 1}{\lambda} y^{\lambda}$$

$$= \sum_{\lambda=0}^{\infty} \frac{(N(N-1)/2)_{\lambda-1}}{\lambda!} y^{\lambda} \quad ,$$

(where $(a)_n = a\cdot(a+1)\cdot(a+2)\cdots(a+n)$),

$$= (1-y)^{-N(N-1)/2} \quad .$$

We can determine $T_N(y)$ similarly for classes VI, VII, and VIII. The results can be summarized as:

$$T_N(y) = (1 + \alpha y)^{\alpha\beta(N)}$$

$$\text{where } \alpha = \begin{cases} -1 & \text{classes V to VIII} \\ 1 & \text{classes I to IV} \end{cases}$$

$$\beta = \begin{cases} N(N-1)/2 & \text{classes I,V} \\ N(N+1)/2 & \text{classes II,VI} \\ N(N-1) & \text{classes III,VII} \\ N^2 & \text{classes IV,VIII} \end{cases} \quad . \tag{3.18}$$

Equation (3.18) can be used in theorem (3.17) and we have then determined $C_N(y)$ for each of the eight classes of graphs. If we wish to obtain $C_{N\lambda}$'s then, we use (3.4), and substitute $C_N(y)$ and $T_N(y)$ as sums $\sum_{\lambda=0}^{\infty} C_{N\lambda}\, y^\lambda$ and $\sum_{\lambda=0}^{\infty} T_{N\lambda}\, y^\lambda$ into equation (3.16) and by equating coefficients of like powers of x and y we obtain the $C_{N\lambda}$'s.

We shall now summarize the results in Gilbert's later paper entitled "Random Graphs"[2]. In this paper Gilbert obtained a generating function for $P_N(\mathcal{C})$, a recursion relation for $P_N(\mathcal{C})$, and also examined the asymptotic behaviour of $P_N(\mathcal{C})$ as $N\to\infty$.

From the previous results on $C_{N\lambda}$ we can easily obtain an explicit expression for $P_N(\mathscr{C})$ in terms of a generating function. Recall equation (3.2) which was

$$P_N(\mathscr{C}) = \sum_{\lambda=0}^{\bar{\lambda}_N} C_{N\lambda}\, p^{\lambda}\, q^{\bar{\lambda}_N - \lambda} \quad .$$

We will let $y = p/q$ and then we can rewrite the above as

$$P_N(\mathscr{C}) = \sum_{\lambda=0}^{\bar{\lambda}_N} C_{N\lambda}\, y^{\lambda}\, q^{\bar{\lambda}_N} \quad ,$$

or, using (3.4)

$$P_N(\mathscr{C}) = C_N(y)\, q^{\bar{\lambda}_N} \quad .$$

(Where we have used $\sum_{\lambda=0}^{\bar{\lambda}_N} C_{N\lambda}\, y^{\lambda} = \sum_{\lambda=0}^{\infty} C_{N\lambda}\, y^{\lambda}$ because $C_{N\lambda} = 0$ if $\lambda > \bar{\lambda}_N$.) We now divide each side by $q^{\bar{\lambda}_N}$, multiply by $\frac{x^N}{N!}$, and sum over N, to obtain

$$\sum_{N=0}^{\infty} q^{-\bar{\lambda}_N}\, P_N(\mathscr{C})\, \frac{x^N}{N!} = \sum_{N=0}^{\infty} C_N(y)\, \frac{x^N}{N!} \quad .$$

Using equation (3.16) we can write

$$\sum_{N=0}^{\infty} q^{-\bar{\lambda}_N}\, P_N(\mathscr{C})\, \frac{x^N}{N!} = \log \sum_{N=0}^{\infty} T_N(y)\, \frac{x^N}{N!} \quad . \qquad (3.19)$$

For each class of graphs I through IV (and corresponding $P_N(\mathscr{C})$) we only need to substitute the appropriate $T_N(y)$ into (3.19). We can further rewrite (3.19) as

$$\sum_{N=0}^{\infty} q^{-\overline{\lambda}_N} P_N(\mathscr{C}) \frac{x^N}{N!} = \log\left(1 + \sum_{N=1}^{\infty} (1+y)^{\overline{\lambda}_N} \frac{x^N}{N!}\right)$$

$$= \log\left(1 + \sum_{N=1}^{\infty} (1+p/q)^{\overline{\lambda}_N} \frac{x^N}{N!}\right)$$

$$= \log\left(1 + \sum_{N=1}^{\infty} q^{-\overline{\lambda}_N} \frac{x^N}{N!}\right) \quad . \qquad (3.19)'$$

Where we have used $\overline{\lambda}_0 = 0$ and $p+q = 1$. Recall that $\overline{\lambda}_N$ is given by equations (1.6), (1.7), (1.8), or (1.9) for graphs of class I, II, III, or IV respectively.

As N increases, the difficulty of finding $P_N(\mathscr{C})$ from (3.19)' increases rapidly.

We can also find a recursion relation for $P_N(\mathscr{C})$.

The vertex labelled 1 must be connected (recall that connected means not necessarily by a single edge) to some number 0,1, ... or N-1, of the other vertices with probability 1. For class I and class II graphs, the probability that vertex 1 is connected to a specified k-1 other vertices is the probability of the simultaneous and independent events: 1) k vertices are connected, which has probability $P_k(\mathscr{C})$, 2) each of these k vertices does not have an edge joining to any of the N-k other vertices. There are (N-k)k independent events involved here and each of them has probability q. The probability that they all occur simultaneously is $q^{(N-k)k}$. So, 1) and 2) give us, $P_k(\mathscr{C})\ q^{(N-k)k}$ is the probability that a given k-1 points are connected to 1. Further, there are $\binom{N-1}{k-1}$ ways of choosing k-1 points to have connected to 1. So, the probability that 1 is connected to exactly k-1 other points is just $\binom{N-1}{k-1}\ P_k(\mathscr{C})\ q^{k(N-k)}$ for class I,II graphs. A similar analysis gives for class III, and class IV graphs the probability that 1 is connected to exactly k-1 other vertices is $\binom{N-1}{k-1}\ P_k(\mathscr{C})\ q^{2k(N-k)}$. Summing over k we get

$$1 - P_N(\mathscr{C}) = \sum_{k=1}^{N-1} \binom{N-1}{k-1} P_k(\mathscr{C})\ q^{ck(N-k)}$$

$$\text{where } c = \begin{cases} 1 & \text{class I,II graphs,} \\ 2 & \text{class III,IV graphs} \end{cases} \quad . \qquad (3.20)$$

The same consideration for obtaining a recursion relation applies to another probability, possibly of interest, that is, the probability R_N, that two specific vertices i and j are connected. The probability that vertex i is connected to exactly k-1 of N-2, other vertices, none of them vertex j, is just $\binom{N-2}{k-1}\ P_k(\mathscr{C})\ q^{ck(N-k)}$, and if we sum from k-1 = 0 to k-1 = N-2, we sum all the ways that vertex i may be not connected to vertex j. So,

$$1-R_N = \sum_{k-1=0}^{N-2} \binom{N-2}{k-1} P_k(\mathscr{C})\, q^{ck(N-k)} \,,$$

where c is as in (3.20) . (3.21)

We can rewrite the sum as

$$1-R_N = \sum_{k=1}^{N-1} \binom{N-2}{k-1} P_k(\mathscr{C})\, q^{ck(N-k)} \quad .$$

The equations (3.20) and (3.21) solve the problem of connectedness for a random graph of class I to IV, chosen according to the probability function P_N. Theoretically $P_N(\mathscr{C})$ can be calculated for any N. Now it is very interesting to examine what happens to $P_N(\mathscr{C})$ as N increases without bound.

As N increases the number of paths by which two given points may be connected increases. It would, therefore, not be surprising if R_N approaches one as N increases without bound. That $P_N(\mathscr{C})$ also goes to one, which is shown in the following, is more surprising since increasing N increases the number of vertices which must be connected and the minimum number of edges which must appear.

An upper bound on $1 - P_N(\mathscr{C})$ and on $1 - R_N$ is found by noting that $P_k(\mathscr{C}) \leq 1$ for all k, so from (3.20) and (3.21),

$$1 - P_N(\mathscr{C}) \leq \sum_{k=1}^{N-1} \binom{N-1}{k-1} q^{ck(N-k)} \quad ; \tag{3.22}$$

$$1 - R_N \leq \sum_{k=1}^{N-1} \binom{N-2}{k-1} q^{ck(N-k)} \quad . \tag{3.23}$$

We shall use the inequality

$$k(N-k) \geq \frac{(N-2)k + N}{2} \qquad \text{for } 1 \leq k \leq N/2 \quad , \tag{3.24}$$

which can be verified by noting that: The only zeroes of $k(N-k) - [\frac{(N-2)k + N}{2}]$ in the interval $1 \leq k \leq N/2$ occur at the endpoints, therefore the function cannot change sign in the interval. It is easy to check that the sign of this function is positive in the given interval. (For example, when $k = 2$, $2(N-2) - [\frac{(N-2)2 + N}{2}] \geq 0$.)

We shall also use

$$k(n-k) \geq \frac{(N-2)(N-k) + N}{2} \qquad \text{for } N/2 \leq k \leq N-1, \tag{3.25}$$

which can be proved from (3.24) by just interchanging k and N-k in (3.24).

Using (3.24) and (3.25) we can write

$$q^{ck(N-k)} \le q^{cN/2} \{q^{c(N-2)k/2} + q^{c(N-2)(N-k)/2}\}$$

for $1 \le k \le N-1$.

Substituting this bound on $q^{ck(N-k)}$ into (3.22), we obtain

$$1 - P_N(\mathcal{E}) \le \sum_{k=1}^{N-1} \binom{N-1}{k-1} q^{cN/2} q^{c(N-2)k/2}$$

$$+ \sum_{k=1}^{N-1} \binom{N-1}{k-1} q^{cN/2} q^{c(N-2)(N-k)/2} ,$$

which can be written as

$$1 - P_N(\mathcal{E}) \le q^{cN/2} \sum_{k=1}^{N-1} \binom{N-1}{k-1} q^{c(N-2)k/2}$$

$$+ q^{cN/2} \sum_{k=1}^{N-1} \binom{N-1}{k-1} q^{c(N-2)(N-k)/2} . \tag{3.26}$$

The first sum on the right hand side of (3.26), we can rearrange as follows,

$$q^{cN/2} \sum_{k=1}^{N-1} \binom{N-1}{k-1} q^{c(N-2)k/2}$$

$$= q^{cN/2} \left(\left[\sum_{k-1=0}^{N-1} \binom{N-1}{k-1} q^{c(N-2)(k-1)/2} \, q^{c(N-2)/2} \right] \right.$$

$$\left. - \binom{N-1}{N-1} \; q^{c(N-2)N/2} \right)$$

$$= q^{cN/2} \left((1 + q^{c(N-2)/2})^{N-1} \, q^{c(N-2)/2} - q^{c(N-2)N/2} \right)$$

$$= q^{cN/2} \, q^{c(N-2)/2} \left((1 + q^{c(N-2)/2})^{N-1} - q^{c(N-2)(N-1)/2} \right).$$

The second sum on the right hand side of (3.26), we can rearrange as follows,

$$q^{cN/2} \sum_{k=1}^{N-1} \binom{N-1}{k-1} \; q^{c(N-2)(N-1-(k-1))/2}$$

$$= q^{cN/2} \left(\sum_{k-1=0}^{N-1} \binom{N-1}{k-1} q^{c(N-2)(N-1-(k-1))/2} \right.$$

$$\left. - \binom{N-1}{N-1} q^{c(N-2)/2 \cdot 0} \right)$$

$$= q^{cN/2} \left((1 + q^{c(N-2)/2})^{N-1} - 1 \right) .$$

We can now rewrite (3.26) using the above. So,

$$1 - P_N(\not{C}) \le q^{cN/2}((1 + q^{c(N-2)/2})^{N-1} - 1)$$

$$+$$

$$+ q^{cN/2} q^{c(N-2)/2}((1 + q^{c(N-2)/2})^{N-1}$$

$$- q^{c(N-2)(N-1)/2}). \qquad (3.27)$$

Using the same approximations (3.24) and (3.25) in (3.23), and performing a similar rearrangement of sums and summation eventually yields

$$1 - R_N \leq 2q^{cN/2} q^{c(N-2)/2}(1 + q^{c(N-2)/2})^{N-2} \quad . \qquad (3.28)$$

We will now find lower bounds on $(1 - R_N)$ and $(1 - P_N(\mathscr{E}))$. Let E_i denote the event that vertex i is connected to no other point. Let E_iE_j denote the event that vertex is connected to no other point and vertex j is connected to no other point, and so on for $E_iE_jE_k$ etcetera. In general, suppose we have M events E_i and we want to know the probability that at least one of these events occurs (any number from 1 to M may occur simultaneously and still satisfy this requirement), in symbols, we want to find $P_N(\bigcup_{i=1}^{M} E_i)$. This problem is solved in most texts on probability and is not difficult; we refer to a text by W. Feller[12], chapter IV to obtain

$$P_N (\bigcup_{i=1}^{M} E_i) = \sum_{i=1}^{M} P_N(E_i) - \sum_{i<j}^{M} P_N(E_iE_j)$$

$$+ \sum_{i<j<k}^{M} P_N(E_iE_jE_k) - \ldots .$$

$$\pm \sum_{\substack{i<j\ldots<l \\ M \text{ of these}}}^{M} P_N(E_iE_j\ldots E_l) \quad . \tag{3.29}$$

It is also shown in the text just referred to, that

$$\sum_{i=1}^{M} P_N(E_i) - \sum_{i<j} P_N(E_iE_j) \leq P_N(\bigcup_{i=1}^{M} E_i) \quad . \tag{3.30}$$

To obtain a lower bound on $1 - R_N$ we observe that, the probability that two vertices i and j are not connected is bounded below by the probability of any event wherein i and j are not connected. One such event, is the event that at least one of i and j is connected to <u>no</u> other vertex. Let E_i be the event that vertex i is connected to no other vertex. The probability $P_N(E_i)$ is easy to determine. In order for i to be connected by some path to another vertex, it must be directly connected by one edge to at least one vertex. So, in order to have i not connected to any other vertex, it is both necessary and sufficient that none of the c(N-1) possible edges, that would directly connect i by one edge to some other vertex, should appear. The probability of this event is the product of the probabilities of the c(N-1) simultaneous independent events that a possible direct connecting edge does not appear. So,

$$P_N(E_i) = q^{c(N-1)} \quad . \tag{3.31}$$

And similarly, of course,

$$P_N(E_iE_j) = q^{c(N-1)}\ q^{c(N-2)}$$

$$= q^{c(2N-3)} \quad . \qquad (3.31)'$$

Using (3.29), M=2, we obtain the probability that at least one of i and j is connected to no other vertex which is

$$\sum_{i=1}^{2} q^{c(N-1)} - q^{c(2N-3)} \quad .$$

And so, (writing $q^{c(2N-3)} = q^{c(N-1)}\ q^{c(N-2)}$).

$$(2 - q^{c(N-2)})q^{c(N-1)} \leq 1 - R_N \quad . \qquad (3.32)$$

A lower bound for $1 - P_N(\mathscr{C})$ is the probability of the event T, that at least one of the vertices 1,2,...,N is connected to no other vertex. This time, instead of (3.29) we use the approximation (3.30) to obtain

$$\sum_{i=1}^{N} P_N(E_i) - \sum_{i<j}^{N} P_N(E_iE_j) \leq 1 - P_N(\mathscr{C}) \quad ,$$

and using (3.31) and (3.31)' we obtain

$$\sum_{i=1}^{N} q^{c(N-1)} - \sum_{i<j} q^{c(2N-3)}$$

$$= Nq^{c(N-1)} \left(1 - \frac{N-1}{2} q^{c(N-2)}\right)$$

$$\leq 1 - P_N(\not{C}) \quad . \tag{3.33}$$

For large N we can show that the upper and lower bounds on $1 - R_N$ and $1 - P_N(\not{C})$ both become very close to zero. Let us examine (3.27) and (3.33) together, we may write

$$Nq^{c(N-1)}\left(1 - \frac{N-1}{2} q^{c(N-2)}\right) \leq 1 - P_N(\not{C})$$

$$\leq q^{cN/2}\left((1 + q^{c(N-2)/2})^{N-1} - 1) \right)$$

$$+ q^{cN/2} q^{c(n-2)/2} \left((1 + q^{c(N-2)/2})^{N-1}\right.$$

$$\left. - q^{c(N-2)(N-1)/2} \right) \quad . \tag{3.34}$$

Suppose A is a finite positive integer and $0<q<1$. It is easy to verify, using L'Hospital's Rule, that $\lim_{N\to\infty} M^A q^M = 0$. Most of the terms in (3.34) are obviously dominated by quantities of the form $M^A q^M$, where $M(N)\to\infty$ as $N\to\infty$. The terms involving $(1 + q^{c(N-2)/2})^{N-1}$ need closer inspection. However, if we expand $\ln(1 + q^{c(N-2)/2})^{N-1}$ we obtain:

$$\ln(1 + q^{c(N-2)/2})^{N-1}$$

$$= (N-1)\ [q^{c(n-2)/2} - q^{(c(N-2)/2)^2} + q^{(c(N-2)/2)^3} \ldots].$$

For large N the first term in the expansion is clearly dominant and, as N increases this term is approaching zero. So, $(1 + q^{c(N-2)/2})^{N-1}$ approaches 1 as N becomes very large, how quickly it does so will depend on the value of q. We can now see that the terms involving $(1 + q^{c(N-2)/2})^{N-1}$ are also dominated by quantities of the form $M^A q^M$ with $M(N)\to\infty$ as $N\to\infty$.

By letting N become large enough we can make either side of the inequality (3.34) as small as we wish (greater than zero), and therefore, of course we can make both sides of the inequality agree as closely as we wish. It follows that we can (using the left hand side of the inequality (3.34)), make the statement, valid for large N:

$$P_N(\mathcal{C}) = 1 - Nq^{c(N-1)} + O(N^2 q^{c3N/2}) \quad .$$

where $O(f(x)) = y(x)$ means that the ratio $y(x)/f(x)$ remains bounded as $x\to\infty$.

(3.35)

Similarly we can find that

$$R_N = 1 - 2q^{c(N-1)} + O(Nq^{3N/2}) \quad . \tag{3.36}$$

Gilbert has checked these approximations against calculations of $P_N(\mathscr{C})$ and R_N up to N=6 for certain q values, and concludes that $Nq^{c(N-1)}$ and $2q^{c(N-1)}$ will represent $1 - P_N(\mathscr{C})$ and $1 - R_N$ respectively, to within three percent accuracy when $q \leq 0.3$ and $N \geq 6$. For the same accuracy of approximation, larger values of q will require larger values of N. Equations (3.35) and (3.36), using $\lim_{M\to\infty} M^A q^M = 0$, give us the conclusion that as N increases without bound, both $P_N(\mathscr{C})$ and R_N go to one.

Chapter IV

In this chapter we shall examine bipartite graphs of type M,N. In sections 4.2 and 4.3 we examine $P_{M,N}(\mathcal{C})$, where $\mathcal{C}$ is the set of all connected graphs belonging to $\Omega_{M,N}$ and the probability space $(\Omega_{M,N}, \mathcal{F}_{o M,N}, P_{M,N})$ is the one we discussed in chapter I, with $P_{M,N}$ not time dependent.

§ 4.1

We will use $T_{M,N,\lambda}$ to denote the number of bipartite graphs of type M,N and having λ edges. $T_{M,0,\lambda} = \begin{cases} 1 & \text{if } \lambda=0 \\ 0 & \text{if } \lambda>0 \end{cases}$.
Notice, that we have adopted the convention $T_{0,0,0} = 1$. We will use $C_{M,N,\lambda}$ to denote the number of connected bipartite graphs of type M,N having λ edges. Of course, $C_{M,N,\lambda} \leq T_{M,N,\lambda}$ so $C_{M,0,\lambda} = 0$ if $\lambda>0$; further, we know $C_{M,0,0} = 0$ if $M>1$, and we choose the convention $C_{1,0,0} = 1$ and $C_{0,0,0} = 0$. We may note here that $C_{M,N,\lambda}$ must equal zero if $\lambda<M+N-1$, (this is easy to check). Finally, we note that $T_{M,N,\lambda} = 0$ if $\lambda>MN$.

Consider a graph belonging to $\Omega_{M,N+1}$ and having λ edges. The vertex labelled M+N+1 is connected (not necessarily by a single edge) to some number of other vertices (possibly 0). Let us consider that M+N+1 is connected to b other vertices belonging to the same part, (that is, chosen from M+1, ...

M+N), and is connected to a other vertices belonging to the other part (that is, labelled between 1 and M), and there are some number μ of edges in this connected subgraph. The remaining subgraph not connected to M+N+1, has M-a+N-b vertices and $\lambda-\mu$ edges. There are $\binom{M}{a}\binom{N}{b}$ ways of choosing the a+b vertices of the connected subgraph involving M+N+1, and $C_{a,b+1,\mu}$ different connected subgraphs with these vertices, and there are $T_{M-a,N-b,\lambda-\mu}$ distinct choices of the remaining subgraph for each case. For each a, b and μ then, there are

$$\binom{M}{a}\binom{N}{b} \; C_{a,b+1,\mu} \; T_{M-a,N-b,\lambda-\mu} \tag{4.1}$$

possible graphs with M+N+1 vertices and λ edges. We sum over a, b, and μ to get

$$T_{M,N+1,\lambda} = \sum_{a=0}^{M} \sum_{b=0}^{N} \sum_{\mu=0}^{\lambda} \binom{M}{a}\binom{N}{b} C_{a,b+1,\mu} T_{M-a,N-b,\lambda-\mu} \; , \tag{4.2}$$

or

$$T_{M,N+1,\lambda} = \sum_{a=0}^{M} \binom{M}{a} \sum_{b=0}^{N} \binom{N}{b} \sum_{\mu=0}^{\lambda} C_{a,b+1,\mu} T_{M-a,N-b,\lambda-\mu} \; , \tag{4.2'}$$

To simplify (4.2) we introduce the generating functions $C_{M,N}(y)$ and $T_{M,N}(y)$ defined by

$$C_{M,N}(y) = \sum_{\lambda=0}^{\infty} C_{M,N,\lambda}\, y^{\lambda}$$

and

$$T_{M,N}(y) = \sum_{\lambda=0}^{\infty} T_{M,N,\lambda}\, y^{\lambda} \quad . \tag{4.3}$$

We note that both the sums of (4.3) are actually finite and so converge for any y. We now multiply both sides of (4.2)' by y^{λ} and sum over λ to get

$$\sum_{\lambda=0}^{\infty} T_{M,N+1,\lambda}\, y^{\lambda}$$

$$= \sum_{a=0}^{M} \binom{M}{a} \sum_{b=0}^{N} \binom{N}{b} \sum_{\lambda=0}^{\infty} \sum_{\mu=0}^{\lambda} C_{a,b+1,\mu}\, T_{M-a,N-b,\lambda-\mu}\, y^{\lambda} \, . \tag{4.4}$$

The right hand side of (4.4) is the product of two power series. We can rewrite (4.4) as

$$\sum_{\lambda=0}^{\infty} T_{M,N+1,\lambda}\, y^{\lambda}$$

$$= \sum_{a=0}^{M} \binom{M}{a} \sum_{b=0}^{N} \binom{N}{b} \sum_{\lambda=0}^{\infty} C_{a,b+1,\lambda}\, y^{\lambda} \sum_{\lambda=0}^{\infty} T_{M-a,N-b,\lambda} y^{\lambda} \quad . \tag{4.5}$$

We use the definitions (4.3) to further simplify the above, and obtain

$$T_{M,N+1}(y) = \sum_{a=0}^{M} \binom{M}{a} \sum_{b=0}^{N} \binom{N}{b} C_{a,b+1}(y)\ T_{M-a,N-b}(y) \quad . \tag{4.6}$$

We can now solve (4.6) for $C_{M,N}(y)$ explicitly in terms of $T_{M,N}(y)$. We will make our calculations in the algebra of double formal power series. Umbral calculus could be used as an alternative method of solution (see appendix).

We will let C and T be the double formal power series:

$$C = \sum_{M=0}^{\infty} \sum_{N=0}^{\infty} C_{M,N}(y) \frac{x^M}{M!} \frac{z^N}{N!} \quad ,$$

and

$$T = \sum_{M=0}^{\infty} \sum_{N=1}^{\infty} T_{M,N}(y) \frac{x^M}{M!} \frac{z^N}{N!} + \sum_{M=1}^{\infty} T_{M,0}(y) \frac{x^M}{M!} \quad . \tag{4.7}$$

We note that T as defined by (4.7) is just

$$\sum_{M=0}^{\infty} \sum_{N=0}^{\infty} T_{M,N}(y) \frac{x^M}{M!} \frac{z^N}{N!} \quad ,$$

with the term $T_{0,0}(y)$ removed.

We observe that

$$\frac{\partial C}{\partial z} = \sum_{M=0}^{\infty} \sum_{N=0}^{\infty} N\, C_{M,N}(y)\, \frac{x^M}{M!} \frac{z^{N-1}}{N!}$$

$$= \sum_{M=0}^{\infty} \sum_{N=0}^{\infty} C_{M,N+1}(y)\, \frac{x^M}{M!} \frac{z^N}{N!} \quad ,$$

and

$$\frac{\partial T}{\partial z} = \sum_{M=0}^{\infty} \sum_{N=1}^{\infty} N\, T_{M,N}(y)\, \frac{x^M}{M!} \frac{z^{N-1}}{N!} \quad + \quad 0$$

$$= \sum_{M=0}^{\infty} \sum_{N=0}^{\infty} T_{M,N+1}(y)\, \frac{x^M}{M!} \frac{z^N}{N!} \quad . \tag{4.8}$$

Now we multiply equation (4.6) by $\frac{x^M}{M!} \frac{z^N}{N!}$ and sum over M and M to obtain

$$\sum_{M=0}^{\infty} \sum_{N=0}^{\infty} T_{M,N+1}(y)\, \frac{x^M}{M!} \frac{z^N}{N!}$$

$$= \sum_{M=0}^{\infty} \sum_{N=0}^{\infty} \sum_{a=0}^{M} \frac{M\,!}{a!\,(M-a)\,!} \frac{1}{M!} \sum_{b=0}^{N} \frac{N!}{b!\,(N-b)\,!} \frac{1}{N!} \quad \cdot$$

$$C_{a,b+1}(y)\; T_{M-a,N-b}(y)\; x^M\, z^N. \tag{4.9}$$

The right hand side of (4.9) is the product of two series and we can rewrite (4.9) as

$$\sum_{M=0}^{\infty} \sum_{N=0}^{\infty} T_{M,N+1}(y) \frac{x^M}{M!} \frac{z^N}{N!}$$

$$= \left(\sum_{M=0}^{\infty} \sum_{N=0}^{\infty} C_{M,N+1}(y) \frac{x^M}{M!} \frac{z^N}{N!} \right) .$$

$$\left(\sum_{M=0}^{\infty} \sum_{N=0}^{\infty} T_{M,N}(y) \frac{x^M}{M!} \frac{z^N}{N!} \right) . \qquad (4.9)'$$

We now use the definitions of (4.7) and equations (4.8) to rewrite (4.9)' as

$$\frac{\partial T}{\partial z} = \frac{\partial C}{\partial z} (1+T) \quad ,$$

or

$$\frac{\frac{\partial T}{\partial z}}{(1+T)} = \frac{\partial C}{\partial z} \quad . \qquad (4.10)$$

If we integrate both sides of (4.10) from 0 to z we obtain

$$\int_0^z \frac{\frac{\partial T}{\partial z}}{(T+1)} \partial z = \int_0^z \frac{\partial C}{\partial z} \partial z \quad .$$

We can now perform this integration (see chapter II) to obtain

$$\log_e(1+T) = C \quad . \tag{4.11}$$

Equation (4.11) can be written in double formal power series form. Using (4.7) we obtain

$$\log\left(\sum_{M=0}^{\infty} \sum_{N=0}^{\infty} T_{M,N}(y) \frac{x^M}{M!} \frac{z^N}{N!} \right)$$

$$= \sum_{M=0}^{\infty} \sum_{N=0}^{\infty} C_{M,N}(y) \frac{x^M}{M!} \frac{z^N}{N!} \quad . \tag{4.12}$$

We could also obtain the following directly from equation (4.9)' :

$$\frac{\sum_{M=0}^{\infty} \sum_{N=0}^{\infty} T_{M,N+1}(y) \frac{x^M}{M!} \frac{z^N}{N!}}{\sum_{M=0}^{\infty} \sum_{N=0}^{\infty} T_{M,N}(y) \frac{x^M}{M!} \frac{z^N}{N!}} = \sum_{M=0}^{\infty} \sum_{N=0}^{\infty} C_{M,N+1}(y) \frac{x^M}{M!} \frac{z^N}{N!} \quad . \tag{4.13}$$

The conclusions regarding $C_{M,N}(y)$ are summarized in the theorem below.

<u>Theorem 4.14</u> From equation (4.12) we conclude that $C_{M,N}(y)$ is $M!\ N!$ times the coefficient of $x^M z^N$ in

the double formal power series expansion,

$\log\left(\sum_{M=0}^{\infty} \sum_{N=0}^{\infty} T_{M,N}(y) \frac{x^M}{M!} \frac{z^N}{N!} \right)$. From equation (4.13) we

conclude that also, $C_{M,N}(y)$ equals M! (N-1)! times the

coeffient of $x^M z^N$ in the double formal power series for

$$\frac{\sum_{M=0}^{\infty} \sum_{N=0}^{\infty} T_{M,N+1}(y) \frac{x^M}{M!} \frac{z^N}{N!}}{\sum_{M=0}^{\infty} \sum_{N=0}^{\infty} T_{M,N}(y) \frac{x^M}{M!} \frac{z^N}{N!}} .$$

Of course, if we wish to obtain $C_{M,N,\lambda}$, we can write $C_{M,N}(y)$ using (4.3) and equate coefficients of like powers of y in theorem (4.14).

It is useful to determine $T_{M,N}(y)$ so that it can be used in all these equations. From the defining equation (4.3),

$$T_{M,N}(y) = \sum_{\lambda=0}^{\infty} T_{M,N,\lambda}\, y^\lambda$$

$$= \sum_{\lambda=0}^{MN} \binom{MN}{\lambda} y^\lambda$$

$$= (1+y)^{MN} . \qquad (4.15)$$

This is because the maximum number of edges allowed is MN, as we have previously noted.

§ 4.2

Using the results of section 4.1, we can now obtain a generating function for $P_{M,N}(\mathscr{C})$. From equations (4.12) and (4.3) we obtain

$$\sum_{M=0}^{\infty} \sum_{N=0}^{\infty} \sum_{\lambda=0}^{\infty} C_{M,N,\lambda}\, y^{\lambda} \frac{x^M}{M!} \frac{z^N}{N!}$$

$$= \log \left(\sum_{M=0}^{\infty} \sum_{N=0}^{\infty} (1 + y)^{MN} \frac{x^M}{M!} \frac{z^N}{N!} \right) . \qquad (4.16)$$

As before, with A as in equation (3.1), we have

$$P_{M,N}(\mathscr{C}) = \sum_{\lambda=0}^{\bar{\lambda}_{MN}} P_{M,N} \, (C \cap A_{\lambda})$$

$$= \sum_{\lambda=0}^{\bar{\lambda}_{MN}} C_{M,N,\lambda}\, p^{\lambda} q^{MN-\lambda} . \qquad (4.17)$$

We shall let $y = p/q$, and then we multiply both sides of (4.17) by $\frac{x^M}{M!} \frac{z^N}{N!}$ and sum over M and N to obtain

$$\sum_{M=0}^{\infty} \sum_{N=0}^{\infty} P_{M,N}(\mathscr{C}) \frac{q^{-MN}}{N!\; M!} x^M z^N$$

$$= \sum_{M=0}^{\infty} \sum_{N=0}^{\infty} \sum_{\lambda=0}^{\infty} C_{M,N,\lambda} \left(\frac{p}{q}\right)^{\lambda} \frac{x^M}{M!} \frac{z^N}{N!} . \qquad (4.18)$$

Using (4.16) we obtain from the above

$$\sum_{M=0}^{\infty} \sum_{N=0}^{\infty} P_{M,N}(\mathcal{C})\, q^{-MN} \frac{x^M}{M!} \frac{z^N}{N!}$$

$$= \log \left(\sum_{M=0}^{\infty} \sum_{N=0}^{\infty} q^{-MN} \frac{x^M}{M!} \frac{z^N}{N!} \right) . \tag{4.19}$$

So, $P_{M,N}(\mathcal{C})$ is q^{MN} N! M! times the coefficient of $x^M z^N$ in the power series expansion of $\log \left(\sum_{M=0}^{\infty} \sum_{N=0}^{\infty} q^{-MN} \frac{x^M}{M!} \frac{z^N}{N!} \right)$.

We will now find a recursion relation for calculating $P_{M,N}(\mathcal{C})$ and then use it in examining the behaviour of $P_{M,N}(\mathcal{C})$ as M and/or N become unbounded.

To obtain a recursion relation we consider the graph in the following manner. The vertex labelled M+N must be connected to some number of other vertices, in fact it may be connected to 0,1,...or up to M+N-1 other vertices. The probabilities of each of the cases (M+N is connected to i other vertices) must add to give 1. The probability that M+N is connected to some number b-1 of other vertices labelled between M+1 and M+N-1, and to some number a of vertices labelled between 1 and M is just: the number of ways the b-1 vertices may be chosen, $\binom{N-1}{b-1}$, multiplied by the number of ways the a vertices may be chosen, $\binom{M}{a}$, multiplied by the probability that these vertices form a connected graph, $P_{a,b}(\mathcal{C})$, multiplied by the probability that M+N is not connected to any of the other vertices. The event, M+N is not connected to any other vertices, is the intersection of (M-a)b + (N-b)a independent events each having probability q. These events are: Each of the b vertices labelled between M+1 and M+N and connected to M+N is not joined by an edge to any of the M-a vertices labelled between 1 and M and which are not in the connected subgraph containing M+N. (This is a total of (M-a)b such events). Each of the a vertices labelled between 1 and M and connected to M+N is not joined by an edge to any of the N-b vertices labelled between M+1 and M+N and not in the connected subgraph containing M+N. (This is a total of (N-b)a such events). So, the probability that M+N is not connected to any of the other vertices is $q^{(M-a)b}\, q^{(N-b)a}$. Therefore, we find the probability that M+N is connected to b-1 other vertices labelled between M+1 and M+N, and to a other vertices labelled between 1 and M, is

$$\binom{M}{a}\binom{N-1}{b-1} P_{a,b}(\mathcal{C})\; q^{(M-a)b}\; q^{(N-b)a} \quad .$$

If we sum the probabilities over all possible a and b we must obtain 1 as the sum. And so,

$$1 = \sum_{b-1=0}^{N-1} \binom{N-1}{b-1} \sum_{a=0}^{M} \binom{M}{a} P_{a,b}(\mathscr{C})\, q^{(N-b)a} q^{(M-a)b} \quad . \tag{4.20}$$

We can rearrange (4.20) and write

$$P_{M,N}(\mathscr{C}) = 1 - \sum_{b-1=0}^{N-1} \sum_{a=0}^{M-1} \binom{M}{a} \binom{N-1}{b-1} P_{a,b}(\mathscr{C}) \cdot q^{(N-b)a}\, q^{(M-a)b}$$

$$- \sum_{b-1=0}^{N-2} \binom{N-1}{b-1} P_{M,b}(\mathscr{C})\; q^{(N-b)M} \quad . \tag{4.21}$$

In (4.21) we have taken $P_{M,N}(\mathscr{C})$ out of the sum and set it equal to 1 minus the remainder of the sum.

An explanation of how the recursion formula (4.21) must be used is in order. The problem which may arise is the following. Suppose we know $P_{a,b}(\mathscr{C})$ for all $a \leq M-1$, and $b \leq N$, and we try to determine $P_{M,N}(\mathscr{C})$ from equation (4.21). Then in (4.21), if $M \geq N$ there are N-1 quantities $P_{M,b}(\mathscr{C})$ which are not yet known, or if $M < N$ there are N-M quantities $P_{M,b}(\mathscr{C})$ which are not yet known.

The way in which (4.21) must be used can be made clear by writing the $P_{M,N}(\mathscr{C})$'s in a matrix array and giving a simple example.

<u>Example</u> Scheme for calculating $P_{4,3}(\mathcal{C})$. Here P_{MN} stands for $P_{M,N}(\mathcal{C})$, and the arrows indicate what to calculate next. Note that $P_{MN} = P_{NM}$ and that $P_{MO} = \begin{cases} 1 & M=1, \\ 0 & \text{otherwise} \end{cases}$, and $P_{M1} = p^M$ for all M and calculations are simplified accordingly.

$$\begin{pmatrix} P_{00} & P_{01} & P_{02} & P_{03} & P_{04} & P_{05} & P_{06} & \cdot & \cdots \\ P_{10} & P_{11} & P_{12} & P_{13} & P_{14} & P_{15} & P_{16} & \cdot & \cdots \\ P_{20} & P_{21} & P_{22} & P_{23} & P_{24} & P_{25} & P_{26} & \cdot & \cdots \\ P_{30} & P_{31} & P_{32} & P_{33} & P_{34} & P_{35} & P_{36} & \cdot & \cdots \\ P_{40} & P_{41} & P_{42} & P_{43} & P_{44} & P_{45} & P_{46} & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

We notice that the more closed formulas we know for the entries of a row (or column), say P_{Mx}, x fixed M varying, the shorter the calculations are, but these closed formulas become complicated to obtain as x increases. Since these closed formulas are of interest we include here a closed formula for $P_{2,N}(\mathcal{C})$ and a closed formula for $P_{3,N}(\mathcal{C})$, valid for all N. We have obtained these from elementary considerations.

Proposition 4.22

$P_{2,N}(\mathcal{C}) = p^N \ [(p + 2q)^N - (2q)^N]$.

Proof: Let a graph $\omega\varepsilon\Omega_{2,N}$ be labelled 1,2,...N+2.

We can state a necessary condition for ω to be connected.

Consider the vertices labelled j, $j \ \varepsilon\{3,...N+2\}$.

For $\omega\varepsilon\Omega_{2,N}$:

Let r be the number of vertices j, such that

$$\omega_{(1,j)} = \omega_{(2,j)} = 1 \quad .$$

Let x be the number of vertices j, such that

$$\omega(1,j) = 1 \quad \text{and} \quad \omega_{(2,j)} = 0 \quad .$$

Let y be the number of vertices j, such that

$$\omega_{(2,j)} = 1 \quad \text{and} \quad \omega_{(1,j)} = 0 \quad .$$

A necessary condition for ω to be connected (that is, for $\omega\varepsilon\mathcal{C}$) is:

$y = N-r-x$, <u>and</u> $r \neq 0$.

From equation (4.17) we know that

$$P_{2,N}(\mathcal{C}) = \sum_{\lambda=N+1}^{2N} C_{2,N,\lambda} \ p^{\lambda} \ q^{2N-\lambda} \quad ,$$

where we have used $\bar{\lambda}_N = 2N$, and $C_{2,N,\lambda} = 0$ if $\lambda<N+1$.

However, for connectedness, we must have $\lambda = N+r$, so instead of summing over λ, we could sum over r from 1 to N, that is, we can write

$$P_{2,N}(\mathcal{C}) = \sum_{r=1}^{N} C_{2,N,N+r}\, p^{N+r}\, q^{2N-(N+r)} \quad .$$

And, we can determine $C_{2,N,N+r}$ from the following elementary considerations. For each r, there are $\binom{N}{r}$ possible choices of the r vertices that will be joined to both vertices 1 and 2, and there are $\sum_{x=0}^{N-r} \binom{N-r}{x}$ possible choices of the x vertices joined to vertex 1 only. The graph is then completely determined (because for connectedness $y = N-r-x$). So, we may write

$$P_{2,N}(\mathcal{C}) = \sum_{r=1}^{N} \binom{N}{r} \sum_{x=0}^{N-r} \binom{N-r}{x}\, p^{N+r}\, q^{2N-(N+r)} \quad . \tag{4.23}$$

To the sum in (4.23) we can add and subtract the term $r = 0$, and then (4.23) becomes

$$P_{2,N}(\mathcal{C}) = \sum_{r=0}^{N} \binom{N}{r} \sum_{x=0}^{N-r} \binom{N-r}{x}\, p^{N+r}\, q^{2N-(N+r)} - \binom{N}{0}\, p^N\, q^N \sum_{x=0}^{N} \binom{N}{x} \, ,$$

which simplifies to

$$P_{2,N}(\mathcal{C}) = \sum_{r=0}^{N} \binom{N}{r} p^N\, p^r\, q^{N-r}\, 2^{N-r} - 2^N\, p^N\, q^N$$

$$= p^N \, [(p + 2q)^N - (2q)^N] \quad .$$

Proposition 4.24

$$P_{3,N}(\mathcal{C}) = p^N q^N \, [3^N - 3(p + 3q)^N + 2(3q)^N]$$

$$+ p^N \, [(p^2 + 3pq + 3q^2)^N - 3(pq + 3q^2)^N + 2(3q^2)^N]$$

Proof: Let a graph $\omega \varepsilon \Omega_{3,N}$ be labelled 1,2,3,...N+3. We can state a necessary condition for ω to be connected. Consider the vertices labelled j, $j \varepsilon \{4,\ldots N+3\}$. For $\omega \varepsilon \Omega_{3,N}$:

Let ξ be the number of vertices j, such that

$$\omega_{(1,j)} + \omega_{(2,j)} + \omega_{(3,j)} \geq 2 \quad .$$

Let r be the number of vertices j, such that

$$\omega_{(1,j)} = \omega_{(2,j)} = \omega_{(3,j)} = 1 \quad .$$

Let s be the number of vertices j, such that

$$\omega_{(1,j)} = \omega_{(2,j)} = 1 \quad , \text{ and } \omega_{(3,j)} = 0 \quad .$$

Let t be the number of vertices j, such that

$$\omega_{(2,j)} = \omega_{(3,j)} = 1 \quad , \text{ and } \omega_{(1,j)} = 0 \quad .$$

Let u be the number of vertices j, such that

$$\omega_{(3,j)} = \omega_{(1,j)} = 1 \quad , \text{ and } \omega_{(2,j)} = 0 \quad .$$

Let x be the number of vertices j, such that

$$\omega_{(1,j)} = 1 \quad , \text{ and } \omega_{(2,j)} = \omega_{(3,j)} = 0 \quad .$$

Let y be the number of vertices j, such that

$$\omega_{(2,j)} = 1 \quad , \text{ and } \omega_{(1,j)} = \omega_{(3,j)} = 0 \quad .$$

Let z be the number of vertices j, such that

$$\omega_{(3,j)} = 1 \quad , \text{ and } \omega_{(1,j)} = \omega_{(2,j)} = 0 \quad .$$

A necessary condition for ω to be connected (that is, for $\omega \varepsilon \mathcal{C}$) is:

$u = \xi\text{-}s\text{-}t$, and $z = N\text{-} \xi\text{-}x\text{-}y$, and either

1) $r \neq 0$, or

2) $r = 0$ and $\xi \geq 2$ and s and $t \geq 1$, or

3) $r = 0$ and $\xi \geq 2$ and s and $\xi\text{-}s\text{-}t \geq 1$, or

4) $r = 0$ and $\xi \geq 2$ and t and $\xi\text{-}s\text{-}t \geq 1$.

From equation (4.17) we know that

$$P_{3,N}(\mathcal{C}) = \sum_{\lambda = N+ \ +r}^{3N} C_{3,N,\lambda}\, p^{\lambda}\, q^{3N-\lambda} \ . \qquad (4.25)$$

We can rewrite the sum in (4.25) in terms of the numbers ξ,r,s,... and the necessary condition for connectedness instead of as a sum over λ. First we split the sum into two parts corresponding to the two ways, $r = 0$ or $r \neq 0$, to achieve connectedness. We shall treat these sums separately then add the results at the conclusion.

Case 1, $r \neq 0$

The contribution to the sum on the right hand side of (4.25) may be rewritten as a sum in ξ, r, x, y, s and t. Since $r < \xi$, we must have $\xi \geq 1$. For each possible ξ, (from 1 to N) there are $\binom{N}{\xi}$ possible ways to choose the ξ vertices from 4,5,...N+3. For each choice of the ξ vertices there are $\sum_{x=0}^{N-\xi} \binom{N-\xi}{x}$ ways to choose the x vertices joined to 1 only, and for each of these ways there are $\sum_{y=0}^{N-\xi-x} \binom{N-\xi-x}{y}$ ways to choose the y vertices joined to 2 only. For each choice of ξ, x and y there are $\sum_{r=1}^{\xi} \binom{\xi}{r}$ possible choices of r vertices joined to 1,2 and 3, and for each of these there are $\sum_{s=0}^{\xi-r} \binom{\xi-r}{s}$ ways to choose the s vertices joined to 1 and 2 and for each of these choices of s vertices there are $\sum_{t=0}^{\xi-r-s} \binom{\xi-r-s}{t}$ ways to choose the t vertices joined to 2 and 3. This completely determines the graph, and so the total sum gives us the number of such connected graphs. The probability of each such graph, for each ξ, is $p^{N+\xi+r} q^{3N-(N+\xi+r)}$. The contribution to (4.25) from case 1, $r \neq 0$ is

$$\sum_{\xi=1}^{N}\left(\binom{N}{\xi}\sum_{x=0}^{N-\xi}\binom{N-\xi}{x}\sum_{y=0}^{N-\xi-x}\binom{N-\xi-x}{y}\sum_{r=1}^{\xi}\binom{\xi}{r}\cdot\right.$$

$$\left.\sum_{s=0}^{\xi-r}\binom{\xi-r}{s}\sum_{t=0}^{\xi-r-s}\binom{\xi-r-s}{t}\;p^{N+\xi+r}\;q^{3N-(N+\xi+r)}\right) \quad . \tag{4.26}$$

We can simplify (4.26) by performing the sums (from the inside out) beginning with the sum over t.

$$\sum_{\xi=1}^{N}\left(\binom{N}{\xi}\sum_{x=0}^{N-\xi}\binom{N-\xi}{x}\sum_{y=0}^{N-\xi-x}\binom{N-\xi-x}{y}\sum_{r=1}^{\xi}\binom{\xi}{r}\sum_{s=0}^{\xi-r}\binom{\xi-r}{s}\cdot\right.$$

$$\left.2^{\xi-r-s}\;p^{N+\xi+r}\;q^{3N-(N+\xi+r)}\right)$$

$$=\sum_{\xi=1}^{N}\left(\binom{N}{\xi}\sum_{x=0}^{N-\xi}\binom{N-\xi}{x}\sum_{y=0}^{N-\xi-x}\binom{N-\xi-x}{y}\sum_{r=1}^{\xi}\binom{\xi}{r}\,3^{\xi-r}\cdot\right.$$

$$\left.p^{N+\xi+r}\;q^{3N-(N+\xi+r)}\right)$$

$$=\sum_{\xi=1}^{N}\left(\binom{N}{\xi}\sum_{x=0}^{N-\xi}\binom{N-\xi}{x}\sum_{y=0}^{N-\xi-x}\binom{N-\xi-x}{y}\,q^{2(N-\xi)}p^{N+\xi}\cdot\right.$$

$$\left.\sum_{r=1}^{\xi}\binom{\xi}{r}\,p^{r}(3q)^{\xi-r}\right),$$

We add and subtract the term r = 0 to the inside sum and obtain

$$\sum_{\xi=1}^{N}\left(\binom{N}{\xi}\sum_{x=0}^{N-\xi}\binom{N-\xi}{x}\sum_{y=0}^{N-\xi-x}\binom{N-\xi-x}{y}q^{2(N-\xi)}\ p^{N+\xi}\ \cdot\right.$$

$$\left.[(p+3q)^{\xi}-3q^{\xi}]\right) .$$

Performing the sums over y and over x we get

$$\sum_{\xi=1}^{N}\binom{N}{\xi}q^{2(N-\xi)}p^{N+\xi}\ 3^{N-\xi}\ [(p+3q)^{\xi}-3q^{\xi}]$$

$$= p^{N}\sum_{\xi=1}^{N}\binom{N}{\xi}(3q^2)^{N-\xi}\ p^{\xi}\ [(p+3q)^{\xi}-3q^{\xi}] \quad .$$

We add and subtract the term $\xi = 0$ and perform the sum to get the contribution from case 1, $r \neq 0$ is

$$p^{N}\ [(p(p+3q)+3q^2)^{N}-3(pq+3q^2)^{N}$$

$$-(3q^2)^{N}+3(3q^2)^{N}]$$

$$= p^{N}\ [(p(p+3q)+3q^2)^{N}-3(pq+3q^2)^{N}+2(3q^2)^{N}] \tag{4.27}$$

Case 2, $r = 0$

We wish to write the contribution to the sum in (4.25) as a sum in ξ,x,y,s and t. By considerations similar to those in the case $r \neq 0$ we obtain the contribution

$$\sum_{\xi=2}^{N} \left(\binom{N}{\xi} \sum_{x=0}^{N-\xi} \binom{N-\xi}{x} \sum_{y=0}^{N-\xi-x} \binom{N-\xi-x}{y} p^{N+\xi} q^{2N-\xi} \cdot \right.$$

$$\left. \sum_{s=1}^{\xi-1} \binom{\xi}{s} \sum_{t=0}^{\xi-s} \binom{\xi-s}{t} + \sum_{t=1}^{\xi-1} \binom{\xi}{t} \right) . \qquad (4.28)$$

The last sums in s and t take the above form to ensure that one of the necessary conditions for connectedness 2), 3), or 4) is always satisfied, that is, we can allow $t = 0$ if $s \geq 1$ and $\xi-s-t \geq 1$, we can allow $s = 0$ if $t \geq 1$ and $\xi-s-t \geq 1$, and we can allow $\xi-s-t = 0$ if $s \geq 1$ and $t \geq 1$. We can simplify (4.28). First we sum over t, to the sum in t we add and subtract the terms $t=0$ and $t= \xi$,

$$\sum_{\xi=2}^{N} \left(\binom{N}{\xi} \sum_{x=0}^{N-\xi} \binom{N-\xi}{x} \sum_{y=0}^{N-\xi-x} \binom{N-\xi-x}{y} p^{N+\xi} q^{2N-\xi} \cdot \right.$$

$$\left. \sum_{s=1}^{\xi-1} \binom{\xi}{s} \sum_{t=0}^{\xi-s} \binom{\xi-s}{t} + \sum_{t=0}^{\xi} \binom{\xi}{t} - \binom{\xi}{\xi} - \binom{\xi}{0} \right)$$

$$= \sum_{\xi=2}^{N} \left(\binom{N}{\xi} \sum_{x=0}^{N-\xi} \binom{N-\xi}{x} \sum_{y=0}^{N-\xi-x} \binom{N-\xi-x}{y} p^{N+\xi} q^{2N-\xi} \cdot \right.$$

$$\left. \sum_{s=1}^{\xi-1} \binom{\xi}{s} 2^{\xi-s} + 2^{\xi} - 2 \right) .$$

To the sum in s we add and subtract the term $s = 0$ and perform the sum:

$$\sum_{\xi=2}^{N}\left(\binom{N}{\xi}\sum_{x=0}^{N-\xi}\binom{N-\xi}{x}\sum_{y=0}^{N-\xi-x}\binom{N-\xi-x}{y} p^{N+\xi} q^{2N-\xi} \cdot \right.$$

$$\left.\sum_{s=0}^{\xi}\binom{\xi}{s} 2^{\xi-s} - \binom{\xi}{\xi} - \binom{\xi}{0} 2^{\xi} + 2^{\xi} - 2\right)$$

$$= \sum_{\xi=2}^{N}\left(\binom{N}{\xi}\sum_{x=0}^{N-\xi}\binom{N-\xi}{x}\sum_{y=0}^{N-\xi-x}\binom{N-\xi-x}{y} p^{N+\xi} q^{2N-\xi} (3^{\xi} - 3)\right) .$$

Now we simplify the sums in y and x

$$\sum_{\xi=2}^{N}\left(\binom{N}{\xi} p^{N+\xi} q^{2N-\xi} (3^{\xi} - 3) \sum_{x=0}^{N-\xi}\binom{N-\xi}{x}\sum_{y=0}^{N-\xi-x}\binom{N-\xi-x}{y}\right)$$

$$= \sum_{\xi=2}^{N}\left(\binom{N}{\xi} p^{N+\xi} q^{2N-\xi} (3^{\xi} - 3)\sum_{x=0}^{N-\xi}\binom{N-\xi}{x} 2^{N-\xi-x}\right)$$

$$= \sum_{\xi=2}^{N}\binom{N}{\xi} p^{N+\xi} q^{2N-\xi} (3^{\xi} - 3)\, 3^{N-\xi}$$

$$= p^{N} q^{N} \sum_{\xi=2}^{N}\binom{N}{\xi} (3p)^{\xi}(3q)^{N-\xi} - 3\sum_{\xi=2}^{N}\binom{N}{\xi} p^{\xi} (3q)^{N-\xi} .$$

To each of the sums over ξ we add and subtract the terms $\xi = 0$, and $\xi = 1$, obtaining

$$p^N q^N \left(\sum_{\xi=0}^{N} \binom{N}{\xi} (3p)^{\xi} (3q)^{N-\xi} - \binom{N}{1} (3p)^1 (3q)^{N-1} \right.$$

$$- \binom{N}{0} (3q)^N - 3\sum_{\xi=0}^{N} \binom{N}{\xi} p^{\xi} (3q)^{N-\xi} + 3 \binom{N}{1} p (3q)^{N-1}$$

$$\left. + 3 \binom{N}{0} (3q)^N \right)$$

$$= p^N q^N \left((3p + 3q)^N - N3p(3q)^{N-1} - (3q)^N - 3(p + 3q)^N \right.$$

$$\left. + 3Np(3q)^{N-1} + 3(3q)^N \right)$$

$$= p^N q^N ((3p + 3q)^N - 3(p + 3q)^N + 2(3q)^N)$$

$$= p^N q^N (3^N - 3(p + 3q)^N + 2(3q)^N) \quad . \qquad (4.29)$$

We now add the contributions from case 1, (4.27), and from case 2, (4.29), to obtain finally,

$$P_{3,N}(\zeta) = p^N q^N (3^N - 3(p + 3q)^N + 2(3q)^N)$$

$$+ p^N ((p^2 + 3pq + 3q^2)^N - 3(pq + 3q^2)^N$$

$$+ 2(3q^2)^N) \quad .$$

Comments

Multipartite graphs of 3, 4 or more parts can be defined similarly to bipartite graphs. For a similar probability function it is still possible to obtain, by similar methods, an explicit generating function for the number of connected graphs having a given number of vertices and a given number of edges. Also, a recursion relation for the probability of connectedness $P_{...}(\mathcal{G})$ can be obtained by similar considerations. The extension is quite easy in concept but the details are, of course, increasingly lengthy as the number of parts in the multipartite graph is increased.

§ 4.3 Asymptotic behaviour of $P_{M,N}(\mathscr{C})$

Some qualitative information can be obtained directly from (4.21). Equation (4.21) was

$$1 - P_{M,N}(\mathscr{C}) = \sum_{b-1=0}^{N-1} \sum_{a=0}^{M-1} \binom{M}{a}\binom{N-1}{b-1} P_{a,b}(\mathscr{C})\; q^{(M-a)b} q^{(N-b)a}$$

$$+ \sum_{b=1}^{N-1} \binom{N-1}{b-1} P_{M,b}(\mathscr{C})\; q^{(N-b)M} \quad .$$

Since $P_{M,N}(\mathscr{C})$ is a probability, we know that $0 \leq 1-P_{M,N}(\mathscr{C}) \leq 1$, and so, certainly each term appearing on the right hand side of (4.21) must be less that one, because they are each positive. Let us suppose first, that N is held fixed. For each M, a term b=N and M-1=a appears on the right hand side of (4.21). This term is

$$M\, P_{M-1,N}\; q^{N}\, 1 \quad .$$

This term must be less than one. We have supposed N fixed and so q^N is a constant. Therefore as M increases, eventually becoming larger than $1/q^N$, $P_{M-1,N}$ must, at least for $M > 1/q^N$, be less than one and must also be decreasing as M increases. Therefore, when N is held fixed and M allowed to increase there is a finite number k, such that for $M > k$, we know that $P_{M,N} < 1$ and is decreasing as M increases. In fact, as $M\to\infty$

$P_{M,N} \to 0$ if N is fixed. Equation (4.21) is symmetric in M and N so a similar conclusion holds if M is fixed and N allowed to increase.

To examine the behaviour of $P_{M,N}(\mathcal{C})$ when both M and N increase without bound we obtain an upper bound on $1 - P_{M,N}(\mathcal{C})$ and examine its behaviour. For all M and N, $P_{M,N}(\mathcal{C}) \leq 1$ and so we can obtain an upper bound on $1 - P_{M,N}(\mathcal{C})$ just by substituting one for $P_{a,b}(\mathcal{C})$ in equation (4.21). This is not necessarily a least upper bound. This gives us

$$0 \leq 1 - P_{M,N}(\mathcal{C}) \leq \sum_{b-1=0}^{N-1} \sum_{a=0}^{M-1} \binom{M}{a} \binom{N-1}{b-1} q^{(M-a)b} q^{(N-b)a}$$

$$+ \sum_{b=1}^{N-1} \binom{N-1}{b-1} q^{(N-b)M} \quad . \qquad (4.30)$$

To the last term on the right of (4.30) we add and subtract the term b=N as follows:

$$\sum_{b=1}^{N-1} \binom{N-1}{b-1} q^{(N-b)M} = \sum_{b-1=0}^{N-1} \binom{N-1}{b-1} q^{(N-b)M} - q^{M(N-N)}$$

$$= (1 + q^{M})^{N-1} - 1 \quad .$$

To the first term on the right of (4.30) we add and subtract the term a=M, as follows:

$$\sum_{b-1=0}^{N-1} \sum_{a=0}^{M} \binom{M}{a} \binom{N-1}{b-1} q^{(M-a)b} q^{(N-b)a} - \sum_{b-1=0}^{N-1} \binom{N-1}{b-1} q^{(N-b)M}$$

$$= \sum_{b-1=0}^{N-1} \binom{N-1}{b-1} [(\sum_{a=0}^{M} \binom{M}{a} q^{(M-a)b} q^{(N-b)a}) - q^{(N-b)M}]$$

$$= \sum_{b-1=0}^{N-1} \binom{N-1}{b-1} [(q^b + q^{N-b})^M - q^{(N-b)M}] \quad .$$

So, equation (4.30) can now be rewritten

$$1 - P_{M,N}(\mathcal{C}) \leq \sum_{b-1=0}^{N-1} \binom{N-1}{b-1} \left((q^b + q^{N-b})^M - q^{(N-b)M}\right) + (1 + q^M)^{N-1} - 1 \quad . \tag{4.31}$$

We can emphasize the symmetry in M and N by writing the term b=N in the summation, separately. Then we have

$$1 - P_{M,N}(\mathcal{C}) \leq \sum_{b-1=0}^{N-2} \binom{N-1}{b-1} \left((q^b + q^{N-b})^M - q^{(N-b)M}\right) + (1 + q^N)^M - 1 + (1 + q^M)^{N-1} - 1 \quad . \tag{4.32}$$

We note that if N is held fixed and M increased without bound the term $(1 + q^N)^M$ in (4.32) approaches infinity so the right hand side of (4.32) is certainly not a least upper bound on $1 - P_{M,N}(\mathcal{C})$ in this case, and (4.32) gives us no additional information on the behaviour of $P_{M,N}(\wp)$ when N is fixed and $M\to\infty$. Similarly if M is fixed and N increased without bound (4.32) yields no additional information because $(1 + q^M)^{N-1}$ approaches infinity. Further, to obtain information from (4.32) we will insist that M and N increase at approximately the same rate, that is, $M-N \leq K$ for some constant K , as M and N are allowed to increase.

Now we suppose that $M-N \leq K$, K a constant, as M and N are increased without bound and we examine the terms appearing in (4.32).

To see how $(1 + q^N)^M$ behaves, we examine

$$\lim_{M,N\to\infty} \ln((1 + q^N)^M) = \lim_{M,N\to\infty} M \ln(1 + q^N) \quad .$$

Since we have supposed $M-N \leq K$, then

$$\begin{aligned}\lim_{M,N\to\infty} M \ln(1 + q^N) &\leq \lim_{M,N\to\infty} N \ln(1 + q^N) \\ &\quad + \lim_{M,N\to\infty} K \ln(1 + q^N) \\ &\leq \lim_{M,N\to\infty} N \ln(1 + q^N) \quad .\end{aligned}$$

(It is easy to see that $\lim_{M,N\to\infty} K \ln(1 + q^N) = 0$.)

Now, we apply L'Hospital's rule to $\lim_{M,N\to\infty} N \ln(1 + q^N)$ as follows:

$$\lim_{M,N\to\infty} M \ln(1 + q^N) \leq \lim_{M,N\to\infty} N \ln(1 + q^N)$$

$$= \lim_{M,N\to\infty} \frac{\ln(1 + q^N)}{1/N}$$

$$= \lim_{M,N\to\infty} \frac{-\ln q}{1 + q^N} \cdot \lim_{M,N\to\infty} q^N N^2 \quad .$$

We can apply L'Hospital's rule twice to $\lim_{M,N\to\infty} q^N N^2$ to find that this limit is 0. So, since the other limit is bounded

$$\lim_{M,N\to\infty} M \ln(1 + q^N) = 0 \quad .$$

This implies that

$$\lim_{M,N\to\infty} (1 + q^N)^M = 1 \quad .$$

Similarly,

$$\lim_{M,N\to\infty} (1 + q^M)^{N-1} = 1 \quad .$$

So,

$$\lim_{M,N\to\infty} 1 - P_{M,N}(\mathscr{C}) \leq \lim_{M,N\to\infty} \left[\sum_{b-1=0}^{N-2} \binom{N-1}{b-1} (q^b + q^{N-b})^M - \sum_{b-1=0}^{N-2} \binom{N-1}{b-1} q^{(N-b)M} \right] \quad . \tag{4.33}$$

Now we examine the term $\sum_{b-1=0}^{N-2} \binom{N-1}{b-1} (q^b + q^{N-b})^M$ in (4.33).

For the range $1 \leq b \leq N-1$, the maximum of $(q^b + q^{N-b})$ occurs at $b = 1$ or $b = N-1$, and is $(q + q^{N-1})$. And $\binom{N-1}{b-1}$ has its maximum when $b-1 = [(N-1)/2]$, where $[x]$ means greatest integer $\leq x$. To simplify slightly we can assume that $(N-1)/2$ is an integer for the N's we consider. So each term in the sum is certainly less than $(q + q^{N-1})^M \binom{N-1}{(N-1)/2}$, and there are N-1 terms, so

$$\sum_{b=1}^{N-1} (q^b + q^{N-b})^M \binom{N-1}{b-1}$$

$$\leq (N-1)(q + q^{N-1})^M \binom{N-1}{(N-1)/2} . \qquad (4.34)$$

Since we are assuming N is large and increasing without bound we can use Stirling's approximation for N!. We can then say that

$$\lim_{M,N\to\infty} \sum_{b=1}^{N-1} (q^b + q^{N-b})^M \begin{matrix} N-1 \\ b-1 \end{matrix}$$

$$\lesssim \lim_{M,N\to\infty} (N-1)(q + q^{N-1})^M \, 2^{N-1} \sqrt{\frac{2}{\pi(N-1)}}$$

$$= \lim_{M,N\to\infty} \sqrt{N-1}\, (2q(1 + q^{N-2}))^M \, 2^{N-M-1} \sqrt{2/\pi} .$$

Since we are assuming that $M-N \leq K$, we know that 2^{N-M-1}

remains bounded. We also know that $\sqrt{N-1}\,(2q(1 + q^{N-2}))^M$ will approach 0 as $M,N\to\infty$ provided that $2q < 1$. Therefore, if $2q < 1$

$$\lim_{M,N\to\infty} \sum_{b=1}^{N-1} (q^b + q^{N-b})^M \binom{N-1}{b-1} = 0 \quad .$$

The sum $\sum_{b-1}^{N-2} \binom{N-1}{b-1} q^{(N-b)M}$ is always less than the sum $\sum_{b-1=0}^{N-2} \binom{N-1}{b-1} (q^b + q^{N-b})^M$. Referring to (4.33) we can now make this qualified statement: At least if $q < 1/2$, when M and N increase without bound in such a way that $M-N \le K$, $P_{M,N}(\mathcal{C})$ approaches 1.

We can also obtain a lower bound for $1 - P_{M,N}(\mathcal{C})$ in a way similar to the way we obtained a lower bound on $1 - P_N(\mathcal{C})$ in chapter III. Again let E_i be the event that vertex i is connected to no other vertex, and let E_iE_j be the event $E_i \cap E_j$. Equation (3.30) becomes for our probability function,

$$\sum_{i=1}^{M+N} P_{M,N}(E_i) - 2\sum_{i<j}^{M+N} P_{M,N}(E_iE_j)$$

$$\le P_{M,N}(\bigcup_i E_i) \le 1 - P_{M,N}(\mathcal{C}) \quad . \tag{4.35}$$

If we do not want vertex i to be connected to any other vertex then at least we can allow no direct edge to appear between the vertex labelled i and any of the other vertices. This condition is both necessary and sufficient and so it is easy to see that

$$P_{M,N}(E_i) = \begin{cases} q^M & \text{if } M+1 \leq i \leq M+N \\ q^N & \text{if } i \leq M \end{cases} .$$

And

$$P_{M,N}(E_iE_j) = \begin{cases} q^{2N} & i < j \text{ and } i,j \leq M \\ q^{2M} & i < j,\ i,j \leq M+N,\ i,j \geq M+1 \\ q^{M+N} & 1 < j \text{ and } i \leq M,\ j \geq M+1, \\ & j \leq M+N . \end{cases}$$

Using the $P_{M,N}(E_i)$ and $P_{M,N}(E_iE_j)$, (4.35) becomes,

$$\sum_{i=1}^{M} P_{M,N}(E_i) + \sum_{i=M+1}^{M+N} P_{M,N}(E_i) - 2\sum_{\substack{i<j \\ i,j\leq M}} P_{M,N}(E_iE_j)$$

$$- 2\sum_{\substack{i<j \\ M+1\leq i,j\leq M+N}} P_{M,N}(E_iE_j) - 2\sum_{\substack{i<j \\ i\leq M \\ M+1\leq j\leq M+N}} P_{M,N}(E_iE_j)$$

$$\leq 1 - P_{M,N}(\mathscr{C}) .$$

Or,

$$Nq^M + Mq^N - (M(M-1)q^{2N} + N(N-1)q^{2M} + NMq^{N+M})$$

$$\leq 1 - P_{M,N}(\mathscr{C}) . \tag{4.36}$$

When M and N are allowed to approach infinity with $M-N \leq K$, the left hand side of (4.36) approaches zero. (Recall that $\lim_{N\to\infty} N^A q^N = 0$ for finite A.) Since the upper bound (4.32) and the lower bound of $1 - P_{M,N}(\mathcal{E})$ are both approaching zero (if $q < 1/2$), we can make them agree as closely as we like by taking M and N large enough. We can therefore use either bound as an approximation to $1 - P_{M,N}(\mathcal{E})$ for large enough M and N, and $q < 1/2$. So,

$$P_{M,N}(\mathcal{E}) = 1 - Nq^M - Mq^N + O(M^2 q^{2M}) \quad ,$$

when $q < 1/2$, and $M-N \leq K$, K a constant.

Chapter V

The intention of this chapter is to acquaint the reader with work which has been done specifically pertaining to the connectedness of a random graph (for various probability functions). We have already reviewed in detail the work of E.N. Gilbert. In chronological order, here, we will mention the work of P. Erdos and A. Renyi, see [3], [4], and [5], of V. E. Stepanov[6] and [7], and of E. M. Wright[8]. We are not attempting to explain any details of the work but only to indicate what kind of results have been obtained.

§ 5.1

P. Erdos and A. Renyi have written two joint papers on the subject of random graphs. The first (chronologically) of these contains the results of a separate paper by Renyi[3].

We state first the results of Renyi's paper[3]. Let N, λ and $C_{N\lambda}$ be as before, and $d = \lambda - N+1$. If a graph is connected then $d \geq 0$ and we call d the "degree of connectivity of a graph". Although explicit formulae and recursive relations were previously known for $C_{N\lambda}$, no simple explicit formula for $C_{N\lambda}$ was known. Renyi attempted to find a simple explicit formula for $C_{N\lambda}$ at least for some restricted N and λ and to determine the asymptotic behaviour of $C_{N,N+d-1}$ for d fixed and N approaching infinity. Renyi was able to find a simple explicit formula for $C_{N\lambda}$ when $\lambda = N$, or $d = 1$, in other words for $C_{N,N}$, and to determine the asymptotic behaviour of $C_{N,N}$ as $N\to\infty$. His results are:

$$C_{N,N} = \frac{N^{N-1}}{2} \sum_{k=3}^{N} \binom{N}{k} \frac{k!}{N^k} , \tag{5.1}$$

or

$$C_{N,N} = \frac{N^{N-1}}{2} \sum_{k=3}^{N} \prod_{j=1}^{k-1} (1 - j/N) . \tag{5.1'}$$

As $N\to\infty$,

$$C_{N,N} \sim \sqrt{\frac{\pi}{8}}\, N^{N-1/2} . \tag{5.2}$$

In the first joint paper[4], Erdos and Renyi consider graphs of class one. They pose the following four questions (some of these we have rephrased in our notation and in terms of our P_N of chapter I) and attempt to answer them:

1) What is the asymptotic behaviour of $C_{N\lambda}/T_{N\lambda}$? (For our probability function P_N, $C_{N\lambda} / T_{N\lambda} = P_N(\mathscr{C}/A_\lambda)$. Recall that $A_\lambda = \{\omega\varepsilon\Omega_N |\ \sum_\omega \omega_{(i,j)} = \lambda\}$.)

2) What is the asymptotic behaviour of: the number of graphs (with N vertices and λ edges) with greatest connected component having N-k vertices, divided by $T_{N\lambda}$? (k=0,1,...). Let us denote this ratio by $P(k,N,\lambda_c)$.

3) What is the asymptotic behaviour of: the number of graphs (with N vertices and λ edges) consisting of K+1 distinct connected subgraphs, divided by $T_{N\lambda}$? (k=0,1,...). Let us denote this ratio by $\Pi(k,N,\lambda)$.

4) Let G be the subset of Ω_N consisting of all connected graphs which can be made disconnected by removing one edge. What is the asymptotic behaviour of $P_N(A_\lambda/G)$? (Where $P_N(A_\lambda/G)$ is the conditional probability of A_λ given G.)

As partial answers to 1) to 4), Erdos and Renyi prove four theorems. In the following, λ_c is used to denote $[1/2N\log N + cN]$ where c is an arbitrary fixed constant, [.] denotes the greatest integer function, N is the number of vertices.

<u>Theorem 5.3</u>

$$\lim_{N\to\infty} \frac{C_{N,\lambda_c}}{T_{N,\lambda_c}} = e^{-e^{-2c}} .$$

Theorem 5.4

$$\lim_{N\to\infty} P(k,N,\lambda_c) = \frac{(e^{-2c})^k \, e^{-e^{-2c}}}{k!} \quad .$$

That is, the number of vertices outside the greatest connected component of a graph with N vertices and λ_c edges, is distributed in the limit according to Poisson's law with mean value e^{-2c}.

Theorem 5.5

$$\lim_{N\to\infty} \Pi(k,N,\lambda_c) = \frac{(e^{-2c})^k \, e^{-e^{-2c}}}{k!} \quad .$$

That is, the number of connected components diminished by one is in the limit distributed according to Poisson's law with mean value e^{-2c}.

Theorem 5.6

$$P_N(A\ /G) \sim \frac{2}{N}\, e^{-\frac{2\ell}{N} - e^{-\frac{2\ell}{N}}}$$

where $= [1/2N\log N + \ell]$ and $|\ell| = O(N)$.

Also,

$$\lim_{N\to\infty} P_N\Big(\bigcup_{\lambda<Nx+1/2N\log N} A_\lambda/G\Big) = e^{-e^{-2x}} \quad .$$

In their later paper [5], Erdos and Renyi give a much more complete exposition of the probable structure of a random graph of class I. As we mentioned in the introduction, for any structural property X (some condition on the graphs $\varepsilon\Omega_N$), we could ask, assuming again our probability function P_N, on class I graphs, what is $P_N(\chi)$? Here χ is the set of all ω in Ω_N with property X. For various structural properties X, this paper examines $P_N(\chi/A_\lambda)$ and examines the changes in $P_N(\chi/A_\lambda)$ as λ is increased. Also, if we assume that we are interested in graphs with N vertices and $\lambda(N)$ edges where $\lambda(N)$ is some function of N, then for certain restrictions on $\lambda(N)$ the paper examines $P_N(\chi/A_{\lambda(N)})$ as $N\to\infty$. This paper contains many detailed specific results.

We are mainly interested in the probability of connectedness of a graph, so we mention only the results referring to connectedness.

The paper proves that a graph of class I with N labelled vertices and $\lambda(N)$ edges ($\lambda(N)$ some function of N), has greatest connected component of size G(c)N, for $\lambda(N) \sim cN$ with $c > 1/2$, with probability tending to 1 as $N\to\infty$, where

$$G(c) = 1 - \frac{\sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} (2ce^{-2c})^k}{2c} .$$

Combining some of their other results in this paper, the information obtained on connectedness is as follows: The largest component of a graph with N vertices, and $\lambda(N)$ edges is of order logN for $\lambda(N)/N \sim c < 1/2$, is of order $N^{2/3}$ for $\lambda(N)/N \sim 1/2$, and of order N for $\lambda(N)/N \sim c > 1/2$. The size of the largest component makes a startling "double jump" as $\lambda(N)/N$ passes the value 1/2. Further, when $\frac{(N) - 1/2\ N \log N}{N} \to \infty$ the whole graph will, with probability tending to 1, be connected. ($P_N(\mathcal{C}/A_{\lambda(N)}) \to 1$ as $N \to \infty$ if $\frac{(N) - 1/2\ N \log N}{N} \to \infty$ as $N \to \infty$.)

§ 5.2

In 1969, V.E. Stepanov published two papers concerned with the probability of connectedness of a random graph. The first of the two papers is entitled "Combinatorial Algebra and Random Graphs"[6], and the second is entitled "On the Probability of Connectedness of a Random Graph"[7]. We will discuss them in chronological order.

Stepanov was the first author to consider a probability function P defined on Ω, the sample space of class I graphs, which did not assume equivalence of the vertices of the graphs. (We will say that a probability P assumes equivalence of vertices if the probability assigned to a subset of Ω of the form $\{\omega \varepsilon \Omega | \ \omega_{(i,j)} = 1\}$ does not depend on i and j.) The aim of Stepanov's first paper is to develop appropriate mathematical tools and then study $P(\mathcal{E})$ for the probability function P he defines on class I graphs (and which does not assume equivalence of vertices).

Recall that for class I graphs, $\Omega = \{0,1\}^B$ where $B = \{(i,j) | i < j, \ i,j \varepsilon I^+\}$. Stepanov defines a measure on the set of possible vertices of a graph, that is on I^+, and we denote this measure by $L(I^+)$. The measure $L(I^+)$ is defined by its values on vertices, the measure of a vertex i, is $L(\{i\}) = \ell_i > 0$ and ℓ_i is called the intensity of vertex i. The probability function on Ω is denoted $P_L(t)$ and is a function of the measure $L(I^+)$ and of time. The event, an edge appears between vertex i and vertex j, is taken to be independent of all similar events including those involving one of the vertices i and j. Further, we define

$$P_L(t)(\omega_{(i,j)}=1) = 1 - e^{-\ell_i \ell_j t} ,$$

and

$$P_L(t)(\omega_{(i,j)}=0) = e^{-\ell_i \ell_j t} .$$

The above definition together with the statement of independence completely determines the probability of all possible events.

The probability function $P_L(t)$ on Ω is similar to $\bar{P}^{*}$ on Ω

defined in chapter I, with p time dependent; the major difference is the measure $L(I^+)$ defined on the set of vertices so that p is now time dependent and a function of the vertices in question.

Stepanov's paper is divided into three sections.

The first section develops formal tools useful in the study of random graphs. These tools are essentially a modification of the method of generating functions. Stepanov defines an algebra, $\mathcal{F}$, of numerical valued functions on the set of all finite subsets of I^+. The algebra $\mathcal{F}$ is defined so that it allows a homomorphism of the algebra of power series in a formal variable with numerical coefficients, into the algebra $\mathcal{F}$. Of course, the algebra $\mathcal{F}$ is carefully designed so that probabilistic statements (which might otherwise be made in complicated power series expressions) can be made easily in $\mathcal{F}$ and desired manipulations performed easily in $\mathcal{F}$.

A finite subset A of I^+ will be called a skeleton. An ordered sequence of skeletons $(A_1,\ldots A_n)$ such that $A_i \cap A_j = \emptyset$ and $\bigcup_{i=1}^{n} A_i = A$, is an n-partition of the skeleton A. A partition $(A_1,A_2,\ldots A_n)$ for which all skeletons are non-empty is a strict n-partition. Strict n-partitions of a skeleton A differing only in the order of their component skeletons are identified as a single class and each such class is called an unordered n-partition. The collection of all n-partitions, strict n-partitions and unordered n-partitions of a skeleton A are denoted, respectively by $(A)_n$, $[A]_n$ and $\{A\}_n$.

The probability of connectedness of a graph with the set of vertices A, would normally be denoted by $P_L(t)(\mathcal{C}/A)$. For convenience we shorten the notation to just $P_L(A/t)$.

In the second section of his paper, Stepanov uses his combinatorial algebra $\mathcal{J}$ in obtaining information about $P_L(A/t)$. The main results he obtained are given below.

An explicit expression for $P_L(A/t)$ is given by

$$P_L(A/t) = e^{-tL^2(A)/2}\left(\sum_{n=1}^{\infty} (-1)^{n-1}(n-1)! \cdot \sum_{\{A\}_n} \exp(t\sum_{i=1}^{n} L^2(A_i)/2)\right) . \qquad (5.7)$$

Recursion relations for $P_L(A/t)$ can be obtained from

$$\sum_{(A)_2} L'(A_1)P_L(A_1/t)\, e^{-tL(A_1)L(A_2)} = L'(A) \quad ,$$

where L' is any additive function on I^+. (5.8)

A measure is called simple if it assumes the value 1 on exactly one vertex. If the vertex is i, we denote the corresponding simple measure by L_i. The next important general result is:

$$\frac{\partial^2 \left(e^{tL^2(A)/2} P_L(A/t)\right)}{\partial L_i \quad L'^{(i)}}$$

$$= \frac{\partial\left(e^{t\ell_i L} - 1\right)\left(e^{tL^2(A)/2} P_L(A/t)\right.}{\partial L'} * \frac{\partial\left(e^{tL^2(A)/2} P_L(A/t)\right.}{\partial L_i}$$

where L' is any additive function,

$L'^{(i)} = L' - \ell_i' L_i$,

* indicates the composition operation in $\mathcal{F}$,

the differentiation is in $\mathcal{F}$.

(5.9)

The final general result is

$$\frac{d\, P_L(A/t)}{d\, t} = {}^1\!/_2 \sum_{(A_1, A_2)} \Big(L(A_1)\, L(A_2)\, P_L(A_1/t) \cdot$$

$$P_L(A_2/t)\, e^{-tL(A_1)L(A_2)} \Big) . \qquad (5.10)$$

From these four main general results interesting special cases can be obtained. Stepanov gives quite a few details in this area.

Stepanov applies the general results to the special case, previously considered by other authors, of L(i) = 1 for each vertex i. That is, $P_L(t)$ is now $\overline{P}$ as in chapter I, with p time dependent and $p = 1 - e^{-t}$. Since the probabilities of various events now depend only on numbers of vertices and numbers of edges $P_L(A/t)$ is the same for all skeletons A of the same size and we can refer to it as $P_N(t)(\mathcal{C})$ as previously, where N is the number of vertices in A. Stepanov applies his general results to this case and obtains explicit expressions and recursion relations for $P_N(t)(\mathcal{C})$. These results agree with previously known results where any exist for comparison.

In the third section of his paper, Stepanov examines the asymptotic behaviour of the probability $P_L(A/t)$. His main result is the following theorem.

<u>Theorem 5.11</u> Let L be a measure defining the intensity of the vertices in the basic set (say I^+) and let skeleton A and time t vary in such a way that the quantity $\mu(A/t) = \sum_{i\varepsilon A} e^{-t\ell_i L(A)}$ remains bounded.

If there exists an L_o varying along with A and t such that

$$t L_o^2 \to 0 ,$$

$$\frac{1}{tL(A)L_o} \sum_{i\varepsilon A} e^{-tL(A)\ell_i/2} \to 0 ,$$

$$\sum_{i\varepsilon A''} e^{-t\ell_i L(A)} \to 0$$ uniformly for all subsets $A'' \subset I^+$ such that $L(A'') \leq L_o$,

then

$$P_L(A/t)\ e^{\mu(A/t)} \to 1 \quad .$$

Stepanov goes on to explain the implications of his theorem and also shows that it yields previously known results for special cases.

In the second paper by Stepanov[7], the probability space consists of class I graphs with $\bar{P}$ as defined in chapter I with p time dependent and $p = 1 - e^{-t}$. As we have already noted, this probability function is a restricted case of the one considered in [6], where $L(I^+)$ is defined by $L(i) = 1$ for each vertex i so that the vertices are equivalent under $\bar{P}$. The work in this paper is based on the results of his earlier paper. (Recall that $\bar{P}$ agrees with P_N for all N, where N is the number of vertices in the skeleton to be considered and that by $P_N(\mathcal{C})$ we really mean $\bar{P}\ (\mathcal{C}(N))$.)

Stepanov notes that in his previous paper when he applied his general results to the special case now being considered, he obtained the result that as $N\to\infty$ and $t = [\log N + c + o(1)]/N$ varies with N, (c a constant), then

$$\lim_{N\to\infty} P_N(t)(\mathcal{C}) = e^{-e^{-c}} \quad .$$

From this it follows that $\lim_{N\to\infty} P_N(t)(\mathcal{C}) = 1$ when t = constant, and $\lim_{N\to\infty} P_N(t)(\mathcal{C}) = 0$ when $t \to 0$ in such a way that $Nt = o(\log N)$. In this second paper, Stepanov examines the probability $P_N(t)(\mathcal{C})$ and finds an asymptotic expression for $P_N(t)(\mathcal{C})$ when $N\to\infty$ and t is small satisfying $Nt = O(1)$. He then uses the asymptotic expression to explain some particularities in the construction of a random graph with N vertices. The asymptotic expression is given by

<u>Theorem 5.12</u> When $N\to\infty$

$$P_N(t)(\mathcal{C}) = \left(1 - \frac{Nt}{e^{Nt} - 1}\right)(1 - e^{-Nt})^N \; (1 + o(1))$$

uniformly for all t such that $Nt \geq y_o > 0$.

The result pertaining to the structure of a random graph is

<u>Theorem 5.13</u> When $N\to\infty$ and $t = y/N$, $y > 1$, the random graph on N vertices contains, with probability tending to 1, one "large" subgraph the number of whose vertices has a normal distribution with mean

$$N(1 - \theta)(y\ e^{-y})/y$$

and variance

$$\frac{N\ \theta\ (y\ e^{-y})}{y\ [1 - \theta(y\ e^{-y})]^2}\left(1 - \frac{\theta\ (y\ e^{-y})}{y}\right) .$$

Where $\Theta(w)$ stands for the main branch of the inverse function of $w = z\,e^{-z}$.

§ 5.3

An investigation by E.M. Wright[8] is related to our topic. Wright has found asymptotic expressions for $C_{N\lambda}$ in terms of binomial coefficients, for the eight classes of graphs I to VIII we have mentioned. The expansions are valid as N approaches infinity if λ is always larger than N in such a way that as N approaches infinity, $\frac{\lambda - 1/2\ N \log N}{N}$ also approaches infinity. So the necessary for validity of the expansion is dependent upon N. For the range of λ for which Wright's results hold, we have $C_{N\lambda} \sim T_{N\lambda}$ and we are interested in the further terms in the asymptotic expansion of $C_{N\lambda}$ which also give the asymptotic expansion of $T_{N\lambda} - C_{N\lambda}$, the number of disconnected graphs with N labelled vertices and λ edges. (These asymptotic expansions of $C_{N\lambda}$ do not immediately give us $P_N(\mathscr{C}) = \sum_{\lambda=N-1}^{\bar{\lambda}_N} C_{N\lambda}\, p^{\lambda}\, q^{\bar{\lambda}N-\lambda}$, for the classes one to four graphs. Because of the restrictions on λ for validity of expansions, $P_N(\mathscr{C})$ cannot be written as just a sum involving asymptotic expansions.) However, $C_{N\lambda}/\ T_{N\lambda}$ is a quantity which may be of interest, (Erdos and Renyi considered this, see §5.1, also, $C_{N\lambda}\, p^{\lambda}\, q^{\bar{\lambda}_N - \lambda}$ might be of interest and the asymptotic expansion for $C_{N\lambda}$ might yield an asymptotic expansion of this.)

The proofs involved in obtaining the results are based on properties of series, some of the proofs are lengthy. We will give just enough detail here as is necessary to state the results. Wright begins his development from the equation (3.9) relating $C_N(y)$ and $T_N(y)$ found by Gilbert. Recall that $C_N(y)$ and $T_N(y)$ are as defined by (3.4) and (3.5). Equation (3.9) may be written as

$$T_N(y) = \sum_{j=1}^{N} \binom{N-1}{j-1} C_j(y)\, T_{N-j}(y) ,$$

where $N \geq 1$ and $T_0(y) = 1$.

Set $T_N = N!\, G_N$ and $C_N = N!\, g_N$ and this becomes

$$NG_N = \sum_{s=1}^{N} s\, g_s\, G_{N-s}$$

$$\text{where } N \geq 1 \text{ and } G_0 = 1 \quad . \tag{5.14}$$

By repeated use of (5.14) we find that

$$G_N = G_N(g_1, g_2, \ldots g_N) = \Sigma\, \omega g_1^{\,j_1} \cdots g_N^{\,j_N} \tag{5.15}$$

where the sum is over all partitions of N, that is, all terms for which $\sum j_i i = N$.

Every $\omega = \omega(j_1,\ldots j_N) \geq 0$ and $\omega(0,0,\ldots,1) = 1$.

Now introduce another series of polynomials α_N defined by

$$N\alpha_N = -\sum_{s=1}^{N} s\, g_s\, \alpha_{N-s} \quad , \text{ for } N \geq 1$$

$$\text{and } \alpha_0 = 1 \quad . \tag{5.16}$$

This may be obtained from (5.14) by changing the sign of every g_s and replacing G_s by α_s. Hence, as in (5.15)

$$\alpha_N = G_N(-g_1,-g_2,\ldots-g_N) \quad . \tag{5.17}$$

In what follows, C denotes a positive number, not always the same whenever it appears, but always independent of N. The notations $O(\cdot)$ and $o(\cdot)$ refer to when $N\to\infty$ and each implied constant is a C.

Two lemmas lead up to the first general result, which is

<u>Theorem 5.18</u>

$$C_N = \sum_{t=0}^{R-1} a_t(N,\lambda) + O(N! \sum_{t=0}^{R-1} \{S_t\}_\lambda) \quad ,$$

where

$$a_t(N,\lambda) = \frac{N!}{(N-t)!}\{\alpha_t\, T_{N-t}(y)\}_\lambda ,$$

($T_N(y)$ is as before)

and

$$S_t = G_t \sum_{s=R-t}^{[(N-t)/2]} G_s\, G_{N-t-s} \quad ,$$

and

$\{*\}_q$ denotes the coefficient of y^q in $*$.

There are five more theorems proved by Wright, which apply only to graphs of classes I to VIII (they are not general results). First we define a few more notations and then we will state these results.

1) Let Ω_N be one of classes I to VIII. Suppose $\omega\varepsilon\Omega_N$ has $\sum\limits_{\omega} \omega_{(i,j)} = \lambda$. Let $M(N,\lambda)$ be the number of positions (i,j) which may possibly have $\omega_{(i,j)} \neq 0$. For classes I to IV graphs, $M(N,\lambda)$ is just $\bar{\lambda}_N$. For classes V to VIII, $M(N,\lambda)$ as we have already shown in §3.1 is just

$$M(N,\lambda) = \begin{cases} N(N-1)/2 + \lambda - 1 & \text{for class V graphs,} \\ N(N+1)/2 + \lambda - 1 & \text{for class VI graphs,} \\ N(N-1) + \lambda - 1 & \text{for class VII graphs,} \\ N^2 + \lambda - 1 & \text{for class VIII graphs.} \end{cases}$$

2) $Q(M)$ refers to the binomial coefficient $\binom{M}{\lambda}$ for M any given number and λ the given number of edges.

3) $M_t = M(N,\lambda) - 2\alpha tN + 2\alpha t^2$, where α depends on the class of graph and is the same α as in (3.17), t is any positive number.

To evaluate $a_t(N,\lambda)$ in (5.18) we have the theorem

<u>Theorem 5.19</u>

$$a_t(N,\lambda) = \binom{N}{t} \sum_{r=0}^{t(t-1)/2} \gamma_t(r) Q(M_t - 2\alpha r) \tag{5.19a}$$

where the $\gamma_t(r)$ are coefficients in a polynomial

$$B_t = B_t(w) = \sum_{r=0}^{t(t-1)/2} \gamma_t(r)\ w^r \tag{5.19b}$$

defined successively by

$$B_o = 1, \text{ and } \sum_{t=0}^{s} \binom{s}{t} w^{t(s-t)} B_t(w) = 0,\ s \geq 1. \tag{5.19c}$$

In particular, if $\lambda < (1 - C)M_t$, then

$$a_t(N,\lambda) = \binom{N}{t}\binom{M_t}{\lambda}\left\{B_t\left(\frac{M_t - \lambda}{M_t}\right)^{2\alpha} + O\left(\frac{1}{N}\right)\right\} \tag{5.19d}$$

and, if $\lambda = o(M)$,

$$a_t(N,\lambda) = (-1)^t \binom{N}{t}\binom{M_t}{\lambda} \{1 + O\left(\frac{\lambda}{N}\right)\}. \tag{5.19e}$$

On the behaviour of the error term we have the following theorem.

<u>Theorem 5.20</u> If $\psi = \psi(N) \to \infty$ as $N \to \infty$, and $\lambda > N(1/2 \log N + \psi)$ then

$$N! \cdot \{S_t\}_\lambda = O\left(\binom{N}{R} Q(M_R)\right) \qquad 0 \leq t \leq R .$$

The final results are the following theorems:

<u>Theorem 5.21</u> If $(\lambda - 1/2N\log N)/N \to \infty$ as $N \to \infty$, then

$$C_{N\lambda} = \sum_{t=0}^{R-1} a_t(N,\lambda) + O(N^R Q(M_R)) .$$

<u>Theorem 5.22</u> If $N(1/2 \log N + \psi) < \lambda = o(M)$, and $\psi \to \infty$ as $N \to \infty$, then

$$C_{N\lambda} = \sum_{t=0}^{R-1} a_t(N,\lambda) + O(a_R(N,\lambda)) .$$

<u>Theorem 5.23</u> For graphs of classes I to IV only, if $M_R < \lambda \leq M$, then

$$C_{N\lambda} = \sum_{t=0}^{R-1} a_t(N,\lambda) .$$

Theorem (5.22) follows from (5.19e). Theorem (5.23) follows from theorem (5.21) if we observe that, when $\lambda > M_R$, then $Q(M_R) = 0$. So for this limited range of $\lambda > M_R$ we have an exact value for $C_{N\lambda}$.

Appendix

In random graph theory, the idea of transforming equations to another algebra for solution appears to be very useful. We have used a transformation to formal power series; Gilbert[2] used a transformation to umbral calculus; Stepanov[6] defined a new combinatorial algebra in which he writes and solves his equations.

We chose to solve equations (3.8) and (4.6) using formal power series rather than to follow Gilbert's use of umbral calculus because we felt the formal power series method was more straightforward. We feel, however, that the use of umbral calculus is an interesting method and it is conceivable that in the study of graphs, equations may arise which could be handled most conveniently using umbral calculus. The use made of umbral calculus by Gilbert did not really exploit the umbral calculus but was more a modification of formal power series methods. In § A.1 we give briefly the basic definitions of umbral algebra, and in § A.2 we mention functions of umbrae which are isomorphic to certain power series. In § A.3 we solve (3.8) and (4.6) using umbral calculus.

§ A.1

We shall follow the development of umbral calculus given by E. T. Bell[9]. We give the definitions for an algebra of umbrae arising from uni-suffixed scalars. Umbral algebras may be similarly defined for umbrae arising from multi-suffixed scalars. We shall make a few comments at the end of § A.1.

In the following, the field of scalars we use will be $\mathbb{R}$. The sign $\equiv$ will be used to express both definitions and identity as in algebra. Scalars are denoted by $\alpha, \beta, \ldots$ or by small Latin letters with non-negative suffixes denoted by Latin capitals, for example, x_N, $(N=0,1,\ldots)$.

Definition A.1 If x_M $(M=0,1,\ldots)$ are any scalars, the ordered array $\langle x_M \rangle_M^\infty$ is denoted by x and called an umbra. The M+1 th element of x is x_M and, in our system, $x^M \equiv x_M$.

We now proceed to defining operations on umbrae which will make the set of all umbrae a vector space over $\mathbb{R}$.

Definition A.2 Two umbrae, x and y are said to be equal if and only if

$$x^M \equiv x_M = y_M \equiv y^M \quad \text{for all } M.$$

Equality of umbrae is easily seen to be reflexive, symmetric and transitive.

<u>Definition A.3</u> The scalar product of α and an umbra x is denoted by juxtaposition αx. We will use $\{\cdot\}_M$ to denote the Mth element of an umbra. We define αx by

$$\{\alpha x\}_M = \alpha x_M \equiv \alpha x^M \quad \text{for all } M.$$

<u>Definition A.4</u> The addition of umbrae, say of $\alpha a, \ldots, \xi x$, is defined by

$$\{\alpha a + \ldots + \xi x\}_M = \alpha a_M + \ldots + \xi x_M \; .$$

It is easy to see that the set of all umbrae is a vector space over the scalar field $\mathcal{R}$. Further, it is easy to see that the umbrae under the operations defined so far, are isomorphic to the vector space of formal power series.

The following is our motivation for the rest of the definitions we make of operations on umbrae. Consider the power series $\sum_{N=0}^{\infty} x_N \frac{\theta^N}{N!}$ where θ is a formal variable, and the x_N are scalars. If we could simply make a transformation wherein $x_N \to x^N$ the new power series would just be $\exp(x\theta)$. There is, however, no such scalar x which would allow the transformation to be an isomorphism, a necessity if calculations are to be transformed, carried out and the answer transformed back with validity. We can, however, arrange to have our umbrae behave so that the scalars x_N transform to an umbra x to the N^{th} power $(x)^N$. The new power series we transformed to would be a function of the umbra x, called exp $x\theta$. We would want the transformation to be an isomorphism and furthermore we would want exp $x\theta$, as an umbral expression to behave in complete formal analogy with exponents of scalars so that calculations involving umbral exponents would be correspondingly easy. The following definitions of umbral operations are made so that umbrae will behave in the way we want.

The next operation we define in the umbral algebra is a special operation peculiar to umbral algebra which we call taking the N^{th} power of a sum of umbrae. This operation we will define is not related to a multiplication performed N times. In fact we have not, as yet, defined a multiplication for our algebra. The special operation is defined in formal analogy with the multinomial expansion of the N^{th} power of a sum of scalars.

If a, ...,x are umbrae such that no two are equal by (A.2), they are said to be distinct.

Definition A.5 If a, ..., x are T distinct umbrae, then $(\alpha a + \ldots + \xi x)^N$ denotes the scalar p_N ,

$$p_N = (\alpha a + \ldots + \xi x)^N \equiv \Sigma\, Q_{s_1 \ldots s_T}\, \alpha^{s_1} \ldots \xi^{s_T} a_{s_1} \ldots x_{s_T},$$

where $Q_{s_1 \ldots s_T}$ is the multinomial coefficient

$\frac{N!}{s_1! \ldots s_T!}$, and the sum is over all $s_1 + \ldots + s_T = N$.

We note that $\{\alpha a + \ldots + \xi x\}_N \neq (\alpha a + \ldots + \xi x)^N$. We can rewrite definition A.5 using , from A.1, $x^N \equiv x_N$,

$$(\alpha a + \ldots + \xi x)^N = \Sigma\, Q_{s_1 \ldots s_T}\, \alpha^{s_1} \ldots \xi^{s_T}\, a^{s_1} \ldots x^{s_T} .$$

This expression appears exactly similar to the multinomial expansion of a sum of scalars to the N^{th} power.

<u>Definition A.6</u> If in A.5 we replace N by N+R the resulting scalar $(\alpha a + \ldots + \xi x)^{N+R}$ is called the <u>product</u> (denoted by a dot),

$$(\alpha a + \ldots + \xi x)^N \cdot (\alpha a + \ldots + \xi x)^R$$

$$= (\alpha a + \ldots + \xi x)^{N+R} .$$

It follows that this dot product is commutative, associative, and has identity $(\alpha a + \ldots + \xi x)^0$. Ordinary scalar multiplication has no dot, so that $(\alpha a + \ldots + \xi x)^N (\alpha a + \ldots + \xi x)^R \neq (\alpha a + \ldots \xi x)^N \cdot (\alpha a + \ldots + \xi x)^R$. As a convenience of notation we write

$$(\xi x)^N \cdot [(\alpha a)^M + (\beta b)^R + \ldots + (\gamma c)^S]$$

$$\equiv (\xi x)^N \cdot (\alpha a)^M + (\xi x)^N \cdot (\beta b)^R + \ldots + (\xi x)^N \cdot (\gamma c)^S \ .$$

If we have $(\alpha x + \ldots + \alpha x)^N$ where the αx appears A times we will use the notation $(A \cdot \alpha x)^N = (\alpha x + \ldots + \alpha x)^N$, where αx appears A times.

Definition A.5 required the T umbrae to be distinct. The definition A.5 will be said to hold for any T umbrae if the calculation is made as though the umbrae were distinct. For example:

$$(\alpha x + \xi x)^N = \Sigma \, s_1 s_2 \, \alpha^{s_1} x^{s_1} \, \xi^{s_2} x^{s_2}$$

$$= \Sigma \, s_1 s_2 \, \alpha^{s_1} \xi^{s_2} \, x_{s_1} x_{s_2} \quad .$$

We now proceed to define multiplication of umbrae. Many different definitions are possible. We choose the following definition which makes use of the operation of taking the N^{th} power of a sum of umbrae. First, we note that any umbra, $x = (x_0, x_1, \ldots)$ can be written as $x = (\overline{x_0}/0!, \overline{x_1}/1!, \ldots)$ where $\overline{x_N} = x_N \, N!$. Then $\overline{x}$ is used to mean the umbra $(\overline{x_0}, \overline{x_1}, \ldots)$. The form $x = (\overline{x_0}/0!, \overline{x_1}/1!, \ldots)$ we will call the exponential form of the umbra x.

<u>Definition A.7</u> Let x and y be umbrae (written in exponential form). The product of x and y denoted by juxtaposition xy, is the umbra p such that

$$\{p\}_N = \frac{(\bar{x} + \bar{y})^N}{N!} \quad .$$

That is,

$$xy \equiv \left(\frac{(\bar{x} + \bar{y})^0}{0!} , \frac{(\bar{x} + \bar{y})^1}{1!} , \ldots \right) \quad .$$

Umbral multiplication is easily shown to be commutative and associative. Powers under this multiplication can be defined as usual, the A^{th} power of x, that is, xxx...x, A times, will be denoted $x^{(A)}$ to distinguish it from $(x)^A$. Notice that x and y behave under multiplication as though they were sequences of coefficients of (for $\bar{x}$ and $\bar{y}$ scalars) exp $\bar{x}\theta$ and exp $\bar{y}\theta$ respectively. We will make this connection more precise by defining exponents of umbrae in §A.2 .

Comments on Umbral Algebras from multi-suffixed scalars

The algebra of umbra from doubly suffixed scalars can be defined similarly to A.1 through A.7. Instead of A.1 we have

Definition A.8 If x_{MN} , (M=0,1,...), (N=0,1,...) are any scalars, the ordered array $\left\langle x_{M,N} \right\rangle_{M,N=0}$, is denoted by x and called an umbra. The suffixes are treated as ordered pairs. The M+1,N+1 th element of x is $x_{M,N}$ and in the umbral algebra $x^{M,N} \equiv x_{M,N}$.

Umbral equality and the operations of scalar multiplication an umbral addition are extended in the obvious way. The umbrae defined by A.8 are then a vector space over $\mathbb{R}$. If we follow the same development as for the uni-suffixed case, we would define the M,N^{th} power of a sum of umbrae, similarly to A.5.

<u>Definition A.9</u> If a,...,x are T distinct umbrae, then $(\alpha a + \ldots + \xi x)^{M,N}$ denotes the scalar $p_{M,N}$,

$$p_{M,N} = (\alpha a + \ldots + \xi x)^{M,N}$$

$$= \Sigma\, \Sigma Q_{S_1 \ldots S_T}\, Q_{R_1 \ldots R_T} \Big(\alpha^{S_1} \ldots \xi^{S_T}\, \alpha^{R_1} \ldots \xi^{R_T} \cdot a_{S_1,R_1} \ldots x_{S_T,R_T} \Big),$$

where $Q_{S_1 \ldots S_T}$ denotes the multinomial coefficient $\frac{M!}{S_1! \ldots S_T!}$, and $Q_{R_1 \ldots R_T}$ denotes the multinomial coefficient $\frac{N!}{R_1! \ldots R_T!}$, and the double sum is over

$S_1 + \ldots + S_T = M$, and $R_1 + \ldots + R_T = N$.

We note that $\{\alpha a + \ldots + \xi x\}_{M,N} \neq (\alpha a + \ldots + \xi x)^{M,N}$. Definition A.6 and the comments following it are extended in the obvious way.

An umbra $x = \left\langle x_{M,N} \right\rangle_{M,N=0}^{\infty}$ will be written in exponential form by writing, similarly to before, $x = \left\langle \frac{\overline{x_{M,N}}}{M!N!} \right\rangle_{M,N=0}^{\infty}$,

where $\overline{x_{M,N}} = x_{M,N}\ M!N!$. We define the product of x and y similarly to definition A.8.

<u>Definition A.10</u> Let x and y be umbrae written in exponential form. The product of x and y denoted by xy, is the umbra p such that

$$\{p\}_{M,N} = \frac{(\overline{x} + \overline{y})^{M,N}}{M!\ N!} .$$

That is,

$$xy = \left\langle \frac{(\overline{x} + \overline{y})^{M,N}}{M!\ N!} \right\rangle_{M,N=0}^{\infty} .$$

Umbral multiplication is again commutative and associative. The Ath power of x will be denoted $x^{(A)}$.

These comments on the doubly-suffixed case should be sufficient to make it easy to see how umbral algebras of umbra arising from 'higher' multi-suffixed scalars would be defined.

§ A.2 The exponential function (of umbrae)

Definition A.11 Let x be an umbra, $x = (x_0, x_1 \ldots)$. We define the function, exponent of x and a formal scalar variable θ to be

$$\exp x\theta = \sum_{N=0}^{\infty} (x)^N \frac{\theta^N}{N!} = e^{x\theta} ,$$

where e has its usual meaning (2.7...).

We note that the function exp xθ is just a formal power series and so the usual operations in formal power series apply.

Definition A.12 For x and y umbrae,

$$e^{x\theta} \cdot e^{y\theta} = \sum_{N=0}^{\infty} (x + y)^N \frac{\theta^N}{N!} = e^{(x+y)\theta} ,$$

where $(x + y)^N$ is defined by A.5. Generally, for any number of factors on the left

$$e^{\xi x\theta} \ldots e^{\eta y\theta} = e^{(\xi x + \ldots + \eta y)\theta}$$

$$= \sum_{N=0}^{\infty} (\xi x + \ldots + \eta y)^N \frac{\theta^N}{N!} .$$

Powers of exponent functions can be obtained from (A.12) or, more conveniently from

$$[e^{\xi x\theta}]^{A} = e^{(A\cdot\xi x)\theta} = \sum_{N=0}^{\infty} (A\cdot\xi x)^{N} \frac{\theta^N}{N!} .$$

Exponent functions are added according to:

<u>Definition A.13</u> Let x and y be umbrae,

$$e^{x\theta} + e^{y\theta} = e^{\{x+y\}\theta} = \sum_{N=0}^{\infty} \{x + y\}_N \frac{\theta^N}{N!} .$$

This definition extends to any number of summands in the obvious way. From definitions (A.12) and (A.13)

$$e^{\xi x\theta} [e^{\alpha a\theta} + \ldots + e^{\gamma c\theta}] = e^{(\xi x + \{\alpha a + \ldots + \gamma c\})\theta} .$$

<u>Definition A.14</u> The N^{th} derivative with respect to θ, denoted $\frac{d}{d\theta^N}$, of $e^{\xi x\theta}$ is

$$\frac{d}{d\theta^N} e^{\xi x\theta} = \frac{d}{d\theta^N} \sum_{M=0}^{\infty} \xi^M x^M \frac{\theta^M}{M!}$$

$$= \sum_{M=0}^{\infty} \xi^{M+N} x^{M+N} \frac{\theta^M}{M!}$$

$$= \sum_{M=0}^{\infty} (\xi x)^N \cdot (\xi x)^M \frac{\theta^M}{M!}$$

$$= (\xi x)^N \cdot \sum_{M=0}^{\infty} (\xi x)^M \frac{\theta^M}{M!}$$

$$= (\xi x)^N \cdot e^{\xi x \theta} \quad .$$

The above definition is just the definition of the N^{th} derivative of a formal power series. The definitions of umbral calculus are then used in rearranging the quantities appearing until the final form shown above is obtained.

The exponential function of an umbra behaves under differentiation in formal analogy with the ordinary exponential function of a scalar. From (A14) and previous comments we also have

$$\frac{d}{d\theta^N} [e^{\xi x \theta}]^A = (A\cdot\xi x)^N e^{(A\cdot\xi x)\theta} \quad . \tag{A.15}$$

We have already noted that $e^{\xi x\theta}$ is a formal power series expression. Expressions such as $\frac{d}{d\theta}(e^{x\theta})d\theta$, $\log(e^{x\theta})$, and so on can be considered as formal power series expressions and have meanings analagous to the usual scalar expressions.

In practice, special notations such as { }, ( )·( ), $A\cdot\alpha x$ and so on are usually dropped as the meaning of expressions is still clear enough.

Comments on the exponential function (of umbrae from multi-suffixed scalars

Again we make a few comments on umbra from doubly-suffixed scalars and this should point out the way to extend to other multi-suffixed cases.

Definition A.16 If x is an umbra, $x = \left\langle x_{M,N} \right\rangle_{M,N=0}^{\infty}$, we define the exponent of x and two formal scalar variables Θ and Λ by

$$e^{x(\Theta,\Lambda)} = \sum_{M=0}^{\infty} \sum_{N=0}^{\infty} (x)^{M,N} \frac{\Theta^M}{M!} \frac{\Lambda^N}{N!},$$

where e has its usual meaning (2.7...).

As before, the exponent function is a formal power series, in this case a double formal power series. The properties of $e^{x(\Theta,\Lambda)}$ are similarly defined to A.12 through A.15, for example

$$e^{x(\Theta,\Lambda)} e^{y(\Theta,\Lambda)} = e^{(x + y)(\Theta,\Lambda)},$$

and so on.

The definition of derivatives must be modified slightly since we now have two scalar variables.

Definition A.17 The A^{th} derivative with respect to Θ, denoted $\frac{\partial}{\partial \Theta^A}$, of $e^{\xi x(\Theta,\Lambda)}$ is given by

$$\frac{\partial}{\partial \Theta^A} e^{\xi x(\Theta,\Lambda)} = \frac{\partial}{\partial \Theta^A} \sum_{M=O}^{\infty} \sum_{N=O}^{\infty} \xi^{M+N} (x)^{M,N} \frac{\Theta^M}{M!} \frac{\Lambda^N}{N!}$$

$$= \sum_{M=O}^{\infty} \sum_{N=O}^{\infty} \xi^{M+N+A} (x)^{M+A,N} \frac{\Theta^M}{M!} \frac{\Lambda^N}{N!}$$

$$= \sum_{M=O} \sum_{N=O} (\xi x)^{A,O} \cdot (\xi x)^{M,N} \frac{\Theta^M}{M!} \frac{\Lambda^N}{N!}$$

$$= (\xi x)^{A,O} \cdot \sum_{M=O}^{\infty} \sum_{N=O}^{\infty} (\xi x)^{M,N} \frac{\Theta^M}{M!} \frac{\Lambda^N}{N!}$$

$$= (\xi x)^{A,O} \cdot e^{\xi x(\Theta,\Lambda)} \quad .$$

The A^{th} derivative with respect to Λ is defined similarly, that is,

$$\frac{\partial}{\partial \Lambda^A} e^{\ x(\Theta,\Lambda)} = (\xi x)^{O,A} \cdot e^{\xi x(\Theta,\Lambda)} \quad .$$

The comments following (A.15) apply here also (with appropriate modifications).

§ A.3

It is easy to see that the transformation of power series τ:

$\sum_{N=0}^{\infty} x_N \frac{\theta^N}{N} \to \exp x\theta$, where x is an umbra, is an onto isomorphism. That it is one to one and onto is easy. The homomorphism properties are guaranteed by definition (A.13) and definition (A.12).

We shall now solve (3.8) using umbral calculus. Recall that (3.8) was

$$\sum_{N=0}^{\infty} T_{N+1} \frac{x^N}{N} = \sum_{N=0}^{\infty} \sum_{b=0}^{N} \binom{N}{b} C_{b+1} T_{N-b} \frac{x^N}{N}$$

$$= \sum_{N=0}^{\infty} C_{N+1} \frac{x^N}{N} \cdot \sum_{N=0}^{\infty} T_N \frac{x^N}{N} \quad .$$

Under the transformation just mentioned this becomes the umbral equation

$$T\cdot\exp Tx = C\cdot\exp Cx \cdot\exp Tx \quad , \tag{A.18}$$

where x is the formal variable and C and T are umbrae.

We can rearrange (A.18) as

$$\frac{T \exp Tx}{\exp Tx} = C \exp Cx \quad . \tag{A.19}$$

Although we did not go into the formal details of the definitions of the integral of umbral expressions such as exp $x\theta$, these details can be obtained by recalling the definition of integration of formal power series and noting that we want τ to be homomorphic with respect to this operation also. These formal details justify the usual manipulations. We can rewrite (A.19) using (A.14) and integrate both sides of the equation with respect to x from O to x,

$$\int_0^x \frac{\frac{d}{dx}\exp Tx}{\exp Tx}\, dx = \int_0^x \frac{d}{dx}\exp Cx \, dx \; . \tag{A.20}$$

Integrating (A.20) we obtain

$$\log\left(\exp Tx\right)\Big|_0^x = \exp Cx \Big|_0^x \; . \tag{A.21}$$

We can then transform (A.21) using τ^{-1} to

$$\log\left(\sum_{N=0}^{\infty} T_N(y)\, \frac{x^N}{N!}\right) - \log T_0(y)$$

$$= \sum_{N=0}^{\infty} C_N(y)\, \frac{x^N}{N!} - C_0(y) \quad , \tag{A.22}$$

which is the same as equation (3.16) when we note that $T_0(y) = 1$ and $C_0(y) = 0$, and the conclusions are drawn by theorem (3.17).

We can also solve (4.6) using umbral calculus. Recall equation (4.6) was

$$\sum_{M=O}^{\infty} \sum_{N=O}^{\infty} T_{M,N+1} \frac{x^M}{M!} \frac{z^N}{N!}$$

$$= \sum_{M=O}^{\infty} \sum_{N=O}^{\infty} C_{M,N+1} \frac{x^M}{M!} \frac{z^N}{N!} \cdot \sum_{M=O}^{\infty} \sum_{N=O}^{\infty} T_{M,N} \frac{x^M}{M!} \frac{z^N}{N!} ,$$

where x and z are formal variables.

This equation transformed to an umbral equation (the umbral algebra is from doubly-suffixed scalars) becomes

$$T^{O,1} \cdot \exp T(x,z) = C^{O,1} \cdot \exp C(x,z) \cdot \exp T(x,z) \quad , \quad (A.23)$$

where x, z are formal variables, C and T are umbrae.

We can integrate both sides of (A.23) with respect to z from O to z , and use definition A.17 to write

$$\int_O^z \frac{\frac{\partial}{\partial z} \exp T(x,z)}{\exp T(x,z)} \partial z = \int_O^z \frac{\partial}{\partial z} \exp C(x,z) \, \partial z \quad . \quad (A.24)$$

By performing the integration we obtain

$$\log \left(\exp T(x,z) \right) \Big|_O^z = \exp C(x,z) \Big|_O^z \quad . \quad (A.25)$$

The result (A.25) can be transformed using τ^{-1} to

$$\log\left(\sum_{M=0}^{\infty}\sum_{N=0}^{\infty} T_{M,N}(y)\,\frac{x^M}{M!}\,\frac{z^N}{N!}\right) - \log\left(\sum_{M=0}^{\infty} T_{M,0}(y)\,\frac{x^M}{M!}\right)$$

$$= \sum_{M=0}^{\infty}\sum_{N=0}^{\infty} C_{M,N}(y)\,\frac{x^M}{M!}\,\frac{z^N}{N!} - \sum_{M=0}^{\infty} C_{M,0}(y)\,\frac{x^M}{M!}\ , \qquad \text{(A.26)}$$

which is the same as (4.12) if we note that $\log\left(\sum_{M=0}^{\infty} T_{M,0}(y)\,\frac{x^M}{M!}\right)$ $= \log\left(\sum_{M=0}^{\infty} \frac{x^M}{M!}\right) = x$, and $\sum_{M=0}^{\infty} C_{M,0}(y)\,\frac{x^M}{M!} = x$, and the conclusions are drawn by theorem (4.14).

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